Inhomogeneous solutions to the Einstein equations

by

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Abstract

In this dissertation we consider spherically symmetric gravitational fields that arise in relativistic astrophysics and cosmology. We first present a general review of static spherically symmetric spacetimes, and highlight a particular class of exact solutions of the Einstein-Maxwell system for charged spheres. In the case of shear-free spacetimes with heat flow, the integration of the system is reduced to solving the condition of pressure isotropy. This condition is a second order linear differential equation with variable coefficients. By choosing particular forms for the gravitational potentials, several classes of new solutions are generated. We regain known solutions corresponding to conformal flatness when tidal forces are absent. We also consider expanding, accelerating and shearing models when the heat flux is not present. A new general class of models is found. This new class of shearing solutions contains the model of Maharaj *et al* (1993) when a parameter is set to zero. Our new solution does not contain a singularity at the stellar centre, and it is therefore useful in modelling the interior of stars. Finally, we demonstrate that the shearing models obtained by Marklund and Bradley (1999) do not satisfy the Einstein field equations.

To

God and my family,

for their guidance and support.

Preface and Declaration

The study described in this dissertation was carried out in the School of Mathematical Sciences, University of KwaZulu-Natal, Durban. This dissertation was completed under the supervision of Professor S D Maharaj.

The research contained in this dissertation represents original work by the author and has not been submitted in any form to another University nor has it been published previously. Where use was made of the work of others it has been duly acknowledged in the text.

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Contents

Chapter 1

Introduction

The theory of general relativity, developed by Einstein, thus far has been the most successful model in describing the phenomenon of gravity especially for strong gravitational fields. For a long time, the gravitational interaction between heavenly bodies was described by the classical Newtonian theory of gravity but there were some astronomical observations that Newtonian gravity failed to explain. It was due to this fact that Einstein developed a new theory of gravity which would not only explain the observations, but also redefine our understanding of the concept of gravity and the important role it plays in shaping our universe. Not only does general relativity describe the interaction between objects but it also defines the interaction due to the gravitational field of the various interacting bodies. Gravity is not just defined as a simple force but rather as being part of a more powerful and richer structure, the four-dimensional spacetime manifold. It is the understanding of this gravitational field that enables us to study the gravitational nature and behaviour of various astrophysical and cosmological objects. In order to analyse the evolution of celestial objects, such as stars and galaxies, and the impact they have on the evolving universe, we first need to understand the behaviour of their gravitational fields. Models that are generated in the context of general relativity are important as they enable us to interpret observations made on the scale of the universe and, also, for strong local gravitational fields. For a comprehensive guide on the basic principles of general relativity and its role in astrophysics and cosmology the reader is referred to Gron and Hervik (2007) and Narlikar (2002).

We aim to find exact solutions to the Einstein field equations in spherically symmetric manifolds which form the basis of a relativistic model in astrophysics and cosmology. Even though there exist many classes of exact solutions, only a few classes are physically acceptable. Certain solutions that are found may be mathematically interesting but may not be appropriate for describing the physics of the problem. However, any exact model helps to provide a deeper insight into the behaviour of the gravitational field; they may provide qualitative features which are present in more complex models in physical scenarios. Exact solutions should be used in conjunction with other fundamental theories of physics, such as thermodynamics and electromagnetism, to make specific predictions and to study the physical features of the model. Hence finding exact solutions is a crucial starting point in the modelling process. Exact solutions to the Einstein field equations may be generated using a number of different techniques and assumptions: ad hoc choices for some of the matter and gravitational variables; imposing an equation of state; utilizing symmetries on the spacetime manifold, e.g. conformal transformations; using the Lie analysis of differential equations; applying generation techniques such as harmonic maps; and transforming nonlinear equations into familiar forms using Backlund transformations, etc. A comprehensive review of the methods and procedures utilized in generating solutions is provided by Stephani *et al* (2003).

