Coherence for Categorified Operadic Theories

by

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To my father

$$1 + 2 + \cdots + n = \frac{n}{2}(n + 1)$$

Abstract

Given an algebraic theory which can be described by a (possibly symmetric) operad P, we propose a definition of the *weakening* (or *categorification*) of the theory, in which equations that hold strictly for P-algebras hold only up to coherent isomorphism. This generalizes the theories of monoidal categories and symmetric monoidal categories, and several related notions defined in the literature. Using this definition, we generalize the result that every monoidal category is monoidally equivalent to a strict monoidal category, and show that the "strictification" functor has an interesting universal property, being left adjoint to the forgetful functor from the category of strict P-categories to the category of weak P-categories. We further show that the categorification obtained is independent of our choice of presentation for P, and extend some of our results to many-sorted theories, using multicategories.

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Declaration

I declare that this thesis is my own original work, except where credited to others. This thesis does not include work forming part of a thesis presented for another degree.

Chapter 0

Introduction

Many definitions exist of categories with some kind of "weakened" algebraic structure, in which the defining equations hold only up to coherent isomorphism. The paradigmatic example is the theory of weak monoidal categories, as presented in [ML98], but there are also definitions of categories with weakened versions of the structure of groups [BL04], Lie algebras [BC04], crossed monoids [Age02], sets acted on by a monoid [Ost03], rigs [Lap72], vector spaces [KV94] and others. A general definition of such categories-with-weakened-structure is obviously desirable, but hard in the general case. In this thesis, we restrict our attention to the case of theories that can be described by (possibly symmetric) operads, and present possible definitions of weak P-category and weak P-functor for any symmetric operad P. We show that this definition is independent (up to equivalence) of our choice of presentation for P; this generalizes the equivalence of classical and unbiased monoidal categories. In support of our definition, we present a generalization of Joyal and Street's result from [JS93] that every weak monoidal category is monoidally equivalent to a strict monoidal category: this holds straightforwardly when P is a plain operad. This generalization includes the classical theorem that every symmetric monoidal category is equivalent via symmetric monoidal functors and transformations to a symmetric monoidal category whose associators and unit maps are identities.

The idea is to consider the strict models of our theory as algebras for an operad, then to obtain the weak models as (strict) algebras for a weakened version of that operad (which will be a **Cat**-operad). In particular, we do not make use of the pseudo-algebras of Blackwell, Kelly and Power, for which see [BKP89]. Their definition is related to ours in the non-symmetric case, however: we explore the connections in Chapter 6. We weaken the operad using a similar approach to that used in Penon's definition of n-category:

see [Pen99], or [CL04] for a non-rigorous summary.

In Chapters 1, 2 and 3, we review some essential background material on theories, operads and factorization systems. Most of this is well-known, and only one result (in Section 2.8) is new. In Chapter 4, we present our definitions of weak P-category, weak P-functor and P-transformation. We start with a naïve, syntactic definition that is only effective for strongly regular (plain-operadic) theories. We then re-state this definition using the theory of factorization systems, which allows us to apply it to the more general symmetric operads. Section 4.6 uses this definition to explicitly calculate the categorification of the theory of commutative monoids with their standard signature, and shows that this is exactly the classical theory of symmetric monoidal categories. In Chapter 5, we treat the problem of different presentations of a given operad: we use this to prove that the weakening of a given theory is independent of the choice of presentation. We also prove some theorems about strictification of weak P-categories. In Chapter 6, we compare our approach to other approaches to categorification which have been proposed in the literature.

Material in this thesis has appeared in two previous papers: the material on strictification for strongly regular theories was in my preprint [Gou06], and the material on signature-independence was in my paper [Gou07], which was presented at the 85th Peripatetic Seminar on Sheaves and Logic in Nice in March 2007, and at CT 2007 in Carvoeiro, Portugal.

0.1 Remarks on notation

Throughout this thesis, the set $\mathbb{N}$ of natural numbers is taken to include 0. We shall occasionally adopt the $_\bullet$ notation from chain complexes and write, for instance, $p_\bullet$ for a finite sequence $p_1, \ldots, p_n$ and $p_\bullet^\bullet$ for a double sequence. We shall use the notation $\underline{n}$ to refer to the set $\{1, \ldots, n\}$ for all $n \in \mathbb{N}$: the set $\underline{0}$ is the empty set. We shall use the symbol 1 to refer to terminal objects of categories and identity arrows, as well as to the first nonzero natural number; it is my hope that no confusion results.

Chapter 1

Theories

The first step will be to obtain a mathematical description of the notion of an algebraic theory, of which the familiar theories of groups, rings, modules etc. are examples. In this chapter, we present some standard ways of doing this, and prove that they are equivalent. The most convenient description for our purposes will be the notion of *clone*, which appears to have been introduced by Philip Hall in unpublished lecture notes in the 1960s, and may be found on [Coh65] page 132, under the name "abstract clone". The treatment here follows [Joh94]. The remainder of the material in this chapter is all well-known, and may be found in e.g. [Bor94] chapters 3 and 4, or [AR94] chapter 3.

In the next chapter, we shall describe *operads*, which allow us to capture certain algebraic theories in an especially simple way, suitable for categorification, and we shall show how operads relate to the clones described in this chapter.

1.1   Syntactic approach

The most traditional way of formalizing algebraic theories is syntactic. In this approach, we abstract from the standard "operations plus equations" description (used to describe e.g. the theory of groups) to create **presentations of algebraic theories**, and define a notion of an **algebra** for a presentation.

Definition 1.1.1. A **signature** Φ is an object of $\mathbf{Set}^{\mathbb{N}}$.

In other words, a signature is a sequence of sets $\Phi_0, \Phi_1, \Phi_2, \ldots$.

Fix a countably infinite set $X = \{x_1, x_2, \ldots, \}$, whose elements we call **variables**. Throughout, let Φ be a signature.

Definition 1.1.2. Let $n \in \mathbb{N}$. An n-**ary** Φ-**term** is defined by the following inductive clauses:

- $x_1, x_2, \ldots, x_n$ are n-ary terms.

- If $\phi \in \Phi_m$ and $t_1, \ldots, t_m$ are n-ary terms, then $\phi(t_1, \ldots, t_m)$ is an n-ary term.

A Φ-**term** is an n-ary Φ-term for some $n \in \mathbb{N}$.

Definition 1.1.3. Let t be an n-ary Φ-term. Then $\mathrm{var}(t)$ is the sequence of elements of $\{x_1, \ldots, x_n\}$ given as follows:

- $\mathrm{var}(x_i) = (x_i)$,

- $\mathrm{var}(\phi(t_1, \ldots, t_n)) = \mathrm{var}(t_1) ++ \mathrm{var}(t_2) ++ \ldots ++ \mathrm{var}(t_n)$,

where $++$ is concatenation.

Definition 1.1.4. Let t be an n-ary Φ-term. Then $\mathrm{supp}(t)$, the **support of** t, is the subset of $\{x_1, \ldots, x_n\}$ given as follows:

- $\mathrm{supp}(x_i) = \{x_i\}$,

- $\mathrm{supp}(\phi(t_1, \ldots, t_n)) = \mathrm{supp}(t_1) \cup \mathrm{supp}(t_2) \cup \ldots \cup \mathrm{supp}(t_n)$,

Definition 1.1.5. Let t be an n-ary Φ-term, with $\mathrm{var}(t) = (x_{i_1}, \ldots, x_{i_m})$. The **labelling function** $\mathrm{label}(t)$ of t is the function $\underline{m} \to \underline{n}$ sending j to i_j.

Definition 1.1.6. An n-**ary** Φ-**equation** is a pair (s, t) of n-ary Φ-terms. A Φ-**equation** is an n-ary Φ-equation for some $n \in \mathbb{N}$.

Definition 1.1.7. An n-ary term t is **linear** if $\mathrm{label}(t)$ is a bijection, and **strongly regular** if $\mathrm{label}(t)$ is an identity. An equation (s, t) is **linear** if both s and t are linear, and **strongly regular** if both s and t are strongly regular.

In other words, a term is linear if every variable is used exactly once, and strongly regular if every variable is used exactly once in the order $x_1, \ldots, x_n$. Up to trivial relabellings, an equation is linear if every variable is used exactly once on both sides, though not necessarily in the same order: an example is the commutative equation $x_1.x_2 = x_2.x_1$. An equation is strongly regular if every variable is used exactly once in the same order on both sides. An example is the associative equation $x_1.(x_2.x_3) = (x_1.x_2).x_3$, though some

care is needed. Strictly, a Φ-equation is a pair $(n, (s, t))$ where $n \in \mathbb{N}$ and s, t are n-ary Φ-terms. The equation $(3, ((x_1.x_2).x_3, x_1.(x_2.x_3)))$ is strongly regular, but the equation $(4, ((x_1.x_2).x_3, x_1.(x_2.x_3)))$ is not.

Classically, an n-ary equation (s, t) is **regular** if label(t) and label(s) are surjections: however, we will not consider regular equations further. The term "linear" is borrowed from linear logic, and the term "strongly regular" is due to Carboni and Johnstone (from [CJ95]).

Definition 1.1.8. A **presentation of a (one-sorted) algebraic theory** is

- a signature Φ,

- a set E of Φ-equations.

Elements of Φ_n are called (n-ary) **generating operations**.

Definition 1.1.9. Let $P = (\Phi, E)$ be a presentation of an algebraic theory. P is **linear** if every equation in E is linear, and **strongly regular** if every equation in E is strongly regular.

We will return to the consideration of linear and strongly regular presentations once we have defined operads.

Definition 1.1.10. Let Φ be a signature. An **algebra** for Φ is

- a set A,

- for each n-ary operation ϕ, a map $\phi_A : A^n \to A$. These are called the **primitive operations** of the algebra A.

Let Φ be a signature, and A a Φ-algebra. Each n-ary Φ-term t gives rise to an n-**ary derived operation** $t_A : A^n \to A$, defined recursively as follows:

- if $t = x_i$, then t_A is projection onto the ith factor,

- if $t = \phi(t_1, \ldots, t_m)$, then t_A is the composite

$$A^n \xrightarrow{((t_1)_A, \ldots, (t_m)_A)} A^m \xrightarrow{\phi_A} A \ .$$

Let $\mathrm{term}_n \, \Phi$ denote the set of n-ary derived operations over Φ. Then $\mathrm{term} \, \Phi$ is a signature for every signature Φ. A morphism of signatures $f : \Phi \to \Psi$ induces a map $\bar{f} : \mathrm{term} \, \Phi \to \mathrm{term} \, \Psi$. Indeed, term is an endofunctor on $\mathbf{Set}^{\mathbb{N}}$, and in Section 2.8 we shall show that it is actually a monad.

Definition 1.1.11. Let $P = (\Phi, E)$ be a presentation of an algebraic theory. A P**-algebra** is a Φ-algebra A such that, for every equation (s, t) in E, the derived operations s_A, t_A are equal.

An algebra for Φ is an algebra for $(\Phi, \{\})$. Conversely, every algebra for (Φ, E) is an algebra for Φ.

Definition 1.1.12. Let Φ be a signature, and A and B be Φ-algebras. A **morphism of Φ-algebras** $f : A \to B$ is a map $f : A \to B$ which commutes with every primitive operation:

$$
\begin{array}{ccc}
A^n & \xrightarrow{\ f^n\ } & B^n \\
\phi_A \downarrow & & \downarrow \phi_B \\
A & \xrightarrow{\ f\ } & B
\end{array}
$$

for every $n \in \mathbb{N}$ and every n-ary primitive operation ϕ. If $P = (\Phi, E)$ is a presentation, then a **morphism of P-algebras** is a morphism of Φ-algebras.

By an easy induction, a morphism of Φ-algebras will commute with every derived operation too.

Given a presentation P, there is a category $\mathbf{Alg}(P)$ whose objects are P-algebras and whose arrows are P-algebra morphisms. We shall call a category $\mathcal{C}$ a **variety of algebras** (or simply a **variety**) if $\mathcal{C}$ is isomorphic to $\mathbf{Alg}(P)$ for some presentation P.

We will need to consider **closures** of sets of equations; the idea is that the closure of E contains the members of E and all of their consequences.

Definition 1.1.13. Let t be an n-ary Φ-term, and $t_1, \ldots, t_n$ be Φ-terms. Then the **graft** $t(t_1, \ldots, t_n)$ is the Φ-term defined recursively as follows.

- If $t = x_i$, then $t(t_1, \ldots, t_n) = t_i$.

- If $t = \phi(s_1, \ldots, s_m)$, where $\phi \in \Phi_m$ and $s_1, \ldots, s_m$ are n-ary Φ-terms, then

    $$t(t_1, \ldots, t_n) = \phi((s_1(t_1, \ldots, t_n)), \ldots, s_m(t_1, \ldots, t_n)).$$

Definition 1.1.14. Let Φ be a signature and E be a set of Φ-equations. The **closure** $\bar{E}$ of E is the smallest equivalence relation on term Φ which contains E and is closed under grafting of terms:

- if $(s, t) \in \bar{E}$, then $(s(t_1, \ldots, t_n), t(t_1, \ldots, t_n)) \in \bar{E}$ for all $t_1, \ldots, t_n$.

- if $(s_i, t_i) \in \bar{E}$ for $i = 1, \ldots, n$, then $(t(s_1, \ldots, s_n), t(t_1, \ldots, t_n)) \in \bar{E}$ for all t.

1.2 Clones

Clones attempt to capture theories directly: a clone is to a presentation of an algebraic theory as a group is to a presentation of that group.

Definition 1.2.1. A **clone** K is

- a sequence of sets $K_0, K_1, \ldots$,

- for all $m, n \in \mathbb{N}$, a function $\bullet : K_n \times (K_m)^n \to K_m$,

- for each $n \in \mathbb{N}$ and each $i \in \{1, \ldots, n\}$, an element $\delta_n^i \in K_n$

such that

- for each $f \in K_n$, $g_1, \ldots, g_n \in K_m$, $h_1, \ldots, h_m \in K_p$,

$$f \bullet (g_1 \bullet (h_1, \ldots, h_m), \ldots, g_n \bullet (h_1, \ldots, h_m)) = (f \bullet (g_1, \ldots, g_n)) \bullet (h_1, \ldots, h_m)$$

- for all n, all $i \in 1, \ldots, n$ and all $f_1, \ldots, f_n \in K_m$,

$$\delta_n^i \bullet (f_1, \ldots, f_n) = f_i$$

- for all n and $f \in K_n$,

$$f \bullet (\delta_n^1, \ldots, \delta_n^n) = f$$

Example 1.2.2. Let $\mathcal{C}$ be a finite product category, and A be an object of $\mathcal{C}$. The **endomorphism clone** of A, $\mathrm{End}(A)$, is defined as follows:

- $\mathrm{End}(A)_n = \mathcal{C}(A^n, A)$ for each $n \in \mathbb{N}$,

- for all $n \in \mathbb{N}$ and $i \in \{1, \ldots, n\}$, the map δ_n^i is the projection of A^n onto its ith factor,

- for all $n, m \in \mathbb{N}$, all $f \in \mathrm{End}(A)_n$, and all $g_1, \ldots, g_n \in \mathrm{End}(A)_m$, the morphism $f \bullet (g_1, \ldots, g_n)$ is the composite fh, where h is the unique arrow $A^m \to A^n$ induced by the maps $g_1, \ldots, g_n$ and the universal property of A^n.

Definition 1.2.3. A **morphism** of clones $f : K \to K'$ is a map in $\mathbf{Set}^{\mathbb{N}}$ which commutes with the composition operations and δs.

Definition 1.2.4. Let K be a clone, and $\mathcal{C}$ a finite product category with specified finite powers. An **algebra** for K in $\mathcal{C}$ is an object $A \in \mathcal{C}$ and a morphism of clones $K \to \mathrm{End}(A)$.

Equivalently, an algebra for a clone K in a finite product category $\mathcal{C}$ with specified powers is

- an object A of $\mathcal{C}$,

- for each $n \in \mathbb{N}$ and each $k \in K_n$, a morphism $\hat{k} : A^n \to A$

such that

- for all $n \in \mathbb{N}$ and all $i \in \{1, \ldots n\}$, the morphism $\widehat{\delta_n^i}$ is the projection of A^n onto its ith factor;

- for all $n, m \in \mathbb{N}$, all $f \in K_n$, and all $g_1, \ldots, g_n \in K_m$, the diagram
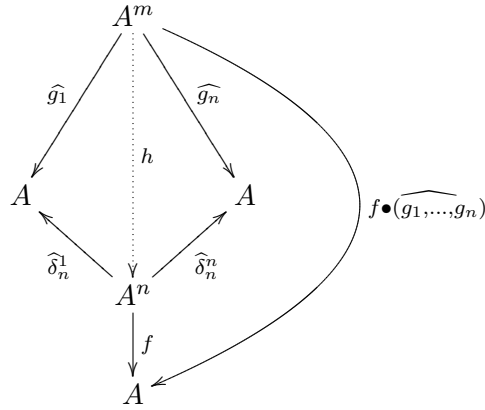
commutes, where h is the unique arrow induced by the universal property of A^n.

Definition 1.2.5. Let A and B be algebras for a clone K in a finite product category $\mathcal{C}$ with specified finite powers. A **morphism** of algebras $A \to B$ is a morphism $F : A \to B$ in $\mathcal{C}$ such that the diagram

commutes for all $n \in \mathbb{N}$ and all $k \in K_n$.

Algebras for a clone and their morphisms form a category: we call this category $\mathbf{Alg}_{\mathcal{C}}(K)$, or $\mathbf{Alg}(K)$ in the case where $\mathcal{C} = \mathbf{Set}$.

Clones can be enriched in any finite product category $\mathcal{V}$ in an obvious way: the sequence of sets $K_0, K_1, \ldots$ becomes a sequence of objects of $\mathcal{V}$, and so on.

1.3 Lawvere theories

Lawvere theories are a particularly elegant approach to describing algebraic theories, introduced by Lawvere in his thesis [Law63]. Like a clone, a Lawvere theory (sometimes called a **finite product theory**) is an object that represents the semantics of the theory directly; in Lawvere theories, the data are encoded into a category. Algebras for the theory are then certain functors from the Lawvere theory to **Set**.

Definition 1.3.1. A **Lawvere theory** is a category $\mathcal{T}$ whose objects form a denumerable set $\{\mathbf{0}, \mathbf{1}, \mathbf{2}, \dots\}$, such that $\mathbf{n}$ is the n-th power of $\mathbf{1}$. A **morphism** of Lawvere theories $\mathcal{T} \to \mathcal{S}$ is an identity-on-objects functor $\mathcal{T} \to \mathcal{S}$ which preserves projection maps. The category of Lawvere theories and their morphisms is called **Law**. An **algebra** for $\mathcal{T}$ is a functor $F : \mathcal{T} \to \mathbf{Set}$ which preserves finite products. A **morphism** of algebras is a natural transformation. The **category of $\mathcal{T}$-algebras** is the full subcategory of $[\mathcal{T}, \mathbf{Set}]$ whose objects are finite-product-preserving functors.

Lawvere theories encode algebraic theories by storing the n-ary operations of the theory as morphisms $\mathbf{n} \to \mathbf{1}$.

We can consider algebras for Lawvere theories in categories other than **Set**: an algebra for a Lawvere theory $\mathcal{T}$ in a finite product category $\mathcal{C}$ is just a finite-product-preserving functor $\mathcal{T} \to \mathcal{C}$. This captures our usual notions of, for instance, topological groups: a topological group is just an algebra for the Lawvere theory of groups in the category **Top**. Much the same could be said for clones and presentations, of course, but in this case the definition is especially economical.

We may generalize this definition as follows:

Definition 1.3.2. Let S be a set. An S-**sorted finite product theory** is a small finite product category whose underlying monoidal category is strict and whose monoid of objects is the free monoid on S. Elements of S will be called **sorts**. Algebras and morphisms of algebras are defined as above.

1.4 Finitary monads

Recall that a **monad** on a category $\mathcal{C}$ is a monoid object in the category $[\mathcal{C}, \mathcal{C}]$ of endofunctors on $\mathcal{C}$. Concretely, a monad is a triple (T, μ, η) where

- $T : \mathcal{C} \to \mathcal{C}$ is a functor,

- $\mu : T^2 \to T$ is a natural transformation,

- $\eta : 1_\mathcal{C} \to T$ is a natural transformation,

and μ, η satisfy coherence axioms which are analogues of the usual associativity and unit laws for monoids, namely

$$\begin{array}{ccc}
& T^3 & \\
{}^{\mu T}\swarrow & & \searrow^{T\mu} \\
T^2 & & T^2 \\
{}_{\mu}\searrow & & \swarrow_{\mu} \\
& T &
\end{array} \tag{1.1}$$

$$\begin{array}{ccccc}
T & \xrightarrow{T\eta} & T^2 & \xleftarrow{\eta T} & T \\
& {}_{1_T}\searrow & \downarrow^{\mu} & \swarrow_{1_T} & \\
& & T & &
\end{array} \tag{1.2}$$

We shall often abuse notation and refer to the monad (T, μ, η) as simply T.

Definition 1.4.1. Let $(T_1, \mu_1, \eta_1), (T_2, \mu_2, \eta_2)$ be monads on a category $\mathcal{C}$. A **morphism of monads** $(T_1, \mu_1, \eta_1) \to (T_2, \mu_2, \eta_2)$ is a natural transformation $\alpha : T_1 \to T_2$ such that the diagrams

$$\begin{array}{ccc}
T_1^2 & \xrightarrow{\mu_1} & T_1 \\
{}_{\alpha * \alpha}\downarrow & & \downarrow^{\alpha} \\
T_2^2 & \xrightarrow{\mu_2} & T_2
\end{array} \tag{1.3}$$

$$\begin{array}{ccc}
1 & \xrightarrow{\eta_1} & T_1 \\
& {}_{\eta_2}\searrow & \downarrow^{\alpha} \\
& & T_2
\end{array} \tag{1.4}$$

commute.

Monads on $\mathcal{C}$ and monad morphisms form a category $\mathbf{Mnd}(\mathcal{C})$. This notion (or rather, a 2-categorical version) was introduced and studied by Street in [Str72].

Definition 1.4.2. A category $\mathcal{C}$ is **filtered** if every finite diagram in $\mathcal{C}$ admits a cocone.

Equivalently, $\mathcal{C}$ is filtered if:

- $\mathcal{C}$ is nonempty;

- for every pair of parallel arrows $A \underset{g}{\overset{f}{\rightrightarrows}} B$ in $\mathcal{C}$, there is an arrow $h : B \to C$ such that $hf = hg$;

- for every pair of objects A, B, there is an object C and arrows

$$
\begin{array}{c}
A \\
\quad \searrow f \\
\qquad C \\
\quad \nearrow g \\
B
\end{array}
$$
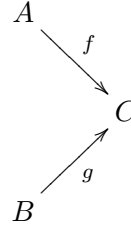
Filteredness generalizes the notion of directedness for posets (a **directed poset** is a poset in which every finite subset has an upper bound). A filtered category which is also a poset is precisely a directed poset.

Definition 1.4.3. A **filtered colimit** in a category $\mathcal{C}$ is the colimit of a diagram $D : \mathbb{I} \to \mathcal{C}$, where $\mathbb{I}$ is a filtered category.

Theorem 1.4.4. *Every object in* **Set** *is a filtered colimit of finite sets.*

Proof. Let $X \in$ **Set**, and consider the subcategory $\mathbb{I}$ of **Set** whose objects are finite subsets of X and whose morphisms are inclusions. This is a directed poset, and thus a filtered category. X is the colimit of the inclusion of $\mathbb{I}$ into **Set**. $\qquad\square$

Theorem 1.4.5. *Let $\mathbb{I}$ be a small category. Colimits of shape $\mathbb{I}$ in* **Set** *commute with all finite limits iff $\mathbb{I}$ is filtered.*

Proof. See [MLM92], Corollary VII.6.5. $\qquad\square$

Definition 1.4.6. A functor $F : \mathcal{C} \to \mathcal{D}$ is **finitary** if it preserves filtered colimits.

Definition 1.4.7. A monad (T, μ, η) on $\mathcal{C}$ is **finitary** if T is finitary.

A finitary monad on **Set** is determined by its behaviour on finite sets, in the following sense: since every set X is a filtered colimit of its finite subsets, then TX has to be the colimit of the images under T of the finite subsets of X.

1.5 Equivalences

Let (Φ, E) be a presentation of an algebraic theory. We define $K_{(\Phi,E)}$ to be the clone whose operations are elements of the quotient signature $(\text{term } \Phi)/\bar{E}$, with composition given by grafting, and $\delta_n^i = x_i$ for all $i, n \in \mathbb{N}$. By definition of $\bar{E}$, grafting gives a well-defined family of composition functions on $K_{(\Phi,E)}$. Conversely, given a clone K, we may define a

presentation of an algebraic theory (Φ_K, E_K), by taking $(\Phi_K)_n = K_n$ for all $n \in \mathbb{N}$, and for all $n, m \in \mathbb{N}$, all $k \in K_n$ and all $k_1, \ldots, k_n \in K_m$, letting E_m contain the equation

$$k(k_1(x_1, \ldots, x_m), \ldots, k_n(x_1, \ldots, x_m)) = k \bullet (k_1, \ldots, k_n)(x_1, \ldots, x_m).$$

**Lemma 1.5.1.** *Let K be a clone. Then $K_{(\Phi_K, E_K)}$ is isomorphic to K.*

Proof. See [Joh94], Lemma 1.7. $\square$

Lemma 1.5.2. *Let (Φ, E) be a presentation of an algebraic theory. Let (Φ', E') be the presentation obtained from the clone $K_{(\Phi, E)}$. Then the category $\mathbf{Alg}(\Phi, E)$ is isomorphic to the category $\mathbf{Alg}(\Phi', E')$*

Proof. See [Joh94], Lemma 1.8. $\square$

Definition 1.5.3. Let K be a clone. We say that K is **strongly regular** (resp. **linear**) if there exists a strongly regular (resp. linear) presentation P such that $K = K_{(\Phi, E)}$.

Given a clone K, we construct a Lawvere theory $\mathcal{T}_K$ for which $\mathcal{T}_K(\mathbf{n}, \mathbf{m}) = (K_n)^m$. Suppose $f = (f_1, \ldots, f_m) \in \mathcal{T}_K(\mathbf{n}, \mathbf{m})$ and $g = (g_1, \ldots, g_p) \in \mathcal{T}_K(\mathbf{m}, \mathbf{p})$, then the composite gf is $(g_1 \bullet (f_1, \ldots, f_m), \ldots, g_p \bullet (f_1, \ldots, f_m))$. By the axioms for a clone, this is a category, with the identity map on $\mathbf{n}$ being $(\delta_n^1, \ldots, \delta_n^n)$. It remains to show that $\mathbf{n}$ is the nth power of $\mathbf{1}$ for every $n \in \mathbb{N}$. The ith projection of $\mathbf{n}$ onto 1 is evidently δ_n^i: we must show that these have the requisite universal property. Take $m, n \in \mathbb{N}$, and n maps $f_1, \ldots, f_n : \mathbf{m} \to \mathbf{1}$ in $\mathcal{T}_K$. The diagram
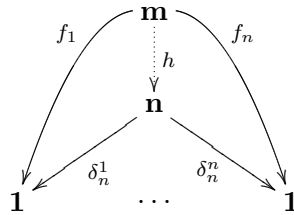
commutes if and only if $h = (f_1, \ldots, f_n)$, and hence $\mathbf{n}$ is indeed the nth power of $\mathbf{1}$, and so $\mathcal{T}_K$ is a Lawvere theory.

The Lawvere theories so constructed evidently respect isomorphisms of clones. Furthermore, the diagram
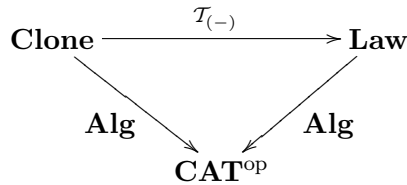
commutes up to equivalence:

Theorem 1.5.4. *Let K be a clone. Then $\mathbf{Alg}(K) \simeq \mathbf{Alg}(\mathcal{T}_K)$.*

Proof. Let A be a K-algebra. We define a $\mathcal{T}_K$-algebra F_A as follows:

- $F_A \mathbf{n} = A^n$ for all $n \in \mathbb{N}$;

- If $k \in \mathcal{T}_K(\mathbf{n}, 1) = K_n$, then $F_A k = \hat{k}$;

- if $(k_1, \ldots, k_n) : \mathbf{m} \to \mathbf{n}$ in $\mathcal{T}_K$, then $F_A(k_1, \ldots, k_n)$ is the unique arrow $A^m \to A^n$ such that the diagram

$$
\begin{array}{ccc}
 & A^m & \\
\widehat{k_1} \swarrow & \downarrow \quad \widehat{k_n} & \\
 & A^n & \\
\swarrow & \widehat{\delta_n^1} \quad \widehat{\delta_n^n} & \searrow \\
A & \cdots & A
\end{array}
$$

commutes.

Let $f : A \to B$ be a morphism of K-algebras. Then the diagram

$$
\begin{array}{ccc}
A^n & \xrightarrow{f^n} & B^n \\
\hat{k} \downarrow & & \downarrow \hat{k} \\
A & \xrightarrow{f} & B
\end{array}
$$

commutes for all $n \in \mathbb{N}$ and all $k \in K_n$. By the universal property of B^m, the diagram

$$
\begin{array}{ccc}
A^n & \xrightarrow{f^n} & B^n \\
F_A(k_1,\ldots,k_m) \downarrow & & \downarrow F_B(k_1,\ldots,k_m) \\
A^m & \xrightarrow{f^m} & B^m
\end{array}
$$

commutes for all $n, m \in \mathbb{N}$ and all $(k_1, \ldots, k_m) : \mathbf{n} \to \mathbf{m}$ in $\mathcal{T}_K$. Hence $F_f = (f^n)_\mathbf{n}$ is a natural transformation $F_A \to F_B$, and hence a morphism of $\mathcal{T}_K$-algebras. This defines a functor $F_{(-)} : \mathbf{Alg}(K) \to \mathbf{Alg}(\mathcal{T}_K)$; we wish to show that it is an equivalence.

For every $\mathcal{T}_K$-algebra G, we may define a K-algebra A by setting $A = G\mathbf{1}$ and $\hat{k} = G(\mathbf{n} \xrightarrow{k} \mathbf{1})$ for all $k \in K_n$ and all $n \in \mathbb{N}$. Then F_A is isomorphic as a $\mathcal{T}_K$-algebra to G, and hence the functor $F_{(-)} : \mathbf{Alg}(K) \to \mathbf{Alg}(\mathcal{T}_K)$ is essentially surjective on objects. We shall show further that it is full and faithful. Let A and B be K-algebras, and let $\alpha_n : F_A \to F_B$ be a morphism between their associated $\mathcal{T}_K$-algebras. Since the diagram

$$
\begin{array}{ccc}
A^n & \xrightarrow{\alpha_n} & B^n \\
\widehat{\delta_n^i} \downarrow & & \downarrow \widehat{\delta_n^i} \\
A & \xrightarrow{\alpha_1} & B
\end{array}
$$

commutes for all $n \in \mathbb{N}$ and all $i \in \{1, \ldots, n\}$, it must be the case that $\alpha_n = \alpha_1^n$ for all n. Hence, the diagram

$$
\begin{array}{ccc}
A^n & \xrightarrow{\alpha_1^n} & B^n \\
\widehat{k} \downarrow & & \downarrow \widehat{k} \\
A & \xrightarrow{\alpha_1} & B
\end{array}
$$

must commute for all $n \in \mathbb{N}$ and all $k \in K_n$. So α_1 is a K-algebra morphism, and $\alpha_n = F_{\alpha_1}$. Hence $F_{(-)}$ is full. Suppose $F_f = F_g$; then $(F_f)_{\mathbf{1}} = (F_g)_{\mathbf{1}}$, so $f = g$. Hence $F_{(-)}$ is faithful; and hence it is an equivalence of categories. $\qquad \square$

Given a Lawvere theory $\mathcal{T}$, we can construct a clone $K_{\mathcal{T}}$, as follows:

- Let $(K_{\mathcal{T}})_n = \mathcal{T}(\mathbf{n}, \mathbf{1})$ for all $n \in \mathbb{N}$.

- For all $n, m \in \mathbb{N}$, all $f \in (K_{\mathcal{T}})_n$ and all $g_1, \ldots, g_n \in (K_{\mathcal{T}})_m$, let $f \bullet (g_1, \ldots, g_n) = f \circ (g_1 + \cdots + g_n) \circ \Delta$, where $(g_1 + \cdots + g_n)$ is the unique map $\mathbf{mn} \to \mathbf{n}$ in $\mathcal{T}$ such that the diagram
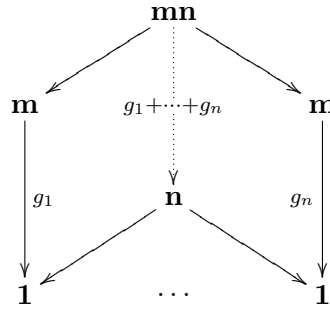
  commutes, and $\Delta : \mathbf{m} \to \mathbf{mn}$ is the diagonal map (or equivalently, the image of the codiagonal function $\mathbf{mn} \to \mathbf{m}$ under the contravariant embedding of $\mathbb{F}$ into $\mathcal{T}$).

- For all $n \in \mathbb{N}$ and all $i \in 1, \ldots, n$, let δ_n^i be the ith projection $\mathbf{n} \to \mathbf{1}$.

This extends to a functor $K_{(-)} : \mathbf{Law} \to \mathbf{Clone}$, as follows: given Lawvere theories $\mathcal{T}_1$ and $\mathcal{T}_2$, and a morphism of Lawvere theories $F : \mathcal{T}_1 \to \mathcal{T}_2$, let K_F be the map of signatures sending $k \in (K_{\mathcal{T}_1})_n = \mathcal{T}_1(\mathbf{n}, \mathbf{1})$ to $Fk \in (K_{\mathcal{T}_2})_n = \mathcal{T}_2(\mathbf{n}, \mathbf{1})$. Since F is a functor, and thus commutes with composition in $\mathcal{T}_1, \mathcal{T}_2$, then K_F must commute with composition in $K_{\mathcal{T}_1}$ and $K_{\mathcal{T}_2}$. Since F preserves finite products, it commutes with the projection maps in $K_{\mathcal{T}_1}$ and $K_{\mathcal{T}_2}$. Thus, K_F is a morphism of clones.

Theorem 1.5.5. *The functor $K_{(-)}$ is pseudo-inverse to the functor $\mathcal{T}_{(-)}$.*

Proof. Since every object in a Lawvere theory is a copower of $\mathbf{1}$, a Lawvere theory $\mathcal{T}$ is entirely determined (up to isomorphism) by the hom-sets $\mathcal{T}(\mathbf{n}, \mathbf{1})$, and thus by $K_{\mathcal{T}}$. The theorem follows straightforwardly. $\qquad\square$

Given a Lawvere theory $\mathcal{T}$, we construct a monad (T, μ, η) on **Set** as follows:

- If X is a set, let $TX = \int^{n \in \mathbb{F}} \mathcal{T}(\mathbf{n}, \mathbf{1}) \times X^n$.

- If $x \in X$, then $\eta(x) = (1, x) \in TX$.

- If $f : \mathbf{n} \to \mathbf{1}$ in T and $(f_i, x_\bullet^i) \in \mathcal{T}(\mathbf{k_i}, \mathbf{1}) \times X^{k_i}$ for $i = 1, \ldots, n$, then

$$\mu(f, ((f_1, x_\bullet^1), \ldots, (f_n, x_\bullet^n))) = (f \circ (f_1 + \cdots + f_n), x_\bullet^\bullet)$$

Theorem 1.5.6. *The monad so constructed is finitary.*

Proof. See [AR94], Theorems 3.18 and 1.5, and Remarks 3.4(4) and 3.6(6). $\qquad\square$

Given a finitary monad T on **Set**, we can construct a Lawvere theory $\mathcal{T}$. Take the full subcategory $\mathbb{F}_T$ of the Kleisli category $\mathbf{Set}_T$ whose objects are finite sets. Now let $\mathcal{T}$ be the skeleton of the dual of $\mathbb{F}_T$. The monad induced by this Lawvere theory is isomorphic to the original monad: see [AR94], Remark 3.17 and Theorem 3.18.

The moral of the above theorems is that presentations, clones, Lawvere theories and finitary monads on **Set** all capture the same notion, and may be used interchangeably. Further, the notion that is captured corresponds to our usual intuitive understanding of equational algebraic theories.

The equivalence between (finitary monads on $\mathcal{C}$) and (monads on $\mathcal{C}$ that may be described by a finitary presentation) may actually be generalized to the case where $\mathcal{C}$ is an arbitrary finitely presentable category: see [KP93].

Chapter 2

Operads

Operads arose in the study of homotopy theory with the work of Boardman and Vogt [BV73], and May [May72]. In that field they are an invaluable tool: [MSS02] describes a diverse range of applications. Independently, multicategories (which are to operads as categories are to monoids) had arisen in categorical logic with the work of Lambek [Lam69]. Multicategories are sometimes called "coloured operads".

