Canonical Structure and Conservation Laws of General Relativity on Null Surfaces and at Null Infinity

by

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This thesis consists of material all of which I authored or co-authored: see Statement of Contributions included in the thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Statement of Contributions

Chapters 2, 3 and 4 are based on the publications [1] and [2] co-authored with Dr. Laurent Freidel. The projects were suggested and supervised by Dr. Freidel, and I did the technical work and a large part of the writing. An earlier version of some of the technical contents of chapter 3, roughly some of the equations until equation (3.43), was also presented in my Perimeter Scholars International thesis [3].

The problem addressed in chapter 6 was suggested to me by Dr. Freidel, and early work, predating the work going into chapter 5, was begun under his supervision. The definitions of χ and $\mathcal{N}_{ab}$ in section 6.1 are from unpublished notes [4] of Dr. Riello and Dr. Freidel on the asymptotic equations of motion of asymptotically flat gravity. I suggested the general approach to removing the divergences and did all of the technical work. Dr. Riello made some important suggestions. I developed the final form of the derivations, as well as the calculation of Hamiltonians and the connection to published results, independently.

Chapter 5 is based on [5] co-authored with Dr. Freidel and Dr. Riello. The project was suggested by Dr. Freidel. I did much of the technical work. The analysis of the asymptotic equations of motion is inspired in part by unpublished notes of Dr. Riello and Dr. Freidel [4]. Both Dr. Riello and Dr. Freidel made technical contributions, and the writing for the publication was done jointly. The presentation in chapter 5 differs from the published version in some aspects.

During the course of my PhD, I also co-authored the publications [6, 7, 8] which are not included in this thesis.

Abstract

The first part of this thesis addresses the canonical structure of general relativity on generic null surfaces. The pre-symplectic potential of metric general relativity, evaluated on a null surface, is decomposed into variables describing the intrinsic and extrinsic geometry of the null surface, without fixing the gauge. Canonical pairs on the null surface and its boundary are identified and interpreted. Boundary contributions to the action corresponding to Dirichlet boundary conditions are identified. The constraints on a null surface are written in the same variables, and naturally take the form of conservation laws mirroring those at null infinity, equating the divergence of a relativistic current intrinsic to the null surface to a flux which has a canonical form. The conservation laws are interpreted canonically and related to Noether's theorem. The second part of this thesis addresses the problem that in asymptotically simple spacetimes, such as asymptotically flat spacetimes, the pre-symplectic potential of field theories on constant-radius surfaces generically diverges in the radius. A scheme is introduced for electromagnetism and gravity which, on-shell, for any spacetime dimension, allows one to absorb the divergences into counterterms, which correspond to the ambiguities of the pre-symplectic potential. The counterterms are local including in the radius, and render also the action and the pre-symplectic potential on constant-time surfaces finite. The scheme employs Penrose's conformal compactification of spacetime. The scheme is introduced and explored for electromagnetism in $D \geq 5$ dimensions. The equations of motion are analyzed in an asymptotic expansion for asymptotically flat spacetimes, and the free data and dependencies among the data entering the symplectic potential are identified. The gauge generators are identified, and are rendered independent of subleading orders of the gauge parameter by introducing further local, finite counterterms. The generators and their fluxes coincide with expressions derived from soft theorems of quantum electrodynamics in even dimensions. Finally, the renormalization scheme is developed for general relativity in asymptotically simple spacetimes, where it applies in $D \geq 3$ dimensions and for any cosmological constant, and does not require any boundary or gauge conditions beyond asymptotic simplicity and some degree of regularity. The resulting expression for the pre-symplectic potential is specialized to a relaxation of the Bondi gauge conditions for four-dimensional asymptotically flat spacetimes, and an existing result is recovered. The scheme is compared to holographic renormalization in four and five spacetime dimensions, and the renormalized stress-energy tensors on asymptotically AdS space are recovered up to scheme-dependent terms. The canonical generators of diffeomorphisms under the renormalized symplectic form are computed.

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Chapter 1

Introduction

The aim of this thesis is to elucidate aspects of the canonical structure of general relativity, on general null surfaces, and for asymptotically flat spacetimes.

1.1 Background

While it was assumed until the sixties that the asymptotic symmetry group of four-dimensional asymptotically flat spacetimes is just the Poincaré group, it was shown by Bondi, van der Burg, Metzner [9] and Sachs [10] that this is not the case: Motivated by defining gravitational waves, they defined "asymptotically flat" by introducing a physically motivated coordinate system and asymptotic boundary conditions, and found that the symmetry group preserving the coordinate and boundary conditions is what is now known as the Bondi-Metzner-Sachs (BMS) group. In addition to the three rotations and three boosts of the Lorentz group, the four translations contained in the Poincaré group are replaced by an infinite number of angle dependent translations called supertranslations, which may be labeled by a function on an asymptotic two-sphere.

The BMS group has since catalyzed and featured in many developments in gravity and quantum gravity.

Through Noether's theorem, the enlargement of the asymptotic symmetry group leads to the appearance of additional conserved charges: the Bondi mass and momentum are generalized to a whole "sphere's worth" of supertranslation charges. More recently, those additional conservation laws have been connected with a seemingly different fact: Weinberg's soft graviton theorem of perturbative quantum gravity [11] was shown to be equivalent to the Ward identity associated with the conservation of supertranslation charges [12, 13]. This development connects asymptotic symmetries with the infrared sector of perturbative quantum gravity: the BMS group is then conjectured to be a symmetry of the quantum gravity S–matrix [13].

The enlargement of the asymptotic symmetry group is also connected to more immediately observable consequences: The memory effect [14, 15, 16], which describes the permanent displacement of detectors after the passage of a gravitational wave, can be understood as a rewriting of the soft theorem [17], or as the transition between classical vacua labeled by different supertranslation charges. The connection between asymptotic symmetries, memory effects and soft theorems has been dubbed the "infrared triangle", see [18] for a pedagogical overview.

In connecting asymptotic symmetries to the infrared sector of perturbative quantum gravity, a careful analysis of the symplectic form at null infinity becomes important. This is because of the correspondence between the Dirac brackets, which are determined by the symplectic form, and the commutators of quantum field operators. Next to the radiative phase space, which has been identified by Ashtekar and Streubel [19], the phase space has a "Coulombic" part, containing the Bondi mass aspect, which also features in the supertranslation charges, and a field which may be interpreted as the Goldstone boson associated to the broken supertranslation symmetry [12, 20]. The two sectors are connected through conservation laws, which arise from the constraints [21], and express the time evolution of the Bondi mass aspect in terms of the radiative data.

While the connection to soft theorems has led to a resurgence of interest in the BMS group, it has also challenged its status as the symmetry group of asymptotically flat spacetimes: That is due to the existence of subleading [22] and double [23] soft graviton theorems. It has been suggested that they also could be obtained as the Ward identities of asymptotic symmetries, thereby requiring an enlargement of the asymptotic symmetry group, larger than BMS, to accommodate them [24, 25]. This has led to much interest to develop more lenient boundary conditions for asymptotically flat spacetimes, with larger asymptotic symmetry groups, and there are competing proposals in the literature [26, 25], some of which predate the connection to soft theorems. Subleading soft theorems have also been related to conservation laws at subleading orders around null infinity [27].

A difficulty that arises is that for lenient boundary conditions, the usual expression for the symplectic form diverges as null infinity is approached, which would lead to divergent expressions for Hamiltonians and fluxes. This was remedied in [28] for the specific case of the relaxation of the Bondi gauge boundary conditions proposed by [25], by using an ambiguity inherent in the definition of field theory symplectic forms, of adding terms which are integrals on the boundary of the initial value surface. A general prescription to remove such divergences, or arguments about under which conditions they can be removed, have not been given for gravity in asymptotically flat spacetimes to the best of our knowledge. In [29] it was shown that a scheme to remove such divergences, if it exists, cannot be fully spacetime covariant. Brackets on a phase space carrying the same symmetry group as [28] have been derived using a different approach in [30]. Earlier work, showing finiteness of canonical quantities in four dimensions using stricter conditions, includes [31, 32].

On asymptotically anti-de Sitter spacetimes, a similar problem arises: the action and the momenta, which are commonly derived using the Brown-York description [33], also diverge in the radius. They can be renormalized using the holographic renormalization scheme [34], which is an important component of the holographic dictionary for theories with a holographic dual. The ideas of holographic renormalization can be expressed in a canonical form, and extended to theories which do not admit a holographic dual [35]. Also, holographic renormalization of asymptotically flat general relativity has been considered in [36, 37].

For asymptotically AdS spacetimes, it is well appreciated that too much regularity of the metric around infinity cannot be required [38]. For asymptotically flat spacetimes, there are also interesting spacetimes leading to asymptotic metrics with only limited regularity [39, 40, 41, 42]. In much of the asymptotic symmetries literature for asymptotically flat spacetimes however, a large degree of regularity is assumed.

The analysis of asymptotically flat spacetimes and their symmetries has been put on a more geometric footing by Penrose [43]. He defined asymptotic flatness and related notions by introducing an auxiliary, unphysical spacetime, in which the original spacetime is compactified by adding a boundary at infinity, and the original metric is rescaled such that is asymptotically finite. This elegant definition makes powerful conformal techniques applicable. The BMS group, the definition of radiation, the Bondi mass and its conservation, have all been translated to this setting. Quantization and numerical calculations have also been studied in this arena. See [44, 45] for overviews.

Asymptotically flat spacetimes, their symmetries, connections to quantum gravity and to memory effects have been fruitful avenues of investigation, particularly in recent years, including many developments not covered here. The work in the second part of this thesis is motivated by some of the open questions and challenges in this area.

The first part is concerned instead with the canonical structure of general relativity at generic null surfaces at finite distance.

Null surfaces are physically special locations in spacetime: The boundaries of light cones are null surfaces, and thus null surfaces are a natural arena to study causality and information in gravity. Many event horizons are null surfaces, making their study broadly applicable for example to questions about black holes. The distinguished physical status of null surfaces also has mathematical consequences, and the constraints and equations of motion have special properties at null surfaces. Null surfaces also provide a possible route to quantization [46, 47].

By analyzing the equations of motion, Sachs first showed that the initial data could be given in an unconstrained form using a double null sheet as an initial value surface [48], which, among others, has spurred many developments in numerical relativity ([49] for an overview). The analysis has been extended using twistor variables by Penrose [50]. The equations of motion and constraints have also been examined in many guises, using geometric variables pertaining to a "null 2+2" foliation of spacetimes, into spacelike codimension-two surfaces with two null normals [51, 52, 53, 54]. From an explicitly canonical perspective, generic null surfaces have been addressed by Epp [55], who gave the symplectic structure in a partially gauge fixed setting. Reisenberger in [56] also investigated the symplectic form, and used it to give Poisson brackets of free data [57].

In order to reach a well defined variational principle under Dirichlet boundary conditions on the metric, a boundary term has to be added to the Einstein-Hilbert Lagrangian. Such boundary terms are important beyond the question of making the Lagrangian field theory consistent, as they contribute to the value of the on–shell action, which features for example in the "Complexity = Action" conjecture of Brown et al [58] in the context of holography. Boundary and corner actions at null boundaries were examined in [59, 60, 61, 62].

Much work related to null surfaces has focused on black hole horizons. In the membrane paradigm, Damour [63], Znajek [64] and others described gravitational effects at and close to black hole horizons using language from fluid mechanics and thermodynamics [65].

An important concept related to black holes is their entropy, which is widely believed to provide a conceptual probe of quantum gravity. Black hole entropy, and questions on the information content of black holes, are also related to asymptotic symmetries, as argued by Hawking, Perry and Strominger in [66]. A classical, canonical perspective on black hole entropy was given by Wald [67], and is also discussed in the framework of isolated horizons [68].

The connection between horizons and null infinity through black hole entropy has motivated research on the symmetries and conserved quantities of black holes, and null surfaces more generally, along the lines of how BMS symmetries and conservation laws are obtained, for example in [69, 70, 71, 72, 73, 74]. Symmetry groups, charges and conservation laws appear that mirror those at null infinity. Besides the application to asymptotically flat spacetimes with an inner null boundary such as a black hole horizon, by analyzing null surfaces in this way one may hope to transfer conceptual lessons from finite distance to null infinity, and vice versa.

As we have seen, at null infinity an important question is which symmetries should be regarded as physical symmetries relating inequivalent solutions, and which as as unphysical "pure gauge" redundancies. At finite distance, there is a more general approach to the similar question of identifying the degrees of freedom of a gauge theory, in a region with boundaries. Edge modes are introduced, which live on the boundary of the initial value surface. These modes, even though they can be removed by acting with an element of the local gauge group, are argued to be physical, and necessary to "glue" a region to its complement, e.g. in [75, 76, 77], see also [6]. They acquire dynamics, for example in [8, 78], and contribute to field theoretic entanglement entropy [79]. The edge modes are valued into the gauge group, and under gauge transformations, transform by group multiplication.

At null infinity, fields have been introduced that transform similarly, for example the Goldstone boson of spontaneously broken supertranslation invariance in [18]. It has been conjectured that edge modes could be related to such fields (see, e.g., [80]), but so far there is only little work on what happens to symmetries in the limit as null infinity is approached [81, 82].

1.2 Outline and Main Results

In this thesis, we will use canonical methods from the covariant Hamiltonian formalism [83, 31] (briefly introduced in chapter 2) to investigate general relativity for finite distance null surfaces, and electromagnetism and general relativity at null infinity.

Throughout, we will avoid fixing the gauge freedom as far as possible. This approach has several advantages: From a pragmatic standpoint, it leads to expressions from which various gauge-fixed expressions can be derived, and can thus unify work in different gauges and inform the decision which gauge fixing is appropriate to different situations. Since many different gauge fixings are used, such expressions that allow bridging between them may be useful. For example, at null infinity there is work in Bondi [9] gauge, in relaxations thereof [28], in Newman-Unti gauge [84] and in harmonic gauge (e.g., [85]). For finite distance null surfaces, gauge fixings have been used in which the free variables are the shear of the null generators [48], or the shift that encodes how the null directions relate to lines of constant spatial coordinates [55], and a wide array of conditions in the phase space sector containing the area of spatial cross-sections, their expansion and the acceleration of the null generators have been used for different purposes.

More conceptually, from a relativist's viewpoint, physical statements about general relativity should not depend on a particular gauge choice, so it is preferable to derive them without a choice of gauge where possible. Moreover, in the context of the "edge modes" literature, any diffeomorphism (or internal gauge transformation) that is not in the kernel of the pre-symplectic[1] form is elevated to a physical degree of freedom. In that view, in order to identify the physical phase space, the safest route is to calculate the symplectic form without any gauge fixing, and only then use diffeomorphisms in its kernel to introduce coordinate conditions. While of course many diffeomorphisms are clearly "pure gauge", exactly which ones depends on many details. It varies with otherwise equivalent formulations of the same theory [86, 87, 88], and is also subject to the ambiguities in the symplectic form.

One may worry that this insistence to leave the gauge unfixed leads to complicated, bloated derivations and expressions, but that turns out not to be the case. On the contrary, taking along all degrees of freedom allows using covariant methods, to a greater degree than working in partially gauge fixed coordinates.

However, leaving the gauge free leads to degenerate symplectic forms, which cannot readily be inverted to give Poisson brackets. Similarly, it is a step removed from identifying free initial data for evolution problems, or from considering scattering, which requires matching conditions connecting the past and future boundaries. However, as we saw, physically relevant questions often pertain to identifying which symmetry transformations are canonical transformations, what are the Hamiltonians generating them, and what are the fluxes describing the non-conservation of those Hamitonians. In the covariant Hamiltonian formalism, one can make much headway on these questions without fixing the gauge.

In a similar vein, most of the work in this thesis is for spacetimes of arbitrary dimension. This also does not introduce much additional difficulty, but brings out some structures more clearly.

We begin by summarizing some aspects of the covariant Hamiltonian formalism in section 2. A central object is the symplectic potential, which is a codimension-one form on spacetime, and may thus be integrated on hypersurfaces. In the simplest case, it is of the form $\sum_i P_i \delta Q_i$, i.e., it pairs the variation of configuration variables Q with their momenta P, and those pairs form the initial data of the Hamiltonian evolution problem.

After collecting the relevant definitions, we show how the action of symmetry transformations is connected with conservation laws. That leads to the introduction of the

[1] We will drop the prefix pre- in the following. All the symplectic potentials and forms in this thesis have degenerate directions, i.e. are pre-symplectic.

Noether charge aspect, which is a codimension-two form on spacetime, and can be interpreted as a relativistic current on a boundary. Its conservation equation, which arises from the constraints associated with the symmetry transformations, involves fluxes, which have a canonical form.

As not all our pre-symplectic potentials will be fully spacetime covariant, we introduce some technology to deal with the non-covariance. That leads to an expression relating the Hamiltonian generating an infinitesimal diffeomorphism, if it exists, to the Noether charge of the same diffeomorphism, while allowing for some non-covariance. Lastly, we collect explicit expressions for general relativity for the objects of the covariant Hamiltonian formalism, which will form the starting points of much of the remainder of this thesis.

1.2.1 Part I: Canonical Structure at Finite Distance Null Surfaces

The first part of this thesis aims to provide a complete accounting of the canonical degrees of freedom of gravity on a null surface, and how they enter the constraints.

Chapter 3 is devoted to identifying and analyzing the pre-symplectic potential on a null portion of an initial value surface, without fixing the gauge, in variables with geometric and physical meaning. We start by defining those variables: The null surface is foliated by a family of spacelike codimension-two spheres, which may be understood as the equal time surfaces of a clock. A general metric is expressed in terms of such a codimension-two foliation. A set of tensors describing the extrinsic geometry of the foliation is given, some of which will feature as momenta in the symplectic potential. The definitions utilize the two null normals to the codimension-two foliation, a choice that is well adapted to the analysis of null surfaces.

The symplectic potential is then brought into a form where the configuration variables depend only on the induced metric of the null hypersurface, and do not contain derivatives – corresponding to Dirichlet boundary conditions on the null metric. We recover that the conformal metric on the spatial cross–sections is conjugate to the shear of the null generators, a fact well-known for both finite distance null surfaces, and null infinity. The shift, which encodes the relationship of the null generators of the null surface with the lines of constant spatial coordinates, is paired with what we call the twist. The twist appears in the membrane paradigm as the linear momentum of the "horizon fluid", and integrating it against a rotation vector field on a black hole horizon gives the black hole angular momentum. Perhaps the most subtle sector is the one comprising the area element of the spatial cross-sections. It is conjugate to a linear combination of the surface gravity and expansion. In the case of a non-expanding horizon, this reduces to the relationship between area and surface gravity familiar from black hole thermodynamics. In addition to the canonical pairs in the interior of the initial value surface, the standard choice of symplectic potential contains canonical pairs on its corner.

The "Dirichlet" requirement that the configuration variables depend only on the induced metric, in the interior and at the corner of the initial value surface, requires adding boundary and corner terms to the action. We identify them from our analysis and compare with existing proposals.

Throughout chapters 3 and 4, the initial value surface is restricted to be null, and to remain so under variations of the metric. After the appearance of the published versions of these chapters, this restriction has been lifted by Aghapour, Jafari and Golshani in [89].

Chapter 4 is concerned with the gravitational constraints on a null surface, which are interpreted as canonical conservation laws, analogous to the conservation laws for gauge generators at null infinity. Attention is restricted to the equivalent of the momentum constraints, i.e., constraints smeared with vector fields tangential to the null surface.

As a first step of marrying the symplectic analysis of chapter 3 with conservation laws, the action of diffeomorphisms on the variables in the symplectic potential is summarized. That action is non-trivial because of the non-covariance introduced by foliating the null-surface into codimension-two surfaces.

An analysis of the constraints associated with infinitesimal diffeomorphisms tangential to the null hypersurface then allows identifying a current associated with each diffeomorphism. The current can be understood as a vector field on the null surface. The constraints then become conservation laws, equating the divergence of this boundary current to a flux. The flux has the same form as the energy-momentum tensor of a generic field theory, and can thus be understood as the flux of gravitational energy-momentum. The time-component of the current, integrated on a codimension-two sphere, is interpreted to describe the energy and momentum contained within the sphere. The variables used in this analysis are the same ones appearing in the symplectic potential of chapter 3.

Those conservation laws are then tied more explicitly to the symplectic analysis of chapter 3. The boundary current is identified with the Noether charge aspect, and the flux is obtained by evaluating the symplectic potential on variations of the metric induced by the infinitesimal diffeomorphism. However, to match the analysis of the constraints, the symplectic potential needs to be modified, and differs from the standard expression: In particular, the pairs on the boundary of the initial value surface need to be dropped, yielding what we call the intrinsic symplectic potential. This modification, which is within the ambiguities of the symplectic potential, also leads to expressions for the boundary current and diffeomorphism generators which are independent of the extension of the infinitesimal diffeomorphism outside the initial value surface.

Lastly, the canonical generators of infinitesimal diffeomorphisms are analyzed. For diffeomorphisms tangential to the spacelike spheres, the result is simple and expected: The canonical generator is the twist, which at the same time is the momentum conjugate to the shifts encoding spatial displacement, and the time-component of the boundary current for those vector fields. Its conservation equation, Damour's Navier-Stokes-like equation, involves the gradient of the momentum conjugate to the area element. This momentum is hence naturally interpreted as a pressure term in a fluid analogy, an interpretation compatible with its role in the boundary current for "null time" translations. Because of the corner modification of the symplectic potential, the generator for spatial diffeomorphisms does not coincide with the Komar charge in general.

For diffeomorphisms with a component parallel to the null generators, the situation is more subtle. Boundary conditions at the boundary of the initial value surface are needed if the symmetry is to be realized as a Hamiltonian symmetry: the shear needs to be zero, reminiscent of a "no-radiation" condition at null infinity. In addition, a constitutive relation needs to be provided in the sector containing the area element and the pressure, and we give the Hamiltonians arising from some such relations.

In summary, we decompose the degrees of freedom and constraints on a null surface in terms of geometrical variables, and read the constraints as canonical conservation laws, similar to those at null infinity. Since the appearance of the published version of chapters 3 and 4, a similar approach has been taken by [74], focusing also on the algebra of symmetries and Hamiltonians.

1.2.2 Part II: Canonical Structure at Null Infinity

In the second part of this thesis, we turn our attention to the asymptotic canonical structure, first of electrodynamics in five or more spacetime dimensions, then of gravity in three or more dimensions.

The central problem addressed in these chapters is that the expressions for the symplectic potential that are usually used at finite distance diverge with the radius as infinity is approached, except in low enough dimension and for strict enough boundary conditions. In the canonical theory at infinity, these divergences would lead to an ill-defined symplectic form and brackets, as well as divergent Hamiltonians and fluxes.

Resolving this issue is a necessary step on the way to developing a canonical viewpoint on symmetries on phase spaces where the boundary conditions do not take care of the divergences. As mentioned above, considering such lenient boundary conditions is necessary to accommodate subleading soft theorems as the Ward identities of asymptotic symmetries. Removing the divergences is also necessary in the context of the "edge modes" literature, where fixing gauge conditions risks removing physical degrees of freedom.

We remove the divergences for electromagnetism and general relativity by introducing a renormalization scheme, exploiting the ambiguities in the definition of the symplectic potential of a field theory. The divergences are absorbed into counterterms, which translate into boundary contributions to the action, and contributions from the corner of the initial value surface to the symplectic potential. We focus on divergences in the radius, and do not address divergences arising from infinite integration domains in other directions.

The counterterms are local, including in the radius, rather than being defined order by order. The resulting renormalized symplectic potential is thus also local, and has a finite limit onto the boundary at infinity on-shell. The scheme is viable for a large class of boundary conditions: all that is required is asymptotic simplicity, i.e., the existence of a conformal compactification in the sense of Penrose, and some regularity of the fields at infinity. No gauge fixing or additional condition on the conformal factor are required. In particular, the recipe works for any cosmological constant, and allows for some non-analyticity in the expansion of the fields around the conformal boundary.

The scheme relies on two pieces of background structure: the conformal factor, which doubles as the canonical time, and a radial vector field, which asymptotically is used to "take orders" in the radius. The background structure introduces some non-covariance, and in the gravitational case allows the resulting symplectic structure to evade the no–go result of [29], that an asymptotically finite, local, and fully spacetime-covariant symplectic structure for general relativity does not exist.

In chapter 5, the basic idea of the renormalization scheme is outlined, using electrodynamics in five or more spacetime dimensions as an example. The motivation for studying higher dimensional electrodynamics as a toy example is the similarity in the structure of asymptotic divergences of six–dimensional electrodynamics with that of four-dimensional gravity. That similarity comes from the fact that, in both cases, the dimension is two greater than the dimension in which the theory is conformal.

We start by working out the consequences of asymptotic simplicity, in general and covariant terms. That sets the stage for deriving an identity for the radial evolution of the symplectic potential on a constant-radius hypersurface, which follows directly from the implicit definition of the symplectic potential in the covariant Hamiltonian formalism. This identity is then used to iteratively remove factors of the radius in the symplectic potential to render it finite, allowing for some non-analyticity in the expansion of fields around infinity. In D spacetime dimensions, there are $D-5$ factors of the radius to be removed.

It is shown that the counterterms arising from that procedure simultaneously renormalize the radial divergence of the action, and of the symplectic potential integrated on surfaces transverse to the constant-radius surfaces. Asymptotically, assuming analyticity, the renormalized symplectic potential coincides with the finite order of a Laurent expansion in the radius of the original symplectic potential. It exhibits a layer structure: In six dimensions, there are two layers. The leading components of the gauge field are paired with the subleading components of the radial components of the field strength tensor, and the subleading components of the gauge field with the leading components of the field strength.

The expression for the renormalized symplectic potential is valid on any "compactifiable" background, but there are dependencies among its constituents, on-shell and even off-shell. To resolve the dependencies, the asymptotic equations of motion are specialized to Minkowski space, and analyzed to identify the free data, assuming analyticity. In the course of that analysis, some consequences of analyticity are identified, which indirectly impose further conditions on the fields. For example, the fact is recovered that assuming analyticity, radiative solutions exist only in even spacetime dimensions.

In six dimensions, the radiative free data are the first subleading order of the gauge potential tangential to the asymptotic spheres. In the symplectic potential, they are paired with a certain order of the magnetic field tangential to the spheres, which we dub the "Maxwell news" and which coincides with the time derivative of the free data.

In addition to the radiative data, to determine all the data entering the symplectic potential, a field on null infinity needs to be specified which encodes the gauge part of the leading order gauge potential. We call it the soft potential, and it is present because the gauge has not been fixed. In the symplectic potential, it is paired with the charge aspect, which is the radial electric field at the order corresponding to the falloff of the Coulombic field of a charged point particle in the bulk. The time evolution of the charge aspect is controlled by the free radiative data via the Gauss law. A special role in the symplectic potential is played by the zero-mode of the radiative data, which appears in both the radiative and the "Coulombic" pair.

In higher (even) dimensions, the radiative data are given by more subleading orders of the sphere components of the gauge field. While in six dimensions, the radiative data directly determines the time derivative of the charge aspect, in higher dimensions, it determines only its higher time derivatives. We give an explicit expression for that evolution.

We compute the generators of asymptotic symmetries under the renormalized symplectic form. They can be rendered independent of subleading orders of the gauge parameter by adding further local, finite counterterms to the symplectic potential. The symmetry generator contains the charge aspect.

Soft theorems in higher dimensional quantum electrodynamics were considered in [90]. We connect to those results, by showing that the flux of the renormalized generator of asymptotic symmetries coincides with an expression derived starting from the soft theorem. This is in contrast to the generator derived from the un-renormalized symplectic potential, which diverges, and thus supports the necessity of our scheme, and its compatibility with other approaches.

In chapter 6, we address the renormalization of the symplectic potential in the case of gravity in three or more dimensions, on asymptotically simple spacetimes of any cosmological constant. Extra steps are necessary compared to electromagnetism. One cause of this is that for gravity, the Lagrangian does not transform homogeneously under conformal transformations, leading to additional, higher divergences.

We start, again, by summarizing the consequences of asymptotic simplicity, and impose falloffs that solve the two most divergent orders of the equations of motion. The renormalization of the symplectic potential now proceeds in two steps: First, its two most divergent orders are absorbed into counterterms. The remainder of the symplectic potential diverges as r^{D-3}. Then, analogously to electromagnetism, an identity for the radial evolution of that remainder is found, and used to iteratively remove the remaining divergences. This does not require choosing a gauge. The result for the renormalized symplectic potential on constant radius surfaces is local, asymptotically finite on-shell, and valid in any dimension and for any cosmological constant.

A renormalized symplectic potential for a specific relaxation of the Bondi coordinate conditions has been given in [28], starting from the usual expression from the covariant Hamiltonian formalism and removing the divergences order by order. The authors also identified the need for a more geometric prescription to render the symplectic potential asymptotically finite, and we provide such a prescription here. We connect to the results of [28] by imposing their gauge conditions on our result, and find that our renormalized symplectic form essentially agrees with theirs.

Similarly to our scheme, in holographic renormalization of pure gravity on asymptotically AdS space, the on-shell action and the stress-energy tensor, which is closely related to the canonical momentum, are renormalized to be asymptotically finite. Since asymptotically AdS space is asymptotically simple, our scheme applies, raising the question what the relationship between the two prescriptions is. We specialize to Fefferman-Graham coordinates in four and five spacetime dimensions, and include a Gibbons-Hawking term in our scheme. In that case, our renormalized symplectic potential pairs the leading order metric with the renormalized stress-energy tensor of holographic renormalization, up to scheme dependent terms which are part of the ambiguity of the holographic renormalization prescription.

Finally, the “would-be” Hamiltonians which generate infinitesimal diffeomorphisms under the renormalized symplectic form are computed. That can be done efficiently by exploiting that our scheme is relatively covariant.

Chapter 2

The Covariant Hamiltonian Formalism

The pre–symplectic geometry of field space can be obtained in a covariant way (see, e.g., [83, 91, 76]), some aspects of which we briefly review here.

2.1 Pre-symplectic Form and Potential

A central object is the pre–symplectic form Ω_B, which is a closed two–form on field space and an integral over a (possibly partial) Cauchy hypersurface B in space–time. The prefix "pre" refers to the fact that Ω_B on field space has degenerate directions, so it does not qualify as "symplectic". The degenerate directions are the gauge degrees of freedom, which have to be ultimately quotiented out to obtain the physical phase space.

Schematically, Ω_B can be written as

$$\Omega_B = \int_B \delta P \curlywedge \delta Q + \int_{\partial B} \delta p \curlywedge \delta q. \tag{2.1}$$

Here, δ is the exterior derivative on field space, and $\curlywedge$ is the wedge product on field space. The pairs (Q, P) of configuration and momentum variables are the canonical pairs. We have allowed for the presence of *corner* degrees of freedom (q, p) on the codimension two boundary of B. There is not a unique way to split Ω_B into P, Q, p and q: The freedom of doing canonical transformations is left. In order to fix that freedom, i.e., fix a choice of polarization, one may require that the Qs depend only on the pullback of the fields to B, and contain no derivatives. We will do so in chapter 3.

Ω_B is the field space exterior derivative $\Omega_B = \delta\Theta_B$ of the *pre-symplectic potential* Θ_B. The symplectic potential Θ_B is the integral of the *pre-symplectic potential current* θ, which is a one–form on field space and a $(D-1)$–form on spacetime, where D is the dimension of spacetime. In the covariant Hamiltonian formalism, θ is defined implicitly through the equation

$$\delta L =: \mathrm{d}\theta - E. \tag{2.2}$$

Here d is the spacetime exterior derivative, and L is the Lagrangian density. E are the equations of motion, which may be obtained from the Euler-Lagrange equations, and they are a one–form on field space and a D–form on spacetime. By definition they do not contain derivatives of the variations of the fields, and they are uniquely determined by the Lagrangian.

(2.2) determines θ only up to the addition of a closed $(D-1)$–form on M. A criterion that is often cited to fix those ambiguities is spacetime covariance. While that criterion keeps with the spirit of the covariant Hamiltonian formalism, often the spacetime region under consideration has boundaries or even corners. That is certainly the case with initial value problems, and also at null infinity in Penrose's framework. Those boundaries, and any additional background structure, break diffeomorphism invariance, so demanding full spacetime covariance may not always be appropriate.

We will fix this ambiguity in several different ways in this thesis: In chapter 3, we will use the most common expression for θ for general relativity, which is covariant under diffeomorphisms. In chapter 4, we will use the ambiguity to make sure that there are no corner pairs in Ω_B. Finally, in chapters 5 and 6, we will use the ambiguity in such a way that the integrand of Ω_B has a finite limit onto the conformal infinity of asymptotically simple spacetimes.

Schematically the symplectic potential is of the form

$$\Theta_B = \Theta_B^{\text{bulk}} + \Theta_{\partial B} + \delta A_B + \delta A_{\partial B}, \tag{2.3}$$

where $\Theta_B^{\text{bulk}} = \int_B P\delta Q$ and $\Theta_{\partial B} = \int_{\partial B} p\delta q$. The total variation terms δA_B and $\delta a_{\partial B}$ do not contribute to the symplectic form, because $\delta\delta = 0$. These terms can be reabsorbed into a redefinition of the action $S \to S - A_B - A_{\partial B}$: Sending $L \mapsto L + \mathrm{d}a$ modifies θ as $\theta \mapsto \theta + \delta a$. The inclusion of these terms corresponds to a choice of polarization, and for general relativity is necessary if one demands that the configuration variables do not include metric derivatives.

Finally, we will assume throughout that the field space exterior derivative δ and the integral $\int_B$ commute. That means the location of the hypersurface B must be specified in a field independent way.

2.2 Noether Charge Aspect

For a diffeomorphism covariant Lagrangian density[1] L, the *diffeomorphism Noether charge aspect* Q_ξ on spacetime M is defined implicitly by:

$$I_\xi \theta - \iota_\xi L = C_\xi + \mathrm{d}Q_\xi. \tag{2.4}$$

Here I_ξ is the contraction on field space, i.e., $I_\xi \theta(\phi, \delta\phi) = \theta(\phi, \mathcal{L}_\xi \phi)$ with the spacetime Lie derivate $\mathcal{L}$, while ι is the space-time contraction.

The LHS of (2.4) is the Noether current density associated with diffeomorphism symmetry. The first Noether theorem is the statement that this current is conserved on-shell, i.e., $\mathrm{d}(I_\xi\theta - \iota_\xi L) = 0$. The RHS of (2.4) expresses this current as the sum of a bulk piece C_ξ and a boundary piece $\mathrm{d}Q_\xi$. The bulk piece C_ξ is the constraint $(D-1)$-form, and vanishes when the equations of motion are satisfied.

The Noether charge aspect Q_ξ is a $(D-2)$ form that can be integrated on codimension two spheres S, such as the boundary of a partial Cauchy hypersurface B. In general relativity, it is sometimes referred to as the gravitational "superpotential" [92]. When pulled back onto B, it may be interpreted as a current, and the pullback of (2.4) becomes its conservation law. We will do so in chapter 4.

Under the modifications $L \mapsto L + \mathrm{d}a$ with a covariant under the class of ξ considered, and $\theta \mapsto \theta + \delta a + \mathrm{d}\alpha$, the Noether charge aspect changes as

$$Q_\xi \to Q_\xi + \iota_\xi a + I_\xi \alpha. \tag{2.5}$$

Lastly, since Q_ξ is defined implicitly through (2.4), it may also be modified by the addition of a closed $(D-2)$-form. The three ambiguities encoded by a, α and the closed ambiguity in Q_ξ are known as the JKM ambiguities [93].

For metric gravity with minimally coupled matter, it is useful write the full Noether current density on the LHS of (2.4) as the sum of a matter contribution associated with

[1] Note that the Gibbons-Hawking boundary term, and existing proposals for null generalizations thereof [62, 60] are not fully covariant.

L^M and a gravitational contribution associated with the Einstein-Hilbert Lagrangian L^G. The matter contribution to the Noether current density is

$$I_\xi \theta^M - \iota_\xi L^M = T_\xi{}^a \epsilon_a - \mathrm{d}Q^M_\xi. \tag{2.6}$$

We have introduced the directed volume $(D-1)$–form

$$\epsilon_a = \iota_a \epsilon = \frac{\sqrt{|g|}}{(D-1)!}\epsilon_{aa_2\ldots a_D}\mathrm{d}x^{a_2} \wedge \ldots \wedge \mathrm{d}x^{a_D}. \tag{2.7}$$

The LHS is, by definition, the canonical matter energy-momentum tensor, and the RHS contains the gravitational matter energy-momentum tensor $T_\xi{}^a$ and possibly a total derivative $\mathrm{d}Q^M_\xi$. Q^M_ξ is the matter contribution to the gravitational Noether charge aspect. The fact that there could be a difference between the canonical energy momentum tensor and the gravitational energy momentum tensor is well known (see [94] and references therein for an elementary review). The presence of a corner contribution to the canonical energy-momentum tensor is due to the presence of a spin current, which vanishes for scalar fields, but not for non-zero spin fields such as gauge fields. For Yang-Mills with Lagrangian $L^M = \frac{1}{g^2}\mathrm{tr}(*F \wedge F)$ the diffeomorphism Noether charge aspect Q^M_ξ coincides with the gauge Noether charge aspect associated with the gauge parameter $\iota_\xi A$ and reads $Q^M_\xi = \frac{1}{g^2}\mathrm{Tr}(*F\iota_\xi A)$. It is thus natural to accompany the infinitesimal diffeomorphism ξ with a field dependent gauge transformation with parameter $-\iota_\xi A$. Under this combined transformation, the matter Noether charge aspect vanishes and the canonical and gravitational energy-momentum tensors agree. The total Noether charge aspect then only involves the gravitational fields, i.e., the metric.

2.3 Covariance

As we saw, the covariant Hamiltonian formalism admits several ambiguities, which we will fix using various criteria. Demanding that all objects are covariant under spacetime diffeomorphisms is a common criterion used to fix those ambiguities, but we will have reason to depart from it. Let us thus formalize the notion of covariance, so that we can deal with the resulting non-covariant objects cleanly.

As an example, consider a scalar field ϕ on spacetime, defined to be the 00-component g_{00} of the metric, in a fixed coordinate system. On the one hand, as a scalar field it transforms under an infinitesimal diffeomorphism as

$$\phi \mapsto \phi + \mathcal{L}_\xi \phi = \phi + \xi^a \partial_a \phi. \tag{2.8}$$

On the other hand, the metric transforms as $g_{ab} \to g_{ab} + \mathcal{L}_\xi g_{ab}$. If we transform ϕ according to its field space dependence on the metric, we are lead to the transformation

$$\phi \mapsto \phi + (\mathcal{L}_\xi g)_{00} = \phi + \xi^a \partial_a \phi - 2 g_{0a} \partial_0 \xi^a, \tag{2.9}$$

which is not the same. The discrepancy arises because there is fixed background structure that is not transformed under diffeomorphisms. Here that background structure is the vector field $\partial/\partial x^0$.

We introduce the field space Lie derivative $\mathfrak{L}_\xi$, which acts on field space scalars according to their dependence on the metric: In our previous example, $\mathfrak{L}_\xi \phi = I_\xi \delta\phi = \frac{\delta\phi}{\delta g_{ab}} \mathcal{L}_\xi g_{ab}$. The field space Lie derivative is extended to higher field space forms via the Cartan formula $\mathfrak{L}_\xi = I_\xi \delta + \delta I_\xi$, and the Leibniz rule.

From that, we may introduce the *anomaly* as the difference between the field space and spacetime Lie derivatives:

$$\Delta_\xi := \mathfrak{L}_\xi - \mathcal{L}_\xi - I_{\delta\xi}. \tag{2.10}$$

We have also allowed the vector field ξ to be itself dependent on the fields. The term $I_{\delta\xi}$ compensates the dependence on $\delta\xi$ of $\mathfrak{L}_\xi$, which arises from the contribution δI_ξ to $\mathfrak{L}_\xi$ if acting on a field space form. The anomaly Δ_ξ thus does not depend on $\delta\xi$. Note that $I_{\delta\xi}$ satisfies an anti-Leibniz rule, and does not contribute if acting on a field space scalar.

We call an object covariant if its anomaly vanishes. As we will see, the anomaly is a useful bookkeeping device to calculate transformations of non-covariant objects: For example, it commutes with the covariant derivative, which will be useful in practice. It is itself a derivation. For more details on related technology, see, e.g., [76, 95].

2.4 Hamiltonians

Let us lastly consider the relation between the Noether charge and the Hamiltonian generating the symmetry ξ. The Hamiltonian for ξ, if it exists, should satisfy

$$\{H_\xi, F\} = \mathfrak{L}_\xi F, \tag{2.11}$$

for any functional F of the fields.

The Poisson brackets are given in terms of the inverse symplectic form (which is a bivector on field space, and of course ill-defined before performing the symplectic reduction) as $\{F, G\} = \Omega_B^{-1}(\delta F, \delta Q)$. Using that F is arbitrary and contracting both sides with the symplectic form Ω_B one obtains the equivalent expression

$$\delta H_\xi = -I_\xi \Omega_B. \tag{2.12}$$

In general, there is no guarantee that $-I_\xi \Omega_B$ is an exact variation and thus that a Hamiltonian exists. Wald and Zoupas [31] have given a prescription which defines a charge even when $I_\xi \Omega_B$ is not a total variation, and gives a unique result in some circumstances, but we will not use their prescription here.

Let us calculate $-I_\xi \Omega_B$, for a symplectic potential $\theta = \theta_0 + \mathrm{d}\alpha$, where θ_0 is covariant under diffeomorphisms, but α need not be.

We have

$$-I_\xi \Omega = -\int_B I_\xi \delta\theta \tag{2.13}$$

$$= \int_B \delta(I_\xi \theta) - \mathfrak{L}_\xi \theta, \tag{2.14}$$

where we used the definition of the symplectic form on B, $\Omega = \int_B \delta\theta$, and the Cartan formula $\mathfrak{L}_\xi = I_\xi \delta + \delta I_\xi$ for the field space Lie derivative. Next, use identity (2.4) for $I_\xi \theta$, and the definition of the anomaly (2.10).

$$-I_\xi \Omega = \int_B \delta(C_\xi + \iota_\xi L + \mathrm{d}Q_\xi) - \mathcal{L}_\xi \theta - \Delta_\xi \theta - I_{\delta\xi} \theta. \tag{2.15}$$

Now use $\delta \iota_\xi L = \iota_\xi(-E + \mathrm{d}\theta) + \iota_{\delta\xi} L$, and $\mathcal{L}_\xi \theta = \iota_\xi \mathrm{d}\theta + \mathrm{d}\iota_\xi \theta$, and $I_{\delta\xi}\theta = C_{\delta\xi} + \iota_{\delta\xi} L + \mathrm{d}Q_{\delta\xi}$. Setting all constraint terms to zero, we get

$$-I_\xi \Omega \hat{=} \int_B -\Delta_\xi \theta + \int_{\partial B} \delta(Q_\xi) - Q_{\delta\xi} - \iota_\xi \theta. \tag{2.16}$$

Using our assumption that $\theta = \theta_0 + \mathrm{d}\alpha$, where θ_0 is covariant under diffeomorphisms, the anomaly of θ comes from the boundary modification: $\Delta_\xi \theta = \mathrm{d}(\Delta_\xi \alpha)$. Thus we get:

$$-I_\xi \Omega \hat{=} \int_{\partial B} \big(\delta(Q_\xi) - Q_{\delta\xi} - \iota_\xi \theta - \Delta_\xi \alpha\big). \tag{2.17}$$

We will use this expression in sections 4.4 and 6.5.

2.5 Expressions for General Relativity

Let us specialize to the case of metric general relativity with cosmological constant Λ, with the Einstein-Hilbert Lagrangian $L = \frac{1}{2}\epsilon(R - 2\Lambda)$. Here ϵ is the volume form of the metric g_{ab}. We will consider also a total Lagrangian $L^T = L + L^M$, with a minimally coupled matter Lagrangian L^M. The equations of motion are

$$E^T = \frac{1}{2}\epsilon(G^{ab} + \Lambda g^{ab} - T^{ab})\delta g_{ab}, \tag{2.18}$$

with T^{ab} the gravitational matter energy-momentum tensor (we work in units where $8\pi G = 1$).

The standard choice (see, e.g., [96]) for the gravitational symplectic potential reads

$$\theta = \frac{1}{2}(g^{bc}\delta\Gamma^a_{bc} - g^{ab}\delta\Gamma^c_{bc})\epsilon_a = \frac{1}{2}\nabla_b(\delta g^{ab} - g^{ab}\delta g)\epsilon_a. \tag{2.19}$$

$\delta g^{ab} = g^{ac}g^{bd}\delta g_{bd}$ denotes the metric variation with indices raised, not the variation of the inverse metric. $\delta g = g^{ab}\delta g_{ab}$ is its trace.