Static spherically symmetric gravitational fields form the basis of the description for models of highly dense objects in astrophysics. Normally the matter distribution is considered to be a static perfect fluid which may be either neutral or charged. The most familiar exact solutions which are of physical importance are the Schwarzschild exterior and interior solutions (Schwarzschild 1916a, 1916b) and the Reissner-Nordstrom solution (Nordstrom 1918, Reissner 1916). The exterior Schwarzschild solution, which was the first solution of the Einstein field equations to be found, describes the gravitational field in the exterior spacetime of a body; the interior Schwarzschild solution models the gravitational field in the interior spacetime with constant density. The Reissner-Nordstrom solution is charged and describes the exterior spacetime. There are many interior stellar solutions which are known; some recent new interior solutions are the charged models obtained by Hansraj and Maharaj (2006), the solutions for charged superdense stars obtained by Komathiraj and Maharaj (2007) and the result for charged compact spheres found by Thirukkanesh and Maharaj (2006, 2008). These solutions contain the well known models obtained by Durgapal and Bannerji (1983), Finch and Skea (1989) and Tikekar (1990). There are also particular solutions of the Einstein field equations which are known for shear-free spacetimes. The earliest model is due to Kustaanheimo and Qvist (1948). Shear-free models may also include heat flow in the form of a nonvanishing heat flux across the boundary for a radiating star. Some recent results with nonvanishing heat flux were obtained by Deng and Mannheim (1990, 1991), in cosmology, and Naidu *et al* (2006), in astrophysics. Conformally flat radiating solutions were found by Banerjee *et al* (1989). These solutions were applied to radiating relativistic stars by Herrera *et al* (2004, 2006), Maharaj and Govender (2005), and Misthry *et al* (2008). The most general case involves spacetimes which have nonzero shear, acceleration and expansion. In these models the Einstein field equations are highly nonlinear, and only two classes of solutions have been reported in the literature. The first solution is by Maharaj *et al* (1993) and the other is by Marklund and Bradley (1999). These results are applicable in cosmological processes in the absence of heat flux. The shearing models may be easily adapted to include heat flux for particular physical applications.

This dissertation is organised as follows:

- Chapter 1: Introduction.

- Chapter 2: In this chapter we present a review and background on the fundamental concepts of differential geometry which are essential for constructing the relativistic models to be studied. A number of key definitions and formalisms are highlighted. The Einstein-Maxwell system of field equations are presented.

- Chapter 3: We set up the model for static spherically symmetric spacetimes, and derive the Einstein field equations for both neutral and charged matter distributions. We review the two classes of exact solutions, of the Einstein-Maxwell system, in the form of elementary functions obtained by Thirukkanesh and Maharaj (2008). We regain previous other solutions for both charged and uncharged stars obtained by various other researchers.

- Chapter 4: This chapter forms a substantial part of this study. We generate the field equations for the shear-free model with heat flow. We make use of the condition of pressure isotropy to generate a linear differential equation with variable coefficients which we solve by choosing various forms for the gravitational potentials. A number of new solutions to the pressure isotropy condition are found in terms of elementary functions. It is interesting to note that the special case of conformal flatness is contained in our models.

- Chapter 5: We construct the model for a spacetime with nonzero expansion, acceleration and shear. The Einstein field equations that are generated are highly nonlinear. The shearing solutions obtained by Maharaj *et al* (1993) are discussed in detail, and we present a new solution which does not contain the singularity at the stellar centre that is present in their results. We also demonstrate an inconsistency in the shearing solutions obtained by Marklund and Bradley (1999), and indicate the flaw in their reasoning.

- Chapter 6: Conclusion

Chapter 2

Basic theory

2.1 Introduction

Einstein's theory of general relativity is successful in describing spherically symmetric matter distributions in strong gravitational fields. A review of the physics of compact objects, black holes and relativistic stellar processes is provided by Shapiro and Teukolsky (1983). For a recent treatment of cosmological models see Gron and Hervik (2007). In this chapter, we present the background theory that enables us to generate a model of a relativistic star or a cosmological system. We present a brief outline of the relevant differential geometry, the Einstein-Maxwell system of equations for charged matter distributions and the essential physical criteria for a stellar model. For more extensive details on differential manifolds and tensor analysis, and related topics, the reader is referred to Bishop and Goldberg (1968), Misner *et al* (1973) and Wald (1984). In §2.2, the essential components of differential geometry such as the Riemann tensor, the Ricci tensor, the Ricci scalar and the Einstein tensor are introduced. These components are required to generate the Einstein field equations which are the primary area of investigation in this dissertation. We introduce the energy momentum tensor and the special case of a perfect fluid, for modelling astrophysical and cosmological situations, in §2.3. Then we present a covariant formulation of Maxwell's laws of electromagnetism. This allows us to formulate the Einstein-Maxwell system of equations in which the electromagnetic and matter fields are coupled. In §2.4, the physical conditions necessary for interior solutions for relativistic stellar systems are considered.