We will use multicategories and operads as tools to approach universal algebra: while operads are not as expressive as Lawvere theories, they can be easily extended to be so, and the theories that *can* be represented by operads provide a useful "toy problem" to help us get started.

Informally, categories have objects and arrows, where an arrow has one source and one target; multicategories have objects and arrows with one target but multiple sources (see Fig. 2.1); and operads are one-object multicategories. Multicategories (and thus operads) have a composition operation that is associative and unital.
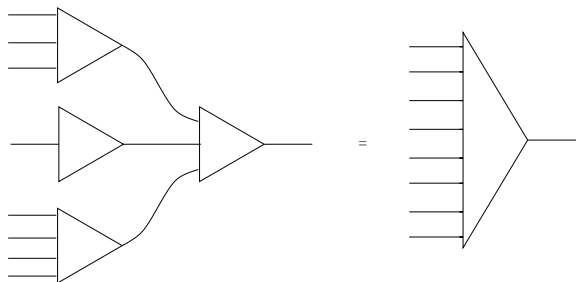

Figure 2.1: Composition in a multicategory

2.1 Plain operads

Definition 2.1.1. A **plain multicategory** (or simply "multicategory") $\mathcal{C}$ consists of the following:

- a collection $\mathcal{C}_0$ of *objects*,

- for all $n \in \mathbb{N}$ and all $c_1, \ldots, c_n, d \in \mathcal{C}_0$, a set of *arrows* $\mathcal{C}(c_1, \ldots, c_n; d)$,

- for all $n, k_1, \ldots k_n \in \mathbb{N}$ and $c_1^1 \ldots, c_{k_n}^n, d_1, \ldots, d_n, e \in \mathcal{C}_0$, a function called *composition*

$$\circ : \mathcal{C}(d_1, \ldots, d_n; e) \times \mathcal{C}(c_1^1, \ldots, c_{k_1}^1; d_1) \times \cdots \times \mathcal{C}(c_1^n, \ldots, c_{k_n}^n; d_n) \to \mathcal{C}(c_1^1, \ldots, c_{k_n}^n; e)$$

- for all $c \in \mathcal{C}$, an *identity arrow* $1_c \in \mathcal{C}(c; c)$

satisfying the following axioms:

- *Associativity:* $f \circ (g_\bullet \circ h_\bullet^\bullet) = (f \circ g_\bullet) \circ h_\bullet^\bullet$ wherever this makes sense (we borrow the $_\bullet$ notation for sequences from chain complexes)

- *Units:* $1 \circ f = f = f \circ (1, \ldots, 1)$ for all f.

A plain multicategory $\mathcal{C}$ is **small** if $\mathcal{C}_0$ forms a set. In line with the definition above, we shall take all our multicategories to be locally small: this restriction is not essential.

We say that an arrow in $\mathcal{C}(c_1, \ldots, c_n; d)$ is n-**ary**, or has **arity** n. We remark that taking $n = 0$ gives us *nullary* arrows. This is in contrast to the definition used by some authors, who do not allow nullary arrows.

Definition 2.1.2. A **morphism** of multicategories $F : \mathcal{C} \to \mathcal{D}$ is a map $F : \mathcal{C}_0 \to \mathcal{D}_0$ together with maps $F : \mathcal{C}(c_1, \ldots, c_n; c) \to \mathcal{D}(Fc_1, \ldots, Fc_n; Fc)$ which commute with $\circ$ and identities. A **transformation** of multicategory maps $\alpha : F \to G$ is a family of arrows $\alpha_c \in \mathcal{D}(Fc; Gc)$, one for each $c \in \mathcal{C}$, satisfying the analogue of the usual naturality squares: for all maps $f : c_1, \ldots, c_k \to c$ in $\mathcal{C}$, we must have

$$\alpha_c \circ Ff = Gf \circ (\alpha_{c_1}, \ldots, \alpha_{c_k})$$

One is tempted to write this last condition as

$$
\begin{array}{ccc}
Fc_1, \ldots, Fc_k & \xrightarrow{\;Ff\;} & Fc \\
{\scriptstyle \alpha_{c_1}, \ldots, \alpha_{c_k}} \downarrow & & \downarrow {\scriptstyle \alpha_c} \\
Gc_1, \ldots, Gc_k & \xrightarrow[\;Gf\;]{} & Gc
\end{array}
$$

but care must be taken: in a general multicategory, $\alpha_{c_1}, \ldots, \alpha_{c_k}$ does not correspond to any single map, as it would in a monoidal category.

Small plain multicategories, their morphisms and their transformations form a 2-category: we shall use the notation **Multicat** for both this 2-category and its underlying 1-category.

To simplify the presentation of our first example, we recall the notion of unbiased monoidal category from [Lei03] section 3.1:

Definition 2.1.3. An **unbiased weak monoidal category** $(\mathcal{C}, \otimes, \gamma, \iota)$ consists of

- a category $\mathcal{C}$,

- for each $n \in \mathbb{N}$, a functor $\otimes_n : \mathcal{C}^n \to \mathcal{C}$ called n-**fold tensor** and written

$$(a_1, \ldots, a_n) \mapsto (a_1 \otimes \cdots \otimes a_n)$$

- for each $n, k_1, \ldots, k_n \in \mathbb{N}$, a natural isomorphism

$$\gamma : \otimes_n \circ (\otimes_{k_1} \times \cdots \times \otimes_{k_n}) \longrightarrow \otimes_{\sum k_i}$$

- a natural isomorphism

$$\iota : 1_A \to \otimes_1$$

satisfying

- associativity: for any triple sequence $a_{\bullet\bullet}^\bullet$ of objects in $\mathcal{C}$, the diagram

$$(((\otimes a_{11}^\bullet) \otimes \cdots \otimes (\otimes a_{1k_1}^\bullet)) \otimes \cdots \otimes ((\otimes a_{n1}^\bullet) \otimes \cdots \otimes (\otimes a_{nk_n}^\bullet)))$$

$$((\otimes a_{1\bullet}^\bullet) \otimes \cdots \otimes (\otimes a_{n\bullet}^\bullet)) \qquad\qquad ((\otimes a_{11}^\bullet) \otimes \cdots \otimes (\otimes a_{nk_n}^\bullet))$$

$$(a_{11}^1 \otimes \cdots \otimes a_{nk_n}^{m_{k_n}})$$

commutes.

- identity: for any $n \in \mathbb{N}$ and any sequence $a_1, \ldots, a_n$ of objects in $\mathcal{C}$, the diagrams

$$(a_1 \otimes \cdots \otimes a_n) \xrightarrow{(\iota \otimes \cdots \otimes \iota)} ((a_1) \otimes \cdots \otimes (a_n))$$

$$\downarrow^{\gamma}$$

$$\xrightarrow{1} (a_1 \otimes \cdots \otimes a_n)$$

$$(a_1 \otimes \cdots \otimes a_n) \xrightarrow{\iota} ((a_1 \otimes \cdots \otimes a_n))$$

$$\downarrow^{\gamma}$$

$$\xrightarrow{1} (a_1 \otimes \cdots \otimes a_n)$$

commute.

Example 2.1.4. Let $\mathcal{C}$ be a locally small unbiased weak monoidal category. The **underlying multicategory** $\mathcal{C}'$ of $\mathcal{C}$ has

- objects: objects of $\mathcal{C}$;

- arrows: $\mathcal{C}'(a_1, \ldots, a_n; b) = \mathcal{C}(a_1 \otimes \cdots \otimes a_n, b)$;

- composition given as follows: if $f_i \in \mathcal{C}'(a_1^i, \ldots, a_{k_i}^i; b_i)$ for $i = 1, \ldots, n$ and $g \in \mathcal{C}'(b_1, \ldots, b_n; c)$, then we define $g \circ (f_1, \ldots, f_n)$ as

$$\bigotimes_{i,j} a_j^i \xrightarrow{\gamma \otimes \cdots \otimes \gamma} \bigotimes_i (\bigotimes_j a_j^i)$$

$$f \circ (g_1, \ldots, g_n) \downarrow \qquad \qquad \downarrow \bigotimes_i f_i$$

$$c \xleftarrow{\quad g \quad} \bigotimes_i b_i$$

Definition 2.1.5. Let M and $\mathcal{C}$ be plain multicategories. An **algebra** for M in $\mathcal{C}$ is a morphism of multicategories $M \to \mathcal{C}$.

Definition 2.1.6. Let M be a plain multicategory, and $\mathcal{C}$ be an unbiased monoidal category. An **algebra** for M in $\mathcal{C}$ is a morphism of multicategories from M to the underlying multicategory of $\mathcal{C}$.

A **plain operad** (or simply "operad") is now a one-object multicategory. Morphisms and transformations of operads are defined as for general multicategories. As before,

we use the notation **Operad** for both the 2-category of operads, morphisms and transformations, and its underlying 1-category. Operads are to multicategories as monoids are to categories: just as with monoids, this allows us to present the theory of operads in a simplified way.

Lemma 2.1.7. *An operad P can be given by the following data:*

- *A sequence $P_0, P_1, \ldots$ of sets*

- *For all $n, k_1, \ldots, k_n \in \mathbb{N}$, a function $\circ : P_n \times P_{k_1} \times \cdots \times P_{k_n} \to P_{\sum k_i}$*

- *An **identity element** $1 \in P_1$*

satisfying the following axioms:

- Associativity: $f \circ (g_\bullet \circ h_\bullet^\bullet) = (f \circ g_\bullet) \circ h_\bullet^\bullet$ *wherever this makes sense*

- Units: $1 \circ f = f = f \circ (1, \ldots, 1)$ *for all f.*

Proof. Using the symbol $*$ for the unique object, let $P_n = P(*, \ldots, *; *)$, where the input is repeated n times. The rest of the conditions follow trivially from the definition of a multicategory. $\qquad\square$

Lemma 2.1.8. *Let P and Q be operads. A morphism $f : P \to Q$ consists of a function $f_n : P_n \to Q_n$ for each $n \in \mathbb{N}$ such that, for all $n, k_1, \ldots, k_n$, the diagram*

$$
\begin{array}{ccc}
P_n \times P_{k_1} \times \cdots \times P_{k_n} & \xrightarrow{\ \circ\ } & P_{\sum k_i} \\
{\scriptstyle f_n \times f_{k_1} \times \cdots \times f_{k_n}} \downarrow & & \downarrow {\scriptstyle f_{\sum k_i}} \\
Q_n \times Q_{k_1} \times \cdots \times Q_{k_n} & \xrightarrow[\ \circ\ ]{} & Q_{\sum k_i}
\end{array}
$$

commutes, and that f_1 preserves the identity object.

If f and g are morphisms of operads from P to Q, then a transformation from f to g is an element $\alpha \in Q_1$ such that $\alpha \circ Fp = Gp \circ (\alpha, \ldots, \alpha)$ for all $n \in \mathbb{N}$ and all $p \in P_n$.

Proof. Trivial. $\qquad\square$

Definition 2.1.9. If a morphism of operads $f : P \to Q$ is such that f_n has some property X for all $n \in \mathbb{N}$, we say that f is **levelwise** X.

Example 2.1.10. Let A be an object of a multicategory $\mathcal{C}$. The **endomorphism operad** of A is the full sub-multicategory $\mathrm{End}(A)$ of $\mathcal{C}$ whose only object is A. In terms of the

description in Lemma 2.1.7, $\mathrm{End}(A)_n$ is the set of n-ary arrows from $A, \ldots, A$ to A. Composition is as in $\mathcal{C}$.

In particular, if $\mathcal{C}$ is the underlying multicategory of some monoidal category $\mathcal{C}'$, then $\mathrm{End}(A)_n = \mathcal{C}'(A \otimes \cdots \otimes A, A)$. This is the case we shall use most frequently.

Example 2.1.11. There is an operad $\mathcal{S}$ for which each $\mathcal{S}_n$ is the symmetric group S_n. Operadic composition is given as follows: if $\sigma \in S_n$, and $\tau_i \in S_{k_i}$ for $i = 1, \ldots, n$, then

$$\sigma \circ (\tau_1, \ldots, \tau_n) : \sum_{i=1}^{j} k_i + m \mapsto \sum_{i:\sigma(i)<\sigma(j+1)} k_i + \tau_{j+1}(m)$$

for all $j \in \{1, \ldots, n\}$ and $m \in \{0, \ldots, k_{j+1} - 1\}$. Informally, the inputs are divided into "blocks" of length $k_1, k_2, \ldots, k_n$, which are then permuted by σ: the elements of each block are then permuted by the appropriate τ_i. For an example, see Figure 2.2.
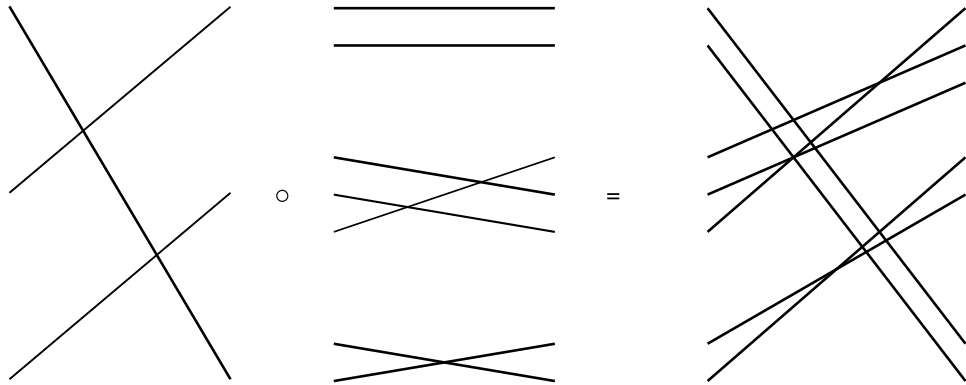

Figure 2.2: Composition in the operad $\mathcal{S}$ of symmetries

Example 2.1.12. There is an operad $\mathcal{B}$ for which each $\mathcal{B}_n$ is the Artin braid group B_n. Composition is analogous to that for $\mathcal{S}$: the inputs are divided into blocks, which are braided, and then the elements of the blocks are braided.

Example 2.1.13. Fix an $m \in \mathbb{N}$. There is an operad LD for which each LD_n is an embedding of n copies of the closed unit disc D_m into D_m. Composition is by gluing – see Figure 2.3.

LD is known as the **little m-discs operad**.

Since we wish to use operads to represent theories, we need to have some way of describing the models of those theories.

Definition 2.1.14. Let P be an operad. An **algebra** for P in a multicategory $\mathcal{C}$ is an object $A \in \mathcal{C}$ and a morphism of operads $(\hat{\ }) : P \to \mathrm{End}(A)$.
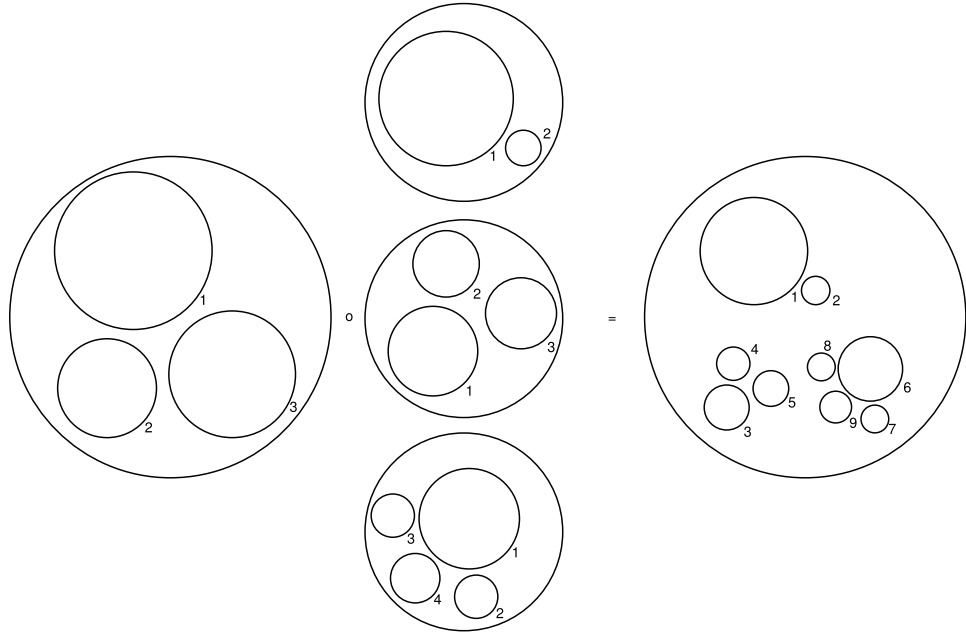
Figure 2.3: Composition in the little 2-discs operad

Where $\mathcal{C}$ is a monoidal category, this is equivalent to requiring an object $A \in \mathcal{C}$, and for each $p \in P_n$ a morphism $\hat{p} : A^{\otimes n} \to A$ such that $\hat{1} = 1_A$ and $\hat{p} \circ (\hat{q}_1 \otimes \cdots \otimes \hat{q}_n) = p \circ \widehat{(q_1 \otimes \cdots \otimes q_n)}$ for all $p, q_1, \ldots, q_n \in P$. A third equivalent definition is, for each $n \in \mathbb{N}$, a map $h_n : P_n \otimes A^{\otimes n} \to A$, such that $h_n(p, h_n(q_\bullet, -)) = h_{\sum k_i}(p \circ q_\bullet, -)$ for all $p \in P_n$, $q_i \in P_{k_i}$, and $h_1(1, -) = 1_A$. We leave the proofs of these equivalences as an easy exercise for the reader, and will make use of whichever formulation is most convenient at the time.

Definition 2.1.15. Let P be a plain operad, and $(A, (\hat{\ }))$ and $(B, (\check{\ }))$ be algebras for P in a multicategory $\mathcal{C}$. A **morphism** of algebras is an arrow $F : A \to B$ in $\mathcal{C}$ such that, for all $n \in \mathbb{N}$, the diagram

$$
\begin{array}{ccc}
P_n & \xrightarrow{\;(\check{\ })\;} & \mathrm{End}(B)_n \\
{\scriptstyle (\hat{\ })} \downarrow & & \downarrow {\scriptstyle -\circ(F,\ldots,F)} \\
\mathrm{End}(A)_n & \xrightarrow{\;F\circ-\;} & \mathcal{C}(A,\ldots,A;B)
\end{array}
$$

commutes.

The definition of morphism may be stated equivalently in terms of any of the three characterizations of algebras given above.

2.2 Symmetric operads

Definition 2.2.1. A **symmetric multicategory** is a multicategory $\mathcal{C}$ and, for every $n \in \mathbb{N}$, every $\sigma \in S_n$, and every $A_1, \ldots A_n, B \in \mathcal{C}$, a map

$$\sigma \cdot - : \quad \mathcal{C}(A_1, \ldots, A_n; B) \quad \longrightarrow \quad \mathcal{C}(A_{\sigma 1}, \ldots, A_{\sigma n}; B)$$
$$f \quad \longmapsto \quad \sigma \cdot f$$

such that

- For each $f \in \mathcal{C}(A_1, \ldots, A_n; B)$, $1 \cdot f = f$.

- For each $\sigma, \rho \in S_n$, and each $f \in \mathcal{C}(A_1, \ldots, A_n; B)$,

$$\rho \cdot (\sigma \cdot f) = (\rho\sigma) \cdot f$$

- For each permutation $\sigma \in S_n$, all objects $A_1^1, \ldots, A_{k_n}^n, B_1, \ldots, B_n, C \in \mathcal{C}$ and all arrows $f_i \in \mathcal{C}(A_1^i, \ldots, A_{k_i}^i; B_i)$ and $g \in \mathcal{C}(B_1, \ldots, B_n; C)$,

$$(\sigma \cdot g) \circ (f_{\sigma 1}, \ldots, f_{\sigma n}) = (\sigma \circ (1, \ldots, 1)) \cdot (g \circ (f_1, \ldots f_n)).$$

- For each $A_1^1, \ldots, A_{k_n}^n, B_1, \ldots, B_n, C \in \mathcal{C}$, $\sigma_i \in S_{k_i}$ for $i = 1, \ldots, n$, and each $f_i \in \mathcal{C}(A_1^i, \ldots, A_{k_i}^i; B_i), g \in \mathcal{C}(B_1, \ldots, B_n; C)$,

$$g \circ (\sigma_1 \cdot f_1, \ldots, \sigma_n \cdot f_n) = (1 \circ (\sigma_1, \ldots, \sigma_n)) \cdot (g \circ (f_1, \ldots, f_n)).$$

where $\sigma \circ (1, \ldots, 1)$ and $1 \circ (\sigma_1, \ldots, \sigma_n)$ are as defined in Example 2.1.11.

This definition is unusual in that the symmetric groups act on the left rather than on the right as is more common: however, this change is essential for our later generalization to finite product multicategories in Section 2.3.

Definition 2.2.2. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be symmetric multicategories. A **morphism** (or **map**) F of symmetric multicategories is a map $F : \mathcal{C}_1 \to \mathcal{C}_2$ of multicategories such that $F(\sigma \cdot f) = \sigma \cdot F(f)$ for all $n \in \mathbb{N}$, all n-ary f in $\mathcal{C}_1$, and all $\sigma \in S_n$.

Definition 2.2.3. Let M and $\mathcal{C}$ be symmetric multicategories. An **algebra** for M in $\mathcal{C}$ is a morphism of symmetric multicategories $M \to \mathcal{C}$.

Definition 2.2.4. A **symmetric operad** is a symmetric multicategory with only one object.

In this case, the definition is equivalent to the following:

Definition 2.2.5. A **symmetric operad** is an operad P together with an action of the symmetric group S_n on each P_n, which is compatible with the operadic composition:

$$
\begin{array}{ccc}
P_n \times \prod P_{k_i} & \xrightarrow{(\sigma \cdot -) \times 1 \times \cdots \times 1} & P_n \times \prod P_{k_i} \\
{\scriptstyle 1 \times \sigma_*}\downarrow & & \\
P_n \times \prod P_{\sigma k_i} & & \\
{\scriptstyle \circ}\downarrow & & \downarrow{\scriptstyle \circ} \\
P_{\sum k_i} & \xrightarrow{(\sigma \circ (1,\ldots,1)) \cdot -} & P_{\sum k_i}
\end{array}
\qquad
\begin{array}{ccc}
P_n \times \prod P_{k_i} & \xrightarrow{1 \times (\rho_1 \cdot -) \times \cdots \times (\rho_n \cdot -)} & P_n \times \prod P_{k_i} \\
{\scriptstyle \circ}\downarrow & & \downarrow{\scriptstyle \circ} \\
P_{\sum k_i} & \xrightarrow{(1 \circ (\rho_1,\ldots,\rho_n)) \cdot -} & P_{\sum k_i}
\end{array}
$$

$$
P_n \overset{1 \cdot -}{\underset{1}{\rightrightarrows}} P_n
$$

Maps of symmetric operads are just maps of symmetric multicategories.

Example 2.2.6. The operad $\mathcal{S}$ of symmetric groups, as given in Example 2.1.11. The action of S_n on $\mathcal{S}_n$ is given by $\sigma \cdot \tau = \tau \sigma^{-1}$.

Example 2.2.7. Let $\mathcal{C}$ be a symmetric multicategory, and $A \in \mathcal{C}$. The **symmetric endomorphism operad** $\mathrm{End}(A)$ of A is the full sub-(symmetric multicategory) of $\mathcal{C}$ whose only object is A.

If $\mathcal{C}$ is the underlying symmetric multicategory of a symmetric monoidal category, then $\mathrm{End}(A)_n = \mathcal{C}(A^{\otimes n}; A)$ for each $n \in \mathbb{N}$, and the actions of the symmetric groups are given by composition with the symmetry maps.

Definition 2.2.8. Let P be a symmetric operad. An **algebra** for P in a multicategory $\mathcal{C}$ is an object A and a map $h : P \to \mathrm{End}(A)$ of symmetric operads. A **morphism** $(A, h) \to (A', h')$ of P-algebras is an arrow $F : A \to B$ in $\mathcal{C}$ such that $h'F = Fh$.

As with plain operads, the definitions of an algebra for a symmetric operad P and of morphisms between those algebras may be stated in several equivalent ways.

2.3 Finite product operads

The definition of categorification in Chapter 4 is couched in terms of operads. To generalize it, therefore, we might generalize the definition of operad so that it is capable of expressing every (one-sorted) algebraic theory.

This generalization is not new: our "finite product operads" were presented by Tronin under the name "**FinSet**-operads". Our Theorem 2.3.12 appears in [Tro02], and Theorem 2.3.13 appears as Theorem 1.2 in [Tro06]. A fuller treatment was given by T. Fiore (who called them "the functional forms of theories") in [Fio06]. Tronin's paper constructs an isomorphism between the category of finite product operads and the category of algebraic clones which commutes with the forgetful functors to $\mathbf{Set}^{\mathbb{N}}$; Fiore's constructs an equivalence between the category of finite product operads and that of Lawvere theories, and also shows that this equivalence preserves the categories of algebras.

Let $\mathbb{F}$ be a skeleton of the category of finite sets and functions, with objects the sets $\underline{0}$, $\underline{1}$, $\underline{2}$, ..., where $\underline{n} = \{1, 2, \ldots, n\}$.

Definition 2.3.1. A **finite product multicategory** is:

- A plain multicategory $\mathcal{C}$;

- for every morphism $f : \underline{n} \to \underline{m}$ in $\mathbb{F}$, and for all objects $C_1, \ldots, C_n, D \in \mathcal{C}$, a function
$$f \cdot - : \mathcal{C}(C_1, \ldots, C_n; D) \to \mathcal{C}(C_{f(1)}, \ldots, C_{f(n)}; D)$$

satisfying the following axioms:

- the $\mathbb{F}$-action is functorial: $f \cdot (g \cdot p) = (f \circ g) \cdot p$, and $\mathrm{id}_{\underline{n}} \cdot p = p$ wherever these equations make sense;

- the $\mathbb{F}$-action and multicategorical composition interact by "combing out":

$$(f \cdot p) \circ (f_1 \cdot p_1, \ldots, f_n \cdot p_n) = (f \circ (f_1, \ldots, f_n)) \cdot (p \circ (p_{f(1)}, \ldots, p_{f(n)}))$$

where $(f \circ (f_1, \ldots, f_n))$ is given as follows:

Let $f : \underline{n} \to \underline{m}$, and $f_i : \underline{k}_i \to \underline{j}_i$ for $i = 1, \ldots, n$. Then

$$
\begin{aligned}
f \circ (f_1, \ldots, f_n) \quad &: \quad & \textstyle\sum k_i \quad &\to \quad \textstyle\sum j_i \\
f \circ (f_1, \ldots, f_n) \quad &: \quad & \left(\textstyle\sum_{i=1}^{p-1} k_{f(i)}\right) + h \quad &\mapsto \quad \left(\textstyle\sum_{i=1}^{f(p)-1} j_i\right) + f_p(h)
\end{aligned}
$$

for all $p \in \{1, \ldots, n\}$ and all $h \in \{1, \ldots, k_{f(p)}\}$. See Figure 2.4. The small specks represent inputs to the arrow that are ignored.

It is now possible to see why we chose to have our symmetries acting on the left in Definition 2.2.1: in this more general case, only a left action is possible.
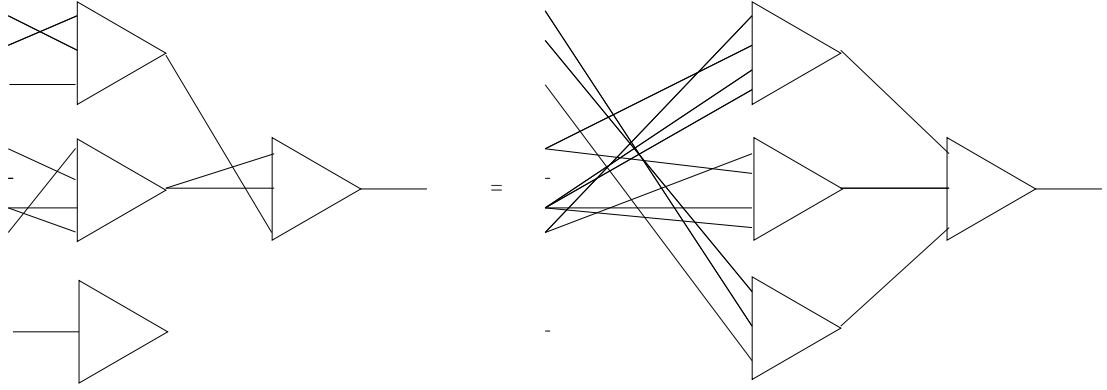
Figure 2.4: "Combing out" the $\mathbb{F}$-action

Definition 2.3.2. A **finite product operad** is a finite product multicategory with only one object.

We will see in Section 2.8 that finite product operads are equivalent in expressive power to Lawvere theories or clones: hence, every finitary algebraic theory provides an example of a finite product operad. As before, the sets P_n contain the n-ary operations in the theory. For illustrative purposes, we work out two examples now:

Example 2.3.3. Let R be a ring, and $P_n = R[x_1, \ldots, x_n]$ (the set of polynomials in n commuting variables over R) for all $n \in \mathbb{N}$. If $p \in P_n$ and $q_i \in P_{k_i}$ for $i = 1, \ldots, n$, then

$$(p \circ (q_1, \ldots, q_n))(x_1, \ldots, x_{\sum_{i=1}^n k_i}) = p(q_1(x_1, \ldots, x_{k_1}), \ldots, q_n(x_{(\sum_{i=1}^{n-1} k_i)+1}, \ldots, x_{\sum_{i=1}^n k_i}))$$

and if $f : \underline{n} \to \underline{m}$, then

$$(f \cdot p)(x_1, \ldots, x_m) = p(x_{f(1)}, \ldots, x_{f(n)})$$

Example 2.3.4. Let P_n be the set of elements of the free commutative monoid on n variables $x_1, \ldots, x_n$. Elements of P_n are in one-to-one-correspondence with elements of $\mathbb{N}^n$. We call the nth component of $p \in P_n$ the **multiplicity of the nth argument**. Composition is defined as follows:

$$\begin{bmatrix} p_1 \\ \vdots \\ p_n \end{bmatrix} \circ \left(\begin{bmatrix} q_1^1 \\ \vdots \\ q_{k_1}^1 \end{bmatrix}, \ldots, \begin{bmatrix} q_1^n \\ \vdots \\ q_{k_n}^n \end{bmatrix} \right) = \begin{bmatrix} p_1 q_1^1 \\ \vdots \\ p_n q_{k_n}^n \end{bmatrix}$$

and if $f : \underline{n} \to \underline{m}$,

$$f \cdot \begin{bmatrix} p_1 \\ \vdots \\ p_n \end{bmatrix} = \begin{bmatrix} \sum_{f(i)=1} p_i \\ \vdots \\ \sum_{f(i)=m} p_i \end{bmatrix}$$

Or, in more familiar notation:

$$
\begin{aligned}
(x_1^{p_1} \ldots x_n^{p_n}) \circ (x_1^{q_1^1} \ldots x_1^{q_{k_1}^1}, \ldots x_1^{q_1^n} \ldots x_{(\sum_{i=1}^{n-1} k_i)+1}^{q_{k_n}^n} \ldots x_{\sum_{i=1}^{n} k_i}^{q_{k_n}^n}) &= x_1^{p_1 q_1^1} x_2^{p_1 q_1^1} \ldots x_{\sum_{i=1}^{n} k_i}^{p_n q_{k_n}^n} \\
f \cdot (x_1^{p_1} \ldots x_n^{p_n}) &= x_{f(1)}^{p_1} \ldots x_{f(n)}^{p_n} \\
&= x_1^{\sum_{f(i)=1} p_i} \ldots x_m^{\sum_{f(i)=m} p_i}
\end{aligned}
$$

Example 2.3.5. Let $\mathcal{C}$ be a finite product category, and A be an object of $\mathcal{C}$. Then there is a finite product operad $\mathrm{End}(A)$, the **endomorphism operad** of A, where $\mathrm{End}(A)_n = \mathcal{C}(A^n, A)$, and $f \cdot p$ is p composed with the appropriate combination of projections to relabel its arguments by f.

Definition 2.3.6. Let M, N be finite product multicategories. A **morphism** $F : M \to N$ consists of

- for each object $m \in M$, an object $Fm \in N$;

- for each $n \in \mathbb{N}$ and all $m_1, \ldots, m_n, m \in M$, a map

$$F_{m_1,\ldots,m_n,m} : M(m_1,\ldots m_n; m) \to N(Fm_1, \ldots, Fm_n; Fm)$$

commuting with the $\mathbb{F}$-action, the unit and composition.

Definition 2.3.7. Let M be a finite product multicategory. An **algebra** for M in a finite product multicategory $\mathcal{C}$ is a map of finite product multicategories $M \to \mathcal{C}$. An **algebra** for M in a finite product category $\mathcal{C}$ is a map of finite product multicategories from M to the underlying finite product multicategory of $\mathcal{C}$. Finite product multicategories and their morphisms form a category called **FP-Multicat**.

In the special case of finite product operads, these definitions are equivalent to the following:

Definition 2.3.8. Let P, Q be finite product operads. A **morphism** $F : P \to Q$ is a sequence of maps $F_i : P_i \to Q_i$ commuting with the $\mathbb{F}$ action, the unit and composition.

Definition 2.3.9. Let P be a finite product operad. An **algebra** for P in a finite product category $\mathcal{C}$ is an object $A \in \mathcal{C}$ and a map of finite product operads $P \to \mathrm{End}(A)$.

Finite product operads and their morphisms form a category called **FP-Operad**.

Example 2.3.10. The algebras in $\mathcal{C}$ for the operad described in Example 2.3.3 are associative R-algebras in $\mathcal{C}$.

Example 2.3.11. The algebras for the operad described in Example 2.3.4 are commutative monoid objects in $\mathcal{C}$.

Theorem 2.3.12. FP-Operad $\cong$ Clone.

Proof. We shall construct a functor $K_{(-)} : \textbf{FP-Operad} \to \textbf{Clone}$, and show that it is bijective on objects, full and faithful.

If P is a finite product operad, let K_P be the following clone:

- $(K_P)_n = P_n$ for all $n \in \mathbb{N}$,

- composition is given by composition in P: if $p \in P_n$ and $p_1, \ldots, p_n \in P_m$, then $p \bullet (p_1, \ldots, p_n) \in (K_P)_m$ is $f \cdot (p \circ (p_1, \ldots, p_n)) \in P_m$, where

$$
\begin{aligned}
f : \quad \underline{nm} \quad &\to \quad \underline{m} \\
x \quad &\mapsto \quad ((x-1) \mod m) + 1,
\end{aligned}
\tag{2.1}
$$

- for all $n \in \mathbb{N}$ and all $i \in \underline{n}$, the projection δ_n^i is $f_n^i \cdot 1$, where

$$
\begin{aligned}
f_n^i : \quad \underline{1} \quad &\to \quad \underline{n} \\
1 \quad &\mapsto \quad i.
\end{aligned}
\tag{2.2}
$$

It is easily checked that K_P satisfies the axioms for a clone given in Definition 1.2.1.

On morphisms, $K_{(-)}$ acts trivially: morphisms of clones and of finite product operads are simply maps of signatures commuting with the extra structure, and $K_{(-)}$ preserves the underlying map of signatures.

Let K be a clone. Let P_K be the finite product operad for which

- $(P_K)_n = K_n$ for all $n \in \mathbb{N}$,

- $1 = \delta_1^1$,

- $p \circ (p_1, \ldots, p_n) = p \bullet (p_1 \bullet (\delta_m^1, \ldots, \delta_{\sum k_i}^{k_1}), \ldots, p_n \bullet (\delta_{\sum k_i}^{k_1 + \cdots + k_{n-1} + 1}, \ldots, \delta_{\sum k_i}^{\sum k_i}))$ for all n and $k_1, \ldots, k_n \in \mathbb{N}$, all $p \in K_n$, and all $p_1 \in K_{k_1}, \ldots, p_n \in K_{k_n}$,

- $f \cdot p = p \bullet (\delta_m^{f(1)}, \ldots, \delta_m^{f(n)})$ for all $n, m \in \mathbb{N}$, all $f : \underline{n} \to \underline{m}$ and all $p \in K_n$.

We will show that $K_{P_K} = K$ for all $K \in \textbf{Clone}$, and that $P_{K_P} = P$ for all $P \in \textbf{FP-Operad}$. Let K be a clone. Then $(K_{P_K})_n = (P_K)_n = K_n$ for all $n \in \mathbb{N}$. If $n, m \in \mathbb{N}$, $k \in K_n$ and $k_1, \ldots, k_n \in K_m$, then the composite $k \bullet (k_1, \ldots, k_n)$ in K_{P_K} is given by the

composite $f \cdot (k \circ (k_1, \ldots, k_n))$ in P_K, where f is given by (2.1) above. This in turn is given by the composite

$$(k \quad \bullet \quad (k_1 \bullet (\delta_{nm}^1, \ldots, \delta_{nm}^m), \ldots, k_n \bullet (\delta_{nm}^{(n-1)m+1}, \ldots, \delta_{nm}^{nm})))$$
$$\bullet \quad (\delta_m^1, \ldots, \delta_m^m, \ldots, \delta_m^1, \ldots, \delta_m^m)$$

in K. By the associativity law for clones, this is equal to

$$k \bullet (\quad k_1 \bullet (\delta_{nm}^1, \ldots, \delta_{nm}^m) \bullet (\delta_m^1, \ldots, \delta_m^m, \ldots, \delta_m^1, \ldots, \delta_m^m),$$
$$\ldots,$$
$$k_n \bullet (\delta_{nm}^{(n-1)m+1}, \ldots, \delta_{nm}^{nm}) \bullet (\delta_m^1, \ldots, \delta_m^m, \ldots, \delta_m^1, \ldots, \delta_m^m))$$

which in turn may be simplified to $k \bullet (k_1, \ldots, k_n)$ as required. For every $n \in \mathbb{N}$ and every $i \in \underline{n}$, the projection δ_n^i in K_{P_K} is given by $f_n^i \cdot 1$, where f_n^i is defined in (2.2): this in turn is given by $1 \circ (\delta_n^i) = \delta_1^1 \bullet (\delta_n^i) = \delta_n^i$. Hence $K_{P_K} = K$.