The constraints read

$$C_\xi = \xi^a({G_a}^b + \Lambda\delta^b_a - {T_a}^b)\iota_b\epsilon \hat{=} 0. \tag{2.20}$$

The Komar charge aspect [97], which is a $(D-2)$-form on spacetime M, and is related to the standard symplectic potential by (2.4), is given by

$$Q_\xi = \tfrac{1}{2} * \mathrm{d}g(\xi) = \tfrac{1}{2}\epsilon_{ab}\nabla^a\xi^b, \tag{2.21}$$

where $\epsilon_{ab} = \iota_a\iota_b\epsilon$.

Part I

Canonical Structure at Finite Distance Null Surfaces

Chapter 3

Gravity Degrees of Freedom on a Null Surface

In this section, we evaluate the standard symplectic potential (2.19), pulled back onto a null hypersurface. We use variables describing the intrinsic and extrinsic geometry of the null surface. The freedom to integrate by parts and to extract total variations from the symplectic potential is fixed by demanding that the bulk and corner configuration variables Q and q depend only on the intrinsic geometry of the null surface, and do not contain derivatives.

The remainder of this chapter is organized as follows. Section 3.1 contains the definitions of our variables, and decomposes the variation of the metric. In section 3.2 we perform our central calculation, obtaining the null canonical pairs of gravity in section 3.3. Section 3.4 contains a suggestion for a Lagrangian boundary term.

3.1 Setup

In this section, we introduce the structures and notation we will use to evaluate the symplectic potential on the null hypersurface B. The setup is taken from [98] and [3]. Previous, similar formalisms were set up e.g. in [99, 51, 100].

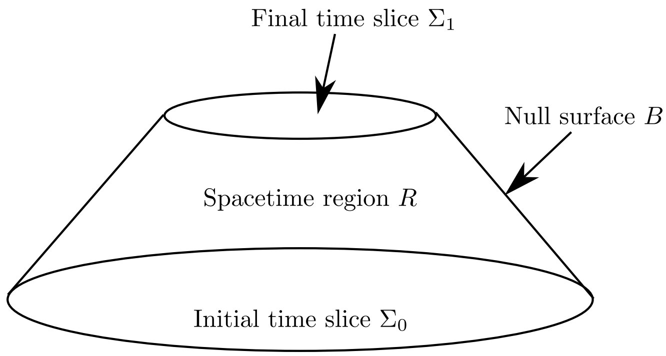

Figure 3.1: A typical situation where the symplectic structure on the null surface B is of interest is when B is part of the boundary of the spacetime region R under consideration. The other parts of the boundary are spacelike surfaces Σ_i.

3.1.1 Foliations, Normal Forms and Coordinates

Let M be the D–dimensional space–time. We are typically interested in a region R of M with boundary $B \cup \Sigma_0 \cup \Sigma_1$ where Σ_i are spacelike hypersurfaces and B a null hypersurface (see figure 3.1). More generally we want to understand the nature of the symplectic potential Θ_B on a null hypersurface B. The location of B is specified by the condition $\phi^1(x) = 0$, where ϕ^1 is a suitable scalar field on M that increases towards the past of B. B is a finite hypersurface with a boundary ∂B that we will call a "corner". It is a member of the foliation specified by $\phi^1 = \text{const}$. We do not assume that every member of the foliation is a null hypersurface, but assume that ϕ^1 is a good foliation function in a neighbourhood of B, i.e., $\mathrm{d}\phi^1 \neq 0$ on B.

We introduce another foliation given by $\phi^0 = \text{const.}$ of spacelike hypersurfaces, where ϕ^0 is a field that increases towards the future. We require that ϕ^0 is a good foliation function in a neighbourhood of B, and that nowhere $\mathrm{d}\phi^0$ is a multiple of $\mathrm{d}\phi^1$. At the intersections of the two foliations lies a two–parameter family of spacelike codimension 2 surfaces S. Coordinates $\sigma^A(x)$ are also chosen on each surface S. They are not required to be constant on the null generators of B. Doing so would be a partial gauge fixing which we want to avoid, since the direction of the null generators is metric dependent.

Using also the foliation fields as coordinates, we introduce a frame $(X^a)(x) = (\phi^i, \sigma^A)(x)$ on M. This frame represents an invertible mapping $X : U \to M$, from a domain $U \in \mathbb{R}^D$ to M. The metric G on M can be represented as a metric on U via the pullback: $X^*G = g$. Here and in the following, $a \in \{0, ..., D-1\}, i \in \{0, 1\}$ and $A \in \{2, ..., D-1\}$. x represents a choice of coordinates while $X^a(x)$ represents points of M. We will refer to X^a as the *foliation frame*. In the foliation frame, the tangent vectors e_A to the surfaces S become

$$e_A = e_A^a \partial_a, \text{ where } e_A^a = \frac{\partial x^a}{\partial \sigma^A} = \delta_A^a, \tag{3.1}$$

while the metric in the foliation frame can be parametrised as

$$\mathrm{d}s^2 = g_{ab}\mathrm{d}x^a \otimes \mathrm{d}x^b = H_{ij}\mathrm{d}\phi^i \otimes \mathrm{d}\phi^j + q_{AB}(\mathrm{d}\sigma^A - A_i^A \mathrm{d}\phi^i) \otimes (\mathrm{d}\sigma^B - A_j^B \mathrm{d}\phi^j). \tag{3.2}$$

Here we have defined the *shift connection* $A^A := A_i^A \mathrm{d}\phi^i$, which is a one–form in the normal plane to S valued into TS. We also defined the *normal metric* H_{ij}, which determines the geometry of the normal two–planes $(TS)^\perp$ to S, while q is the *tangential metric* which determines the geometry of the sphere S. The metric g contains $\frac{1}{2}D(D+1)$ parameters and this parametrization is completely general. No gauge fixing has taken place, and we have not yet specialized to the case of a null hypersurface B.

The inverse metric is

$$g^{ab}\partial_a \otimes \partial_b = H^{ij}(\partial_i + A_i^A \partial_A) \otimes (\partial_j + A_j^B \partial_B) + q^{AB}\partial_A \otimes \partial_B, \tag{3.3}$$

where H^{ij} and q^{AB} are the inverses of H_{ij} and q_{AB}, respectively. We introduce the covariant normal derivatives

$$D_i := (\partial_i + A_i^A \partial_A). \tag{3.4}$$

They can be understood as normal derivatives covariant under the gauge group Diff(S) of diffeomorphisms on S. That is because under an infinitesimal change of foliation frame $\delta_V \phi^i = 0$ and $\delta_V \sigma^A = V^A(x)$, the normal metric transforms as a scalar $\delta H_{ij} = V^C \partial_C H_{ij}$, the tangential metric transforms as a tensor $\delta_V q_{AB} = \mathcal{L}_V q_{AB}$, while A_i^A transforms as a connection:

$$\delta_V A_i^A = \partial_i V^A + [A_i, V]_S^A, \tag{3.5}$$

where $[.,.]_S$ is the Lie bracket on S. Then, the derivative D_i transforms covariantly as a scalar under the gauge group Diff(S): $\delta_V(D_i f) = V^A \partial_A(D_i f)$ for a field f on M. The curvature of the normal connection is the vector field

$$[D_0, D_1]^A = \partial_0 A_1^A - \partial_1 A_0^A + [A_0, A_1]_S^A. \tag{3.6}$$

We will use ∇_a for the covariant derivative of g, and d_A for the covariant derivative of q on S.

We introduce the logarithmic normal volume element h as

$$e^h = \sqrt{|H|}. \tag{3.7}$$

It will play an important role in the symplectic structure and the boundary action. The determinants of the normal metric H_{ij}, the induced metric q_{AB} and the full metric g_{ab} are therefore linked by

$$\sqrt{|g|} = e^h \sqrt{q}. \tag{3.8}$$

In order to write the symplectic potential using quantities intrinsic to the surfaces S we need to be able to project along its two normal directions. We therefore have to choose a basis of one–forms normal to S. There is a simple choice of basis which is metric independent, and depends only on the choice of foliation. It is given by $(\mathrm{d}\phi^0, \mathrm{d}\phi^1) \in (TS)^\perp$. However, since the surfaces S are part of a null hypersurface, the most convenient choice is to use a null co–frame $(\boldsymbol{\ell}, \bar{\boldsymbol{\ell}})$ consisting of two null forms normal to the family of surfaces S, one of which will be normal also to B. This is what we do here.

Let $\boldsymbol{\ell} = \ell_a \mathrm{d}x^a$ and $\bar{\boldsymbol{\ell}} = \bar{\ell}_a \mathrm{d}x^a$ be two smooth, null one–form fields normal to the surfaces S (here and in the following bold–face letters denote one–forms). Let $\boldsymbol{\ell}$ be such that at B, $\boldsymbol{\ell}$ is normal to B, and $g^{-1}(\boldsymbol{\ell}) = \ell^a \partial_a$ is future pointing. On the forms $(\boldsymbol{\ell}, \bar{\boldsymbol{\ell}})$, we impose the normalization condition that $g^{-1}(\boldsymbol{\ell}, \bar{\boldsymbol{\ell}}) = 1$. These conditions uniquely determine $\boldsymbol{\ell}$ and $\bar{\boldsymbol{\ell}}$ in a neighbourhood of B, up to a rescaling $(\boldsymbol{\ell} \to e^\epsilon \boldsymbol{\ell}, \bar{\boldsymbol{\ell}} \to e^{-\epsilon} \bar{\boldsymbol{\ell}})$, where ϵ is an arbitrary function. Our choice of a null dyad diagonalizes the $SO(1,1)$–symmetry of the plane normal to S, and the rescaling is the action of a $SO(1,1)$ transformation. Since $(\boldsymbol{\ell}, \bar{\boldsymbol{\ell}})$ and $(\mathrm{d}\phi^0, \mathrm{d}\phi^1)$ both form a basis of $(TS)^\perp$ their relationships can be parametrized in terms of four fields $\alpha, \bar{\alpha}, \beta$ and $\bar{\beta}$ which form a set of generalized lapses. We set

$$\begin{aligned} \boldsymbol{\ell} &= e^{\alpha}(\mathrm{d}\phi^1 - \beta \mathrm{d}\phi^0), \\ \bar{\boldsymbol{\ell}} &= \frac{e^{\bar{\alpha}}}{1+\beta\bar{\beta}}(\mathrm{d}\phi^0 + \bar{\beta}\mathrm{d}\phi^1). \end{aligned} \tag{3.9}$$

The condition that the slices $\phi^1 = \text{const.}$ are timelike or null and that the slices $\phi^0 = \text{const.}$ are spacelike is encoded in the inequalities $\beta \geq 0, \bar{\beta} > 0$. The four functions $(\alpha, \bar{\alpha}, \beta, \bar{\beta})$ determine the inverse normal metric H through the conditions $H^{ij} = g^{-1}(\mathrm{d}\phi^i, \mathrm{d}\phi^j)$. We get

$$H^{ij} = \frac{e^{-h}}{1+\beta\bar{\beta}} \begin{pmatrix} -2\bar{\beta} & 1-\beta\bar{\beta} \\ 1-\beta\bar{\beta} & 2\beta \end{pmatrix}, \qquad H_{ij} = \frac{e^{h}}{1+\beta\bar{\beta}} \begin{pmatrix} -2\beta & 1-\beta\bar{\beta} \\ 1-\beta\bar{\beta} & 2\bar{\beta} \end{pmatrix}, \tag{3.10}$$

where the normal volume element h is

$$h = \alpha + \bar{\alpha}. \tag{3.11}$$

The quantity $\alpha - \bar{\alpha}$ does not enter the metric, and encodes the rescaling freedom in $\boldsymbol{\ell}$ and $\bar{\boldsymbol{\ell}}$ alluded to above. $\alpha - \bar{\alpha}$ is therefore not physical, it is pure gauge freedom. We will refer to it as the *boost gauge*, because a boost transformation in the normal plane to S will change $\alpha - \bar{\alpha}$, keeping h and the directions of $(\boldsymbol{\ell}, \bar{\boldsymbol{\ell}})$ fixed. A boost transformation $(\ell, \bar{\ell}) \to (e^{\epsilon}\ell, e^{-\epsilon}\bar{\ell})$ acts as $(\alpha, \bar{\alpha}) \to (\alpha + \epsilon, \bar{\alpha} - \epsilon)$.

Even though it is pure gauge, we will not fix $\alpha - \bar{\alpha}$ for now. In the literature different choices are made, and the generality of our boost gauge can be a starting point to connect them. For instance, [61] and some work at null infinity work in the gauge $\alpha = 0$ while [62] works in the gauge $\bar{\alpha} = 0$. We will see that it is more convenient for the problem at hand to choose $\bar{\alpha} = 0$ such that $\alpha = h$. Note that the boost gauge can be fixed only with reference to the foliation functions ϕ^0, ϕ^1, and a boost gauge fixing thus depends on how we parametrize the family of surfaces S.

While the forms are denoted by bold letters, we denote the corresponding vectors with non-bold letters as $\ell = g^{-1}(\boldsymbol{\ell})$ and $\bar{\ell} = g^{-1}(\bar{\boldsymbol{\ell}})$. They are obtained by raising the index on $\boldsymbol{\ell}$ and $\bar{\boldsymbol{\ell}}$ and are given by

$$\ell = \ell^a \partial_a = e^{-\bar{\alpha}}(D_0 + \beta D_1), \qquad \bar{\ell} = \bar{\ell}^a \partial_a = \frac{e^{-\alpha}}{1 + \beta\bar{\beta}}(D_1 - \bar{\beta} D_0). \tag{3.12}$$

Note that the forms $(\boldsymbol{\ell}, \bar{\boldsymbol{\ell}})$ as well as the vectors $(\ell, \bar{\ell}, D_i)$ contain metric parameters and are thus metric dependent.

For notational convenience, we will mostly work with tensors that have D–dimensional indices, even if they are intrinsic to S. Vectors v^A and contravariant tensors on S are pushed forward into M along the inclusion, yielding in the foliation frame $v^a = e^a_A v^A = \delta^a_A v^A$. Covectors and covariant tensors like q_{AB} are pushed forward using the forms

$$\boldsymbol{e}^A := e^A_a \mathrm{d}x^a = (q^{AB} g_{ab} e^b_B)\mathrm{d}x^a = (\delta^A_a - A^A_a)\mathrm{d}x^a, \tag{3.13}$$

yielding, e.g., $q_{ab} = (\delta^A_a - A^A_a)(\delta^B_b - A^B_b) q_{AB}$.

Using that notation, we can write the components of the shifted derivative in foliation coordinates as $D^a_i = \delta^a_i + A^a_i$. It can be checked that $q_{ab} D^b_i = q_{ab}\ell^a = q_{ab}\bar{\ell}^a = 0$. The two vectors (D_i) span the same space as the vectors $(\ell, \bar{\ell})$, and all four are indeed orthogonal to S. It can easily be checked that the induced metric q on S satisfies the completeness relation

$$q_{ab} + \ell_a \bar{\ell}_b + \bar{\ell}_a \ell_b = g_{ab}. \tag{3.14}$$

q also satisfies $q_a{}^b q_{bc} = q_{ac}$.

We now have a variety of ways to repackage the information contained in the metric g. The basic variables are the matrices (H_{ij}, q_{AB}, A^A_i) in the parametrization (3.2). Using

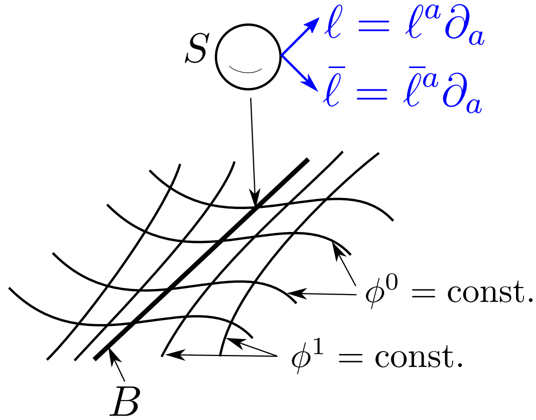

Figure 3.2: The geometry of our setup is depicted. The null hypersurface B is a member of the foliation $\phi^1 =$ const. that need not be null everywhere. It is ruled into codimension two surfaces S by a second foliation $\phi^0 =$ const. The vectors ℓ and $\bar{\ell}$ are null and normal to S. ℓ is normal also to B, and since B is null it is at the same time tangential to B. $\bar{\ell}$ is transverse to B, and the vectors are normalized as $\ell^a \bar{\ell}^b g_{ab} = 1$.

(3.10), H_{ij} can be rewritten as $(h, \beta, \bar{\beta})$, which contains the same number of independent components. After introducing the quantity $\alpha - \bar{\alpha}$, we can rewrite $(\alpha - \bar{\alpha}, H)$ as the one–forms $(\boldsymbol{\ell}, \bar{\boldsymbol{\ell}})$ using (3.9). The one–forms have two independent components each, because they are constrained to be orthogonal to S. Finally, the variables $(\boldsymbol{\ell}, \bar{\boldsymbol{\ell}}, A_i^A)$ can be rewritten as the vectors $(\ell, \bar{\ell})$ using (3.12). The vectors $(\ell, \bar{\ell})$ contain the shifts A_i^A, and determine $2D$ independent variables, one of which is $(\alpha - \bar{\alpha})$. The variables $(\ell, \bar{\ell}, q_{AB})$ thus fully determine the metric, and the quantity $(\alpha - \bar{\alpha})$. They combine covariance with an intuitive picture adapted to null structures, and we will use them in the following[1].

So far, the setup we described works for any two foliations (ϕ^0, ϕ^1) with spacelike intersections. Let us now specialize to the case that $\phi^1 = 0$ describes a null hypersurface B. For an illustration of the null geometry, see figure 3.2. The nullness condition reads $g^{ab}(\nabla_a \phi^1)(\nabla_b \phi^1) = H^{11} \overset{B}{=} 0$, and from (3.10) we see that this is equivalent to the condition $\beta \overset{B}{=} 0$. So we get

$$\beta \overset{B}{=} 0, \qquad \boldsymbol{\ell} \overset{B}{=} e^{\alpha} \mathrm{d}\phi^1, \qquad \ell \overset{B}{=} e^{-\bar{\alpha}} D_0. \tag{3.15}$$

Note that also the derivatives $\nabla_\ell \beta$ and $q_a{}^b \nabla_b \beta$ vanish on B. We see that as expected,

[1] Except for the ambiguity $\alpha - \bar{\alpha}$, the situation is analogous to the spacelike case, where the metric is parametrized in terms of induced metric, lapse and shift. These variables can be repackaged into the induced metric and the unit normal vector n.

the vector ℓ is parallel to B since on B it does not contain a transverse derivative ∂_{ϕ^1}. Its integral curves are the null generators of B. If we had chosen the coordinates σ to be constant along the null generators of B, then the shift A_0^a would vanish on B yielding $\ell^a = e^{\bar{\alpha}}\partial_{\phi^0}$. The induced metric on B is

$$\mathrm{d}s^2|_B = q_{AB}(\mathrm{d}\sigma^A - A_0^A\mathrm{d}\phi^0) \otimes (\mathrm{d}\sigma^B - A_0^B\mathrm{d}\phi^0), \tag{3.16}$$

where we have used that $h_{00} \overset{B}{=} 0$. Its parameters are q_{AB} and A_0^A, and the number of parameters is $\frac{1}{2}D(D-1) - 1$, as expected for the induced metric of a codimension 1 hypersurface that satisfies one condition.

Since the metric on B is degenerate, it does not have a preferred volume $(D-1)$-form. However, there is an covariant area $(D-2)$-form ϵ_{S}. Let ϵ be the volume form[2] on M. The area $(D-2)$-form on B is given by[3]

$$\epsilon_{\mathrm{S}} := \jmath_B^*(\iota_{\bar{\ell}}\iota_\ell\epsilon) \tag{3.17}$$

where again ι is the contraction of vectors with forms. The pullback of ϵ_{S} to any cross-section S_u of B coincides with the induced volume form $\mathrm{d}S := \sqrt{q}\mathrm{d}^{(D-2)}\sigma$ on the cross-section, i.e., we have $i_S^*\epsilon_{\mathrm{S}} = \mathrm{d}S$ with the inclusion $i_S : S \hookrightarrow B$. The area form is orthogonal to the null directions, we have $\iota_\ell\epsilon_{\mathrm{S}} = 0$. It is covariant under diffeomorphisms of B.

We also introduce a (non-covariant) volume $(D-1)$-form on B given by

$$\epsilon_{\mathrm{B}} := \mathrm{d}\phi^0 \wedge \epsilon_{\mathrm{S}} \tag{3.18}$$

It is related to ϵ_{S} as $\iota_{D_0}\epsilon_{\mathrm{B}} = \epsilon_{\mathrm{S}}$. A $(D-1)$-form $\iota_\xi\epsilon$ then pulls back to B as

$$\jmath_B^*(\iota_\xi\epsilon) = -e^{\bar{\alpha}}\ell_a\xi^a\epsilon_{\mathrm{B}}. \tag{3.19}$$

The combination $L_a = e^{\bar{\alpha}}\ell_a$ that enters here will play a special role in our construction, as we will see.

The introduction of the auxiliary foliation ϕ^0 on S should be thought of as a choice of reference frame on B. It avoids dealing with the degenerate induced metric on B and makes

[2]In our coordinates it is explicitly given by $\epsilon = e^h\sqrt{q}\mathrm{d}\phi^0 \wedge \mathrm{d}\phi^1 \wedge \mathrm{d}^{D-2}\sigma$.

[3]In our coordinates it reads

$$\epsilon_{\mathrm{S}} = \sqrt{q}\tfrac{1}{(D-2)!}\epsilon_{A_3\ldots A_D}(\mathrm{d}\sigma^{A_3} - A_0^{A_3}\mathrm{d}\phi^0) \wedge \ldots \wedge (\mathrm{d}\sigma^{A_D} - A_0^{A_D}\mathrm{d}\phi^0).$$

It is invariant under the redefinitions $\ell \to e^\epsilon\ell$ and $\bar{\ell} \to e^{-\epsilon}(\bar{\ell} + v)$ where v is tangent to B, and covariant under diffeomorphisms of B.

calculations more straightforward, but comes at the cost of introducing some additional structure into the setup: the decomposition of B into spheres S. Note, however, that we need an auxiliary foliation ϕ^0 in order to locate the position of the corner ∂B, so we cannot avoid introducing such extra data, at least near the boundary of B. This data can be interpreted as edge modes in the setting of [76].

3.1.2 Decomposition of Metric Variations

The symplectic potential contains the variation of the space–time metric, δg_{ab}. For now, we will consider a completely general metric variation, but later we will specialize to the case that the metric variations leave the hypersurface B null. We view the foliations (ϕ^0, ϕ^1) and the coordinates σ as fixed, so they do not vary: $\delta\phi^i = \delta\sigma^A = 0$. Since the position of B is described using the foliations, this also ensures that B does not move, while its geometry varies, so that integral signs and variations commute. We write $\delta g = g^{ab}\delta g_{ab}$ for the trace of the metric variation, and $\delta g^{ab} = g^{ac}g^{bd}\delta g_{cd}$ is the variation of the metric with the indices raised.

The variation of the metric will be decomposed into tensors intrinsic to S, using the structure of the two foliations. We then express it using the variations of q, ℓ and $\bar{\ell}$. Note that since the forms $(\boldsymbol{\ell}, \bar{\boldsymbol{\ell}})$ are linear combinations of the $\mathrm{d}\phi^i$ which do not vary, their variations stays orthogonal to the surfaces S, i.e., $q^{ab}\delta\ell_b = 0$ (and similarly for $\bar{\boldsymbol{\ell}}$). The relationships among ℓ and $\bar{\ell}$, which are implemented by the definition of the metric dependent coefficients $(\alpha, \bar{\alpha}, \beta, \bar{\beta})$, are also preserved under variations: We have $\delta(\ell_a\ell^a)\delta(\bar{\ell}_a\bar{\ell}^a) = \delta(\ell^a\bar{\ell}_a) = 0$.

Our first variation quantity,

$$\delta q_{ab} := q_a{}^c q_b{}^d \delta g_{ab} = \delta q_{AB} e_a^A e_b^B \tag{3.20}$$

is the variation of the induced metric, pushed forward into M. Its trace $\delta q := q^{ab}\delta q_{ab} = q^{AB}\delta q_{AB}$ is related to the change of the area element on S as $\delta\sqrt{q} = \frac{1}{2}\sqrt{q}\delta q$. Note that $\delta q_{ab} \neq \delta(q_{ab})$, because the latter expression contains the variation of the tensors e_a^A.

The vector ℓ is null and normal to S by definition, but both of these properties are metric dependent. When the metric varies, ℓ will therefore change to restore the properties. The change in ℓ parallel to S is $q^a{}_b\delta\ell^b$. It can be written as

$$q^a{}_b\delta\ell^b = -q^{ab}\ell^c\delta g_{bc} = e^{-\bar{\alpha}}(\delta A_0^a + \beta\delta A_1^a) \stackrel{B}{=} e^{-\bar{\alpha}}\delta A_0^a, \tag{3.21}$$

For the first identity, we have used $\ell^a \delta g_{ab} = \delta \ell_b - g_{ab}\delta\ell^a$, and that the variation $q_a{}^b \delta \ell_b = 0$, since $\boldsymbol{\ell}$ is fixed to be normal to S. For the second identity, we varied the expression (3.12), and used that $\delta D_i^a = \delta A_i^a$ and that $D_i^a q_{ab} = 0$. Similarly, we get

$$q^a{}_b \delta \bar{\ell}^b = -q^{ab}\bar{\ell}^c \delta g_{bc} = \frac{e^{-\alpha}}{1+\beta\bar{\beta}}(\delta A_1^a - \bar{\beta}\delta A_0^a). \tag{3.22}$$

The change of the normal volume element e^h is given by $\ell^a \bar{\ell}^b \delta g_{ab}$:

$$\ell^a \bar{\ell}^b \delta g_{ab} = -\bar{\ell}_a \delta \ell^a - \ell_a \delta \bar{\ell}^a = \delta(\alpha + \bar{\alpha}) = \delta h. \tag{3.23}$$

The second equality can be checked explicitly using the expressions (3.9) and varying them. Remembering that $\sqrt{|g|} = \sqrt{q} e^h$, and noting that $\delta \sqrt{|g|} = \frac{1}{2}\sqrt{|g|}\delta g$, we get

$$\delta g = g^{ab}\delta g_{ab} = \delta q - 2(\bar{\ell}_a \delta \ell^a + \ell_a \delta \bar{\ell}^a). \tag{3.24}$$

The part of the change in ℓ that lies in the normal plane to S and is not parallel to ℓ is given by $\ell_a \delta \ell^a$. We obtain

$$\ell_a \delta \ell^a = -\frac{1}{2}\ell^a \ell^b \delta g_{ab} = e^{\alpha - \bar{\alpha}}\delta\beta, \tag{3.25}$$

so on B, $\ell_a \delta \ell^a$ encodes how much B changes away from being null. We will later fix this quantity to zero on B. Similarly, we get

$$\bar{\ell}_a \delta \bar{\ell}^a = -\frac{1}{2}\bar{\ell}^a \bar{\ell}^b \delta g_{ab} = -\frac{e^{\bar{\alpha}-\alpha}}{(1+\beta\bar{\beta})^2}\delta\bar{\beta}. \tag{3.26}$$

In (3.20) through (3.26), we have listed all possible projections of δg_{ab} with $\ell, \bar{\ell}$ and q, and expressed them using the variations $\delta\ell^a, \delta\bar{\ell}^a$ and δq_{AB}. We have also given them as variations of the parameters $(\alpha, \bar{\alpha}, \beta, \bar{\beta}, q_{AB}, A_i^A)$. The set of variations

$$(\delta q_{ab}, q^a{}_b \delta \ell^b, q^a{}_b \delta \bar{\ell}^b, \bar{\ell}_a \delta \ell^a, \ell_a \delta \bar{\ell}^a, \ell_a \delta \ell^a, \bar{\ell}_a \delta \bar{\ell}^a) \tag{3.27}$$

are independent, as can be seen from their expressions in metric parameters. Using the completeness relation (3.14) the metric variation can be expressed fully in terms of the variations we have given, as

$$\begin{aligned}\delta g_{ab} = \delta q_{ab} &- (\ell_a q_{bc}\delta\bar{\ell}^c + \ell_b q_{ac}\delta\bar{\ell}^c) - (\bar{\ell}_a q_{bc}\delta\ell^c + \bar{\ell}_b q_{ac}\delta\ell^c) - (\ell_a \bar{\ell}_b + \bar{\ell}_a \ell_b)(\bar{\ell}_c \delta\ell^c + \ell_c \delta\bar{\ell}^c) \\ &- 2\ell_a \ell_b (\bar{\ell}_c \delta \bar{\ell}^c) - 2\bar{\ell}_a \bar{\ell}_b (\ell_c \delta \ell^c).\end{aligned} \tag{3.28}$$

The change of normalization of ℓ is $\bar{\ell}_a \delta \ell^a$, and the change of the normalization of $\bar{\ell}$ is $\ell_a \delta \bar{\ell}^a$. They enter the metric variation only through the symmetric combination $\bar{\ell}_a \delta \ell^a + \ell_a \delta \bar{\ell}^a$. This is the variational expression of the fact that the boost gauge of ℓ and $\bar{\ell}$ is indeed pure gauge.

3.1.3 Extrinsic Geometry

The momenta conjugate to the metric are the extrinsic geometry of S. As it was the case with our variations, all of the extrinsic geometry is expressed in tensors intrinsic to S, which we push forward onto M. We will not give a complete list here, but just define the ones that will appear in our calculations.

The *extrinsic curvature* associated with ℓ is

$$\theta_\ell^{ab} := q^{ac}q^{bd}\nabla_c \ell_d = \frac{1}{2}q^{ac}q^{bd}\pounds_\ell q_{cd} \tag{3.29}$$

It describes how the induced two–metric changes along the vectors ℓ and is symmetric. Its trace $\theta_\ell = q_{ab}\theta_\ell^{ab} = q^{ab}\nabla_a \ell_b$ is the *expansion*. It measures how the area element $\sqrt{q}$ on S changes along ℓ, corrected for the divergence of the coordinate lines $\sigma =$ const. relative to ℓ and the normalization of ℓ, and can be written as

$$\sqrt{q}e^{\bar{\alpha}}\theta_\ell = \partial_a(\sqrt{q}D_0^a) - \beta\partial_a(\sqrt{q}D_1^a) \overset{B}{=} \partial_a(\sqrt{q}D_0^a). \tag{3.30}$$

(see appendix A.1). If the shift A_0^A is set zero, and the boost gauge $\bar{\alpha} = 0$ is chosen such that $\ell \overset{B}{=} \partial/\partial_{\phi^0}$, the last expression reduces on B to the usual $\sqrt{q}\theta = \partial_\ell \sqrt{q}$. The barred expansion is analogously defined as $\bar{\theta}_{\bar{\ell}} = q^{ab}\nabla_a \bar{\ell}_b$.

The *tangential acceleration* a_a is defined as

$$a_a := q_a{}^b \nabla_\ell \ell_b. \tag{3.31}$$

It vanishes on B. That can be seen by writing $a_a = q_a{}^c \ell^b (\mathrm{d}\boldsymbol{\ell})_{bc}$, and using that $(\mathrm{d}\boldsymbol{\ell})_{bc}$ is orthogonal to S in both indices, or by explicitly evaluating $a_a = e^{-\bar{\alpha}} q_a{}^b \nabla_b \beta \overset{B}{=} 0$. Since $a_a \overset{B}{=} 0$ and also $\ell^a \nabla_\ell \ell_a = 0$ because ℓ has constant modulus, we obtain that $\nabla_\ell \ell_a$ on B must be parallel to ℓ_a: ℓ is geodesic. The proportionality factor is the *normal acceleration*

$$\gamma := \bar{\ell}^a \nabla_\ell \ell_a. \tag{3.32}$$

Although $\bar{\ell}$ is in general not geodesic, we introduce the "barred" normal acceleration

$$\bar{\gamma} := \ell^a \nabla_{\bar{\ell}} \bar{\ell}_a. \tag{3.33}$$

Lastly, we introduce the *twists* η_a and $\bar{\eta}_a$, and the *normal connection* ω_a.

$$\begin{aligned} \eta_a &:= -\, q_a{}^b \nabla_{\bar{\ell}} \ell_b \\ \bar{\eta}_a &:= -\, q_a{}^b \nabla_\ell \bar{\ell}_b \\ \omega_a &:= q_a{}^b \bar{\ell}^c \nabla_b \ell_c \end{aligned} \tag{3.34}$$

The combination $\eta - \bar{\eta}$ which computes the commutator $q_{ab}[\ell, \bar{\ell}]^b$ is essentially the curvature of the Diff(S)–connection:

$$\eta^a - \bar{\eta}^a = e^{-h}[D_0, D_1]^a. \tag{3.35}$$

This completes our geometrical setup. We will have more to say about some of these variables in section 3.3.

Let us note that under the boost transformations $(\ell, \bar{\ell}) \to (e^{\epsilon}\ell, e^{-\epsilon}\bar{\ell})$, the tensors $(\eta, \bar{\eta}, \theta_\ell, \bar{\theta}_{\bar{\ell}})$ transform covariantly to become $(\eta, \bar{\eta}, e^{\epsilon}\theta_\ell, e^{-\epsilon}\bar{\theta}_{\bar{\ell}})$, while $(\gamma, \bar{\gamma}, \omega_a)$ transform inhomogeneously as connections and become $(e^{\epsilon}(\gamma + \nabla_\ell \epsilon), e^{-\epsilon}(\bar{\gamma} - \nabla_{\bar{\ell}}\epsilon), (\omega_a + q_a{}^b \nabla_b \epsilon))$. We now turn to our main task of evaluating the symplectic potential on a null hypersurface.

3.2 The Symplectic Potential on a Null Hypersurface

The symplectic potential current integrated on B is

$$\Theta_B = -\int_B \Big(e^{\bar{\alpha}} \Theta^a \ell_a \Big) \epsilon_{\rm B} = \frac{1}{2} \int_B \Big(e^{\bar{\alpha}} \big(\nabla_\ell \delta g - \ell_a \nabla_b \delta g^{ab} \big) \Big) \epsilon_{\rm B}, \tag{3.36}$$

where we have used the expression (2.19) for the symplectic potential current and our expression (3.19) for the pullback of a contraction with the volume form. Let us first evaluate

$$-\Theta^a \ell_a = \frac{1}{2} \big(\nabla_\ell \delta g - \ell_a \nabla_b \delta g^{ab} \big), \tag{3.37}$$

using the decomposition of variations and the extrinsic geometry introduced in section 3.1.

3.2.1 Evaluation of $\Theta^a \ell_a$

The second term $-\frac{1}{2}\ell_a \nabla_b \delta g^{ab}$ of the last equation requires some work. We integrate it by parts, and using that $\delta g_{ab}\ell^b = \delta \ell_a - g_{ab}\delta \ell^b$ obtain:

$$-\frac{1}{2}\ell_a \nabla_b \delta g^{ab} = \frac{1}{2}\big(\delta g_{ab} \nabla^a \ell^b + \nabla_a (\delta \ell^a - g^{ab}\delta \ell_b) \big). \tag{3.38}$$

Let us consider the last term of the last equation, and insert the completeness relation (3.14) inside the derivative.

$$\begin{aligned}\frac{1}{2}\nabla_a(\delta\ell^a - g^{ab}\delta\ell_b) &= \frac{1}{2}\nabla_a\big(q^a{}_b\delta\ell^b + \ell^a(\bar\ell_b\delta\ell^b + \ell_b\delta\bar\ell^b) + 2\bar\ell^a(\ell_b\delta\ell^b)\big)\\ &= \frac{1}{2}\Big(\nabla_a(q^a{}_b\delta\ell^b) + (\bar\ell_b\delta\ell^b + \ell_b\delta\bar\ell^b)(\theta_\ell + \gamma) + 2(\ell_b\delta\ell^b)(\bar\theta_{\bar\ell} + \bar\gamma)\\ &\qquad + \nabla_\ell(\bar\ell_b\delta\ell^b + \ell_b\delta\bar\ell^b) + 2\nabla_{\bar\ell}(\ell_b\delta\ell^b)\Big).\end{aligned} \tag{3.39}$$

In the first line we have used that the variations stays orthogonal to the surfaces S, i.e. $q^{ab}\delta\ell_b = 0$, for the second line, we have used $\nabla_a\ell^a = \theta_\ell + \gamma$ and $\nabla_a\bar\ell^a = \bar\theta_{\bar\ell} + \bar\gamma$.

The first term in (3.38) is $\delta g_{ab}\nabla^a\ell^b$. It is already of the form $P\delta Q$. To evaluate it, we insert the decomposition of the metric twice. Comparing with the projected variations and the definitions of extrinsic geometry from section 3.1, it becomes the sum of six terms which are not identically zero:

$$\begin{aligned}\delta g_{ab}q^{ac}q^{bd}\nabla_c\ell_d &= \delta q_{ab}\theta_\ell^{ab}\\ \delta g_{ab}q^{ac}\ell^b\bar\ell^d\nabla_c\ell_d &= -\,q^a{}_b\delta\ell^b\omega_a\\ \delta g_{ab}\ell^a\bar\ell^c q^{bd}\nabla_c\ell_d &= q^a{}_b\delta\ell^b\eta_a\\ \delta g_{ab}\ell^a\bar\ell^c\ell^b\bar\ell^d\nabla_c\ell_d &= 2\ell_a\delta\ell^a\bar\gamma\\ \delta g_{ab}\bar\ell^a\ell^c q^{bd}\nabla_c\ell_d &= -\,q^a{}_b\delta\bar\ell^b a_a\\ \delta g_{ab}\bar\ell^a\ell^c\ell^b\bar\ell^d\nabla_c\ell_d &= -\,(\bar\ell_a\delta\ell^a + \ell_a\delta\bar\ell^a)\gamma.\end{aligned} \tag{3.40}$$

We have used that $q^{ab}\delta\ell_b = 0$ and that the remaining terms are zero because $\ell^a\nabla_b\ell_a = 0$. Adding this up yields

$$\frac{1}{2}\delta g_{ab}\nabla^a\ell^b = \frac{1}{2}\big(\delta q_{ab}\theta_\ell^{ab} + \delta\ell^a(\eta_a - \omega_a) - \delta\bar\ell^a a_a - (\bar\ell_a\delta\ell^a + \ell_a\delta\bar\ell^a)\gamma - 2\ell_a\delta\ell^a\bar\gamma\big). \tag{3.41}$$

We have dropped some projectors $q_a{}^b$ where they are unnecessary. Now all that is left to evaluate is the term $\frac{1}{2}\nabla_\ell\delta g$ in (3.37). Using (3.24), it becomes just

$$\frac{1}{2}\nabla_\ell\delta g = \frac{1}{2}\nabla_\ell\big(\delta q - 2(\bar\ell_a\delta\ell^a + \ell_a\delta\bar\ell^a)\big). \tag{3.42}$$

We add everything up to obtain

$$\begin{aligned}-\Theta^a\ell_a = \frac{1}{2}\Big(&\delta q_{ab}\theta_\ell^{ab} + \delta\ell^a(\eta_a - \omega_a) + \theta_\ell(\bar\ell_a\delta\ell^a + \ell_a\delta\bar\ell^a)\\ &+ \nabla_\ell\big(\delta q - (\bar\ell_a\delta\ell^a + \ell_a\delta\bar\ell^a)\big) + \nabla_a(q^a{}_b\delta\ell^b) + 2\nabla_{\bar\ell}(\ell_a\delta\ell^a)\\ &+ 2(\ell_a\delta\ell^a)(\bar\theta_{\bar\ell} + 2\bar\gamma) - \delta\bar\ell^a a_a\Big)\end{aligned} \tag{3.43}$$

Remembering that $\Theta_B = -\int_B (e^{\bar{\alpha}}\Theta^a \ell_a)\epsilon_{\rm B}$, this is our first version of the symplectic potential on a null hypersurface. It is obtained by integrating by parts, inserting the completeness relation, and substituting the variations and extrinsic geometry we defined, and was first derived in [3] We have not yet assumed $\delta\beta \stackrel{B}{=} 0$ or $\beta \stackrel{B}{=} 0$.

In this form, the result is not suited yet to read off the canonical pairs of gravity, since it still contains derivatives of variations. It is not of the form $P\delta Q$ with the configuration variable Q depending only on the geometry of B.

3.2.2 Splitting the Symplectic Potential into Bulk, Boundary and Total Variation

In the following, we will restrict attention to metrics for which B is null and metric variations that keep the hypersurface B null, i.e., we set

$$\beta \stackrel{B}{=} 0, \qquad \delta\beta \stackrel{B}{=} 0. \tag{3.44}$$

This nullness condition means that we restrict attention to the submanifold of field space on which B is null, and consider only metric variations which are tangential to this submanifold. In other words, what is calculated in the present section is the pullback of the pre–symplectic potential Θ_B to the submanifold of field space where B is null. If B is part of the boundary of spacetime, fixing β and its variation is a partial boundary condition. To avoid removing any degree of freedom by fixing β, one might "turn on" the frame field ϕ_1, hence making the location of B dynamical. We will not consider that here.

Using those conditions, the expressions a_a and $\ell^a\delta\ell_a$ vanish on B (but not the transverse derivative $\nabla_{\bar{\ell}}(\ell^a\delta\ell_a)$ of the latter expression). These conditions therefore lead to a simpler expression for the symplectic potential:

$$\begin{aligned}\Theta_B = \frac{1}{2}\int_B e^{\bar{\alpha}}\Big(&\delta q_{ab}\theta_\ell^{ab} + \delta\ell^a(\eta_a - \omega_a) - \theta_\ell\delta h \\ &+ \nabla_\ell\big(\delta q + \delta h\big) + \nabla_a(q^a{}_b\delta\ell^b) + 2\nabla_{\bar{\ell}}(\ell_a\delta\ell^a)\Big)\epsilon_{\rm B}.\end{aligned} \tag{3.45}$$

We have written δh for $(-\bar{\ell}^a\delta\ell_a - \ell^a\delta\bar{\ell}_a)$.

This expression is not fully satisfactory yet for two reasons: It is not manifestly boost gauge invariant, and it still contains derivatives of variations. From (3.36) we see that the integrand of Θ_B is boost invariant because the combination $e^{\bar{\alpha}}\ell_a$ is, but in the equation above the invariance is not insured term by term. In order to achieve this it is worthwhile to notice that the combination $e^{\bar{\alpha}}\ell$ enters the symplectic potential in many instances. We therefore introduce the boost invariant (but non-covariant) combination

$$L^a := e^{\bar{\alpha}}\ell^a, \qquad L^a\partial_a = D_0 + \beta D_1 \overset{B}{=} D_0, \qquad \delta L^a \overset{B}{=} q^a{}_b\delta L^b = \delta A^a_0. \tag{3.46}$$

We also denote its extrinsic curvature θ^{ab}_L simply by θ^{ab}, which is equal to $\theta^{ab} = e^{\bar{\alpha}}\theta^{ab}_\ell$. Now using the identity $\bar{\eta}_a - \omega_a = q_a{}^b\nabla_b\bar{\alpha}$, we can evaluate

$$e^{\bar{\alpha}}\nabla_a(q^a{}_b\delta\ell^b) = \nabla_a\delta L^a - (\bar{\eta}_a - \omega_a)\delta L^a. \tag{3.47}$$

We can also use that $e^{\bar{\alpha}}\nabla_{\bar{\ell}}(\ell_a\delta\ell^a) \overset{B}{=} \nabla_{\bar{\ell}}(e^{\bar{\alpha}}\ell_a\delta\ell^a)$ since $\ell_a\delta\ell^a \overset{B}{=} 0$. The symplectic potential can then be written as

$$\begin{aligned}\Theta_B = \frac{1}{2}\int_B \Big(&\delta q_{ab}\theta^{ab} + \delta L^a(\eta_a - \bar{\eta}_a) - \theta\delta h \\ &+ \nabla_L\big(\delta q + \delta h\big) + \nabla_a\delta L^a + 2\nabla_{\bar{\ell}}(e^{\bar{\alpha}}\ell_a\delta\ell^a)\Big)\epsilon_{\mathrm{B}}.\end{aligned} \tag{3.48}$$

In this form all the terms are now individually boost invariant. For the last term this is due to the fact that $\ell^a\delta\ell_a \overset{B}{=} 0$. We have also discovered that the most convenient boost gauge for the symplectic structure is $\bar{\alpha} = 0$, since it identifies $\ell = L$. Note that the induced metric (3.16) on B is determined by (q_{AB}, L^a).