2.2 Spacetime geometry

In general relativity, we assume that the spacetime $\boldsymbol{M}$ is a four-dimensional differentiable manifiold endowed with a symmetric, nonsingular metric tensor field $\boldsymbol{g}$. In local regions the manifold has the structure of Euclidean space which implies that it may be covered by overlapping coordinate patches so that special relativity is regained in the relevant limit. The manifold of general relativity, with an indefinite metric tensor field, is called a pseudo-Riemannian manifold. The tensor field $\boldsymbol{g}$ represents the gravitational field and it has signature $(-+++)$. Individual points in the manifold are labelled by the real coordinates $(x^a) = (x^0, x^1, x^2, x^3)$, where $x^0 = ct$ (c is the speed of light in vacuum) is the timelike coordinate and x^1, x^2, x^3 are spacelike coordinates. In this dissertation, we use the convention that the speed of light $c = 1$. For more comprehensive treatments of spacetime geometry, the reader is referred to the standard text books in differential geometry such as Bishop and Goldberg (1968), de Felice and Clark (1990), Hawking and Ellis (1973), Misner *et al* (1973) and Wald (1984).

The invariant distance between neighbouring points in $\boldsymbol{M}$ is defined by the line element

$$ds^2 = g_{ab}dx^a dx^b \tag{2.2.1}$$

The metric connection Γ is defined in terms of the metric tensor and its derivatives by

$$\Gamma^a{}_{bc} = \frac{1}{2}g^{ad}(g_{cd,b} + g_{db,c} - g_{bc,d}) \tag{2.2.2}$$

where commas denote partial differentiation. There exists a unique symmetric connection Γ that preserves inner products under parallel transport (do Carmo 1992). The Riemannian (curvature or Riemann-Christoffel) tensor $\boldsymbol{R}$ is given by

$$R^d{}_{abc} \quad = \quad \Gamma^d{}_{ac,b} - \Gamma^d{}_{ab,c} + \Gamma^e{}_{ac}\Gamma^d{}_{eb} - \Gamma^e{}_{ab}\Gamma^d{}_{ec} \tag{2.2.3}$$

On contraction of (2.2.3) we obtain the Ricci tensor

$$\begin{aligned} R_{ab} &= R^c{}_{acb} \\ &= \Gamma^c{}_{ab,c} - \Gamma^c{}_{ac,b} + \Gamma^c{}_{dc}\Gamma^d{}_{ab} - \Gamma^c{}_{db}\Gamma^d{}_{ac} \end{aligned} \tag{2.2.4}$$

which is symmetric. On contracting the Ricci tensor (2.2.4) we obtain

$$\begin{aligned} R &= R^a{}_a \\ &= g^{ab}R_{ab} \end{aligned} \tag{2.2.5}$$

which is the Ricci (or curvature) scalar.

With these definitions it is now possible to construct the Einstein tensor $\boldsymbol{G}$, in terms of the Ricci tensor (2.2.4) and the Ricci scalar (2.2.5), as follows

$$G^{ab} = R^{ab} - \frac{1}{2}Rg^{ab} \tag{2.2.6}$$

Clearly the Einstein tensor $\boldsymbol{G}$ is symmetric. The Einstein tensor has zero divergence so that

$$G^{ab}{}_{;b} = 0 \tag{2.2.7}$$

which follows from the definition of the Einstein tensor (2.2.6). This property is sometimes called the Bianchi identity, and it is a necessary condition to generate the conservation of energy momentum via the Einstein field equations.