Conversely, let P be a finite product operad we shall show that $P_{K_P} = P$. For every $n \in \mathbb{N}$, the set $(P_{K_P})_n$ is equal to P_n. The unit element is given by $1 = \delta_1^1 = f_1^1 \cdot 1 = 1$. If $p \in P_n$ and $p_i \in P_{k_i}$ for $i = 1, \ldots, n$, then the composite $p \circ (p_1, \ldots, p_n)$ is given by $p \bullet (p_1 \bullet (\delta_m^1, \ldots, \delta_{\sum k_i}^{k_1}), \ldots, p_n \bullet (\delta_{\sum k_i}^{k_1 + \cdots + k_{n-1}+1}, \ldots, \delta_{\sum k_i}^{\sum k_i}))$ in K_P, which in turn is given by (after simplification) $p \circ (p_1, \ldots, p_n)$ in P. Hence $P = P_{K_P}$, and $K_{(-)}$ is bijective on objects. The reasoning above also suffices to show that $K_{(-)}$ is well-defined on morphisms and full (since preserving a finite product operad structure amounts exactly to preserving the associated clone structure). Since the morphisms of both categories are simply maps of signatures with extra properties and $K_{(-)}$ commutes with the forgetful functors to $\mathbf{Set}^{\mathbb{N}}$, then $K_{(-)}$ is faithful. Hence $K_{(-)}$ is an isomorphism of categories, and $\mathbf{FP\text{-}Operad} \cong \mathbf{Clone}$. $\qquad\square$

Theorem 2.3.13. *Let P be a finite product operad. Then $\mathbf{Alg}(P) \cong \mathbf{Alg}(K_P)$.*

Proof. Let $(A, (\hat{\ }))$ be a P-algebra. Since the elements of the finite product endomorphism operad $\mathrm{End}(A)$ are endomorphisms of A, and composition is given by composition of morphisms, then $K_{\mathrm{End}(A)} = \mathrm{End}(A)$, the endomorphism clone of A. Since the functor $K_{(-)} : \mathbf{FP\text{-}Operad} \to \mathbf{Clone}$ is an isomorphism, a morphism of finite product operads $P \to \mathrm{End}(A)$ is exactly a map of clones $K_P \to \mathrm{End}(A)$. Hence an algebra for P is exactly an algebra for K_P. A morphism between P-algebras is a morphism between their underlying objects that commutes with $\hat{p}$ for every $p \in P_n$ and every $n \in \mathbb{N}$; this is true iff it commutes with $\hat{k}$ for every $k \in (K_P)_n$ and every $n \in \mathbb{N}$. $\qquad\square$

2.4 Adjunctions

In the next few sections, we shall show that there is a chain of monadic adjunctions

$$(2.3)$$

The notation is chosen such that $F_y^x \dashv U_x^y$, and $U_x^y U_y^z = U_x^z$. The notation is inspired by the exponential notation used for hom-objects: the source category of one of these functors is determined by its superscript, and the target category is determined by its subscript. The "pl" stands for "plain". A similar chain of adjunctions (for PROPs rather than operads) was discussed in [Bae], pages 51–59.

We refer to the monad $U_x^y F_y^x$ as T_y^x. The right adjoints $U^{\mathrm{pl}}, U_{\mathrm{pl}}^\Sigma$ and U_Σ^{fp} are found by forgetting respectively the compositional structure, the symmetric structure, and the actions of all non-bijective functions, and will not be described further. By standard properties of adjunctions, the composite functors are adjoint: $F_\Sigma \dashv U^\Sigma$ etc.

2.5 Existence and monadicity

All the left adjoints in (2.3) are examples of a more general construction. We shall now investigate this general case, and show that the adjunction which arises is always monadic.

But first, we have so far only asserted that $U^{\mathrm{fp}}, U_\Sigma^{\mathrm{fp}}$ and $U_{\mathrm{pl}}^{\mathrm{fp}}$ have left adjoints. We shall show that these left adjoints must exist for general reasons.

Let **FP** be the category of small categories with finite products and product-preserving functors.

Lemma 2.5.1. *Let $\mathbb{C}$ and $\mathbb{D}$ be small finite-product categories, let $\mathcal{C}$ be cartesian closed and have all small colimits, and let $Q : \mathbb{C} \to \mathbb{D}$ preserve finite products. Then the adjunction*

$$[\mathbb{D}, \mathcal{C}] \underset{Q^*}{\overset{Q_!}{\underset{\perp}{\rightleftarrows}}} [\mathbb{C}, \mathcal{C}] \ ,$$

where Q^* is composition with Q and $Q_! = \mathrm{Lan}_Q$, restricts to an adjunction

$$\mathbf{FP}(\mathbb{D},\mathcal{C}) \underset{Q^*}{\overset{Q_!}{\underset{\perp}{\rightleftarrows}}} \mathbf{FP}(\mathbb{C},\mathcal{C}) \ ,$$

Proof. Certainly Q^* restricts in this way, since Q preserves finite products. $\mathbf{FP}(\mathbb{C},\mathcal{C})$ and $\mathbf{FP}(\mathbb{D},\mathcal{C})$ are full subcategories of $[\mathbb{C},\mathcal{C}]$ and $[\mathbb{D},\mathcal{C}]$, so if we can show that $Q_!$ restricts to a functor $\mathbf{FP}(\mathbb{C},\mathcal{C}) \to \mathbf{FP}(\mathbb{D},\mathcal{C})$, then it is automatically left adjoint to the restriction of Q^*.

Let $X : \mathbb{C} \to \mathcal{C}$ preserve finite products. We must show that $Q_! X : \mathbb{D} \to \mathcal{C}$ preserves finite products. We shall proceed by showing that $Q_! X$ preserves terminal objects and binary products.

Recall that

$$(Q_! X)(b) \cong \int^a \mathbb{D}(Qa,b) \times Xa$$

for all $b \in \mathbb{D}$. Hence, using 1 for the terminal objects in $\mathbb{D}$ and $\mathcal{C}$,

$$
\begin{aligned}
(Q_! X)(1) \ &\cong \ \int^a \mathbb{D}(Qa,1) \times Xa \\
&\cong \ \int^a 1 \times Xa \\
&\cong \ \int^a Xa \\
&\cong \ X1 \\
&\cong \ 1
\end{aligned}
$$

since X preserves finite products and the colimit of a diagram D over a category with a terminal object 1 is simply $D1$.

Now, let $b_1, b_2 \in \mathbb{D}$. Then

$$
\begin{aligned}
&(Q_! X)(b_1 \times b_2) \\
&\cong \ \int^a \mathbb{D}(Qa, b_1 \times b_2) \times Xa & (2.4) \\
&\cong \ \int^a \mathbb{D}(Qa, b_1) \times \mathbb{D}(Qa, b_2) \times Xa & (2.5) \\
&\cong \ \int^a \left(\int^{c_1} \mathbb{D}(Qc_1, b_1) \times \mathbb{C}(a, c_1) \right) \times \left(\int^{c_2} \mathbb{D}(Qc_2, b_2) \times \mathbb{C}(a, c_2) \right) \times Xc & (2.6) \\
&\cong \ \int^{a,c_1,c_2} \mathbb{D}(Qc_1, b_1) \times \mathbb{D}(Qc_2, b_2) \times \mathbb{C}(a, c_1) \times \mathbb{C}(a, c_2) \times Xa & (2.7) \\
&\cong \ \int^{a,c_1,c_2} \mathbb{D}(Qc_1, b_1) \times \mathbb{D}(Qc_2, b_2) \times \mathbb{C}(a, c_1 \times c_2) \times Xa & (2.8)
\end{aligned}
$$

$$\cong \int^{c_1,c_2} \mathbb{D}(Qc_1,b_1) \times \mathbb{D}(Qc_2,b_2) \times \left(\int^a \mathbb{C}(a,c_1 \times c_2) \times Xa \right) \tag{2.9}$$

$$\cong \int^{c_1,c_2} \mathbb{D}(Qc_1,b_1) \times \mathbb{D}(Qc_2,b_2) \times X(c_1 \times c_2) \tag{2.10}$$

$$\cong \int^{c_1,c_2} \mathbb{D}(Qc_1,b_1) \times \mathbb{D}(Qc_2,b_2) \times Xc_1 \times Xc_2 \tag{2.11}$$

$$\cong \int^{c_1,c_2} \mathbb{D}(Qc_1,b_1) \times Xc_1 \times \mathbb{D}(Qc_2,b_2) \times Xc_2 \tag{2.12}$$

$$\cong \left(\int^{c_1} \mathbb{D}(Qc_1,b_1) \times Xc_1 \right) \times \left(\int^{c_2} \mathbb{D}(Qc_2,b_2) \times Xc_2 \right) \tag{2.13}$$

$$\cong (Q_!X)(b_1) \times (Q_!X)(b_2) \tag{2.14}$$

(2.4) is the definition of $Q_!$; (2.5), (2.8) and (2.11) are from the definition of products; (2.6) and (2.10) are applications of the Density Formula; (2.7), (2.9) and (2.14) use the distributivity of products over colimits in $\mathcal{C}$ (since $\mathcal{C}$ is cartesian closed), and (2.12) uses the fact that X preserves finite products.

So $Q_!$ preserves terminal objects and binary products, and hence all finite products. $\square$

Corollary 2.5.2. *The functors $U_\Sigma^{\mathrm{fp}}, U_{\mathrm{pl}}^{\mathrm{fp}}$ and U^{fp} all have left adjoints.*

Lemma 2.5.3. *Let S be a set, whose elements we will call **sorts**. Let T and T' be S-sorted finite product theories, such that T' is a subcategory of T and the inclusion of T' into T preserves finite products. Let $\mathbf{Alg}(T)$ be the category of T-algebras and morphisms in some finite product category $\mathcal{C}$, and $\mathbf{Alg}(T')$ be the category of T'-algebras and morphisms in $\mathcal{C}$. Then the free/forgetful adjunction*

$$\mathbf{Alg}(T') \underset{U}{\overset{F}{\rightleftarrows}} \mathbf{Alg}(T)$$

is monadic, provided the left adjoint F exists.

Proof. We will make use of Beck's theorem to prove monadicity: precisely, we shall make use of the version in [ML98] VI.7.1, which states that U is monadic if it has a left adjoint and it strictly creates coequalizers for U-absolute coequalizer pairs. Recall that a functor $G : \mathcal{C} \to \mathcal{D}$ **strictly creates coequalizers** for a diagram $A \underset{g}{\overset{f}{\rightrightarrows}} B$ in $\mathcal{C}$ if, for every coequalizer $e : GB \to E$ of Gf and Gg in $\mathcal{D}$, there are a unique object E' in $\mathcal{C}$ and a unique arrow $e' : B \to E'$ such that $GE' = E$ and $Ge' = e$, and moreover that e' is a coequalizer of $A \underset{g}{\overset{f}{\rightrightarrows}} B$.

Let $A \underset{g}{\overset{f}{\rightrightarrows}} B$ be a U-absolute coequalizer pair in $\mathbf{Alg}(T)$, and $e : UB \to E$ be the coequalizer of $UA \underset{Ug}{\overset{Uf}{\rightrightarrows}} UB$. We wish to extend E to a functor $E' : T \to \mathcal{C}$. Define E'

to be equal to E on objects. On arrows, we shall define E' using the universal property of E and the U-absolute property of $A \overset{f}{\underset{g}{\rightrightarrows}} B$.

For each arrow $\phi : s_1 \times \cdots \times s_n \to r_1 \times \cdots \times r_m$ in T (where $s_i, r_j \in S$), consider the diagram

$$
\begin{array}{ccccc}
\prod UAs_i & \overset{\prod Uf_{s_i}}{\underset{\prod Ug_{s_i}}{\rightrightarrows}} & \prod UBs_i & \overset{\prod e_{s_i}}{\longrightarrow} & \prod Es_i \\
\Big\downarrow {\scriptstyle A\phi} & & \Big\downarrow {\scriptstyle B\phi} & & \Big\downarrow {\scriptstyle E'\phi} \\
\prod UAr_j & \overset{\prod f_{r_j}}{\underset{\prod g_{r_j}}{\rightrightarrows}} & \prod UBr_j & \overset{\prod e_{r_j}}{\longrightarrow} & \prod Er_j
\end{array}
\tag{2.15}
$$

in $\mathcal{C}$, where $e : UB \to E$ is a coequalizer for $UA \overset{Uf}{\underset{Ug}{\rightrightarrows}} UB$.

Since $A \overset{f}{\underset{g}{\rightrightarrows}} B$ is a U-absolute coequalizer pair, $\prod e_{s_i} : \prod UBs_i \to \prod Es_i$ is a coequalizer. Since f and g are T-homomorphisms, (2.15) serially commutes, so $(\prod e_{r_j})\phi$ factors uniquely through $\prod e_{s_i}$. Define $E'\phi$ to be this map, as shown (and note that $E'\phi = E\phi$ if ϕ is in T'). This definition straightforwardly makes E' into a functor $T \to \mathcal{C}$. Since E is a T'-algebra, and products in T are the same as products in T', we may deduce that $E' : T \to C$ preserves finite products, and thus is a T-algebra. Clearly, E' is the unique extension of E to a T-algebra such that e is a T-algebra morphism. It remains to show that e is a coequalizer map for $A \overset{f}{\underset{g}{\rightrightarrows}} B$ in $\mathbf{Alg}(T)$.

Suppose $A \overset{f}{\underset{g}{\rightrightarrows}} B \overset{d}{\to} D$ is a fork in $\mathbf{Alg}(T)$. Then $UA \overset{Uf}{\underset{Ug}{\rightrightarrows}} UB \overset{Ud}{\to} UD$ is a fork in $\mathbf{Alg}(T')$, so Ud factors through e; say $Ud = he$. We must show that h is a T-homomorphism. As before, take $\phi : s_1 \times \cdots \times s_n \to r_1 \times \cdots \times r_m$ in T, and consider the diagram

$$
\begin{array}{ccccc}
\prod UAs_i & \overset{\prod Uf_{s_i}}{\underset{\prod Ug_{s_i}}{\rightrightarrows}} & \prod UBs_i & \overset{\prod e_{s_i}}{\longrightarrow} & \prod E's_i \\
\Big\downarrow {\scriptstyle A\phi} & & \Big\downarrow {\scriptstyle B\phi} & \searrow_{\prod Ud_{s_i}} \quad \downarrow {\scriptstyle \prod h_{s_i}} & \\
& & & \prod UDs_i & {\scriptstyle E'\phi} \\
\prod UAr_j & \overset{\prod f_{r_j}}{\underset{\prod g_{r_j}}{\rightrightarrows}} & \prod UBr_j & \overset{\prod e_{r_j}}{\longrightarrow} & \prod E'r_j \\
& & & \searrow_{\prod Ud_{r_j}} \quad \downarrow {\scriptstyle \prod h_{r_j}} & {\scriptstyle D\phi} \\
& & & & \prod Dr_j
\end{array}
\tag{2.16}
$$

We must show that the curved square on the far right commutes. Now $(D\phi) \circ (\prod d_{s_i}) = (D\phi) \circ (\prod h_{s_i}) \circ (\prod e_{s_i})$, and $(\prod Ud_{r_j}) \circ (UB\phi) = h \circ e \circ (UB\phi) = h \circ (E\phi) \circ (\prod e_{s_i})$, since e

is a T-algebra homomorphism. But $(\prod U d_{r_j}) \circ \phi = \phi \circ (\prod d_i)$, so $D\phi \circ (\prod h_{s_i}) \circ (\prod e_{s_i}) = h \circ (E\phi) \circ (\prod e_{s_i})$. And $\prod e_{s_i}$ is (regular) epic, so $h \circ (E\phi) = (D\phi) \circ (\prod h_{s_i})$.

So h is a T-algebra homomorphism. Hence U strictly creates coequalizers for U-absolute coequalizer pairs, and hence is monadic. $\square$

This result could also have been deduced from the Sandwich Theorem of Manes: see [Man76] Theorem 3.1.29 (page 182).

Theorem 2.5.4. *All the adjunctions in diagram 2.3, namely* $F_{\mathrm{fp}}^{\Sigma} \dashv U_{\Sigma}^{\mathrm{fp}}, F_{\mathrm{fp}}^{\mathrm{pl}} \dashv U_{\mathrm{pl}}^{\mathrm{fp}}, F_{\mathrm{fp}} \dashv U^{\mathrm{fp}}, F_{\Sigma}^{\mathrm{pl}} \dashv U_{\mathrm{pl}}^{\Sigma}, F_{\Sigma} \dashv U^{\Sigma}$ *and* $F_{\mathrm{pl}} \dashv U^{\mathrm{pl}}$, *are monadic.*

Proof. Each category mentioned is a category of algebras for some $\mathbb{N}$-sorted theory, and the monadicity of each adjunction mentioned is obtained by a simple application of Lemma 2.5.3. For instance, symmetric operads are algebras for the theory presented by

- *operations*: one of the appropriate arity for each composition operation in Definition 2.2.5, and an operation $\sigma \cdot -$ for each $n \in \mathbb{N}$ and each σ in S_n.

- *equations*: one for each instance of the axioms in Definition 2.2.5, and an equation $(\sigma \cdot -) \circ (\rho \cdot -) = \sigma\rho \cdot -$ for each $\sigma, \rho \in S_n$ and every $n \in \mathbb{N}$.

$\square$

2.6 Explicit construction of F_{pl} and F_{Σ}^{pl}

The previous section showed that F_{pl} and F_{Σ}^{pl} exist for general reasons, but it will be useful later to have an explicit construction of these functors. For this reason, we shall now explicitly construct functors $\mathbf{Set}^{\mathbb{N}} \to \mathbf{Operad}$ and $\mathbf{Operad} \to \Sigma\text{-}\mathbf{Operad}$, and prove that they are left adjoint to U^{pl} and U_{pl}^{Σ}.

Definition 2.6.1. Let Φ be a signature. An n**-ary strongly regular tree labelled by** Φ is an element of the set $\mathrm{tr}_n \Phi$, which is recursively defined as follows:

- $|$ is an element of $\mathrm{tr}_1 \Phi$.

- If $\phi \in \Phi_n$, and $\tau_1 \in \mathrm{tr}_{k_1} \Phi, \dots, \tau_n \in \mathrm{tr}_{k_n} \Phi$, then $\phi \circ (\tau_1, \dots, \tau_n) \in \mathrm{tr}_{\sum k_i} \Phi$.

  In graph-theoretic terms, all our trees are planar and rooted. They need not be level. We shall abuse notation and write ϕ instead of $\phi \circ (|, \dots, |)$, for $\phi \in \Phi_n$.

  Given a signature Φ, the objects of the plain operad $(F_{\mathrm{pl}}\Phi)_n$ are the elements of $\mathrm{tr}_n \Phi$, and composition is given by grafting of trees:
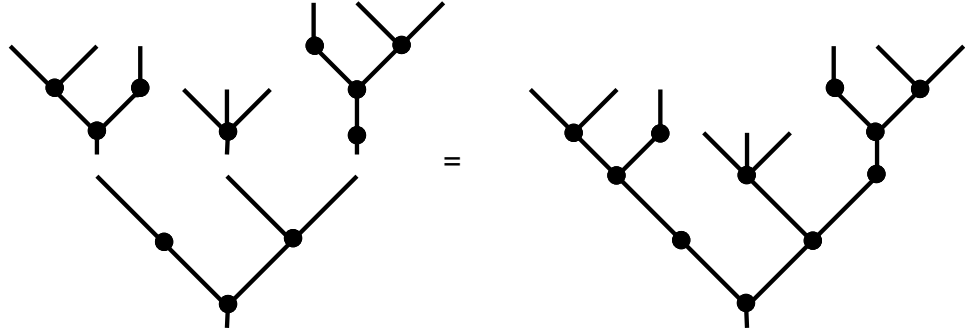
Figure 2.5: Grafting of trees

- $| \circ (\tau) = \tau$

- If $\tau_1 \in \mathrm{tr}_{k_1} \Phi, \ldots, \tau_n \in \mathrm{tr}_{k_n} \Phi$, then

$$(\phi \circ (\tau_1, \ldots, \tau_n)) \circ (\sigma_1, \ldots, \sigma_{\sum k_i}) = \phi \circ (\tau_1 \circ (\sigma_1, \ldots, \sigma_{k_1}), \ldots, \tau_n \circ (\sigma_{(\sum k_i) - k_n + 1}, \ldots, \sigma_{\sum k_i}))$$

See Figure 2.5.

The unary tree $|$ is thus the identity in $(F_{\mathrm{pl}} \Phi)$.

F_{pl} acts on arrows as follows. Let $f : \Phi \to \Psi$ be a map of signatures. Then:

- $(F_{\mathrm{pl}} f)| = |$

- $(F_{\mathrm{pl}} f)(\phi \circ (\tau_1, \ldots, \tau_n)) = (f\phi) \circ ((F_{\mathrm{pl}} f)\tau_1, \ldots, (F_{\mathrm{pl}} f)\tau_n)$

It is readily verified that with this definition F_{pl} is a functor $\mathbf{Set}^{\mathbb{N}} \to \mathbf{Operad}$.

We define natural transformations $\eta : 1_{\mathbf{Set}^{\mathbb{N}}} \to U^{\mathrm{pl}} F_{\mathrm{pl}}$ and $\epsilon : F_{\mathrm{pl}} U^{\mathrm{pl}} \to 1_{\mathbf{Operad}}$ as follows:

$$\eta_\Phi(\phi) = \phi \circ (|, \ldots, |) \tag{2.17}$$

$$\epsilon_P(|) = 1_P \tag{2.18}$$

$$\epsilon_P(\phi \circ (\tau_1, \ldots, \tau_n)) = \phi \circ (\epsilon_P(\tau_1), \ldots, \epsilon_P(\tau_n)) \tag{2.19}$$

where $P \in \mathbf{Operad}, \Phi \in \mathbf{Set}^{\mathbb{N}}, \phi \in \Phi$, and $\tau_1, \ldots, \tau_n$ are arrows of P.

In other words, ϵ_P is given by applying composition in P to the formal tree $F_{\mathrm{pl}} U^{\mathrm{pl}} P$.

Lemma 2.6.2. $(F_{\mathrm{pl}}, U^{\mathrm{pl}}, \eta, \epsilon)$ *is an adjunction.*

Proof. We proceed by checking the triangle identities. We require to show that

$$
\begin{array}{ccc}
F_{\mathrm{pl}} & \xrightarrow{\ F_{\mathrm{pl}}\eta\ } & F_{\mathrm{pl}} U^{\mathrm{pl}} F_{\mathrm{pl}} \\
& {}_{1_{F_{\mathrm{pl}}}}\searrow & \downarrow {\scriptstyle \epsilon F_{\mathrm{pl}}} \\
& & F_{\mathrm{pl}}
\end{array}
\tag{2.20}
$$

$$U^{\mathrm{pl}} \xrightarrow{\eta U^{\mathrm{pl}}} U^{\mathrm{pl}} F_{\mathrm{pl}} U^{\mathrm{pl}} \tag{2.21}$$

$$1_{U^{\mathrm{pl}}} \searrow \quad \downarrow U^{\mathrm{pl}}\epsilon$$

$$U^{\mathrm{pl}}$$

commute. For (2.20), we proceed by induction on trees. We shall suppress all subscripts on natural transformations in the interest of legibility. For the base case:

$$
\begin{aligned}
\epsilon F_{\mathrm{pl}}(F_{\mathrm{pl}}\eta(|)) &= \epsilon F_{\mathrm{pl}}(| \circ (|)) \\
&= | \circ (\epsilon(|)) \\
&= | \\
&= 1_{F_{\mathrm{pl}}}(|).
\end{aligned}
$$

For the inductive step, let Φ be a signature, ϕ be an n-ary element of Φ, and $\tau_1, \ldots, \tau_n$ be trees labelled by Φ. Then:

$$
\begin{aligned}
(\epsilon F_{\mathrm{pl}})((F_{\mathrm{pl}}\eta)(\phi \circ (\tau_1, \ldots, \tau_n))) &= \epsilon F_{\mathrm{pl}}(\phi \circ (\tau_1, \ldots, \tau_n) \circ (|, \ldots, |)) \\
&= \phi \circ (\tau_1, \ldots, \tau_n) \circ ((\epsilon F_{\mathrm{pl}})(|), \ldots, (\epsilon F_{\mathrm{pl}})(|)) \\
&= \phi \circ (\tau_1, \ldots, \tau_n) \\
&= 1_{F_{\mathrm{pl}}}(\phi \circ (\tau_1, \ldots, \tau_n))
\end{aligned}
$$

Hence $\epsilon F_{\mathrm{pl}} \circ F_{\mathrm{pl}}\eta = 1 F_{\mathrm{pl}}$, as required.

For (2.21), let P be a plain operad, and let p be an n-ary arrow in P.

$$
\begin{aligned}
(U^{\mathrm{pl}}\epsilon)((\eta U^{\mathrm{pl}})(p)) &= U^{\mathrm{pl}}\epsilon(p \circ (|, \ldots, |)) \\
&= p \circ ((U^{\mathrm{pl}}\epsilon)(|), \ldots, (U^{\mathrm{pl}}\epsilon)(|)) \\
&= p \circ (1, \ldots, 1) \\
&= p \\
&= 1_{U^{\mathrm{pl}}}(p)
\end{aligned}
$$

So $U^{\mathrm{pl}}\epsilon \circ \eta U^{\mathrm{pl}} = 1_{U^{\mathrm{pl}}}$, as required. $\square$

We now consider the "free symmetric operad" functor F^{pl}_{Σ}. We shall explicitly define a functor $\mathcal{S} \times - : \mathbf{Operad} \to \Sigma\text{-}\mathbf{Operad}$ and show that it is left adjoint to U^{Σ}_{pl}, and hence isomorphic to F^{pl}_{Σ}.

If P is a plain operad, an element of $(\mathcal{S} \times P)_n$ is a pair (σ, p), where $p \in P_n$ and $\sigma \in S_n$; i.e., $(\mathcal{S} \times P)_n = S_n \times P_n$. Composition is given as follows:

$$(\sigma, p) \circ ((\tau_1, q_1), \ldots, (\tau_n, q_n)) = (\sigma \circ (\tau_1, \ldots, \tau_n), p \circ (q_{\sigma(1)}, \ldots, q_{\sigma(n)}))$$

The symmetric group action is given by $\rho \cdot (\sigma, p) = (\rho\sigma, p)$.
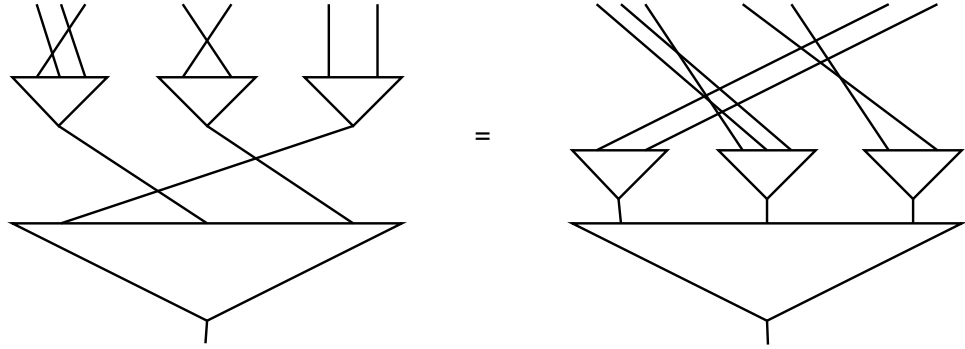

Figure 2.6: Composition in $\mathcal{S} \times P$

Lemma 2.6.3. $(\mathcal{S} \times -)$ *is left adjoint to* U_{pl}^{Σ}. *The unit of the adjunction is given by*

$$\eta : 1 \;\;\rightarrow\;\; U_{\mathrm{pl}}^{\Sigma}(\mathcal{S} \times -)$$

$$\eta_P : p \;\;\mapsto\;\; (1, p),$$

and the counit is given by

$$\epsilon : (\mathcal{S} \times -)U_{\mathrm{pl}}^{\Sigma} \;\;\rightarrow\;\; 1$$

$$\epsilon_P : (\sigma, p) \;\;\mapsto\;\; \sigma \cdot p.$$

Proof. As before, we proceed by checking the triangle identities. First, let P be a plain operad, p be an n-ary arrow in P, and $\sigma \in S_n$. Then (σ, p) is an element of $\mathcal{S} \times P$.

$$
\begin{aligned}
(\epsilon(\mathcal{S} \times -))(\mathcal{S} \times \eta)(\sigma, p)) &= (\epsilon(\mathcal{S} \times -))(\sigma, (1, p)) \\
&= \sigma \cdot (1, p) \\
&= (\sigma \circ 1, p) \\
&= (\sigma, p)
\end{aligned}
$$

Now let P' be a symmetric operad, and p' be an n-ary arrow in P'.

$$
\begin{aligned}
(U_{\mathrm{pl}}^{\Sigma}\epsilon)(\eta U_{\mathrm{pl}}^{\Sigma}(p')) &= (U_{\mathrm{pl}}^{\Sigma}\epsilon)((1, p')) \\
&= 1 \cdot p' \\
&= p'
\end{aligned}
$$

So the triangle identities are indeed satisfied, and $(\mathcal{S} \times -) \dashv U_{\mathrm{pl}}^{\Sigma}$. □

Hence, $F_{\Sigma}^{\mathrm{pl}} = \mathcal{S} \times -$.

Definition 2.6.4. Let Φ be a signature. An n**-ary permuted tree labelled by** Φ is an element of $(F^{\mathrm{pl}}_\Sigma F_{\mathrm{pl}}\Phi)_n = (F_\Sigma \Phi)_n$. An n**-ary finite product tree labelled by** Φ is an element of $(F_{\mathrm{fp}}\Phi)_n$.

By Lemma 2.6.3, a permuted tree is a pair (σ, t), where $t \in \mathrm{tr}_n \Phi$ and $\sigma \in S_n$, and (by analogous reasoning) an n-ary finite product tree is a pair (f, t), where $t \in \mathrm{tr}_m \Phi$ and $f : \underline{m} \to \underline{n}$.

2.7 Syntactic characterization of the forgetful functors

There is also a syntactic characterization of the forgetful functor U^Σ_{pl}. Given a symmetric operad P, we take the signature given by all operations in P (in other words, the signature $U^\Sigma P$). We then impose all the plain-operadic equations that are true in P, and take the plain operad corresponding to this strongly regular theory. This operad is $U^\Sigma_{\mathrm{pl}}P$.

We start by making this precise.

Definition 2.7.1. Let Φ be a signature. A **plain-operadic equation over** Φ **in** n **variables** is an element of $((U^{\mathrm{pl}}F_{\mathrm{pl}}\Phi)_n)^2$ (that is, a pair of n-ary strongly regular trees over Φ), and a **plain-operadic equation over** Φ is an element of $\sum_n ((U^{\mathrm{pl}}F_{\mathrm{pl}}\Phi)_n)^2$.

We shall show that a plain-operadic equation over Φ is the same thing as a strongly regular equation over Φ.

Definition 2.7.2. Let P be a plain operad. A **presentation** for P is a signature Φ, a signature E, and maps $e_1, e_2 : F_{\mathrm{pl}}E \to F_{\mathrm{pl}}\Phi$ such that, for some ϕ,

$$F_{\mathrm{pl}}E \xrightarrow[e_2]{e_1} F_{\mathrm{pl}}\Phi \xrightarrow{\phi} P$$

is a coequalizer. We say that a regular epi $\phi : F_{\mathrm{pl}}\Phi \to P$ **generates** P, or that ϕ (or, where the choice of ϕ is clear, Φ) is a **generator** of P.

Presentations and generators for symmetric and finite product operads are defined analogously.

We will see how these "presentations" are related to presentations of algebraic theories in Section 2.8. We now wish to describe the family of all strongly regular equations that are true in a given symmetric operad P: we will then show that this, together with the signature given by $U^\Sigma P$, is a presentation for $U^\Sigma_{\mathrm{pl}}P$ as claimed.

Definition 2.7.3. Let P be a plain operad, and $\phi : F_{\mathrm{pl}}\Phi \to P$ be a generator for P. Let E be a subsignature of $(U^{\mathrm{pl}}F_{\mathrm{pl}}\Phi)^2$, so that each E_n is a set of n-ary Φ-equations. Let i be the inclusion map $E \lhook\joinrel\longrightarrow (U^{\mathrm{pl}}F_{\mathrm{pl}}\Phi)^2$, and π_1, π_2 be the projection maps $(U^{\mathrm{pl}}F_{\mathrm{pl}}\Phi)^2 \to U^{\mathrm{pl}}F_{\mathrm{pl}}\Phi$. Then P **satisfies all equations in** E if the diagram

$$F_{\mathrm{pl}}E \underset{\overline{\pi_2 i}}{\overset{\overline{\pi_1 i}}{\rightrightarrows}} F_{\mathrm{pl}}\Phi \overset{\phi}{\longrightarrow} P$$

is a fork.

We say that a symmetric or finite product operad satisfies a signature of equations if the analogous condition holds in Σ-**Operad** or **FP-Operad**.

Recall the notion of the "kernel pair" of a morphism:

Definition 2.7.4. Let $f : A \to B$ in some category $\mathcal{C}$. The **kernel pair** of f is the pair $W \underset{q}{\overset{p}{\rightrightarrows}} A$ of maps in the pullback square

$$
\begin{array}{ccc}
W & \overset{p}{\longrightarrow} & A \\
{\scriptstyle q}\downarrow & \lrcorner & \downarrow{\scriptstyle f} \\
A & \underset{f}{\longrightarrow} & B
\end{array}
$$

if this pullback exists.

Lemma 2.7.5. *Let ϵ be the counit of the adjunction $F_{\mathrm{pl}} \dashv U^{\mathrm{pl}}$. Let $Q \underset{\pi_2}{\overset{\pi_1}{\rightrightarrows}} F_{\mathrm{pl}}(U^{\Sigma}P)$ be the kernel pair of the component*

$$\epsilon_{U^{\Sigma}_{\mathrm{pl}}P} : F_{\mathrm{pl}}U^{\Sigma}P = F_{\mathrm{pl}}U^{\mathrm{pl}}U^{\Sigma}_{\mathrm{pl}}P \to U^{\Sigma}_{\mathrm{pl}}P,$$

of ϵ. Let h be the unique map $Q \to (F_{\mathrm{pl}}U^{\Sigma}P)^2$ induced by π_1, π_2. Then the image of $U^{\mathrm{pl}}h$ is the largest signature of plain-operadic $U^{\Sigma}P$-equations satisfied by P.

Proof. Q, π_1, π_2 are given by the diagram

$$
\begin{array}{ccc}
Q & \overset{\pi_1}{\longrightarrow} & F_{\mathrm{pl}}U^{\Sigma}P \\
{\scriptstyle \pi_2}\downarrow & \lrcorner & \downarrow{\scriptstyle \epsilon} \\
F_{\mathrm{pl}}U^{\Sigma}P & \underset{\epsilon}{\longrightarrow} & U^{\Sigma}_{\mathrm{pl}}P
\end{array}
$$

As a right adjoint, U^{pl} preserves pullbacks; we take the standard construction of pullbacks in $\mathbf{Set}^{\mathbb{N}}$ as subobjects of products, in which case h is an inclusion map. An element of $(U^{\mathrm{pl}}Q)_n$ is then a pair (e_1, e_2) of n-ary strongly regular $U^{\Sigma}P$ trees such that $\epsilon(e_1) = \epsilon(e_2)$. Hence, Q is a signature of plain-operadic $U^{\Sigma}P$-equations satisfied by P. Conversely, let E be a signature of plain-operadic $U^{\Sigma}P$-equations satisfied by P, and let (e_1, e_2) be an element of E_n: then $\epsilon(e_1) = \epsilon(e_2)$ and so (e_1, e_2) is an element of $(U^{\mathrm{pl}}Q)_n$. $\qquad\square$

Corollary 2.7.6. *Let R be the plain operad generated by $U^\Sigma P$, satisfying exactly those plain-operadic equations satisfied by P. Then*

$$F_{\mathrm{pl}}U^{\mathrm{pl}}Q \underset{\overline{U^{\mathrm{pl}}\pi_2}}{\overset{\overline{U^{\mathrm{pl}}\pi_1}}{\rightrightarrows}} F_{\mathrm{pl}}U^\Sigma P$$

is a presentation for R, where the overbars refer to transposition with respect to the adjunction $F_{\mathrm{pl}} \dashv U^{\mathrm{pl}}$.