The last term $\nabla_{\bar{\ell}}(e^{\bar{\alpha}}\ell_a\delta\ell^a)$ is still problematic though. Indeed even if $\ell_a\delta\ell^a$ vanishes on B, its derivative $\nabla_{\bar{\ell}}(\ell_a\delta\ell^a)$ does not, since the derivative is in a direction transverse to B. The challenge we are facing is to find a way to eliminate this transverse derivative. In the case where the boundary is spacelike a similar issue arises, and there it is possible to eliminate the transverse derivative by including it into the variation of the densitized extrinsic curvature, which leads to the Gibbons–Hawking term. This is therefore exactly the strategy we are now going to follow: We show that it is possible to absorb the transverse derivative $\nabla_{\bar{\ell}}(e^{\bar{\alpha}}(\ell_a\delta\ell^a))$ into a total variation.

Using that $\delta\beta \overset{B}{=} 0$, we can evaluate that $\nabla_{\bar{\ell}}[e^{\bar{\alpha}}(\ell_a\delta\ell^a)] \overset{B}{=} D_1(\delta\beta)$. Note that (even outside of B) the normal acceleration can be written as $\gamma{=}e^{-h}[D_0e^{\alpha} + D_1(e^{\alpha}\beta)]$. This suggests that we can extract from its variation the transverse derivative up to tangential derivatives. Before doing so, one has to remember that the normal acceleration transforms as a connection under boosts, while we want to keep boost invariance manifest. Under the rescaling $(\ell, \bar{\ell}) \to (e^{\epsilon}\ell, e^{-\epsilon}\bar{\ell})$, γ transforms as

$$\gamma \to e^{\epsilon}(\gamma + \nabla_\ell\epsilon). \tag{3.49}$$

This suggests introducing the *surface gravity* which is the boost invariant combination

$$\kappa := e^{\bar{\alpha}}(\gamma + \nabla_\ell \bar{\alpha}). \tag{3.50}$$

It is boost invariant, since the transformation of $\bar{\alpha}$ and $\nabla_\ell \bar{\alpha}$ cancels the non–invariant terms in γ. It corresponds to the normal acceleration $\kappa = \bar{L}_a \nabla_L L^a$ of the vector $L{=}D_0 + \beta D_1$. Using metric parameters, the surface gravity κ can be written as

$$\kappa \stackrel{B}{=} D_0 h + D_1 \beta, \tag{3.51}$$

and is manifestly boost gauge invariant (see appendix A.1).

In appendix A.2, we calculate the total variation of the surface gravity on B for variations that preserve the nullness of B, i.e., such that $\delta\beta \stackrel{B}{=} 0$. It is given by

$$\delta\kappa \stackrel{B}{=} \delta L^a(\eta_a + \bar{\eta}_a) + \nabla_L \delta h + \nabla_{\bar{\ell}}(e^{\bar{\alpha}} \ell_a \delta \ell^a). \tag{3.52}$$

By using these results we can now write down the symplectic potential in a form intrinsic to B which does not involve any transverse derivatives. It reads

$$\begin{aligned} \Theta_B = \frac{1}{2} \int_B \Big(& \delta q_{ab} \theta^{ab} - \delta L^a(\eta_a + 3\bar{\eta}_a) - \theta \delta h \\ & + \nabla_L \big(\delta q - \delta h\big) + \nabla_a \delta L^a + 2\delta\kappa \Big) \epsilon_{\rm B}. \end{aligned} \tag{3.53}$$

In order to finalize the expression we first need to integrate by parts the derivative along L, producing a total derivative. We use that for any ρ,

$$\sqrt{q} \nabla_L \rho \stackrel{B}{=} \partial_a \left[\sqrt{q} D_0^a \rho\right] - \sqrt{q}\, \theta \rho. \tag{3.54}$$

where we used that $L^a \stackrel{B}{=} D_0^a$ and that $\partial_a(\sqrt{q} D_0^a) = \sqrt{q}\theta$. We can also express divergences of vectors tangential to S as

$$\sqrt{q} \nabla_a \delta L^a = \partial_a(\sqrt{q} \delta L^a) + \delta L^a(\eta_a + \bar{\eta}_a). \tag{3.55}$$

These identities are proven in appendix A.3. We also need to convert the last term into a total variation, using that $\delta(2\kappa\epsilon_{\rm B}) = (2\delta\kappa + \kappa\delta q)\epsilon_{\rm B}$. This gives us

$$\begin{aligned} \Theta_B = & \frac{1}{2} \int_B \Big(\delta q_{ab} \theta^{ab} - 2\delta L^a \bar{\eta}_a - (\kappa + \theta)\delta q\Big) \epsilon_{\rm B} \\ & + \frac{1}{2} \int_{\partial B} \Big(L^a \big(\delta q - \delta h\big) + \delta L^a \Big) \iota_a \epsilon_{\rm B} + \delta\Big(\int_B \kappa\, \epsilon_{\rm B} \Big). \end{aligned} \tag{3.56}$$

Here the directed volume element $\iota_a\epsilon_{\rm B}$ on ∂B appears. Recall that in the foliation frame we have $\iota_L\epsilon_{\rm B} = \epsilon_{\rm S}$.

This expression is the sum of three terms, a bulk symplectic potential, a boundary symplectic potential and a total variation. The variational terms in the bulk symplectic potential only involve δq_{ab} and δL^a, which form the intrinsic geometry of B. In particular we see that the term involving the variation δh has cancelled from the bulk part. This term is still present in the boundary contribution of the symplectic potential. In order to remove it we introduce another total variation

$$-\delta h\, L^a\iota_a\epsilon_{\rm B} \;\;=\;\; -\delta\big(hL^a\iota_a\epsilon_{\rm B}\big) + \big[\delta L^a h + \tfrac{1}{2}h\delta q L^a\big]\,\iota_a\epsilon_{\rm B}, \tag{3.57}$$

where we have used $\delta\epsilon_{\rm B} = \frac{1}{2}\delta q\epsilon_{\rm B}$. We get

$$\begin{aligned}\Theta_B = \frac{1}{2}\int_B &\Big(\delta q_{ab}\theta^{ab} - 2\delta L^a\bar{\eta}_a - (\kappa+\theta)\delta q\Big)\epsilon_{\rm B}\\ &+ \frac{1}{2}\int_{\partial B}\Big(\big(1+\tfrac{1}{2}h\big)L^a\,\delta q + \big(1+h\big)\delta L^a\Big)\iota_a\epsilon_{\rm B}\\ &+ \delta\Big(\int_B \kappa\,\epsilon_{\rm B} - \frac{1}{2}\int_{\partial B} h\epsilon_{\rm S}\Big).\end{aligned} \tag{3.58}$$

The boundary part and the total variation part of the symplectic potential can be written in a variety of different ways, all keeping with our principle that Θ_B should just contain variations of induced geometry and total variations. First using $\sqrt{q}\theta = \partial_a(\sqrt{q}D_0^a)$ and $D_0 \stackrel{B}{=} L$, it is important to note that the expansion θ is a boundary term on B:

$$\int_B \theta\epsilon_{\rm B} = \int_{\partial B} L^a\iota_a\epsilon_{\rm B} = \int_{\partial B}\epsilon_{\rm S}. \tag{3.59}$$

The variation of the last equation becomes

$$\delta\Big(\int_B \theta\epsilon_{\rm B}\Big) = \delta\Big(\int_{\partial B}\epsilon_{\rm S}\Big) = \int_{\partial B}(\delta L^a + \tfrac{1}{2}\delta q L^a)\iota_a\epsilon_{\rm B}. \tag{3.60}$$

We can thus rewrite the last expression as

$$\begin{aligned}\Theta_B = \frac{1}{2}\int_B &\Big(\delta q_{ab}\theta^{ab} - 2\delta L^a\bar{\eta}_a - (\kappa+\theta)\delta q\Big)\epsilon_{\rm B} + \frac{1}{2}\int_{\partial B}\big(\tfrac{1}{2}hL^a\delta q + (h-1)\delta L^a\big)\iota_a\epsilon_{\rm B}\\ &+ \delta\Big(\int_B(\theta+\kappa)\epsilon_{\rm B} - \frac{1}{2}\int_{\partial B} h\epsilon_{\rm S}\Big).\end{aligned} \tag{3.61}$$

Also noting that $\nabla_a L^a = \theta + \kappa$, we see that in this form, the bulk total variation $A_B = \int_B (\theta + \kappa)\epsilon_{\rm B}$ is a close null analog of the Gibbons–Hawking term which features the divergence $K = \nabla_a n^a$ of the unit normal to the hypersurface. A boundary action of this form is given in [60].

Note that the momenta for the trace $\delta q = \delta q_{ab} q^{ab}$ and the traceless part $\delta q_{\langle ab \rangle}$ of the variation of the induced metric on S have different forms. It is therefore natural to split the induced metric into a conformal part and the volume element. We parametrize the sphere metric q_{AB} in terms of a conformal factor φ and a conformal metric γ of unit determinant:

$$q_{ab} = e^{2\varphi}\gamma_{ab}, \qquad \det(\gamma) = 1. \tag{3.62}$$

The conformal factor determines the luminosity distance $R = e^{\varphi}$.

The variation of the conformal metric $\delta\gamma_{ab} = e^{-2\varphi}(\delta q_{ab} - 2\delta\varphi q_{ab})$ is traceless. Its momentum is the *conformal shear*

$$\tilde{\sigma}^{ab} = e^{2\varphi}(\theta^{ab} - \tfrac{1}{D-2} q^{ab}\theta) \tag{3.63}$$

which is also traceless, and captures the change of the conformal inverse metric $\gamma^{ab} = e^{2\varphi} q^{ab}$ along L. Splitting the term $\delta q_{ab}\theta^{ab}$ into its trace and traceless parts then yields

$$\delta q_{ab}\theta^{ab} = \delta\gamma_{ab}\tilde{\sigma}^{ab} + \tfrac{1}{D-2}\delta q\, \theta. \tag{3.64}$$

Lastly, we will substitute $\delta q = 2(D-2)\delta\varphi$ to produce an exact variation. These replacements give the symplectic potential as the sum of a bulk term, a boundary term, and the variations of a boundary action and a corner action:

$$\Theta_B = \Theta_B^{\rm bulk} + \Theta_{\partial B} + \delta A_B + \delta a_{\partial B}, \tag{3.65}$$

where

$$\boxed{\begin{aligned} \Theta_B^{\rm bulk} &= \int_B \left(\frac{1}{2}\delta\gamma_{ab}\tilde{\sigma}^{ab} - \left(\tfrac{D-3}{D-2}\theta + \kappa\right)\cdot (D-2)\delta\varphi - \delta L^a \bar{\eta}_a\right)\epsilon_{\rm B}, \\ \Theta_{\partial B} &= \frac{1}{2}\int_{\partial B} \Big(\big(1+h\big)L^a \cdot (D-2)\delta\varphi + h\delta L^a\Big)\iota_a \epsilon_{\rm B} \\ A_B &= \int_B \kappa\, \epsilon_{\rm B} \\ a_{\partial B} &= \frac{1}{2}\int_{\partial B} \big(1-h\big)\epsilon_{\rm S}. \end{aligned}} \tag{3.66}$$

This is the final expression we are looking for. We analyze it in the next sections.

3.3 Canonical Pairs

We now read off the null canonical pairs of gravity from (3.66), comparing with the schematic expression (2.3).

Bulk configuration		Bulk momentum	
Conformal metric:	γ_{ab}	$\frac{1}{2}\tilde{\sigma}^{ab}$	Conformal shear
Normal vector:	L^a	$-\bar{\eta}_a$	Twist
Volume element:	$(D-2)\varphi$	$-\big(\kappa + \frac{D-3}{D-2}\theta\big)$	Expansion, surface gravity

(3.67)

Note that what we call momenta are B-densities $P\epsilon_{\mathrm{B}}$. The boundary canonical pairs can also be read off from (3.66) and are

$$\big(L^a, \tfrac{1}{2}h q_a{}^b \iota_b \epsilon_{\mathrm{S}}\big) \;\text{ and }\; \big((D-2)\varphi, \tfrac{1}{2}(1+h)\epsilon_{\mathrm{S}}\big)\,. \tag{3.68}$$

Written in this form, the configuration variables in the bulk of B contain only variations of the induced metric (3.16) on B, and no variations of the normal metric, nor derivatives of variations. That is analogous to the symplectic structure on spacelike and timelike surfaces, and it was not obvious from the outset that there is such a form. Also the configuration variables on the corner ∂B are a subset of the induced geometry. Let us analyze each of the pairs in turn. We will refer to them as the "spin-2", "spin-1" and "spin-0" pairs in accordance to the number of their independent components.

3.3.1 Spin-2

The spin-2 configuration variable is the conformal metric γ_{AB}, which allows measuring angles, but not lengths, on the surfaces S. Its momentum is the conformal shear, which is given by the Lie derivative along L of the conformal metric: $\tilde{\sigma}^{AB} := \gamma^{AA'}\gamma^{BB'}\frac{1}{2}\mathcal{L}_L\gamma_{A'B'}$ where γ^{AB} is the inverse of γ_{AB}. That the shear is conjugate to the conformal metric was first established by Ashtekar et al. [19] in the context of asymptotic null infinity. The shear is automatically trace free, and can be defined from the trace free part of the extrinsic curvature[4] as $\tilde{\sigma}^{AB} = e^{2\varphi}\theta^{<AB>}$ with $\theta^{AB} = q^{Aa}q^{Bb}\nabla_a L_b$. In metric parameters, the shear

[4] We denote the trace free components of a tensor as $\theta^{<AB>} = \theta^{AB} - \frac{q^{AB}}{(D-2)}q^{CD}\theta_{CD}$

becomes (see appendix A.1)

$$\begin{aligned}\tilde{\sigma}^{AB} &= -\tfrac{1}{2}\partial_0\gamma^{AB} + e^{2\varphi} d^{<A} A_0^{B>} \\ &= -\tfrac{1}{2}\partial_0\gamma^{AB} + \tfrac{1}{2}\big(\gamma^{AC}\partial_C A_0^B + \gamma^{BC}\partial_C A_0^A - A_0^C\partial_C\gamma^{AB}\big) - \tfrac{1}{D-2}\partial_C A_0^C\gamma^{AB}.\end{aligned} \tag{3.69}$$

Note that the shear is independent of the conformal factor φ. It is determined by the intrinsic geometry of B. If we interpreted A_0^A as the velocity field of a fluid on B, the term $e^{-2\varphi}\mathrm{d}_{<A}A_{0B>}$ is naturally interpreted as the rate of strain tensor. It is complemented by the time derivative of the metric in the case where the metric is explicitly time dependent, which is not usually the case in fluid dynamics. Also note that both γ_{AB} and $\tilde{\sigma}^{AB}$ are invariant under conformal rescalings of the metric.

3.3.2 Spin-1

The spin 1 configuration variable is L. Since its ∂_{ϕ^0}–component is fixed, its variation is $\delta L^a = \delta A_0^a$ and is purely tangential to S. The momentum $\bar{\eta}$ conjugate to L is given by $\bar{\eta}_a = -q_a{}^b\nabla_\ell\bar{\ell}_b$. Since $\bar{\ell}$ determines the orientation of S within B, $\bar{\eta}_a$ describes how the cross–sections S of B tilt and twist when parallel–transported along ℓ. It can be expressed as the sum of two terms (see appendix A.1):

$$\bar{\eta}_A = \frac{1}{2}\big(\partial_A h - F_A\big), \qquad F_A := q_{AB}e^{-h}\left(\partial_0 A_1^B - \partial_1 A_0^B + [A_0, A_1]^B\right) \tag{3.70}$$

Here F_A is the curvature of the shift connection, and measures the non-integrability of the normal two–planes. The other term $\partial_A h$ measures the rate of change of the normal volume element h along the cross–section S.

Using the boost gauge $\bar{\alpha} = 0$, Damour ([63]) first interpreted $\omega_a = q_a{}^b\bar{\ell}^c\nabla_b\ell_c$ as a momentum density. He was motivated by the fact that for a cylindrically symmetric black hole, an integral of ω_a is the total angular momentum, and that in the Navier–Stokes–like equation $q_a{}^bL^cR_{bc} = 0$, ω_a plays the role of a linear momentum. However, ω_a is not boost gauge invariant and transforms as a connection under the boost gauge. The twist $\bar{\eta}$ is boost gauge invariant and coincides with ω in the boost gauge $\bar{\alpha} = 0$ since $\bar{\eta}_a - \omega_a = q_a{}^b\nabla_b\bar{\alpha}$. The twist $\bar{\eta}$ is thus the proper boost gauge invariant generalization of ω. In the light of the fluid interpretation of null surfaces, it is thus very natural that we found $\bar{\eta}$ as the momentum conjugate to the "displacement" A_0. We have thus confirmed Damour's interpretation of ω from a symplectic analysis.

Under conformal rescaling of the metric $g \to e^{2\epsilon}g$, the size of the normal geometry transforms as $h \to h + 2\epsilon$ while A_0 and A_1 do not change. The twist then transforms by a total derivative, $\bar{\eta}_A \to \bar{\eta}_A + \partial_A\epsilon$, and its curvature $\mathrm{d}_{[A}\bar{\eta}_{B]}$ is conformally invariant.

3.3.3 Spin-0

The spin-0 sector is especially interesting, since it carries information about mass and energy. The first ingredient is the expansion. Even though it is part of the extrinsic geometry of S, it is determined by the intrinsic geometry of B. That can also be seen noting that the divergence of the area form on B is $\mathrm{d}\epsilon_{\mathrm{S}} = \theta\epsilon_{\mathrm{B}}$. More generally, we have for any function g on B:

$$\mathrm{d}(g\epsilon_{\mathrm{S}}) = (L^a\partial_a g + g\theta)\epsilon_{\mathrm{B}}, \tag{3.71}$$

The expansion also plays a central role in the Raychaudhuri equation. In terms of our parametrization of the metric, the expansion becomes

$$\theta = (D-2)D_0\varphi + \partial_A A_0^A. \tag{3.72}$$

In the case of a non–expanding null surface, the spin-0 sector becomes the pair from black hole thermodynamics: the volume element $\sqrt{q}$ is conjugate to the surface gravity κ.

For a general null surface, κ can be interpreted as an infinitesimal redshift: Suppose $g^{00} = -1$, such that ϕ^0 corresponds to the clock of a family of geodesic observers. Consider a lightray propagating along L. It may be described by an affinely parametrized null geodesic $r = fL$, where for r to be affine, f needs to satisfy $L^a\partial_a(\ln f) = -\kappa$. The light's frequency will be measured by the observers to be $fL^a(\mathrm{d}\phi_0)_a = f$. The infinitesimal redshift then becomes

$$\mathrm{d}z = \frac{f(\phi^0 - \mathrm{d}\phi^0) - f(\phi^0)}{f(\phi^0)} = \kappa\mathrm{d}\phi^0. \tag{3.73}$$

The coordinate expression for the surface gravity κ is derived in appendix A.1 and reads

$$\kappa = (D_0 + \beta D_1)h + D_1\beta \overset{B}{=} D_0 h + \partial_1\beta. \tag{3.74}$$

The coefficients κ and θ are not invariant under local rescalings of the metric. Under a change $g_{ab} \to e^{2\alpha}g_{ab}$, holding L^a fixed, we have on B

$$(\kappa, \theta) \to (\kappa + 2D_0\alpha, \theta + (D-2)D_0\alpha). \tag{3.75}$$

A wide variety of different linear combinations of the spin-0 variables κ and θ appear in the literature. The conformally invariant combination is $\kappa - \frac{2}{D-2}\theta$, it is constant on conformal Killing horizons [101]. The combination $\kappa - \frac{1}{D-2}\theta$ features in the null Raychaudhuri equation written as

$$G_{LL} = -L^a\partial_a\theta + \left(\kappa - \tfrac{1}{D-2}\theta\right)\theta - \sigma_A{}^B\sigma_B{}^A, \tag{3.76}$$

so if one gauge fixes that combination to zero and knows the shear ${\sigma_A}^B$, the equation can straightforwardly integrated for θ (such as in [53]). The combination $\kappa + \theta$ is obtained as $\kappa + \theta = \nabla_a L^a$ and has been suggested as the null analogue of the Gibbons-Hawking-York term [60]. However the combination that is of crucial interest for us is the combination

$$\mu := \kappa + \tfrac{D-3}{D-2}\theta \tag{3.77}$$

which we call the *spin-0 momentum*. It enters our analysis as the canonical variable conjugate to the conformal factor φ, and naturally appears in the densitized Raychaudhuri and Damour equations as we will see in the next chapter. This combination appeared in dimension 4 in the canonical analysis of Torre [52] and of Epp [55], see also [102] for its interpretation in the first order formalism. It combines the pressure and bulk viscosity terms from the membrane paradigm [65]. The coordinate expression for the spin-0 momentum becomes

$$\mu \stackrel{B}{=} D_0(h + (D-3)\varphi) + D_1\beta + \tfrac{D-3}{D-2}\partial_A A_0^A. \tag{3.78}$$

3.3.4 Geometrical Interpretation

Remarkably, the elements of extrinsic geometry that form the momenta appear naturally in the comparison between two different ways of transporting a vector field ξ on B: the Lie transport $\mathcal{L}_L\xi$, which is purely intrinsic to B, and the parallel transport $\nabla_L\xi$, which through the Christoffel symbols contains information about the metric components transverse to B. Let us decompose the vector field $\xi \in \Gamma(TB)$ as

$$\xi^a = fL^a + v^a, \tag{3.79}$$

with $v \parallel S$. The difference between the parallel and Lie transport along L of a vector ξ tangent to B is given by

$$\nabla_L\xi^a - [L,\xi]^a = \nabla_\xi L^a = v^b(\omega_b L^a + {\sigma_b}^a + \tfrac{\delta_b^a}{D-2}\theta) + f\kappa L^a. \tag{3.80}$$

This difference is encoded into the so-called Weingarten map $\nabla_\xi L^a$, see e.g. [100]. Note that in our boost gauge, $\omega_a = \bar{\eta}_a$, so all three momenta are contained in the Weingarten map.

3.4 Lagrangian Boundary Terms

In the expression (3.65) for the symplectic potential, we have extracted a total variation from the symplectic potential. That corresponds to a choice of polarization: It tells us which are the configuration and which the momentum variables. This can be seen most clearly by noting that such a total variation can be used to interchange to roles of configuration and momentum variables:

$$\Theta = P\delta Q = -Q\delta P + \delta(PQ). \tag{3.81}$$

The choice of polarization we have made is that the configuration variables Q should not contain derivatives of the metric. As discussed in chapter 2, the total variation in $\Theta_B = \Theta_B^{\text{bulk}} + \Theta_{\partial B} + \delta(A_B + a_{\partial B})$ can be canceled by adding a boundary term and a corner term to the action:

$$S = \int_M L - A_B - a_{\partial B} \tag{3.82}$$

From (3.66), we therefore make the following suggestion for the action of a space–time region with null boundaries that may possess corners:

$$\boxed{S = \frac{1}{2}\int_M R\epsilon - \int_B \kappa\epsilon_{\text{B}} - \frac{1}{2}\int_{\partial B}(1-h)\epsilon_{\text{S}}.} \tag{3.83}$$

Note that the corner term vanishes for segments of ∂B that contain the null direction L, since in that case the pullback of ϵ_{S} vanishes.

A similar line of reasoning to this section was followed in the recent paper by Lehner et al. [62], and a proposal for the boundary and corner action for null boundaries was given. We have thus reproduced one of the results there with a different calculation.[5]

Parattu et al. ([61]) also gave a suggestion for the boundary action and the canonical structure. They mostly work in the boost gauge $\alpha = 0$, and extract a total variation containing the normal acceleration γ rather than the surface gravity κ. As can be seen from (3.50), γ and κ are inequivalent unless $\bar{\alpha} = 0$. For that reason Parattu et al. obtain an extra canonical pair on B, which contains a piece of normal geometry as a configuration variable. As we have seen, that pair can be removed by choosing κ rather than γ in the total variation extracted.

[5] Up to the summand 1 in the corner term, which is a choice of corner polarization.

Chapter 4

Null Conservation Laws for Gravity

In this chapter, we analyze the gravitational constraints on a null surface, reading them as conservation laws and connecting to the symplectic analysis of chapter 3.

The plan for the chapter is as follows: Section 4.1 recalls the relevant pieces of intrinsic and extrinsic geometry of the null surface and gives the action of the field space Lie derivative. Section 4.2 rearranges the constraint equations as a canonical conservation equation, and derives the boundary current J_ξ and the flux terms F_ξ. Section 4.3 addresses the rationale and consequences of modifying the symplectic potential away from the standard one, giving the intrinsic symplectic potential, and derives the canonical conservation equation starting from the intrinsic symplectic potential. Section 4.4 addresses the question how the boundary current J_ξ is related to the Hamiltonian generators of the infinitesimal diffeomorphism ξ. Technical manipulations have been relegated to appendices.

4.1 Geometry

This section recalls the pieces of intrinsic and extrinsic geometry that will appear, and gives their transformations under diffeomorphisms.

4.1.1 Intrinsic and Extrinsic Geometry of a Null Surface

The null surface B is located at the level surface $\phi^1 = 0$. It is foliated by a "time" coordinate ϕ^0 into spacelike codimension-two spheres S with coordinates x^A. The pieces of intrinsic geometry of B that we will use in this chapter are the conformal $(D-2)$ metric γ_{AB}, the conformal factor φ and the null direction L^a. The metric on the spheres S is $q_{AB} = e^{2\varphi}\gamma_{AB}$. As we saw, these data determine a covariant vector valued $(D-1)$-form $L^a\epsilon_{\rm B}$, and a covariant area $(D-2)$-form $\epsilon_{\rm S}$. We also introduce the form $\bar{L}$, which is normal to the $(D-2)$-spheres S and satisfies $L^a\bar{L}_a = 1$. We have parameterized

$$L = D_0 + \beta D_1, \qquad \bar{L} = \frac{1}{1+\beta\bar{\beta}}(\mathrm{d}\phi^0 + \bar{\beta}\mathrm{d}\phi^1)$$
$$g(L) = e^h(\mathrm{d}\phi^1 - \beta\mathrm{d}\phi^0), \qquad g^{-1}(\bar{L}) = \frac{e^{-h}}{1+\beta\bar{\beta}}(D_1 - \bar{\beta}D_0). \tag{4.1}$$

The variable β vanishes on the null surface.

The extrinsic geometry of the null surface B which we will encounter here are the shear σ^{AB}, the twist $\bar{\eta}_A$, the expansion θ and the surface gravity κ. They are defined and explained in section 3.3. We use $\sigma_{AB} = \theta_{<AB>}$ for the shear, and $\tilde{\sigma}^{AB} = e^{2\varphi}\sigma^{AB}$ for the conformal shear, which is invariant under rescaling of the metric.

The spin-0 momentum is the combination $\mu = \kappa + \frac{D-3}{D-2}\theta$. We will write $\xi[f] := \xi^a\partial_a f$ for the directional derivative.

4.1.2 Transformations of Intrinsic and Extrinsic Geometry under Diffeomorphisms

We now turn to the transformations under infinitesimal diffeomorphisms of the pieces of intrinsic and extrinsic geometry of the null surface B, which we will need to understand the conservation laws. The expressions we have introduced make reference to the coordinate fields, especially to the "time" variable ϕ^0 in the normalization of L. As explained in section 2.3, we thus cannot expect them to transform covariantly under diffeomorphisms. The non-covariance is made precise by the anomaly, which acts on field-space scalars T, which may be spacetime tensors of arbitrary rank, as $\Delta_\xi T = \mathfrak{L}_\xi T - \mathcal{L}_\xi T$. Here $\mathfrak{L}_\xi T = I_\xi \delta T = \frac{\delta T}{\delta g_{ab}}\mathcal{L}_\xi g_{ab}$ derives T according to its metric dependence, and $\mathcal{L}_\xi$ is the ordinary Lie derivative, acting according to the index structure of T.

The covariance of the covariant derivative is encoded as

$$\Delta_\xi(\nabla_a T) = \nabla_a(\Delta_\xi T) \tag{4.2}$$

for any tensor T, which may have a non-vanishing anomaly. The identity may be checked explicitly using the standard identity $\delta\Gamma^a_{bc} = \frac{1}{2}(\nabla_b\delta g^a{}_c + \nabla_c\delta g^a{}_b - \nabla^a\delta g_{bc})$. We make use of this in appendix B.3 to derive the diffeomorphism transformations of extrinsic geometry.

The anomaly of the null vector L can be understood as the source of all the anomalous diffeomorphism transformations, and it will be instructive to calculate it here. The anomaly stems from the normalization condition $L[\phi^0] = 1$, which introduces the field ϕ^0 as background structure and thus breaks covariance.

Restricting attention to vector fields parallel to B, we have $(\mathcal{L}_\xi L)^a = \xi^b \partial_b L^a - L^b \partial_b \xi^a$. However, viewing L^a as a function of the metric, we may write

$$L^a[g_{bc}] \stackrel{B}{=} \frac{g^{1a}}{g^{10}}. \tag{4.3}$$

Using the Leibniz rule, we get

$$\mathfrak{L}_\xi L^a = \frac{1}{g^{10}} \mathfrak{L}_\xi g^{1a} - \frac{g^{1a}}{(g^{10})^2} \mathfrak{L}_\xi g^{10}. \tag{4.4}$$

Now use that the components of the inverse metric transform by the spacetime Lie derivative: $\mathfrak{L}_\xi (g^{ab}) = \xi^c \partial_c g^{ab} - g^{ac} \partial_c \xi^b - g^{cb} \partial_c \xi^a$. Parametrizing $\xi^a = f L^a + v^a$ with $v \parallel S$, we get after a short calculation that $\mathfrak{L}_\xi L^a = [v, L]^a$, and hence, on B, the anomaly becomes

$$\Delta_\xi L^a = [v, L]^a - [\xi, L]^a = L[f] L^a. \tag{4.5}$$

The field space Lie derivatives of all the data we have defined so far are derived in the appendix B.3, and we summarize the relevant results now. As before, let $\xi^a = f L^a + v^a$ be a vector field on B, and let $v \parallel S$. In addition to the transformation of L, we will have the following: The conformal metric transforms as

$$\mathfrak{L}_\xi \gamma_{AB} = 2(f \sigma_{AB} + e^{-2\varphi} \mathrm{d}_{<A} v_{B>}), \tag{4.6}$$

where we recall that d_A is the covariant derivative of $q_{AB} = e^{2\varphi} \gamma_{AB}$, and we have written $T_{<AB>}$ for the trace free part of a tensor. Note that $e^{-2\varphi} \mathrm{d}_{<A} v_{B>}$ is independent of φ, and that the RHS is trace free as expected of derivatives of a unimodular matrix. The conformal factor transforms as

$$\mathfrak{L}_\xi \varphi = \tfrac{1}{D-2}(f\theta + \mathrm{d}_A v^A). \tag{4.7}$$

As argued earlier, the combination $L^a \epsilon_{\mathrm{B}}$ is covariant; it transforms under diffeomorphisms of B as

$$\mathfrak{L}_\xi (L^a \epsilon_{\mathrm{B}}) = \mathcal{L}_\xi (L^a \epsilon_{\mathrm{B}}) = \big([v, L]^a + (\theta f + \mathrm{d}_B v^B) L^a\big) \epsilon_{\mathrm{B}}. \tag{4.8}$$

Here, the spacetime Lie derivative acts on $L^a\epsilon_{\mathrm{B}}$ as a vector valued top form, i.e., $\mathcal{L}_\xi(L^a\epsilon_{\mathrm{B}}) = [\xi, L]^a\epsilon_{\mathrm{B}} + L^a \mathrm{d}(\iota_\xi\epsilon_{\mathrm{B}})$. The area $(D-2)$-form is covariant, and by contracting the previous equation we obtain

$$\mathfrak{L}_\xi\epsilon_{\mathrm{S}} = \mathcal{L}_\xi\epsilon_{\mathrm{S}} = \iota_\xi \mathrm{d}\epsilon_{\mathrm{S}} + \mathrm{d}\iota_\xi\epsilon_{\mathrm{S}} = \big([v, L]^a\iota_a\epsilon_{\mathrm{B}} + (\theta f + \mathrm{d}_B v^B)\big)\epsilon_{\mathrm{S}}. \tag{4.9}$$

Finally, we need the transformation of the spin-0 momentum, which is more subtle. Under finite rescalings of the null generators $L \to gL$ (or equivalently under redefinition of the coordinate ϕ^0 with $\partial\phi^0/\partial\phi^{0\prime} = g$), the spin-0 momentum transforms as a connection and goes to

$$\mu \to \mu_g := (L + \mu)[g]. \tag{4.10}$$

The spin-0 momentum can thus be fixed to any value by controlling the "clock" ϕ^0. We will make use of that fact in section 4.4. Infinitesimally, the transformation involves a second derivative of the vector field ξ, and reads:

$$\mathfrak{L}_\xi\mu = v[\mu] + L\big[(L + \mu)[f]\big]. \tag{4.11}$$

Note the appearance of the differential operator $L+\mu$, which is a covariant derivative with respect to local rescaling of the null generators[1].

Two remarks are in order: Firstly, the transformations of $(\varphi, \gamma, L, \mu, \bar\eta_A, \sigma^{AB})$ only involve ξ as a vector field on B, and do not depend on how (and if) it is extended to a vector field on M. This is not obvious looking at the coordinate expressions given in section 3.3, and is an important and desirable feature of those variables. Secondly, note that $\mu, L, \epsilon_{\mathrm{B}}$ and $\bar\eta_A$ transform covariantly under diffeomorphisms v parallel to the cross-sections S: Anomalies arise only for diffeomorphisms transverse to S.

We will also need the transformation of h, the logarithmic determinant of the metric in directions normal to S. It depends on the extension of ξ, and we parametrize an arbitrary extension as $\xi^a = fL^a + \bar f\bar L^a + v^a$ with $\bar f$ vanishing on B. Using $\delta h = L^a\bar L^b\delta g_{ab}$, we get

$$\mathfrak{L}_\xi h = (L + \kappa)[f] + (g^{-1}(\bar L) + \bar\kappa)[\bar f] + (\eta_A + \bar\eta_A)v^A, \tag{4.12}$$

where $\eta_A = -q_A{}^a\nabla_{\bar L}L_a$ and $\bar\kappa = L_a\nabla_{\bar L}\bar L^a$. Note that η and $\bar\kappa$ are not part of the extrinsic geometry of B, but rather part of the extrinsic geometry of S as embedded in M.

To summarize, the tensors that make up the intrinsic and extrinsic geometry of B can be Lie-derived in two ways: The spacetime Lie derivative views them as tensors and Lie-derives

[1] Under $L \to e^\alpha L$ holding fL fixed, we have $f \to e^{-\alpha}f$ and $\mu \to e^\alpha(\mu + L[\alpha])$, so $(L+\mu)[f]$ is invariant.

them according to their index structure, and the field space Lie derivative views them as functionals of the metric and derives them according to their metric dependence. The difference between the two prescriptions is the anomaly Δ_ξ. We have given the field space Lie derivatives that we will need in the following; more transformations are in appendix B.3.

4.2 Einstein Equations as Conservation Equations

Having completed the setup, let us turn to our central task of interpreting the Einstein constraint equations as conservation equations. We are looking for a conservation equation intrinsic to the null surface B which is of the form "Divergence of current = gravitational flux + matter energy-momentum flux". Both the current and the gravitational flux will depend on a vector field ξ parallel to B. The conservation equation is an equality of $(D-1)$-forms, and can be integrated on portions of the null surface B.

The current on the LHS is the *boundary current* j^a_ξ. It is a vector tangent to B and we can associate with it a $(D-2)$-form $J_\xi = \iota_{j_\xi}\epsilon_{\rm B}$. J_ξ is a codimension one form on B and the divergence of the current corresponds to $\mathrm{d}J_\xi$. In the following we will interchangeably use the denomination boundary current for j_ξ or J_ξ even if the later is the dual boundary form. The boundary current j_ξ can be expanded in terms of a time component, i.e., the component along ϕ^0, and a component tangential to the sphere. Its time component may be thought of as the gravitational charge aspect, and the spatial components as the finite boundary analogue of soft currents.

The Einstein equations we consider are the null Raychaudhuri equation [103] for G_{LL} and the Damour equation [104] for $q_a{}^b G_{Lb}$.

They are derived, using our variables, in appendix B.2. This set of equations are the null analogue of the ADM momentum constraint equations. Since we are looking for a conservation law that can be integrated on the null surface B, and since the canonical constraints are best thought of as densities on a hypersurface, we multiply them with the density $\epsilon_{\rm B}$. The densitized expressions are

$$\boxed{\begin{aligned} G_{LL}\,\epsilon_{\rm B} &= \quad -\mathcal{L}_L(\theta\epsilon_{\rm B}) + (\mu\theta - \sigma_b{}^a\sigma_a{}^b)\epsilon_{\rm B}, \\ q_a{}^b G_{Lb}\epsilon_{\rm B} &= \;\; q_a{}^b\mathcal{L}_L(\bar{\eta}_b\epsilon_{\rm B}) - (\mathrm{d}_a\mu + \mathrm{d}_b\sigma_a{}^b)\,\epsilon_{\rm B}. \end{aligned}} \tag{4.13}$$

Note that densitizing with $\epsilon_{\rm B}$, and performing the trace-traceless split on θ and $\sigma_a{}^b$, naturally leads to the appearance of the spin-0 momentum $\mu = \kappa + \frac{D-3}{D-2}\theta$ in both equations.

Let us analyze them as conservation equations on B, first when contracted with a "constant" vector field, and then for a general vector field.

4.2.1 Conservation Law for "Constant" Vector Fields

To gain a first understanding, consider the Raychaudhuri and Damour equations smeared with a vector field $\xi = fL + v$ parallel to B which is Lie dragged along L, i.e.,

$$[L, \xi] = 0. \tag{4.14}$$

This simplifying assumption means that ξ is "constant in time" and implies $L[f] = 0$ and $[L, v] = 0$. It is sensitive to the choice of normalization of L, i.e., to a choice of clock. Let us go on-shell and set $G_{ab} = T_{ab}$ (in units where $8\pi G = 1$). Contracting with ξ, we can rewrite our two equations as

$$-\mathcal{L}_L(f\theta\epsilon_{\rm B}) = f\left[T_{LL} - \mu\theta + \sigma_b{}^a\sigma_a{}^b\right]\epsilon_{\rm B}, \tag{4.15}$$

$$\mathcal{L}_L(v^a\bar{\eta}_a\epsilon_{\rm B}) = v^a\left[T_{aL} + ({\rm d}_a\mu - {\rm d}_b\sigma_a{}^b)\right]\epsilon_{\rm B}, \tag{4.16}$$

where we used $\mathcal{L}_L(g\epsilon_{\rm S}) = (L[g] + g\theta)\epsilon_{\rm B}$ for any function g. Since $\mathcal{L}_L(g\epsilon_{\rm B}) = {\rm d}(g\epsilon_{\rm S})$, the LHSs of both equations are total derivatives.

Written in this manner the Raychaudhuri equation (4.15) can be understood as a conservation equation for an energy $E_f := -\int_S f\theta {\rm d}S$. Indeed, by integrating the Raychaudhuri equation on a portion of B delimited by S_i and S_f, one gets the balance equation $\Delta E_f = \int_{S_i}^{S_f} f(T_{LL} + T^G_{LL})\epsilon_{\rm B}$, which expresses that the change in energy E_f is due to exchange of material and gravitational energy with the exterior. This allows us to identify the gravitational energy momentum tensor

$$T^G_{LL} := (\sigma_b{}^a\sigma_a{}^b - \mu\theta), \tag{4.17}$$

which appears alongside the matter energy-momentum tensor and measures the amount of gravitational energy that leaves the region enclosed by S per unit time and unit area, according to the observer ξ. Part of the gravitational energy is carried out by the gravitational waves or spin 2 components $\sigma_b{}^a\sigma_a{}^b$, but another part is carried out by the spin zero component and measures the work done by the rescaling of the surface through the term $-\mu\theta$. Since θ is the rate of change of area, this naturally leads to the interpretation of μ as a boundary pressure term.

In the Damour equation (4.16), $P_v := \int_S (v \cdot \bar{\eta}) \mathrm{d}S$ is interpreted as the super-momentum enclosed by the region S. We can identify a gravitational momentum flux $T^G_{\boldsymbol{v}L}$ given by

$$T^G_{\boldsymbol{v}L} = v^a(\mathrm{d}_a\mu - \mathrm{d}_b\sigma_a{}^b). \tag{4.18}$$

This expression confirms the interpretation of μ as a pressure term, while the shear σ appears as a viscous stress component. Integrating the Damour equation then gives the balance equation $\Delta P_v = \int_S^{S'}(T_{Lv} + T^G_{Lv})\epsilon_{\mathrm{B}}$.

4.2.2 The Boundary Current and its Conservation

We would now like to understand the conservation equations more covariantly and locally. This requires that we use a general vector field $\xi \in TB$, and combine the Raychaudhuri and Damour equations as components of one equation.

In order to decide which terms on the RHS of (4.13) should be seen as part of the boundary current and which as part of the fluxes, let us recall the form of the energy-momentum flux for a scalar field with Lagrangian $L_{\text{scalar}} = \frac{1}{2}g^{ab}\partial_a\phi\partial_b\phi - V(\phi)$. On a null surface, the canonical momentum density P conjugate to ϕ is $P = L[\phi]\epsilon_{\mathrm{B}}$, and the energy momentum tensor becomes

$$T_{LL}\epsilon_{\mathrm{B}} = L[\phi]L[\phi]\epsilon_{\mathrm{B}}, \qquad T_{Lv}\epsilon_{\mathrm{B}} = L[\phi]v[\phi]\epsilon_{\mathrm{B}}. \tag{4.19}$$

Those components combine into $T_{L\xi}\epsilon_{\mathrm{B}} = L[\phi]\xi[\phi]\epsilon_{\mathrm{B}}$ for $\xi \parallel B$. For a scalar field, the flux that controls the flow of energy and momenta through B thus has a natural canonical expression given by the product of the momenta with the field transforms

$$T_{L\xi}\epsilon_{\mathrm{B}} = P\mathfrak{L}_\xi\phi. \tag{4.20}$$

We therefore expect the gravitational flux term to have a similar canonical form $\sum_i P_i\mathfrak{L}_\xi Q_i$.

In order to establish this we need to isolate terms that can be interpreted in a canonical form $P\mathfrak{L}_\xi Q$, from the equations (4.13). Lets first recall the action of diffeomorphisms on our data (section 4.1.2): We have

$$\begin{aligned} \mathfrak{L}_\xi\gamma_{AB} &= 2\big(f\sigma_{AB} + e^{-2\varphi}\mathrm{d}_{<A}v_{B>}\big), & \mathfrak{L}_\xi\epsilon_{\mathrm{B}} &= (f\theta + \mathrm{d}_A v^A)\epsilon_{\mathrm{B}}, \\ \mathfrak{L}_\xi L^a &= [v, L]^a, & \mathfrak{L}_\xi\mu &= L[(L+\mu)[f]] + v[\mu]. \end{aligned} \tag{4.21}$$

We can now express the Raychaudhuri equation contracted with fL^a as a canonical conservation law. Using again $\mathcal{L}_L(g\epsilon_{\mathrm{B}}) = \mathrm{d}(g\epsilon_{\mathrm{S}}) = (L[g] + \theta g)\epsilon_{\mathrm{B}}$, we get

$$(fL)^a G_{aL}\epsilon_{\mathrm{B}} = -\mathrm{d}(f\theta\epsilon_{\mathrm{S}}) + (f\mu + L[f])\mathrm{d}\epsilon_{\mathrm{S}} - f\sigma_a{}^b\sigma_b{}^a\epsilon_{\mathrm{B}}. \tag{4.22}$$

The second term on the RHS is not of the canonical form $P\mathfrak{L}_\xi Q$, so we integrate by parts and use $(\mathrm{d}g)\wedge\epsilon_\mathrm{S} = L[g]\epsilon_\mathrm{B}$ to get

$$(fL)^a G_{aL}\epsilon_\mathrm{B} = \mathrm{d}\big(-f\theta\epsilon_\mathrm{S} + (L+\mu)[f]\epsilon_\mathrm{S}\big) - L[(L+\mu)[f]]\epsilon_\mathrm{B} - f\sigma_a{}^b\sigma_b{}^a\epsilon_\mathrm{B} \tag{4.23}$$

$$= \mathrm{d}\big((L+\mu-\theta)[f]\epsilon_\mathrm{S}\big) - \epsilon_\mathrm{B}(\mathfrak{L}_{fL}\mu + \tfrac{1}{2}\sigma^{AB}\mathfrak{L}_{fL}\gamma_{AB}). \tag{4.24}$$

The RHS is now written as the sum of the differential of a $(D-2)$-form and two canonical flux terms. This is the form we want.