2.3 Fluids and electromagnetic fields

For applications in astrophysics and cosmology the matter distribution is described by a relativistic fluid. The energy momentum tensor for uncharged matter is described by the symmetric tensor $\boldsymbol{T}$ where

$$T^{ab} = (\rho + p)u^a u^b + pg^{ab} + q^a u^b + q^b u^a + \pi^{ab} \tag{2.3.1}$$

where ρ is the energy density, p is the isotropic (kinetic) pressure, q^a is the heat flux vector $(q^a u_a) = 0$ and π^{ab} is the anisotropic pressure (stress) tensor $(\pi^{ab}u_a = 0 = \pi^a{}_a)$. These quantities are measured relative to a comoving fluid four-velocity $\boldsymbol{u}$ which is unit and timelike $(u^a u_a = -1)$. In perfect fluids there are no heat conduction and stress terms $(q^a = 0, \pi^{ab} = 0)$. For a perfect fluid the energy momentum tensor, equation (2.3.1) becomes

$$T^{ab} = (\rho + p)u^a u^b + pg^{ab} \tag{2.3.2}$$

For many applications we require that the matter distribution satisfies a barotropic equation of state

$$p = p(\rho) \tag{2.3.3}$$

on physical grounds. Sometimes the particular equation of state

$$p = (\gamma - 1)\rho$$

where $0 \leq \gamma \leq 1$, is assumed in cosmology to describe matter distributions. This is called the linear γ equation of state. The case $\gamma = 1$ corresponds to dust; $\gamma = 2$ gives a stiff equation of state in which the speed of sound is equal to the speed of light ; $\gamma = 4/3$ corresponds to radiation. Often the particular equation of state

$$p = k\rho^{1+\frac{1}{n}}$$

where k and n are constants, is assumed in relativistic astrophysics. This is called a polytropic equation of state.

The Einstein field equations

$$G^{ab} = T^{ab} \tag{2.3.4}$$

governs the interaction between curvature and the matter content in the absence of charge. We have set the coupling constant to be unity in (2.3.4). From (2.2.7) and (2.3.4) we obtain

$$T^{ab}{}_{;b} = 0 \tag{2.3.5}$$

which is the conservation of matter.

We define the electromagnetic field tensor $\boldsymbol{F}$ in terms of the four-potential $\boldsymbol{A}$ by

$$F_{ab} = A_{b;a} - A_{a;b}$$

which is skew-symmetric. The electromagnetic field tensor can be written in terms of the electric field $\boldsymbol{E} = (E^1, E^2, E^3)$ and the magnetic field $\boldsymbol{B} = (B^1, B^2, B^3)$ as follows

$$F^{ab} = \begin{pmatrix} 0 & E^1 & E^2 & E^3 \\ -E^1 & 0 & B^3 & -B^2 \\ -E^2 & -B^3 & 0 & B^1 \\ -E^3 & B^2 & -B^1 & 0 \end{pmatrix} \tag{2.3.6}$$

The electromagnetic contribution $\boldsymbol{E}$ to the total energy momentum is given by the result

$$E_{ab} = F_{ac}F_b{}^c - \frac{1}{4}g_{ab}F_{cd}F^{cd} \tag{2.3.7}$$

To consider the effect of $\boldsymbol{E}$ on the gravitational field it is necessary to express the fundamental equations of electromagnetism, namely Maxwell's laws, in covariant form. The governing equations are given by

$$F_{ab;c} + F_{bc;a} + F_{ca;b} = 0 \tag{2.3.8a}$$

$$F^{ab}{}_{;b} = J^a \tag{2.3.8b}$$

where $\boldsymbol{J}$ is the four-current density defined by

$$J^a = \sigma u^a \tag{2.3.9}$$

and σ is the proper charge density. For further information on Maxwell's field equations (2.3.8) see Misner *et al* (1973) and Narlikar (2002). Note that the Maxwell equations (2.3.8) are the basic equations that govern the behaviour of the electromagnetic field in a curved background.

We point out that the total energy momentum tensor is the sum of $\boldsymbol{T}$ and $\boldsymbol{E}$. We are now in a position to introduce the Einstein-Maxwell system of equations for a charged fluid in a gravitational field. The interaction between $\boldsymbol{T}$, $\boldsymbol{E}$ and $\boldsymbol{g}$ is governed by the Einstein-Maxwell system of equations