We recall some standard results.

Lemma 2.7.7. *The counit of a monadic adjunction is componentwise regular epi.*

Proof. See [AHS04] 20.15. □

Lemma 2.7.8. *If $X \overset{f}{\underset{g}{\rightrightarrows}} Y \overset{h}{\longrightarrow} Z$ is a coequalizer in some category, and if $e : W \to X$ is epi, then $W \overset{fe}{\underset{ge}{\rightrightarrows}} Y \overset{h}{\longrightarrow} Z$ is a coequalizer.*

Proof. Suppose $W \overset{fe}{\underset{ge}{\rightrightarrows}} Y \overset{i}{\longrightarrow} A$ is a fork. Then $ife = ige$, so $if = ig$ since e is epi. So $X \overset{f}{\underset{g}{\rightrightarrows}} Y \overset{i}{\longrightarrow} A$ is a fork, and hence i factors uniquely through h. □

Lemma 2.7.9. *In categories with all kernel pairs, every regular epi is the coequalizer of its kernel pair.*

Proof. Let $\mathcal{C}$ have all kernel pairs, and $A \overset{f}{\underset{g}{\rightrightarrows}} B \overset{e}{\longrightarrow} C$ be a coequalizer diagram in $\mathcal{C}$. Let $W \overset{p}{\underset{q}{\rightrightarrows}} B$ be the kernel pair of e. We will show that $W \overset{p}{\underset{q}{\rightrightarrows}} B \overset{e}{\longrightarrow} C$ is a coequalizer diagram. Since $ef = eg$, we may uniquely factor (f, g) through W:

Suppose $W \overset{p}{\underset{q}{\rightrightarrows}} B \overset{h}{\longrightarrow} D$ is a fork. $hp = hq$, so $hpi = hqi$, so $hf = hg$. By the universal property of e, we may factor h uniquely through e. So $W \overset{p}{\underset{q}{\rightrightarrows}} B \overset{e}{\longrightarrow} C$ is a coequalizer diagram, as required. □

Lemma 2.7.10. *Let P be a symmetric operad, and let Q, π_1, π_2 be as in Lemma 2.7.5. Then the coequalizer of the diagram*

$$Q \underset{\pi_1}{\overset{\pi_1}{\rightrightarrows}} F_{\mathrm{pl}} U^{\Sigma} P$$

is $U_{\mathrm{pl}}^{\Sigma} P$.

Proof. Let ϵ' be the unit of the adjunction $F_{\Sigma}^{\mathrm{pl}} \dashv U_{\mathrm{pl}}^{\Sigma}$. This adjunction is monadic, so $\epsilon_{U_{\mathrm{pl}}^{\Sigma} P}$ is regular epi by Lemma 2.7.7. By Lemma 2.7.9, $\epsilon_{U_{\mathrm{pl}}^{\Sigma} P}$ is the coequalizer of its kernel pair, i.e.

$$Q \underset{\pi_2}{\overset{\pi_1}{\rightrightarrows}} \sum_n (U^{\mathrm{pl}} F_{\mathrm{pl}} U^{\Sigma} P)_n \xrightarrow{\epsilon_{U_{\mathrm{pl}}^{\Sigma} P}} U_{\mathrm{pl}}^{\Sigma} P$$

is a coequalizer diagram. $\square$

Theorem 2.7.11. *Let P be a symmetric operad. Then $U_{\mathrm{pl}}^{\Sigma} P$ is the plain operad whose operations are those in P, satisfying exactly those plain-operadic equations which are true in P.*

Proof. The adjunction $F_{\mathrm{pl}} \dashv U^{\mathrm{pl}}$ is monadic, so if ϵ' is its counit, then $\epsilon'_Q : F_{\mathrm{pl}} U^{\mathrm{pl}} Q \to Q$ is (regular) epi by Lemma 2.7.7. Hence, by Lemma 2.7.8,

$$F_{\mathrm{pl}} U^{\mathrm{pl}} Q \underset{\pi_2 \epsilon'_Q}{\overset{\pi_1 \epsilon'_Q}{\rightrightarrows}} F_{\mathrm{pl}} U^{\Sigma} P \xrightarrow{\epsilon_{U_{\mathrm{pl}}^{\Sigma} P}} U_{\mathrm{pl}}^{\Sigma} P$$

is a coequalizer. But $\pi_1 \epsilon'_Q = \overline{U^{\mathrm{pl}} \pi_1}$, and similarly $\pi_2 \epsilon'_Q = \overline{U^{\mathrm{pl}} \pi_2}$. Hence, by Corollary 2.7.6, $U_{\mathrm{pl}}^{\Sigma} P$ is the plain operad generated by $U^{\Sigma} P$, satisfying all plain-operadic equations true in P. Since $U^{\Sigma} P = U^{\mathrm{pl}} U_{\mathrm{pl}}^{\Sigma} P$, the n-ary operations of $U_{\mathrm{pl}}^{\Sigma} P$ are exactly the n-ary operations of P. $\square$

We may generalize this as follows:

Theorem 2.7.12. *Let $\mathcal{C}$ be a category with pullbacks, and T be a monad on $\mathcal{C}$. Let $(TA \xrightarrow{a} A) \in \mathcal{C}^T$. Let $E \underset{\phi_2}{\overset{\phi_1}{\rightrightarrows}} TA$ be the kernel pair of a in $\mathcal{C}$. Then*

$$F_T E \underset{\bar{\phi}_2}{\overset{\bar{\phi}_1}{\rightrightarrows}} F_T A \xrightarrow{\epsilon_{(A,a)}} (A, a)$$

is a coequalizer in $\mathcal{C}^T$, where $F_T : \mathcal{C} \to \mathcal{C}^T$ is the free functor, and ϵ is the counit of the adjunction $F_T \dashv U_T$.

Proof. As above. $\square$

Corollary 2.7.13. *Let P be a finite product operad. Then $U_\Sigma^{\mathrm{fp}} P$ is the symmetric operad whose operations are given by those of P, satisfying all linear equations that are true in P, and $U_{\mathrm{pl}}^{\mathrm{fp}} P$ is the plain operad whose operations are given by those in P, satisfying all strongly regular equations that are true in P.*

2.8 Operads and syntactic classes of theories

We have defined notions of algebras for plain, symmetric and finite product operads. We might ask how these are related to the algebraic theories of Chapter 1: are the algebras for an operad P algebras for some algebraic theory $\mathcal{T}_P$? If so, what can we say about the theories that so arise?

We will show the following:

- Plain operads are equivalent in expressive power to *strongly regular* theories.

- Symmetric operads are equivalent in expressive power to *linear* theories.

- Finite product operads are equivalent in expressive power to general algebraic theories.

The first equivalence is proved in [Lei03]. The second has long been folklore (see, for instance, [Bae] page 50), but as far as I know no proof has appeared before. An (independently found) proof does appear in an unpublished paper of Adámek and Velebil, who also consider the enriched case. The third equivalence was proved in two stages by Tronin, in [Tro02] and [Tro06].

Recall the definitions of strongly regular and linear terms from Definition 1.1.7, and the definitions of strongly regular, permuted and finite product trees (Definitions 2.6.1 and 2.6.4).

Let Φ be a signature. We will show that there is an isomorphism between the set $(T_{\mathrm{fp}}\Phi)_n$ and the set of n-ary words in Φ, and that this isomorphism restricts to further isomorphisms as follows:

$$
\begin{array}{ccc}
(T_{\mathrm{fp}}\Phi)_n & \cong & \{n\text{-ary words in }\Phi\} \\
\uparrow & & \uparrow \\
(T_{\Sigma}\Phi)_n & \cong & \{n\text{-ary linear words in }\Phi\} \\
\uparrow & & \uparrow \\
(T_{\mathrm{pl}}\Phi)_n & \cong & \{n\text{-ary strongly regular words in }\Phi\}
\end{array}
\tag{2.22}
$$

The maps in the left-hand column can be viewed as inclusions between different sets of finite product trees, or equivalently as maps arising from the units of the adjunctions $F_{\mathrm{fp}}^{\Sigma} \dashv U_{\Sigma}^{\mathrm{fp}}$ and $F_{\Sigma}^{\mathrm{pl}} \dashv U_{\mathrm{pl}}^{\Sigma}$.

Let Φ be a signature. Observe that trees in Φ give rise to terms according to the following recursive algorithm:

- Let τ be an n-ary strongly regular tree, and $Y = (y_1, y_2, \ldots, y_n)$ a finite sequence of variables. The **term** $\mathrm{term}(\tau, Y)$ **arising from τ with working alphabet Y** is given as follows:

  - If $\tau = \,\mid$, then $\mathrm{term}(\tau, Y) = y_1$.

  - If $\tau = \phi \circ (\tau_1, \ldots, \tau_n)$, then

    $$\mathrm{term}(\tau, Y) = \phi(\mathrm{term}(\tau_1, (y_1, \ldots, y_{i_1})), \ldots, \mathrm{term}(\tau_n, (y_{1+i_{n-1}}, \ldots, y_{i_n}))),$$

    where i_1 is the arity of τ_1, and $i_j - i_{j-1}$ is the arity of τ_j for $j > 1$.

- The **term** $\mathrm{term}(\tau)$ **arising from τ** is $\mathrm{term}(\tau, (x_1, x_2, \ldots, x_n))$.

- Let $\sigma \cdot \tau$ be a permuted tree. Then $\mathrm{term}(\sigma \cdot \tau) = \mathrm{term}(\tau, (x_{\sigma 1}, x_{\sigma 2}, \ldots, x_{\sigma n}))$.

- Let $f \cdot \tau$ be a finite product tree. Then $\mathrm{term}(f \cdot \tau) = \mathrm{term}(\tau, (x_{f(1)}, x_{f(2)}, \ldots, x_{f(n)}))$

Definition 2.8.1. Let t be a Φ-term. We define a plain-operadic tree $\mathrm{tree}(t)$ recursively:

- if t is a variable, let $\mathrm{tree}(t) = \,\mid$.

- if $t = \phi(t_1, \ldots, t_n)$, let $\mathrm{tree}(t) = \phi \circ (\mathrm{tree}(t_1), \ldots, \mathrm{tree}(t_n))$.

Lemma 2.8.2. *Every Φ-term t is equal to $\mathrm{term}(f \cdot \tau)$ for a unique finite product tree $(f \cdot \tau)$.*

Proof. We will show

1. if t is a Φ-term, then $\mathrm{term}(\mathrm{label}(t) \cdot \mathrm{tree}(t)) = t$;

2. if $(f \cdot \tau)$ is a finite product tree, then $f = \text{label}(\text{term}(f \cdot \tau))$ and $\tau = \text{tree}(\text{term}(f \cdot \tau))$.

(1) Let t be a Φ-term. Let $f = \text{label}(t)$, and $\tau = \text{tree}(t)$. Then $\text{term}(f \cdot \tau)$ is $\text{term}(\tau, (x_{f(1)}, \ldots, x_{f(n)}))$. We proceed by induction.

- if $t = x_i$, then $\text{term}(f \cdot \tau)$ is $\text{term}(|, (x_i))$, which is x_i.

- if $t = \phi(t_1, \ldots, t_n)$, where each t_i has arity k_i, then

$$
\begin{aligned}
\text{term}(f \cdot \tau) &= \text{term}(\phi \circ (\text{tree}(t_1), \ldots, \text{tree}(t_n)), (x_{f(1)}, \ldots, x_{f(n)})) \\
&= \phi(\text{term}(\text{tree}(t_1), (x_{f(1)}, \ldots, x_{f(k_1)})), \ldots, \\
&\qquad \text{term}(\text{tree}(t_n), (x_{f((\sum_{i=1}^{n-1} k_i)+1)}, \ldots, x_{f(\sum_{i=1}^{n} k_i)}))) \\
&= \phi(t_1, \ldots, t_n) \\
&= t.
\end{aligned}
$$

(2) Let τ be an n-ary plain-operadic tree in Φ, and f a function of finite sets with codomain $\underline{m}$. Let $t = \text{term}(f \cdot \tau)$. We proceed as usual by induction on τ.

- If $\tau = |$, then $t = x_{f(1)}$; then $\text{tree}(t) = | = \tau$ and $\text{label}(t)$ is the function $\underline{1} \to \underline{m}$ sending 1 to $f(1)$, i.e. $\text{label}(t) = f$.

- If $\tau = \phi \circ (\tau_1, \ldots, \tau_n)$, then

$$
\begin{aligned}
t &= \text{term}(\phi \circ (\tau_1, \ldots, \tau_n), (x_{f(1)}, \ldots, x_{f(\sum k_i)})) \\
&= \phi(\text{term}(\tau_1, (x_{f(1)}, \ldots, x_{f(k_1)})), \ldots, \text{term}(\tau_n, (x_{f((\sum_{i=1}^{n-1} k_i)+1)}) \ldots, x_{f(\sum_{i=1}^{n} k_i)})))
\end{aligned}
$$

By induction, $\text{var}(t) = (x_{f(1)}, \ldots, x_{f(\sum k_i)})$, so $\text{label}(t) = f$, and $\text{tree}(t) = \phi \circ (\tau_1, \ldots, \tau_n) = \tau$ as required.

$\square$

We have now established the isomorphism in the top line of 2.22. If we use this isomorphism to identify finite product operads with finitary monads on **Set**, we may view the functor $F_{\text{fp}}^{\text{pl}}$ as the well-known functor sending a plain operad to its associated monad on **Set**.

Lemma 2.8.3. *Let t be a Φ-term. Then t is linear iff $t = \text{term}(\tau)$ for some permuted tree τ, and strongly regular iff $t = \text{term}(\tau)$ for some strongly regular tree τ.*

Proof. In Lemma 2.8.2, we factored every Φ-term t into a strongly regular tree tree(t) and a labelling function label(t). By definition, t is linear iff label(t) is a bijection, which occurs iff $t = \mathrm{term}(\sigma \cdot \tau)$ for some plain-operadic tree τ and some bijection σ. Hence, the linear terms and permuted trees are in one-to-one correspondence. Similarly, strongly regular terms and plain-operadic trees are in one-to-one correspondence. $\square$

The commutativity of 2.22 now follows from our explicit construction of F_Σ and F_{pl} in Section 2.4.

Lemma 2.8.4. *Let (Φ, E) be a presentation of an algebraic theory. Then (Φ, E) is linear if and only if the projection maps $E \overset{\pi_1}{\underset{\pi_2}{\rightrightarrows}} \mathrm{term}\,(\Phi)$ may be factored through the map $T_\Sigma\Phi \overset{\eta}{\longrightarrow} T_{\mathrm{fp}}\Phi \overset{\sim}{\longrightarrow} \mathrm{term}\,\Phi$:*

$$
\begin{array}{ccccc}
E & \overset{\pi_1}{\underset{\pi_2}{\rightrightarrows}} & T_{\mathrm{fp}}\Phi & \overset{\sim}{\longrightarrow} & \mathrm{term}\,\Phi \\
& \searrow \eta & \nearrow & \nwarrow & \nearrow \\
& T_\Sigma\Phi & \overset{\sim}{\longrightarrow} & \{\text{linear }\Phi\text{-terms}\} &
\end{array}
$$

Proof. By definition, the presentation is linear iff π_1, π_2 factor through the signature of linear Φ-terms. By Lemma 2.8.3, this signature is isomorphic to $T_\Sigma\Phi$, so we are done. $\square$

Theorem 2.8.5. *Let $Q \in$ **FP-Operad**. Then*

1. *Q is strongly regular iff there exists a $P \in$ **Operad** such that $Q \cong F_{\mathrm{fp}}^{\mathrm{pl}}P$;*

2. *Q is linear iff there exists a $P \in \Sigma$-**Operad** such that $Q \cong F_{\mathrm{fp}}^{\Sigma}P$;*

Proof. We will consider the linear case; the strongly regular case is proved analogously.

If Q is linear, then there exists a linear presentation $E \rightrightarrows F_{\mathrm{fp}}\Phi$ for Q. We may regard E as a subobject of the signature of Φ-equations. By assumption, E consists only of linear equations; by Lemma 2.8.3, every $(s, t) \in E$ is $(\mathrm{term}(\sigma_1 \cdot \tau_1), \mathrm{term}(\sigma_2 \cdot \tau_2))$ for some pair $(\sigma_1 \cdot \tau_1, \sigma_2 \cdot \tau_2)$ of permuted trees. So the diagram $E \rightrightarrows F_{\mathrm{fp}}\Phi$ in **FP-Operad** is the image under F_{fp}^{Σ} of a diagram

$$
E' \rightrightarrows F_\Sigma\Phi
$$

in Σ-**Operad**. This diagram has a coequalizer: call it P. The functor F_{fp}^{Σ} is a left adjoint, and thus preserves coequalizers: hence, Q is the image under F_{fp}^{Σ} of P.

Now suppose $Q = F_{\mathrm{fp}}^\Sigma P$ for some symmetric operad P. We may take the canonical presentation of P:

$$F_\Sigma U^\Sigma F_\Sigma U^\Sigma P \underset{F_\Sigma U^\Sigma \epsilon}{\overset{\epsilon F_\Sigma U^\Sigma}{\rightrightarrows}} F_\Sigma U^\Sigma P \xrightarrow{\ \epsilon\ } P$$

and apply F_{fp}^Σ to it:

$$F_{\mathrm{fp}}(U^\Sigma F_\Sigma U^\Sigma P) \underset{F_{\mathrm{fp}} U^\Sigma \epsilon}{\overset{F_{\mathrm{fp}}^\Sigma \epsilon F_\Sigma U^\Sigma}{\rightrightarrows}} F_{\mathrm{fp}} U^\Sigma P \xrightarrow{\ F_{\mathrm{fp}}^\Sigma \epsilon\ } F_{\mathrm{fp}}^\Sigma P = Q$$

Since F_{fp}^Σ is a left adjoint, it preserves coequalizers, so the transpose of this parallel pair is a presentation for Q. Take this transpose:

$$U^\Sigma F_\Sigma U^\Sigma P \underset{\overline{F_\Sigma U^\Sigma \epsilon}}{\overset{\overline{\epsilon F_\Sigma U^\Sigma}}{\rightrightarrows}} U^\Sigma F_\Sigma U^\Sigma P \xrightarrow{\ \eta'\ } U^{\mathrm{fp}} F_{\mathrm{fp}} U^\Sigma P \xrightarrow{\ U^{\mathrm{fp}} F_{\mathrm{fp}}^\Sigma \epsilon\ } U^{\mathrm{fp}} Q$$

where η' is the unit of the adjunction $F_{\mathrm{fp}} \dashv U^{\mathrm{fp}}$, and the bars refer to transposition with respect to the adjunction $F_\Sigma \dashv U^\Sigma$.

This is in precisely the form required for Lemma 2.8.4. $\square$

Example 2.8.6. The theories of monoids and pointed sets are strongly regular, because the finite product operads corresponding to these theories are in the image of $F_{\mathrm{fp}}^{\mathrm{pl}}$; the theory of commutative monoids is linear but not strongly regular, because the finite product operad whose algebras are commutative monoids is in the image of F_{fp}^Σ but not in the image of $F_{\mathrm{fp}}^{\mathrm{pl}}$.

There is a little more to be said about these classes of theories.

Definition 2.8.7. A **wide pullback** is a limit of a (possibly infinite) diagram of the form

Definition 2.8.8. A natural transformation $\alpha : F \to G$ is **cartesian** if every naturality square

$$
\begin{array}{ccc}
FA & \xrightarrow{\ Ff\ } & FB \\
{\scriptstyle \alpha_A}\downarrow & & \downarrow{\scriptstyle \alpha_B} \\
GA & \xrightarrow[\ Gf\ ]{} & GB
\end{array}
$$

is a pullback square.

Definition 2.8.9. A monad (T, μ, η) is **cartesian** if T preserves pullbacks and μ, η are cartesian natural transformations.

Theorem 2.8.10. *A plain operad is equivalent to a cartesian monad on* **Set** *equipped with a cartesian map of monads to the free monoid monad.*

Proof. See [Lei03] 6.2.4. Let 1 be the terminal plain operad; algebras for 1 are monoids. Since 1 is terminal, every plain operad P comes equipped with a map $! : P \to 1$. This induces a cartesian map of monads $T_! : T_P \to T_1$, and T_1 is the free monoid monad. $\square$

Lemma 2.8.11. *Let T, S be endofunctors on a category $\mathbb{A}$, let $\alpha : T \to S$ be a cartesian natural transformation, and let S preserve wide pullbacks. Then T preserves wide pullbacks.*

Proof. This follows from the facts that wide pullbacks are products in slice categories and that the functor $f^* : \mathbb{A}/B \to \mathbb{A}/A$ induced by a map $f : A \to B$ is product-preserving. $\square$

Corollary 2.8.12. *Let P be a plain operad. Then the functor part of the monad T_P arising from P preserves wide pullbacks.*

Definition 2.8.13. A functor $F : \mathcal{C} \to$ **Set** is **familially representable** if F is a coproduct of representable functors. A monad (T, μ, η) on **Set** is **familially representable** if T is familially representable.

Theorem 2.8.14. *(Carboni-Johnstone) Let $\mathcal{C}$ be a complete, locally small, well-powered category with a small cogenerating set, and let $F : \mathcal{C} \to$ **Set** be a functor. The following are equivalent:*

1. *F is familially representable;*

2. *F preserves wide pullbacks.*

Proof. See [CJ95], Theorem 2.6. $\square$

Corollary 2.8.15. *The monad associated to a strongly regular theory is familially representable.*

However, the inclusion is only one-way: there exist cartesian monads (T, μ, η) such that T is familially representable but the induced theory is not strongly regular. For instance, take the theory of involutive monoids:

Definition 2.8.16. An **involutive monoid** (or **monoid with involution**) is a monoid $(M, ., 1)$ equipped with an involution $i : M \to M$, satisfying $i(a.b) = i(b).i(a)$.

The theory of involutive monoids is familially representable, but not strongly regular — see [CJ04].

2.9 Enriched operads and multicategories

In the previous sections we considered operads P where $P_0, P_1, \cdots \in \mathbf{Set}$, and composition was given by functions. It is possible to consider operads where $P_0, P_1, \ldots$ lie in some other category; the resulting objects are called *enriched operads*. Enriched operads have many applications and a rich theory: for instance, topologists often consider operads enriched in **Top** or in some category of vector spaces. Our treatment here will be brief, sufficient only to set up the definitions of Chapter 4: for more on enriched operads, see [MSS02].

Throughout this section, let $(\mathcal{V}, \otimes, I, \alpha, \lambda, \rho, \tau)$ be a symmetric monoidal category.

Definition 2.9.1. A $\mathcal{V}$-**multicategory** $\mathcal{C}$ consists of the following:

- a collection $\mathcal{C}_0$ of *objects*,

- for all $n \in \mathbb{N}$ and all $c_1, \ldots, c_n, d \in \mathcal{C}_0$, an object $\mathcal{C}(c_1, \ldots, c_n; d) \in \mathcal{V}$ called the *arrows* from $c_1, \ldots, c_n$ to d,

- for all $n, k_1, \ldots k_n \in \mathbb{N}$ and $c_1^1 \ldots, c_{k_n}^n, d_1, \ldots, d_n, e \in \mathcal{C}_0$, an arrow in $\mathcal{V}$ called *composition*

$$\circ : \mathcal{C}(d_1, \ldots, d_n; e) \otimes \mathcal{C}(c_1^1, \ldots, c_{k_1}^1; d_1) \otimes \cdots \otimes \mathcal{C}(c_1^n, \ldots, c_{k_n}^n; d_n) \to \mathcal{C}(c_1^1, \ldots, c_{k_n}^n; e)$$

- for all $c \in \mathcal{C}$, a *unit* $u_c : I \to \mathcal{C}(c; c)$

satisfying the following axioms:

- *Associativity:* For all $b_{\bullet\bullet}^\bullet, c_\bullet^\bullet, d_\bullet, e \in \mathcal{C}$, the following diagram commutes:

$$
\begin{array}{ccc}
\begin{matrix} \mathcal{C}(d_\bullet; e) \otimes \mathcal{C}(c_\bullet^1; d_1) \otimes \cdots \otimes \mathcal{C}(c_\bullet^n; d_n) \\ \otimes \mathcal{C}(b_{1\bullet}^1; c_1^1) \otimes \cdots \otimes \mathcal{C}(b_{k_n\bullet}^n; c_{k_n}^n) \end{matrix} & \xrightarrow{\ \circ\ } & \mathcal{C}(d_\bullet; e) \otimes \mathcal{C}(b_{\bullet\bullet}^1; d_1) \otimes \cdots \otimes \mathcal{C}(b_{\bullet\bullet}^n; d_n) \\
\Big\downarrow{\circ} & & \Big\downarrow{\circ} \\
\mathcal{C}(c_\bullet^\bullet; e) \otimes \mathcal{C}(b_{1\bullet}^1; c_1^1) \otimes \cdots \otimes \mathcal{C}(b_{k_n\bullet}^n; c_{k_n}^n) & \xrightarrow{\ \circ\ } & \mathcal{C}(b_{\bullet\bullet}^\bullet; e)
\end{array}
$$

- *Units:* For all $c_\bullet, d \in \mathcal{C}$, the following diagram commutes:

$$\begin{array}{ccc}
\mathcal{C}(c_\bullet; d) & \xrightarrow{\quad \lambda \quad} & I \otimes \mathcal{C}(c_\bullet; d) \\
\end{array}$$

(diagram)

$\mathcal{C}(c_\bullet; d) \xrightarrow{\lambda} I \otimes \mathcal{C}(c_\bullet; d)$

$\rho^n \downarrow$, $u \otimes 1$

$\mathcal{C}(c_\bullet; d) \otimes I \otimes \cdots \otimes I$, 1 , $\mathcal{C}(d; d) \otimes \mathcal{C}(c_\bullet; d)$

$1 \otimes u^{\otimes n}$, $\circ$

$\mathcal{C}(c_\bullet; d) \otimes \mathcal{C}(c_1; c_1) \otimes \cdots \otimes \mathcal{C}(c_n; c_n) \xrightarrow{\quad \circ \quad} \mathcal{C}(c_\bullet; d)$

(We suppress the symmetry maps in $\mathcal{V}$ for clarity).

Definition 2.9.2. A **symmetric $\mathcal{V}$-multicategory** is a $\mathcal{V}$-multicategory $\mathcal{C}$ and, for every $n \in \mathbb{N}$, every $\sigma \in S_n$, and every $a_1, \ldots a_n, b \in \mathcal{C}$, an arrow

$$\sigma \cdot - : \mathcal{C}(a_1, \ldots, a_n; b) \longrightarrow \mathcal{C}(a_{\sigma 1}, \ldots, a_{\sigma n}; b)$$

in $\mathcal{V}$ such that

- for each $n \in \mathbb{N}$ and each $a_1, \ldots, a_n, b \in \mathcal{C}$, the arrow $1_n \cdot - : \mathcal{C}(a_1, \ldots, a_n; b) \to \mathcal{C}(a_1, \ldots, a_n; b)$ is the identity arrow on $\mathcal{C}(a_1, \ldots, a_n; b)$,

- for each $\sigma, \rho \in S_n$,
$$(\rho \cdot -)(\sigma \cdot -) = (\rho\sigma) \cdot -$$

- for each $n, k_1, \ldots, k_n \in n$, each $\sigma \in S_n$ and $\rho_i \in S_{k_i}$ for $i = 1, \ldots, n$, and for all $a_1^1, \ldots, a_{k_n}^n, b_1, \ldots, b_n, c \in \mathcal{C}$, the diagram

(diagram)

$\mathcal{C}(b_1, \ldots, b_n; c) \otimes \bigotimes_{i=1}^n \mathcal{C}(a_1^i, \ldots, a_n^i; b_i)$

$(\sigma \cdot -) \otimes \bigotimes_{i=1}^n (\rho_i \cdot -)$, $\circ$

$\mathcal{C}(b_{\sigma 1}, \ldots, b_{\sigma n}; c) \otimes \bigotimes_{i=1}^n \mathcal{C}(a_{\rho_i 1}^i, \ldots, a_{\rho_i n}^i; b_i)$

$1 \otimes \sigma_*$, $\mathcal{C}(a_1^1, \ldots, a_{k_n}^n; c)$

$\mathcal{C}(b_{\sigma 1}, \ldots, b_{\sigma n}; c) \otimes \bigotimes_{i=1}^n \mathcal{C}(a_{\rho_{\sigma i} 1}^{\sigma i}, \ldots, a_{\rho_{\sigma i} n}^{\sigma i}; b_{\sigma i})$

$\circ$, $\sigma \circ (\rho_1, \ldots, \rho_n)$

$\mathcal{C}(a_{\rho_{\sigma 1} 1}^{\sigma 1}, \ldots, a_{\rho_{\sigma n} k_n}^{\sigma n}; c)$

commutes, where $\sigma \circ (\rho_1, \ldots, \rho_n)$ is as defined in Example 2.1.11.

In the case $\mathcal{V} = \mathbf{Set}$ (with the cartesian monoidal structure), this is equivalent to Definition 2.2.1.

Let $\mathbb{F}$ be a skeleton of the category of finite sets and functions, with objects the sets $\underline{0}, \underline{1}, \underline{2}, \ldots$, where $\underline{n} = \{1, 2, \ldots, n\}$.

Definition 2.9.3. A **finite product $\mathcal{V}$-multicategory** is

- A plain $\mathcal{V}$-multicategory $\mathcal{C}$;

- for every function $f : \underline{n} \to \underline{m}$ in $\mathbb{F}$, and for all objects $C_1, \ldots, C_n, D \in \mathcal{C}$, a morphism

  $f \cdot - : \mathcal{C}(C_1, \ldots, C_n; D) \to \mathcal{C}(C_{f(1)}, \ldots, C_{f(n)}; D)$ in $\mathcal{V}$

satisfying the conditions given in Definition 2.9.2, where $(f \circ (f_1, \ldots, f_n))$ is as given in Definition 2.3.1.

In the case $\mathcal{V} = \mathbf{Set}$, this is equivalent to Definition 2.3.1.

Definition 2.9.4. A (**plain**, **symmetric**, **finite product**) $\mathcal{V}$-**operad** is a (plain, symmetric, finite product) $\mathcal{V}$-multicategory with only one object.

Definition 2.9.5. Let $\mathcal{C}, \mathcal{D}$ be plain $\mathcal{V}$-multicategories. A **morphism** $F : \mathcal{C} \to \mathcal{D}$ is

- for each object $C \in \mathcal{C}$, a choice of object $FC \in \mathcal{D}$,

- for each $n \in \mathbb{N}$ and all collections of objects $A_1, \ldots, A_n, B \in \mathcal{C}$, an arrow

  $\mathcal{C}(A_1, \ldots, A_n; B) \to \mathcal{D}(FA_1, \ldots, FA_n; FB)$ in $\mathcal{V}$

such that

- for all $A \in \mathcal{C}$, the diagram

$$
\begin{array}{ccc}
 & I & \\
 {}^{u}\swarrow & & \searrow^{u} \\
\mathcal{C}(A; A) & \xrightarrow{\quad F \quad} & \mathcal{D}(FA; FA)
\end{array}
$$

  commutes;

- for all $n, k_1, \ldots, k_n \in \mathbb{N}$ and all $C, B_1, \ldots, B_n, A_1^1, \ldots, A_{k_n}^n \in \mathcal{C}$, the diagram

$$
\begin{array}{ccc}
\mathcal{C}(B_\bullet; C) \otimes \bigotimes_{i=1}^n \mathcal{C}(A_\bullet^i; B_i) & \xrightarrow{\quad \circ \quad} & \mathcal{C}(A_\bullet^\bullet; C) \\
{\scriptstyle F \otimes \cdots \otimes F} \downarrow & & \downarrow {\scriptstyle F} \\
\mathcal{D}(FB_\bullet; FC) \otimes \bigotimes_{i=1}^n \mathcal{D}(FA_\bullet^i; FB_i) & \xrightarrow{\quad \circ \quad} & \mathcal{D}(FA_\bullet^\bullet; FC)
\end{array}
$$

commutes.

Suppose that $\mathcal{V}$ is cocomplete. Let Q be a (plain, symmetric, finite product) $\mathcal{V}$-operad, and A an object of $\mathcal{V}$. Let $Q \circ A$ denote the coend

$$\int^{n \in \mathcal{C}} Q_n \times A^n$$

where $\mathcal{C}$ is

- the discrete category on $\mathbb{N}$ if Q is a plain operad;

- a skeleton $\mathbb{B}$ of the category of finite sets and bijections if Q is a symmetric operad;

- a skeleton $\mathbb{F}$ of the category of finite sets and all functions if Q is a finite product operad.

This notation is taken from Kelly's papers [Kel72a] and [Kel72b] on clubs.

The various endomorphism operads defined in Examples 2.1.10, 2.2.7 and 2.3.5 transfer straightforwardly to the $\mathcal{V}$-enriched setting. An **algebra** for a (plain, symmetric, finite product) $\mathcal{V}$-operad P in a (plain, symmetric, finite product) $\mathcal{V}$-multicategory $\mathcal{C}$ is an object $A \in \mathcal{C}$ and a morphism $(\hat{\ }) : P \to \mathrm{End}(A)$ of the appropriate type. Equivalently, an algebra for P in $\mathcal{C}$ is an object $A \in \mathcal{C}$ and a morphism $h : P \circ A \to A$ such that the diagram

$$
\begin{array}{ccc}
P \circ P \circ A & \xrightarrow{1 \circ h} & P \circ A \\
{\scriptstyle \circ} \downarrow & & \downarrow {\scriptstyle h} \\
P \circ A & \xrightarrow{\quad h \quad} & A
\end{array}
$$

commutes, and $h(1_P, -)$ is the identity on A.

Remark 2.9.6. There is another possibility, that of considering *internal* multicategories in the category $\mathcal{V}$, which gives a different notion: now $\mathcal{C}_0$ is an object in $\mathcal{V}$ rather than a collection. An internal operad in $\mathcal{V}$ is an internal multicategory $\mathcal{C}$ such that $\mathcal{C}_0$ is terminal in $\mathcal{V}$. We shall not consider internal multicategories or operads further.

We shall in particular consider the case $\mathcal{V} = \mathbf{Cat}$, and $\mathbf{Cat}$-operads again have a simple concrete description:

Lemma 2.9.7. *A (plain)* $\mathbf{Cat}$*-operad Q is a sequence of categories $Q_0, Q_1, \ldots,$ a family of composition functors $\circ : Q_n \times Q_{k_1} \times \ldots \times Q_{k_n} \to Q_{\sum k_i}$ and an identity $1_Q \in Q_1$, satisfying (strict) functorial versions of the axioms given in 2.1.7.*

Lemma 2.9.8. *A symmetric* **Cat**-*operad is a plain* **Cat**-*operad Q with a left group action of each symmetric group S_n on the corresponding category Q_n, strictly satisfying equations as in Definition 2.2.5.*

Lemma 2.9.9. *A finite product* **Cat**-*operad is a plain* **Cat**-*operad Q equipped with functors $f \cdot - : Q_n \to Q_m$ for each function $f : \underline{n} \to \underline{m}$ of finite sets, strictly satisfying equations as in Definition 2.3.1.*

All of these lemmas can be established by a straightforward check of the definitions.

Just as 2-category theory has a special flavour distinct from the theory of $\mathcal{V}$-categories in the case $\mathcal{V} = \mathbf{Cat}$, so the theories of $\mathbf{Cat}$-operads and $\mathbf{Cat}$-multicategories have unique features:

Definition 2.9.10. Let Q be a finite product $\mathbf{Cat}$-operad, and let $Q \circ A \xrightarrow{\alpha} A, Q \circ B \xrightarrow{\beta} B$ be algebras for Q in $\mathbf{Cat}$. A **lax morphism of Q-algebras** $A \to B$ consists of a 1-cell $F : A \to B$ and a 2-cell $\phi : \beta F \to F\alpha$ satisfying the following conditions:

$$\tag{2.23}$$

$$\tag{2.24}$$

$$
\begin{array}{c}
Q_m \times A^n \\
\xrightarrow{1\times f^*} \quad \xrightarrow{1\times F^n} \\
Q_m \times A^m \quad = \quad Q_m \times B^n \\
1\times F^m \quad 1\times f^* \\
\alpha \quad Q_n \times B^m \quad f_* \times 1 \quad = \\
\Downarrow_\phi \quad \nearrow\!\!/ \\
A \quad \beta \quad Q_n \times B^n \\
F \quad \beta \\
B
\end{array}
\qquad
\begin{array}{c}
Q_m \times A^n \\
\xrightarrow{1\times f^*} \quad \xrightarrow{1\times F^n} \\
Q_m \times A^m \quad f_* \times 1 \quad Q_m \times B^n \\
/\!/ \quad \quad \backslash\!\backslash \\
\alpha \quad Q_n \times A^n \quad f_* \times 1 \\
\alpha \quad 1\times F^n \\
A \quad \underset{\phi}{\Leftarrow} \quad Q_n \times B^n \\
F \quad \beta \\
B
\end{array}
\tag{2.25}
$$

for all functions $f : \underline{m} \to \underline{n}$.