Let us turn to the densitized Damour equation (4.13). Contracting with v^a and using $\mathrm{d}_a v^a\epsilon_\mathrm{B} = \mathrm{d}(\iota_v\epsilon_\mathrm{B})$, we can rewrite it as

$$v^a G_{La}\epsilon_\mathrm{B} \overset{B}{=} \mathrm{d}\big(v^a\bar{\eta}_a\epsilon_\mathrm{S} + (v^a\sigma_a{}^b)\iota_b\epsilon_\mathrm{B}\big) + \big(\bar{\eta}_a[v,L]^a - (v^a\mathrm{d}_a\mu) - (e^{-2\varphi}\sigma^{ab}\mathrm{d}_a v_b)\big)\epsilon_\mathrm{B} \tag{4.25}$$

$$= \mathrm{d}\big(v^a\bar{\eta}_a\epsilon_\mathrm{S} + (v^a\sigma_a{}^b)\iota_b\epsilon_\mathrm{B}\big) - \epsilon_\mathrm{B}(\mathfrak{L}_v\mu - \bar{\eta}_a\mathfrak{L}_v L^a - \tfrac{1}{2}\sigma^{AB}\mathfrak{L}_\xi\gamma_{AB}). \tag{4.26}$$

The RHS is also written as the sum of a differential plus three canonical flux terms.

We can now combine the Raychaudhuri and Damour equations and express the Einstein equations $G_{\xi L} = T_{\xi L}$ as a canonical conservation equation. Let us again parametrize $\xi = fL + v$, and define the boundary current j_ξ^a as

$$\boxed{j_\xi^a := \big((L+\mu-\theta)[f] + v^b\bar{\eta}_b\big)L^a + v^b\sigma_b{}^a.} \tag{4.27}$$

The corresponding boundary current form is the $(D-2)$-form $J_\xi := \iota_{j_\xi}\epsilon_\mathrm{B}$ given by

$$\boxed{J_\xi = (L+\mu-\theta)[f]\epsilon_\mathrm{S} + v^b\bar{\eta}_b\epsilon_\mathrm{S} + v^b\sigma_b{}^a\iota_a\epsilon_\mathrm{B}.} \tag{4.28}$$

Setting $G_{ab} = T_{ab}$, we can then write the Raychaudhuri and Damour equations as

$$\boxed{\mathrm{d}J_\xi \overset{B}{=} \big(T_{\xi L} + \tfrac{1}{2}\tilde{\sigma}^{ab}(\mathfrak{L}_\xi\gamma_{ab}) - \bar{\eta}_a(\mathfrak{L}_\xi L^a) + \mathfrak{L}_\xi\mu\big)\epsilon_\mathrm{B}.} \tag{4.29}$$

In this expression the gravitational flux is now expressed in a canonical form. The equations (4.28, 4.29) summarize the null gravitational constraint equations. The expression for the boundary current J_ξ is determined by this analysis up to a total differential $J_\xi \to J_\xi + \mathrm{d}\beta_\xi$.

The gravitational flux terms, which appear alongside the matter flux terms on the RHS of (4.29), are of the canonical form $P\mathfrak{L}_\xi Q$. The canonical pairs are usually identified using the symplectic potential or related technology, but this analysis provides an alternative route towards their identification. We see that the gravitational canonical pairs (P,Q)

are the spin-2 pair $(\frac{1}{2}\tilde{\sigma}^{AB}\epsilon_{\rm B}, \gamma_{AB})$ of densitized shear and conformal metric, the spin-1 pair $(-\bar{\eta}_a\epsilon_{\rm B}, L^a)$ consisting of the twist and the null directions, and the spin-0 pair $(\epsilon_{\rm B}, \mu)$ consisting of the area form and spin-0 momentum.

Let us interpret the boundary current vector (4.27). In a given reference frame, the time component of a current vector is interpreted as the charge density and the spatial components as non-relativistic currents. In analogy, we may interpret the components along L of j^a_ξ as charge aspects. First, consider a vector field $\xi = fL$ parallel to L, which we interpret as a "null time" translation. The conserved charge of time translations is energy, and we thus find the *gravitational energy aspect*

$$\boxed{e_f = (-\theta + \mu + L)[f]\epsilon_{\rm S}.} \tag{4.30}$$

It can be rewritten as $e_f = (\kappa - \frac{1}{D-2}\theta + L)[f]\epsilon_{\rm S}$, and features the combination $\kappa - \frac{1}{D-2}\theta$, which also appears in the non-densitized Raychaudhuri equation. Note that the gravitational energy aspect e_f differs from the previous energy density $-f\theta\epsilon_{\rm B}$ by the addition of a pressure term $\mu_f\epsilon_{\rm B}$ with $\mu_f := f\mu + L[f]$. We can therefore interpret $-f\theta\epsilon_{\rm B}$ as an internal energy of the sphere S while e_f is its enthalpy.

The conserved charge for spatial vector fields is the momentum. We can thus identify from (4.27) the *momentum aspect*

$$\boxed{p_v = v^b\bar{\eta}_b\,\epsilon_{\rm S}.} \tag{4.31}$$

The term $v^b\sigma_b{}^a$ then finds interpretation as a spatial momentum current.

In (4.27), we have written the boundary current j_ξ using the split $\xi \to (f, v)$ and the extrinsic geometry of B. It can also be written more covariantly and geometrically if we recognize (see eq. 3.80) that

$$\nabla_L\xi^a = -[v, L]^a + \big((\kappa + L)[f] + v^b\bar{\eta}_b\big)L^a + v^b\theta_b{}^a, \tag{4.32}$$

and use $\kappa - \frac{1}{D-2}\theta = \mu - \theta$. We get

$$\boxed{j^a_\xi = \nabla_L\xi^a - \tfrac{1}{D-2}\theta\xi^a + [v, L]^a.} \tag{4.33}$$

The dependence on the extrinsic geometry of B is now captured by the spacetime covariant derivative $\nabla_L\xi^a$.

To summarize, in (4.29) we have rewritten the null Raychaudhuri and the Damour equations as a conservation law on the null surface B, equating the divergence of the gravity boundary current (4.28) to the matter energy-momentum flux $T_{\xi L}$ plus a gravitational flux of the canonical form $\sum_i P_i\mathfrak{L}_\xi Q_i$.

4.3 Charges from Symplectic Potential

We will now give a more canonical derivation of the conservation equation, starting from the explicit expression for the null gravity symplectic potential in terms of the intrinsic and extrinsic geometry of B derived in chapter 3, and using technology from the covariant Hamiltonian formalism.

In chapter 2, we saw that the Noether charge density satisfies (2.4),

$$I_\xi \theta_0 - \iota_\xi L = C_\xi + \mathrm{d} J^0_\xi. \tag{4.34}$$

Here, we will write θ_0 for the standard symplectic potential (4.38), whose pullback Θ_B was computed in chapter 3, and J^0_ξ for the associated Noether charge aspect.

Pulling back onto B and contracting with a vector ξ tangential to B, the term $\iota_\xi L$ and the cosmological constant contribution to the constraints do not contribute, and if the canonical and gravitational matter energy-momentum agree, we get

$$\boxed{\xi^a G_{aL} \epsilon_{\mathrm{B}} = \jmath^*_B(\mathrm{d} J^0_\xi - I_\xi \Theta_B).} \tag{4.35}$$

Since the symplectic potential contains the terms $\Theta \overset{B}{=} P\delta Q$, we expect that $I_\xi \Theta = P \mathfrak{L}_\xi Q$ reproduces the flux terms of the last section. Then, we can identify the pullback of the Noether charge aspect $\jmath^*_B(J^0_\xi)$ with the boundary current of the conservation law (4.29), and (4.35) and (4.29) become the same canonical conservation equation.

4.3.1 Null Symplectic Potential and Intrinsic Symplectic Potential

The Einstein-Hilbert Lagrangian leads to the well-known standard symplectic potential current

$$\hat{\Theta}[g, \delta g] = \tfrac{1}{2} \nabla_b (\delta g^{ab} - g^{ab} \delta g) \epsilon_a. \tag{4.36}$$

The standard symplectic potential current is covariant in the sense that it does not make reference to any background structure. Hence, its anomaly vanishes: $\Delta_\xi \hat{\Theta} = 0$ for all ξ. We recall that the anomaly is defined on field-space forms such as Θ as

$$\Delta_\xi = \mathfrak{L}_\xi - \mathcal{L}_\xi - I_{\delta \xi}, \tag{4.37}$$

and the field space Lie derivative acts on field space forms via the Cartan formula $\mathfrak{L}_\xi = \delta I_\xi + I_\xi \delta$.

In chapter 3, we rewrote the pullback of θ_0 onto the null surface B along the embedding $\jmath_B : B \hookrightarrow M$ in terms of the intrinsic and extrinsic geometry of B. There we took the view that the configuration variables should not contain derivatives of the metric. However, in (4.29), the derivative $\mathfrak{L}_\xi \mu$ appears. Thus, in order to interpret the conservation law canonically, we will change the polarization so that the symplectic potential contains $\delta\mu$. A further argument favoring that polarization is given in section 4.4.

In appendix B.1 we write the symplectic potential $\Theta_B = \jmath_B^* \theta_0$ as the sum of a bulk and a corner term:

$$\Theta_B = \Theta_{\rm int} - {\rm d}\alpha_S \tag{4.38}$$

The bulk term is

$$\boxed{\Theta_{\rm int} = \left(\tfrac{1}{2}\delta\gamma_{ab}\sigma^{ab} - \bar{\eta}_a \delta L^a + \delta\mu\right)\epsilon_{\rm B}} \tag{4.39}$$

We recognize the canonical Ps and Qs appearing in the flux terms of the conservation law (4.29). The boundary contribution is

$$\alpha_S = \tfrac{1}{2}\left(\delta h\, \epsilon_{\rm S} + \imath_{\delta L}\epsilon_{\rm B}\right) - \tfrac{1}{D-2}\delta\epsilon_{\rm S}. \tag{4.40}$$

Recall that e^h is the scale of the normal metric, $e^h = \sqrt{|g|}/\sqrt{q}$. The expression (4.38) is valid for variations δg_{ab} of the metric that keep the surface B null, i.e. such that $\delta\beta \overset{B}{=} 0$ in the parametrization (3.10).

The boundary contribution can be rewritten as follows: using $\epsilon_{\rm S} = \iota_L \epsilon_{\rm B}$ and $\delta\epsilon_{\rm B} = (D-2)\delta\varphi\epsilon_{\rm B}$ we have $\delta\epsilon_{\rm S} = \imath_{\delta L}\epsilon_{\rm B} + (D-2)\delta\varphi\epsilon_{\rm S}$, so

$$\alpha_S = \frac{1}{2}\left(\delta h - 2\delta\varphi\right)\,\epsilon_{\rm S} + \tfrac{1}{2}\tfrac{D-4}{D-2}\,\imath_{\delta L}\epsilon_{\rm B}. \tag{4.41}$$

It is interesting that the combination $h - 2\varphi$ is invariant under local rescaling of the metric, and that the extra contribution vanishes in dimension $D = 4$.

A further possible rewriting is

$$\alpha_S = \left(\frac{1}{2}L^a \delta g_{ac} g^{bc} + \delta L^b\right)\iota_b \epsilon_{\rm B} - \tfrac{1}{D-2}\delta(\epsilon_{\rm S}). \tag{4.42}$$

To pass from (4.40) to the last line, use that $L^a \delta g_{ac} g^{bc} = \delta(L_c) g^{bc} - \delta(L^b)$, and $\delta(L_c) \overset{B}{=} \delta h L_c$.

As we have seen the symplectic potential current is defined up to the addition of a closed form. That means that the corner term α_S can be removed by exploiting the ambiguity. We define the *intrinsic symplectic potential current* on B as

$$\boxed{\Theta_{\text{int}} := \hat{\Theta}_B + \mathrm{d}\alpha_S.} \tag{4.43}$$

Its explicit form is given in (4.39). This choice fixes the closed ambiguity in the symplectic potential in such a way that the modified symplectic potential has no boundary pairs. Our symplectic potential current contains only the intrinsic geometry $\{\gamma_{ab}, L^a, \varphi\}$ and extrinsic geometry $\{\sigma^{ab}, \bar{\eta}_a, \mu\}$ of B. This is in contrast to the usual expression (4.38) that also contains δh, which fits in neither of these categories. We choose the name "intrinsic" because as we saw in section 4.1.2, the transformation under diffeomorphisms of the data contained in the intrinsic symplectic potential does not depend on how the diffeomorphism is extended outside B. Again, h is not intrinsic in this sense.

Let us take a closer look at the three terms that appear in the boundary contribution (4.40). Removing the first term $-\frac{1}{2}\mathrm{d}(\delta h \epsilon_{\mathrm{S}})$ from Θ_B is central to our analysis, as we will see in the next subsection. The second term $-\frac{1}{2}\mathrm{d}(\iota_{\delta L}\epsilon_{\mathrm{B}})$ does not enter the integral $\int_B \Theta_B$ if the boundaries ∂B are aligned with the foliation S, since δL is parallel to S. Its removal thus does not influence the boundary current integrated on S, but rather modifies the parts of the boundary current that vanish when pulled back to S. Lastly, the term $\mathrm{d}\frac{1}{D-2}\delta\epsilon_{\mathrm{S}}$ is both a total derivative and a total variation, it could thus also be understood as arising from a codimension two corner action proportional to the corner area. Removing it does not change the symplectic form.

Having fixed the ambiguity of the symplectic potential current, we can now evaluate the flux term $I_\xi \Theta$ on the RHS of (4.35). It becomes

$$\boxed{I_\xi \Theta_{\text{int}} = \left(\tfrac{1}{2}\sigma^{AB}\mathfrak{L}_\xi \gamma_{AB} - \bar{\eta}_a \mathfrak{L}_\xi L^a + \mathfrak{L}_\xi \mu\right)\epsilon_{\mathrm{B}},} \tag{4.44}$$

which coincides with the flux term from the conservation equation (4.29).

4.3.2 Noether Charge and Conservation Law

Let us now turn to the boundary current term on the RHS of (4.35). To evaluate it, we start from the boundary current J^0_ξ of the standard symplectic potential (4.36). J^0_ξ is the well-known Komar charge form [97], which is a $(D-2)$-form on spacetime M, and given by

$$J^0_\xi = \tfrac{1}{2} * \mathrm{d}g(\xi) = \tfrac{1}{2}\epsilon_{ab}\nabla^a\xi^b, \tag{4.45}$$

where $\epsilon_{ab} = \iota_a\iota_b\epsilon$. Pulling back onto S and parameterizing $\xi = fL + \bar{f}\bar{L} + v$, one gets

$$i^*_S(J^0_\xi) = \tfrac{1}{2}((L+\kappa)[f] - (\bar{L}+\bar{\kappa})[\bar{f}] + (\bar{\eta}_a - \eta_a)v^a)\mathrm{d}S, \tag{4.46}$$

where as before $\eta_a = -q_a{}^b\nabla_{\bar{L}}L_b$ and $\bar{\kappa} = L_a\nabla_{\bar{L}}\bar{L}^a$. The Komar charge form has the advantage of being simple, and covariant under all diffeomorphisms. However, it possesses two features that make it unsatisfactory for the analysis of conservation laws along a null hypersurface B: It depends not only on ξ as a vector field on B, but on its extension outside of B through the transverse derivative $\bar{L}[\bar{f}]$. Therefore even a vector field which vanishes on B may have non-zero charge. We also see that in addition to the variables $\kappa, \bar{\eta}_a$ that form part of the extrinsic geometry of B as embedded in M, $\hat{J}$ involves the variables $\bar{\kappa}$ and η, which cannot be interpreted in terms of the intrinsic or extrinsic geometry of B. In trying to describe physics from the viewpoint of the null surface B, both of these features are undesirable.

We now show directly that the Noether boundary J^{int}_ξ associated with the intrinsic symplectic potential Θ_{int} of (4.39) resolves both issues affecting the Komar boundary current: The boundary current J^{int}_ξ does not contain derivatives of ξ transverse to B, so it vanishes if ξ vanishes on B. Furthermore, J^{int}_ξ is entirely determined by the intrinsic and extrinsic geometry of B. We also show that J^{int}_ξ coincides, up to a total differential, with the boundary current (4.28) which we found from analyzing the constraints.

By (2.5), the boundary current J_ξ of the modified symplectic potential is related to the Komar charge form $\hat{J}_\xi$ as

$$J^{\mathrm{int}}_\xi = J^0_\xi + I_\xi\alpha_S. \tag{4.47}$$

The core reason ensuring the properties of J^{int}_ξ is that we removed δh from the symplectic potential current. Using the transformation of h given in (4.12), we have

$$I_\xi\big(\tfrac{1}{2}\delta h\mathrm{d}S\big) = \tfrac{1}{2}\big((L+\kappa)[f] + (\bar{L}+\bar{\kappa})[\bar{f}] + (\bar{\eta}_a + \eta_a)v^a\big)\mathrm{d}S. \tag{4.48}$$

It is then clear that adding this term to (4.46) removes both the transverse derivative acting on $\bar{f}$ as well as the dependence on $\bar{\kappa}$ and η_a.

In detail, J^{int}_ξ is obtained as follows: As a form on B, the Komar charge form reads

$$\jmath^*_B(J^0_\xi) = \tfrac{1}{2}L_a(\nabla^a\xi^b - \nabla^b\xi^a)\iota_b\epsilon_{\mathrm{B}}, \tag{4.49}$$

where we used $\jmath_B^*(\epsilon_{ab}) = L_a(\iota_b\epsilon_{\rm B}) - L_b(\iota_a\epsilon_{\rm B})$.

The expression $I_\xi\alpha_S$ is computed most efficiently from the form (4.42) for α_S. Recall from 4.1.2 that for $\xi = fL + v$ we have

$$\begin{aligned} I_\xi\delta\epsilon_{\rm S} &= \mathcal{L}_\xi\epsilon_{\rm S} = \iota_\xi(\theta\epsilon_{\rm B}) + {\rm d}(\iota_\xi\epsilon_{\rm S}) \\ I_\xi\delta L^a &= [v, L]^a. \end{aligned} \tag{4.50}$$

Using also $I_\xi\delta g_{ab} = \nabla_a\xi_b + \nabla_b\xi_a$, altogether we get

$$I_\xi\alpha_S = \frac{1}{2}L_a(\nabla^a\xi^b + \nabla^b\xi^a)\iota_b\epsilon_{\rm B} + [v, L]^a\iota_a\epsilon_{\rm B} - \tfrac{1}{D-2}(\iota_\xi\theta\epsilon_{\rm B} + {\rm d}\iota_\xi\epsilon_{\rm S}). \tag{4.51}$$

Adding (4.49) and (4.51) yields the Noether boundary current $J_\xi^{\rm int} = J_\xi^0 + I_\xi\alpha_S$:

$$\boxed{J_\xi^{\rm int} = \big((\nabla_L\xi^a) + [v, L]^a\big)\iota_a\epsilon_{\rm B} - \tfrac{1}{D-2}\iota_\xi\theta\epsilon_{\rm B} - {\rm d}(\tfrac{1}{D-2}\iota_\xi\epsilon_{\rm S})} \tag{4.52}$$

As claimed, this coincides with the boundary current (4.28) that we found analyzing the constraints, up to the total derivative term ${\rm d}(\frac{1}{D-2}\iota_\xi\epsilon_{\rm S})$, which is within the ambiguity of both prescriptions, since both Js are defined implicitly by specifying ${\rm d}J$, and vanishes when integrated on any closed surface.

Let us reiterate the results of this section: The general identity (4.35) on B reads

$$G_{\xi L}\epsilon_{\rm B} = {\rm d}J_\xi - I_\xi\Theta_{\rm int}. \tag{4.53}$$

This equation coincides exactly with the conservation equation (4.29) for the edge mode current — if we use the intrinsic symplectic potential current $\Theta_{\rm int}$ of (4.39), which differs from the standard one by a total derivative and has no corner pairs. The gravitational flux terms are given by $I_\xi\Theta_{\rm int}$ and are of the same form as the canonical energy-momentum of matter, i.e., $P\mathfrak{L}_\xi Q$. The boundary current J_ξ is (essentially) given by the Noether boundary current of the intrinsic symplectic potential current $\Theta_{\rm int}$. As a further consequence of modifying the symplectic potential current, everything is expressed in terms of the intrinsic and extrinsic geometry of the null surface B, and independent of the extension of the vector field ξ outside of B.

4.4 Hamiltonians

What makes conserved charges interesting, especially in the quantum theory, is that they are the generators, i.e., the Hamiltonians, of symmetries. We now turn to the question how the charge $\int_{\partial B} J_\xi$ is related to the Hamiltonian generating the infinitesimal transformation ξ. See [105] for a related discussion. Some related results have also appeared in [69], focusing mainly on isolated horizon boundary conditions.

We saw in section (2.4) that in our present situation with a non-covariant corner modification, the Hamiltonian and the Noether charge aspect are related as

$$-I_\xi \Omega \hat{=} \int_{\partial B} \Big(\delta(J_\xi) - J_{\delta\xi} - \iota_\xi \Theta_{\rm int} - \Delta_\xi \alpha_S \Big). \tag{4.54}$$

We will thus need the anomaly of the modification $\alpha_S = \frac{1}{2} L_a \delta g^{ab} \iota_b \epsilon_{\rm B} + \delta L^a \iota_a \epsilon_{\rm B} - \frac{1}{D-2} \delta\epsilon_{\rm S}$. The anomalies of $\delta\epsilon_{\rm S}$, of δg^{ab} and of $L_a \epsilon_{\rm B}$ vanish for $\xi \parallel B$, so we are left with

$$\Delta_\xi \alpha_S = (\mathfrak{L}_\xi - \mathcal{L}_\xi - I_{\delta\xi})(\delta L^a \iota_a \epsilon_{\rm B}) \tag{4.55}$$

$$= (\Delta_\xi \delta L^a) \iota_a \epsilon_{\rm B} + \delta L^a \iota_a (\Delta_\xi \epsilon_{\rm B}), \tag{4.56}$$

where for the second line we have used that Δ_ξ satisfies the Leibniz rule. Now use that $\delta \Delta_\xi = \Delta_\xi \delta + \Delta_{\delta\xi}$. Using the results $\Delta_\xi L^a = L[f] L^a$ and $\Delta_\xi \epsilon_{\rm B} = -L[f]\epsilon_{\rm B}$, as well as $\Delta_{\delta\xi} L^a = L[\delta f] L^a$, one obtains

$$\Delta_\xi \alpha_S = (\delta L)[f] \epsilon_{\rm S}, \tag{4.57}$$

where as before $\xi = fL + v$ with $v \parallel S$.

To proceed further, let us fix the metric dependence of ξ, and set $\xi = fL + v$ with $\delta f = 0$ and $\delta v = 0$, such that $\delta\xi = f\delta L$. Intuitively this means that the direction of ξ relative to the null generators is fixed, or from a fluid perspective that ξ is fixed in a "co-moving" frame. As before, we also assume that the boundaries ∂B are aligned with the cross-sections S. Using that $i_S^*(\iota_\xi \epsilon_{\rm B}) = f \mathrm{d}S$, the ingredients are[2]:

$$\begin{aligned} i_S^* \delta(J_\xi) &= \delta\Big(\mathrm{d}S \big((-\theta + \mu) f + L[f] + v^a \bar{\eta}_a \big) \Big) \\ -i_S^*(J_{\delta\xi}) &= -\, f \delta L^a \bar{\eta}_a \mathrm{d}S \\ -i_S^*(\iota_\xi \Theta_{\rm int}) &= -\, f \left(\frac{1}{2} \sigma^{ab} \delta\gamma_{ab} - \delta L^a \bar{\eta}_a + \delta\mu \right) \mathrm{d}S \\ -i_S^*(\Delta_\xi \alpha_S) &= -\, \delta L[f] \mathrm{d}S. \end{aligned} \tag{4.58}$$

Let us denote $\mu_f = f\mu + L[f]$. μ_f can be understood as the spin-0 momentum in the frame of the observer ξ. Noting $\delta\mu_f = f\delta\mu + \delta L[f]$ we get the following, remarkably compact

[2]Recall that $\mathrm{d}S$ is the induced volume element on a cross-section S of B.

result:

$$\boxed{-I_\xi \Omega = \delta\left(\int_{\partial B} J_\xi\right) - \int_{\partial B}\left(\tfrac{1}{2} f \sigma^{ab}\delta\gamma_{ab} + \delta\mu_f\right)\mathrm{d}S.} \tag{4.59}$$

The transformation $\mathfrak{L}_\xi$ is a Hamiltonian symmetry with Hamiltonian H_ξ if and only if the RHS is a total variation δH_ξ. Due to the presence of the second term we see that boundary conditions are needed to ensure the existence of a Hamiltonian. The simple form of (4.59) is a consequence of using the modified symplectic structure which has the corner pair (h, ϵ_S) removed. Similar equations have also been given in [69], and from a first order perspective in [47].

Let us analyze the result (4.59) for some different cases. First consider a "superrotation-like" transformation, i.e., a vector field v which is parallel to the foliation S to first order around ∂B. Then, no boundary conditions are needed, and the Hamiltonian is just the charge:

$$H_v = \int_{\partial B} J_v = \int_{\partial B} v^a \bar{\eta}_a \epsilon_\mathrm{S}. \tag{4.60}$$

The charge, the momentum conjugate to the null directions L and the Hamiltonian coincide. This simple situation is a consequence of using the modified symplectic potential without boundary pairs.

Next, consider null dilatations that "stretch" the null surface in the null direction at its corners, i.e., $\xi = fL$ with $f = 0$ at ∂B. We get $\delta\mu_f = (\delta L)[f] = 0$, since δL is parallel to ∂B, so

$$H_{fL} = \int_{\partial B} L[f]\epsilon_\mathrm{S}. \tag{4.61}$$

The null dilatations are thus generated by the corner area element, they are Hamiltonian symmetries even if no boundary conditions are fixed.

The case of null translations $\xi = fL$ with f non-vanishing at the corners ∂B is more subtle. Boundary conditions are needed to ensure the existence of a Hamiltonian. The boundary conditions can be split up into conditions on the pair $(\sigma^{ab}, \gamma_{ab})$ and the pair (ϵ_S, μ_f). No boundary conditions are needed for the spin-1 pair $(L^a, \bar{\eta}_a)$, this is because we have chosen ξ to vary with L.

For the spin-2 pair $(\sigma^{ab}, \gamma_{ab})$, a possible boundary condition is fixing the shear $\sigma^{ab} = 0$ at ∂B. That is done, e.g., at isolated, Killing and conformal Killing horizons, and in the far past and far future of future null infinity. More generally we can impose at the boundary of B any relationship of the form $\sigma_{ab} = F(\gamma_{ab})$. Alternatively, one can fix the conformal metric to be the conformal metric of the unit sphere such that $\delta\gamma_{ab} = 0$. Note that in four spacetime dimensions, ∂B is two-dimensional. If it has spherical topology, every metric is diffeomorphic to a metric conformal to the unit sphere metric. Thus, fixing γ_{ab} can be interpreted as a condition on the coordinates, rather than on the metric degrees of freedom. The residual transformations preserving the condition are the conformal Killing vectors of the unit sphere.

For the spin-0 pair $\epsilon_S \delta\mu_f$, one possible boundary condition is fixing the area element such that the term becomes the total variation $\delta(\epsilon_S \mu_f)$. This leads to a Hamiltonian for null translations

$$H_\xi^{\text{area}} = -\int_{\partial B} \theta f \epsilon_{\text{S}}. \tag{4.62}$$

Under boundary conditions fixing the area element of the corner, the generator of translations along L is minus the expansion.[3] See [106] for related results. Fixing the area element can also be viewed as a condition on the location of the spheres ∂B, rather than a condition on the metric, such as in Bondi gauge at null infinity. A null translation then has to be accompanied by a radial diffeomorphism to restore the size of the spheres.

As a more general spin-0 boundary condition, one could provide a constitutive relation linking ϵ_S and μ_f. This situation arises in black hole thermodynamics [67], where (4.59) becomes the "Hamiltonian first law" of black hole thermodynamics.

As another spin-0 boundary condition, one can fix μ_f. We remind the reader that $\mu = \kappa + \frac{D-3}{D-2}\theta$ and $\mu_f = f\mu + L[f]$. For an isolated horizon, where the expansion vanishes, this conditions amounts to fixing the horizon "temperature" κ.

Since any value for μ_f can be reached by choice of the coordinate field ϕ^0 or by choosing the coefficient f, fixing μ_f can be interpreted as a condition on the clock ϕ^0 or the vector field ξ rather than on the metric degrees of freedom. The most obvious choice is fixing $\mu_f = c$ with a fixed constant c. The residual transformations preserving this condition satisfy $L[f]/f = -c$. The null translation Hamiltonian for fixed μ_f becomes

$$H_\xi = \int_{\partial B} (\mu_f - \theta f)\epsilon_S. \tag{4.63}$$

It coincides with the energy aspect (4.30), which we found by analyzing the constraints.

[3] However, ξ does not preserve this condition unless $\theta = 0$. It is thus an outer symmetry.

Since fixing μ_f gives a condition on how the coordinates are extended around the corners ∂B, while fixing the area element ϵ_S requires moving the corners, from the viewpoint of a null surface at finite distance fixing μ_f seems a more natural condition than fixing ϵ_S. It is well known that boundary conditions are linked to a choice of boundary action: the symplectic potential of the full action should be made to vanish by the boundary conditions. As we saw, the symplectic potential of the Einstein-Hilbert action contains $\delta\mu\epsilon_{\mathrm{B}}$, which vanishes when μ is fixed. If a Gibbons-Hawking like null boundary action containing $\int_B \mu\epsilon_{\mathrm{B}}$ is added, the term in the symplectic potential becomes $-(\delta\epsilon_{\mathrm{B}})\mu$, which vanishes if $\mathrm{d}S$ is fixed. From the perspective of a null hypersurface at finite distance, it thus seems more natural to work with the pure Einstein-Hilbert Lagrangian, rather than adding a "null Gibbons-Hawking" boundary action to switch to the metric polarization.

Different conditions on the spin-0 sector have appeared in the literature. The "time" ϕ^0 can be linked to the total area of the cross-section S at ϕ^0 as in [56], which fixes the expansion θ. One can use an affine parameter along the null geodesics, which fixes $\kappa = 0$. As stated earlier, the combination $\kappa - \frac{1}{D-2}\theta = \mu - \theta$ can be set zero to simplify the Raychaudhuri equation [53]. For a generic expanding null surface, the condition $\mu_f = 0$ is different from all of those.

To summarize, we asked for which symmetries ξ and under which boundary conditions there exists a Hamiltonian generating the symmetry, using the intrinsic symplectic form. For spatial transformations $\xi^a = v^a$, a Hamiltonian always exists and is given by the twist field $\bar{\eta}_a$. Null dilatations are generated by the area element ϵ_S. For null translations, spin-2 and spin-0 boundary conditions are needed for the existence of a Hamiltonian. The most natural spin-0 boundary condition seems to be fixing the spin-0 momentum μ, and the resulting Hamiltonian is the energy aspect (4.30).

Part II

Canonical Structure at Null Infinity

Chapter 5

Asymptotic Renormalization of the Canonical Structure of Electromagnetism in Flat Space

In this chapter, we present a general scheme to remove the divergences in the radius of the asymptotic symplectic potential (SP) and canonical charges of electromagnetism on flat space in $D \geq 5$ dimensions.

Section 5.1 contains the definitions of the conformally compactified spacetime, asymptotic simplicity and a radial vector field. The equations of motion are analyzed in general terms in section 5.2. The SP is then renormalized in section 5.3, and the asymptotic equations of motion are analyzed in detail in 5.4 to identify the free data. Section 5.5 goes towards writing the renormalized SP in terms of the free data. Finally, section 5.6 calculates the canonical generators, comments on their layer structure, and connects them to previous work on soft theorems in higher dimensional electromagnetism.

5.1 Spacetime Structure

This section lays out the basic spacetime structures used in the remainder. We consider vacuum Maxwell theory on Minkowski spacetime of spacetime dimension $D \geq 5$.

We will find useful to work in an auxiliary spacetime, obtained through a conformal compactification of Minkowski spacetime à la Penrose [107, 108, 109]. The key advantage of this approach is that asymptotic infinity presents itself as a boundary at finite coordinate distance in the unphysical spacetime. The structure of infinity is reflected in the behavior of the conformally rescaled fields near this boundary. To avoid technicalities, we restrict our analysis to a coordinate patch, that of retarded Bondi coordinates, which covers only future null infinity.

The Minkowski metric in retarded Bondi coordinates, with $u = t - r$, reads $\hat{g}_{ab}\mathrm{d}x^a\mathrm{d}x^b = -\mathrm{d}u^2 - 2\mathrm{d}u\mathrm{d}r + r^2 q_{AB}\mathrm{d}x^A\mathrm{d}x^B$, q_{AB} being the metric of a unit round $(D-2)$-sphere. We introduce the coordinate $\Omega = 1/r$, and work in the conformally compactified spacetime with the rescaled metric $g_{ab} := \Omega^2\hat{g}_{ab}$ and inverse $g^{ab} = \Omega^{-2}\hat{g}^{ab}$:

$$g_{ab}\mathrm{d}x^a\mathrm{d}x^b = -\Omega^2\mathrm{d}u^2 + 2\mathrm{d}u\mathrm{d}\Omega + q_{AB}\mathrm{d}x^A\mathrm{d}x^B \tag{5.1a}$$

$$g^{ab}\partial_a\partial_b = \Omega^2\partial_\Omega^2 + 2\partial_\Omega\partial_u + q^{AB}\partial_A\partial_B. \tag{5.1b}$$

All indices will be contracted with respect to this metric unless otherwise specified by the use of hats, $\hat{\cdot}$. Note that in these coordinates $\sqrt{|g|} = \sqrt{q} = \Omega^D\sqrt{\hat{g}}$. In these coordinates, future null infinity $\mathscr{I}^+ \cong S^{D-2}\times\mathbb{R}$ (we will drop the plus in the following) is located at $\Omega = 0$, corresponding to the limit at $r \to \infty$ of the timelike level surfaces of r. We define the conormal to the surfaces at $\Omega =$ const.,

$$N_a = \partial_a\Omega. \tag{5.2}$$

The normal N_a becomes null in the unphysical metric g at $\Omega = 0$. Note that N_a is the inward-pointing normal, this will lead to a sign in the Stokes theorem. Some of the equations in the following will be simplified by the introduction of the normalized normal, which has unit modulus with respect to the conformal metric g_{ab}:

$$n_a = \frac{1}{\Omega}N_a. \tag{5.3}$$

Working with the coordinate Ω and the metric g_{ab}, rather than r and $\hat{g}_{ab}$, is useful because the components of g_{ab} in the coordinates (Ω, u, x^A) are asymptotically finite. This framework will also automatically provide natural fall-offs for the fields, and allow for a systematic analysis of the finiteness of asymptotic canonical quantities.

The Bondi coordinates determine a coordinate vector field ∂_Ω, which is defined throughout the spacetime. This vector field will play a crucial role in the following. At $\mathscr{I}$, ∂_Ω can be used to "take orders in $1/r$" of tensors.

Let $x^i = (u, x^A)$ be the coordinates on $\Omega = \text{const.}$ surfaces. The retarded Bondi coordinates (Ω, u, x^A) determine a coordinate projector P^i_a, which maps spacetime vectors to vectors on $\Omega = \text{const.}$ by dropping their ∂_Ω component. Vectors may then be decomposed as

$$V^a \partial_a = V^i \partial_i + (N_a V^a)\partial_\Omega. \tag{5.4}$$

We will suppress the projector in the notation. For example $F^{ij} = P^i_a P^j_b g^{aa'} g^{bb'} F_{a'b'}$ is the projection of the field strength with raised spacetime indices, and not the pulled back field strength with indices raised by the inverse of the induced metric (which does not have a finite limit since $\mathscr{I}$ is null in the conformal metric). Because $g^{u\Omega} = 1$, F^{ij} contains $F_\Omega{}^j$, and depends not only on the pullback of the gauge potential A_i but also on A_Ω and the transverse derivatives of A_i. The projector commutes with coordinate derivatives, including ∂_Ω.

We will focus on finite intervals in the retarded time u, and will not consider any possible divergences in u. $I = I(\Omega = 0) \subset \mathscr{I}$ will denote the limit of the following hypersurfaces (with boundary)

$$I(\omega) = \{(u, \Omega, x^A) : \Omega = \omega, u_i \le u \le u_f\} \quad \text{with} \quad \partial I(\omega) = S_i(\omega) \sqcup S_f(\omega), \tag{5.5}$$

where $S_i(\omega)$ ($S_f(\omega)$) is a codimension-2 sphere obtained as the cut of the hypersurface $\Omega = \omega$ at $u = u_i$, ($u = u_f$), respectively.

5.2 Equations of Motion: Asymptotic Simplicity and the Conformal Current

Utilizing Penrose's idea of asymptotic simplicity, we will assume that the components of the gauge field A_a in the coordinates (u, Ω, x^A) have finite values at $\mathscr{I}$ and admit an expansion in powers and logarithms of Ω:

$$A_a = \sum_{k=0} A_{a(k)} \Omega^k + \sum_{k=0, l=1} A_{a(k,l)} \Omega^k (\ln \Omega)^l. \tag{5.6}$$

We will make further assumptions on the coefficients $A_{a(k,l)}$ of the logarithmic terms as we go along, and summarize them in subsection 5.3.4.

The Lagrangian of vacuum electrodynamics is

$$\hat{\mathbf{L}} := -\frac{1}{4}\sqrt{\hat{g}}\hat{F}^{ab}\hat{F}_{ab} = \Omega^{-(D-4)}\mathbf{L}, \quad \text{where} \quad \mathbf{L} := -\frac{1}{4}\sqrt{q}F^{ab}F_{ab}. \tag{5.7}$$

Since $A_a \equiv \hat{A}_a$, we get $\hat{F}_{ab} \equiv F_{ab} = \partial_a A_b - \partial_b A_a$ and $\hat{F}^{ab} = \hat{g}^{aa'}\hat{g}^{bb'}F_{a'b'} = \Omega^4 F^{ab}$. We assume that $\mathbf{L}$ has a finite limit onto $\mathscr{I}$.

Varying $\hat{\mathbf{L}}$ with respect to A_a gives[1]

$$\delta\hat{\mathbf{L}} = \hat{\mathbf{E}}^a \delta A_a + \partial_a \hat{\boldsymbol{\theta}}^a \tag{5.8}$$

where $\hat{\boldsymbol{\theta}}^a$ is the SP current density, which we will come back to shortly, and $\hat{\mathbf{E}}^a$ are the equations of motion (EoM):

$$\hat{\mathbf{E}}^a = \partial_b(\sqrt{\hat{g}}\hat{F}^{ba}) \tag{5.9a}$$

$$= \Omega^{-(D-4)}\Big(\partial_b(\sqrt{q}F^{ba}) - (D-4)\frac{1}{\Omega}N_b F^{ba}\Big). \tag{5.9b}$$

We assume that $\partial_b(\sqrt{q}F^{ba})$ has a finite limit at large r. Then, in $D \neq 4$, the dominant asymptotic order of the equations of motion comes from the second term. The dominant order of the equations of motion is hence solved by requiring that $N_b F^{ba}$ is of order Ω. We call these the *asymptotic Maxwell conditions*

$$(D-4)N_b F^{ba} \overset{\mathscr{I}}{=} 0, \tag{5.10}$$

and will require that they are implemented as a restriction on the field space itself.

The asymptotic Maxwell conditions allow us to define what we call the *conformal current* as

$$\mathcal{J}^a := \frac{1}{\Omega}N_b F^{ba} \equiv F^{na}, \tag{5.11}$$

where we recall $n_a = \Omega^{-1}N_a = \Omega^{-1}\partial_a\Omega$. The conformal current is defined in a finite neighbourhood of $\mathscr{I}$, not just on $\mathscr{I}$ itself or order by order. By the antisymmetry of F, it is tangential to the level surfaces of Ω. By the asymptotic Maxwell conditions, it has a finite limit onto $\mathscr{I}$.

[1] We use boldface letters for forms and "hats" for unrescaled quantities referring to the physical spacetime. Hatted quantities can diverge in the limit $\Omega \to 0$, while bareheaded quantities are defined so that they will not. Geometrically, $\mathbf{E}^a$ and $\hat{\boldsymbol{\theta}}^a$ are codimension-1 forms, or current densities.

We can then rescale the EoM to remove negative powers of Ω, obtaining

$$\mathbf{E}^a := \Omega^{(D-4)}\hat{\mathbf{E}}^a = \partial_b(\sqrt{q}F^{ba}) - (D-4)\sqrt{q}\mathcal{J}^a. \tag{5.12}$$

In $D \geq 5$, the vacuum EoM take the form of Maxwell equations in presence of an external source: the conformal current.[2] The origin of the conformal current is the fact that the EoM transform inhomogeneously under the conformal rescaling of the metric, or alternatively, that in the conformal frame the Lagrangian is non-minimally coupled to a background scalar field Ω. The normal component of the EoM reads

$$\mathbf{E}^n = -\partial_i(\sqrt{q}\mathcal{J}^i), \tag{5.13}$$

The conformal current is thus conserved on-shell. This concludes the analysis of the EoM for now, we will come back to them in more explicit detail in section 5.4.

5.3 Renormalizing the Symplectic Potential

5.3.1 The Standard Symplectic Potential and its Ambiguities

The SP current density $\hat{\boldsymbol{\theta}}^a$ determines the canonical structure of the theory. In the covariant Hamiltonian formalism [110, 83, 111, 91], which we use here, it is related to the Lagrangian through the equation (5.8), which is usually taken to imply

$$\hat{\boldsymbol{\theta}}^a = \Omega^{-(D-4)}\boldsymbol{\theta}^a \qquad \text{where} \qquad \boldsymbol{\theta}^a := -\sqrt{q}F^{ab}\delta A_b. \tag{5.14}$$

We refer to $\hat{\boldsymbol{\theta}}^a$ as the *standard SP*. Its normal component, which determines the standard symplectic form on $I(\Omega)$, is

$$\hat{\boldsymbol{\theta}}^\Omega = \Omega^{-(D-5)}\boldsymbol{\theta}^n \qquad \text{where} \qquad \boldsymbol{\theta}^n = -\sqrt{q}\mathcal{J}^i\delta A_i. \tag{5.15}$$

The symplectic form on the $\Omega =$ const. surfaces is the integral of the (antisymmetrized) variation of $\hat{\boldsymbol{\theta}}^\Omega$.[3] Since N_a is the inward facing normal, the integration comes with a sign. Hence, the contribution to the symplectic form from an interval $I(\Omega)$ is

$$\omega(\Omega) := \delta\hat{\Theta}(\Omega) \tag{5.16}$$

[2]To avoid specifying further asymptotic properties, we neglect any matter contribution to the current. It seems however natural to require that the conformal current is a well defined quantity at $\mathscr{I}$ even in presence of matter.

[3]In the language of differential forms, of the pullback of $\hat{\boldsymbol{\theta}}$ onto $I(\Omega)$.

where

$$\hat{\Theta}(\Omega) := -\int_{I(\Omega)} \hat{\boldsymbol{\theta}}^{\Omega} = -\Omega^{-(D-5)} \int_{I(\Omega)} \boldsymbol{\theta}^{n}. \tag{5.17}$$

Notice that since Ω has a double role as "canonical time" for the radial evolution and as the conformal factor, the conformal current appears both as a source term in the EoM and in the SP as the momentum canonically conjugate to the tangential connection A_i . $\boldsymbol{\theta}^n$ has a finite, non-zero limit onto $\mathscr{I}$.

In $D \geq 6$, the SP on the level surfaces of Ω diverges as $\Omega^{-(D-5)}$ when approaching $\mathscr{I}^+$. The divergence seems like bad news for the canonical theory, signifying potentially infinite Hamiltonians, infinite charge generators and ill-defined Poisson brackets. However, as is well known [93], the SP is ambiguous. Firstly, adding a boundary term to the action adds a total variation to the SP (which does not change the symplectic form). Secondly, since $\hat{\boldsymbol{\theta}}^a$ is defined only implicitly through (5.8), it is ambiguous by the divergence of an antisymmetric tensor. The ambiguities are

$$\hat{\boldsymbol{\theta}}^a \mapsto \hat{\boldsymbol{\theta}}^a + \partial_b \hat{\boldsymbol{\alpha}}^{ab} + \delta \hat{\boldsymbol{\beta}}^a \tag{5.18a}$$

and thus

$$\hat{\boldsymbol{\theta}}^{\Omega} \mapsto \hat{\boldsymbol{\theta}}^{\Omega} + \partial_i \hat{\boldsymbol{\alpha}}^{\Omega i} + \delta \hat{\boldsymbol{\beta}}^{\Omega}. \tag{5.18b}$$

Here, $\hat{\boldsymbol{\alpha}}^{ab} = \hat{\boldsymbol{\alpha}}^{[ab]}$ is the *corner counterterm*. It is a $(D-2)$-form and it modifies the canonical expression of the charges. $\hat{\boldsymbol{\beta}}^a$ is a change of polarization coming from a choice of *boundary action*, $\hat{\mathbf{L}} \mapsto \hat{\mathbf{L}} + \partial_a \hat{\boldsymbol{\beta}}^a$.