$$G^{ab} = T^{ab} + E^{ab} \tag{2.3.10a}$$

$$F_{ab;c} + F_{bc;a} + F_{ca;b} = 0 \tag{2.3.10b}$$

$$F^{ab}{}_{;b} = J^a \tag{2.3.10c}$$

The system (2.3.10) is a highly nonlinear system of coupled, partial differential equations governing the behaviour of gravitating systems in the presence of an electromagnetic field. In (2.3.10a), we use units in which the coupling constant in the Einstein equations is unity. We need to solve the system (2.3.10) to generate an exact solution; one approach is to specify a particular form for the matter distribution and electromagnetic field on physical grounds and then integrate the partial differential equations to find the metric tensor field $\boldsymbol{g}$. For uncharged matter, the only equation that has to be satisfied is the Einstein field equation (2.3.10a) with $\boldsymbol{E} = 0$. Note that from (2.2.7) and (2.3.10a) we obtain

$$(T^{ab} + E^{ab})_{;b} = 0 \tag{2.3.11}$$

which is the total conservation of matter and charge which generalises (2.3.5).

2.4 Physical conditions

We briefly consider the physical conditions applicable to a relativistic stellar model. For physical viability, any solution applicable to the interior of the stellar body should match smoothly to the appropriate exterior spacetime. The gravitational field outside a static spherically symmetric body, in the absence of charge, is given by

$$ds^2 = -\left(1 - \frac{2m}{r}\right)dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{2.4.1}$$

which is the exterior Schwarzschild solution. Here the quantity m is the mass of the stellar body as measured by an observer at infinity. The exterior gravitational field to a static spherically symmetric body, in the presence of charge, has the form

$$ds^2 = -\left(1 - \frac{2m}{r} + \frac{q^2}{r^2}\right) dt^2 + \left(1 - \frac{2m}{r} + \frac{q^2}{r^2}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \quad (2.4.2)$$

In the above q is the constant related to the total charge of the sphere. The line element (2.4.2) is the exterior Reissner-Nordstrom solution. The radial electric field is

$$E = \frac{q}{r^2}$$

and, consequently, the proper charge density is $\sigma = 0$. Consequently, the four current density $J = 0$ which is consistent with an exterior spacetime with no barotropic matter. When $q = 0$, (2.4.2) reduces to the exterior Schwarzschild line element (2.4.1).

Physical conditions will restrict the solutions of the Einstein-Maxwell system (2.3.10) for a realistic star. It is often assumed by researchers that realistic stellar models for isotropic matter should satisfy the following conditions:

(a) The energy density ρ and the pressure p should be positive and finite throughout the interior of the star. The radial pressure should vanish at the boundary $r = b$:

$$0 < \rho < \infty, \quad 0 < p < \infty, \quad p(b) = 0$$

(b) The energy density ρ and the pressure p should be monotonic decreasing functions from the centre to the boundary:

$$\tfrac{d\rho}{dr} \leq 0, \quad \tfrac{dp}{dr} \leq 0$$

(c) Causality should be satisfied. The speed of sound should remain less than the speed of light throughout the interior of the star which leads to the condition:

$$0 \leq \frac{dp}{d\rho} \leq 1$$

(d) The metric functions $e^{2\nu}$ and $e^{2\lambda}$ and the electric field intensity E should be positive and nonsingular throughout the interior of the star.

(e) At the boundary the interior gravitational potentials should match smoothly to the exterior line elements (2.4.1) and (2.4.2) for neutral and charged matter, respectively. This generates the following conditions on the gravitational potentials:

$$e^{2\nu(b)} = e^{-2\lambda(b)} = 1 - \frac{2m}{b}, \quad (E = 0)$$

$$e^{2\nu(b)} = e^{-2\lambda(b)} = 1 - \frac{2m}{b} + \frac{q^2}{b^2}, \quad (E \neq 0)$$

(f) The electric field intensity E should be continuous across the boundary for the case of charged models:

$$E(b) = \frac{q}{b^2}$$

(g) The models should be stable with respect to radial perturbations.

It should be observed that not all relativistic stellar models satisfy the full set of the conditions listed above throughout the stellar interior; particular solutions may be valid only in some regions of spacetime. For example, the Schwarzschild interior solution becomes singular at the centre. Such solutions need to be treated as an envelope of the star and should be matched to another solution valid for the core. An example of a core-envelope model is provided by Thomas *et al* (2005). Some of the conditions (a)-(g) may be very restrictive. For example, observational evidence suggests that in some stars the energy density ρ may be not a strictly decreasing function. However, many researchers, for example Delgaty and Lake (1998), require that an exact solution satisfy these conditions. In addition, it is interesting to study the behaviour of anisotropic matter distributions with radial pressures different from tangential pressures. Such cases were studied by Chaisi and Maharaj (2005), and Dev and Gleiser (2002, 2003) in the case of neutral spheres; Herrera and Ponce de Leon (1985) analysed tangential pressures in the presence of charge. Anisotropic matter and charge distributions may be relevant in the description of quark stars as pointed out by Sharma and Maharaj (2007) and Komathiraj and Maharaj (2007), respectively. Exact solutions to the field equations which do not satisfy all of the conditions (a)-(g) are still of value because they provide useful information which assist in the qualitative analysis of relativistic stars.