A morphism (F, ϕ) is **weak** if ϕ is invertible, and **strict** if ϕ is an identity.

Lax morphisms for algebras of plain **Cat**-operads are required to satisfy 2.23 and 2.24, and lax morphisms for algebras of symmetric **Cat**-operads are required to satisfy 2.23, 2.24 and the restriction of 2.25 to the case where f is a bijection.

We shall make use of a more explicit formulation in the plain case.

Lemma 2.9.11. *Let Q be a plain **Cat**-operad, and let (A, h) and (B, h') be Q-algebras. A lax map of Q-algebras $(A, h) \to (B, h')$ is a pair (G, ψ), where $G : A \to B$ is a functor and ψ is a sequence of natural transformations $\psi_i : h'_i(1 \times G^i) \to Gh_i$, called the **coherence maps**, such that the following equation holds, for all $n, k_1, \ldots, k_n \in \mathbb{N}$:*

$$
\begin{array}{ccccc}
Q_n \times Q_{k_1} \times \cdots \times Q_{k_n} \times A^{\sum k_i} & \xrightarrow{h_{k_1}\times\cdots\times h_{k_n}} & Q_n \times A^n & \xrightarrow{h_n} & A \\
{\scriptstyle 1\times 1^n \times G^{\sum k_i}}\Big\downarrow & {}^{\psi_{k_1}\times\cdots\times\psi_{k_n}}\!\!\Nearrow & \Big\downarrow{\scriptstyle 1\times G^n} & {}^{\psi_n}\!\!\Nearrow & \Big\downarrow{\scriptstyle G} \\
Q_n \times Q_{k_1} \times \cdots \times Q_{k_n} \times A^{\sum k_i} & \xrightarrow[h'_{k_1}\times\cdots\times h'_{k_n}]{} & Q_n \times B^n & \xrightarrow[h']{} & B
\end{array}
$$

$$
= \quad
\begin{array}{ccc}
Q_n \times Q_{k_1} \times \cdots \times Q_{k_n} \times A^{\sum k_i} & \xrightarrow{h_{\sum k_i}} & A \\
{\scriptstyle 1\times 1^n \times G^{\sum k_i}}\Big\downarrow & {}^{\psi_{\sum k_i}}\!\!\Nearrow & \Big\downarrow{\scriptstyle G} \\
Q_n \times Q_{k_1} \times \cdots \times Q_{k_n} \times B^{\sum k_i} & \xrightarrow[h'_{\sum k_i}]{} & B
\end{array}
\tag{2.26}
$$

and the diagram

$$
\begin{array}{ccc}
Ga & \xrightarrow{\ \delta_1'\ } & h'(1_P, Ga) \\
\downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle \psi_1} \\
Ga & \xrightarrow[\ G\delta_1\ ]{} & Gh(1_P, a)
\end{array}
\tag{2.27}
$$

commutes. The morphism is weak if every ψ is invertible, and strict if every ψ is an identity.

Proof. This can be established by a straightforward check of the definition. $\qquad\square$

Definition 2.9.12. Let Q, A, B etc. be as above, and let $(F, \phi), (G, \gamma)$ be lax morphisms of Q-algebras $A \to B$. A Q-**transformation** $F \to G$ is a natural transformation $\sigma : F \to G$ such that

$$
\begin{array}{ccc}
Q \circ A \overset{\overset{1 \circ F}{\Downarrow \sigma}}{\underset{1 \circ G}{\longrightarrow}} Q \circ B & & Q \circ A \xrightarrow{\ 1 \circ F\ } Q \circ B \\
{\scriptstyle \alpha}\downarrow \quad {\Swarrow}{\scriptstyle \gamma} \quad \downarrow{\scriptstyle \beta} \qquad = & & {\scriptstyle \alpha}\downarrow \quad {\Swarrow}{\scriptstyle \phi} \quad \downarrow{\scriptstyle \beta} \\
A \xrightarrow[\ G\ ]{} B & & A \overset{F}{\underset{G}{\rightrightarrows}} B
\end{array}
\tag{2.28}
$$

Lemma 2.9.13. *A Q-transformation $\sigma : (F, \phi) \to (G, \psi)$ is invertible as a natural transformation if and only if it is invertible as a Q-transformation.*

Proof. "If" is obvious: we concentrate on "only if". It is enough to show that σ^{-1} is a Q-transformation, which is to say that

$$
\begin{array}{ccc}
h(q, Ga_\bullet) & \xrightarrow{\ \psi\ } & Gh(q, a_\bullet) \\
{\scriptstyle h(q, \sigma_{a_\bullet}^{-1})}\downarrow & & \downarrow{\scriptstyle \sigma_{h(q, a_\bullet)}^{-1}} \\
h(q, Fa_\bullet) & \xrightarrow[\ \phi\ ]{} & Fh(q, a_\bullet)
\end{array}
\tag{2.29}
$$

commutes for all $(q, a_\bullet) \in Q \circ A$, and this follows from the fact that $\sigma_{h(q, a_\bullet)} \circ \phi = \psi \circ h(q, \sigma_{a_\bullet})$.
$\qquad\square$

Finite product **Cat**-operads, their morphisms and transformations form a 2-category called **Cat-FP-Operad**. Similarly, there is a 2-category **Cat-Operad** of plain **Cat**-operads, their morphisms and transformations, and a 2-category **Cat-Σ-Operad**, of symmetric operads, their morphisms and transformations.

Theorem 2.9.14. *There is a chain of monadic adjunctions*

$$(2.30)$$

Proof. This follows from Lemmas 2.5.1 and 2.5.3, via an application of the argument of Theorem 2.5.4. □

Since operads can be considered as one-object multicategories, a **Cat**-operad P (of whatever type) is really a 2-dimensional structure. We will therefore refer to the objects and morphisms of the categories P_i as **1-cells** and **2-cells** of P, respectively.

2.10 Maps of algebras as algebras for a multicategory

Let P be a plain operad. We form a multicategory $\bar{P} = \mathbf{2} \times P$, where $\mathbf{2}$ is the category $(\,\cdot \longrightarrow \cdot\,)$. We may describe $\bar{P}$ as follows: there are two objects, labelled 0 and 1; the hom-sets $\bar{P}(0,\ldots,0;0)$ and $\bar{P}(x_1,\ldots,x_n;1)$ are copies of P_n, for $x_i \in \{0,1\}$, and $\bar{P}(x_1,\ldots,x_n;0) = \emptyset$ if any of the x_is are 1. Composition is given by composition in P. An algebra for $\bar{P}$ is a pair A_0, A_1 of P-algebras, and a morphism of P-algebras $A_0 \to A_2$. See [Mar], Example 2.4 for more details.

We can extend this construction by defining a multicategory $\bar{\bar{P}} = \mathbf{3} \times P$, whose algebras are composable pairs of maps of P-algebras, a multicategory $\bar{\bar{\bar{P}}} = \mathbf{4} \times P$ whose algebras are composable triples of maps of P-algebras, and so on. With the obvious face and degeneracy maps, these multicategories form a cosimplicial object in the category of plain multicategories.

The same construction can be performed for symmetric and enriched operads, and the result continues to hold.

Chapter 3

Factorization Systems

The theory of factorization systems was introduced by Freyd and Kelly in [FK72] (though it was implicit in work of Isbell in the 1950s). We shall use it in subsequent chapters to define the weakening of an algebraic theory. Here, we recall the basic definitions and some relevant theorems.

The material in this chapter is standard, and may be found in (for instance) [Bor94] or [AHS04]; for an alternative perspective and some more historical background (as well as the interesting generalization to *weak* factorization systems), see [KT93].

Definition 3.0.1. Let $e : a \to b$ and $m : c \to d$ be arrows in a category $\mathcal{C}$. We say that e is **left orthogonal** to m, written $e \perp m$, if, for all arrows $f : a \to c$ and $g : b \to d$ such that $mf = ge$, there exists a unique map $t : b \to c$ such that the following diagram commutes:

$$
\begin{array}{ccc}
a & \xrightarrow{\;\forall f\;} & c \\
{\scriptstyle e}\downarrow & {\scriptstyle \exists! t}\nearrow & \downarrow{\scriptstyle m} \\
b & \xrightarrow[\;\forall g\;]{} & d
\end{array}
$$

Definition 3.0.2. Let $\mathcal{C}$ be a category. A **factorization system** on $\mathcal{C}$ is a pair $(\mathcal{E}, \mathcal{M})$ of classes of maps in $\mathcal{C}$ such that

1. for all maps f in $\mathcal{C}$, there exist $e \in \mathcal{E}$ and $m \in \mathcal{M}$ such that $f = me$;

2. $\mathcal{E}$ and $\mathcal{M}$ contain all identities, and are closed under composition with isomorphisms on both sides;

3. $\mathcal{E} \perp \mathcal{M}$, i.e. $e \perp m$ for all $e \in \mathcal{E}$ and $m \in \mathcal{M}$.

Example 3.0.3. Let $\mathcal{C} = \mathbf{Set}$, $\mathcal{E}$ be the epimorphisms, and $\mathcal{M}$ be the monomorphisms. Then $(\mathcal{E}, \mathcal{M})$ is a factorization system.

Example 3.0.4. More generally, let $\mathcal{C}$ be some variety of algebras, $\mathcal{E}$ be the regular epimorphisms (i.e., the surjections), and $\mathcal{M}$ be the monomorphisms. Then $(\mathcal{E}, \mathcal{M})$ is a factorization system.

Example 3.0.5. Let $\mathcal{C} = \mathbf{Digraph}$, the category of directed graphs and graph morphisms. Let $\mathcal{E}$ be the maps bijective on objects, and $\mathcal{M}$ be the full and faithful maps. Then $(\mathcal{E}, \mathcal{M})$ is a factorization system.

In deference to Example 3.0.3, we shall use arrows like $\longrightarrow\!\!\!\!\rightarrow$ to denote members of $\mathcal{E}$ in commutative diagrams, and arrows like $\rightarrowtail$ to denote members of $\mathcal{M}$, for whatever values of $\mathcal{E}$ and $\mathcal{M}$ happen to be in force at the time.

We will use without proof the following standard properties of factorization systems:

Lemma 3.0.6. *Let $\mathcal{C}$ be a category, and $(\mathcal{E}, \mathcal{M})$ be a factorization system on $\mathcal{C}$.*

1. *$\mathcal{E} \cap \mathcal{M}$ is the class of isomorphisms in $\mathcal{C}$.*

2. *The factorization in 3.0.2 (1) is unique up to unique isomorphism.*

3. *The factorization in 3.0.2 (1) is functorial, in the following sense: if the square*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ C & \xrightarrow{f'} & D \end{array}$$

commutes, and $f = me, f' = m'e'$, then there is a unique morphism i making

$$\begin{array}{ccccc} A & \xrightarrow{e} & \rightarrowtail & \xrightarrow{m} & B \\ {\scriptstyle g}\downarrow & & {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle h} \\ C & \xrightarrow{e'} & \rightarrowtail & \xrightarrow{m'} & D \end{array}$$

commute. Thus, given a choice of $e \in \mathcal{E}$ and $m \in \mathcal{M}$ for each f in $\mathcal{C}$ (such that $f = me$), we may construct functors $\mathcal{E}_*, \mathcal{M}_* : [\mathbf{2}, \mathcal{C}] \to [\mathbf{2}, \mathcal{C}]$:

$$\mathcal{E}_* : f \mapsto e$$
$$\mathcal{E}_* : (g, h) \mapsto (g, i)$$
$$\mathcal{M}_* : f \mapsto m$$
$$\mathcal{M}_* : (g, h) \mapsto (i, h).$$

These functors are determined by $\mathcal{E}$ and $\mathcal{M}$ uniquely up to unique isomorphism.

4. $\mathcal{E}$ and $\mathcal{M}$ are closed under composition.

5. $\mathcal{E}^{\perp} = \mathcal{M}$ and $^{\perp}\mathcal{M} = \mathcal{E}$, where $\mathcal{E}^{\perp} = \{f \text{ in } \mathcal{C} : e \perp f \text{ for all } e \in \mathcal{E}\}$ and $^{\perp}\mathcal{M} = \{f \text{ in } \mathcal{C} : f \perp m \text{ for all } m \in \mathcal{M}\}$.

Proofs of these statements may be found in [AHS04] section 14.

We will also use the following fact:

Lemma 3.0.7. *Let $\mathcal{C}$ be a category with a factorization system $(\mathcal{E}, \mathcal{M})$. Let T be a monad on $\mathcal{C}$ and let $\overline{\mathcal{E}} = \{f \text{ in } \mathcal{C} : Uf \in \mathcal{E}\}$ and $\overline{\mathcal{M}} = \{f \text{ in } \mathcal{C} : Uf \in \mathcal{M}\}$, where U is the forgetful functor $\mathcal{C}^T \to \mathcal{C}$. Then $(\overline{\mathcal{E}}, \overline{\mathcal{M}})$ is a factorization system on $\mathcal{C}^T$ if T preserves $\mathcal{E}$-arrows.*

Proof. This is established in [AHS04], Proposition 20.24: however, we shall provide a proof for the reader's convenience. We shall establish the axioms listed in Definition 3.0.2.

1. Take an algebra map

$$
\begin{array}{ccc}
TA & \xrightarrow{Tf} & TB \\
{\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\
A & \xrightarrow{f} & B
\end{array}
$$

Applying axiom 1 to the factorization system $(\mathcal{E}, \mathcal{M})$, we obtain a decomposition $f = me$, where $e : A \to I$ and $m : I \to B$. We wish to lift this to a decomposition of f as an algebra map. In other words, we need a map $i : TI \to I$ making the diagram

$$
\begin{array}{ccccc}
TA & \xrightarrow{Te} & TI & \xrightarrow{Tm} & TB \\
{\scriptstyle a}\downarrow & & {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle b} \\
A & \xrightarrow{e} & I & \xrightarrow{m} & B
\end{array}
$$

commute, such that (I, i) is a T-algebra. Since T preserves $\mathcal{E}$-arrows, $Te \perp m$, and we may obtain i by applying this orthogonality to the diagram

$$
\begin{array}{ccccc}
TA & \xrightarrow{a} & A & \xrightarrow{e} & I \\
{\scriptstyle Te}\downarrow & & {\scriptstyle \exists!i} & \nearrow & \downarrow{\scriptstyle m} \\
TI & \xrightarrow{Tm} & TB & \xrightarrow{b} & B.
\end{array}
$$

It remains to show that (I, i) is a T-algebra. For the unit axiom, consider the diagram

$$
\begin{array}{ccccc}
A & \xrightarrow{e} & I & \xrightarrow{m} & B \\
{\scriptstyle \eta_A}\downarrow & & {\scriptstyle \eta_I}\downarrow & & \downarrow{\scriptstyle \eta_B} \\
TA & \xrightarrow{Te} & TI & \xrightarrow{Tm} & TB \\
{\scriptstyle a}\downarrow & & {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle b} \\
A & \xrightarrow{e} & E & \xrightarrow{m} & B
\end{array}
$$

The top squares commute by naturality, and the outside triangles commute since (A, a) and (B, b) are T-algebras. Hence the diagram

$$
\begin{array}{ccc}
 & I & \\
{}^{e}\nearrow & \vdots & \searrow^{m} \\
A & \vdots & B \\
{}_{e}\searrow & \vdots & \nearrow_{m} \\
 & I &
\end{array}
$$
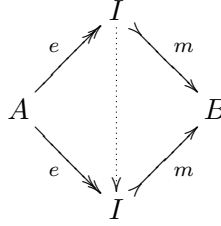
commutes if the dotted arrow is either 1_I or $i\eta_I$. By orthogonality, $i\eta_I = 1_I$.

For the multiplication axiom, observe that the diagrams

$$
\begin{array}{ccc}
T^2A \xrightarrow{T^2e} T^2I \xrightarrow{T^2m} T^2B \\
\mu_A\downarrow \qquad \mu_I\downarrow \qquad \mu_B\downarrow \\
TA \xrightarrow{Te} TI \xrightarrow{Tm} TB \\
a\downarrow \qquad i\downarrow \qquad b\downarrow \\
A \xrightarrow{e} E \xrightarrow{m} B
\end{array}
\qquad
\begin{array}{ccc}
T^2A \xrightarrow{T^2e} T^2I \xrightarrow{T^2m} T^2B \\
Ta\downarrow \qquad Ti\downarrow \qquad Tb\downarrow \\
a\mu_A\; TA \xrightarrow{Te} TI \xrightarrow{Tm} TB \;b\mu_B \\
a\downarrow \qquad i\downarrow \qquad b\downarrow \\
A \xrightarrow{e} E \xrightarrow{m} B
\end{array}
$$
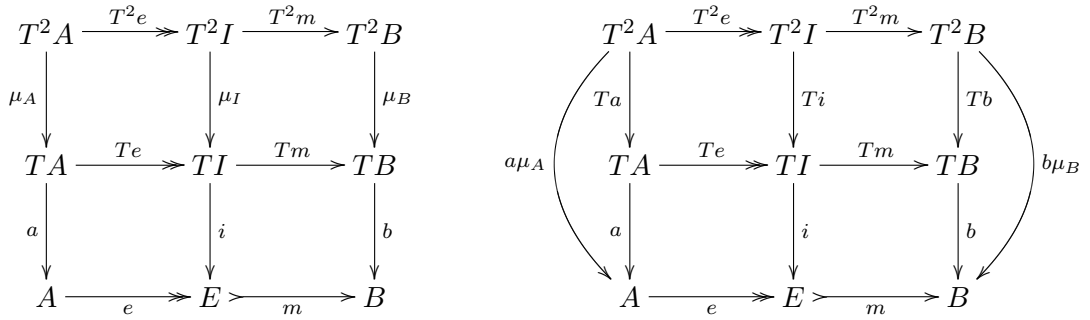
both commute. So the diagram

$$
\begin{array}{ccccc}
T^2A \xrightarrow{\mu_A} TA \xrightarrow{a} A \xrightarrow{e} I \\
T^2e\downarrow \qquad\qquad\qquad\qquad m\downarrow \\
T^2I \xrightarrow{T^2m} T^2B \xrightarrow{\mu_B} TB \xrightarrow{b} B
\end{array}
$$
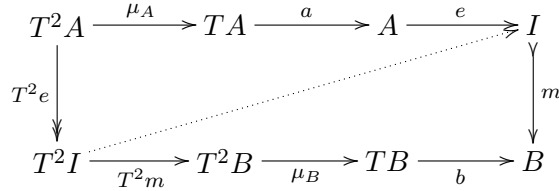
commutes if we take the dotted arrow to be either $i\mu_I$ or $i(Ti)$. By orthogonality, $i\mu_I = i(Ti)$.

2. The image under U of an isomorphism in $\mathcal{C}^T$ is an isomorphism in $\mathcal{C}$. The class $\mathcal{E}$ contains all isomorphisms in $\mathcal{C}$, so $\overline{\mathcal{E}} = U^{-1}(\mathcal{E})$ contains all isomorphisms in $\mathcal{C}^T$. By similar reasoning, $\overline{\mathcal{E}}$ is closed under composition with isomorphisms, and $\overline{\mathcal{M}}$ also satisfies these conditions.

3. We wish to show that $\overline{\mathcal{E}} \perp \overline{\mathcal{M}}$. Take T-algebras

$$
\begin{pmatrix} TA \\ \downarrow a \\ A \end{pmatrix}, \begin{pmatrix} TB \\ \downarrow b \\ B \end{pmatrix}, \begin{pmatrix} TI \\ \downarrow i \\ I \end{pmatrix}, \begin{pmatrix} TJ \\ \downarrow j \\ J \end{pmatrix}
$$

and algebra maps

$$TA \xrightarrow{Te} TI \; , \; TJ \xrightarrow{Tm} TB \; , \; TA \xrightarrow{Tf} TJ \; , \; TI \xrightarrow{Tg} TB$$

where the first two maps are in $\overline{\mathcal{E}}$ and $\overline{\mathcal{M}}$ respectively. Suppose that $ge = mf$. Now, $e \in \mathcal{E}$ and $m \in \mathcal{M}$, so $e \perp m$, and there is a unique map t in $\mathcal{C}$ such that

$$
\begin{array}{ccc}
A & \xrightarrow{f} & J \\
{\scriptstyle e}\downarrow & {\scriptstyle \exists! t}\nearrow & \downarrow{\scriptstyle m} \\
I & \xrightarrow{g} & B
\end{array}
$$

commutes. We wish to show that t is a map of T-algebras.

Consider the diagram

We wish to show that the middle square commutes: the assumptions tell us that all other squares commute. Recall that $Te \perp m$, and apply orthogonality to the square

$$
\begin{array}{ccc}
TA & \xrightarrow{jTf} & J \\
{\scriptstyle Te}\downarrow & {\scriptstyle \exists! u}\nearrow & \downarrow{\scriptstyle m} \\
TI & \xrightarrow{gi} & B
\end{array}
$$

Now,

$$
\begin{aligned}
j(Tf) &= fa & &(f \text{ is a map of } T\text{-algebras}) \\
&= tea & &(\text{Definition of } t) \\
&= ti(Te) & &(e \text{ is a map of } T\text{-algebras})
\end{aligned}
$$

and $mti = gi$ by definition of t, so $ti = u$ by uniqueness. Similarly,

$$
\begin{aligned}
gi &= b(Tg) & &(g \text{ is a map of } T\text{-algebras}) \\
&= b(Tm)(Tt) & &(\text{Definition of } t) \\
&= mj(Tt) & &(m \text{ is a map of } T\text{-algebras})
\end{aligned}
$$

and $j(Tf) = j(Tt)(Te)$ by definition of t, so $j(Tt) = u$ by uniqueness. Hence $j(Tt) = ti$, and t is a map of T-algebras.

By construction, t is unique. So $e \perp m$ in $\mathcal{C}^T$, so $\overline{\mathcal{E}} \perp \overline{\mathcal{M}}$. All the axioms are satisfied, and so $(\overline{\mathcal{E}}, \overline{\mathcal{M}})$ is a factorization system on $\mathcal{C}^T$. $\qquad\square$

Example 3.0.8. Let $(\mathcal{E}, \mathcal{M})$ be the factorization system on **Digraph** described in Example 3.0.5 above, and let T be the free category monad. **Cat** is monadic over **Digraph**, and T preserves the property of being bijective on objects. Hence, this gives a factorization system $(\overline{\mathcal{E}}, \overline{\mathcal{M}})$ on **Cat** where $\overline{\mathcal{E}}$ is the collection of bijective-on-objects functors, and $\overline{\mathcal{M}}$ is the collection of full and faithful functors.

Example 3.0.9. Similarly, there is a factorization system on **Digraph**$^{\mathbb{N}}$, where $\mathcal{E}$ is the class of maps that are pointwise bijective on objects, and $\mathcal{M}$ is the class of maps that are pointwise full and faithful. This lifts to a factorization system $(\overline{\mathcal{E}}, \overline{\mathcal{M}})$ on **Cat**$^{\mathbb{N}}$, in which $\overline{\mathcal{E}}$ is the class of pointwise bijective-on-objects arrows, and $\overline{\mathcal{M}}$ is the class of pointwise full-and-faithful arrows.

Example 3.0.10. Let $\mathcal{C} = \mathbf{Cat}^{\mathbb{N}}$, $\mathcal{E}$ be the pointwise bijective-on-objects maps, and $\mathcal{M}$ be those that are pointwise full and faithful. Since **Cat-Operad** is monadic over **Cat**$^{\mathbb{N}}$ and the monad preserves bijective-on-objects maps, this gives a factorization system $(\overline{\mathcal{E}}, \overline{\mathcal{M}})$ on **Cat-Operad** where $\overline{\mathcal{E}}$ is the class of levelwise bijective-on-objects maps, and $\overline{\mathcal{M}}$ is the class of levelwise full and faithful ones. Similarly, there is a factorization system $(\overline{\mathcal{E}}', \overline{\mathcal{M}}')$ on **Cat-Σ-Operad** where $\overline{\mathcal{E}}'$ is the class of bijective-on-objects maps, and $\overline{\mathcal{M}}'$ is the class of levelwise full and faithful ones.

We shall need one final piece of background:

Theorem 3.0.11. *If X is a set and T is a monad on $\mathbf{Set}^X$ then the regular epis in $(\mathbf{Set}^X)^T$ are the pointwise surjections. In other words, the forgetful functor $U : (\mathbf{Set}^X)^T \to \mathbf{Set}^X$ preserves and reflects regular epis.*

Proof. See again [AHS04] section 20, in particular Definition 20.21 and Proposition 20.30. $\square$

Chapter 4

Categorification

4.1 Desiderata

Many categorifications of individual theories have been proposed in the literature. We aim to replace these with a general definition, which should satisfy the following criteria insofar as possible:

- **Broad:** it should cover as large a class of theories as possible.

- **Consistent with earlier work:** where a categorification of a given theory is known, ours should agree with this categorification or be demonstrably better in some way.

- **Canonical:** it should be free of arbitrary tunable parameters (and if possible should be given by some universal property).

We shall return to these criteria in Section 4.9 and evaluate how close we have come to achieving them.

Our strategy is as follows: we start with a naïve version of categorification for strongly regular theories, which closely parallels Mac Lane and Benabou's categorification of the theory of monoids. This will be an *unbiased* categorification, which treats all operations equally, without regarding any as "primitive": for instance, if P is the terminal operad (whose strict algebras are monoids), then the weak P-categories will have tensor products of all arities, not just 0 and 2. We then re-express our definition of categorification in terms of factorization systems, which allows us to generalize our definition in two directions simultaneously: to symmetric operads, and to operads with presentations. We then use this new definition to recover the classical theory of symmetric monoidal categories (at

which several other proposed general definitions of categorification fail), and investigate what it yields in the case of some other linear theories.

4.2 Categorification of strongly regular theories

The idea is to consider the strict models of our theory as algebras for an operad, then to obtain the weak models as (strict) algebras for a weakened version of that operad (which will be a **Cat**-operad). We weaken the operad using a similar approach to that used in Penon's definition of n-category, as described in [Pen99]. A non-rigorous summary of Penon's construction can be found in [CL04].

Throughout this section, let P be a plain (**Set**-)operad.

Let $D_* : \mathbf{Operad} \to \mathbf{Cat\text{-}Operad}$ be the functor which takes discrete categories levelwise; i.e., $(D_*P)_n$ is the discrete category on the set P_n. In terms of the "n-cell" terminology introduced in Chapter 3, the 1-cells of $(D_*P)_n$ are n-ary arrows in P, and the only 2-cells are identities.

Definition 4.2.1. The **unbiased weakening of** P, $\mathrm{Wk}(P)$, is the following **Cat**-operad:

- *1-cells:* 1-cells of $D_*F_{\mathrm{pl}}U^{\mathrm{pl}}P$;

- *2-cells:* if $A, B \in (F_{\mathrm{pl}}U^{\mathrm{pl}}P)_n$, there is a single 2-cell $A \to B$ if $\epsilon(A) = \epsilon(B)$ (where ϵ is the counit of the adjunction $F_{\mathrm{pl}} \dashv U^{\mathrm{pl}}$), and no 2-cells $A \to B$ otherwise;

- *Composition of 2-cells:* the composite of two arrows $A \to B \to C$ is the unique arrow $A \to C$, and in particular, the arrows $A \to B$ and $B \to A$ are inverses;

- *Operadic composition:* on 1-cells, as in $F_{\mathrm{pl}}U^{\mathrm{pl}}P$, and on 2-cells, determined by the uniqueness property.

See Fig. 4.1, which illustrates a fragment of the unbiased weakening of the terminal operad 1. Since 1_n is a singleton set for every $n \in \mathbb{N}$, then $\mathrm{Wk}(1)_n$ is the indiscrete category whose objects are unlabelled n-ary strongly regular trees for all $n \in \mathbb{N}$. We may embed the discrete category on each P_n in $\mathrm{Wk}(P)_n$, via the map $p \mapsto p \circ (|, \ldots, |)$. We shall occasionally abuse notation and consider some $p \in P_n$ as a 1-cell of $\mathrm{Wk}(P)_n$.

Theorem 4.2.2. $\mathrm{Wk}(P)$ *is the unique* **Cat**-*operad with the following properties:*

- $\mathrm{Wk}(P)$ *has the same 1-cells as* $D_*F_{\mathrm{pl}}U^{\mathrm{pl}}P$;
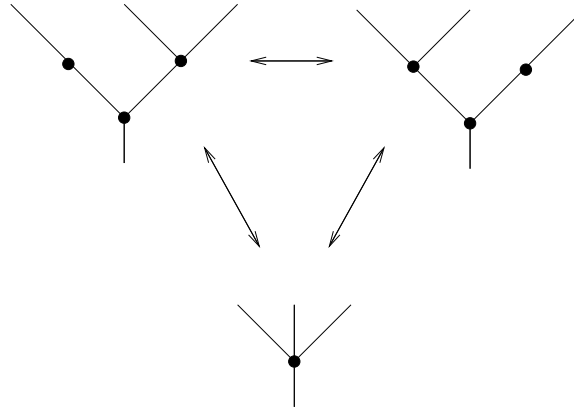
Figure 4.1: Part of $\mathrm{Wk}(1)_3$

- *we may extend the counit $\epsilon_P : F_{\mathrm{pl}}U^{\mathrm{pl}}P \to P$ to a map of* **Cat**-*operads* $\mathrm{Wk}(P) \to D_*P$, *which is full and faithful levelwise.*

Proof. Immediate. □

We may now make the following definition:

Definition 4.2.3. A **weak P-category** is an algebra for $\mathrm{Wk}(P)$.

In the case $P = 1$, this reduces exactly to Leinster's definition of unbiased monoidal category in [Lei03] section 3.1. There, two 1-cells ϕ and ψ have the same image under ϵ iff they have the same arity, so the categories $\mathrm{Wk}(1)_i$ are indiscrete. If $h : \mathrm{Wk}(P) \circ A \to A$ is a weak P-category, we refer to the image under h of a 2-cell $q \to q'$ in $\mathrm{Wk}(P)$ as $\delta_{q,q'}$. This is clearly a natural transformation $h(q, -) \to h(q', -)$. As a special case, we write δ_q for $\delta_{q,\epsilon(q)}$ (where we consider $\epsilon(q)$ as a 1-cell of $\mathrm{Wk}(P)$ as described above).

Definition 4.2.4. A **strict P-category** is an algebra for D_*P.

Equivalently, a strict P-category is a weak P-category in which every component of δ is an identity arrow.

Definition 4.2.5. Let (A, h) and (B, h') be weak P-categories. A **weak P-functor** from (A, h) to (B, h') is a weak map of $\mathrm{Wk}(P)$-algebras. A **strict P-functor** from (A, h) to (B, h') is a strict map of $\mathrm{Wk}(P)$-algebras.

Equivalently, a strict P-functor is a weak P-functor for which all the coherence maps are identities. These definition are natural generalizations of the definition of weak and strict unbiased monoidal functors given in [Lei03] section 3.1.

Definition 4.2.6. Let (F, ϕ) and (G, ψ) be weak P-functors $(A, h) \to (B, h')$. A P-**transformation** $\sigma : (F, \phi) \to (G, \psi)$ is a $\mathrm{Wk}(P)$-transformation $(F, \phi) \to (G, \psi)$, in the sense of Definition 2.9.12.

Note that there is only one possible level of strictness here.

There is a 2-category, **Wk-P-Cat**, whose objects are weak P-categories, whose 1-cells are weak P-functors, and whose 2-cells are P-transformations. Similarly, there is a 2-category **Str-P-Cat** of strict P-categories, strict P-functors, and P-transformations, which can be considered a sub-2-category of **Wk-P-Cat**.

Definition 4.2.7. A P-**equivalence** is an equivalence in the 2-category **Wk-P-Cat**.

Lemma 4.2.8. *Let P be a plain operad, (A, h) and (B, h') be weak P-categories, and $(F, \phi), (G, \psi) : (A, h) \to (B, h')$ be weak P-functors. A P-transformation $\sigma : (F, \phi) \to (G, \psi)$ is invertible as a P-transformation if and only if it is invertible as a natural transformation.*

Proof. This is a straightforward application of Lemma 2.9.13. $\square$

4.3 Examples

Unfortunately, few well-studied theories are strongly regular. We will consider the following examples:

1. the trivial theory (in other words, the theory of sets);

2. the theory of pointed sets;

3. the theory of monoids;

4. the theory of M-sets, for a monoid M.

While we could easily invent a new strongly regular theory to categorify, this would not help us to see how well our definition of weakening accords with our intuitions. Further examples will be considered later, when the machinery to categorify theories-with-generators and linear theories has been developed.

We will first need to introduce an auxiliary definition:

Definition 4.3.1. Let $\mathcal{C}$ be a category, and (T, μ, η) be a monad on $\mathcal{C}$. We say that (T, μ, η) is **trivial** if η is a natural isomorphism.

Lemma 4.3.2. *The identity monad on $\mathcal{C}$ is initial in the category $\mathbf{Mnd}(\mathcal{C})$ of monads on $\mathcal{C}$, with the unique morphism of monads $(1_{\mathcal{C}}, 1, 1) \rightarrow (T, \mu, \eta)$ being η.*

Proof. First we show that η is a morphism of monads in the sense of Street (Definition 1.4.1). One axiom corresponds to the outside of the diagram

$$
\begin{array}{ccccc}
1 & \xrightarrow{\ \eta\ } & T & \xrightarrow{\ \eta T\ } & T^2 \\
{\scriptstyle 1}\downarrow & & {\scriptstyle 1}\downarrow & {\scriptstyle 1}\searrow & \downarrow{\scriptstyle \mu} \\
1 & \xrightarrow{\ \eta\ } & T & \xrightarrow{\ 1\ } & T
\end{array}
$$

commuting; all the inner segments commute (the top right triangle by the unit axiom for monads), so the outside must commute. The other axiom corresponds to the diagram

$$
\begin{array}{ccc}
1 & \xrightarrow{\ 1\ } & 1 \\
& {\scriptstyle \eta}\searrow & \downarrow{\scriptstyle \eta} \\
& & T
\end{array}
$$

and this commutes trivially. Hence, η is a morphism of monads $1 \rightarrow T$.

Now suppose that $\alpha : 1 \rightarrow T$ is a morphism of monads. From the unit axiom for monad morphisms, the diagram

$$
\begin{array}{ccc}
1 & \xrightarrow{\ 1\ } & 1 \\
& {\scriptstyle \eta}\searrow & \downarrow{\scriptstyle \alpha} \\
& & T
\end{array}
$$

must commute, so $\eta = \alpha$. $\qquad\square$

Corollary 4.3.3. *A monad (T, μ, η) on $\mathcal{C}$ is trivial if and only if it is isomorphic to the identity monad on $\mathcal{C}$.*

Proof. If T is isomorphic to the identity monad, then by Lemma 4.3.2 the isomorphism concerned must be η, so η must be invertible. It is readily checked that if η is invertible, then η^{-1} must be a morphism of monads, so if T is trivial then it is isomorphic to the identity monad. $\qquad\square$

Example 4.3.4. *The trivial theory:* Let 0 be the initial operad, whose algebras are sets. An unbiased weak 0-category is a category equipped with a specified trivial monad, for the following reason. 0 has only one operator (call it I), of arity one. Hence $(F_{\mathrm{pl}}U^{\mathrm{pl}}0)_1 \cong \mathbb{N}$, and all other $(F_{\mathrm{pl}}U^{\mathrm{pl}}0)_n$'s are empty. All derived operations in the theory of sets are composites of identities, and thus equivalent to the identity. So all objects of $\mathrm{Wk}(0)_1$ are isomorphic. Hence, $I \cong \mathrm{id}$. All diagrams commute: in particular, those giving the monad

and monad morphism axioms commute, so in any weak 0-category $(\mathcal{C}, (\hat{\ }))$, the functor $\hat{I}$ is a monad, and the isomorphism $\hat{I} \to 1_{\mathcal{C}}$ is an isomorphism of monads. By Corollary 4.3.3, $\hat{I}$ must be trivial.

Conversely, suppose T is a trivial monad on a category $\mathcal{C}$. We wish to show that μ is also invertible, and thus that $\mathcal{C}$ is an unbiased weak 0-category. From the monad axioms, we have that

$$
\begin{array}{ccc}
T & \xrightarrow{\eta T} & T^2 \\
 & \text{id} \searrow & \downarrow \mu \\
 & & T
\end{array}
$$

commutes. But ηT is invertible, so μ must be its inverse. So $\text{id} \cong T \cong T^2 \cong T^3 \cong \ldots$, and all diagrams commute. Hence $\mathcal{C}$ is an unbiased weak 0-category.

If $(\mathcal{C}, S)$ and $(\mathcal{D}, T)$ are weak 0-categories, then a weak 0-functor $(\mathcal{C}, S) \to (\mathcal{D}, T)$ is a functor $F : \mathcal{C} \to \mathcal{D}$ and a natural isomorphism $\phi : TF \to FS$, such that the equations

and

are satisfied.

Example 4.3.5. *Pointed sets:* Let P be the operad with a single element of arity 0 (call it $*$) and a single element of arity 1 (the identity). Strict algebras for P in **Set** are pointed sets. The set $(F_{\mathrm{pl}} U^{\mathrm{pl}} P)_0$ is countable (it has elements $*, I*, I^2*, I^3*, \ldots$, and so is $(F_{\mathrm{pl}} U^{\mathrm{pl}} P)_1$ (it has elements $\text{id}, I, I^2, \ldots$). So an unbiased weak P-category is a category $\mathcal{C}$ equipped with a distinguished object $\hat{*}$ and a trivial monad $\hat{I}$.