We can now phrase the main idea behind our construction. In order to have a well defined action on an asymptotically simple spacetime and a finite symplectic structure at $\mathscr{I}$, what really matters physically is that it is possible to reabsorb all the divergences of $\hat{\boldsymbol{\theta}}^a$ into a divergent boundary action and divergent corner terms. We now show that this is exactly the case.

5.3.2 Radial Equation and its Solution

Splitting the divergence in the defining relation (5.8) for the SP, $\delta \hat{\mathbf{L}} = \hat{\mathbf{E}}^a \delta A_a + \partial_a \hat{\boldsymbol{\theta}}^a$, into a divergence on the $\Omega =$ const. surfaces and a transverse derivative by using the decomposition (5.4) of the identity, one obtains

$$\delta\hat{\mathbf{L}} = \delta(\Omega^{-(D-4)}\mathbf{L}) = \Omega^{-(D-4)}\mathbf{E}^a\delta A_a + \Omega^{-(D-4)}\partial_i\boldsymbol{\theta}^i + \partial_\Omega(\Omega^{-(D-5)}\boldsymbol{\theta}^n). \tag{5.19}$$

Rearranging the terms and extracting the factor in Ω, one obtains

$$\big[(D-5) - \Omega\partial_\Omega\big]\boldsymbol{\theta}^n = \partial_i\boldsymbol{\theta}^i - \delta\mathbf{L} + \mathbf{E}^a\delta A_a. \tag{5.20}$$

Up to EoM terms, the RHS contains only a total derivative and a total variation which are part of the ambiguity in $\hat{\boldsymbol{\theta}}^\Omega$. We call equations involving the operators $(n - \Omega\partial_\Omega)$ *radial equations*.

The radial equation (5.20) implies that $\hat{\boldsymbol{\theta}}^\Omega$ can be made finite on-shell by subtracting counterterms that fall under the ambiguities (5.18b). As a first way to see it, if fields are analytic, note that at each order of a Laurent series for $\hat{\boldsymbol{\theta}}^\Omega$

$$\hat{\boldsymbol{\theta}}^\Omega = \Omega^{-(D-5)}\boldsymbol{\theta}^n_{(0)} + \Omega^{-(D-6)}\boldsymbol{\theta}^n_{(1)} + ... + \Omega^{-1}\boldsymbol{\theta}^n_{(D-6)} + \boldsymbol{\theta}^n_{(D-5)} + \Omega\boldsymbol{\theta}^n_{(D-4)} + ... \tag{5.21}$$

the radial equation reads

$$(D-5-k)\boldsymbol{\theta}^n_{(k)} \hat{=} \partial_i\boldsymbol{\theta}^i_{(k)} - \delta\mathbf{L}_{(k)}, \tag{5.22}$$

where $\hat{=}$ denotes on-shell equality. The orders $k < D-5$ of $\boldsymbol{\theta}^n$, which are the ones that come with divergent prefactors in $\hat{\boldsymbol{\theta}}^\Omega$, are fixed on-shell by the radial equation to be total derivatives plus total variations, while $\boldsymbol{\theta}^n_{(D-5)}$, which gives the finite order of $\hat{\boldsymbol{\theta}}^\Omega$, is not determined. The remaining terms do not contribute in the asymptotic limit $\Omega \to 0$. Thus, it is clear that the divergences in $\hat{\boldsymbol{\theta}}^\Omega$ can be removed order by order in the Laurent series.

Rather than working order by order, we perform the counterterm subtraction at finite distance and take the limit in the end. In this way we obtain the asymptotic SP as the finite limit of a renormalized SP.

At any Ω, applying ∂^k_Ω to (5.20) gives

$$(D-5-k)\partial^k_\Omega\boldsymbol{\theta}^n - \Omega\partial^{k+1}_\Omega\boldsymbol{\theta}^n = \partial^k_\Omega R, \tag{5.23}$$

where R is the RHS of (5.20), which is made up from EoM terms, and total variations and derivatives which may be absorbed into the ambiguities of the SP. We can thus replace $\boldsymbol{\theta}^n$ by $\Omega\partial_\Omega\boldsymbol{\theta}^n$ up to ambiguity terms, and then replace $\Omega\partial_\Omega\boldsymbol{\theta}^n$ by $\Omega^2\partial^2_\Omega\boldsymbol{\theta}^n$ up to ambiguities, and so forth until $k = D-6$.

To do this all at once, we act with the operator $\frac{(D-5-(k+1))!}{(D-5)!}\Omega^k\partial_\Omega^k$ on (5.20) to obtain:

$$\frac{(D-5-k)!}{(D-5)!}\Omega^k\partial_\Omega^k\boldsymbol{\theta}^n - \frac{(D-5-(k+1))!}{(D-5)!}\Omega^{k+1}\partial_\Omega^{k+1}\boldsymbol{\theta}^n = \frac{(D-5-(k+1))!}{(D-5)!}\Omega^k\partial_\Omega^k R. \tag{5.24}$$

This relation holds for integers $0 \leq k \leq D-6$, with the convention $0! = 1$. Taking the sum $\sum_{k=0}^{D-6}$, the middle terms cancel and we are left with

$$\boldsymbol{\theta}^n = \frac{1}{(D-5)!}\Omega^{D-5}\partial_\Omega^{D-5}\boldsymbol{\theta}^n + \sum_{k=0}^{D-6}\frac{(D-5-(k+1))!}{(D-5)!}\Omega^k\partial_\Omega^k R. \tag{5.25}$$

Remembering that $\hat{\boldsymbol{\theta}}^\Omega = \Omega^{-(D-5)}\boldsymbol{\theta}^n$, if we assume that $\partial_\Omega^{D-5}\boldsymbol{\theta}^n$ has a finite limit, this relation is enough to absorb the divergences in $\hat{\boldsymbol{\theta}}^\Omega$ into the ambiguities of the SP on-shell.

However, requiring $\partial_\Omega^{D-5}\boldsymbol{\theta}^n$ to have a finite limit may be too restrictive: In terms of the expansion (5.6), it would mean that $\boldsymbol{\theta}^n_{(k,l)} = 0$ for $k \leq (D-5)$. On the other hand, ∂_Ω^{D-5} of (5.20) gives:

$$\Omega\partial_\Omega^{D-4}\boldsymbol{\theta}^n = -\partial_\Omega^{D-5}R. \tag{5.26}$$

The condition $\boldsymbol{\theta}^n_{(k,l)} = 0$ for $k \leq (D-5)$ implies that $\Omega\partial_\Omega^{D-4}\boldsymbol{\theta}^n$ limits to zero, and hence that $\partial_\Omega^{D-5}R$ limits to zero, which may be overly restrictive.

To allow a non-zero limit of $\partial_\Omega^{D-5}R$, we need to allow a term $\boldsymbol{\theta}^n_{(D-5,1)}\Omega^{D-5}\ln\Omega$ in $\boldsymbol{\theta}^n$. That term is determined by the radial equation as

$$(D-5)!\boldsymbol{\theta}^n_{(D-5,1)} \overset{\mathscr{I}}{=} -\partial_\Omega^{D-5}R, \tag{5.27}$$

so it is itself part of the ambiguity of the SP. It spoils the finiteness of $\partial_\Omega^{D-5}\boldsymbol{\theta}^n$, however, instead we get

$$\partial_\Omega^{D-5}\boldsymbol{\theta}^n = (D-5)!\boldsymbol{\theta}^n_{(D-5,1)}\ln\Omega + \text{finite}. \tag{5.28}$$

The combination $\partial_\Omega^{D-5}\boldsymbol{\theta}^n - \Omega\ln\Omega\partial_\Omega^{D-4}\boldsymbol{\theta}^n$ has a finite limit even when $\boldsymbol{\theta}^n_{(D-5,1)}$ is "on". To complete the renormalization of the SP, we can thus subtract this logarithmic piece. With these arguments, we can then deduce from (5.20) that:

$$\begin{aligned}\boldsymbol{\theta}^n &= \frac{1}{(D-5)!}\Omega^{D-5}(\partial_\Omega^{D-5}\boldsymbol{\theta}^n - \Omega\ln\Omega\partial_\Omega^{D-4}\boldsymbol{\theta}^n) + \mathcal{D}(R), \qquad \text{where}\\ \mathcal{D}(R) &= \sum_{k=0}^{D-6}\frac{(D-5-(k+1))!}{(D-5)!}\Omega^k\partial_\Omega^k R - \frac{\Omega^{D-5}}{(D-5)!}\ln\Omega\partial_\Omega^{D-5}R \qquad \text{and}\\ R &= \partial_i\boldsymbol{\theta}^i - \delta\mathbf{L} + \mathbf{E}^a\delta A_a.\end{aligned} \tag{5.29}$$

We have introduced the operator $\mathcal{D}$ to summarize the radial derivatives.

Now, we can finally absorb the divergences of $\hat{\boldsymbol{\theta}}^\Omega$ into a boundary action and corner counterterm: We set

$$\hat{\boldsymbol{\theta}}^a_R := \hat{\boldsymbol{\theta}}^a + \partial_b \hat{\boldsymbol{\alpha}}^{ab} + \delta \hat{\boldsymbol{\beta}}^a, \qquad \text{where} \tag{5.30}$$

$$\hat{\boldsymbol{\alpha}}^{\Omega i} = -\hat{\boldsymbol{\alpha}}^{i\Omega} = -\Omega^{-(D-5)}\mathcal{D}(\boldsymbol{\theta}^i) \qquad \text{and} \qquad \hat{\boldsymbol{\alpha}}^{ij} = 0; \tag{5.31}$$

$$\hat{\boldsymbol{\beta}}^\Omega = \Omega^{-(D-5)}\mathcal{D}(\mathbf{L}) \qquad \text{and} \qquad \hat{\boldsymbol{\beta}}^i = 0. \tag{5.32}$$

The choice $\hat{\boldsymbol{\alpha}}^{ij} = 0 = \hat{\boldsymbol{\beta}}^i$ is not unique, and can be modified without interfering with the renormalization of the SP on $\mathscr{I}$. Using the first $(D-5)$ orders of the EoM, the normal component of the renormalized SP then becomes

$$\hat{\boldsymbol{\theta}}^\Omega_R \hat{=} \frac{1}{(D-5)!}\big(\partial_\Omega^{D-5}\boldsymbol{\theta}^n - \Omega \ln\Omega\, \partial_\Omega^{D-4}\boldsymbol{\theta}^n\big). \tag{5.33}$$

This expression has a finite limit, even if there is a term $\boldsymbol{\theta}^n_{(D-5,1)}\Omega^{D-5}\ln\Omega$ in $\boldsymbol{\theta}^n$. Let us reiterate that the counterterms are defined at any value of Ω, and are local in Ω.

5.3.3 Renormalization of the Action and Transverse Symplectic Potential

The addition (5.30) of counterterms also modifies the transverse components $\hat{\boldsymbol{\theta}}^i$ of the SP, and the Lagrangian $\hat{\mathbf{L}}$: we get $\hat{\mathbf{L}}_R = \hat{\mathbf{L}} + \partial_\Omega \hat{\boldsymbol{\beta}}^\Omega$, and $\hat{\boldsymbol{\theta}}^i_R = \hat{\boldsymbol{\theta}}^i + \partial_\Omega \hat{\boldsymbol{\alpha}}^{i\Omega}$. To understand the effect of this modification, note that for any function F,

$$\partial_\Omega\big(\Omega^{-(D-5)}\mathcal{D}(F)\big) = -\Omega^{-(D-4)}F - \frac{1}{(D-5)!}\ln\Omega\,\partial_\Omega^{D-4}F. \tag{5.34}$$

The renormalized Lagrangian and transverse SP current thus become:

$$\begin{aligned} \hat{\mathbf{L}}_R &= -\frac{1}{(D-5)!}\ln\Omega\,\partial_\Omega^{D-4}\mathbf{L} \\ \hat{\boldsymbol{\theta}}^i_R &= -\frac{1}{(D-5)!}\ln\Omega\,\partial_\Omega^{D-4}\boldsymbol{\theta}^i. \end{aligned} \tag{5.35}$$

We recall that $\mathbf{L} = \Omega^{D-4}\hat{\mathbf{L}}$ and $\boldsymbol{\theta}^i = \Omega^{D-4}\hat{\boldsymbol{\theta}}^i$ are rescaled to be asymptotically finite. Let us assume that $\partial_\Omega^{D-4}\mathbf{L}$ and $\partial_\Omega^{D-4}\boldsymbol{\theta}^i$ diverge at most as a power of $(\ln\Omega)$. Then, since $\lim_{\Omega\to 0}\int_{\omega=\Omega}\mathrm{d}\omega\,\ln^k(\omega)$ is finite, the action and the integrated transverse SP are finite in the limit. Our scheme to renormalize the SP $\hat{\boldsymbol{\theta}}^\Omega$ has thus automatically renormalized the action and the remainder of the components of the SP too.

5.3.4 Analyticity Requirements

Let us summarize which degree of analyticity is sufficient to obtain a finite SP in our scheme. For $\hat{\boldsymbol{\theta}}^{\Omega}_{R}$ in (5.33) to have a finite limit, we must have:

$$\boldsymbol{\theta}^n = \sum_{k\geq 0} \Omega^k \boldsymbol{\theta}^n_{(k)} + \boldsymbol{\theta}^n_{(D-5,1)} \Omega^{D-5} \ln \Omega + \sum_{k\geq (D-4),\, l\geq 1} \boldsymbol{\theta}^n_{(k,l)} \Omega^k (\ln \Omega)^l. \tag{5.36}$$

Let us work out sufficient conditions on the gauge fields A to make that true. We have $\boldsymbol{\theta}^n = -\frac{1}{\Omega}\sqrt{q} F^{\Omega i} \delta A_i$. The most leading possible term involving $\ln \Omega$ comes from $F^{\Omega i}$, which involves a ∂_Ω–derivative. A sufficient condition, which implies the finiteness of $\hat{\boldsymbol{\theta}}^{\Omega}_{R}$, is thus

$$F_{ab} = \sum_{k\geq 0} \Omega^k F_{ab,(k)} + \Omega^{D-4} \ln \Omega F_{ab,(D-4,1)} + \sum_{k\geq (D-3),\, l\geq 1} \Omega^k (\ln \Omega)^l F_{ab,(k,l)}. \tag{5.37}$$

Because F_{ab} involves an Ω–derivative, the conditions on A_a need to be stricter. The following is sufficient:

$$A_a = \sum_{k\geq 0} \Omega^k A_{a,(k)} + \Omega^{D-3} \ln \Omega A_{a,(D-3,1)} + \sum_{k\geq (D-2),\, l\geq 1} \Omega^k (\ln \Omega)^l A_{a,(k,l)}. \tag{5.38}$$

The conditions on F_{ab} and A_a also ensure that $\partial_\Omega^{D-4} \mathbf{L}$ and $\partial_\Omega^{D-4} \boldsymbol{\theta}^i$ diverge at most as $\ln \Omega$, which ensures the renormalized action and transverse SPs are finite.

In the remainder of this chapter, we will however discard all logarithmic terms, and work in the fully analytic setting.

5.3.5 Layers

In the absence of any logarithms, in (5.33) the term $\Omega \ln \Omega \partial_\Omega^{D-4} \boldsymbol{\theta}^n$ does not contribute in the limit. We get, on $\mathscr{I}$

$$\hat{\boldsymbol{\theta}}^{\Omega}_{R} \overset{\mathscr{I}}{=} \boldsymbol{\theta}^n_{(D-5)} \overset{\mathscr{I}}{=} -\sum_{k=0}^{D-5} \sqrt{q} \mathcal{J}^i_{(D-5-k)} \delta A_{i(k)}. \tag{5.39}$$

We see that the renormalized SP coincides with the finite part in a Laurent series expansion of the original SP $\hat{\boldsymbol{\theta}}^{\Omega}$. Asymptotically, it is therefore correct to simply drop the divergent orders of the SP.

The SP splits into layers: The most leading gauge potential, which is affected by leading gauge transformations, is conjugate to the $(D-5)$th order of the conformal current. As we will see in section 5.4, the constraints involve exactly that order. Intermediate gauge potentials are conjugate to intermediate orders of the conformal current, and as we will see from the analysis of the EoM, the radiative data for Minkowski space live primarily in the pair $\mathcal{J}^i_{(\ (D-6)/2\ )}\delta A_{i((D-4)/2)}$.

One may wonder why, instead of the expected $D-1$ degrees of freedom (before gauge fixing), we now have $(D-4)$ layers of $(D-1)$ degrees of freedom each. The answer is that the quantities appearing at different layers are not independent: They are related through equations of motion, and are not even independent off-shell – knowledge of some components of $A_{i(k+1)}$ and $A_{i(k+2)}$ determines $\mathcal{J}^i_{(k)}$.

The upshot of not using those relationships until now is the generality of the renormalization scheme: So far, we have not used the explicit form of the metric (5.1a) at all, and the scheme would work as well for other metrics, for example, on AdS space. To relate to soft theorems, however, requires expressing the SP in terms of free radiative data, and thus needs a detailed analysis of the equations of motion.

5.4 Asymptotic Equations of Motion

In this section, we give the complete set of relations between the quantities entering the renormalized SP. Specifically, we will identify the free data needed to solve the EoM asymptotically. Computations are performed in general $D \geq 6$ (even).

5.4.1 Asymptotic Expansion of the Equations of Motion

The first step is to split the EoM into their radial, retarded-time, and sphere components, and to develop them in orders of Ω. We will write the equations in "radial-time" gauge[4]

$$A_\Omega \equiv 0. \tag{5.40}$$

[4]Since $\mathscr{I}$ is null and transverse to ∂_Ω, the radial gauge $A_\Omega \equiv 0$ shares there various features with the usual time gauge $A_t \equiv 0$ fixed at a standard Cauchy surface $\Sigma_{t=\text{const.}}$.

The residual gauge transformations satisfy $\partial_\Omega \epsilon = 0$. The gauge fixing may be un-done by setting

$$A_a^{\text{g.f.}} = A_a - \partial_a \int_0^\Omega \mathrm{d}\omega A_\Omega, \tag{5.41}$$

which satisfies $A_\Omega^{\text{g.f.}} = 0$. Order by order, we have

$$A_{i(k)}^{\text{g.f.}} = A_i - \frac{1}{k}\partial_i A_{\Omega(k-1)}. \tag{5.42}$$

The equations of this section may then be regarded as equations for $A^{\text{g.f.}}$. Equations for A_a itself may be derived by performing the replacement, and regarding all orders of A_Ω as free data.

Consider first the conformal current $\mathcal{J}^a = \Omega^{-1} N_b F^{ba}$. We write the definitions of $\mathcal{J}^u$ as a radial evolution equation for $\mathcal{A}_u$ and the definition of $\mathcal{J}^A$ as a retarded time evolution equation for A_A:

$$\partial_\Omega A_u = -\Omega \mathcal{J}^u \tag{5.43a}$$
$$\partial_u A_A = \partial_A A_u + \Omega \mathcal{J}_A - \Omega^2 \partial_\Omega A_A. \tag{5.43b}$$

The EoM, $\mathbf{E}^a = \partial_b(\sqrt{q} F^{ba}) - (D-4)\sqrt{q}\mathcal{J}^a$, can be decomposed as:

$$\mathbf{E}^n = -\partial_u(\sqrt{q}\mathcal{J}^u) - \partial_A(\sqrt{q}\mathcal{J}^A), \tag{5.44a}$$
$$\mathbf{E}^u = -\Big[(D-5) - \Omega\partial_\Omega\Big](\sqrt{q}\mathcal{J}^u) - \partial_A\partial_\Omega(\sqrt{q}A^A), \tag{5.44b}$$
$$\mathbf{E}^A = -\Big[(D-5) - \Omega\partial_\Omega\Big](\sqrt{q}\mathcal{J}^A) + \partial_u\partial_\Omega(\sqrt{q}A^A) + \partial_B(\sqrt{q}F^{BA}) \tag{5.44c}$$
$$= -\Big[(D-6) - 2\Omega\partial_\Omega\Big](\sqrt{q}\mathcal{J}^A) + \partial_B(\sqrt{q}F^{BA}) - \sqrt{q}(1+\Omega\partial_\Omega)(\Omega\partial_\Omega A^A) - \Omega\sqrt{q}\partial^A\mathcal{J}^u, \tag{5.44d}$$

where in the last line we have rewritten $\mathbf{E}^A$ as a purely radial evolution equation, by means of (5.43). Notice the factor of 2 which appeared in the radial derivative operator as a consequence of this manipulation.

We now develop the equations in orders of Ω. First, consider the normal component of the EoM,

$$\mathbf{E}^n_{(k)} = -\partial_u(\sqrt{q}\mathcal{J}^u_{(k)}) - \partial_A(\sqrt{q}\mathcal{J}^A_{(k)}). \tag{5.45}$$

Note that the identity $\partial_a \partial_b(\sqrt{\hat{g}}\hat{F}^{ab}) = 0$ can be written as

$$\Big[(D-5) - \Omega\partial_\Omega\Big]\mathbf{E}^n = \partial_u \mathbf{E}^u + \partial_A \mathbf{E}^A \tag{5.46}$$

Asymptotically, this implies that the only independent information contained in $\mathbf{E}^n$ lies in its $k = (D-5)$ order. The rest of its orders automatically vanish once the tangential EoM are solved, and do not need to be considered separately. Thus we define

$$\mathbf{G} := \mathbf{E}^n_{(D-5)} = -\partial_u(\sqrt{q}\mathcal{J}^u_{(D-5)}) - \partial_A(\sqrt{q}\mathcal{J}^A_{(D-5)}). \tag{5.47a}$$

As it will become clear shortly, this is the Gauss law on the $\Omega = $ const. slices. The orders of the remainder EoM and the definitions of the conformal current are

$$A_{u(k+1)} = -\tfrac{1}{(k+1)}\mathcal{J}^u_{(k-1)} \tag{5.47b}$$

$$\partial_u A_{A(k)} = \partial_A A_{u(k)} + \mathcal{J}_{A(k-1)} - (k-1)A_{A(k-1)} \tag{5.47c}$$

$$\mathbf{E}^u_{(k)} = -(D-5-k)(\sqrt{q}\mathcal{J}^u_{(k)}) - (k+1)\partial_A(\sqrt{q}A^A_{(k+1)}) \tag{5.47d}$$

$$\mathbf{E}^A_{(k)} = -(D-6-2k)(\sqrt{q}\mathcal{J}^A_{(k)}) + \partial_B(\sqrt{q}F^{BA}_{(k)}) - \sqrt{q}k(1+k)A^A_{(k)} - \sqrt{q}\partial^A\mathcal{J}^u_{(k-1)}. \tag{5.47e}$$

These equations hold for $k \geq 0$ if we set negative orders of $\mathcal{J}$ and A to zero. The equations (5.47) are the complete set of asymptotic EoM.

These equations contain the asymptotic Maxwell conditions $N_a F^{ab} \overset{\mathscr{I}}{=} 0$, which are explicitly given by

$$A_{u(1)} = 0, \qquad \mathcal{J}^u_{(-1)} = 0, \qquad \partial_u A_{A(0)} = \partial_A A_{u(0)}. \tag{5.48}$$

The last equation can be conveniently solved by introducing a Hodge decomposition of

$$A_{A(0)} = \epsilon_A{}^{BC\cdots}\partial_B\mu_{C\cdots} + \partial_A\varphi =: \alpha_{A(0)} + \partial_A\varphi. \tag{5.49}$$

Then, equation (5.48) says that the purely magnetic part $\alpha_{A(0)}$ must be u-independent and that the purely electric part φ is related to $A_{u(0)}$ by

$$A_{u(0)} = \partial_u\varphi. \tag{5.50}$$

We call φ the *soft potential*.[5]

[5] Notice that φ in (5.50) is not fully determined by the Hodge decomposition of $A_{A(0)}$, but only up to a time-dependent sphere-constant term. We will see that φ is in an appropriate sense canonically conjugated to the local electric flux. Thus, since in absence of charged matter the total flux vanishes, this sphere-constant term does not play much of a role, see section 5.5.

5.4.2 Free Data

We are now going to analyze the asymptotic EoM to identify the asymptotic free data. As before, we focus on a finite region $I \subset \mathscr{I}$, with $u_i \le u \le u_f$. The boundary of I is the union of two codimension-two spheres, denoted $\partial I = S_i \sqcup S_f$, where S_i (S_f) is the cut of $\mathscr{I}$ at $u = u_i$ ($u = u_f$, respectively).

We define $\alpha_{A(k)} := A_{A(k)}(u_i)$ which is a corner variable evaluating the value of A_A at the initial slice. The value of A_A on a arbitrary time slice can then be obtained as

$$A_{A(k)}(u) = \alpha_{A\,(k)} + \int_{u_i}^{u} \partial_u A_{A(k)}(u')\mathrm{d}u'. \tag{5.51}$$

For later convenience we introduce the new symbol

$$\ell := \frac{D-6}{2}. \tag{5.52}$$

Note the obvious relations $D = 6 + 2\ell$ and $D - 5 = 2\ell + 1$.

We will now show that the free canonical data which enter the renormalized SP are given by

$$\left\{\varphi,\, \mathcal{J}^A_{(\ell)}\right\} \text{ on } \mathscr{I}, \qquad \left\{\mathcal{J}^u_{(D-5)}, \alpha_{A(0)}, \cdots, \alpha_{A(D-5)}\right\} \text{ on } S_i. \tag{5.53}$$

We view the conformal current $\mathcal{J}^i$ as an a priori independent variable from the gauge field, such that the definition of $\mathcal{J}^i$ in terms of components of the gauge field has the same status as the EoM. The key to identifying the free data is that the factor $(D-5-k)$ in (5.47d) becomes zero for $k = D-5$, and the factor $(D-6-2k)$ in (5.47d) becomes zero for $k = \frac{D-6}{2}$.

We prove (5.53) by recurrence. We start the recurrence by assuming that we are given the variables $A_{u(0)}$ (determined by φ) and $\alpha_{A(0)}$. (5.48) then determines $\partial_u A_{A(0)}$, and together with $\alpha_{A(0)}$ determines $A_{A\,(0)}$. To continue the recurrence it is convenient to lay out the equation of motions as follows[6]

$$\begin{aligned}
(D-4-k)\mathcal{J}^u_{(k-1)} &= -kD_A A^A_{(k)}, && (5.54a)\\
(D-6-2k)\mathcal{J}^A_{(k)} &= D_B F^{BA}_{(k)} - [\partial^A \mathcal{J}^u_{(k-1)} + k(k+1)A^A_{(k)}], && (5.54b)\\
\partial_u A^A_{(k+1)} &= \mathcal{J}^A_{(k)} - \tfrac{1}{k+1}[\partial^A \mathcal{J}^u_{(k-1)} + k(k+1)A^A_{(k)}]. && (5.54c)
\end{aligned}$$

[6] D_A is the covariant derivative on the sphere S, so that e.g. $\partial_A(\sqrt{q}v^A) = \sqrt{q}D_A v^A$.

We now assume that $A^A_{(k)}$ is known on I. The first equation allows to obtain $\mathcal{J}^u_{(k-1)}$ from $A^A_{(k)}$, as long as $k \neq (D-4)$. It does not determine $\mathcal{J}^u_{(D-5)} = \mathcal{J}^u_{(2\ell+1)}$, which we assume as given for now. The second equation allows to obtain $\mathcal{J}^A_{(k)}$ from $(A^A_{(k)}, \mathcal{J}^u_{(k-1)})$, as long as $k \neq \ell$. The third equation determines $\partial_u A^A_{(k+1)}$ from $(A^A_{(k)}, \mathcal{J}^u_{(k-1)}, \mathcal{J}^A_{(k)})$. This in turns determines $A^A_{(k+1)}$ from $\alpha^A_{(k+1)}$ and (5.51) and we can start a new cycle of recurrence.[7]

This establishes that the free data is $\{\varphi, \mathcal{J}^A_{(\ell)}, \mathcal{J}^u_{(D-5)}\}$ on I and $\{\alpha^A_{(k)}\}$ on S. One can then use the Gauss law (5.47a) to deduce the value of $\partial_u \mathcal{J}^u_{(D-5)}$. This effectively reduces the free part of $\mathcal{J}^u_{(D-5)}$ to its initial value on S_i.

Of course, knowledge of more subleading orders of $\alpha_{A,(k>D-5)}$ would allow to solve the equations of motion to higher orders into the bulk. These more subleading orders, however, do not appear in the SP at $\mathscr{I}$. Since a portion of $\mathscr{I}$ is not on its own a Cauchy surface, to obtain a solution in the bulk the data at $\mathscr{I}$ has to be complemented by data on a spacelike hypersurface, and there will be a contribution to the total SP from that hypersurface. The subleading orders $\alpha_{A,(k>D-5)}$ should thus be regarded as belonging to the free data on such a spacelike hypersurface.

5.4.3 News, Charge Aspects, and Radiative Modes

As we have seen there are two pieces of data that are exceptional in the sense that they are not determined recursively by the rest of the data. The first exception appears at order $k = \ell = \frac{D-6}{2}$: the variable $\mathcal{J}^A_{(\ell)}$ is not determined by (5.54b), contrarily to its other orders which are algebraically determined. It is free data on all of $\mathscr{I}$. We call it the *Maxwell news*:

$$\mathcal{N}^A := \mathcal{J}^A_{(\ell)}, \tag{5.55}$$

for its role in the asymptotic EoM analogous to the Bondi news in 4D General Relativity. It is the free radiative data. Let us further introduce, the *radiative modes*

$$\mathcal{A}_A := A_{A(\ell+1)}. \tag{5.56}$$

[7] The knowledge of $A_{u(k+1)}$ for $k > 1$ is not explicitly required, one just deduce its value from $A_{u(k)} = -\mathcal{J}_{u(k-1)}/(k+1)$

Using (5.54c), $\mathcal{A}_A$ is determined by $\mathcal{N}^A$, up to an integration constant. Explicitly, we have

$$\mathcal{N}^A = \partial_u \mathcal{A}^A + \ell \left(A^A_{(\ell)} - \frac{\partial^A (D_B A^B_{(\ell)})}{(\ell+1)(\ell+2)} \right). \tag{5.57}$$

In odd spacetime dimensions, all orders of $\mathcal{J}^A_{(k)}$ are algebraically determined by (5.47e). We thus see from an asymptotic perspective that in odd spacetime dimensions, solutions that are analytic around $\mathscr{I}$ do not have free radiative data. This is why we restrict our analysis to even dimensions. A similar statement has been made, albeit from a different perspective, for gravity e.g. in [112, 42].

For the last exception, consider the order $k = D - 5$, where the factor in (5.47d) vanishes. $\mathcal{J}^u_{(D-5)}$ is hence not determined by (5.47d), unlike the other orders of $\mathcal{J}^u_{(k)}$ which are algebraically determined. The retarded time evolution is, however, determined by the Gauss law (5.47a). We hence call

$$\sigma := \mathcal{J}^u_{(D-5)} \tag{5.58}$$

the *charge aspect*, for its role analogous to the (Bondi) mass aspect in general relativity. Note also that asymptotic Coulombic fields, such as the spherically symmetric Coulombic field of a finite point charge in the interior of spacetime, fall off such that they contribute to σ, but not to the more leading orders $\mathcal{J}^u_{(k<D-5)}$.

The charge aspect conservation is controlled by the Gauss law (5.47a),

$$\partial_u \sigma + D_A \mathcal{J}^A_{(D-5)} = 0. \tag{5.59}$$

This can be more explicitly expressed by using (5.54), and taking the divergence of (5.54b), as

$$\partial_u \sigma = \frac{D-5}{D-4} \Big(D^A D_A - (D-4) \Big) (D_B A^B_{(D-5)}). \tag{5.60}$$

In $D = 6$, this readily gives a relation between the conservation of the charge aspect and the radiative modes:

$$\partial_u \sigma = \tfrac{1}{2} \left(D^A D_A - 2 \right) \left(D_B \mathcal{A}^B \right) \qquad (D = 6). \tag{5.61}$$

However, in general $A_{A(D-5)}$ does not correspond to the radiative modes, and one might wonder whether a relation analogous to this one still holds in general dimensions (this relation is crucial for the derivation of the soft theorems, see [90]). Indeed, a similar relation exists, but it rather expresses the higher time derivatives $\partial_u^{\ell+1}\sigma$ in terms of higher spatial derivatives of $\mathcal{A}_A$. This relation can be found by taking the divergences of equations (5.54b) and (5.54c). To see this, it is convenient to rewrite equations (5.54) for $k \neq 2\ell+2$ as

$$\begin{aligned} \mathcal{J}^u_{(k-1)} &= -\tfrac{k}{(2\ell+2-k)} D_A A^A_{(k)}, && (5.62a)\\ 2(\ell-k)\mathcal{J}^A_{(k)} &= D_B F^{BA}_{(k)} + \tfrac{k}{(2\ell+2-k)} \left[D^A D_B - c^\ell_k \delta^A_B\right] A^B_{(k)}, && (5.62b)\\ \partial_u A^A_{(k+1)} &= \mathcal{J}^A_{(k)} + \tfrac{k}{(k+1)(2\ell+2-k)} \left[D^A D_B - c^\ell_k \delta^A_B\right] A^B_{(k)}. && (5.62c) \end{aligned}$$

where

$$c^\ell_k := (k+1)(2\ell+2-k) \tag{5.63}$$

is a symmetric coefficient under the exchange $k \leftrightarrow D-5-k$.

Thus, the divergences of (5.54b) and (5.54c) readily give a recursion relation[8] for $D_A A^A_{(k)}$:

$$\begin{aligned} D_A \mathcal{J}^A_{(k)} &= \tfrac{k}{2(\ell-k)(2\ell+2-k)} \Delta^\ell_k (D_A A^A_{(k)}) && (5.64a)\\ \partial_u (D_A A^A_{(k+1)}) &= \tfrac{k(2\ell+1-k)}{2(\ell-k)(k+1)(2\ell+2-k)} \Delta^\ell_k (D_A A^A_{(k)}) && (5.64b) \end{aligned}$$

where we introduced the elliptic negative-definite differential operator

$$\Delta^\ell_k := \Big(D_A D^A - c^\ell_k\Big). \tag{5.65}$$

Using the above recursion relation, we find

$$\begin{aligned} \partial_u^{\ell+1}\sigma &= -\partial_u^\ell (D_A \mathcal{J}^A_{(2\ell+1)}) = \tfrac{2\ell+1}{2(\ell+1)} \Delta^\ell_{2\ell+1} \partial_u^\ell (D_A A^A_{(2\ell+1)}) \\ &= \cdots = \frac{(-1)^\ell}{2^{(\ell+1)}} \frac{1}{(\ell+1)!} \Big(\Delta^\ell_{2\ell+1}\Delta^\ell_{2\ell}\cdots\Delta^\ell_{\ell+1}\Big)(D_A \mathcal{A}^A). && (5.66) \end{aligned}$$

Thus, we see that in dimensions $D > 6$ (even), i.e. $\ell > 0$, the radiative potential only controls the higher time derivative $\partial^{\ell+1}\sigma$ of the charge aspect.

[8] With similar methods, a recursion relation can be found for $F^{AB}_{(k)}$ by taking the antisymmetric derivative of equations (5.54b) and (5.54c), instead of their divergences.

5.4.4 Consequences of Analyticity

Even though the equations $\mathbf{E}^A_{(\frac{D-6}{2})} \equiv \mathbf{E}^A_{(\ell)} = 0$ and $\mathbf{E}^u_{(D-5)} \equiv \mathbf{E}^u_{(2\ell+1)} = 0$ do not determine the Maxwell news and the charge aspect, they of course still hold true. Their status is similar to (5.26): assuming analyticity, they indirectly require their RHS to vanish. In order to avoid that requirement, one would have to allow a logarithmic terms in the Maxwell news and the charge aspect.

$\mathbf{E}^u_{(D-5)}$ gives

$$D_A A^A_{(D-4)} = 0. \tag{5.67}$$

Because only the values of the potential $\{A^A_{(0)}, \cdots, A^A_{(D-5)}\}$ enter the SP at $\mathscr{I}$, this condition should be regarded as a condition on the data on a spacelike hypersurface, which together with I forms a Cauchy surface.

The LHS of (5.54b) becomes zero at $k = \ell = \frac{D-6}{2}$. Its vanishing hence restricts $A^A_{(\ell)}$. Using that $\mathcal{J}^u_{(\ell-1)} = -\frac{\ell}{\ell+2} D_B A^B_{(\ell)}$, one gets

$$D_B F^{AB}_{(\ell)} + \ell \left[(\ell+1) A^A_{(\ell)} - \tfrac{1}{\ell+2} \partial^A (D_B A^B_{(\ell)})\right] = 0. \tag{5.68}$$

In $D \neq 6$, taking the divergence of this equation we get that $[D_A D^A - (\ell+1)(\ell+2)](D_B A^B_{(\ell)}) = 0$. The Laplacian on the sphere has non-positive eigenvalues so this equation implies $D_B A^B_{(\ell)} = 0$ and $D_B F^{AB}_{(\ell)} = -\ell(\ell+1) A^A_{(\ell)}$. Using the relation (5.54a), we can translate the first condition into a restriction on $\mathcal{J}^u_{(\ell-1)}$. Hence,(5.68) implies

$$\mathcal{J}^u_{(\ell-1)} = 0 \quad \text{and} \quad D_B F^{AB}_{(\ell)} = -\ell(\ell+1) A^A_{(\ell)}. \tag{5.69}$$

In dimension $D = 6$, $\ell = 0$ and the above manipulations fail. However, the two equations (5.69) stay true: the first one degenerates with the asymptotic Maxwell condition (5.48), while the second one simply means that $D_B F^{AB}_{(\ell=0)} = 0$, compatible with (5.68). Thus F_{AB} as a form on the spheres is closed and co-closed, and must vanish since there are no non-trivial harmonic forms on the sphere[9].

$$F^{AB}_{(\ell=0)} = 0 \qquad (D = 6), \tag{5.70}$$

[9] In [90], the same condition is derived from a finite energy argument. Since we have here renormalized the symplectic form, the generators of time translations, whose on-shell value is energy, are likewise renormalized and their argument cannot be directly applied.

an equation that holds only in $D = 6$. Furthermore, in all dimensions, (5.69) simplifies the expression (5.57) for the news tensor, giving

$$\mathcal{N}^A = \partial_u \mathcal{A}^A + \ell A^A_{(\ell)} = \partial_u \mathcal{A}^A + \tfrac{1}{(\ell+1)} D_B F^{AB}_{(\ell)}. \tag{5.71}$$

5.5 Renormalized Symplectic Potential

With these results, we can now analyze the renormalized asymptotic SP in the case where $D \geq 6$ and even. We focus on the contribution from a subregion $I \subset \mathscr{I}$ with $u_i \leq u \leq u_f$. As we have shown in section 5.3, the renormalized SP organized itself as a sum of different layers (5.39). It is convenient to rearrange these layers as

$$\hat{\Theta}^R = \Theta_{\text{C}} + \Theta_{\text{rad}} + \sum_{p=1}^{\ell} \left(\Theta^{\text{int}}_{(p)} + \Theta^{\text{int}}_{(-p)} \right), \tag{5.72}$$

where

$$\Theta^{\text{int}}_{(p)} := \int_I \sqrt{q}\; \mathcal{J}^i_{(\ell+p)} \delta A_{i(\ell+1-p)} \tag{5.73}$$

is the contribution of the "intermediate potentials" (present only when $\ell \geq 1$, i.e. $D \geq 8$), while Θ_{C} and Θ_{rad} are the Coulombic and the radiative contributions respectively:

$$\Theta_{\text{C}} = \int_I \sqrt{q}\; \mathcal{J}^i_{(2\ell+1)} \delta A_{i(0)}, \qquad \Theta_{\text{rad}} = \int_I \sqrt{q}\; \mathcal{J}^i_{(\ell)} \delta A_{i(\ell+1)}. \tag{5.74}$$

The Coulombic and radiative contributions are common to all even dimensions $D \geq 6$.

In order to express the radiative component of the SP, one has to remember that analyticity requires [10] that $A_{u(\ell+1)} = -\mathcal{J}^u_{(\ell-1)}/(\ell+1) = 0$. Therefore, the radiative component is purely transverse and pairs the Maxwell news $\mathcal{N}^A := \mathcal{J}^A_{(\ell)}$ to the radiative modes $\mathcal{A}_A := A_{A(\ell+1)}$, hence its name:

$$\Theta_{\text{rad}} = \int_I \sqrt{q}\, \mathcal{N}^A \delta \mathcal{A}_A. \tag{5.75}$$

[10] Recall also that in $D = 6$ the same condition follows from the asymptotic Maxwell conditions.

The study of the Coulombic component is more subtle. Recalling our definitions of the soft potential (5.50) and of the charge aspect, $\sigma := \mathcal{J}^u_{(2\ell+1)}$, and making use of the Gauss law (5.47a), the Coulombic component can be cast in the form

$$\begin{aligned} \Theta_{\mathrm{C}} &:= \int_I \sqrt{q}\mathcal{J}^i_{(2\ell+1)}\delta A_{i(0)} = \int_I \sqrt{q}\mathcal{J}^A_{(D-5)}\delta\alpha_{A(0)} + \int_I \sqrt{q}\mathcal{J}^i_{(D-5)}\partial_i\delta\varphi \\ &= \oint_S \sqrt{q}\langle\mathcal{J}^A_{(D-5)}\rangle\delta\alpha_{A(0)} + \oint_S \sqrt{q}\,[\sigma\delta\varphi]^f_i\,. \end{aligned} \tag{5.76}$$

Here, we used the notation $[X]^f_i := X(u_f) - X(u_i)$, and introduced the Fourier zero-mode

$$\langle X\rangle := \int_{u_i}^{u_f} \sqrt{q}\,X(u)\mathrm{d}u. \tag{5.77}$$

This shows that the charge aspect $\sigma_i = \sigma(u_i)$ (resp.$\sigma_f = \sigma(u_f)$) is canonically conjugated to $\varphi_i = \varphi(u_i)$ (resp. $\varphi_f = \varphi(u_f)$) while the zero-mode of the current $\langle\mathcal{J}^A_{(D-5)}\rangle$ is conjugated to $\alpha_{A(0)}$.

It is convenient to introduce the charge aspect (semi-)sum, σ^+, and difference, σ^-:

$$\sigma^+ := \tfrac{1}{2}(\sigma_f + \sigma_i), \qquad \sigma^- := \sigma(u_f) - \sigma(u_i), \tag{5.78}$$

and similarly for the soft potential. Using

$$\oint_S \sqrt{q}\,[\sigma\delta\varphi]^f_i = \oint_S \sqrt{q}(\sigma^+\delta\varphi^- + \sigma^-\delta\varphi^+), \tag{5.79}$$

the Coulombic part of the soft potential can be finally written as

$$\Theta_{\mathrm{C}} = \oint_S \sqrt{q}\Big(\langle\mathcal{J}^A_{(2\ell+1)}\rangle\delta\alpha_{A(0)} + \sigma^+\delta\varphi^- + \sigma^-\delta\varphi^+\Big). \tag{5.80}$$

What is interesting in this formulation is that $\alpha_{A(0)}$, φ^+, and φ^- have a clear meaning in terms of the leading gauge potential $A_{i(0)}$: the soft potential difference φ^- is equal to $[\varphi]^f_i = \int_{u_i}^{u_f} A_{u(0)}$; the sum φ^+ is the electric component in the Hodge decomposition of $A^+_{A(0)}$ and since this expression does not depend on the retarded time explicitly, it does not enter $A_{u(0)}$. $\alpha_{A(0)}$ is the magnetic component in the Hodge decomposition of $A_{A(0)}$, which the asymptotic Maxwell conditions requires to be time independent (see (5.49)).