Chapter 3

Spherically symmetric static models

3.1 Introduction

Static spherically symmetric spacetimes are used to model the behaviour of compact relativistic spheres. This model caters for both neutral as well as charged matter distributions; the charged case reduces to the neutral case when the electromagnetic field is absent. There exist particular classes of physically reasonable exact solutions that are known for both charged and uncharged matter. These exact solutions for the interior of charged spheres are required to match the Reissner-Nordstrom metric at the boundary, and satisfy the conditions listed in §2.4. The Reissner-Nordstrom metric describes the exterior spacetime for a spherically symmetric, charged matter distribution. Charged relativistic spheres may be used to model core-envelope stellar configurations as shown by Paul and Tikekar (2005), Thomas *et al* (2005), and Tikekar and Thomas (1998). Here the core is an isotropic fluid and the surrounding envelope is taken to be an anisotropic fluid. The role of the electromagnetic field in describing the gravitational behaviour of the quark stars (with a linear equation of state) has been recently investigated by Komathiraj and Maharaj (2007), and Mak and Harko (2004). An interesting fact about the presence of charge is that it may prevent the gravitational collapse of a spherically symmetric matter distribution to a point singularity. Here the inwardly directed gravitational attraction is counterbalanced by the repulsive Coulombic force in addition to the effect of the pressure gradient. In §3.2, we discuss the spacetime geometry for static spherically symmetric gravitational fields. We generate the relevant quantities associated with the curvature. The Einstein field equations are found for neutral fluids in §3.3. This is extended to include the electromagnetic field. Then the Einstein-Maxwell system is transformed to an equivalent form using a transformation of Durgapal and Bannerji (1983). A general class of exact solutions to the Einstein-Maxwell coupled equations is presented in §3.4. Particular solutions found previously for charged and neutral spheres are shown to be contained in this class of solutions. The results of this chapter serve as a basis for the research undertaken in chapters 4 and 5.

3.2 Spacetime geometry

In this section, we describe the spacetime geometry corresponding to static spherically symmetric manifolds. The line element can be written in the form

$$ds^2 = -e^{2\nu(r)}dt^2 + e^{2\lambda(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{3.2.1}$$

in standard coordinates $(x^a) = (t, r, \theta, \phi)$. The quantities $\nu(r)$ and $\lambda(r)$ are associated with the gravitational potentials.

The nonvanishing connection coefficients (2.2.2) are given by

$$\Gamma^0{}_{01} = \nu' \qquad \Gamma^1{}_{00} = \nu' e^{2(\nu-\lambda)}$$

$$\Gamma^1{}_{11} = \lambda' \qquad \Gamma^1{}_{22} = -re^{-2\lambda}$$

$\Gamma^1{}_{33} = -re^{-2\lambda}\sin^2\theta \qquad \Gamma^2{}_{12} = \frac{1}{r}$

$\Gamma^2{}_{33} = -\sin\theta\cos\theta \qquad \Gamma^3{}_{13} = \frac{1}{r}$

$\Gamma^3{}_{23} = \cot\theta$

for the metric (3.2.1). Primes denote differentiation with respect to the radial coordinate r. Substituting the above connection coefficients into the definition (2.2.4) we obtain the nonvanishing Ricci tensor components

$$R_{00} = \left[\nu'' + {\nu'}^2 - \nu'\lambda' + \frac{2\nu'}{r}\right] e^{2(\nu-\lambda)} \tag{3.2.2a}$$

$$R_{11} = -\left[\nu'' + {\nu'}^2 - \nu'\lambda' - \frac{2\lambda'}{r}\right] \tag{3.2.2b}$$

$$R_{22} = 1 - \left[1 + r(\nu' - \lambda')\right] e^{-2\lambda} \tag{3.2.2c}$$

$$R_{33} = \sin^2\theta R_{22} \tag{3.2.2d}$$

Using (3.2.2) and the definition for the Ricci scalar (2.2.5) we obtain the result

$$R = 2\left[\frac{1}{r^2} - \left(\nu'' + {\nu'}^2 - \nu'\lambda' + \frac{2\nu'}{r} - \frac{2\lambda'}{r} + \frac{1}{r^2}\right) e^{-2\lambda}\right] \tag{3.2.3}$$