If $(\mathcal{C}, (\hat{\ }))$ and $(\mathcal{D}, (\bar{\ }))$ are unbiased weak P-categories, then a weak P-functor $(\mathcal{C}, (\hat{\ })) \to (\mathcal{D}, (\bar{\ }))$ is a triple (F, ϕ, ψ), where F and ϕ are as in Example 4.3.4, $\psi : \bar{*} \to F\hat{*}$ is an isomorphism, and there is exactly one natural isomorphism $\bar{I}^n \bar{*} \to F\hat{I}^m \hat{*}$ composed from ϕs and ψs for each m and each $n \in \mathbb{N}$.

Example 4.3.6. *Monoids:* An unbiased weak 1-category is precisely an unbiased weak monoidal category in the sense of Definition 2.1.3. An unbiased weak 1-functor is an unbiased weak monoidal functor. For a proof, see [Lei03] Theorem 3.2.2.

Example 4.3.7. *M-sets:* Let M be a monoid, and N be the operad such that

$$N_1 = M$$
$$N_i = \emptyset \text{ whenever } i \neq 1$$

with composition of arrows of arity 1 given by the multiplication in M. An algebra for N in **Set** is an M-set. An unbiased weak N-category is a category $\mathcal{C}$ with a functor $\hat{m} : \mathcal{C} \to \mathcal{C}$ for each $m \in \mathcal{M}$. For every equation $m_1 m_2 \ldots m_i = n_1 n_2 \ldots n_j$ that is true in M, there is a natural isomorphism $\delta_{m_1 \ldots m_i}^{n_1 \ldots n_j} : \hat{m}_1 \hat{m}_2 \ldots \hat{m}_i \to \hat{n}_1 \hat{n}_2 \ldots \hat{n}_j$. If e is the identity element in M, then $\hat{e}$ is a trivial monad. All diagrams involving these natural isomorphisms commute. Hence, an unbiased weak N-category is a category $\mathcal{C}$ together with a weak monoidal functor $M \to \text{End}(\mathcal{C})$. If $(\mathcal{C}, (\hat{\ }))$ and $(\mathcal{D}, (\bar{\ }))$ are unbiased weak N-categories, an unbiased weak N-functor is a functor $F : \mathcal{C} \to \mathcal{D}$ together with natural transformations $\phi_m : \bar{m}F \to F\hat{m}$ for all $m \in M$, such that if $m_1 m_2 \ldots m_i = n_1 n_2 \ldots n_j$ in M, there is precisely one natural isomorphism $\bar{m}_1 \ldots \bar{m}_i F \to F\hat{n}_1 \ldots \hat{n}_j$ that can be formed by composing δs and ϕs.

4.4 A more general approach: factorization systems

Recall from Definition 2.7.2 the definition of a presentation and a generator for an operad. We will define a categorification of any symmetric operad equipped with a generator, generalizing the unbiased categorification defined in Section 4.2. In particular, we shall consider categorification with respect to the component of the counit $\epsilon_P : F_\Sigma U^\Sigma P \to P$ at a symmetric operad P; this is a generator for P since both

$$F_\Sigma U^\Sigma F_\Sigma U^\Sigma P \underset{F_\Sigma U^\Sigma \epsilon}{\overset{\epsilon F_\Sigma U^\Sigma}{\rightrightarrows}} F_\Sigma U^\Sigma P \overset{\epsilon}{\longrightarrow} P$$

and

$$F_\Sigma U^\Sigma P \times_P F_\Sigma U^\Sigma P \underset{\pi_2}{\overset{\pi_1}{\rightrightarrows}} F_\Sigma U^\Sigma P \overset{\epsilon}{\longrightarrow} P$$

are coequalizer diagrams (the latter by Lemma 2.7.9). We will then show that the categorification is independent of our choice of generator, in the sense that the symmetric **Cat**-operads which arise are equivalent (and thus have equivalent categories of algebras).

Definition 4.4.1. Let Φ be a signature, P be a symmetric operad, and $\phi : F_\Sigma \Phi \to P$ be a regular epi in Σ-**Operad**. Then the **weakening** (or **categorification**) $\mathrm{Wk}_\phi(P)$ of P with respect to ϕ is the (unique-up-to-isomorphism) symmetric **Cat**-operad such that the following diagram commutes:

$$
\begin{array}{ccc}
D_* F_\Sigma \Phi & \xrightarrow{\ \ D_* \phi\ \ } & D_* P \\
& {\scriptstyle b}\searrow \quad \nearrow {\scriptstyle f} & \\
& \mathrm{Wk}_\phi(P) &
\end{array}
$$

where f is full and faithful levelwise, b is levelwise bijective on objects, and D_* is the levelwise discrete category functor Σ-**Operad** $\to$ **Cat**-Σ-**Operad**. The existence and uniqueness of $\mathrm{Wk}_\phi(P)$ follow from Lemma 3.0.6 applied to the factorization system on **Cat**-Σ-**Operad** described in 3.0.10 above.

Definition 4.4.2. Let ϕ, Φ and P be as above. A ϕ-**weak** P-**category** is an algebra for $\mathrm{Wk}_\phi(P)$.

Note that any strict algebra for P can be considered as a ϕ-weak P-category (for any ϕ), via the map $\mathrm{Wk}_\phi(P) \rightarrowtail D_* P$.

Definition 4.4.3. Let ϕ, Φ and P be as above. A ϕ-**weak** P-**functor** is a weak map of $\mathrm{Wk}_\phi(P)$-algebras.

Definition 4.4.4. Let P be a symmetric operad, and $F_\Sigma E \overset{e_1}{\underset{e_2}{\rightrightarrows}} F_\Sigma \Phi$ be a presentation for P, with $\phi : F_\Sigma \Phi \to P$ being the regular epi in Definition 2.7.2. The **weakening of** P **with respect to** (Φ, E) is the weakening of P with respect to ϕ.

Definition 4.4.5. The **unbiased** weakening of P is the weakening arising from the counit $\epsilon : F_\Sigma U^\Sigma P \to P$ of the adjunction $F_\Sigma \dashv U^\Sigma$. Call this symmetric **Cat**-operad $\mathrm{Wk}(P)$.

Lemma 4.4.6. *Let ϕ, Φ and P be as above. Then, for every $n \in \mathbb{N}$, the category $\mathrm{Wk}_\phi(P)_n$ is the equivalence relation $\sim$ on the elements of $(F_\Sigma \Phi)_n$, where $t_1 \sim t_2$ if $\phi(t_1) = \phi(t_2)$.*

Proof. Let $n \in \mathbb{N}$, and $t_1, t_2 \in \mathrm{Wk}_\phi(P)_n$. The objects of $\mathrm{Wk}_\phi(P)_n$ are the elements of $(F_\Sigma \Phi)_n$, by construction. Since ϕ_n factors through a full functor $\mathrm{Wk}_\phi(P)_n \rightarrowtail (D_* P)_n$ and $(D_* P)_n$ is the discrete category on P_n, there is an arrow $t_1 \to t_2$ in $\mathrm{Wk}_\phi(P)_n$ iff $\phi(t_1) = \phi(t_2)$. Since this functor is also faithful, such an arrow must be unique. Hence $\mathrm{Wk}_\phi(P)_n$ is a poset; it is readily checked that it is also an equivalence relation. $\qquad \square$

An obvious question is how this notion of weakening is related to the version defined for plain operads in Section 4.2. In light of Theorem 4.2.2, it is clear that the plain-operadic version can be re-phrased as in Definition 4.4.5 above, but with the factorization occurring in **Cat-Operad** rather than **Cat-Σ-Operad**. We may generalize it to give a definition of the weakening of a plain operad P with respect to a generator ϕ:

Definition 4.4.7. Let P be a plain operad, Φ be a signature, and $\phi : F_{\mathrm{pl}}\Phi \to P$ be a regular epi. The **weakening $\mathrm{Wk}_\phi(P)$ of P with respect to** ϕ is the plain **Cat**-operad given by the bijective on objects/levelwise full and faithful factorization

$$D_* F_{\mathrm{pl}}\Phi \xrightarrow{\quad D_*\phi \quad} D_* P$$
$$\mathrm{Wk}_\phi(P)$$

in **Cat-Operad**. A ϕ-**weak P-category** is an algebra for $\mathrm{Wk}_\phi(P)$.

But do the weak algebras for a strongly regular theory T change if we consider T as a linear theory instead? We now answer that question in the negative.

Theorem 4.4.8. *Let P be a plain operad, let Φ be a signature, and let $\phi : F_{\mathrm{pl}}\Phi \to P$ be a regular epi. Then $\mathrm{Wk}_{F_\Sigma^{\mathrm{pl}}\phi}(F_\Sigma^{\mathrm{pl}}P) \cong F_\Sigma^{\mathrm{pl}}(\mathrm{Wk}_\phi(P))$ in the category* **Cat-Σ-Operad**.

Proof. First note that $\mathrm{Wk}_{F_\Sigma^{\mathrm{pl}}\phi}(F_\Sigma^{\mathrm{pl}}P)$ is well-defined: F_Σ^{pl} is a left adjoint, and hence preserves colimits, so $F_\Sigma^{\mathrm{pl}}\phi$ is a regular epi in Σ-**Operad**.

$\mathrm{Wk}_{F_\Sigma^{\mathrm{pl}}\phi}(F_\Sigma^{\mathrm{pl}}P)$ is defined by its universal property, so it is enough to show that the **Cat**-operad $F_\Sigma^{\mathrm{pl}}(\mathrm{Wk}_\phi(P))$ also has this property. Specifically, it is enough to show that if

$$D_* F_{\mathrm{pl}}\Phi \xrightarrow{\quad D_*\phi \quad} D_* P$$
$$b \searrow \qquad \nearrow f$$
$$\mathrm{Wk}_\phi(P)$$

is the bijective-on-objects/full-and-faithful factorization of ϕ, then in the diagram

$$D_* F_\Sigma\Phi \xrightarrow{\quad D_* F_\Sigma^{\mathrm{pl}}\phi \quad} D_* F_\Sigma^{\mathrm{pl}}P$$
$$F_\Sigma^{\mathrm{pl}}b \searrow \qquad \nearrow F_\Sigma^{\mathrm{pl}}f$$
$$F_\Sigma^{\mathrm{pl}}(\mathrm{Wk}_\phi(P))$$

the arrow $F_\Sigma^{\mathrm{pl}}b$ is bijective on objects and the arrow $F_\Sigma^{\mathrm{pl}}f$ is levelwise full and faithful (note that $D_* F_\Sigma^{\mathrm{pl}} = F_\Sigma^{\mathrm{pl}}D_*$). But this follows straightforwardly from the explicit construction of F_Σ^{pl} in Section 2.6. $\qquad\square$

Corollary 4.4.9. *Let P be a plain operad, $\phi : F_{\mathrm{pl}}\Phi \to P$ generate P, and A be a ϕ-weak P-category in the sense of Definition 4.4.7. Then A is an $F_\Sigma^{\mathrm{pl}}\phi$-weak $F_\Sigma^{\mathrm{pl}}P$-category in the sense of Definition 4.4.2. Conversely, every $F_\Sigma^{\mathrm{pl}}\phi$-weak $F_\Sigma^{\mathrm{pl}}P$-category is a weak P-category.*

Proof. A $F_\Sigma^{\mathrm{pl}}\phi$-weak $F_\Sigma^{\mathrm{pl}}P$-category is a category A and a morphism $\mathrm{Wk}_{F_\Sigma^{\mathrm{pl}}\phi}(F_\Sigma^{\mathrm{pl}}P) \to \mathrm{End}(A)$ of symmetric **Cat**-operads. By Theorem 4.4.8, this is equivalent to a morphism $F_\Sigma^{\mathrm{pl}}(\mathrm{Wk}_\phi(P)) \to \mathrm{End}(A)$ in **Cat-Σ-Operad**, which is equivalent by the adjunction $F_\Sigma^{\mathrm{pl}} \dashv U_{\mathrm{pl}}^\Sigma$ to a morphism of plain **Cat**-operads $\mathrm{Wk}_\phi(P) \to U_{\mathrm{pl}}^\Sigma\mathrm{End}(A)$. This is exactly a ϕ-weak P-category. $\square$

Note that we had to apply F_Σ^{pl} to ϕ to obtain a generator for $F_\Sigma^{\mathrm{pl}}P$. This means that the theorem does not tell us that the unbiased categorification is unaffected by whether we consider our theory as a linear or a strongly regular one. In fact, it is not the case that $\mathrm{Wk}(F_\Sigma^{\mathrm{pl}}P) \cong F_\Sigma^{\mathrm{pl}}(\mathrm{Wk}(P))$ in general.

Example 4.4.10. Consider the terminal plain operad 1 whose algebras are monoids. $F_\Sigma^{\mathrm{pl}}1$ is the operad $\mathcal{S}$ of Example 2.1.11, for which each $\mathcal{S}_n$ is the symmetric group S_n. Then the objects of $\mathrm{Wk}(\mathcal{S})_n$ are n-leafed permuted trees with each node labelled by a permutation, whereas the 1-cells of $(F_\Sigma^{\mathrm{pl}}\mathrm{Wk}(1))_n$ are unlabelled permuted trees. These two sets are not canonically isomorphic. Hence, there is no canonical isomorphism between $\mathrm{Wk}(\mathcal{S})$ and $F_\Sigma^{\mathrm{pl}}\mathrm{Wk}(1)$.

However, we can make a weaker statement: the two candidate unbiased weakenings are *equivalent* in the 2-category **Cat-Σ-Operad**. We shall return to this point in Corollary 5.3.3.

4.5   Examples

Example 4.5.1. Consider the trivial theory (given by the initial operad 0), with the empty generating set. A weak algebra for this theory (with respect to this generating set) is simply a category. F_Σ is a left adjoint, and hence preserves colimits, so $F_\Sigma\emptyset$ is the initial operad, and the coequalizer $\phi : F_\Sigma\emptyset \to 0$ is therefore the identity. Hence $\mathrm{Wk}_\phi(0)$ is also the initial operad, and so a ϕ-weak 0-category is just a category. A ϕ-weak 0-functor is just a functor.

Example 4.5.2. Consider the operad P of Example 4.3.5, generated by one nullary operation $*$. Let ϕ be the associated regular epi. Then $\mathrm{Wk}_\phi(P)$ has one nullary object and no objects of any other arity; the only arrow is the identity on the unique nullary object. In fact, $\mathrm{Wk}_\phi(P) = D_* P$. So a weak algebra for this theory and this generating set is a category $\mathcal{C}$ with a distinguished object $\hat{*} \in \mathcal{C}$. A ϕ-weak P-functor from $(\mathcal{C}, (\hat{\ }))$ to $(\mathcal{D}, (\bar{\ }))$ is a functor $F : \mathcal{C} \to \mathcal{D}$ and an isomorphism $\bar{*} \xrightarrow{\sim} \hat{*}$.

Example 4.5.3. Consider again the operad P of Example 4.3.5, this time generated by four nullary operations A, B, C, D (which are all set equal to each other). Let ϕ be the associated regular epi. Then $\mathrm{Wk}_\phi(P)_0$ is the indiscrete category on the four objects A, B, C, D, and $\mathrm{Wk}_\phi(P)_i$ is empty for all other $i \in \mathbb{N}$. Hence a ϕ-weak P-category is a category $\mathcal{C}$ containing four specified objects $\hat{A}, \hat{B}, \hat{C}$ and $\hat{D}$. These four objects are isomorphic via specified isomorphisms $\delta_{AB}, \delta_{AC}, \delta_{AD}$ etc, and all diagrams involving these isomorphisms commute:

$$
\begin{array}{ccc}
\hat{A} & \xrightarrow{\ \delta_{AB}\ } & \hat{B} \\
\downarrow{\scriptstyle \delta_{AC}} & \begin{smallmatrix} \delta_{AD} & & \delta_{BC} \end{smallmatrix} & \downarrow{\scriptstyle \delta_{BD}} \\
\hat{C} & \xrightarrow{\ \delta_{CD}\ } & \hat{D}
\end{array}
$$

and $\delta_{XY}\delta_{YX} = 1_{\hat{X}}$ for all $X, Y \in \{A, B, C, D\}$.

Let $(\mathcal{C}, (\hat{\ }))$ and $(\mathcal{D}, (\bar{\ }))$ be ϕ-weak P-categories. A ϕ-weak P-functor $(\mathcal{C}, (\hat{\ })) \to (\mathcal{D}, (\bar{\ }))$ consists of

- a functor $F : \mathcal{C} \to \mathcal{D}$,

- an isomorphism $\phi_{XY} : \bar{X} \xrightarrow{\sim} F\hat{X}$ for all $X \in \{A, B, C, D\}$,

such that, for all $X, Y \in \{A, B, C, D\}$, there is precisely one isomorphism $\bar{X} \to F\hat{Y}$ formed by compositions of δs and ϕs.

4.6 Symmetric monoidal categories

Consider the terminal symmetric operad P, whose algebras in **Set** are commutative monoids, and the following linear presentation (Φ, E) for P:

- $\Phi_0 = \{e\}, \Phi_2 = \{.\}$, all other Φ_is are empty;

- E contains the equations

1. $x_1.(x_2.x_3) = (x_1.x_2).x_3$

2. $e.x_1 = x_1$

3. $x_1.e = x_1$

4. $x_1.x_2 = x_2.x_1$

This linear presentation gives rise to a symmetric-operadic presentation (Φ, E), as described in Lemma 2.8.3. Let $\phi : F_\Sigma \Phi \to P$ be the coequalizer in the diagram

$$F_\Sigma E \rightrightarrows F_\Sigma \Phi \xrightarrow{\phi} P$$

We shall now prove that the algebras for $\mathrm{Wk}_\phi(P)$ are classical symmetric monoidal categories. More precisely, we shall show the following:

1. for a given category $\mathcal{C}$, the $\mathrm{Wk}_\phi(P)$-algebra structures on $\mathcal{C}$ are in one-to-one correspondence with the symmetric monoidal category structures on $\mathcal{C}$;

2. there exists an isomorphism (which we construct) between the category **Wk-P-Cat** and the category of symmetric monoidal categories and weak functors;

3. the isomorphism in (2) respects the correspondence in (1).

To fix notation, we recall the classical notions of symmetric monoidal category and symmetric monoidal functor:

Definition 4.6.1. A **symmetric monoidal category** is a 7-tuple $(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho, \tau)$, where

- $\mathcal{C}$ is a category;

- $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is a functor;

- I is an object of $\mathcal{C}$,

- $\alpha : A \otimes (B \otimes C) \to (A \otimes B) \otimes C$ is natural in $A, B, C \in \mathcal{C}$;

- $\lambda : I \otimes A \to A$ and $\rho : A \otimes I \to A$ are natural in $A \in \mathcal{C}$;

- $\tau : A \otimes B \to B \otimes A$ is natural in $A, B \in \mathcal{C}$,

$\alpha, \lambda, \rho, \tau$ are all invertible, and the following diagrams commute:

$$(A \otimes B) \otimes (C \otimes D) \tag{4.1}$$

Diagram (4.1): $A \otimes (B \otimes (C \otimes D)) \xrightarrow{\alpha} (A \otimes B) \otimes (C \otimes D) \xrightarrow{\alpha} ((A \otimes B) \otimes C) \otimes D$, with $A \otimes (B \otimes (C \otimes D)) \xrightarrow{1 \otimes \alpha} A \otimes ((B \otimes C) \otimes D) \xrightarrow{\alpha} (A \otimes (B \otimes C)) \otimes D \xrightarrow{\alpha \otimes 1} ((A \otimes B) \otimes C) \otimes D$.

$$A \otimes (I \otimes C) \xrightarrow{\quad \alpha \quad} (A \otimes I) \otimes C \tag{4.2}$$

Diagram (4.2): $A \otimes (I \otimes C) \xrightarrow{1 \otimes \lambda} A \otimes C \xleftarrow{\rho \otimes 1} (A \otimes I) \otimes C$.

$$A \otimes I \xrightarrow{\quad \tau \quad} I \otimes A \tag{4.3}$$

Diagram (4.3): $A \otimes I \xrightarrow{\rho} A \xleftarrow{\lambda} I \otimes A$.

$$
\begin{array}{ccc}
(A \otimes B) \otimes C \xrightarrow{\ \tau\ } C \otimes (A \otimes B) & \quad & A \otimes (B \otimes C) \xrightarrow{\ \tau\ } (B \otimes C) \otimes A \\
\downarrow{\scriptstyle \alpha^{-1}} \qquad\qquad \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha} \qquad\qquad \downarrow{\scriptstyle \alpha^{-1}} \\
A \otimes (B \otimes C) \qquad (C \otimes A) \otimes B & & (A \otimes B) \otimes C \qquad B \otimes (C \otimes A) \\
\downarrow{\scriptstyle 1 \otimes \tau} \qquad\qquad \downarrow{\scriptstyle \tau \otimes 1} & & \downarrow{\scriptstyle \tau \otimes 1} \qquad\qquad \downarrow{\scriptstyle 1 \otimes \tau} \\
A \otimes (C \otimes B) \xrightarrow{\ \alpha\ } (A \otimes C) \otimes B & & (B \otimes A) \otimes C \xrightarrow{\ \alpha^{-1}\ } B \otimes (A \otimes C)
\end{array}
\tag{4.4}
$$

$$A \otimes B \xrightarrow{\ \tau_{A,B}\ } B \otimes A \tag{4.5}$$

Diagram (4.5): $A \otimes B \xrightarrow{\tau_{A,B}} B \otimes A \xrightarrow{\tau_{B,A}} A \otimes B$, with $A \otimes B \xrightarrow{1} A \otimes B$.

Definition 4.6.2. Let $M = (\mathcal{C}, \otimes, I, \alpha, \lambda, \rho, \tau)$ and $N = (\mathcal{C}', \otimes', I', \alpha', \lambda', \rho', \tau')$ be symmetric monoidal categories. A **lax symmetric monoidal functor** $F : M \to N$ consists of

- a functor $F : \mathcal{C} \to \mathcal{C}'$,

- morphisms $F_2 : (FA) \otimes' (FB) \to F(A \otimes B)$, natural in $A, B \in \mathcal{C}$,

- a morphism $F_0 : I' \to FI$ in $\mathcal{C}'$,

such that the following diagrams commute:

$$
\begin{array}{ccc}
FA \otimes' (FB \otimes' FC) & \xrightarrow{\alpha'} & (FA \otimes' FB) \otimes' FC \\
\downarrow{\scriptstyle 1 \otimes' F_2} & & \downarrow{\scriptstyle F_2 \otimes' 1} \\
FA \otimes' (F(B \otimes C)) & & F(A \otimes B) \otimes' FC \\
\downarrow{\scriptstyle F_2} & & \downarrow{\scriptstyle F_2} \\
F(A \otimes (B \otimes C)) & \xrightarrow{F\alpha} & F((A \otimes B) \otimes C)
\end{array}
\tag{4.6}
$$

$$
\begin{array}{ccccccc}
(FB) \otimes' I'FB & \xrightarrow{\rho'} & FB & \qquad & I' \otimes' (FB) & \xrightarrow{\lambda'} & FB \\
\downarrow{\scriptstyle 1 \otimes' F_0} & & \uparrow{\scriptstyle F\rho} & & \uparrow{\scriptstyle F_0 \otimes' 1} & & \uparrow{\scriptstyle F\lambda} \\
(FB) \otimes' (FI) & \xrightarrow{F_2} & F(B \otimes I) & & (FI) \otimes' (FB) & \xrightarrow{F_2} & F(I \otimes B)
\end{array}
\tag{4.7}
$$

$$
\begin{array}{ccc}
(FA) \otimes' (FB) & \xrightarrow{\tau'} & (FB) \otimes' (FA) \\
\downarrow{\scriptstyle F_2} & & \downarrow{\scriptstyle F_2} \\
F(A \otimes B) & \xrightarrow{F\tau} & F(B \otimes A).
\end{array}
\tag{4.8}
$$

F is said to be **weak** when F_0, F_2 are isomorphisms, and **strict** when F_0, F_2 are identities.

Recall also the coherence theorem for classical symmetric monoidal categories. For any n-ary permuted Φ-tree $(\sigma \cdot t)$, let $(\sigma \cdot t)_M$ be the functor $M^n \to M$ obtained by replacing every . in t by $\otimes$ and every e by I, and permuting the arguments according to σ, so $(\sigma \cdot t)_M(A_1, \ldots, A_n) = t_M(A_{\sigma 1}, \ldots, A_{\sigma n})$ for all $A_1, \ldots, A_n \in M$. In particular, we do not make use of the symmetry maps on M in constructing these functors. Then:

Theorem 4.6.3. *(Mac Lane) In each weak symmetric monoidal category M there is a function which assigns to each pair $(\sigma \cdot t_1, \rho \cdot t_2)$ of permuted Φ-trees of the same arity n a unique natural isomorphism*

$$
\mathrm{can}_M(\sigma \cdot t_1, \rho \cdot t_2) : (\sigma \cdot t_1)_M \to (\rho \cdot t_2)_M : M^n \to M
$$

*called the **canonical map** from $\sigma \cdot t_1$ to $\rho \cdot t_2$, in such a way that the identity of M and all instances of α, λ, ρ and τ are canonical, and the composite as well as the $\otimes$-product of two canonical maps is canonical.*

Proof. See [ML98] XI.1. □

Finally, recall the coherence theorem for weak monoidal functors:

Lemma 4.6.4. *Let M, N be monoidal categories, and $F : M \to N$ be a weak monoidal functor. For every $n \in \mathbb{N}$ and every strongly regular Φ-tree v of arity n, there is a unique map $F_v : v_N(FA_1, \ldots, FA_n) \to Fv_M(A_1, \ldots, A_n)$ natural in $A_1, \ldots, A_n \in M$ and formed by taking composites and tensors of F_0 and F_2, such that the diagram*

$$
\begin{array}{ccc}
v_N(FA_1, \ldots FA_n) & \xrightarrow{\ F_v\ } & Fv_M(A_1, \ldots, A_n) \\
\text{can}_N \downarrow & & \downarrow F\text{can}_M \\
w_N(FA_1, \ldots FA_n) & \xrightarrow{\ F_w\ } & F(w)_M(A_1, \ldots, A_n)
\end{array}
$$

commutes for all $n \in \mathbb{N}$, all $v, w \in (F_{\mathrm{pl}}\Phi)_n$, and all $A_1, \ldots, A_n \in \mathcal{M}$.

Proof. See [ML98], p. 257. $\qquad\square$

We may use this result to sketch a proof of a coherence theorem for weak symmetric monoidal functors:

Theorem 4.6.5. *Let M, N be symmetric monoidal categories, and $F : M \to N$ be a weak symmetric monoidal functor. Let $\sigma \cdot v$ be an n-ary permuted Φ-tree. Then there is a unique natural transformation*

$$
\begin{array}{ccc}
M^n & \xrightarrow{\ F^n\ } & N^n \\
(\sigma \cdot v)_M \downarrow & \overset{F_{\sigma \cdot v}}{\nearrow} & \downarrow (\sigma \cdot v)_N \\
M & \xrightarrow{\ F\ } & N
\end{array}
$$

formed by composing tensor products of F_2 and F_0, possibly with their arguments permuted. Furthermore, if $\rho \cdot w$ is another permuted Φ-tree, then the diagram

$$
\begin{array}{ccc}
(\sigma \cdot v)_N(FA_1, \ldots FA_n) & \xrightarrow{\ F_{\sigma \cdot v}\ } & F(\sigma \cdot v)_M(A_1, \ldots, A_n) \\
\text{can}_N \downarrow & & \downarrow F\text{can}_M \\
(\rho \cdot w)_N(FA_1, \ldots FA_n) & \xrightarrow{\ F_{\rho \cdot w}\ } & F(\rho \cdot w)_M(A_1, \ldots, A_n)
\end{array}
$$

commutes.

Proof. Let $F_{\sigma \cdot v}(A_1, \ldots, A_n) = F_v(A_{\sigma(1)}, \ldots, A_{\sigma(n)})$, and similarly on morphisms. Then $F_{\sigma \cdot v}$ has the required type. We may decompose $\text{can}_M(\sigma \cdot v, \rho \cdot w)$ as $\text{perm}_M(\sigma, \rho) \, \text{can}_M(v, w)$, where $\text{perm}_M(\sigma, \rho) : F_{\sigma \cdot v} \to F_{\rho \cdot v}$ is a composite of τs.

Equation 4.8 and Lemma 4.6.4 together imply that the diagram

$$
\begin{array}{ccc}
(\sigma \cdot v)_N(FA_1, \ldots FA_n) & \xrightarrow{F_{\sigma \cdot v}} & F((\sigma \cdot v)_M(A_1, \ldots, A_n)) \\
\mathrm{perm}_N \downarrow & & \downarrow F\mathrm{perm}_M \\
(\rho \cdot v)_N(FA_1, \ldots FA_n) & \xrightarrow{F_{\rho \cdot v}} & F((\rho \cdot v)_M(A_1, \ldots, A_n)) \\
\mathrm{can}_N \downarrow & & \downarrow F\mathrm{can}_M \\
(\rho \cdot w)_N(FA_1, \ldots FA_n) & \xrightarrow{F_{\rho \cdot w}} & F((\rho \cdot w)_M(A_1, \ldots, A_n))
\end{array}
$$

commutes. It remains to show that $F_{\sigma \cdot v}$ is unique with this property.

Suppose that $F_{\sigma \cdot v}$ is not unique for some $\sigma \cdot v$, and that there exists some natural transformation $G : (\sigma \cdot v)_N(FA_1, \ldots, FA_n) \to F((\sigma \cdot v)_M(A_1, \ldots, A_n))$, composed of tensor products of components of F_0 and F_2, such that $G \neq F_{\sigma \cdot v}$. Suppose further that $\sigma \cdot v$ and G have been chosen to be a minimal counterexample, in the sense that of all such counterexamples, σ may be written as a product of the smallest number of transpositions. If no transpositions are used, then we have a contradiction, because then $\sigma = 1_n$, and Lemma 4.6.4 tells us that $G = F_v$. But suppose $\sigma = t_1 t_2 \ldots t_m$ where each t_i is a transposition: then $t_1 \cdot G$ is a natural transformation $(\sigma \cdot v)_N(FA_{t_1 1}, \ldots, FA_{t_1 n}) \to F((\sigma \cdot v)_M(A_{t_1 1}, \ldots, A_{t_1 n}))$, and thus a transformation $(t_1 \sigma \cdot v)_N(FA_\bullet) \to F((t_1 \sigma \cdot v)_M(A_\bullet))$. But $t_1 \sigma = t_2 t_3 \ldots t_m$, and thus (by minimality of σ), it must be the case that $t_1 \cdot G = F_{t_1 \sigma \cdot v} = t_1 \cdot F_{\sigma \cdot v}$. Hence $G = F_{\sigma \cdot v}$. □

We now proceed to relate the classical theory of symmetric monoidal categories to the more general notion of categorification we developed in previous sections.

By Lemma 4.4.6, if τ_1, τ_2 are n-ary 1-cells in $\mathrm{Wk}_\phi(P)$ (in other words n-ary permuted Φ-trees), there is a (unique) 2-cell $\tau_1 \to \tau_2$ in $\mathrm{Wk}_\phi(P)$ iff $\tau_1 \sim \tau_2$ under the congruence generated by E. By standard properties of commutative monoids, this relation holds iff τ_1 and τ_2 take the same number of arguments, so there is exactly one 2-cell $\tau_1 \to \tau_2$ for every $n \in \mathbb{N}$ and every pair (τ_1, τ_2) of n-ary 1-cells in $\mathrm{Wk}_\phi(P)$.

Let **SMC** denote the category of symmetric monoidal categories and weak maps between them. We shall define functors $S : \textbf{SMC} \to \textbf{Wk-}P\textbf{-Cat}$ and $R : \textbf{Wk-}P\textbf{-Cat} \to \textbf{SMC}$, and show that they are inverses of each other.

Let $M = (\mathcal{C}, \otimes, I, \alpha, \lambda, \rho, \tau)$ be a symmetric monoidal category. Let SM be the weak P-category $(\hat{\ }) : \mathrm{Wk}_\phi(P) \to \mathrm{End}(\mathcal{C})$ defined as follows:

- On 1-cells of $\mathrm{Wk}_\phi(P)$, $(\hat{\ })$ is determined by $\hat{.} = \otimes$ and $\hat{e} = I$.

- If $\delta : \tau_1 \to \tau_2$ is an n-ary 2-cell in $\mathrm{Wk}_\phi(P)$ (i.e. a morphism in the category $\mathrm{Wk}_\phi(P)_n$), let $\hat{\delta}$ be the canonical map $\hat{\tau}_1 \to \hat{\tau}_2$.

Lemma 4.6.6. SM *is a well-defined* $\mathrm{Wk}_\phi(P)$*-algebra for all* $M \in \mathbf{SMC}$.

Proof. The 1-cells of $\mathrm{Wk}_\phi(P)$ are the same as those of $F_\Sigma \Phi$; hence, ($\hat{}$) is entirely determined on 1-cells by a map of signatures $\Phi \to U^\Sigma \mathrm{End}(\mathcal{C})$, which we have given. On 2-cells, Theorem 4.6.3 and the uniqueness property of 2-cells in $\mathrm{Wk}_\phi(P)$ tell us that if δ_1, δ_2 are 2-cells in $\mathrm{Wk}_\phi(P)$, then $\widehat{\delta_1 . \delta_2} = \hat{\delta}_1 \otimes \hat{\delta}_2 = \hat{\delta}_1 . \hat{\delta}_2$, and $\widehat{\delta_1 \delta_2} = \hat{\delta}_1 \hat{\delta}_2$ wherever δ_1, δ_2 are composable. Hence, $(\mathcal{C}, (\hat{}))$ is a well-defined $\mathrm{Wk}_\phi(P)$-algebra. $\square$

Given symmetric monoidal categories M and N, and a weak symmetric monoidal functor $F : M \to N$, we would like to define a weak P-functor $SF = (F, \psi) : SM \to SN$. Let $\psi_{\sigma \cdot v, A_\bullet} = F_{\sigma \cdot v}$ for all $n \in \mathbb{N}$, all $\sigma \cdot v \in (F_\Sigma \Phi)_n$, and all $A_1, \ldots, A_n \in M$. By Theorem 4.6.5, this is natural in $\sigma \cdot v$ and in $A_1, \ldots, A_n$. The other axioms for a weak P-functor are all implied by the coherence theorem (Theorem 4.6.5). This can be generalized: a lax symmetric monoidal functor F determines a lax P-functor SF, and a strict symmetric monoidal functor F determines a strict P-functor SF.

Now, let $\mathcal{C}$ be a $\mathrm{Wk}_\phi(P)$-algebra, with map ($\hat{}$) : $\mathrm{Wk}_\phi(P) \to \mathrm{End}(\mathcal{C})$. We shall construct a symmetric monoidal category $R(\mathcal{C}, (\hat{})) = (\mathcal{C}, \otimes, I, \alpha, \lambda, \rho, \tau)$. Take

- $\otimes = \hat{.}$

- $I = \hat{e}$

- $\alpha = \hat{\delta}_1$, where $\delta_1 : -.(-.-) \to (-.-).-$ in $\mathrm{Wk}_\phi(P)_3$,

- $\lambda = \hat{\delta}_2$, where $\delta_2 : e.- \to -$ in $\mathrm{Wk}_\phi(P)_1$,

- $\rho = \hat{\delta}_3$, where $\delta_3 : -.e \to e$ in $\mathrm{Wk}_\phi(P)_1$,

- $\tau = \hat{\delta}_4$, where $\delta_4 : (-.-) \to (12) \cdot (-.-)$ in $\mathrm{Wk}_\phi(P)_2$.

Lemma 4.6.7. $R(\mathcal{C}, (\hat{}))$ *is a symmetric monoidal category.*

Proof. Because there is at most one 2-cell $\tau_1 \to \tau_2$ for any pair of 1-cells τ_1, τ_2 in $\mathrm{Wk}_\phi(P)$, all diagrams involving these commute. In particular, the axioms for a symmetric monoidal category are satisfied. The 2-cells in $\mathrm{End}(\mathcal{C})$ are natural transformations, so α, λ, ρ and τ (as images of 2-cells in $\mathrm{Wk}_\phi(P)$ under the map ($\hat{}$) : $\mathrm{Wk}_\phi(P) \to \mathrm{End}(\mathcal{C})$) are natural transformations. All 2-cells in $\mathrm{Wk}_\phi(P)$ are invertible, so α, λ, ρ and τ are all natural isomorphisms. Hence $(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho, \tau)$ is a symmetric monoidal category. $\square$

Let $(F, \psi) : (\mathcal{C}, (\hat{\ })) \to (\mathcal{C}', (\check{\ }))$ be a weak morphism of $\mathrm{Wk}_\phi(P)$-algebras. Then let $R(F, \psi) : R(\mathcal{C}, (\hat{\ })) \to R(\mathcal{C}', (\check{\ }))$ be the following symmetric monoidal functor:

- the underlying functor is F,

- F_0 is $\psi_{e\circ 1} : \check{e} \to F\hat{e}$,

- F_2 is $\psi_{.\circ 1} : (\check{\cdot})F^2 \to F(\hat{\cdot})$.