5.5.1 $D = 6$

In $D = 6$, which means $\ell = 0$, the SP contains only two layers. Moreover, while $\mathcal{A}_A = A_{A(1)}$ is the radiative mode, the curvature $F_{AB(0)}$ vanishes by analyticity. We thus get that $\alpha_{A(0)} = 0$, in this case. Hence, the Coulombic component simplifies further. We also have in this case that the charge conservation is directly determined by the radiative zero modes,

$$\sigma^- = \tfrac{1}{2}\big(D_A D^A - 2\big) D_B \langle \mathcal{A}^B \rangle. \tag{5.81}$$

In six dimensions, the SP thus becomes

$$\hat{\Theta}_R = \int_I \Big(\sqrt{q} \mathcal{N}^A \delta \mathcal{A}_A + \sigma^+ \delta \varphi^- + \sigma^- \delta \varphi^+ \Big) \tag{5.82}$$

The zero mode of $\mathcal{A}$ enters both the Coulombic part and the radiative part of the SP. Thus, to find Poisson brackets, it is necessary to carefully disentangle the zero mode contribution of $\mathcal{A}$ from its purely radiative component contained in $\partial_u \mathcal{A}$. For a study of this problem in the case of four-dimensional gravity, see [20].

5.5.2 Coulombic Contribution in Higher Dimensions

Now, we briefly turn to the higher dimensional case, i.e. $D \geq 8$ (even) or equivalently $\ell \geq 1$, and focus on the soft contribution to the renormalized symplectic structure. In this case the key equation is (5.4.3), i.e.

$$\partial_u^{\ell+1} \sigma = \frac{(-1)^\ell}{2^{(\ell+1)}} \frac{1}{(\ell+1)!} \Big(\Delta^\ell_{2\ell+1} \Delta^\ell_{2\ell} \cdots \Delta^\ell_{\ell+1} \Big)] (D_A \mathcal{A}^A). \tag{5.83}$$

where we also recall that

$$\Delta^\ell_k := \Big(D^A D_A - (k+1)(2\ell + 2 - k) \Big). \tag{5.84}$$

From this, *assuming* $D^A \alpha_{A(k)} = 0$ for $k \in \{\ell+1, \ldots, 2\ell+1\}$, one finds that the soft potential sum φ^+ is conjugated to

$$\sigma^- = \int_{u_i}^{u_f} \mathrm{d}u \, \partial_u \sigma = \frac{(-1)^\ell}{2^{(\ell+1)}} \frac{1}{(\ell+1)!} \Big(\Delta^\ell_{2\ell+1} \Delta^\ell_{2\ell} \cdots \Delta^\ell_{\ell+1} \Big) \langle D^A \mathcal{A}_A \rangle_{(\ell)} \tag{5.85}$$

where the "generalized zero-mode" is defined as[11]

$$\langle D^A \mathcal{A}_A \rangle_{(\ell)} = \int_{u_i}^{u_f} \mathrm{d}u_{\ell+1} \int_{u_i}^{u_{\ell+1}} \mathrm{d}u_\ell \cdots \int_{u_i}^{u_2} \mathrm{d}u_1 D^A \mathcal{A}_A(u_1). \tag{5.86}$$

Notice that the neglected contributions proportional to $D^A \alpha_{A(k)}$ contain powers of the interval $(u_f - u_i)$ and therefore require a more subtle analysis. Moreover, in these cases where $D \geq 8$, the intermediate potentials also contribute via $\Theta^{\mathrm{int}}_{(p)}$. These contributions, once fully unraveled in terms of the free data, end up "dressing" the different contributions to the SP while also providing new terms involving $\delta\alpha_{A(k)}$. We do not attempt a full analysis here.

5.6 Generators and Connection to Soft Theorems

5.6.1 Generators

What is the physical role of the counterterms $\hat{\boldsymbol{\alpha}}^{ab}$ and $\hat{\boldsymbol{\beta}}^a$? Whereas the physical interpretation of $\hat{\boldsymbol{\beta}}^a$ is clear—it renormalizes the action—the interpretation of $\hat{\boldsymbol{\alpha}}^{ab}$ may seem more mysterious. However, its role is physical and it renormalizes the symmetry generators, associated with the (asymptotic) gauge symmetries. We now turn to the analysis of those generators.

The asymptotic renormalized symplectic form is[12]

$$\hat{\omega}^R := \delta\hat{\Theta}^R = \int_I \sum_{k=0}^{D-5} \delta\mathcal{J}^i_{(k)} \curlywedge \delta A_{i(D-5-k)}, \tag{5.87}$$

where $\curlywedge$ denotes antisymmetrization of the δ's. The generators of gauge transformations $\hat{H}^R_\alpha$ are related to the symplectic form as

$$\delta\hat{H}^R_\epsilon = -I_\epsilon \hat{\omega}_R \tag{5.88}$$

[11] Thus $\langle\cdot\rangle_{(0)} = \langle\cdot\rangle$ of the previous section.

[12] Recall there is a sign because N_a is the ingoing normal.

where I_ϵ is the action of a gauge transformation, i.e., $I_\epsilon \hat{\omega}_R(\delta A_a, \delta A_a) = \hat{\omega}_R(\partial_a \epsilon, \delta A_a)$. The action of a gauge transformation $\epsilon = \epsilon_{(0)} + \Omega\epsilon_{(1)} + ...$ on the variables is

$$I_\epsilon \delta A_{i(k)} = \partial_i \epsilon_{(k)}, \qquad I_\epsilon \delta A_{\Omega(k)} = (k+1)\epsilon_{(k+1)}, \qquad I_\epsilon \delta \mathcal{J}^i_{(k)} = 0. \tag{5.89}$$

Using the conservation of the conformal current $\partial_i(\sqrt{q}\mathcal{J}^i_{(k)}) = 0$, we obtain for the asymptotic renormalized on-shell generators

$$\hat{H}^R_\epsilon = \left[\hat{Q}^R_\epsilon\right]^f_i \qquad \text{where} \qquad \hat{Q}^R_\epsilon = \sum_{k=0}^{D-5} \oint_S \sqrt{q}\mathcal{J}^u_{(D-5-k)}\epsilon_{(k)} \tag{5.90}$$

and $[X]^f_i := X(u_f) - X(u_i)$. This expression is manifestly finite, and should be contrasted with the generators obtained from the standard symplectic form $\hat{\omega} = \Omega^{-(D-5)} \int_{I(\Omega)} \sqrt{q}\delta\mathcal{J}^i \curlywedge \delta A_i$, which read $\hat{H}_\epsilon = [\hat{Q}_\epsilon]^f_i$ with

$$\hat{Q}_\epsilon(\Omega) = \Omega^{-(D-5)} \oint_{S(\Omega)} \sqrt{q}\mathcal{J}^u \epsilon \tag{5.91}$$

and diverge in the asymptotic limit (unless one puts extra restrictions on the space of asymptotic data).

Observe that just as the renormalized SP coincides asymptotically with the finite part of the Laurent series of the standard SP, the renormalized generators are the finite part of the standard generators. The "layering" structure also transfers from the SP to the charges.

5.6.2 Extension Independent Symplectic Potential and Charges

From (5.90), one may wonder if the dependence of the charges on the radial derivatives of the gauge parameter (or asymptotically, on its subleading orders) is an essential feature of higher dimensions. If so, there would be not one, but $(D-5)$ "sphere's-worth" of physical symmetries.

But as we will see, the extension dependence of the charges can be removed by adding further, finite corner counterterms involving δA_Ω to the SP. In that sense, the extension dependence is not essential. We call the resulting SP, with the extension dependence removed, the *total* SP.

A corner term $\hat{\boldsymbol{\theta}}^a_R \mapsto \hat{\boldsymbol{\theta}}^a_{\rm tot} = \hat{\boldsymbol{\theta}}^a_R + \partial_b \boldsymbol{\alpha}^{ab}_{\rm tot}$ modifies the Hamiltonians as

$$\hat{H}^R_\epsilon \mapsto \hat{H}^{\rm tot}_\epsilon = \hat{H}^R_\epsilon + I_\epsilon \delta \oint_S [\hat{\boldsymbol{\alpha}}^{\Omega u}_{\rm tot}]^f_i . \tag{5.92}$$

Noting that for $k \geq 1$, $\epsilon_{(k)} = \frac{1}{k} I_\epsilon \delta A_{\Omega(k-1)}$, the extension dependence of the charges is easily removed by setting

$$\boldsymbol{\alpha}^{\Omega i}_{\rm tot} \overset{\mathscr{I}}{=} \sum_{k=1}^{D-5} \frac{1}{k} \mathcal{J}^i_{(D-5-k)} \delta A_{\Omega(k-1)}, \tag{5.93}$$

leaving the total Hamiltonian

$$H^{\rm tot}_\epsilon \overset{\mathscr{I}}{=} \oint_S \sqrt{q} \big[\mathcal{J}^u_{(D-5)} \epsilon_{(0)} \big]^f_i . \tag{5.94}$$

Subleading gauge transformations are in the kernel of the total symplectic form, and should thus be regarded as pure gauge transformations. Only one “sphere's-worth” of physical symmetries survive, which are generated by the charge aspect $\mathcal{J}^u_{(D-5)} = \sigma$.

Removing the extension dependence of the charges in this way relies on the gauge invariance of the current $\mathcal{J}^i$. The counterterms $\boldsymbol{\alpha}^{\Omega i}_{\rm tot}$ can also easily be extended to spacetime-local expressions, which cancel the dependence of Hamiltonians on $\partial_\Omega \epsilon$ also at finite distance. Asymptotically and on-shell, the total SP may also be obtained by replacing, for $k \geq 1$, $A_{i(k)} \mapsto A^{\rm g.f.}_{i(k)} = A_i - \frac{1}{k} \partial_i A_{\Omega(k-1)}$ in the expression (5.39) for the layers of the SP. It is thus closely related to going to radial gauge.

5.6.3 Connecting to Soft Theorems

Let us compare the expression (5.94) for the symmetry Hamiltonian to the results of [90]. There, starting from the QED soft theorem in dimensions $D = 6 + 2\ell$, the authors derive the charge expression whose Ward identity encodes the soft theorem. They then fix the classical Poisson brackets, or equivalently the symplectic form, by demanding that the charge expression generate the correct gauge transformations of the gauge field $A_{A(0)}$. Here, we took a different route. We determined the symplectic form using the covariant Hamiltonian formalism and our renormalization procedure, and derived the charge from the symplectic form rather than deriving the symplectic form from the charge.

The charge expression of [90] coincides with our $\hat{H}_\epsilon^{\text{tot},R}$, for $u_i \to -\infty$, $u_f \to +\infty$, and under the assumption made in [90] that $\sigma(u_f) = 0$. That is, using (5.81), in $D = 6$ we find

$$\hat{H}_\epsilon^{\text{tot},R} \to \oint_S \sqrt{q}\, \epsilon_{(0)} \sigma_i = -\frac{1}{2} \oint_S \sqrt{q}\, \epsilon_{(0)} (D^A D_A - 2) \langle D_A \mathcal{A}^A \rangle \tag{5.95}$$

In higher dimensions, the correct generalization is obtained through equations (5.85), and also coincides with the results of [90]

$$\begin{aligned} \hat{H}_\epsilon^{\text{tot},R} &\to \oint_S \sqrt{q}\, \epsilon_{(0)} \sigma_i \\ &= -\frac{(-1)^\ell}{2^{(\ell+1)}} \frac{1}{(\ell+1)!} \oint_S \sqrt{q}\, \epsilon_{(0)} \left[\Delta^\ell_{2\ell+1} \Delta^\ell_{2\ell} \cdots \Delta^\ell_{\ell+p} \cdots \Delta^\ell_{\ell+1} \right] \langle D_A \mathcal{A}^A \rangle_{(\ell)} \end{aligned} \tag{5.96}$$

In particular, the "soft-theorem charge" is not the total radial electric field, which would lead to divergent charges, but only the finite part of its Laurent series, which is the charge aspect σ. The agreement of the charge obtained from the renormalization procedure with the charge obtained from soft theorems supports the physical viability of the asymptotic renormalization procedure in gauge theories.

Chapter 6

Asymptotic Renormalization of the Canonical Structure of General Relativity

In this chapter, we renormalize the asymptotic symplectic potential (SP) and action of metric general relativity on asymptotically simple spacetimes, using a scheme similar to that of chapter 5. See page iv for acknowledgments.

Section 6.1 defines asymptotic simplicity and our variables, writes the Einstein equations in those variables, solves their two most divergent orders, and introduces background structure we will need. The renormalization is carried out in section 6.2. In section 6.3, we connect our results to the SP of [28], who work in a generalized Bondi gauge. In section 6.4, we connect to holographic renormalization in asymptotically AdS spacetimes, using Fefferman-Graham coordinates in four and five spacetime dimensions. In section 6.5 we give the diffeomorphism generators (if they exist) of the renormalized symplectic form. Technical manipulations are in the appendices C.

6.1 Asymptotic Simplicity, Variables and Einstein Equations

Following [43], by an asymptotically simple spacetime $\hat{M}$ with metric $\hat{g}_{ab}$ we shall mean a spacetime which satisfies the following:[1]

- There exists a manifold M with boundary $\mathcal{I}$ and metric g_{ab}, whose interior is conformal to $\hat{M}$, with $g_{ab} = \Omega^2 \hat{g}_{ab}$.
- M and and g_{ab} are suitably regular in a neighbourhood of $\mathcal{I}$.
- $\Omega = 0$ on $\mathcal{I} := \partial M$, $\nabla_a \Omega \neq 0$ on $\mathcal{I}$, and Ω is suitably regular in a neighborhood of $\mathcal{I}$.

Asymptotic simplicity captures asymptotically de Sitter, anti-de Sitter as well as flat spacetimes of any dimension. We will call $\hat{g}$ the physical metric, refer to g as the conformal or unphysical metric, and set $8\pi G = 1$.

Let us first write the Einstein equations using the conformal metric g_{ab}: we get

$$\begin{aligned} 2\hat{E}_{ab} &= \hat{G}_{ab} - \Lambda \hat{g}_{ab} \\ &= G_{ab} + \Omega^{-1}(D-2)(\nabla_a N_b - g_{ab}\nabla_c N^c) + \Omega^{-2}(\Lambda g_{ab} + \frac{1}{2}(D-1)(D-2)N_c N^c g_{ab}), \end{aligned} \tag{6.1}$$

where $N_a = \nabla_a \Omega$ is the unnormalized, inward pointing normal to the $\Omega =$ const. surfaces. For the Einstein equations to be satisfied, in particular $\Omega^2 \hat{E}_{ab}$ must limit to zero on $\mathcal{I}$:

$$\Lambda g_{ab} + \frac{1}{2}(D-1)(D-2)N_c N^c g_{ab} \stackrel{\mathcal{I}}{=} 0. \tag{6.2}$$

We introduce the reduced cosmological constant $\lambda = \frac{1}{(D-1)(D-2)}\Lambda$. Then the last equation may be solved by requiring $\frac{1}{2}N_c N^c + \lambda = O(\Omega)$. We thus introduce

$$\chi := \frac{1}{\Omega}(\frac{1}{2}N_c N^c + \lambda), \tag{6.3}$$

and require that it has a finite limit onto $\mathcal{I}$. With that definition, the Einstein equations become

$$2\hat{E}_{ab} = G_{ab} + \Omega^{-1}(D-2)\big(\nabla_a N_b - g_{ab}\nabla_c N^c + (D-1)g_{ab}\chi\big). \tag{6.4}$$

[1] [43] also includes the global condition that every null geodesic on M has two endpoints on its boundary $\mathcal{I}$. Since we are working locally, we won't need that condition.

The next order may then be implemented by requiring that

$$\nabla_a N_b - \chi g_{ab} \overset{\mathcal{I}}{=} 0. \tag{6.5}$$

As before, we implement this by introducing:

$$\mathcal{N}_{ab} := \frac{1}{\Omega}(\nabla_a N_b - \chi g_{ab}), \tag{6.6}$$

and require that $\mathcal{N}_{ab}$ has a finite limit onto $\mathcal{I}$. Because N_a is an exact form on M, $\mathcal{N}_{ab}$ is symmetric. One may check that $N^a \mathcal{N}_{ab} = \partial_b \chi$. With those definitions and requirements, the Einstein equations finally become

$$\hat{E}_{ab} = \frac{1}{2}\Big(G_{ab} + (D-2)(\mathcal{N}_{ab} - g_{ab}\mathcal{N})\Big), \tag{6.7}$$

where $\mathcal{N} = g^{ab}\mathcal{N}_{ab}$. Let us comment that we view N_a, χ and $\mathcal{N}_{ab}$ as fields on the whole of M, not just at $\mathcal{I}$. We have not employed the Bondi condition on the conformal factor, i.e., we allow $\Omega \nabla_a N^a \overset{\mathcal{I}}{\neq} 0$.

The combination $\mathcal{N}_{ab} - g_{ab}\mathcal{N}$ is conserved on-shell:

$$\nabla_a (\mathcal{N}^a{}_b - \delta^a_b \mathcal{N}) \hat{=} 0. \tag{6.8}$$

That can be seen by taking a divergence, with the unphysical derivative ∇^a, of (6.7), and using the Bianchi identity of G_{ab}.

As in chapter 5, we will regard the conformal factor Ω and the vector field ∂_Ω as fixed background structures. We may then decompose vectors as

$$X^a \partial_a = N_a X^a \partial_\Omega + X^i \partial_i, \tag{6.9}$$

where i indexes coordinates on the $\Omega = \text{const.}$ surfaces. We will also write P^i_a for the coordinate projector onto the $\Omega = \text{const.}$ surfaces, i.e.

$$\xi^a P^i_a \partial_i := \xi^a \partial_a - N_b \xi^b \partial_\Omega. \tag{6.10}$$

6.2 Renormalization of Symplectic Potential and Action

In this section, we will renormalize the normal component $\hat{\boldsymbol{\theta}}^\Omega$ of the SP such that it is finite in Ω, and renormalize the transverse components $\hat{\boldsymbol{\theta}}^i$ as well as the Lagrangian $\hat{\boldsymbol{L}}$ up to terms in $\ln\Omega$, such that their integrals are asymptotically finite. To do so, we will use the ambiguities

$$\hat{\boldsymbol{\theta}}^a \mapsto \hat{\boldsymbol{\theta}}^a + \delta\hat{\boldsymbol{\beta}}^a + \partial_b\hat{\boldsymbol{\alpha}}^{ab}, \hat{\boldsymbol{L}} \mapsto \hat{\boldsymbol{L}} + \partial_a\hat{\boldsymbol{\beta}}^a, \tag{6.11}$$

and construct counterterms $\hat{\boldsymbol{\beta}}^a$ and $\hat{\boldsymbol{\alpha}}^{ab} = -\hat{\boldsymbol{\alpha}}^{ba}$ that are local in the metric and its variations.

The renormalization proceeds in two steps: in the first step, we identify counterterms that ensure $\hat{\boldsymbol{\theta}}^\Omega = \mathcal{O}(\Omega^{-(D-3)}), \hat{\boldsymbol{\theta}}^i = \mathcal{O}(\Omega^{-(D-2)})$ and $\hat{\boldsymbol{L}} = \mathcal{O}(\Omega^{-(D-2)})$. In the second step, the remainder of the negative orders in Ω are removed using a recursion.

The second step is exactly analogous to the procedure for electromagnetism in chapter 5. The first step does not have an analogue for electromagnetism, and becomes necessary because of inhomogeneous transformations under conformal rescalings of the metric, which lead to additional, higher divergences in the Lagrangian and SP.

6.2.1 First Step

The standard choice of SP for the Einstein–Hilbert action $\hat{\boldsymbol{L}} = \frac{1}{2}\sqrt{\hat{g}}(\hat{R} - 2\Lambda)$ reads:

$$\hat{\boldsymbol{\theta}}^\Omega = \frac{1}{2}\sqrt{\hat{g}}N_a(\hat{g}^{bc}\delta\hat{\Gamma}^a_{bc} - \hat{g}^{ab}\delta\hat{\Gamma}^c_{bc}). \tag{6.12}$$

Let us translate this into the unphysical metric. $\delta\hat{\Gamma}$ is related to $\delta\Gamma$ as:

$$\delta\hat{\Gamma}^a_{bc} = \delta\Gamma^a_{bc} + \frac{1}{\Omega}\delta(N^a g_{bc}). \tag{6.13}$$

Plugging that in, and replacing $\sqrt{\hat{g}}\hat{g}^{ab} = \Omega^{-(D-2)}\sqrt{g}g^{ab}$, gives

$$\begin{aligned}\hat{\boldsymbol{\theta}}^\Omega = &\frac{1}{2}\Omega^{-(D-1)}\big(\sqrt{g}N_a g^{bc}\delta(N^a g_{bc})\big)\\ &+ \frac{1}{2}\Omega^{-(D-2)}\sqrt{g}N_a\big(g^{bc}\delta\Gamma^a_{bc} - g^{ab}\delta\Gamma^c_{bc}\big).\end{aligned} \tag{6.14}$$

To treat the first line, we distribute the variation, and use that $\sqrt{g}g^{bc}\delta g_{bc} = 2\delta\sqrt{g}$, and $N_a N^a = 2\Omega\chi - 2\lambda$. We will also need that

$$N_a\delta N^a = \delta(N_a N^a) = 2\Omega\delta\chi, \tag{6.15}$$

which follows using (6.3) because Ω does not vary and hence N_a does not vary, and λ is constant as well. We get:

$$\begin{aligned}\hat{\boldsymbol{\theta}}^{\Omega} &= \Omega^{-(D-1)}\big(-2\lambda\delta\sqrt{g}\big)\\ &\quad + \Omega^{-(D-2)}\Big(D\sqrt{g}\delta\chi + 2\chi\delta\sqrt{g} + \frac{1}{2}\sqrt{g}N_a\big(g^{bc}\delta\Gamma^a_{bc} - g^{ab}\delta\Gamma^c_{bc}\big)\Big). \end{aligned} \tag{6.16}$$

We see that the most divergent piece of the standard SP goes as $\Omega^{-(D-1)}$, and is absent in asymptotically flat spacetimes. It is a total variation which can be absorbed into a boundary action.

To continue, we focus on the piece which looks like the rescaled SP of the Einstein-Hilbert action of the metric g_{ab}:

$$A := \frac{1}{2}\sqrt{g}N_a\big(g^{bc}\delta\Gamma^a_{bc} - g^{ab}\delta\Gamma^c_{bc}\big) = \frac{1}{2}\Big(\sqrt{g}N^a\nabla^b\delta g_{ab} - \sqrt{g}N^a\nabla_a\delta g\Big), \tag{6.17}$$

where $\delta g = g^{ab}\delta g_{ab}$. For the first term, we integrate by parts and "vary by parts", giving

$$N^a\nabla^b\delta g_{ab} = \nabla^b(N^a\delta g_{ab}) - (\nabla^b N^a)\delta g_{ab} = -\nabla_a\delta N^a - (\nabla^b N^a)\delta g_{ab}. \tag{6.18}$$

For the second term of (6.17), we use that

$$N^a\nabla_a\delta g = 2\delta(\nabla_a N^a) - 2\nabla_a\delta N^a, \tag{6.19}$$

which may be shown by noting $\sqrt{g}\nabla_a N^a = \partial_a(\sqrt{g}N^a)$ and varying both sides. Combining the previous two equations yields

$$A = \frac{1}{2}\sqrt{g}\big(\nabla_a\delta N^a - (\nabla^a N^b)\delta g_{ab} - 2\delta(\nabla_a N^a)\big). \tag{6.20}$$

For the first term, let us split the divergence into parts tangential and transverse to $\Omega =$ const. according to (6.9):

$$\begin{aligned}\sqrt{g}\nabla_a\delta N^a &= \partial_a(\sqrt{g}\delta N^a) = \partial_i(\sqrt{g}\delta N^i) + \partial_\Omega(2\Omega\sqrt{g}\delta\chi)\\ &= \partial_i(\sqrt{g}\delta N^i) + 2\sqrt{g}\delta\chi + 2\Omega\partial_\Omega(\sqrt{g}\delta\chi),\end{aligned} \tag{6.21}$$

where we used (6.15) again. For the second and third term of (6.20), we use the definition of $\mathcal{N}_{ab}$, which may be inverted as

$$\nabla_a N_b = \Omega\mathcal{N}_{ab} + \chi g_{ab}, \qquad \nabla_a N^a = \Omega\mathcal{N} + D\chi. \tag{6.22}$$

Together we get

$$A = (-D+1)\sqrt{g}\delta\chi - \chi\delta\sqrt{g} + \frac{1}{2}\partial_i(\sqrt{g}\delta N^i) + \Omega\big(-\frac{1}{2}\sqrt{g}\mathcal{N}^{ab}\delta g_{ab} - \sqrt{g}\delta\mathcal{N} + \partial_\Omega(\sqrt{g}\delta\chi)\big). \quad (6.23)$$

We change polarization in the term proportional to Ω, plug into (6.16), and combine $\chi\delta\sqrt{g} + \sqrt{g}\delta\chi = \delta(\sqrt{g}\chi)$ to give:

$$\begin{aligned}\hat{\boldsymbol{\theta}}^\Omega &= \Omega^{-(D-1)}\big(-2\lambda\delta\sqrt{g}\big)\\ &\quad + \Omega^{-(D-2)}\Big(\delta(\sqrt{g}\chi) + \frac{1}{2}\partial_i(\sqrt{g}\delta N^i)\Big)\\ &\quad + \Omega^{-(D-3)}\boldsymbol{\theta}^n, \quad \text{where}\\ \boldsymbol{\theta}^n &:= -\frac{1}{2}\sqrt{g}(\mathcal{N}^{ab} - g^{ab}\mathcal{N})\delta g_{ab} + \partial_\Omega(\sqrt{g}\delta\chi) - \delta(\sqrt{g}\mathcal{N}). \end{aligned} \quad (6.24)$$

We see that the two most divergent orders of $\hat{\boldsymbol{\theta}}^\Omega$ may be absorbed into counterterms. The remainder goes as $\Omega^{-(D-3)}$ by virtue of the two most divergent orders of the Einstein equations.

Let us turn to the transverse components $\hat{\boldsymbol{\theta}}^i$. Plugging in (6.13) as before, we get

$$\begin{aligned}\hat{\boldsymbol{\theta}}^i &= \frac{1}{2}\sqrt{\hat{g}}(\hat{g}^{bc}\delta\hat{\Gamma}^i_{bc} - \hat{g}^{ib}\delta\hat{\Gamma}^c_{bc})\\ &= \frac{1}{2}\Omega^{-(D-2)}\sqrt{g}\big(g^{bc}\delta\Gamma^i_{bc} - g^{ib}\delta\Gamma^c_{bc} + \frac{1}{\Omega}\sqrt{g}g^{bc}\delta(N^i g_{bc})\big). \end{aligned} \quad (6.25)$$

Distributing the variation in the last term gives

$$\begin{aligned}\hat{\boldsymbol{\theta}}^i &= \Omega^{-(D-1)}(\frac{D}{2}\sqrt{g}\delta N^i + N^i\delta\sqrt{g}) + \Omega^{-(D-2)}\boldsymbol{\theta}^i\\ &= -\frac{1}{2}\partial_\Omega\Big(\Omega^{-(D-2)}\sqrt{g}\delta N^i\Big) + \Omega^{-(D-1)}\delta(\sqrt{g}N^i) + \Omega^{-(D-2)}\big(\boldsymbol{\theta}^i + \frac{1}{2}\partial_\Omega(\sqrt{g}\delta N^i)\big). \end{aligned} \quad (6.26)$$

In the second line, we have extracted a total ∂_Ω-derivative and a total variation. We have abbreviated the SP of the Einstein-Hilbert action of the unphysical metric as

$$\boldsymbol{\theta}^i := \frac{1}{2}\sqrt{g}(g^{bc}\delta\Gamma^i_{bc} - g^{ib}\delta\Gamma^c_{bc}). \quad (6.27)$$

We see that also the most divergent order of the transverse SP may be absorbed into counterterms. To remove the most divergent orders, let us define the counterterms which will lead to the partially renormalized SP $\hat{\boldsymbol{\theta}}_1^a$: We set

$$\begin{aligned}
&\hat{\boldsymbol{\theta}}_1^a := \hat{\boldsymbol{\theta}}^a + \partial_b \hat{\boldsymbol{\alpha}}_1^{ab} + \delta \hat{\boldsymbol{\beta}}_1^a, \qquad \text{where} \\
&\hat{\boldsymbol{\beta}}_1^\Omega = \Omega^{-(D-1)}(2\lambda\sqrt{g}) + \Omega^{-(D-2)}(-\sqrt{g}\chi), \qquad \hat{\boldsymbol{\beta}}_1^i = -\Omega^{-(D-1)}\sqrt{g}N^i \\
&\hat{\boldsymbol{\alpha}}_1^{\Omega i} = -\hat{\boldsymbol{\alpha}}_1^{i\Omega} = -\frac{1}{2}\Omega^{-(D-2)}\sqrt{g}\delta N^i, \qquad \hat{\boldsymbol{\alpha}}_1^{ij} = 0.
\end{aligned} \tag{6.28}$$

Using (6.24) and (6.26), the partially renormalized SP becomes

$$\begin{aligned}
\hat{\boldsymbol{\theta}}_1^\Omega &= \Omega^{-(D-3)}\boldsymbol{\theta}^n \\
\hat{\boldsymbol{\theta}}_1^i &= \Omega^{-(D-2)}\big(\boldsymbol{\theta}^i + \frac{1}{2}\partial_\Omega(\sqrt{g}\delta N^i)\big).
\end{aligned} \tag{6.29}$$

Let us turn to the Lagrangian. By taking the trace of (6.7), we get that

$$\hat{\boldsymbol{L}} = \sqrt{g}(\hat{R} - 2\Lambda) = \Omega^{-D} \cdot 2(D-1)\lambda\sqrt{g} + \Omega^{-(D-2)} \cdot \sqrt{g}\big(\frac{1}{2}R + (D-1)\mathcal{N}\big). \tag{6.30}$$

The Lagrangian will be modified by $\partial_a \hat{\boldsymbol{\beta}}_1^a$. $\hat{\boldsymbol{\beta}}_1$ may also be written as:

$$\hat{\boldsymbol{\beta}}_1^a = -\Omega^{-(D-1)}\sqrt{g}N^a + \Omega^{-(D-2)}\sqrt{g}\chi\delta_\Omega^a. \tag{6.31}$$

Its divergence then becomes, using the definitions of χ and of $\mathcal{N}$:

$$\partial_a \hat{\boldsymbol{\beta}}_1^a = -2(D-1)\Omega^{-D}\lambda\sqrt{g} + \Omega^{-(D-2)}\big(-\sqrt{g}\mathcal{N} + \partial_\Omega(\sqrt{g}\chi)\big) \tag{6.32}$$

The partially renormalized Lagrangian is thus

$$\hat{\boldsymbol{L}}_1 = \hat{\boldsymbol{L}} + \partial_a \hat{\boldsymbol{\beta}}_1^a = \Omega^{-(D-2)}\big(\frac{1}{2}\sqrt{g}R + (D-2)\sqrt{g}\mathcal{N} + \partial_\Omega(\sqrt{g}\chi)\big). \tag{6.33}$$

To summarize, we have identified local counterterms $\hat{\boldsymbol{\alpha}}_1$, $\hat{\boldsymbol{\beta}}_1$, which remove the most leading divergences from the Lagrangian and SP. Asymptotic simplicity implies that the partially renormalized transverse SP $\hat{\boldsymbol{\theta}}_1^i$ and Lagrangian $\hat{\boldsymbol{L}}_1$ go as $\Omega^{-(D-2)}$, and the normal component of the SP $\hat{\boldsymbol{\theta}}^\Omega$ goes as $\Omega^{-(D-3)}$.

6.2.2 Second Step

We will now turn to removing the remaining divergences in the SP and action with a recursive scheme, exactly along the lines of chapter 5 (with the replacement $(D-5) \mapsto (D-3)$, since electrodynamics is conformal in four dimensions and gravity in two). For self-containedness, we briefly repeat the arguments here. The partially renormalized Lagrangian and SP are linked by

$$\delta\hat{\boldsymbol{L}}_1 = -\hat{\boldsymbol{E}}^{ab}\delta\hat{g}_{ab} + \partial_a\hat{\boldsymbol{\theta}}_1^a. \tag{6.34}$$

This follows from $\delta\hat{\boldsymbol{L}} = -\hat{\boldsymbol{E}}^{ab}\delta\hat{g}_{ab} + \partial_a\hat{\boldsymbol{\theta}}^a$, together with the definitions of $\hat{\boldsymbol{L}}_1$ and $\hat{\boldsymbol{\theta}}_1^a$, and from $\partial_a\partial_b\hat{\boldsymbol{\alpha}}_1^{ab} = 0$ by antisymmetry of $\hat{\boldsymbol{\alpha}}_1$. Splitting the divergence in the last equations gives

$$(D-3-\Omega\partial_\Omega)\boldsymbol{\theta}^n \hat{=} \partial_i\boldsymbol{\theta}_1^i - \delta\boldsymbol{L}_1, \tag{6.35}$$

where we have dropped EoM-terms and written

$$\boldsymbol{\theta}_1^i := \Omega^{D-2}\hat{\boldsymbol{\theta}}_1^i = \boldsymbol{\theta}^i + \frac{1}{2}\partial_\Omega(\sqrt{g}\delta N^i) \text{ and} \tag{6.36}$$

$$\boldsymbol{L}_1 := \Omega^{D-2}\hat{\boldsymbol{L}}_1 = \frac{1}{2}\sqrt{g}R + (D-2)\sqrt{g}\mathcal{N} + \partial_\Omega(\sqrt{g}\chi) \tag{6.37}$$

for the finite versions of $\hat{\boldsymbol{\theta}}_1^i$ and $\hat{\boldsymbol{L}}_1$. Recall that $\hat{\boldsymbol{\theta}}_1^\Omega = \Omega^{-(D-3)}\boldsymbol{\theta}^n$. We introduce the operator

$$\mathcal{D} = \sum_{k=0}^{D-4}\frac{\big(D-3-(k+1)\big)!}{(D-3)!}\Omega^k{\partial_\Omega}^k - \frac{\Omega^{D-3}}{(D-3)!}\ln\Omega{\partial_\Omega}^{D-3}. \tag{6.38}$$

Acting with it on (6.35) gives

$$\boldsymbol{\theta}^n - \frac{\Omega^{D-3}}{(D-3)!}({\partial_\Omega}^{D-3} - \Omega\ln\Omega{\partial_\Omega}^{D-2})\boldsymbol{\theta}^n \hat{=} \partial_i(\mathcal{D}\boldsymbol{\theta}_1^i) - \delta(\mathcal{D}\boldsymbol{L}_1). \tag{6.39}$$

This holds on-shell and at any Ω, and allows the absorption of the remaining divergences in $\hat{\boldsymbol{\theta}}_1^\Omega$ into additional counterterms. The counterterms are

$$\begin{aligned} \hat{\boldsymbol{\alpha}}_2^{\Omega i} &= -\hat{\boldsymbol{\alpha}}_2^{i\Omega} = -\Omega^{-(D-5)}\mathcal{D}(\boldsymbol{\theta}_1^i), & \hat{\boldsymbol{\alpha}}_2^{ij} &= 0 \\ \hat{\boldsymbol{\beta}}_2^\Omega &= \Omega^{-(D-5)}\mathcal{D}(\boldsymbol{L}_1), & \hat{\boldsymbol{\beta}}_2^i &= 0 \end{aligned} \tag{6.40}$$

The renormalized SP and Lagrangian become

$$\begin{aligned} \hat{\boldsymbol{\theta}}_R^\Omega &= \hat{\boldsymbol{\theta}}_1^\Omega + \delta\hat{\boldsymbol{\beta}}_2^\Omega\partial_i\hat{\boldsymbol{\alpha}}_2^{\Omega i} \hat{=} \frac{1}{(D-3)!}({\partial_\Omega}^{D-3} - \Omega\ln\Omega{\partial_\Omega}^{D-2})\boldsymbol{\theta}^n \\ \hat{\boldsymbol{\theta}}_R^i &= \hat{\boldsymbol{\theta}}_1^i + \partial_\Omega\hat{\boldsymbol{\alpha}}_2^{i\Omega} = -\frac{1}{(D-3)!}\ln\Omega{\partial_\Omega}^{D-2}\boldsymbol{\theta}_1^i \\ \hat{\boldsymbol{L}}_R &= \hat{\boldsymbol{L}}_1 + \partial_\Omega\hat{\boldsymbol{\beta}}_2^\Omega = -\frac{1}{(D-3)!}\ln\Omega{\partial_\Omega}^{D-2}\boldsymbol{L}_1. \end{aligned} \tag{6.41}$$

For the second and third line, we have used that

$$\partial_\Omega\big(\Omega^{-(D-3)}\mathcal{D}\big) = -\Omega^{-(D-2)} - \frac{1}{(D-3)!}\ln\Omega\,\partial_\Omega{}^{D-3}. \tag{6.42}$$

The component $\hat{\boldsymbol{\theta}}^\Omega_R$, which determines the symplectic form at infinity, is finite in Ω assuming sufficient analyticity. The analyticity requirement is that $\boldsymbol{\theta}^n$ may have a term proportional to $\Omega^{D-3}\ln\Omega$, but any other logarithms must be accompanied by at least Ω^{D-2}. While the transverse components of the SP current $\hat{\boldsymbol{\theta}}^i_R$ and the Lagrangian $\hat{\boldsymbol{L}}_R$ still diverge as $\ln\Omega$, their integrals in Ω are asymptotically finite.

Let us note that if one is only concerned with renormalizing $\hat{\boldsymbol{\theta}}^\Omega$, and assumes analytic field expansions, then the term in $\ln\Omega$ may be safely dropped from $\mathcal{D}$. In that case, the renormalized SP becomes

$$\hat{\boldsymbol{\theta}}^\Omega_{\text{no ln}} = \frac{1}{(D-3)!}\partial_\Omega{}^{D-3}\boldsymbol{\theta}^n. \tag{6.43}$$

In subsection 6.4.3 we will see a case where the logarithmic term is necessary.

6.2.3 The Renormalized Symplectic Potential

We have thus completed the renormalization of the SP and action, by adding total variation and total derivative counterterms local in the fields.

The renormalized SP we have given works for asymptotically AdS, dS, and flat space, and is continuous in the cosmological constant. To achieve this, we needed to treat ∂_Ω and N^a as separate objects, and make no condition on the norm of N. However, the terms in the renormalized SP are not all independent. The structure of their dependencies is different depending on the cosmological constant, but by staying sufficiently off-shell, and not resolving these dependencies, the renormalized SP can be valid for any Λ. The renormalized SP may thus be a useful tool in connecting asymptotic structures of asymptotically flat and AdS spaces - either by finding analogues of BMS and extended BMS symmetries and conservation laws on (A)dS space (see, e.g., [113]) or by finding analogues of holographic structures on flat space (see, e.g., [36]).

In addition to the ADM-like pair $(\mathcal{N}^{ab} - g^{ab}\mathcal{N})\delta g_{ab}$, and the term $\delta(\sqrt{g}\mathcal{N})$ reminiscent of the Gibbons-Hawking term, the SP $\boldsymbol{\theta}^n$ contains the term $\partial_\Omega(\sqrt{g}\delta\chi)$, which is more puzzling. Its role can be understood as follows: Consider a vector field ξ^i, parallel to the level surfaces of Ω. The quantities χ, g_{ab} and $\mathcal{N}_{ab}$ are covariant under such a vector field, i.e., they transform by the Lie derivative. Hence we get

$$I_\xi\boldsymbol{\theta}^n = -\sqrt{g}(\mathcal{N}^{ab} - g^{ab}\mathcal{N})\nabla_a\xi_b + \partial_\Omega(\sqrt{g}\xi^i\partial_i\chi) - \partial_i(\sqrt{g}\mathcal{N}\chi^i). \tag{6.44}$$

Now let us integrate by parts in the first term, and use $\nabla_a(\mathcal{N}^{ab} - \mathcal{N} g^{ab}) \hat{=} 0$. Splitting the resulting divergence $-\partial_a(\sqrt{g}(\mathcal{N}^a{}_b - \delta^a{}_b\mathcal{N})\xi^b)$ as usual, we get

$$I_\xi \boldsymbol{\theta}^n = \; -\partial_i(\sqrt{g}\mathcal{N}^i{}_j\xi^j) - \partial_\Omega(\sqrt{g}\mathcal{N}^\Omega{}_i\xi^i) + \partial_\Omega(\sqrt{g}\xi^i\partial_i\chi). \tag{6.45}$$

Lastly, using that $\mathcal{N}^\Omega{}_i = N^a\mathcal{N}_{ai} = \partial_i\chi$, the last two terms cancel and we are left with $I_\xi\boldsymbol{\theta}^n = -\partial_i(\sqrt{g}\mathcal{N}^i{}_j\xi^j)$. This is a divergence on the $\Omega = $ const. surfaces.

The $\delta\chi$ -term in $\boldsymbol{\theta}^n$ thus ensures that $I_\xi\boldsymbol{\theta}^n$ leads to a conservation law intrinsic to the level surfaces of Ω, equating the divergence of a boundary current to a flux. This is analogous to the conservation laws on a finite distance null surface of chapter 4.

6.2.4 Laurent Expansions and Higher Divergences

One may wonder what the relationship of our renormalized SP with the Laurent expansion of the standard SP is. Let us assume full analyticity, and expand the standard SP as

$$\hat{\boldsymbol{\theta}}^\Omega = \sum_{k=-D}^{\infty} \hat{\boldsymbol{\theta}}^\Omega_{(k)}\Omega^k. \tag{6.46}$$

The partially renormalized SP differs from the standard SP by a corner term and a variation, see (6.28). Its orders are thus

$$\hat{\boldsymbol{\theta}}^\Omega_1 = \sum_{k=-(D-3)}^{\infty} (\hat{\boldsymbol{\theta}}^\Omega_{(k)} + \partial_i\hat{\boldsymbol{\alpha}}^{\Omega i}_{1,(k)} + \delta\hat{\boldsymbol{\beta}}^\Omega_{1,(k)}). \tag{6.47}$$

The orders $(-D)$ to $-(D-2)$ of $\hat{\boldsymbol{\theta}}^\Omega_1$ vanish, by construction of the counterterms $\hat{\boldsymbol{\alpha}}_1$ and $\hat{\boldsymbol{\beta}}_1$. Finally, the asymptotic renormalized SP corresponds to the finite order of $\hat{\boldsymbol{\theta}}^\Omega_1$. This may be seen from (6.41): The $(\Omega \ln \Omega)$-term does not contribute in the limit assuming analyticity, so we are left with

$$\hat{\boldsymbol{\theta}}^\Omega_R \overset{\mathcal{I}}{=} \frac{1}{(D-3)!}{\partial_\Omega}^{D-3}(\Omega^{D-3}\hat{\boldsymbol{\theta}}^\Omega_1), \tag{6.48}$$

which picks out the finite order. To summarize, we have

$$\hat{\boldsymbol{\theta}}^\Omega_R \overset{\mathcal{I}}{=} \hat{\boldsymbol{\theta}}^\Omega_{1,(0)} = \hat{\boldsymbol{\theta}}^\Omega_{(0)} + \partial_i\hat{\boldsymbol{\alpha}}^{\Omega i}_{1,(0)} + \delta\hat{\boldsymbol{\beta}}^\Omega_{1,(0)}. \tag{6.49}$$

Assuming analyticity, the limit of the renormalized SP $\hat{\boldsymbol{\theta}}^\Omega_R$ agrees with the finite order of the partially renormalized SP $\hat{\boldsymbol{\theta}}^\Omega_1$, and differs from the finite order of the standard SP $\hat{\boldsymbol{\theta}}^\Omega$ by the finite orders of the counterterms $\hat{\boldsymbol{\alpha}}_1$ and $\hat{\boldsymbol{\beta}}_1$.

Let us also note that the renormalization can also be carried out when higher divergences are present, as long as the metric admits a Laurent expansion around $\Omega = 0$: If the standard SP diverges as Ω^{-n}, we can simply carry out the second step of the previous section, using the operator

$$\mathcal{D}_{(n)} = \sum_{k=0}^{n-1} \frac{\big(n-(k+1)\big)!}{n!} \Omega^k {\partial_\Omega}^k - \frac{\Omega^n}{n!} \ln \Omega {\partial_\Omega}^n, \tag{6.50}$$

instead of choosing $n = D - 3$ as above. Asymptotically, the renormalized SP is then still the finite order of $\hat{\boldsymbol{\theta}}_1^\Omega$. This may be useful for considering spacetimes with metrics diverging faster than is allowed by asymptotic simplicity.