The Ricci tensor components (3.2.2), together with the Ricci scalar (3.2.3), may be used to generate the nonvanishing Einstein tensor components (2.2.6). These are given by

$$G^{00} = \frac{1}{r^2}e^{-2\nu}\left[r\left(1-e^{-2\lambda}\right)\right]' \tag{3.2.4a}$$

$$G^{11} = e^{-2\lambda}\left[-\frac{1}{r^2}\left(1-e^{-2\lambda}\right)+\frac{2\nu'}{r}e^{-2\lambda}\right] \tag{3.2.4b}$$

$$G^{22} = \frac{1}{r^2}e^{-2\lambda}\left(\nu''+\nu'^2+\frac{\nu'}{r}-\nu'\lambda'-\frac{\lambda'}{r}\right) \tag{3.2.4c}$$

$$G^{33} = \frac{1}{\sin^2\theta}G^{22} \tag{3.2.4d}$$

for the line element (3.2.1).

3.3 Field equations

As the fluid four-velocity is comoving we have $u^a = e^{-\nu}\delta^a_0$ for the metric (3.2.1). Then the perfect fluid energy momentum tensor (2.3.2) has the nonvanishing components

$$T^{00} = e^{-2\nu}\rho \tag{3.3.1a}$$

$$T^{11} = e^{-2\lambda}p \tag{3.3.1b}$$

$$T^{22} = \frac{1}{r^2}p \tag{3.3.1c}$$

$$T^{33} = \frac{1}{r^2\sin^2\theta}p \tag{3.3.1d}$$

On equating the components of the Einstein tensor (3.2.4) to the components of the energy momentum tensor (3.3.1), we obtain the Einstein field equations (2.3.4) in the form

$$\rho = \frac{1}{r^2}\left[r\left(1-e^{-2\lambda}\right)\right]' \tag{3.3.2a}$$

$$p = -\frac{1}{r^2}\left(1-e^{-2\lambda}\right)+\frac{2\nu'}{r}e^{-2\lambda} \tag{3.3.2b}$$

$$p = e^{-2\lambda}\left(\nu''+\nu'^2+\frac{\nu'}{r}-\nu'\lambda'-\frac{\lambda'}{r}\right) \tag{3.3.2c}$$

From the conservation law (2.3.5) we have

$$\frac{dp}{dr} = -\left(\rho+p\right)\frac{d\nu}{dr} \tag{3.3.3}$$

Note that (3.3.3) can also be obtained directly from the field equations (3.3.2); it may be used to replace one of the field equations in the integration process. The system of Einstein field equations (3.3.2) determines the evolution of the static spherically symmetric star which we have modelled as a perfect fluid.

The Einstein equations given above may be generalized to include nonzero electric charge. For the perfect fluid energy momentum tensor (2.3.2), together with the electromagnetic contribution (2.3.7), the Einstein-Maxwell system (2.3.10) can be written as

$$\frac{1}{r^2}(1-e^{-2\lambda})+\frac{2\lambda'}{r}e^{-2\lambda} = \rho+\frac{1}{2}E^2 \qquad (3.3.4a)$$

$$-\frac{1}{r^2}(1-e^{-2\lambda})+\frac{2\nu'}{r}e^{-2\lambda} = p-\frac{1}{2}E^2 \qquad (3.3.4b)$$

$$e^{-2\lambda}\left(\nu''+\nu'^2+\frac{\nu'}{r}-\nu'\lambda'-\frac{\lambda'}{r}\right) = p+\frac{1}{2}E^2 \qquad (3.3.4c)$$

$$\sigma = \frac{1}{r^2}e^{-\lambda}(r^2E)' \qquad (3.3.4d)$$

It is possible to transform (3.3.4) to a simpler form. Durgapal and Bannerji (1983) introduced the following transformation

$$x = Cr^2, \quad Z(x) = e^{-2\lambda(r)}, \quad A^2y^2(x) = e^{2\nu(r)} \qquad (3.3.5)$$