The coherence diagrams (4.6), (4.7) and (4.8) all commute by virtue of the coherence axioms for a weak morphism of $\mathrm{Wk}_\phi(P)$-algebras and the naturality of ψ. Hence (F, F_0, F_2) is a symmetric monoidal functor.

Lemma 4.6.8. *Let* $(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho, \tau)$ *be a symmetric monoidal category. Then*

$$RS(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho, \tau) = (\mathcal{C}, \otimes, I, \alpha, \lambda, \rho, \tau).$$

Proof. Let $RS(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho, \tau) = (\mathcal{C}, \otimes', I', \alpha', \lambda', \rho', \tau')$. Their underlying categories are equal, both being $\mathcal{C}$.

$$
\begin{aligned}
\otimes' &= \hat{\cdot} &= \otimes \\
I' &= \hat{e} &= I \\
\alpha' &= \hat{\delta}_1 &= \alpha, \text{ the unique canonical map of the correct type} \\
\lambda' &= \hat{\delta}_2 &= \lambda \\
\rho' &= \hat{\delta}_3 &= \rho \\
\tau' &= \hat{\delta}_4 &= \tau
\end{aligned}
$$

$\square$

Lemma 4.6.9. *Let* $(\mathcal{C}, (\hat{\ }))$ *be a* $\mathrm{Wk}_\phi(P)$*-algebra, and let* $(\mathcal{C}', (\check{\ })) = SR(\mathcal{C}, (\hat{\ }))$. *Then* $(\mathcal{C}, (\hat{\ })) = (\mathcal{C}', (\check{\ }))$.

Proof. Their underlying categories are the same. As above, $(\check{\ })$ is determined on objects by the values it takes on . and e: these are $\otimes = \hat{\cdot}$ and $I = \hat{e}$ respectively. So $(\check{\ }) = (\hat{\ })$ on objects. If $\delta : \tau_1 \to \tau_2$, then $\check{\delta}$ is the unique canonical map from $\check{\tau}_1 \to \check{\tau}_2$, which, by an easy induction, must be $\hat{\delta}$. So $(\check{\ }) = (\hat{\ })$, and hence $(\mathcal{C}, (\hat{\ })) = SR(\mathcal{C}, (\hat{\ }))$. $\square$

Lemma 4.6.10. *Let* $M = (\mathcal{C}, \otimes, I, \alpha, \lambda, \rho, \tau)$ *and* $N = (\mathcal{C}', \otimes', I', \alpha', \lambda', \rho', \tau')$ *be symmetric monoidal categories, and let* (F, F_0, F_2) *be a weak symmetric monoidal functor* $M \to N$. *Then* $RS(F, F_0, F_2) = (F, F_0, F_2)$.

Proof. Let $(G, G_0, G_2) = RS(F, F_0, F_2)$. Then G is the underlying functor of $S(F, F_0, F_2)$ which is F, and G_0, G_2 are both the canonical maps with the correct types given by Theorem 4.6.5: that is to say, they are F_0 and F_2 respectively. □

Lemma 4.6.11. *Let $(\mathcal{C}, (\hat{\ }))$ and $(\mathcal{C}', (\check{\ }))$ be $\mathrm{Wk}_\phi(P)$-algebras, and let $(F, \phi) : (\mathcal{C}, (\hat{\ })) \to (\mathcal{C}', (\check{\ }))$ be a weak morphism of $\mathrm{Wk}_\phi(P)$-algebras. Then $SR(F, \phi) = (F, \phi)$.*

Proof. Let $(G, \gamma) = SR(F, \phi)$. Then G is the underlying functor of $R(F, \phi)$, which is F. Let $(F, F_0, F_2) = R(F, \phi)$. Each component of γ is then by definition the correct component of the canonical map arising from F_0, F_2 in the process described in Theorem 4.6.5. By the "uniqueness" part of the Theorem, this must be the corresponding component of ϕ. Hence $\gamma = \phi$. □

Theorem 4.6.12. *S and R form an isomorphism of categories $\mathbf{SMC} \cong \mathbf{Wk}\text{-}P\text{-}\mathbf{Cat}$.*

Proof. Lemmas 4.6.8 and 4.6.9 show that R and S are bijective on objects; Lemmas 4.6.10 and 4.6.11 show that R and S are locally bijective on morphisms. Hence, R and S are a pair of mutually inverse isomorphisms of categories. □

4.7 Multicategories

We can tell this whole story for (symmetric) multicategories as well as just operads. We sketch this development briefly here, although the remainder of the thesis will continue to focus on the special case of operads.

Definition 4.7.1. A (directed) **multigraph** consists of

1. a set of **vertices** V,

2. for each $n \in \mathbb{N}$ and each sequence $v_1, v_2, \ldots, v_n, w$ of vertices, a set $E(v_1, \ldots, v_n; w)$ of **funnels from** $v_1, \ldots, v_n$ **to** w.

Definition 4.7.2. Let $M_1 = (V_1, E_1)$ and $M_2 = (V_2, E_2)$ be multigraphs. A **morphism of multigraphs** $f : M_1 \to M_2$ is

1. a function $f_V : V_1 \to V_2$,

2. for each finite sequence $v_1, v_2 \ldots v_n, w$ of vertices in M_1, a function

$$f_w^{v_1, \ldots, v_n} : E_1(v_1, \ldots, v_n; w) \to E_2(f_V(v_1), \ldots, f_V(v_n); f_V(w)).$$

We say that a funnel $f \in E(v_1, \ldots, v_n; w)$ has **source** $v_1, \ldots, v_n$ and **target** w; we say that two funnels are **parallel** if they have the same source and target. The reason for the "funnel" terminology should be clear from Figure 4.2. We shall say that a multigraph has some property P **locally** if every $E(v_1, \ldots, v_n; w)$ is P, and similarly a morphism f of multicategories is locally P if every $f_w^{v_1, \ldots, v_n}$ is P.

Multigraphs and their morphisms form a category which we shall call **Multigraph**.
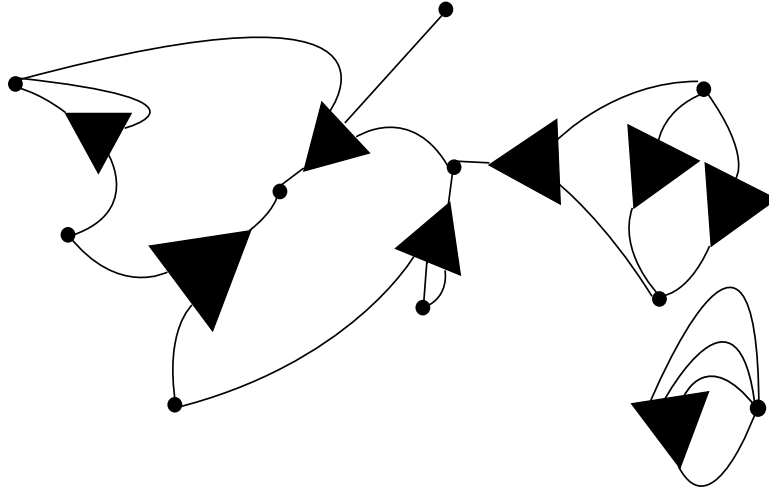
Figure 4.2: A multigraph

In order to proceed with the rest of the construction, we will need to consider subcategories of **Multicat**, **Multigraph** etc.

Definition 4.7.3. Let X be a set. Then **Multigraph**$_X$ is the subcategory of **Multigraph** whose objects are multigraphs with vertex set X, and whose morphisms are identity-on-vertices maps of multigraphs. We define **Multicat**$_X$ and Σ-**Multicat**$_X$ similarly.

For each $X \in$ **Set**, there is a chain of adjunctions similar to that given in Section 2.4:
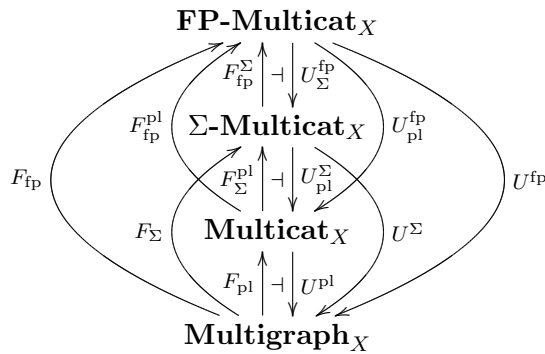
These adjunctions are monadic, by Lemma 2.5.1 and Lemma 2.5.3. Note that **Set**$^{\mathbb{N}}$ can

be regarded as **Multigraph**$_1$: thus, the adjunctions of Section 2.4 are just the restrictions of the adjunctions above to the one-vertex case.

We can consider multigraphs enriched in some category $\mathcal{V}$:

Definition 4.7.4. Let $\mathcal{V}$ be a category. A $\mathcal{V}$**-multigraph** $M = (V, E)$ consists of

1. a set V of **vertices**,

2. for each $n \in \mathbb{N}$ and each finite sequence $v_1, v_2 \ldots v_n, w$ of vertices, an object of $\mathcal{V}$ called $E(v_1, \ldots, v_n; w)$ of **funnels from** $v_1, \ldots, v_n$ **to** w.

Definition 4.7.5. Let $M_1 = (V_1, E_1)$ and $M_2 = (V_2, E_2)$ be $\mathcal{V}$-multigraphs. A **morphism of $\mathcal{V}$-multigraphs** $f : M_1 \to M_2$ is

1. a function $f_V : V_1 \to V_2$,

2. for each $n \in \mathbb{N}$ and each finite sequence $v_1, v_2 \ldots v_n, w$ of vertices in M_1, an arrow
   $$E_1(v_1, \ldots, v_n; w) \to E_2(f_V(v_1), \ldots, f_V(v_n); f_V(w)) \text{ in } \mathcal{V}.$$

The category of $\mathcal{V}$-multigraphs and their morphisms is called $\mathcal{V}$**-Multigraph**. The category whose objects are $\mathcal{V}$-multigraphs with vertex-set X and whose morphisms are identity-on-vertices maps is called $\mathcal{V}$**-Multigraph**$_X$. In particular, we shall consider multigraphs enriched in the category **Digraph** of directed graphs. An object of the category **Digraph-Multigraph** consists of

1. **vertices** (or **objects**);

2. **funnels**, each of which has one object as its target, and a sequence of objects as its source;

3. **edges**, which each have one funnel as a source and one as a target: the source and target of a given edge must be parallel.

The factorization system construction of Example 3.0.10 works in this broader setting too. Let X be a set. The factorization system $(\mathcal{E}, \mathcal{M})$ on **Digraph** of Example 3.0.5 gives rise to a factorization system $(\mathcal{E}', \mathcal{M}')$ on **Digraph-Multigraph**$_X$, where $\mathcal{E}$ consists of maps which are bijective on objects and funnels, and $\mathcal{M}$ consists of maps which are locally full-and-faithful. This lifts to a factorization system $(\mathcal{E}'', \mathcal{M}'')$ on **Cat-Multigraph**$_X$ via
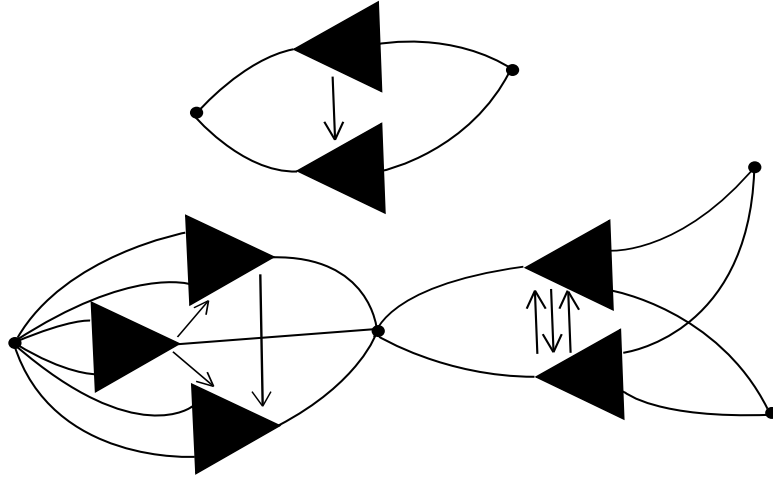
Figure 4.3: A multigraph enriched in directed graphs

Lemma 3.0.7. By the usual argument, there is a chain of monadic adjunctions:
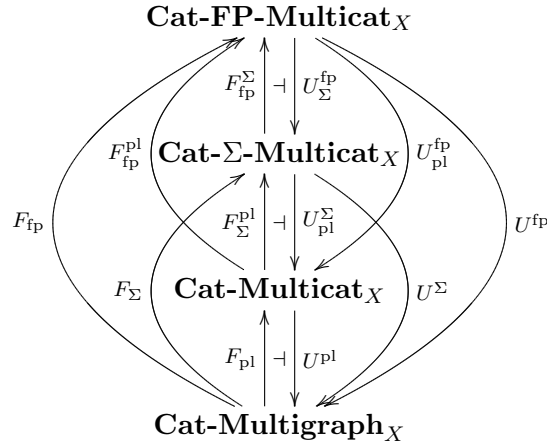
Since **Cat-Multicat**$_X$ is monadic over **Cat-Multigraph**$_X$, this in turn lifts to a factorization system on **Cat-Multicat**$_X$. Similarly, we obtain a factorization system on **Cat-Σ-Multicat**$_X$.

A **generator** for a plain multicategory M with object-set X is a multigraph $\Phi = (X, E)$ together with a regular epi $F_{\mathrm{pl}}\Phi \to M$ in **Multicat**$_X$. Similarly, a **generator** for a symmetric multicategory M with object-set X is a multigraph $\Phi = (X, E)$ together with a regular epi $F_\Sigma\Phi \to M$ in Σ-**Multicat**$_X$.

We can therefore extend Definition 4.4.7 above, in the obvious way. Let D_* be the embedding of **Multicat**$_X$ into **Cat-Multicat**$_X$ via the (full and faithful) discrete category functor applied locally.

Definition 4.7.6. Let M be a plain multicategory with object-set X, and let $\phi : F_{\mathrm{pl}}\Phi \to M$ be a regular epi in **Multicat**$_X$. Then the **weakening of M with respect to ϕ** is the

unique-up-to-isomorphism **Cat**-multicategory $\mathrm{Wk}_\phi(M)$ such that the following diagram commutes:

$$D_* F\Phi \xrightarrow{\quad D_*\phi \quad} D_* M$$

where f is locally full and faithful, and b is locally bijective on objects (i.e., each map of sets of funnels in b is a bijection). The uniqueness of $\mathrm{Wk}_\phi(M)$ follows from properties of the factorization system on **Cat-Multicat**$_X$ given above.

Definition 4.7.7. Let M be a (symmetric) multicategory. The **unbiased weakening** of M is the weakening of M with respect to the counit ϵ of the adjunction $F_{\mathrm{pl}} \dashv U^{\mathrm{pl}}$ (respectively, the adjunction $F_\Sigma \dashv U^\Sigma$).

Definition 4.7.8. Let M be a multicategory, and let $\phi : F_{\mathrm{pl}}\Phi \to M$ be a regular epi in **Multicat**$_X$. A ϕ-**weak** M-**algebra** is an algebra for $\mathrm{Wk}_\phi(M)$. An **unbiased weak** M-**algebra** is an algebra for $\mathrm{Wk}(M)$.

We define weakenings of symmetric multicategories analogously.

4.8 Examples

Example 4.8.1. Let M be the multicategory generated by

$$0 \xrightarrow{\ f\ } 1 \xrightarrow{\ g\ } 2$$

Then a weak algebra for M in **Cat** with respect to this generating set consists of a diagram

$$\hat{0} \xrightarrow{\ \hat{f}\ } \hat{1} \xrightarrow{\ \hat{g}\ } \hat{2}$$

in **Cat**, whereas an unbiased weak M-algebra is a diagram

where $\widehat{I_0}, \widehat{I_1}$ and $\widehat{I_2}$ are trivial monads.

Example 4.8.2. Consider the theory T whose algebras are a monoid M together with an M-set A. Then a weak T-algebra with respect to the standard presentation (a binary and nullary operation on M, and an operation $M \times A \to A$) is a classical monoidal category $\hat{M}$, a category $\hat{A}$, and a weak monoidal functor $\hat{M} \to \mathrm{End}(\hat{A})$. An unbiased weak T-algebra is an unbiased monoidal category $\hat{M}$, a category $\hat{A}$ equipped with a trivial monad $\hat{I}_A$, and an unbiased monoidal functor $\hat{M} \to \mathrm{End}(\hat{A})$ which commutes up to coherent isomorphism with $\hat{I}_A$.

Example 4.8.3. Let P be an operad, and let $\bar{P}$ be the multicategory from Section 2.10 whose algebras are pairs of P-algebras with a morphism between them. It seems clear that an unbiased weak $\bar{P}$-category is a pair of unbiased weak P-categories and an unbiased weak P-functor between them; a rigorous proof would first require a coherence theorem to be proven for weak maps of **Cat**-operad algebras, and currently no such theorem is known.

4.9   Evaluation

At the beginning of this chapter, we proposed three criteria that a successful definition of categorification should satisfy: namely, it should be broad, consistent with earlier work, and canonical. The examples considered throughout the chapter show that our theory agrees with the standard categorifications that are within its scope. It is determined by the universal property given by the factorization system on **Cat-Σ-Operad**: the only tunable parameter is the choice of generator of a given theory, and in Chapter 5 we shall see that the weakening of a given theory is independent (up to equivalence) of the generator used. The main problem is the breadth of our theory: as presented, it is restricted to linear theories, preventing us from categorifying the theories of groups, rings, Lie algebras, and many other interesting nonlinear theories. We shall now show what happens when we try to extend our theory to general algebraic theories.

Lemma 4.9.1. *There is a factorization system $(\mathcal{E}, \mathcal{M})$ on* **Cat-FP-Operad** *where $\mathcal{E}$ is the collection of maps which are bijective on objects, and $\mathcal{M}$ is the collection of maps which are levelwise full and faithful.*

Proof. The proof is exactly as for the proof of existence of the factorization systems on **Cat-Operad** and **Cat-Σ-Operad** given in Example 3.0.10. $\square$

Theorem 4.9.2. *Let P be the finite product operad whose algebras are commutative*

*monoids, and $D_* :$ **FP-Operad** $\rightarrow$ **Cat-FP-Operad** be the levelwise "discrete category"*

functor. Let Q be the finite product **Cat**-*operad given by the factorization*

$$D_* F_{\mathrm{fp}} U^{\mathrm{fp}} P \xrightarrow{\ D_* \epsilon_P\ } D_* P$$

$$Q$$

Then an algebra for Q is an unbiased symmetric monoidal category $\mathcal{C}$ such that, for all

$A \in \mathcal{C}$, the component τ_{AA} of the symmetry map τ is the identity on $A \otimes A$.

Proof. We adopt the notation for elements of P introduced in Example 2.3.4. Let f be

the unique function $\underline{2} \rightarrow \underline{1}$, and let $t : \underline{2} \rightarrow \underline{2}$ be the permutation transposing 1 and 2.

Then $\epsilon(f \cdot \left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]) = [2] \in P_1$. Let $(\mathcal{C}, (\hat{\ }))$ be a Q-algebra. We shall write $\left[\begin{smallmatrix}\hat{1}\\1\end{smallmatrix}\right](A, B)$ as $A \otimes B$.

We may impose a symmetric monoidal category structure on $\mathcal{C}$, where the symmetry map

is the image under $(\hat{\ })$ of the unique map $\delta : \left[\begin{smallmatrix}1\\1\end{smallmatrix}\right] \rightarrow t \cdot \left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]$ in Q_1. All diagrams in Q_1

commute, so in particular, the following diagram commutes:

$$\begin{array}{ccc}
[2] & \longrightarrow & f \cdot \left[\begin{smallmatrix}1\\1\end{smallmatrix}\right] \\
\uparrow & \nearrow_{f \cdot \delta} & \\
f \cdot \left[\begin{smallmatrix}1\\1\end{smallmatrix}\right] & &
\end{array}$$

The two unlabelled arrows are mutually inverse. Applying $(\hat{\ })$ to the entire diagram, and

evaluating the resulting functors at $A \in \mathcal{C}$, we see that the following diagram commutes:

$$\begin{array}{ccc}
[\hat{2}](A) & \longrightarrow & A \otimes A \\
\uparrow & \nearrow_{\tau_{AA}} & \\
A \otimes A & &
\end{array}$$

and hence $\tau_{AA} = 1_{A \otimes A}$. $\square$

This is not the case for most interesting symmetric monoidal categories. Hence this

definition of categorification would fail to be consistent with earlier work.

Chapter 5

Coherence

There are many "coherence theorems" in category theory, but in practice they usually fall into one of two classes:

1. "All diagrams commute", or more precisely, that diagrams in a given class commute if and only if some quantity is invariant.

2. Every "weak" object is equivalent to an appropriate "strict" object.

Since the diagrams of interest in theorems of type 1 will usually commute trivially in a strict object, a coherence theorem of type 2 usually implies one of type 1. However, establishing the converse is usually harder. In the previous chapter, our "weak P-categories" were defined explicitly in terms of an infinite class of commuting diagrams (namely, those diagrams which become identities under the application of the counit of the adjunctions $F_\Sigma \dashv U^\Sigma$ or $F_{\mathrm{pl}} \dashv U^{\mathrm{pl}}$): it is therefore interesting to see if we can prove a theorem of type 2 about them. We do this in Section 5.1, and investigate an interesting property of the strictification functor in Section 5.2.

In Section 5.3, we investigate how the operad defining weak P-categories is affected by our choice of presentation for P. While the independence result obtained is not a coherence theorem of the usual form, it can be seen as a coherence theorem in a higher-dimensional sense: that the process of categorification is itself coherent.

For other related work, see Power's paper [Pow89].

5.1 Strictification

Let P be a plain operad, and $Q = \mathrm{Wk}(P)$, with $\pi : Q \to D_*P$ the levelwise full-and-faithful map in Theorem 4.2.2. We again adopt the $_\bullet$ notation from chain complexes and write, for instance, $p_\bullet$ for a finite sequence of objects in P, and $p_\bullet^\bullet$ for a double sequence. Let $Q \circ A \xrightarrow{h} A$ be a weak P-category. We shall construct a strict P-category $\mathbf{st}(A)$ and a weak P-functor $(F, \phi) : \mathbf{st}(A) \to A$, and show that it is an equivalence of weak P-categories. This "strictification" construction is closely related to that given in [JS93] for monoidal categories; however, it is more general, and since we work for the moment with unbiased weak P-categories, our construction has some additional pleasant properties.

In fact, $\mathbf{st}$ is functorial, and is left adjoint to the forgetful functor $\mathbf{Str}\text{-}P\text{-}\mathbf{Cat} \to \mathbf{Wk}\text{-}P\text{-}\mathbf{Cat}$ (see Section 5.2). The theorem then says that the unit of this adjunction is pseudo-invertible, and that the strict P-categories and strict P-functors form a weakly coreflective sub-2-category of $\mathbf{Wk}\text{-}P\text{-}\mathbf{Cat}$.

If P is a plain operad, let ι be the embedding

$$\iota : U^{\mathrm{pl}} D_* P \;\to\; U^{\mathrm{pl}} \mathrm{Wk}(P)$$
$$\iota(p) \;=\; p \circ (|, \ldots, |)$$

Note that this is a morphism in $\mathbf{Cat}^{\mathbb{N}}$, not in $\mathbf{Cat\text{-}Operad}$.

Definition 5.1.1. Let P, Q, h, A, ι be as above. The **strictification of** A, written $\mathbf{st}(A)$, is given by the bijective-on-objects/full and faithful factorization of $h(\iota \circ 1)$ in $\mathbf{Cat}$:

$$P \circ A \xrightarrow{\iota \circ 1} Q \circ A \xrightarrow{h} A.$$
$$\mathbf{st}(A)$$

We shall show that $\mathbf{st}(A)$ is a strict P-category. We may describe $\mathbf{st}(A)$ explicitly as follows:

- An object of $\mathbf{st}(A)$ is an object of $P \circ A$.

- If $p \in P_n$ and $a_1, \ldots, a_n \in A$, an arrow $(p, a_\bullet) \to (p', a'_\bullet)$ in $\mathbf{st}(A)$ is an arrow $h(p, a_\bullet) \to h(p', a'_\bullet)$ in A. We say that such an arrow is a **lifting** of $h(p, a_\bullet)$. Composition and identities are as in A.

We define an action h' of P on $\mathbf{st}(A)$ as follows:

- On objects, h' acts by $h'(q, (p, a_\bullet)^\bullet) = (\pi(p \circ (p^\bullet)), a_\bullet^\bullet)$ where $p \in P_n$ and $(p^i, a_\bullet^i) \in$ $\mathbf{st}(A)$ for $n \in \mathbb{N}$ and $i = 1, \ldots n$.

- Let $f_i : (p_i, a_i) \to (p_i', a_i')$ for $i = 1, \ldots, n$. Then $h'(p, f_\bullet)$ is the composite

$$
\begin{aligned}
h(p \circ (p_\bullet), a_\bullet) & \xrightarrow{\delta^{-1}_{p \circ (p_\bullet)}} & h(p \circ (p_\bullet), a_\bullet) = h(p, h(p_1, a_1), \ldots, h(p_n, a_n)) \\
& \xrightarrow{h(p, f_\bullet)} & h(p, h(p_1', a_1'), \ldots, h(p_n', a_n')) = h(p \circ (p_\bullet'), a_\bullet') \\
& \xrightarrow{\delta_{p \circ (p_\bullet')}} & h(p \circ (p_\bullet'), a_\bullet').
\end{aligned}
$$

Lemma 5.1.2. $\mathbf{st}(A)$ *is a strict P-category.*

Proof. It is clear that the action we have defined is strict and associative on objects and that 1_P acts as a unit: we must show that the action on arrows is associative. Let $f_i^j : (p_i^j, a_{i\bullet}^j) \to (q_i^j, b_{i\bullet}^j)$, $\sigma \in Q_n$, and $\tau_i \in Q_{k_i}$ for $j = 1, \ldots, k_i$ and $i = 1, \ldots, n$. We wish to show that $h'(\sigma \circ (\tau_\bullet), f_\bullet^\bullet) = h'(\sigma, h'(\tau_1, f_1^\bullet), \ldots, h'(\tau_n, f_n^\bullet))$.

The LHS is

$$
\begin{aligned}
h(\sigma \circ (\tau_\bullet) \circ (p_\bullet^\bullet), a_\bullet^\bullet) & \xrightarrow{\delta^{-1}_{\sigma \circ (\tau_i) \circ (p_\bullet^\bullet)}} & h(\sigma \circ (\tau_\bullet), h(p_1^1, a_{1\bullet}^1), \ldots, h(p_n^{k_n}, a_{n\bullet}^{k_n})) \\
& \xrightarrow{h(\sigma \circ (\tau_\bullet), f_\bullet^\bullet)} & h(\sigma \circ (\tau_\bullet), h(q_1^1, b_{1\bullet}^1), \ldots, h(q_n^{k_n'}, b_{n\bullet}^{k_n'})) \\
& \xrightarrow{\delta_{\sigma \circ (\tau_\bullet) \circ (q_\bullet^\bullet)}} & h(\sigma \circ (\tau_\bullet) \circ (q_\bullet^\bullet), b_\bullet^\bullet).
\end{aligned}
$$

The RHS is

$$
\begin{aligned}
h(\sigma \circ (\tau_\bullet) \circ (p_\bullet^\bullet), a_\bullet^\bullet) & \xrightarrow{\delta^{-1}_{\sigma \circ (\tau_i \circ (p_\bullet^\bullet))}} & h(\sigma, h(\tau_1 \circ (p_1^\bullet), a_{1\bullet}^\bullet), \ldots, h(\tau_n \circ (p_n^\bullet), a_{n\bullet}^\bullet)) \\
& \xrightarrow{h(\sigma, h'(\tau_\bullet, f_\bullet^\bullet))} & h(\sigma, h(\tau_1 \circ (q_1^\bullet), b_{1\bullet}^\bullet), \ldots, h(\tau_n \circ (q_n^\bullet), b_{n\bullet}^\bullet)) \\
& \xrightarrow{\delta_{\sigma \circ (\tau_i \circ (p_\bullet^\bullet))}} & h(\sigma \circ (\tau_\bullet) \circ (q_\bullet^\bullet), b_\bullet^\bullet),
\end{aligned}
$$

where each $h'(\tau_i, f_i^\bullet)$ is

$$
\begin{aligned}
h(\tau_i \circ (p_i^\bullet), a_i^\bullet)) & \xrightarrow{\delta^{-1}_{\tau_i \circ (p_i^\bullet)}} & h(\tau_i, h(p_i^1, a_{i\bullet}^1), \ldots, h(p_i^{k_i}, a_{i\bullet}^{k_i})) \\
& \xrightarrow{h(\tau_i, f_i^\bullet)} & h(\tau_i, h(q_i^1, b_{i\bullet}^1), \ldots, h(q_i^{k_i}, b_{i\bullet}^{k_i})) \\
& \xrightarrow{\delta_{\tau_i \circ (p_i^\bullet)}} & h(\tau_i \circ (q_i^\bullet), b_i^\bullet).
\end{aligned}
$$

So the equation holds if the following diagram commutes:

$$h(\sigma \circ (\tau_\bullet) \circ (p_\bullet^\bullet), a_\bullet^\bullet)$$

with labels $\delta^{-1}_{\sigma \circ (\tau_i) \circ (p_\bullet^\bullet)}$, $\delta^{-1}_{\sigma \circ (\tau_\bullet) \circ (p_\bullet^\bullet)}$, $\delta^{-1}_{\sigma \circ (\tau_i \circ (p_\bullet^\bullet))}$

$$h(\sigma \circ (\tau_\bullet), h(p_\bullet^\bullet, a_\bullet^\bullet)) \xleftarrow{\ \delta_{\sigma \circ (\tau_\bullet)}\ } h(\sigma, h(\tau_\bullet \circ (p_\bullet^\bullet), a_\bullet^\bullet)) \xleftarrow{\ h(\sigma, \delta^{-1}_{\tau_\bullet \circ (p_\bullet^\bullet)})\ } h(\sigma \circ (\tau_\bullet), h(p_\bullet^\bullet, a_\bullet^\bullet))$$

with vertical arrows $h(\sigma \circ (\tau_\bullet), f_\bullet^\bullet)$, $h(\sigma, h(\tau_i, f_i^\bullet))$, $h(\sigma, h'(\tau_\bullet, f_\bullet^\bullet))$ and regions ① and ②

$$h(\sigma \circ (\tau_\bullet), h(q_\bullet^\bullet, b_\bullet^\bullet)) \xleftarrow{\ \delta_{\sigma \circ (\tau_\bullet)}\ } h(\sigma, h(\tau_\bullet \circ (q_\bullet^\bullet), b_\bullet^\bullet)) \xrightarrow{\ h(\sigma, \delta_{\tau_i \circ (p_i^\bullet)})\ } h(\sigma, h(\tau_\bullet \circ (q_\bullet^\bullet), b_\bullet^\bullet))$$

with labels $\delta_{\sigma \circ (\tau_\bullet) \circ (q_\bullet^\bullet)}$, $\delta_{\sigma \circ (\tau_\bullet \circ (q_\bullet^\bullet))}$, $\delta_{\sigma \circ (\tau_\bullet \circ (q_\bullet^\bullet))}$

$$h(\sigma \circ (\tau_\bullet) \circ (q_\bullet^\bullet), b_\bullet^\bullet)$$

The triangles all commute because all δs are images of arrows in Q, and there is at most one 2-cell between any two 1-cells in Q. ② commutes by the definition of $h'(\tau_i, f_i^\bullet)$, and ① commutes by naturality of δ. □

Lemma 5.1.3. *Let $Q \circ A \xrightarrow{h} A$ and $Q \circ B \xrightarrow{h'} B$ be weak P-categories, $(F, \pi) : A \to B$ be a weak P-functor, and (F, G, η, ϵ) be an adjoint equivalence in $\mathbf{Cat}$. Then G naturally carries the structure of a weak P-functor, and (F, G, η, ϵ) is an adjoint equivalence in* $\mathbf{Wk}$-P-$\mathbf{Cat}$.

Proof. We want a sequence $(\psi_\bullet)$ of natural transformations:

$$
\begin{array}{ccc}
Q_i \times B^i & \xrightarrow{h_i'} & B \\
\downarrow{\scriptstyle 1 \times G^i} & {\scriptstyle \psi_i}\ \nearrow & \downarrow{\scriptstyle G} \\
Q_i \times A^i & \xrightarrow{h_i} & A
\end{array}
$$

Let ψ_i be given by

$$
\begin{array}{ccc}
Q_i \times B^i & \xrightarrow{h_i'} & B \\
\downarrow{\scriptstyle 1 \times G^i} & {\scriptstyle \psi_i}\ \nearrow & \downarrow{\scriptstyle G} \\
Q_i \times A^i & \xrightarrow{h_i} & A
\end{array}
\quad = \quad
\begin{array}{ccccc}
Q_i \times B^i & \xrightarrow{1} & Q_i \times B^i & \xrightarrow{h_i'} & B \\
\downarrow{\scriptstyle 1 \times G^i} & {\scriptstyle 1 \times \epsilon^i}\nearrow\ {\scriptstyle 1 \times F^i} & \ \Uparrow{\scriptstyle \pi_i^{-1}}\ {\scriptstyle F} & \nearrow\ {\scriptstyle \eta} & \downarrow{\scriptstyle G} \\
Q_i \times A^i & \xrightarrow{h_i} & A & \xrightarrow{1} & A
\end{array}
$$

We must check that ψ satisfies (2.26) and (2.27) from Lemma 2.9.11. For (2.26):

LHS $=$

$$
\begin{array}{c}
\xrightarrow{\;h'_{k_1}\times\cdots\times h'_{k_n}\;}\qquad\xrightarrow{\;h'_n\;}\\[4pt]
1\times G^{\sum k_i}\Big\downarrow\quad \overset{\psi_{k_1}\times\cdots\times\psi_{k_n}}{\nearrow}\quad 1\times G^n\quad \overset{\psi_n}{\nearrow}\quad \Big\downarrow G\\[4pt]
\xrightarrow{\;h_{k_1}\times\cdots\times h_{k_n}\;}\qquad\xrightarrow{\;h\;}
\end{array}
$$

$=$

$$
\begin{array}{c}
\xrightarrow{1}\;\xrightarrow{h'_{k_1}\times\cdots\times h'_{k_n}}\;\xrightarrow{1}\;\xrightarrow{h'_n}\\[4pt]
1\times G^{\sum k_i}\ \overset{1\times\varepsilon^{\sum k_i}}{\nearrow}\ 1\times F^{\sum k_i}\ \overset{\pi_{k_1}^{-1}\times\cdots\times\pi_{k_n}^{-1}}{\Uparrow}\ 1\times F^n\ 1\times G^n\ \overset{1\times\varepsilon^n}{\nearrow}\ 1\times F^n\ \overset{\pi_n^{-1}}{\Uparrow}\ F\ \overset{\eta}{\nearrow}\ G\\[4pt]
\xrightarrow{h_{k_1}\times\cdots\times h_{k_n}}\;\xrightarrow{1}\;\xrightarrow{h}\;\xrightarrow{1}\\[4pt]
\text{with } 1\times\eta^n
\end{array}
$$

$=$

$$
\begin{array}{c}
\xrightarrow{1}\;\xrightarrow{h'_{k_1}\times\cdots\times h'_{k_n}}\;\xrightarrow{1}\;\xrightarrow{h'_n}\\[4pt]
1\times G^{\sum k_i}\ \overset{1\times\varepsilon^{\sum k_i}}{\nearrow}\ 1\times F^{\sum k_i}\ \overset{\pi_{k_1}^{-1}\times\cdots\times\pi_{k_n}^{-1}}{\Uparrow}\ 1\times F^n\ =\ 1\times F^n\ \overset{\pi_n^{-1}}{\Uparrow}\ F\ \overset{\eta}{\nearrow}\ G\\[4pt]
\xrightarrow{h_{k_1}\times\cdots\times h_{k_n}}\;\xrightarrow{1}\;\xrightarrow{h}\;\xrightarrow{1}
\end{array}
$$

$=$

$$
\begin{array}{c}
\xrightarrow{1}\;\xrightarrow{h'_{k_1}\times\cdots\times h'_{k_n}}\;\xrightarrow{h'_n}\\[4pt]
1\times G^{\sum k_i}\ \overset{1\times\varepsilon^{\sum k_i}}{\nearrow}\ 1\times F^{\sum k_i}\ \overset{\pi_{k_1}^{-1}\times\cdots\times\pi_{k_n}^{-1}}{\Uparrow}\ 1\times F^n\ \overset{\pi_n^{-1}}{\Uparrow}\ F\ \overset{\eta}{\nearrow}\ G\\[4pt]
\xrightarrow{h_{k_1}\times\cdots\times h_{k_n}}\;\xrightarrow{h}\;\xrightarrow{1}
\end{array}
$$

$=$

$$
\begin{array}{c}
\xrightarrow{1}\;\xrightarrow{h'_{\sum k_i}}\\[4pt]
1\times G^{\sum k_i}\ \overset{1\times\varepsilon^{\sum k_i}}{\nearrow}\ 1\times F^{\sum k_i}\ \overset{\pi_{\sum k_i}^{-1}}{\Uparrow}\ F\ \overset{\eta}{\nearrow}\ G\\[4pt]
\xrightarrow{h_{\sum k_i}}\;\xrightarrow{1}
\end{array}
$$

$=$

$$
\begin{array}{c}
\xrightarrow{\;h'_{\sum k_i}\;}\\[4pt]
1\times G^{\sum k_i}\Big\downarrow\quad \overset{\psi_{\sum k_i}}{\nearrow}\quad \Big\downarrow G\\[4pt]
\xrightarrow{\;h_{\sum k_i}\;}
\end{array}
$$

$=$ RHS.