6.3 Connecting to Generalized Bondi Gauge

In [28], Compere, Fiorucci and Ruzziconi, motivated by extending the symmetry algebra of four-dimensional asymptotically flat spacetimes, work on a phase space in which the Bondi gauge conditions are relaxed: In contrast to the BMS phase space, the leading order of the sphere metric is not fixed, but may vary. Its determinant is fixed however, and it is restricted to be independent of the retarded time u. This relaxation allows the inclusion of arbitrary diffeomorphisms of the asymptotic spheres in the symmetry algebra.

On this phase space, the standard SP diverges. The authors give a renormalized SP, by showing that the divergent orders become total variations and total derivatives on-shell. It is instructive to make contact with that, by writing our SP out in their parameterization of the metric, and comparing the results.

After identifying $\Omega = \frac{1}{r}$, the unphysical metric and its inverse, in the parameters of [28], become

$$\begin{aligned} g_{ab} &= \Omega^2(\Omega V)e^{2\beta}\mathrm{d}u^2 + e^{2\beta}(\mathrm{d}u\mathrm{d}\Omega + \mathrm{d}\Omega\mathrm{d}u) + q_{AB}(\mathrm{d}x^A - U^A\mathrm{d}u)(\mathrm{d}x^B - U^B\mathrm{d}u), \\ g^{ab}\partial_a\partial_b &= -e^{-2\beta}\Omega^2(\Omega V){\partial_\Omega}^2 + e^{-2\beta}\big((\partial_u + U^A\partial_A)\partial_\Omega + \partial_\Omega(\partial_u + U^B\partial_B)\big) + g^{AB}\partial_A\partial_B. \end{aligned} \tag{6.51}$$

Our sphere metric q_{AB} is related to the sphere metric g_{AB}^{CFR} of [28] as $q_{AB} = \Omega^2 g_{AB}^{CFR}$. One of the Bondi gauge conditions is that the determinant of (our) q_{AB} is independent of Ω. We have $\sqrt{g} = e^{2\beta}\sqrt{q}$.

The parameters in the metric have asymptotic expansions

$$\begin{aligned}
\Omega V &= \mathring{V} + 2\Omega M + ..., \\
\beta &= \Omega^2 \mathring{\beta} + ..., \\
g_{AB} &= Q_{AB} + \Omega C_{AB} + \Omega^2 D_{AB} ..., \\
U^A &= \Omega^2 \mathring{U}^A + ...,
\end{aligned} \tag{6.52}$$

where we have just given the orders that we need. We have written Q_{AB} for the leading order of the sphere metric, whereas [28] write q_{AB}, i.e. $Q_{AB} = q_{AB}^{CFR}$. The rest of the parameters are the same. C_{AB} is trace free (according to Q_{AB}) by the condition $\partial_\Omega \sqrt{q} = 0$. We will also need the equation of motion

$$D_{AB} = \frac{1}{4} Q_{AB} (C_{CD} C^{CD}). \tag{6.53}$$

Dropping the total variation, to arrive at a Gibbons-Hawking-like polarization, our renormalized SP becomes, in 4D:

$$\begin{aligned}
\hat{\boldsymbol{\theta}}^\Omega_{R,GH} = &-\frac{1}{2}\big(\sqrt{g}(\mathcal{N}^{ab} - g^{ab}\mathcal{N})\big)_{(0)} \delta g^{(1)}_{ab} - \frac{1}{2}\big(\sqrt{g}(\mathcal{N}^{ab} - g^{ab}\mathcal{N})\big)_{(1)} \delta g^{(0)}_{ab} \\
&+ 2\big(\sqrt{g}_{(2)} \delta\chi_{(0)} + \sqrt{g}_{(1)} \delta\chi_{(1)} + \sqrt{g}_{(0)} \delta\chi_{(2)}\big).
\end{aligned} \tag{6.54}$$

Here, as before, the subscripts and superscripts (i) refer to the ith order in an asymptotic expansion, i.e. $X = \sum X_{(i)} \Omega^i$. The calculation is done in C.1 and the result reads:

$$\begin{aligned}
\hat{\boldsymbol{\theta}}^\Omega_{R,GH} = &-2\delta(\sqrt{Q} M) + \frac{1}{2} \partial_A (\sqrt{Q} \mathring{U}_B \delta Q^{AB}) \\
&- \frac{1}{2}\Big(\frac{1}{2}\sqrt{Q}(\partial_u C_{AB})\delta C^{AB} + \frac{1}{2}\mathring{V} C_{AB} \delta Q^{AB} + \mathring{U}_B D_A \delta Q^{AB}\Big).
\end{aligned} \tag{6.55}$$

The first term is a total variation, which does not contribute to the symplectic form and Hamiltonians. It may be absorbed into a boundary action. The second term is a total derivative, and vanishes when $\hat{\boldsymbol{\theta}}_{R,GH}$ is integrated on a portion of $\mathcal{I}$ with boundaries at spheres with u = const. The third line consists of terms that cannot be removed, and coincides with the core piece $\bar{\Theta}_{\text{flux}}$ from [28] (up to a sign because N_a is the ingoing normal).

The agreement with $\bar{\Theta}_{\text{flux}}$ supports the correctness of our renormalization scheme, and shows that coordinate expressions, as they are commonly used at null infinity, can be derived from it. It is also encouraging that there is no total ∂_u–derivative in (6.55): For regions of $\mathcal{I}$ with boundaries at constant u, our scheme and [28] fix the finite corner ambiguity of the SP in the same way, which implies that the symplectic forms derived from $\bar{\Theta}_{\text{flux}}$ and from $\hat{\boldsymbol{\theta}}^\Omega_{R,GH}$, and hence also the Hamiltonians, agree.

6.4 Connecting to Holographic Renormalization

In the holographic renormalization (HR) of pure Einstein gravity on asymptotically AdS spacetimes (see e.g. [114]), the on-shell action is renormalized by the addition of counterterms, rendering it finite. The stress-energy tensor, which is obtained through the Brown-York [33] prescription from the on-shell action, is likewise renormalized and finite. In our case, the action is also renormalized, as is the symplectic potential. Since the Brown-York tensor corresponds closely to the momenta conjugate to the leading-order metric, one may suspect that the renormalized stress-energy tensor of HR coincides with our momenta.

HR starts from the action with a Gibbons-Hawking term, corresponding to Dirichlet boundary conditions. Since the Gibbons-Hawking term effects a canonical transformation among configuration and momentum variables, we can expect our momenta to agree with the HR stress-energy tensor only once we add a Gibbons-Hawking term as well. In addition, since HR proceeds largely on-shell, we can expect agreement between our momenta and the HR stress-energy tensor only on-shell. We will focus on the standard case of HR in Fefferman-Graham coordinates, where partial gauge conditions are chosen to fix the lapse and shift of the radial foliation. Those conditions identify ∂_Ω and N^a up to a fixed factor, while to address also the asymptotically flat case we kept ∂_Ω and N^a separate.

The HR prescription admits an ambiguity in odd spacetime dimensions, of adding a multiple of the holographic anomaly to the renormalized action at finite order. The holographic anomaly may be defined from the trace of the renormalized stress-energy tensor, and enters the logarithmic counterterm to the action. This ambiguity, which goes by the name of scheme dependence, also modifies the renormalized stress-energy tensor by a trace-free piece.

Even on-shell, and in Fefferman-Graham coordinates, it is not obvious that there would be agreement between our momenta and the stress-energy tensor: In HR it is important that the counterterms, on-shell, can be written in a form that depends only on the leading-order metric, and not on the free part of the stress-energy tensor. Here, we have made no such requirement. While the divergent orders of the action counterterms must agree on-shell because both actions are finite, it would be conceivable that the finite order of our action counterterms is not part of the scheme dependence ambiguity of HR.

We will now explicitly show that the momenta of our renormalization scheme coincide with the renormalized stress-energy tensors given in [114], in four and five spacetime dimensions, up to scheme dependent terms.

In Fefferman-Graham coordinates, we have $\chi = 0$ everywhere. In addition, we set $-2\lambda = 1$, corresponding to unit AdS radius. That entails $N_a N_b g^{ab} = 1$. Since also the shift of the radial foliation is zero, we have $N^a\partial_a = \partial_\Omega$.

6.4.1 Switching to Gibbons-Hawking Polarization

In Fefferman-Graham coordinates and with unit AdS radius, the lapse of the physical metric is$\hat{N} = \frac{1}{\Omega}$, and the normal, normalized w.r.t. the physical metric, is

$$\hat{n}^a = \frac{1}{\hat{N}}\hat{g}^{ab}N_b = \Omega N^a. \tag{6.56}$$

The Gibbons-Hawking term is

$$\sqrt{\hat{h}}\hat{K} = \sqrt{\hat{h}}\hat{\nabla}_a\hat{n}^a = \frac{1}{\hat{N}}\partial_a(\sqrt{\hat{g}}\hat{n}^a) = \Omega\partial_a(\Omega^{-(D-1)}\sqrt{g}N^a), \tag{6.57}$$

where $\hat{h}$ denotes the physical induced metric on constant radius surfaces. Writing that out in our variables gives

$$\Omega\partial_a(\Omega^{-(D-1)}\sqrt{g}N^a) = -(D-1)\Omega^{-(D-1)}\sqrt{g} + \Omega^{-(D-3)}\sqrt{g}\mathcal{N}. \tag{6.58}$$

We can thus pass to the Gibbons-Hawking polarization by setting

$$\begin{aligned}&\hat{\boldsymbol{L}}_{GH} = \hat{\boldsymbol{L}} + \partial_a\hat{\boldsymbol{\beta}}^a_{GH}, \text{ where}\\ &\hat{\boldsymbol{\beta}}^\Omega_{GH} = \Omega\partial_a(\Omega^{-(D-1)}\sqrt{g}N^a), \hat{\boldsymbol{\beta}}^i_{GH} = 0,\end{aligned} \tag{6.59}$$

where we recall that i indexes coordinates on the level surfaces of Ω. Of course, one can make the same modification of the action also in the case $\chi \neq 0$, but in that case it is not quite the Gibbons-Hawking term.

In any case, the Gibbons-Hawking term changes the coefficients of the most leading divergences of the action. Thus we need to modify the counterterms of our first step to account for them. Combining (6.58) and (6.28), and using that $-2\lambda = 1, \chi = 0, N^i = 0$, the modified first step is

$$\begin{aligned}&\hat{\boldsymbol{\theta}}^a_{GH,1} = \hat{\boldsymbol{\theta}}^a + \delta\boldsymbol{\beta}^a_{GH} + \delta\boldsymbol{\beta}^a_{GH,1} \text{ where}\\ &\boldsymbol{\beta}^\Omega_{GH,1} = (D-2)\Omega^{-(D-1)}\sqrt{g}, \qquad \boldsymbol{\beta}^i_{GH,1} = 0.\end{aligned} \tag{6.60}$$

The corner term $\hat{\boldsymbol{\alpha}}^{ab}_1$ is eliminated because we have eliminated the shift N^i. The partially renormalized SP and Lagrangian become

$$\begin{aligned}&\hat{\boldsymbol{\theta}}^\Omega_{GH,1} = \Omega^{-(D-3)}\boldsymbol{\theta}^n_{GH} = \Omega^{-(D-3)}\big(\tfrac{1}{2}\sqrt{g}(\mathcal{N}_{ij} - g_{ij}\mathcal{N})\delta g^{ij}\big)\\ &\hat{\boldsymbol{L}}_{GH,1} = \hat{\boldsymbol{L}} + \partial_\Omega(\boldsymbol{\beta}^\Omega_{GH} + \boldsymbol{\beta}^\Omega_{GH,1}) = \Omega^{-(D-2)}\big(\tfrac{1}{2}\sqrt{g}R + \partial_\Omega(\Omega\sqrt{g}\mathcal{N})\big).\end{aligned} \tag{6.61}$$

For the first line, we have used that the lapse and shift do not vary, and thus δg^{ab} is tangential to the constant-radius surfaces. For the second line we have used that in our present setting, $\partial_\Omega \sqrt{g} = \Omega\sqrt{g}\mathcal{N}$.

The partially renormalized Gibbons-Hawking SP in the first line now involves only variations of the inverse metric g^{ab}, in contrast to the previous expression (6.24), which involves a total variation term. We have written the SP in terms of the inverse metric for convenience below. Following the second step of our scheme, the renormalized Gibbons-Hawking SP is then, on-shell,

$$\hat{\boldsymbol{\theta}}^{\Omega}_{R,GH} \hat{=} \frac{1}{(D-3)!}\left(\partial_\Omega{}^{D-3} - \Omega\ln\Omega\partial_\Omega{}^{D-2}\right)\boldsymbol{\theta}^n_{GH}. \tag{6.62}$$

We will need that $\mathcal{N}_{ab}$ is related to g_{ab} as

$$\mathcal{N}_{ij} = \frac{1}{2\Omega}(\mathcal{L}_N g)_{ij} = \frac{1}{2\Omega}\partial_\Omega g_{ij}, \qquad \mathcal{N}_{\Omega a} = 0 = \mathcal{N}^{\Omega a}. \tag{6.63}$$

We see that the asymptotic finiteness of $\mathcal{N}_{ij}$ implies $g_{(1),ij} = 0$.

6.4.2 Four Dimensions

In four (spacetime) dimensions, the asymptotic expansion of the unphysical metric used in [114] reads

$$g_{ij} = g_{(0),ij} + \Omega^2 g_{(2),ij} + \Omega^3 g_{(3),ij} + \mathcal{O}(\Omega^4). \tag{6.64}$$

There is no logarithmic term, and $g_{(3)}$ is trace-free and covariantly conserved. The only term that contributes to the asymptotic limit of the renormalized SP becomes

$$\hat{\boldsymbol{\theta}}^R_{GH} = \frac{1}{2}(\sqrt{g}(\mathcal{N}_{ij} - g_{ij}\mathcal{N}))_{(1)}\delta g^{ij}_{(0)}. \tag{6.65}$$

Now using that $g_{(1),ij} = 0$, that $\mathcal{N}_{(1),ij} = \frac{3}{2}g_{(3),ij}$, and that $g_{(3)}$ is traceless such that $\mathcal{N}_{(1)} = 0$, we get

$$\hat{\boldsymbol{\theta}}^R_{GH} = -\frac{3}{4}\sqrt{g}_{(0)}g^{ij}_{(3)}\delta g_{(0),ij}, \tag{6.66}$$

where the indices on $g_{(3)}$ are raised with $g_{(0)}$. The momentum $-\frac{3}{4}\sqrt{g}_{(0)}g^{ij}_{(3)}$ is in agreement with the results of [114] for the stress-energy tensor, remembering that we get a sign because

N_a is the ingoing normal, and the Brown-York stress tensor is enhanced by a factor of 2 compared to the gravitational momentum.

Our procedure thus agrees with HR in four dimensions. A symplectic form which, in four dimensions, agrees with HR in asymptotically AdS space, and with the symplectic form of [28] in flat space, has previously been given in [115].

6.4.3 Five Dimensions

In five spacetime dimensions, the situation is a little more complex. The expansion of the metric now reads

$$g_{ij} = g_{(0),ij} + \Omega^2 g_{(2),ij} + \Omega^4 g_{(4),ij} + 2\Omega^4 \ln\Omega h_{(4),ij} + \mathcal{O}(\Omega^5). \tag{6.67}$$

It features a logarithmic piece, and $h_{(4)}$ is traceless and covariantly conserved. The trace of $g_{(4)}$ is

$$g^{ij}_{(0)} g_{(4),ij} = \frac{1}{4} g^{ij}_{(2)} g_{(2),ij} = \frac{1}{4}\mathrm{tr} g^2_{(2)}. \tag{6.68}$$

The nonzero contributions to the asymptotic SP are

$$\hat{\boldsymbol{\theta}}^{\Omega}_{R,GH} = \frac{1}{4}({\partial_\Omega}^2 - \Omega\ln\Omega{\partial_\Omega}^3)\Big(\sqrt{g}(\mathcal{N}_{ij} - g_{ij}g^{kl}\mathcal{N}_{kl})\delta g^{ij}\Big) \tag{6.69}$$

$$\begin{aligned} &\overset{\mathcal{I}}{=} \frac{1}{4}\delta g^{ij}_{(0)}\Big(\sqrt{g}_{(0)}(\delta^k_i\delta^l_j - g_{(0),ij}g^{kl}_{(0)})\cdot({\partial_\Omega}^2 - \Omega\ln\Omega{\partial_\Omega}^3)\mathcal{N}_{kl}\Big) \\ &\quad + \frac{1}{2}\delta g^{ij}_{(0)}\Big(\sqrt{g}_{(2)}(\mathcal{N}_{(0),ij} - g_{(0),ij}\mathcal{N}_{(0)}) - \sqrt{g}_{(0)}g_{(2),ij}\mathcal{N}_{(0)} - \sqrt{g}_{(0)}g_{(0),ij}(g^{kl})_{(2)}\mathcal{N}_{(0),kl}\Big) \\ &\quad + \frac{1}{2}\delta(g^{ij})_{(2)}(\sqrt{g}_{(0)}\mathcal{N}_{(0),ij} - g_{(0),ij}\mathcal{N}). \end{aligned} \tag{6.70}$$

We have used that the operator $\Omega\ln\Omega{\partial_\Omega}^3$ gives a non-zero limit only when acting on $\mathcal{N}_{ij}$, because the most leading non-analytic term in the metric is at order $\Omega^4\ln\Omega$. For the first line of (6.70), using (6.63), we get that asymptotically

$$({\partial_\Omega}^2 - \Omega\ln\Omega{\partial_\Omega}^3)\mathcal{N}_{kl} = 4g_{(4),kl} + 14h_{(4),kl}. \tag{6.71}$$

The $\Omega\ln\Omega{\partial_\Omega}^3$ term is necessary to cancel the divergence in $\ln\Omega$ arising from the ${\partial_\Omega}^2$ term. The first line of (6.70) becomes

$$\frac{1}{2}\sqrt{g}_{(0)}\delta g^{ij}_{(0)}\Big(2g_{(4),ij} + 7h_{(4),ij} - \frac{1}{2}g_{(0),ij}\mathrm{tr}g^2_{(2)}\Big), \tag{6.72}$$

where we have also used (6.68) and that $h_{(4)}$ is traceless. For the second line of (6.70), we use $\sqrt{g}_{(2)} = \frac{1}{2}\sqrt{g}_{(0)} g^{kl}_{(0)} g_{(2),kl}$ and $(g^{kl})_{(2)} = -g^{kl}_{(2)}$, and we get

$$\frac{1}{2}\sqrt{g}_{(0)}\delta g^{ij}_{(0)}\Big(-\frac{1}{2}g_{(2),ij}\mathrm{tr}g_{(2)} + g_{(0),ij}(\mathrm{tr}g^2_{(2)} - \frac{1}{2}\mathrm{tr}^2 g_{(2)})\Big). \tag{6.73}$$

The third line of (6.70) involves variations of a subleading order of the metric, which we need to remove by extracting a total variation. Firstly, using $\mathcal{N}_{(0),ij} = g_{(2),ij}$ and $(g^{ij})_{(2)} = -g^{ij}_{(2)}$, we have

$$\frac{1}{2}\sqrt{g}_{(0)}\delta(g^{ij})_{(2)}(\mathcal{N}_{(0),ij} - g_{(0),ij}\mathcal{N}_{(0)}) = -\frac{1}{2}\sqrt{g}_{(0)}\delta(g^{ij}_{(2)})(g_{(2),ij} - g_{(0),ij}\mathrm{tr}g_{(2)}). \tag{6.74}$$

We add and subtract the total variation

$$\begin{aligned}\frac{1}{4}\delta\big(\sqrt{g}_{(0)}(\mathrm{tr}g^2_{(2)} - \mathrm{tr}^2 g_{(2)})\big) &= \frac{1}{4}\delta\big(\sqrt{g}_{(0)}(g_{(0),ik}g_{(0),jl} - g_{(0),ij}g_{(0),kl})(g^{ij}_{(2)}g^{kl}_{(2)})\big)\\ &= \frac{1}{2}\sqrt{g}_{(0)}\delta g^{ij}_{(0)}\big(g_{(2),ij}\mathrm{tr}g_{(2)} - g_{(2),ik}g^{k}_{(2),j} + \frac{1}{4}g_{(0),ij}(\mathrm{tr}^2 g_{(2)} - \mathrm{tr}g^2_{(2)})\big)\\ &\quad + \frac{1}{2}\sqrt{g}_{(0)}\delta(g^{ij}_{(2)})\big(g_{(2),ij} - g_{(0),ij}\mathrm{tr}g_{(2)}\big).\end{aligned} \tag{6.75}$$

The third line of (6.70) becomes

$$\frac{1}{2}\sqrt{g}\delta g^{ij}\big(g_{(2),ij}\mathrm{tr}g_{(2)} - g_{(2),ik}g^{k}_{(2),l} + \frac{1}{4}g_{(0),ij}(\mathrm{tr}^2 g_{(2)} - \mathrm{tr}g^2_{(2)})\big) - \frac{1}{4}\delta\big(\sqrt{g}(\mathrm{tr}g^2_{(2)} - \mathrm{tr}^2 g_{(2)})\big). \tag{6.76}$$

Adding all up gives

$$\begin{aligned}\hat{\boldsymbol{\theta}}^{\Omega}_{R,GH} = &-\sqrt{g}\delta g_{(0),ij}\Big(g^{ij}_{(4)} + \frac{1}{4}g^{ij}_{(2)}\mathrm{tr}g_{(2)} - \frac{1}{2}(g_{(2)}\cdot g_{(2)})^{ij} + \frac{1}{8}g^{ij}_{(0)}(\mathrm{tr}g^2_{(2)} - \mathrm{tr}^2 g_{(2)})\Big)\\ &-\frac{7}{2}\sqrt{g}\delta g_{(0),ij}h^{ij}_{(4)} - \frac{1}{4}\delta\big(\sqrt{g}_{(0)}(\mathrm{tr}g^2_{(2)} - \mathrm{tr}^2 g_{(2)})\big).\end{aligned} \tag{6.77}$$

The momentum in the first line agrees with the stress-energy tensor given in [114], noting again that there is a sign because N_a is the ingoing normal and that the stress-energy tensor has an extra factor of 2.

The second line contains terms which are scheme dependent: In five dimensions, the conformal anomaly is $\frac{1}{2}(\mathrm{tr}g^2_{(2)} - \mathrm{tr}^2 g_{(2)})$. [114] show that the stress-energy tensor of the conformal anomaly is proportional to $h_{(4)}$, or in our language, that $\delta\big(\sqrt{g}_{(0)}(\mathrm{tr}g^2_{(2)} - \mathrm{tr}^2 g_{(2)})\big)$

is proportional to $\sqrt{g}_{(0)} h_{(4)}^{ij} \delta g_{(0),ij}$. Thus both terms on the second line combine into one total variation, and can be absorbed into a finite boundary action proportional to the conformal anomaly. Such a finite counterterm satisfies the requirements that HR puts on the counterterms, and is part of the scheme dependence ambiguity of HR.

To summarize, we have specialized to Fefferman-Graham coordinates on asymptotically AdS space in four and five spacetimes dimensions, and included a Gibbons-Hawking term in our scheme. We obtained that the momenta, which in the renormalized SP are conjugate to the leading order metric, are consistent with the results of HR for the renormalized stress-energy tensor, up to terms which fall under the scheme-dependence ambiguity of HR.

6.5 Hamiltonians

In this section, we find the Hamiltonians associated with $\hat{\boldsymbol{\theta}}_R^{\Omega}$, in any gauge, for any Λ and in any dimension, exploiting that our expressions are fairly covariant. The result may serve as a starting point to derive Hamiltonians in various partial gauges.

The variation of the Hamiltonian (if it exists) is given by $-I_\xi \delta \hat{\boldsymbol{\theta}}_R^{\Omega}$. We will use (6.41) for $\hat{\boldsymbol{\theta}}_R^{\Omega}$, and that I_ξ commutes past the ∂_Ω-derivatives. We have

$$- I_\xi \delta \hat{\boldsymbol{\theta}}_R^{\Omega} = -\frac{1}{(D-3)!} ({\partial_\Omega}^{D-3} - \Omega \ln \Omega {\partial_\Omega}^{D-2}) I_\xi \delta \boldsymbol{\theta}^n. \tag{6.78}$$

Thus it is enough to evaluate $-I_\xi \delta \boldsymbol{\theta}^n$.

We parameterize the vector field as

$$\xi^a \partial_a = \xi^i \partial_i + \Omega p \partial_\Omega. \tag{6.79}$$

This parameterization is motivated by the action of ξ on the unphysical metric g_{ab}, which reads

$$I_\xi \delta g_{ab} = \mathcal{L}_\xi g_{ab} - \frac{2}{\Omega} N_c \xi^c g_{ab} = \mathcal{L}_\xi g_{ab} - 2 p g_{ab}. \tag{6.80}$$

To preserve the finiteness of the unphysical metric, it is thus sufficient (though not necessary) that ξ^i and p are asymptotically finite. We will take the vector field to be field-space constant, i.e. $\delta \xi = 0$, but the dependence on $\delta \xi$ can be simply reinstated since the final result $-I_\xi \delta \hat{\boldsymbol{\theta}}_R^{\Omega}$ should not depend on $\delta \xi$.

Calculating $-I_\xi \delta\boldsymbol{\theta}^n$ directly is quite laborious, because the constituents $\mathcal{N}_{ab}, g_{ab}, \chi$ involve background structure and hence do not transform straightforwardly under diffeomorphisms. Rather, we start from (6.24), which implies

$$\Omega^{-(D-3)}\delta\boldsymbol{\theta}^n = \delta\hat{\boldsymbol{\theta}}^\Omega - \frac{1}{2}\Omega^{-(D-2)}\partial_i\delta(\sqrt{g}\delta N^i). \tag{6.81}$$

The total variation terms from (6.24) drop because the δs are antisymmetrized. The result of $I_\xi\delta\hat{\boldsymbol{\theta}}^\Omega$, written in the physical metric, may be obtained from the differential version of (2.17). It involves the variation of the Komar charge, and symplectic flux terms for diffeomorphisms that displace the codimension two surface where the Hamiltonian is integrated.

To get the Hamiltonians, we need to translate $I_\xi\delta\hat{\boldsymbol{\theta}}^\Omega$ to the unphysical metric g, and evaluate $I_\xi\delta(\sqrt{g}\delta N^i)$. The calculation is performed in appendix C.2. The result is

$$-I_\xi\delta\boldsymbol{\theta}^n = \partial_i \not{\delta}\boldsymbol{h}_1^i, \text{ where}$$
$$\not{\delta}\boldsymbol{h}_1^i = \delta\big(-\sqrt{g}\mathcal{N}^i{}_a\xi^a + \sqrt{g}g^{ia}\partial_a p\big) + p\boldsymbol{\theta}_1^i - \xi^i\boldsymbol{\theta}^n - \sqrt{g}\delta\chi\partial_\Omega\xi^i - \frac{1}{2}\sqrt{g}\delta g^{ij}\partial_j p, \tag{6.82}$$

and $\boldsymbol{\theta}_1^i = \Omega^{D-2}\hat{\boldsymbol{\theta}}_1^i$ is given in (6.29). Here tensors with indices i and j are projected onto the level surfaces of Ω along ∂_Ω, with the coordinate projector P_a^i. δg^{ij} is the variation of the projection of the inverse spacetime metric, not the inverse of the pulled-back metric. The result is valid on-shell for on-shell variations. It holds at any Ω and for any cosmological constant, and no gauge conditions have been chosen, beyond the conditions of asymptotic simplicity.

As expected, the result for $-I_\xi\delta\boldsymbol{\theta}^n$ is asymptotically finite for finite ξ and p, and on-shell is a total derivative on the level surfaces of Ω. In addition to the integrable piece, there are several non-integrable terms: symplectic flux terms associated with the displacement of the codimension-two sphere, and also extra non-integrable terms involving derivatives of the gauge parameters. The Hamiltonians, associated with a portion of an $\Omega = \text{const.}$ surface with boundaries S_i and S_f are then given by

$$\not{\delta}H_R = \int_{S_\text{i}}^{S_\text{f}} \frac{1}{(D-3)!}(\partial_\Omega{}^{D-3} - \Omega\ln\Omega\partial_\Omega{}^{D-2})s_i\boldsymbol{\theta}_1^i, \tag{6.83}$$

where s_i is the normal of $S_\text{i,f}$ embedded into the $\Omega = \text{const.}$ surface.

In chapter 4, we used a counterterm involving δN^i to remove the dependence of charges on the extension of ξ outside the hypersurface considered. Here the sign of the counterterm is opposite, and the "would-be Hamiltonian" contains terms in $\partial_\Omega \xi$. However, if we consider the integral of the Hamiltonian on a sphere at constant u and Ω, and the level surfaces of u are null (i.e., $g^{uu} = 0$ as in Bondi gauge), then the Hamiltonian does not depend on the extension of the gauge parameters outside the $u =$ const. surface. In Fefferman-Graham coordinates, $\not\delta \boldsymbol{h}_1^i$ does not involve radial derivatives of the gauge parameters.

Since we have not fixed any gauge, there are no conditions on ξ. On the face of it, it then seems that there would be a large number of non-zero charges, associated to orders up to $(D-1)$ of the gauge parameters. However, it seems likely that some of that dependence can be removed by adding further finite corner counterterms, along the lines of section 5.6.2. Since not all the terms in the Hamiltonian are functionally independent, it is also possible that there would be non-zero field-dependent vector fields in the kernel of the symplectic form, reducing the dimension of the symmetry algebra.

Chapter 7

Conclusions

In the first part of this thesis, we examined the canonical degrees of freedom and conservation laws of general relativity on generic null surfaces: In chapter 3, we understood the canonical degrees of freedom in terms of the null geometry. In chapter 4, we wrote the constraints in conservation law form, equating the divergence of a relativistic current intrinsic to the null surface to a flux through the surface. The flux describes the passage of gravitational energy and momentum through the surface, and is built from the degrees of freedom. The current is closely related to the generators of diffeomorphisms.

A similar network of relationships between degrees of freedom, constraints, conservation laws and generators is an integral part of the “infrared triangle” at null infinity of asymptotically flat spacetimes [18]. By working out similar relationships at generic null surfaces, we took a step towards connecting the situations at null infinity and generic null surfaces. The work in the first part also fits well with the interpretation in the membrane paradigm of geometrical quantities on null surfaces as charges and fluxes, thereby providing a canonical perspective on the membrane paradigm.

Let us mention some possible future avenues of investigation. Building on the work in chapters 3 and 4, it would be interesting to apply the results to achieve a more detailed canonical perspective on the thermodynamics of spacetime, generalizing black hole thermodynamics to more general situations [116]. It would also be interesting to extend the analysis of constraints to include the null analogue of the Hamiltonian constraint.

In the second part of this thesis, the asymptotic canonical structure of general relativity in $D \geq 3$ and of electrodynamics in $D \geq 5$ was addressed for asymptotically flat spacetimes. In many situations, the symplectic potentials of those theories diverge with the radius as the boundary at infinity is approached. For example, the divergence occurs for gravity in four dimensions, with relaxations of the Bondi gauge conditions which are relevant for the study of spacetimes with asymptotic symmetry groups larger than the BMS group.

We proposed a general scheme to remove such divergences, absorbing them into counterterms by exploiting the ambiguities of the symplectic potential. The scheme works in any dimension, and requires no gauge choice; only the existence of a conformal compactification of spacetime in the sense of Penrose and some degree of analyticity of the fields around infinity are required.

We spelled out the scheme for electrodynamics, where we also removed the remaining dependencies among the constituents of the renormalized symplectic potential by writing everything in terms of free data, and showed agreement with earlier work which derived canonical quantities using the connection to soft theorems [90]. In terms of the structure of asymptotic divergences, electrodynamics in six dimensions was seen to be analogous to gravity in four dimensions. It would be desirable to further develop and exploit that analogy, for example regarding falloffs and matching at spatial infinity, or the role of logarithmic terms in the expansions of the fields in the flat case.

The scheme was applied also to general relativity. We gave an expression for the renormalized symplectic potential in terms of geometrical data, and connected to earlier work considering divergences in a specific partial gauge [28]. The scheme, when applied to asymptotically AdS spacetimes, bears some similarity to holographic renormalization, and we connected the two explicitly.

Our work shows that an asymptotically finite asymptotic symplectic potential exists for gravity under quite general conditions, and gives expressions which may be straightforwardly specialized to (partial) gauges for any cosmological constant. The results may be applied to consider asymptotically flat phase spaces with weaker boundary conditions: For example, one may attempt to turn on the time dependence of the leading order of the metric, which was left as an open problem in [28].

We obtained the canonical structures of asymptotically flat spacetimes in Bondi gauge and aAdS spacetimes in Fefferman-Graham coordinates as specializations of the same expression, thereby connecting the two (see [113, 117, 115] for prior work). Our expressions thus forms a “canonical bridge” between asymptotically flat and aAdS spacetimes; they may be used to consider holography at null infinity, or asymptotic symmetry groups of aAdS spacetimes. It would also be interesting to relate to the Mann-Marolf counterterm at spatial infinity of asymptotically flat spacetimes [36, 37]).

A limitation of the scheme as presented here is that we have not kept track of what the counterterms depend on when written in terms of free data, nor of the precise status of the equations of motion. For example, the renormalized Lagrangian as presented does not have the same equations of motion as the original Lagrangian. The on-shell symplectic form, which forms the basis of much work on asymptotic symmetries, is not affected by these questions. But in order to consider the Hamilton-Jacobi functional starting from our scheme, or to interpret the counterterms more physically, or to connect to holographic renormalization more tightly, more work is needed.

The renormalization scheme for gravity of chapter 6 leaves open the possibility of adding finite counterterms. Such counterterms influence the charges and their algebra, including the number of nonzero charges, which corresponds to the size of the asymptotic symmetry algebra. It would be desirable to fix the ambiguity, for example with an "extension independence" argument as in chapter 5; or at least to restrict the set of allowable finite counterterms and understand their status, as in the "scheme dependence" ambiguity in holographic renormalization. At a technical level, to do so for the gravitational case, it may be useful to "dress" the full metric with a metric dependent diffeomorphism implementing, e.g., Bondi gauge; doing so would mirror the correspondence of the extension independent symplectic potential of chapter 5 with the symplectic potential of the "dressed" gauge field. A calculation along the lines of [95] for the symplectic form in terms of the dressed metric may be useful. u-falloffs may also ameliorate the corner ambiguities.

Once the finite corner ambiguity is fixed, one can pose the question whether any of the diffeomorphisms implementing Bondi gauge have non-zero charges, and should be "upgraded" to physical symmetries. In [28] it was already shown that the sphere diffeomorphisms which fix the leading order sphere metric to be that of the unit sphere correspond to non-zero charges. A further candidate may be dilatations along the null generators of $\mathcal{I}$, which have non-zero generators at finite distance null surfaces as we saw in chapter 4 (a similar result was obtained in [74]). Another candidate may be the radial diffeomorphisms which implement the Bondi condition on the conformal factor.

Lastly, since the asymptotic gravitational symplectic potential of chapter 6 is the limit of a local expression, it may be used to write down an explicit limit of the "edge modes" construction, and see to which extent edge modes and associated structures coincide with similar fields at null infinity.

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APPENDICES

Appendix A

Appendices to Chapter 3

A.1 Extrinsic Geometry Expressed in Metric Parameters

We relate the extrinsic geometry to the derivatives of metric parameters. The versions of these identities which hold true on B were used in section 3.2. We start with the normal acceleration, which is the most involved expression:

$$\begin{aligned}\gamma &= \bar{\ell}^a \nabla_\ell \ell_b = \bar{\ell}^a (\pounds_\ell \boldsymbol{\ell})_a = \nabla_\ell \alpha + e^{\alpha-\bar{\alpha}} \nabla_{\bar{\ell}} \beta + \frac{\bar{\beta}}{1+\beta\bar{\beta}} \nabla_\ell \beta \\ &= e^{-\bar{\alpha}}[(D_0 + \beta D_1)\alpha + D_1 \beta] = e^{-\bar{\alpha}-\alpha}[D_0 e^\alpha + D_1(e^\alpha \beta)] \overset{B}{=} \nabla_\ell \alpha + e^{\alpha-\bar{\alpha}} \nabla_{\bar{\ell}} \beta. \quad \text{(A.1)}\end{aligned}$$

From this we can evaluate the surface gravity

$$\kappa := e^{\bar{\alpha}}(\gamma + \nabla_\ell \bar{\alpha}) = (D_0 + \beta D_1)h + D_1 \beta. \quad \text{(A.2)}$$

The tangential acceleration a, the twists $(\eta, \bar{\eta})$ and normal connection ω are given by:

$$\begin{aligned}a_a &= q_a{}^b \nabla_\ell \ell_b = q_a{}^b (\pounds_\ell \boldsymbol{\ell})_b = e^{-\bar{\alpha}} q_a{}^b \nabla_b \beta \overset{B}{=} 0 \\ \eta_a + \omega_a &= -q_a{}^b \nabla_{\bar{\ell}} \ell_b + q_a{}^b \bar{\ell}^c \nabla_b \ell_c = -q_a{}^b (\pounds_{\bar{\ell}} \boldsymbol{\ell})_c = q_a{}^b \Big(\nabla_b \alpha + \frac{\bar{\beta}}{1+\beta\bar{\beta}} \nabla_b \beta\Big) \overset{B}{=} q_a{}^b \nabla_b \alpha \\ \bar{\eta}_a - \omega_a &= -q_a{}^b \nabla_\ell \bar{\ell}_b + q_a{}^b \ell^c \nabla_\ell \bar{\ell}_c = -q_a{}^b (\pounds_\ell \bar{\boldsymbol{\ell}})_b = q_a{}^b \Big(\nabla_b \bar{\alpha} - \frac{\bar{\beta}}{1+\beta\bar{\beta}} \nabla_a \beta\Big) \overset{B}{=} q_a{}^b \nabla_b \bar{\alpha} \\ \eta_a - \bar{\eta}_a &= -q_a{}^b \nabla_{\bar{\ell}} \ell_b + q_a{}^b \nabla_\ell \bar{\ell}_b = q_{ab} [\ell, \bar{\ell}]^b = e^{-h} q_{ab} [D_0, D_1]^a. \quad \text{(A.3)}\end{aligned}$$

These identities are proven by inserting the parametrizations (3.9) and (3.12), and executing the Lie derivatives. Linear combinations of the last three identities yield

$$\begin{aligned} \eta_a + \bar{\eta}_a &= q_a{}^b \nabla_b h, \\ \bar{\eta}_a &= \frac{1}{2}\big(q_a{}^b \nabla_b h - q_{ab} e^h [D_0, D_1]^b\big). \end{aligned} \tag{A.4}$$

In section 3.2, we used the identity $\partial_a(\sqrt{q} D_0^a) = \sqrt{q}\theta$. Let us prove it. First evaluate

$$\nabla_a L^a = (q^{ab} + \bar{\ell}^a \ell^b + \ell^a \bar{\ell}^b)\nabla_a L_b = \theta_L + \bar{\ell}^a \nabla_\ell L_a = \theta + \kappa. \tag{A.5}$$

Let us evaluate the same object using now the relationship between covariant and regular derivative and that $L^a = D_0 + \beta D_1$:

$$\begin{aligned} \nabla_a L^a &= \frac{1}{\sqrt{|g|}} \partial_a(\sqrt{|g|} L^a) = \frac{e^{-h}}{\sqrt{q}} \partial_a(\sqrt{q} e^{\alpha+\bar{\alpha}} L^a) \\ &= \frac{1}{\sqrt{q}} \partial_a(\sqrt{q} L^a) + \nabla_L h \\ &= \frac{1}{\sqrt{q}} \partial_a(\sqrt{q}(D_0^a + \beta D_1^a)) + \kappa - D_1 \beta \\ &= \frac{1}{\sqrt{q}} \partial_a(\sqrt{q} D_0^a) + \frac{\beta}{\sqrt{q}} \partial_a(\sqrt{q} D_1^a) + \kappa. \end{aligned} \tag{A.6}$$

Comparing (A.6) and (A.5) gives what we wanted to show:

$$\theta = \frac{1}{\sqrt{q}} \partial_a(\sqrt{q} D_0^a) + \frac{\beta}{\sqrt{q}} \partial_a(\sqrt{q} D_1^a) \overset{B}{=} \frac{1}{\sqrt{q}} \partial_a(\sqrt{q} D_0^a). \tag{A.7}$$

Lastly, the extrinsic curvature of L is given by

$$\begin{aligned} \theta^{AB} &= -\tfrac{1}{2}\mathcal{L}_L q^{AB} = -\tfrac{1}{2}\partial_0 q^{AB} + \tfrac{1}{2}\big(q^{AC}\partial_C A_0^B + q^{CB}\partial_C A_0^A - A_0^C \partial_C q^{AB}\big) \\ &= -\tfrac{1}{2}\partial_0 q^{AB} + \tfrac{1}{2}(d^A A_0^B + d^B A_0^A). \end{aligned} \tag{A.8}$$

The bulk momentum for the conformal metric γ_{ab} is the conformal shear, which is related to the traceless part of the extrinsic curvature:

$$\begin{aligned} \tilde{\sigma}^{AB} &= e^{2\varphi}\theta^{<AB>} = -\frac{1}{2}\partial_0 \gamma^{AB} + e^{2\varphi} d^{<A} A_0^{B>} \\ &= -\frac{1}{2}\partial_0 \gamma^{AB} + \frac{1}{2}\big(\gamma^{AC}\partial_C A_0^B + \gamma^{BC}\partial_C A_0^A - A_0^C \partial_C \gamma^{AB}\big) - \frac{1}{D-2}\partial_C A_0^C \gamma^{AB}. \end{aligned} \tag{A.9}$$

A.2 Calculation of the Variation of the Surface Gravity

Let us evaluate the total variation $\delta\kappa$ that was used in 3.52. Using the coordinate expression for κ given in the previous appendix and assuming $\delta\beta \overset{B}{=} 0$, we obtain

$$\delta\kappa \overset{B}{=} \delta\big(D_0 h + D_1\beta\big). \tag{A.10}$$

We distribute the variation, and use that $\delta\beta \overset{B}{=} 0$, that the variations $\delta D_i^a = \delta A_i^a$ are purely tangential to S, that $q_a{}^b\nabla_b h = \eta_a + \bar{\eta}_a$ and that $q_a{}^b\nabla_b\beta = 0$:

$$\delta\kappa \overset{B}{=} D_0\delta h + \delta A_0^a(\eta_a + \bar{\eta}_a) + D_1\delta\beta. \tag{A.11}$$

Substituting $\delta L^a \overset{B}{=} \delta A_0^a$, $\ell_a\delta\ell^a = e^{\alpha-\bar{\alpha}}\delta\beta$ and the coordinate expressions for L and $\bar{\ell}$ yields

$$\delta\kappa \overset{B}{=} \nabla_L\delta h + \delta L^a(\eta_a + \bar{\eta}_a) + \nabla_{\bar{\ell}}(e^{\bar{\alpha}}\ell_a\delta\ell^a). \tag{A.12}$$

That is the expression we used.

A.3 Calculation of Integration by Parts

We prove identities that we used in section 3.2.2 to integrate by parts in Θ_B, producing boundary terms on ∂B. We first use that for any vector V

$$\begin{aligned}\sqrt{q}\nabla_a V^a &= \sqrt{|g|}e^{-h}\nabla_a V^a = e^{-h}\partial_a(\sqrt{|g|}V^a) = e^{-h}\partial_a(\sqrt{q}e^h V^a)\\ &= \partial_a(\sqrt{q}V^a) + \sqrt{q}V^a\partial_a h.\end{aligned} \tag{A.13}$$

If $V^a = q^a{}_b V^b$ is a tangential vector to S this means that

$$\sqrt{q}\nabla_a V^a = \partial_a(\sqrt{q}V^a) + \sqrt{q}V^a(\eta_a + \bar{\eta}_a). \tag{A.14}$$

If on the other hand we take $V^a = \rho L^a$ we obtain the identity

$$\begin{aligned}\sqrt{q}\nabla_L\rho &= \sqrt{q}L^a\nabla_a\rho \overset{B}{=} \sqrt{q}D_0^a\nabla_a\rho = \sqrt{q}\nabla_a(D_0^a\rho) - \rho\sqrt{q}\nabla_a D_0^a]\\ &= \partial_a(\sqrt{q}D_0^a\rho) - \rho\partial_a(\sqrt{q}D_0^a)\\ &= \partial_a(\sqrt{q}D_0^a\rho) - \sqrt{q}\theta.\end{aligned} \tag{A.15}$$

where we used that $L^a \overset{B}{=} D_0^a$ and that $\partial_a(\sqrt{q}D_0^a) = \sqrt{q}\theta$.