For (2.27), consider the following diagram:

$$
\begin{array}{c}
Gb \xrightarrow{\ \delta_{1_Q}\ } h(1_P, Gb) \\
\end{array}
$$

(2.27) is the outside of the diagram. ① commutes by the triangle identities. ② commutes by naturality of η. ③ commutes since (F, π) is a P-functor. ④ commutes by naturality of δ. ⑤ is the definition of ψ. Hence the whole diagram commutes, and (G, ψ) is a P-functor.

To see that (F, G, η, ϵ) is a P-equivalence, it is now enough to show that η and ϵ are P-transformations, since they satisfy the triangle identities by hypothesis.

Write $(GF, \chi) = (G, \psi) \circ (F, \pi)$. We wish to show that η is a P-transformation $(1, 1) \to (GF, \chi)$. Each $\chi_{q, a_\bullet}$ is the composite

$$
h(q, GFa_\bullet) \xrightarrow{\psi_{q, Fa_\bullet}} Gh(q, Fa_\bullet) \xrightarrow{G\pi_{q, a_\bullet}} GFh(q, a_\bullet)
$$

Applying the definition of ψ, this is

$$
h(q, GFa_\bullet) \xrightarrow{\eta} GFh(q, GFa_\bullet) \xrightarrow{G\pi^{-1}} Gh(q, FGFa_\bullet) \xrightarrow{Gh_q\epsilon F} Gh(q, Fa_\bullet) \xrightarrow{G\pi} GFh(q, a_\bullet)
$$

The axiom on η is the outside of the diagram

① commutes by naturality of η, ② commutes by naturality of π^{-1}, and ③ commutes since $G\pi \circ G\pi^{-1} = G(\pi \circ \pi^{-1}) = G1 = 1G$. The triangle commutes by the triangle identities. So the whole diagram commutes, and η is a P-transformation. By Lemma 4.2.8, η^{-1} is also a P-transformation. Similarly, ϵ and ϵ^{-1} are P-transformations.

$\square$

The statement of the lemma is a fragment of the statement that **Wk-P-Cat** is 2-monadic over **Cat**. Compare the fact that monadic functors reflect isos.

Theorem 5.1.4. *Let $Q \circ A \xrightarrow{h} A$ be a weak P-category. Then A is equivalent to $\mathbf{st}(A)$ in the 2-category* **Wk-P-Cat**.

Proof. Let $F : \mathbf{st}(A) \to A$ be given by $F(p, a_\bullet) = h(p, a_\bullet)$ and identification of maps. This is certainly full and faithful, and it is essentially surjective on objects because $\delta_{1_Q}^{-1} : h(1_P, a) \to a$ is an isomorphism. By Lemma 5.1.3, it remains only to show that F is a weak P-functor.

We must find a sequence $(\phi_i : h_i(1 \times F^i) \to Fh')$ of natural transformations satisfying equations (2.26) and (2.27) from Lemma 2.9.11. We can take $(\phi_i)_{q,(p_\bullet, a_\bullet^\bullet)} = (\delta_{q \circ (p_\bullet)})_{a_\bullet^\bullet}$ for $q \in Q_n$ and $(p_1, a_\bullet^1), \ldots, (p_n, a_\bullet^n) \in \mathbf{st}(A)$. For (2.26), we must show that

$$
\begin{array}{ccc}
\xrightarrow{h'_{k_1} \times \cdots \times h'_{k_n}} \xrightarrow{h'_n} & & \xrightarrow{h'_{\sum k_i}} \\
\Big\downarrow{}^{\phi_{k_1} \times \cdots \times \phi_{k_n}} \quad \Big\downarrow{}^{\phi_n} \quad \Big\downarrow{}^{F} & = & \Big\downarrow{}^{\phi_{\sum k_i}} \quad \Big\downarrow{}^{F} \\
\xrightarrow{h_{k_1} \times \cdots \times h_{k_n}} \xrightarrow{h} & & \xrightarrow{h_{\sum k_i}}
\end{array}
$$

All 2-cells in this equation are instances of δ. Since there is at most one 2-cell between two 1-cells in Q, the equation holds.

For (2.27) to hold, we must have

$$
\begin{array}{ccc}
F(p, a_\bullet) & \xrightarrow{\delta_{1_Q}} & h(1_P, F(p, a_\bullet)) \\
{\scriptstyle 1}\Big\downarrow & & \Big\downarrow{}^{\phi_{1_P}} \\
F(p, a_\bullet) & \xrightarrow{F\delta'_{1_Q}} & Fh'(1_P, (p, a_\bullet))
\end{array}
\tag{5.1}
$$

Since $\mathbf{st}(A)$ is a strict monoidal category, $\delta' = 1$. Apply this observation, and the definitions of F, ϕ and h'; then (5.1) becomes

$$
\begin{array}{ccc}
h(p, a_\bullet) & \xrightarrow{\delta_{1_Q}} & h(1_P, h(p, a_\bullet)) \\
{\scriptstyle 1}\Big\downarrow & & \Big\downarrow{}^{\delta_{1_P \circ (p)}} \\
h(p, a_\bullet) & \xrightarrow{1} & h(p, a_\bullet)
\end{array}
$$

Since there is at most one arrow between two 1-cells in Q, this diagram commutes. So (F, ϕ) is a weak P-functor, and hence (by Lemma 5.1.3) an equivalence in **Wk-P-Cat**. $\qquad\square$

Example 5.1.5. Consider the initial operad 0, whose algebras are sets. We saw in Example 4.3.4 that unbiased weak 0-categories are categories equipped with a trivial monad. By Theorem 5.1.4, every unbiased weak 0-category is equivalent via weak 0-functors to a category equipped with a monad which *is* the identity: in other words, a category.

Example 5.1.6. Consider the terminal operad 1, whose algebras are monoids. Theorem 5.1.4 tells us that every unbiased weak monoidal category is monoidally equivalent to a strict monoidal category.

5.2 Universal property of st

Let P be a plain operad.

Theorem 5.2.1. *Let U' be the forgetful functor* **Str-P-Cat** $\rightarrow$ **Wk-P-Cat** *(considering both of these as 1-categories). Then* **st** *is left adjoint to U'.*

Proof. For each $(A, h) \in$ **Wk-P-Cat**, we construct an initial object $A \xrightarrow{(F', \psi)} \mathbf{st}(A)$ of the comma category $(A \downarrow U')$, thus showing that **st** is functorial and that **st** $\dashv U'$ (and that (F', ψ) is the component of the unit at A). Let (B, h'') be a strict P-category, and $(G, \gamma) : A \rightarrow U'B$ be a weak P-functor. We must show that there is a unique strict P-functor H making the following diagram commute:

$$
\begin{array}{ccc}
 & A & \\
{\scriptstyle (F',\psi)} \swarrow & & \searrow {\scriptstyle (G,\gamma)} \\
U'\,\mathbf{st}(A) & \xrightarrow[\quad(H,\mathrm{id})\quad]{} & U'B
\end{array}
\tag{5.2}
$$

(F', ψ) is given as follows:

- If $a \in A$, then $F'(a) = (1, a)$.

- If $f : a \rightarrow a'$ in A then $F'f$ is the lifting of $h(1, f)$ with source $(1, a)$ and target $(1, a')$.

- Each $\psi_{(p, a_\bullet)}$ is the lifting of $(\delta_{1_Q})_{h(p, a_\bullet)} : h(p, a_\bullet) \rightarrow h(1, h(p, a_\bullet))$ to a morphism $h'(p, F'(a)_\bullet) = (p, a_\bullet) \rightarrow (1, h(p, a_\bullet)) = F'(h(p, a_\bullet))$.

For commutativity of (5.2), we must have $H(1, a) = G(a)$, and for strictness of H, we must have $H(p, a_\bullet) = h''(p, H(1, a)_\bullet)$. These two conditions completely determine H on objects.

Now, take a morphism $f : (p, a_\bullet) \to (p', a'_\bullet)$, which is a lifting of a morphism $g :$ $h(p, a_\bullet) \to h(p', a'_\bullet)$ in A. Then Hf is a morphism $h''(p, Ga_\bullet) \to h''(p', Ga'_\bullet)$: the obvious thing for it to be is the composite

$$h''(p, Ga_\bullet) \xrightarrow{\ \gamma\ } Gh''(p, a_\bullet) \xrightarrow{\ Gg\ } Gh''(p', a'_\bullet) \xrightarrow{\ \gamma^{-1}\ } h''(p', Ga'_\bullet)$$

and we shall show that this is in fact the only possibility. Consider the composite

$$(1, h(p, a_\bullet)) \xrightarrow{\ \psi^{-1}\ } (p, a_\bullet) \xrightarrow{\ f\ } (p', a_\bullet) \xrightarrow{\ \psi\ } (1, h(p', a'_\bullet))$$

in $\mathbf{st}(A)$. Composition in $\mathbf{st}(A)$ is given by composition in A, so this is equal to the lifting of $\delta_{1_Q} \circ g \circ \delta_{1_Q}^{-1} = h(1, g)$ to a morphism $(1, h(p, a_\bullet)) \to (1, h(p', a'_\bullet))$, namely $F'g$. So $f = \psi^{-1} \circ F'g \circ \psi$, and $Hf = H\psi^{-1} \circ HF'g \circ H\psi$. By commutativity of (5.2), $HF' = G$ and $H\psi = \gamma$, so $Hf = \gamma^{-1} \circ Gg \circ \gamma$ as required.

This completely defines H. So we have constructed a unique H which makes (5.2) commute and which is strict. Hence $(F', \psi) : A \to U'\mathbf{st}(A)$ is initial in $(A \downarrow U')$, and so $\mathbf{st} \dashv U'$. $\square$

The P-functor $(F, \phi) : \mathbf{st}(A) \to A$ constructed in Theorem 5.1.4 is pseudo-inverse to (F', ψ), which we have just shown to be the A-component of the unit of the adjunction $\mathbf{st} \dashv U'$. We can therefore say that $\mathbf{Str}\text{-}P\text{-}\mathbf{Cat}$ is a weakly coreflective sub-2-category of $\mathbf{Wk}\text{-}P\text{-}\mathbf{Cat}$. Note that the counit is *not* pseudo-invertible, so this is not a 2-equivalence.

Example 5.2.2. Consider again the initial operad 0, whose algebras are sets. We saw in Example 4.3.4 that unbiased weak 0-categories are categories equipped with a specified trivial monad. Let **Triv** denote the category of such categories, with morphisms being functors that preserve the trivial monad up to coherent isomorphism. A strict unbiased 0-category is a category equipped with a monad equal to the identity monad, which is simply a category. So **Cat** is a weakly coreflective sub-2-category of **Triv**.

5.3   Presentation-independence

We will now show that the weakening of a symmetric operad P is essentially independent of the generators chosen. This generalizes Leinster's result (in [Lei03] section 3.2) that the theory of weak monoidal categories is essentially unaffected by the choice of a different presentation for the theory of monoids.

We will need the following lemma:

Lemma 5.3.1. *In* **Cat-Σ-Operad**, *if* $P \overset{\alpha}{\underset{\beta}{\rightrightarrows}} Q \overset{\gamma}{\longrightarrow} R$ *is a fork, and γ is levelwise full and faithful, then $\alpha \cong \beta$.*

Proof. We shall construct an invertible **Cat**-Σ-operad transformation $\eta : \alpha \to \beta$. We form the η_ns as follows: for all $p \in P_n$, let $\gamma\alpha(p) = \gamma\beta(p)$. Since γ is levelwise full, there exists an arrow $(\eta_n)_p : \alpha(p) \to \beta(p)$ such that $\gamma_n((\eta_n)_p) = 1_{\gamma\alpha(p)}$. Since γ is levelwise full and faithful, this arrow is an isomorphism. Each η_n is natural because, for all $n \in \mathbb{N}$ and $f : p \to q$ in P_n, the image under γ of the naturality square

$$\begin{array}{ccc} \alpha(p) & \overset{(\eta_n)_p}{\longrightarrow} & \beta(p) \\ {\scriptstyle \alpha(f)}\downarrow & & \downarrow{\scriptstyle \beta(f)} \\ \alpha(q) & \overset{(\eta_n)_q}{\longrightarrow} & \beta(q) \end{array}$$

is

$$\begin{array}{ccc} \gamma\alpha(p) & \overset{1}{\longrightarrow} & \gamma\beta(p) \\ {\scriptstyle \gamma\alpha(f)}\downarrow & & \downarrow{\scriptstyle \gamma\beta(f)} \\ \gamma\alpha(q) & \overset{1}{\longrightarrow} & \gamma\beta(q) \end{array}$$

which commutes since $\gamma\alpha = \gamma\beta$. Since γ is faithful, the naturality square commutes, and η_n is natural. It remains to show that the collection $(\eta_n)_{n\in\mathbb{N}}$ forms a **Cat**-Σ-operad transformation, in other words that the equations

$$(5.3)$$

$$(\eta_1)_1 \quad = \quad 1 \tag{5.4}$$

$$(5.5)$$

hold, for all $n, k_1 \ldots k_n \in \mathbb{N}$ and every $\sigma \in S_n$. As above, it is enough to show that the images of both sides under γ are equal, and this is trivially true by definition of η. $\square$

Let P be a symmetric operad.

Theorem 5.3.2. *Let $\Phi \in \mathbf{Set}^{\mathbb{N}}$ and let $\phi : F_{\Sigma}\Phi \to P$ be a regular epi. Then $\mathrm{Wk}_{\phi}(P)$ is equivalent as a symmetric $\mathbf{Cat}$-operad to $\mathrm{Wk}(P)$.*

Proof. Let Q be the weakening of P with respect to $\phi : F_{\Sigma}\Phi \to P$. By the triangle identities, we have a commutative square

$$
\begin{array}{ccc}
F_{\Sigma}\Phi & \xrightarrow{\phi} & P \\
{\scriptstyle F_{\Sigma}\bar{\phi}} \downarrow & & \downarrow {\scriptstyle 1} \\
F_{\Sigma}U^{\Sigma}P & \xrightarrow{\epsilon_P} & P
\end{array}
$$

By functoriality of the factorization system, this gives rise to a unique map $\chi : Q \to \mathrm{Wk}(P)$ such that

$$
\begin{array}{ccccc}
 & & \xrightarrow{\phi} & & \\
F_{\Sigma}\Phi & \longrightarrow & Q & \rightarrowtail & P \\
{\scriptstyle F_{\Sigma}\bar{\phi}} \downarrow & & \downarrow {\scriptstyle \chi} & & \downarrow {\scriptstyle 1} \\
F_{\Sigma}U^{\Sigma}P & \twoheadrightarrow & \mathrm{Wk}(P) & \rightarrowtail & P \\
 & & \xrightarrow{\epsilon_P} & &
\end{array}
$$

commutes. We wish to find a pseudo-inverse to χ.

Since $\Sigma\text{-}\mathbf{Operad}$ is monadic over $\mathbf{Set}^{\mathbb{N}}$, a regular epi in $\Sigma\text{-}\mathbf{Operad}$ is a levelwise surjection by Theorem 3.0.11. So we may choose a section ψ_n of $\phi_n : (F_{\Sigma}\Phi)_n \to P_n$ for all $n \in \mathbb{N}$. So we have a morphism $\psi : U^{\Sigma}P \to U^{\Sigma}F_{\Sigma}\Phi$ in $\mathbf{Set}^{\mathbb{N}}$. We wish to show that

$$
\begin{array}{ccc}
F_{\Sigma}U^{\Sigma}P & \xrightarrow{\epsilon_P} & P \\
{\scriptstyle \bar{\psi}} \downarrow & & \downarrow {\scriptstyle 1} \\
F_{\Sigma}\Phi & \xrightarrow{\phi} & P
\end{array}
$$

commutes. This follows from a simple transpose argument:

$$
\frac{F_{\Sigma}U^{\Sigma}P \xrightarrow{\bar{\psi}} F_{\Sigma}\Phi \xrightarrow{\phi} P}{U^{\Sigma}P \xrightarrow{\psi} U^{\Sigma}F_{\Sigma}\Phi \xrightarrow{U^{\Sigma}\phi} U^{\Sigma}P} \qquad = \qquad \frac{U^{\Sigma}P \xrightarrow{1} U^{\Sigma}P}{F_{\Sigma}U^{\Sigma}P \xrightarrow{\epsilon} P.}
$$

This induces a map

$$
\begin{array}{ccccc}
 & & \xrightarrow{\epsilon_P} & & \\
F_{\Sigma}U^{\Sigma}P & \twoheadrightarrow & \mathrm{Wk}(P) & \rightarrowtail & P \\
{\scriptstyle \bar{\psi}} \downarrow & & \downarrow {\scriptstyle \omega} & & \downarrow {\scriptstyle 1} \\
F_{\Sigma}\Phi & \twoheadrightarrow & Q & \rightarrowtail & P \\
 & & \xrightarrow{\phi} & &
\end{array}
$$

We will show that ω is pseudo-inverse to χ. Now,

$$
\begin{array}{ccc}
Q & \rightarrowtail & P \\
\omega \downarrow & & \downarrow 1 \\
\mathrm{Wk}(P) & \rightarrowtail & P \\
\chi \downarrow & & \downarrow 1 \\
Q & \rightarrowtail & P
\end{array}
$$

commutes. So $Q \underset{\chi\omega}{\overset{1_Q}{\rightrightarrows}} Q \rightarrowtail P$ is a fork. By Lemma 5.3.1, $\chi\omega \cong 1_Q$, and similarly $\omega\chi \cong 1_{\mathrm{Wk}(P)}$. So $Q \simeq \mathrm{Wk}(P)$ as a symmetric **Cat**-operad, as required. $\qquad\square$

Corollary 5.3.3. *Let P be a plain operad. Then $F_\Sigma^{\mathrm{pl}}(\mathrm{Wk}(P)) \simeq \mathrm{Wk}(F_\Sigma^{\mathrm{pl}}P)$.*

Proof. Let $\phi : F_{\mathrm{pl}}U^{\mathrm{pl}}P \to P$ be the component at P of the counit of the adjunction $F_{\mathrm{pl}} \dashv U^{\mathrm{pl}}$. Let ϵ be the counit of the adjunction $F_\Sigma^{\mathrm{pl}} \dashv U_{\mathrm{pl}}^\Sigma$.

By Theorem 4.4.8, there is an isomorphism $F_\Sigma^{\mathrm{pl}}(\mathrm{Wk}_\phi(P)) \cong \mathrm{Wk}_{F_\Sigma^{\mathrm{pl}}\phi}(F_\Sigma^{\mathrm{pl}}P)$, and by Theorem 5.3.2, there is an equivalence $\mathrm{Wk}_{F_\Sigma^{\mathrm{pl}}\phi}(F_\Sigma^{\mathrm{pl}}P) \simeq \mathrm{Wk}(F_\Sigma^{\mathrm{pl}}P)$. Hence $F_\Sigma^{\mathrm{pl}}(\mathrm{Wk}(P)) \simeq \mathrm{Wk}(F_\Sigma^{\mathrm{pl}}P)$. $\qquad\square$

Corollary 5.3.4. *Let P be a plain operad. Then the category* **Wk-P-Cat** *is equivalent to the category* **Wk-$F_\Sigma^{\mathrm{pl}}P$-Cat**.

This tells us that the unbiased categorification of a strongly regular theory is essentially unaffected by our treating it as a linear theory instead.

Example 5.3.5. Considering again the trivial theory 0, we see that **Triv** $\simeq$ **Cat**.

This can be generalised to the multi-sorted situation:

Lemma 5.3.6. *Let X be a set, and f be a regular epi in the category* **Cat-Multicat$_X$** *or in the category* **Cat-Σ-Multicat$_X$**. *Then f is locally surjective on objects.*

Proof. **Multicat$_X$** is monadic over **Multigraph$_X$** by Lemma 2.5.1 and Theorem 2.5.3, and an object of **Multigraph$_X$** can be considered as an object of **Set^Y**, where $Y = X \times X^*$, and X^* is the free monoid on X: for each $x \in X$, and each sequence $x_1, \dots, x_n \in X^*$, there is a set of funnels $x_1, \dots, x_n \to x$. Hence, by 3.0.11, every regular epi in **Multicat$_X$** is locally surjective.

The objects functor $O : \mathbf{Cat} \to \mathbf{Set}$ has both a left adjoint D and a right adjoint I. Hence O and I preserve products, and hence by Lemma 2.5.1 they induce an adjunction

$$\mathbf{Cat\text{-}Multicat}_X \underset{I_*}{\overset{O_*}{\rightleftarrows}} \mathbf{Multicat}_X \ .$$

Since O_* is a left adjoint, it preserves colimits, and in particular regular epis: hence, every regular epi in $\mathbf{Cat\text{-}Multicat}_X$ must be locally surjective on objects.

The symmetric case is proved analogously. $\qquad\square$

Theorem 5.3.7. *Let M be a (symmetric) multicategory, and $\phi : F_{\mathrm{pl}}\Phi \to M$ (or in the symmetric case, $\phi : F_\Sigma\Phi \to M$) be a regular epi. Then the weakening of M with respect to ϕ is equivalent as a $\mathbf{Cat}$-multicategory to $\mathrm{Wk}(M)$.*

Proof. The proof is exactly as for Theorem 5.3.2. $\qquad\square$

Chapter 6

Other Approaches

6.1 Pseudo-algebras for 2-monads

We begin by recalling some standard notions of 2-monad theory.

Definition 6.1.1. A **2-monad** is a monad object in the 2-category of 2-categories, in the sense of [Str72]; that is to say, a 2-category $\mathcal{C}$, a strict 2-functor $T : \mathcal{C} \to \mathcal{C}$, and 2-transformations $\mu : T^2 \to T$ and $\eta : 1_{\mathcal{C}} \to T$ satisfying the usual monad laws:

$$
\begin{array}{ccc}
 & T^3 & \\
T\mu \swarrow & & \searrow \mu T \\
T^2 & & T^2 \\
\mu \searrow & & \swarrow \mu \\
 & T &
\end{array}
\tag{6.1}
$$

$$
\begin{array}{ccccc}
T & \xrightarrow{T\eta} & T^2 & \xleftarrow{\eta T} & T \\
 & {}_{1_T}\searrow & \downarrow{\mu} & \swarrow{}_{1_T} & \\
 & & T & &
\end{array}
\tag{6.2}
$$

As is common for ordinary 1-monads, we will usually refer to a 2-monad $(\mathcal{C}, T, \mu, \eta)$ as simply T.

The usual notion of an algebra for a monad carries over simply to this case:

Definition 6.1.2. Let $(\mathcal{C}, T, \mu, \eta)$ be a 2-monad. A **strict algebra** for T is an object

$A \in \mathcal{C}$ and a 1-cell $a : TA \to A$ satisfying the following axioms:

$$
\begin{array}{ccc}
 & T^2A & \\
Ta \swarrow & & \searrow \mu \\
TA & & TA \\
a \searrow & & \swarrow a \\
 & A &
\end{array}
\tag{6.3}
$$

$$
\begin{array}{ccc}
A & \xrightarrow{\;\eta\;} & TA \\
 & {}_{1_A}\searrow & \downarrow a \\
 & & A
\end{array}
\tag{6.4}
$$

For our purposes, it is more interesting to consider the well-known **pseudo-algebras** for a 2-monad. These are algebras "up to isomorphism":

Definition 6.1.3. Let $(\mathcal{C}, T, \mu, \eta)$ be a 2-monad. A **pseudo-algebra** for T is an object $A \in \mathcal{C}$, a 1-cell $a : TA \to A$, and invertible 2-cells

$$
\begin{array}{ccc}
 & T^2A & \\
Ta \swarrow & & \searrow \mu \\
TA & \overset{\alpha}{\Rightarrow} & TA \\
a \searrow & & \swarrow a \\
 & A &
\end{array}
\qquad
\begin{array}{ccc}
A & \xrightarrow{\;\eta\;} & TA \\
 & {}_{\beta\nearrow}{}_{1_A}\searrow & \downarrow a \\
 & & A
\end{array}
$$

satisfying the equations

$$
\tag{6.5}
$$

$$(6.6)$$

Definition 6.1.4. Let $(\mathcal{C}, T, \mu, \eta)$ be a 2-monad, and let $(A, a, \alpha_1, \alpha_2)$ and (B, b, β_1, β_2) be pseudo-algebras for T. A **pseudo-morphism of pseudo-algebras** $(A, a, \alpha_1, \alpha_2)$ to (B, b, β_1, β_2) is a pair (f, ϕ), where $f : A \to B$ is a 1-cell in $\mathcal{C}$ and ϕ is an invertible 2-cell:

$$
\begin{array}{ccc}
TA & \xrightarrow{Tf} & TB \\
a \downarrow & \Swarrow_\phi & \downarrow b \\
A & \xrightarrow{f} & B
\end{array}
$$

satisfying the axioms

$$(6.7)$$

$$(6.8)$$

This gives rise to a category **Ps-Alg**(T) for any 2-monad T.

Every cartesian monad T on **Set** gives rise to a 2-monad $\bar{T}$ on **Cat** in an obvious way, and (as we saw in Theorem 2.8.10) every plain operad P gives rise to a cartesian monad T_P on **Set**. So an alternative definition of "weak P-category" might be "pseudo-algebra for $\bar{T}_P$".

In order to explore the connections between this idea and the notion of weak P-category given in previous chapters, we shall need some theorems from [BKP89] and related papers.

Theorem 6.1.5. *(Blackwell, Kelly, Power) Let T be a 2-monad with rank on a cocomplete 2-category K, let $\mathbf{Alg}(T)_{\mathrm{str}}$ be the 2-category of strict T-algebras and strict morphisms, and $\mathbf{Alg}(T)_{\mathrm{wk}}$ be the 2-category of strict T-algebras and weak morphisms. Then the inclusion $J : \mathbf{Alg}(T)_{\mathrm{str}} \to \mathbf{Alg}(T)_{\mathrm{wk}}$ has a left adjoint L. Thus every strict T-algebra A has a* **pseudo-morphism classifier** $p : A \to A'$ *(where $A' = JLA$), such that for all $B \in K$, and every pseudo-morphism $f : A \to B$, we may express f uniquely as the composite of p and a strict morphism:*

$$
\begin{array}{ccc}
A & \xrightarrow{\ p\ } & A' \\
 & {\scriptstyle f}\searrow & \big\downarrow{\scriptstyle J\bar{f}} \\
 & & B
\end{array}
$$

Proof. See [BKP89], Theorem 3.13. □

Theorem 6.1.6. *(Blackwell, Kelly, Power) Let $f : S \to T$ be a strict map between 2-monads with rank on a cocomplete 2-category K. Then the induced map $f^* : \mathbf{Alg}(T)_{\mathrm{str}} \to \mathbf{Alg}(S)_{\mathrm{str}}$ has a left adjoint, and the induced map $f^* : \mathbf{Alg}(T)_{\mathrm{wk}} \to \mathbf{Alg}(S)_{\mathrm{wk}}$ has a left biadjoint.*

Proof. See [BKP89], Theorem 5.12. □

Corollary 6.1.7. *Composing this left adjoint with the left adjoint of Theorem 6.1.5 gives us an adjunction*

$$
\mathbf{Alg}(S)_{\mathrm{wk}} \underset{U}{\overset{F}{\underset{\perp}{\rightleftarrows}}} \mathbf{Alg}(T)_{\mathrm{str}}
$$

Theorem 6.1.8. *(Power, Lack) Let T be a 2-monad with rank on a cocomplete 2-category K of the form $\mathbf{Cat}^X$ for some set X, and let T preserve pointwise bijectivity-on-objects. Let (A, a) be a strict T-algebra. Then the pseudo-morphism classifier A' for A may be found by factorizing the structure map $a : TA \to A$ as a pointwise bijective-on-objects map followed by a locally full-and-faithful map:*

$$
\begin{array}{ccc}
TA & \xrightarrow{\ a\ } & A \\
 & \searrow \qquad \nearrow & \\
 & A' &
\end{array}
$$

Proof. The construction is given in Power's paper [Pow89], and the universal property of the algebra constructed is proved in Lack's paper [Lac02]. □

This argument is due to Steve Lack (private communication).

Theorem 6.1.9. *Let P be a plain operad. Let T_P be the monad induced by P on* **Set**. *Then a pseudo-algebra for $\bar{T}_P$ is a weak P-category in the sense of Definition 4.2.1. Furthermore, there is an isomorphism of categories* $\mathbf{Ps\text{-}Alg}(\bar{T}_P) \cong \mathbf{Alg}_{\mathrm{wk}}(P)$.

Proof. **Cat-Operad** is monadic over $\mathbf{Cat}^{\mathbb{N}}$ via one of the special monads of Theorem 6.1.8, and hence, for every plain **Cat**-operad P, the pseudo-morphism classifier of P is none other than $\mathrm{Wk}(P)$. Hence, if A is a category, then a strict map of **Cat**-operads $\mathrm{Wk}(P) \to \mathrm{End}(A)$ is precisely a weak map $P \to \mathrm{End}(A)$, or equivalently a $\bar{T}_P$-pseudo-algebra structure on A. □

We may also use these ideas to provide a simple proof of the strictification result in Theorem 5.1.4. The map $P \to \mathrm{Wk}(P)$ given by Theorem 6.1.5 is pseudo, but it has a strict retraction $q : \mathrm{Wk}(P) \to P$. This is equivalent to a strict map of monads $T_{\mathrm{Wk}(P)} \to T_P$. By Corollary 6.1.7, this induces a 2-functor $\mathbf{Alg}(P)_{\mathrm{str}} \to \mathbf{Alg}(\mathbf{Wk}(P))_{\mathrm{wk}}$ with a left adjoint. This functor is simply the inclusion of the 2-category of strict P-categories, strict P-functors and P-transformations into the 2-category of weak P-(categories, functors, transformations), and its left adjoint is the functor **st** constructed in Section 5.1. The fact that any weak P-category A is equivalent to $\mathbf{st}(A)$ is a consequence of the fact that any pseudo P-algebra is equivalent to a strict one, and this holds by the General Coherence Result of Power.

However, pseudo-algebras are less useful in the case of linear theories. Since the monads arising from symmetric operads are not in general cartesian, we may not perform the construction given above. We may, however, use the existence of colimits in **Cat**, and consider the 2-monad

$$A \mapsto \int^{n \in \mathbb{B}} P_n \times A^n$$

for any symmetric operad P. If P is the free symmetric operad on a plain operad P', this 2-monad is equal to $\bar{T}_{P'}$. Yet this coend construction also leads to problems.

Let T be the "free commutative monoid" monad on **Set**, and S be the "free monoid" monad on **Set**. Since these both arise from symmetric operads, we may lift them to 2-monads T', S' on **Cat** as described above. T' is the free commutative monoid monad on **Cat**, which is to say the free strict symmetric monoidal category 2-monad; similarly, S' is the free strict monoidal category 2-monad. For each category A, there is a functor

$\pi_A : S'A \to T'A$ which is full and surjective-on-objects; hence, if $(A, a, \alpha_1, \alpha_2)$ is a pseudo-algebra for T', we obtain an S'-pseudo-algebra structure by precomposing with π_A:
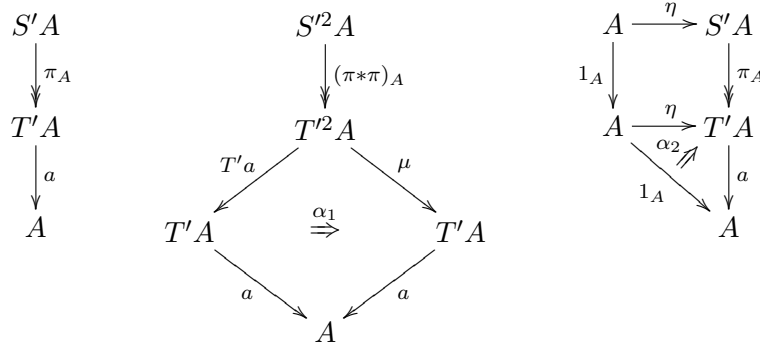
The S'-pseudo-algebra structure so obtained is uniquely determined. Since π_A is full and surjective-on-objects, it is epic, and so every pseudo-algebra for S' is a pseudo-algebra for T' in at most one way. Hence we may view all pseudo-algebras for T' as pseudo-algebras for S' (that is, as monoidal categories) with extra properties. But there exist monoidal categories with several choices of symmetric structure on them. For instance, consider the category of graded Abelian groups, with tensor product $(A \otimes B)_n = \bigoplus_{i+j=n} A_i \otimes B_j$. As well as the obvious symmetry, there is another given by $\tau_{AB}(a \otimes b) = (-1)^{ij} b \otimes a$, where $a \in A_i, b \in B_j$.

We can say at least something about the extra properties that pseudo-algebras for T' must have:

Theorem 6.1.10. *A pseudo-algebra for T' is a symmetric monoidal category A in which $x \otimes y = y \otimes x$ for all $x, y \in A$.*

Proof. Recall our construction of the finite product operad whose algebras are commutative monoids in Example 2.3.4. From this, we may deduce that if A is a set, then an element of $T_P A$ is a function $A \to \mathbb{N}$ assigning each element of A its multiplicity: in other words, a multiset of elements of A. Let (A, a, α, β) be a pseudo-algebra for T' in **Cat**. Then we have a binary tensor product:

$$x \otimes y := a(x^1 y^1)$$

where $x^1 y^1$ is the function $A \to \mathbb{N}$ sending x and y to 1 and all other objects of A to 0. The tensor is defined analogously on morphisms. The components of α and β give us associator, symmetry and unit maps, and it can be shown that they satisfy the axioms for a monoidal category. However, since the function $x^1 y^1$ is equal to the function $y^1 x^1$ for all $x, y \in A$, it must be the case that $x \otimes y = y \otimes x$. $\qquad \square$

Since not all symmetric monoidal categories satisfy this condition, it is apparent that a naïve approach to categorification based on pseudo-algebras is doomed to fail, and that more sophistication is required. In fact, I conjecture that a stronger condition holds: that the symmetry maps are all identities.

In the specific case of symmetric monoidal categories, we may remedy the situation as follows. Let T be the "free symmetric strict monoidal category" 2-monad. Then pseudo-algebras for T are precisely symmetric monoidal categories.

6.2   Laplaza sets

This notion was introduced by T. Fiore, P. Hu and I. Kriz in [FHK], as a generalization of Laplaza's categorification of rigs in [Lap72]. It was introduced as an attempt to correct an error in the earlier definition of categorification proposed in [Fio06]; the error in question is essentially that discussed in Section 4.9 above.

Definition 6.2.1. Let T be a finite product operad. A **Laplaza set** for T is a subsignature of $U^{\mathrm{fp}}T$.

Concretely, a Laplaza set S for T is a sequence $S_0 \subset T_0, S_1 \subset T_1, \ldots$ of subsets of $T_0, T_1, \ldots$.

Definition 6.2.2. Let T be a finite product operad, and S be a Laplaza set for T. A (T, S)**-pseudo algebra** is

- a category $\mathcal{C}$

- for each $\phi \in T_n$, a functor $\hat{\phi} : \mathcal{C}^n \to \mathcal{C}$,

- **coherence morphisms** witnessing all equations that are true in T,

such that, if

- s_1, s_2, t_1 and t_2 are elements of $(F_{\mathrm{fp}}U^{\mathrm{fp}}T)_n$,

- $\delta_1 : \hat{s}_1 \to \hat{t}_1$ and $\delta_2 : \hat{s}_2 \to \hat{t}_2$ are coherence morphisms,

- $\epsilon(s_1) = \epsilon(s_2) \in S$ and $\epsilon(t_1) = \epsilon(t_2) \in S$,

then $\delta_1 = \delta_2$.

This definition can be recast in terms of strict algebras for a finite product **Cat**-operad.

By judicious choice of Laplaza set, one can recover the classical notion of symmetric monoidal category and Laplaza's categorification of the theory of rigs.

6.3 Non-algebraic definitions

Various definitions have appeared that are inspired by the notions of homotopy monoids etc. in topology. In [Lei00], Leinster proposes a definition of a "homotopy P-algebra in M" for any plain operad P and any monoidal category M; his shorter paper [Lei99] explores this definition in the case $P = 1$. Related (but more general) is Rosický's work described in [Ros].

These definitions stand roughly in relation to ours as do the "non-algebraic" definitions of n-category in relation to the "algebraic" ones: see [CL04].

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