Appendix B

Appendices to Chapter 4

B.1 Symplectic Potential

This appendix derives the form of the symplectic potential used in 4.3. From (3.61) we read that $\Theta_B = A + \mathrm{d}B$ where

$$A = \left(\tfrac{1}{2}\delta q_{ab}\theta^{ab} - \bar{\eta}_a \delta L^a + \delta(\kappa + \theta)\right)\epsilon_{\mathrm{B}} \tag{B.1}$$

is the bulk symplectic potential and the boundary potential is given by

$$B = \tfrac{1}{2}\left[\left(\tfrac{1}{2}hL^a\delta q + (h-1)\delta L^a\right)\imath_a\epsilon_{\mathrm{B}} - \delta\left([hL^a]\imath_a\epsilon_{\mathrm{B}}\right)\right]. \tag{B.2}$$

Let us perform a trace-traceless split in A, using

$$\theta^{ab} = e^{-2\varphi}(\tilde{\sigma}^{ab} + \tfrac{1}{D-2}\gamma^{ab}\theta), \qquad \delta q_{ab} = \delta(e^{2\varphi}\gamma_{ab}) = e^{2\varphi}(2\delta\varphi\gamma_{ab} + \delta\gamma_{ab}), \tag{B.3}$$

therefore

$$\begin{aligned}
(\tfrac{1}{2}\delta q_{ab}\theta^{ab})\epsilon_{\mathrm{B}} &= (\tfrac{1}{2}\delta\gamma_{ab}\tilde{\sigma}^{ab})\epsilon_{\mathrm{B}} + \theta\delta\varphi\epsilon_{\mathrm{B}} \\
&= (\tfrac{1}{2}\delta\gamma_{ab}\tilde{\sigma}^{ab})\epsilon_{\mathrm{B}} + \tfrac{1}{(D-2)}\delta(\epsilon_{\mathrm{B}})\theta \\
&= (\tfrac{1}{2}\delta\gamma_{ab}\tilde{\sigma}^{ab})\epsilon_{\mathrm{B}} - \tfrac{1}{(D-2)}(\delta\theta)\epsilon_{\mathrm{B}} + \tfrac{1}{(D-2)}\delta(\epsilon_{\mathrm{B}}\theta).
\end{aligned} \tag{B.4}$$

Plugging in:

$$A = \left(\tfrac{1}{2}\delta\gamma_{ab}\tilde{\sigma}^{ab} - \bar{\eta}_a\delta L^a + \delta(\kappa + \tfrac{D-3}{D-2}\theta)\right)\epsilon_{\mathrm{B}} + \tfrac{1}{(D-2)}\delta(\epsilon_{\mathrm{B}}\theta), \tag{B.5}$$

The boundary term can be rewritten, using that $\frac{1}{2}\delta q\, \imath_a\epsilon_{\rm B} = \delta(\imath_a\epsilon_{\rm B})$ as

$$\begin{aligned} B &= \frac{1}{2}\left((h-1)\delta L^a - \delta[hL^a]\right)\imath_a\epsilon_{\rm B} \\ &= -\frac{1}{2}\left(\delta h L^a + \delta L^a\right)\imath_a\epsilon_{\rm B} \\ &= -\frac{1}{2}\left(\delta h \imath_L\epsilon_{\rm B} + \imath_{\delta L}\epsilon_{\rm B}\right) \end{aligned} \tag{B.6}$$

Noting also $\mathrm{d}\epsilon_{\rm S} = \theta\epsilon_{\rm B}$ and using the definition (3.77) of μ, we get the equation (4.38) used in the main text:

$$\Theta_B = \epsilon_{\rm B}\left(\frac{1}{2}\tilde{\sigma}^{AB}\delta\gamma_{AB} - \delta L^a\omega_a + \delta\mu\right) - \mathrm{d}(\frac{1}{2}\delta h\iota_L\epsilon_{\rm B} + \frac{1}{2}\iota_{\delta L}\epsilon_{\rm B} - \tfrac{1}{D-2}\delta\epsilon_{\rm S}). \tag{B.7}$$

B.2 Derivation of Raychaudhuri and Damour equations

This appendix derives the densitized Damour equation

$$q_a{}^b G_{Lb}\epsilon_{\rm B} = q_a{}^b\mathcal{L}_L(\omega_b\epsilon_{\rm B}) + (\mathrm{d}_b\sigma_a{}^b - \mathrm{d}_a\mu)\epsilon_{\rm B} \tag{B.8}$$

and the densitized null Raychaudhuri equation

$$G_{LL}\epsilon_{\rm B} = -\mathcal{L}_L(\theta\epsilon_{\rm B}) + (\mu\theta - \sigma^a{}_b\sigma^b{}_a)\epsilon_{\rm B}. \tag{B.9}$$

We define

$$\eta_a := -\, q_a{}^b\bar{L}^c\nabla_c L_b \tag{B.10}$$

$$a_a := q_{ab}\nabla_L L^b \tag{B.11}$$

$$\bar{\theta}_{ab} := q_a{}^{a'}q_b{}^{b'}\nabla_{a'}\bar{L}_{b'} \tag{B.12}$$

and recall $\bar{\eta}_a := -q_a{}^b L^c\nabla_c\bar{L}_b$, while $\omega_a = q_a{}^b\bar{L}_c\nabla_b L^c$ and $\mu = \kappa + \frac{D-3}{D-2}\theta$. The tangential acceleration a_a vanishes on B since, L is geodesic on B.

For the Damour equation, we have

$$q_a{}^b G_{Lb} = q_a{}^b R_{Lb} = q_a{}^c(\nabla_b\nabla_c L^b - \nabla_c\nabla_b L^b). \tag{B.13}$$

For the first term, use the decomposition of the identity $\delta^a_b = q_a{}^b + L_a\bar{L}^b + \bar{L}_a L^b$ to obtain

$$\nabla_a L^b = \bar{L}_a L^b \kappa - L_a \eta^b + \omega_a L^b + \theta_a{}^b + \bar{L}_a a^b. \tag{B.14}$$

Taking an additional derivative and projecting with q yields

$$\begin{aligned} q_a{}^{a'}(\nabla_b \nabla_{a'} L^b) &= q_a{}^{a'}\Big((\nabla_L \bar{L}_{a'})\kappa - (\nabla_b L_{a'})\eta^b + \omega_{a'}\nabla_b L^b + \nabla_L \omega_{a'} + \nabla_b \theta_{a'}{}^b + \nabla_b \bar{L}_{a'} a^b\Big) \\ &= -\bar{\eta}_a \kappa - \eta^b \theta_{ba} + \omega_a(\kappa + \theta) + q_a{}^{a'}\nabla_L \omega_{a'} + q_a{}^{a'}\nabla_b \theta_{a'}{}^b + \bar{\theta}_{ba} a^b \\ &= (\omega - \bar{\eta})_a \kappa + q_a{}^{a'}\nabla_L \omega_{a'} + \theta \omega_a + \mathrm{d}_b \theta_a{}^b + \theta_a{}^b \bar{\eta}_b + \bar{\theta}_a{}^b a_b, \end{aligned} \tag{B.15}$$

where for the second line we used $\nabla_a L^a = \kappa + \theta$ and for the third line we used $\nabla_b \theta_a{}^b = \mathrm{d}_b \theta_a{}^b + (\eta + \bar{\eta})_b \theta_a{}^b$. The latter follows from

$$\begin{aligned} \nabla_b v^b &= q^{ab}\nabla_a v_b + L^a \bar{L}^b \nabla_a v_b + L^b \bar{L}^a \nabla_a v_b \\ &= \mathrm{d}_a v^a - \nabla_L \bar{L}^b v_b - \nabla_{\bar{L}} L^b v_b \end{aligned} \tag{B.16}$$

for $v^a = q^a{}_b v^b$. Using that $a_a = 0$ and $\bar{\eta}_a = \omega_a$ on B, and using that $\theta_a{}^b = (\sigma_a{}^b + \frac{1}{D-2}\gamma_a{}^b\theta)$, this gives

$$q_a{}^c \nabla_b \nabla_c L^b \overset{B}{=} q_a{}^c \nabla_L \omega_c + \theta_a{}^c \omega_c + \theta \omega_a + \mathrm{d}_b \sigma_a{}^b + \tfrac{1}{D-2}\mathrm{d}_a \theta \tag{B.17}$$

$$= q_a{}^b(\mathcal{L}_L + \theta)\omega_b + \mathrm{d}_b \sigma_a{}^b + \tfrac{1}{D-2}\mathrm{d}_a \theta. \tag{B.18}$$

For the second line, we used $q_a{}^b \mathcal{L}_L \omega_b = q_a{}^b \nabla_L \omega_b + \theta_a{}^c \omega_c$. The other term in $q_a{}^b R_{Lb}$ is simply $-q_a{}^c \nabla_c \nabla_b L^b = -\mathrm{d}_a(\theta + \kappa)$, yielding

$$q_a{}^b R_{Lb} = q_a{}^b(\mathcal{L}_L + \theta)\bar{\eta}_b + \mathrm{d}_b \sigma_a{}^b - \mathrm{d}_a(\kappa + \tfrac{D-3}{D-2}\theta). \tag{B.19}$$

Finally, using $\mathcal{L}_L \epsilon_{\mathrm{B}} = \theta \epsilon_{\mathrm{B}}$, this may be written in the form

$$q_a{}^b R_{Lb}\epsilon_{\mathrm{B}} = q_a{}^b \mathcal{L}_L(\bar{\eta}_a \epsilon_{\mathrm{B}}) + \mathrm{d}_b \sigma_a{}^b \epsilon_{\mathrm{B}} - \mathrm{d}_a \mu \epsilon_{\mathrm{B}}. \tag{B.20}$$

Let us turn to the null Raychaudhuri equation. We have

$$G_{LL} = R_{LL} = L^a(\nabla_b \nabla_a L^b - \nabla_a \nabla_b L^b). \tag{B.21}$$

For the first term, we use that

$$\begin{aligned} L^a \nabla_b \nabla_a L^b &= \nabla_b(\nabla_L L^b) - (\nabla_b L^a)(\nabla_a L^b) \\ &= \nabla_b(L^b \kappa - a^b) - (L^a(\kappa \bar{L}_b + \omega_b) + \theta_b{}^a + \bar{L}_b a^a)(L^b(\kappa \bar{L}_a + \omega_a) + \theta_a{}^b + \bar{L}_a a^b) \\ &= L[\kappa] + \kappa \nabla_a L^a - \nabla_a a^a - [\kappa^2 + 2\omega_b a^b + \theta_b{}^a \theta_a{}^b] \\ &= L[\kappa] + \kappa\theta - \theta_b{}^a \theta_a{}^b - \mathrm{d}_a a^a - (\eta + \bar{\eta} + 2\omega)_a a^a. \end{aligned} \tag{B.22}$$

We used

$$\nabla_a L^b = L^b(\bar{L}_a\kappa + \omega_a) + \theta_a{}^b + \bar{L}_a a^b - L_a\eta^b, \tag{B.23}$$

and $\nabla_a a^a = \mathrm{d}_a a^a + (\eta_a + \bar{\eta}_a)a^a$. The second term in G_{LL} is simply $L[\kappa+\theta]$ and taking the difference gives

$$G_{LL} = -L[\theta] + \kappa\theta - \theta_b{}^a\theta_a{}^b - \mathrm{d}_a a^a - (\eta + \bar{\eta} + 2\omega)_a a^a. \tag{B.24}$$

Using that $a = 0$ on B and splitting $\theta^a{}_b$ into trace and traceless part as before we get

$$\begin{aligned} G_{LL} &\overset{B}{=} -L[\theta] + \kappa\theta - \sigma_b{}^a\sigma_a{}^b - \tfrac{1}{(D-2)}\theta^2 & \text{(B.25)}\\ &= -(L+\theta)[\theta] + \mu\theta - \sigma_b{}^a\sigma_b{}^a. & \text{(B.26)}\end{aligned}$$

Using $\mathcal{L}_L\epsilon_{\mathrm{B}} = \theta\epsilon_{\mathrm{B}}$, we get

$$G_{LL}\epsilon_{\mathrm{B}} = -\mathcal{L}_L(\theta\epsilon_{\mathrm{B}}) + \mu\theta\epsilon_{\mathrm{B}} - \sigma_b{}^a\sigma_a{}^b\epsilon_{\mathrm{B}}. \tag{B.27}$$

The form of this equation is sensitive to the normalization of the null normal L. Introducing $\mu_f = f\mu + L[f]$ for an arbitrary function f on B, we can write the densitized Raychaudhuri equation for an arbitrary normalization of L:

$$fL^aG_a{}^bL_b\epsilon_{\mathrm{B}} = -\mathcal{L}_{fL}(\theta\epsilon_{\mathrm{B}}) + (\mu_f\theta - f\sigma_b{}^a\sigma_b{}^a)\epsilon_{\mathrm{B}}. \tag{B.28}$$

B.3 Diffeomorphism Actions

This appendix derives the field space Lie derivatives of intrinsic and extrinsic geometry along a spacetime vector field ξ. Since we are interested in which pieces of geometry "talk to" the extension of ξ outside of B, we fix $\xi \parallel B$ at B but allow for an arbitrary extension outside of B, setting $\xi^a = fL^a + \bar{f}\bar{L}^a + v^a$ with $\bar{f} = 0$ on B.

Intrinsic Geometry:

- $\mathfrak{L}_\xi L^a$: We have $-q^{ab}L^c\delta g_{bc} = -q^{ab}(\delta L_b - g_{ab}\delta L^b) = \delta L^a$. We used that the ϕ^0-component of L^a is fixed: $\delta L^0 = 0$, such that $q^a{}_b\delta L^b = \delta L^b$. We also used that $\delta L_a = \delta h L_a$ is normal to S. We get

$$\begin{aligned}\mathfrak{L}_\xi L^a &= I_\xi\delta L^a = -q^{ab}L^c(\nabla_b\xi_c + \nabla_c\xi_b)\\ &= -q^{ab}L^c(\nabla_b v_c + \nabla_c v_b)\\ &= q^a{}_b(v^c\nabla_b L_c - L^c\nabla_c v^b)\\ &= q^a{}_b[v,L]^b = [v,L]^a. \end{aligned}\tag{B.29}$$

We used that $\nabla_a L_b$ is symmetric when pulled back to S, and that $[v, L]^a$ is tangential to S since L preserves the foliation S. Noting $\mathcal{L}_\xi L^a = -L(f)L^a + [v, L]$, we obtain the anomaly $\Delta_\xi L^a = -[fL, L]^a = L(f)L^a$.

- $\mathfrak{L}_\xi h$: We have $\delta h = \delta g_{ab} L^a \bar{L}^b$, as may be checked explicitly in coordinates. Then

$$\begin{aligned}\mathfrak{L}_\xi h &= (L^a \bar{L}^b + \bar{L}^a L^b)\nabla_a (f L_b + \bar{f}\bar{L}_b + v_b) \\ &= (L + \kappa)[f] + (\bar{L} + \bar{\kappa})[\bar{f}] + (\eta_a + \bar{\eta}_a)v^a,\end{aligned} \tag{B.30}$$

where $\eta_a = -q_{ab}\nabla_{\bar{L}} L^b$ and $\bar{\kappa} = L_a \nabla_{\bar{L}} \bar{L}^a$. Since h contains the transverse derivative $\bar{L}[\bar{f}]$, it "talks to" the extension of ξ outside of B.

- $\mathfrak{L}_\xi q_{AB}$: We have $\delta q_{AB} = q_A{}^a q_B{}^b \delta g_{ab}$, hence, also using that $\theta_{AB} = \frac{1}{2} q_A{}^a q_B{}^b \mathcal{L}_L g_{ab}$,

$$\mathfrak{L}_\xi q_{AB} = 2f\theta_{AB} + \mathcal{L}_v q_{AB}. \tag{B.31}$$

Similarly,

$$\mathfrak{L}_\xi q^{AB} = -2f\theta^{AB} + \mathcal{L}_v q^{AB}. \tag{B.32}$$

- $\mathfrak{L}_\xi \varphi$, ϵ_{B} and ϵ_{S}: We have $\varphi = \frac{1}{D-2}\ln\sqrt{q}$, hence $\delta\varphi = \frac{1}{2}\frac{1}{D-2} q^{AB}\delta q_{AB}$. Using the previous we get

$$\begin{aligned}\mathfrak{L}_\xi \varphi &= \frac{1}{D-2} q^{ab}\nabla_a (f L_b + \bar{f}\bar{L}_b + v_b) \\ &= \frac{1}{D-2}(f\theta + \mathrm{d}_A v^A).\end{aligned} \tag{B.33}$$

Using $\delta\epsilon_{\mathrm{B}} = (D-2)\delta\varphi\epsilon_{\mathrm{B}}$, one gets

$$\mathfrak{L}_\xi \epsilon_{\mathrm{B}} = (f\theta + \mathrm{d}_A v^A). \tag{B.34}$$

Using also $\mathcal{L}_\xi \epsilon_{\mathrm{B}} = \mathrm{d}\iota_\xi \epsilon_{\mathrm{B}} = (f\theta + L[f] + \mathrm{d}_A v^A)\epsilon_{\mathrm{B}}$, which may be checked in a coordinate calculation, one gets

$$\Delta_\xi \epsilon_{\mathrm{B}} = -L[f]\epsilon_{\mathrm{B}}. \tag{B.35}$$

Using that Δ satisfies the Leibniz rule, we get

$$\Delta_\xi (L^a \epsilon_{\mathrm{B}}) = 0, \tag{B.36}$$

thus as we claimed $L^a\epsilon_{\rm B}$ is a density on B valued into vectors of B which is covariant under diffeomorphisms of B. Putting together the previous results, we have

$$\mathfrak{L}_\xi(L^a\epsilon_{\rm B}) = \mathcal{L}_\xi(L^a\epsilon_{\rm B}) = [v,L]^a + (f\theta + {\rm d}_A v^A)L^a\epsilon_{\rm B}. \tag{B.37}$$

Since $\epsilon_{\rm S} = L^a\iota_a\epsilon_{\rm B}$, we also get

$$\begin{aligned}&\Delta_\xi\epsilon_{\rm S} = 0\\ &\mathfrak{L}_\xi\epsilon_{\rm S} = \mathcal{L}_\xi\epsilon_{\rm S} = \iota_{[v,L]}\epsilon_{\rm B} + (f\theta + {\rm d}_A v^A)\epsilon_{\rm S}.\end{aligned} \tag{B.38}$$

Using $\mathcal{L}_\xi\epsilon_{\rm S} = \iota_\xi{\rm d}\epsilon_{\rm S} + {\rm d}\iota_\xi\epsilon_{\rm S}$, we have

$$\begin{aligned}&\iota_\xi{\rm d}\epsilon_{\rm S} = f\theta\epsilon_{\rm S} + \theta\iota_v\epsilon_{\rm B}\\ &{\rm d}\iota_\xi\epsilon_{\rm S} = {\rm d}_A v^A\epsilon_{\rm S} + \iota_{[v,L]}\epsilon_{\rm B} - \theta\iota_v\epsilon_{\rm B}.\end{aligned} \tag{B.39}$$

- The derivative $\mathfrak{L}_\xi\gamma_{AB}$ may be derived using $\gamma_{AB} = e^{-2\varphi}q_{AB}$ and the chain and Leibniz rules and reads

$$\mathfrak{L}_\xi\gamma_{AB} = 2f\sigma_{AB} - 2e^{-2\varphi}{\rm d}_{<A}v_{B>}. \tag{B.40}$$

Extrinsic Geometry: The Weingarten map is the tensor $\nabla_a L^b$, which is a tensor on B (i.e., the index a is pulled back onto B and the index b is tangential to B). Its transformation is easily worked out as

$$\begin{aligned}\mathfrak{L}_\xi(\nabla_a L^b) &\stackrel{B}{=} \mathcal{L}_\xi(\nabla_a L^b) + \Delta_\xi(\nabla_a L^b) = \mathcal{L}_\xi(\nabla_a L^b) + \nabla_a(\Delta_\xi L^b)\\ &= \mathcal{L}_\xi(\nabla_a L^b) + \nabla_a(L(f)L^b).\end{aligned} \tag{B.41}$$

We used that the anomaly Δ_ξ commutes with the covariant derivative ∇ as argued in section 4.1.2, and plugged in the anomaly $\Delta_\xi L^a = L[f]L^a$. The Weingarten map is thus non-covariant, but its transformation is independent of the extension of ξ. We will get the transformations of extrinsic geometry by taking components of the transformation of the Weingarten map. Recall that as a tensor on B,

$$\nabla_a L^b = (\omega_a + \bar{L}_a\kappa)L^b + \theta_a{}^b. \tag{B.42}$$

- κ is the ϕ^0-component of $L^a\nabla_a L^b$, thus

$$\begin{aligned}\mathfrak{L}_\xi\kappa &= \bar{L}_b\mathfrak{L}_\xi(L^a\nabla_a L^b)\\ &= \bar{L}_b[\mathcal{L}_\xi(L^a\nabla_a L^b) + (\Delta_\xi L^a)\nabla_a L^b + L^a\Delta_\xi(\nabla_a L^b)]\\ &= v[\kappa] + L[(L+\kappa)[f]].\end{aligned} \tag{B.43}$$

- θ: We have that $(\kappa+\theta)$ is the trace (on B) of the Weingarten map. Taking the trace of B.41 gives:

$$\begin{aligned}\mathfrak{L}_\xi(\kappa+\theta) &= \xi[\kappa+\theta] + L[L[f]] + L[f](\kappa+\theta) \\ &= v[\kappa+\theta] + L[(L+\kappa+\theta)[f]].\end{aligned} \tag{B.44}$$

Then using the result for κ, θ transforms as

$$\mathfrak{L}_\xi\theta = v[\theta] + L[f\theta]. \tag{B.45}$$

We get that $\theta\epsilon_{\mathrm{B}}$ is covariant, which can also be seen because ϵ_{S} is covariant and $\theta\epsilon_{\mathrm{B}} = \mathrm{d}\epsilon_{\mathrm{S}}$.

- μ: combining the previous two results, we get

$$\mathfrak{L}_\xi\mu = v[\mu] + L[[L+\mu]f]. \tag{B.46}$$

- We have $\theta_{ab} = g_{bc}\nabla_a L^c$ (remember that everything is pulled back onto B), and using that g_{ab} has vanishing anomaly and that $g(L)$ vanishes when pulled back,

$$\mathfrak{L}_\xi\theta_{ab} = \mathcal{L}_\xi\theta_{ab} + L[f]\theta_{ab} = (f\mathcal{L}_L + L[f])\theta_{ab} + \mathcal{L}_v\theta_{ab}. \tag{B.47}$$

Thus $\theta_{ab}\epsilon_{\mathrm{B}}$ as a tensor on B is covariant. The transformation of the upstairs extrinsic curvature and shear are obtained by combining with the transformation of q^{AB} and φ. The upstairs extrinsic curvature transforms as

$$\mathfrak{L}_\xi\theta^{AB} = (f\partial_0 + f\mathcal{L}_U + L[f])\theta^{AB} + \mathcal{L}_v\theta^{AB}. \tag{B.48}$$

- ω_A: Note that ${q_A}^b = \partial x^b/\partial\sigma^A$ is independent of the metric, such that $\mathfrak{L}_\xi {q_A}^b = 0$. Also, $\bar{L}_a = (\mathrm{d}\phi^0)_a$ as a tensor on B is independent of the metric, so $\mathfrak{L}_\xi\bar{L}_a = 0$. With that,

$$\begin{aligned}\mathfrak{L}_\xi\omega_A &= \mathfrak{L}_\xi({q_A}^a\bar{L}_b\nabla_a L^b) = {q_A}^a\bar{L}_b\mathfrak{L}_\xi\nabla_a L^b) \\ &= {q_A}^a\bar{L}_b(\mathcal{L}_\xi\nabla_a L^b + \nabla_a(L[f]L^b)) \\ &= {q_A}^a\bar{L}_b\mathcal{L}_\xi(\omega_a L^b + \kappa\bar{L}_a L^b + {\theta_a}^b) + \partial_A L[f] + L[f]\omega_A \\ &= \mathcal{L}_v\omega_A + f{q_A}^a\mathcal{L}_L\omega_a + \partial_A(\partial_L f) + \kappa\partial_A f - {\theta_A}^B\partial_B f.\end{aligned} \tag{B.49}$$

The same transformation has been given in [65].

Appendix C

Appendices to Chapter 6

C.1 Connecting to Bondi Gauge

We want to express (6.54) in the parameterization of [28].

C.1.1 Orders of the Metric and χ

For χ, we get:

$$\chi = \frac{1}{2\Omega} g^{\Omega\Omega} = -\frac{1}{2} e^{-2\beta} \Omega(\Omega V), \qquad \chi_{(0)} = 0, \qquad \chi_{(1)} = -\frac{1}{2} \mathring{V}, \qquad \chi_{(2)} = -M. \tag{C.1}$$

The determinant of q_{AB} is fixed, and β starts only at second order, so we have: $\sqrt{g}_{(0)} = \sqrt{Q}$ and $\sqrt{g}_{(1)} = 0$. The only term that survives from the second line of (6.54) is thus:

$$2\sqrt{g}_{(0)} \delta\chi_{(2)} = -2\sqrt{Q}\delta M. \tag{C.2}$$

It is a total variation and may thus be dropped.

The only components that vary at the leading and first subleading order of g_{ab} are:

$$g^{(0)}_{AB} = Q_{AB} \qquad \text{and} \qquad g^{(1)}_{AB} = C_{AB}. \tag{C.3}$$

To get the symplectic potential in this partial gauge, we thus need to compute the leading and first subleading order of the sphere-sphere components of $\sqrt{g}(\mathcal{N}^{AB} - q^{AB}\mathcal{N})$. To do so, we will need

$$N^a \partial_a = e^{-2\beta}(\partial_u + U^A \partial_A - \Omega^2(\Omega V)\partial_\Omega). \tag{C.4}$$

C.1.2 Orders of Some Components of $\mathcal{N}$

Let us start with the leading and first subleading orders of $\sqrt{g}\mathcal{N}$: we have

$$\Omega\sqrt{g}\mathcal{N} = \partial_a(\sqrt{g}N^a) - 4\sqrt{g}\chi = \partial_u\sqrt{q} + \partial_A(\sqrt{q}U^A) - \sqrt{q}\Omega^2\partial_\Omega(\Omega V). \tag{C.5}$$

Now $\sqrt{q}$ is independent of Ω, and U^A starts only at second order. We get

$$(\sqrt{g}\mathcal{N})_{(0)} = 0 \qquad (\sqrt{g}\mathcal{N})_{(1)} = \sqrt{Q}(D_A\mathring{U}^A - 2M), \tag{C.6}$$

where D is the covariant derivative of Q.

Finding the orders of $\mathcal{N}^{AB}$ is more involved. We start from

$$\Omega\mathcal{N}^{ab} = \nabla^a N^b - g^{ab}\chi = -\frac{1}{2}\mathcal{L}_N g^{ab} - g^{ab}\chi = \frac{1}{2}(-N^c\partial_c g^{ab} + g^{ac}\partial_c N^b + g^{bc}\partial_c N^a) - g^{ab}\chi. \tag{C.7}$$

The sphere-sphere components become

$$\begin{aligned}\Omega\mathcal{N}^{AB} = \frac{1}{2}\Big(&- e^{-2\beta}\partial_u q^{AB} - e^{-2\beta}U^C\partial_C q^{AB} + q^{AC}\partial_C(e^{-2\beta}U^B) + q^{BC}\partial_C(e^{-2\beta}U^A) \\ &+ \partial_\Omega(e^{-4\beta}U^AU^B) + e^{-2\beta}\Omega^2(\Omega V)\partial_\Omega g^{AB} + e^{-2\beta}\Omega(\Omega V)g^{AB}\Big).\end{aligned} \tag{C.8}$$

The leading order is

$$\mathcal{N}^{AB}_{(0)} = \partial_\Omega(\Omega\mathcal{N}^{AB})|_{\Omega=0} = \frac{1}{2}\partial_u C^{AB} + \frac{1}{2}\mathring{V}Q^{AB}. \tag{C.9}$$

We have used that $\partial_\Omega q^{AB}|_{\Omega=0} = -C^{AB}$ where $C^{AB} = Q^{AA'}Q^{BB'}C_{A'B'}$. The subleading order is

$$\mathcal{N}^{AB}_{(1)} = \frac{1}{2}\partial_\Omega{}^2(\Omega\mathcal{N}^{AB})|_{\Omega=0} = -\frac{1}{2}\partial_u(q^{AB}_{(2)}) + \frac{1}{2}(D^AU^B + D^BU^A) - \mathring{V}C^{AB} + MQ^{AB}. \tag{C.10}$$

We have used that Q is independent of u, so that $(\partial_\Omega{}^2 e^{-2\beta})\partial_u q^{AB}$ does not contribute. $q^{AB}_{(2)} = \frac{1}{2}\partial_\Omega{}^2 q^{AB}|_{\Omega=0}$ is the second order of the inverse sphere metric.

C.1.3 Symplectic Potential in Generalized Bondi Gauge

Let us put these pieces together. For the first term in (6.54), we get

$$
\begin{aligned}
&-\frac{1}{2}\big(\sqrt{g}(\mathcal{N}^{ab}-g^{ab}\mathcal{N})\big)_{(0)}\delta g^{(1)}_{ab} = -\frac{1}{4}\sqrt{Q}(\partial_u C^{AB}+V_{(0)}Q^{AB})\delta C_{AB}\\
&=\frac{1}{4}\big(-\delta\big(\sqrt{Q}(\partial_u C^{AB})C_{AB}\big)+\partial_u(\sqrt{Q}C_{AB}\delta C^{AB})-\sqrt{Q}\partial_u C_{AB}\delta C^{AB}+\sqrt{Q}\mathring{V}C_{AB}\delta Q^{AB}\big).
\end{aligned}
\tag{C.11}
$$

Here, δQ^{AB} is the variation of the inverse of Q_{AB}, and δC^{AB} is the variation of $C^{AB} = Q^{AA'}Q^{BB'}C_{A'B'}$, which is the negative of the first order of q^{AB}: $C^{AB} = -q^{AB}_{(1)}$. For the second line, we have integrated and varied by parts, using that $\partial_u\sqrt{Q} = \delta\sqrt{Q} = Q^{AB}C_{AB} = 0$.

For the second term, using that $\mathcal{N}_{(0)} = 0$, that $\sqrt{g}_{(1)} = 0$, and that $Q^{AB}\delta Q_{AB} = 0$, we get:

$$
\begin{aligned}
&-\frac{1}{2}\big(\sqrt{g}(\mathcal{N}^{ab}-g^{ab}\mathcal{N})\big)_{(1)}\delta g^{(0)}_{ab} = -\frac{1}{2}\sqrt{Q}\mathcal{N}^{AB}_{(1)}\delta Q_{AB}\\
&=\frac{1}{2}\sqrt{Q}\Big(-D^A\mathring{U}^B+\mathring{V}C^{AB}+\frac{1}{2}\partial_u q^{AB}_{(2)}\Big)\delta Q_{AB}\\
&=\frac{1}{2}\partial_A(\sqrt{Q}\mathring{U}_B\delta Q^{AB})+\frac{1}{2}\partial_u(\frac{1}{2}\sqrt{Q}q^{AB}_{(2)}\delta Q_{AB})-\frac{1}{2}\mathring{U}_B D_A\delta Q^{AB}-\frac{1}{2}\mathring{V}C_{AB}\delta Q^{AB}.
\end{aligned}
\tag{C.12}
$$

For the last line, we have used $\delta Q_{AB} = -Q_{AA'}Q_{BB'}\delta Q^{A'B'}$, and that Q is constant in u.

Altogether, we get

$$
\begin{aligned}
\hat{\boldsymbol{\theta}}^{\Omega}_{R,GH} = {}&\delta\big(-\frac{1}{4}\sqrt{Q}(\partial_u C^{AB})C_{AB}-2\sqrt{Q}M\big)\\
&+\partial_u\big(\frac{1}{4}\sqrt{Q}C_{AB}\delta C^{AB}+\frac{1}{4}\sqrt{Q}q^{AB}_{(2)}\delta Q_{AB})+\partial_A\big(\frac{1}{2}\sqrt{Q}\mathring{U}_B\delta Q^{AB}\big)\\
&-\frac{1}{2}\Big(\frac{1}{2}\sqrt{Q}(\partial_u C_{AB})\delta C^{AB}+\frac{1}{2}\mathring{V}C_{AB}\delta Q^{AB}+\mathring{U}_B D_A\delta Q^{AB}\Big).
\end{aligned}
\tag{C.13}
$$

In the first line, note that $-\frac{1}{4}\sqrt{Q}(\partial_u C^{AB})C_{AB} = -\frac{1}{8}\partial_u(\sqrt{Q}C_{AB}C^{AB})$. In the second line, use that $q^{AB}_{(2)} = -D^{AB}+C^{AC}C^{B}_{C}$. By the equation of motion (6.53), D^{AB} is proportional to Q^{AB} and does not contribute since $Q^{AB}\delta Q_{AB} = 0$. Using also that $\frac{1}{8}\delta(\sqrt{Q}C^{AB}C_{AB}) = \frac{1}{4}\sqrt{Q}C_{AB}\delta C^{AB}+\frac{1}{4}\sqrt{Q}C^{AC}C^{B}_{C}\delta Q_{AB}$, the ∂_u-term in the second line becomes a total variation and cancels against the term in the first line. We are left with

$$\begin{aligned}\hat{\boldsymbol{\theta}}^{\Omega}_{R,GH} = & -2\delta(\sqrt{Q}M) + \frac{1}{2}\partial_A(\sqrt{Q}\mathring{U}_B\delta Q^{AB}) \\ & -\frac{1}{2}\Big(\frac{1}{2}\sqrt{Q}(\partial_u C_{AB})\delta C^{AB} + \frac{1}{2}\mathring{V}C_{AB}\delta Q^{AB} + \mathring{U}_B D_A\delta Q^{AB}\Big).\end{aligned} \tag{C.14}$$

C.2 Hamiltonians

We want to evaluate $-I_\xi\delta\boldsymbol{\theta}^n$. The starting point is

$$\Omega^{-(D-3)}\delta\boldsymbol{\theta}^n = \delta\hat{\boldsymbol{\theta}}^{\Omega} - \frac{1}{2}\Omega^{-(D-2)}\partial_i\delta(\sqrt{g}\delta N^i). \tag{C.15}$$

The expression for $I_\xi\delta\hat{\boldsymbol{\theta}}^{\Omega}$ reads, translating (2.17) to our notation and dropping terms in the constraint and its variation:

$$\begin{aligned}-I_\xi\delta\hat{\boldsymbol{\theta}}^{\Omega} &= \partial_i \not{\delta}\hat{\boldsymbol{h}}^i, \text{ where} \\ \not{\delta}\hat{\boldsymbol{h}}^i &= N_a P^i_b\Big(\tfrac{1}{2}\delta\big(\sqrt{\hat{g}}(\hat{\nabla}^b\xi^a - \hat{\nabla}^a\xi^b)\big) + \xi^a\hat{\boldsymbol{\theta}}^b - \xi^b\hat{\boldsymbol{\theta}}^a\Big).\end{aligned} \tag{C.16}$$

It features the variation of the Komar charge, where the index on $\hat{\nabla}$ is raised with the physical metric. We have made the coordinate projector P^i_b explicit, however, the expression is automatically tangential to the level surfaces of Ω because of the antisymmetry of the terms in parentheses. Geometrically, $\not{\delta}\hat{\boldsymbol{h}}^i$ is a $(D-2)$-form, pulled back onto an $\Omega = $ const. surface.

The correction term can be written as

$$\frac{1}{2}\Omega^{-(D-2)}I_\xi\delta(\sqrt{g}\delta N^i) = \frac{1}{2}N_a P^i_b I_\xi\delta\big(\sqrt{\hat{g}}\delta(\hat{g}^{ab})\big). \tag{C.17}$$

We have used that N_a and P^i_b do not vary, and $\delta(\hat{g}^{ab})$ is the variation of the inverse metric. The term $\sqrt{\hat{g}}\delta(\hat{g}^{ab})$ is covariant, in the sense that the action $I_\xi\delta$ coincides with the spacetime Lie derivative, since it involves no background structure. Using $\mathcal{L}_\xi = \delta I_\xi + I_\xi\delta$, we get

$$\frac{1}{2}N_a P^i_b I_\xi\delta\big(\sqrt{\hat{g}}\delta(\hat{g}^{ab})\big) = \frac{1}{2}N_a P^i_b(\mathcal{L}_\xi - \delta I_\xi)\big(\sqrt{\hat{g}}\delta(\hat{g}^{ab})\big) \tag{C.18}$$

$$= \frac{1}{2}N_a P^i_b\delta\big(\sqrt{\hat{g}}(\hat{\nabla}^a\xi^b + \hat{\nabla}^b\xi^a)\big) + \frac{1}{2}N_a P^i_b\mathcal{L}_\xi(\sqrt{\hat{g}}\delta\hat{g}^{ab}). \tag{C.19}$$

Together, we get

$$-\Omega^{-(D-3)} I_\xi \delta \boldsymbol{\theta}^n = \partial_i \oint \hat{\boldsymbol{h}}_1^i, \text{ where}$$
$$\oint \hat{\boldsymbol{h}}_1^i = N_a P^i_b \Big(\delta(\sqrt{\hat{g}} \hat{\nabla}^b \xi^a) + \xi^a \hat{\boldsymbol{\theta}}^b - \xi^b \hat{\boldsymbol{\theta}}^a + \frac{1}{2} \mathcal{L}_\xi (\sqrt{\hat{g}} \delta \hat{g}^{ab}) \Big). \tag{C.20}$$

Let us now consider each of the pieces of (C.20) in turn. In the term $N_a P^i{}_b \sqrt{\hat{g}} \hat{\nabla}^b \xi^a$, translating to the unphysical metric gives

$$N_a P^i{}_b \sqrt{\hat{g}} \hat{\nabla}^b \xi^a = \Omega^{-(D-2)} \sqrt{g} N_a P^i_b g^{bb'} \nabla_b \xi^a + \Omega^{-(D-1)} \sqrt{g} (-2N^i N_a \xi^a + N^a N_a \xi^i). \tag{C.21}$$

Now integrate by parts in the first term, and use $\nabla_b N_a = \Omega \mathcal{N}_{ab} + \chi g_{ab}$, as well as $N_a \xi^a = \Omega p$ and $N_a N^a = 2\Omega\chi - 2\lambda$. We get

$$\begin{aligned} N_a P^i{}_b \sqrt{\hat{g}} \hat{\nabla}^b \xi^a = & -2\Omega^{-(D-1)} \sqrt{g} \xi^i \lambda + \Omega^{-(D-2)} \sqrt{g} \big(\chi \xi^i - N^i p \big) \\ & + \Omega^{-(D-3)} \sqrt{g} \big(-\mathcal{N}^i_a \xi^a + P^i_b g^{bb'} \nabla_{b'} p \big). \end{aligned} \tag{C.22}$$

Here $N^i = P^i_b N^b$, and $\mathcal{N}^i_a = P^i_b \mathcal{N}^b_a$.

Using (6.26), the next term of (C.20) becomes

$$N_a P^i_b \xi^a \hat{\boldsymbol{\theta}}^b = \Omega p \hat{\boldsymbol{\theta}}^i = -\frac{1}{2} \Omega p N_a P^i_b \partial_\Omega (\sqrt{\hat{g}} \delta \hat{g}^{ab}) + \Omega^{-(D-2)} p \delta(\sqrt{g} N^i) + \Omega^{-(D-3)} p \boldsymbol{\theta}^i_1. \tag{C.23}$$

The next term of (C.20) is, using (6.24):

$$\begin{aligned} -N_a P^i_b \xi^b \hat{\boldsymbol{\theta}}^a = -\xi^i \hat{\boldsymbol{\theta}}^\Omega = & \, 2\Omega^{-(D-1)} \xi^i \lambda \delta \sqrt{g} + \Omega^{-(D-2)} \big(-\xi^i \delta(\sqrt{g}\chi) - \tfrac{1}{2} \xi^i \partial_j (\sqrt{g} \delta N^j) \big) \\ & - \Omega^{-(D-3)} \xi^i \boldsymbol{\theta}^n. \end{aligned} \tag{C.24}$$

Lastly, the Lie derivative may be written as

$$\frac{1}{2} N_a P^i_b \mathcal{L}_\xi (\sqrt{\hat{g}} \delta \hat{g}^{ab}) = \frac{1}{2} N_a P^i_b \Big(\partial_c (\sqrt{\hat{g}} \delta \hat{g}^{ab} \xi^c) - \sqrt{\hat{g}} \delta \hat{g}^{ac} \partial_c \xi^b - \sqrt{\hat{g}} \delta \hat{g}^{bc} \partial_c \xi^a \Big). \tag{C.25}$$

Splitting the summation over c into tangential and transverse contributions in all three terms, we get

$$\begin{aligned} \frac{1}{2} N_a P^i_b \mathcal{L}_\xi (\sqrt{\hat{g}} \delta \hat{g}^{ab}) = \frac{1}{2} N_a P^i_b \Big(& \partial_\Omega (\Omega p \sqrt{\hat{g}} \delta \hat{g}^{ab}) + \partial_j (\sqrt{\hat{g}} \delta \hat{g}^{ab} \xi^j) - \sqrt{\hat{g}} P^j_c \delta \hat{g}^{ac} \partial_j \xi^b \\ & - \sqrt{\hat{g}} N_c \delta \hat{g}^{ac} \partial_\Omega \xi^b - \sqrt{\hat{g}} \delta \hat{g}^{bc} P^i_c \partial_i \xi^a - \sqrt{\hat{g}} N_c \delta \hat{g}^{bc} \partial_\Omega \xi^a \Big). \end{aligned} \tag{C.26}$$

Now, we use that $\sqrt{\hat{g}}\delta\hat{g}^{ab} = \Omega^{-(D-2)}\sqrt{g}\delta g^{ab}$, that N_a and P^i_b commute with the coordinate derivatives ∂_i and ∂_Ω, that $N_a\delta\sqrt{g}^{ab} = \delta N^a$, and that $N_a\delta N^a = 2\Omega\delta\chi$. Altogether, that gives

$$\frac{1}{2}N_a P^i_b \mathcal{L}_\xi(\sqrt{\hat{g}}\delta\hat{g}^{ab}) = \frac{1}{2}\Omega p N_a P^i_b \partial_\Omega(\sqrt{\hat{g}}\delta\hat{g}^{ab}) + \Omega^{-(D-2)}\big(\frac{1}{2}\partial_j(\sqrt{g}\xi^j\delta N^i) - \frac{1}{2}\sqrt{g}\delta N^j\partial_j\xi^i\big) + \Omega^{-(D-3)}\big(-\sqrt{g}\delta\chi\partial_\Omega\xi^i - \frac{1}{2}\sqrt{g}\delta g^{ij}\partial_j p\big). \tag{C.27}$$

Putting the previous equations together gives

$$\oint\hat{\boldsymbol{h}}^i_1 = \frac{1}{2}\Omega^{-(D-2)}\big(\partial_j(\sqrt{g}\xi^j\delta N^i) - \sqrt{g}\delta N^j\partial_j\xi^i - \xi^i\partial_j(\sqrt{g}\delta N^j)\big) + \Omega^{-(D-3)}\Big(\delta\big(-\sqrt{g}\mathcal{N}^i{}_a\xi^a + \sqrt{g}g^{ia}\partial_a p\big) + p\boldsymbol{\theta}^a_1 - \xi^i\boldsymbol{\theta}^n - \sqrt{g}\delta\chi\partial_\Omega\xi^i - \frac{1}{2}\sqrt{g}\delta g^{ij}\partial_j p\Big). \tag{C.28}$$

The terms in the first line combine to $\partial_j(\sqrt{g}\xi^j\delta N^i - \sqrt{g}\xi^i\delta N^j)$. Since ultimately only $\partial_i\oint\hat{\boldsymbol{h}}^i_1$ is determined, and ∂_i of the first line is zero, they may be dropped. We thus get

$$-I_\xi\delta\boldsymbol{\theta}^n = \partial_i\oint\boldsymbol{h}^i_1, \text{ where}$$
$$\oint\boldsymbol{h}^i_1 = \delta\big(-\sqrt{g}\mathcal{N}^i{}_a\xi^a + \sqrt{g}g^{ia}\partial_a p\big) + p\boldsymbol{\theta}^a_1 - \xi^i\boldsymbol{\theta}^n - \sqrt{g}\delta\chi\partial_\Omega\xi^i - \frac{1}{2}\sqrt{g}\delta g^{ij}\partial_j p. \tag{C.29}$$

Here tensors with indices i and j are projected onto the level surfaces of Ω with the coordinate projector along ∂_Ω. g^{ij} is the projection of the inverse spacetime metric, not the inverse of the pulled-back metric.