The metric (3.2.1) now has the equivalent form

$$ds^2 = -A^2y^2dt^2+\frac{1}{4CxZ}dx^2+\frac{x}{C}(d\theta^2+\sin^2\theta d\phi^2)$$

The corresponding Einstein-Maxwell system may be written as

$$\frac{1-Z}{x} - 2\frac{dZ}{dx} = \frac{\rho}{C} + \frac{E^2}{2C} \tag{3.3.6a}$$

$$4Z\frac{1}{y}\frac{dy}{dx} + \frac{Z-1}{x} = \frac{p}{C} - \frac{E^2}{2C} \tag{3.3.6b}$$

$$4Zx^2\frac{d^2y}{dx^2} + 2\frac{dZ}{dx}x^2\frac{dy}{dx} + \left(\frac{dZ}{dx}x - Z + 1 - \frac{E^2x}{C}\right)y = 0 \tag{3.3.6c}$$

$$\frac{\sigma^2}{C} = \frac{4Z}{x}\left(x\frac{dE}{dx} + E\right)^2 \tag{3.3.6d}$$

which follows from (3.3.4) and (3.3.5). Equation (3.3.6c) is the condition of pressure isotropy generalised to include the electromagnetic field. It is the master equation that must be integrated to provide a solution to the system (3.3.6).

3.4 Exact solutions

There exist many exact solutions to the systems (3.3.2) and (3.3.4) for neutral and charged matter, respectively. However, only a few of the known solutions are physically reasonable. For comprehensive reviews of the known solutions and their physical properties the reader is referred to Delgaty and Lake (1998), Finch and Skea (1989) and Ivanov (2002). In recent treatments there have been attempts to find general classes of exact solutions which unify particular cases found previously. Examples of these treatments are provided by Komathiraj and Maharaj (2007), Maharaj and Komathiraj (2007), Maharaj and Thirukkanesh (2006) and Thirukkanesh and Maharaj (2006, 2008). These treatments provide new generalised classes of Einstein-Maxwell solutions in closed form which are physically acceptable. For example, the speed of sound is less than the speed of light in these models.

A general class of models is the charged perfect fluid solution found by Thirukkanesh and Maharaj (2008) to the Einstein-Maxwell system (3.3.6). The gravitational potential Z was chosen to be of the particular form

$$Z = \frac{1+ax}{1+bx}$$

where a and b are constants. The electric field was chosen to be

$$E^2 = \frac{\alpha C bx}{(1+bx)^2}$$

where α is a constant. These choices result in the following differential equation

$$4(1+ax)(1+bx)\frac{d^2y}{dx^2} + 2(a-b)\frac{dy}{dx} + [b(b-a)-ab]\,y = 0 \tag{3.4.1}$$

which is the condition of pressure isotropy (3.3.6c).

Equation (3.4.1) can be integrated and it was found that the general solution to the above system comprises two classes of elementary functions. The first class of solutions may be written as follows

$$y =$$

$$d_1(1+ax)^{\frac{1}{2}}\left[1-(n+1)\sum_{i=1}^{n+1}\left(\frac{4a}{b-a}\right)^i \frac{(2i-1)(n+i)!}{(2i)!(n-i+1)!}(1+bx)^i\right]$$

$$+d_2(1+bx)^{\frac{3}{2}}\left[1+\frac{3}{(n+1)}\sum_{i=1}^{n}\left(\frac{4a}{b-a}\right)^i \frac{(2i+2)(n+i+1)!}{(n-i)!(2i+3)!}(1+bx)^i\right]$$

where $a - b + \alpha = a(2n+3)(2n+1)$. The second class of solutions has the form

$$y =$$

$$d_1(1+ax)^{\frac{1}{2}}(1+bx)^{\frac{3}{2}}\left[1+\frac{3}{n(n-1)}\sum_{i=1}^{n-2}\left(\frac{4a}{b-a}\right)^i\frac{(2i+2)(n+i)!}{(2i+3)!(n-i-2)!}(1+bx)^i\right]$$

$$+d_2\left[1-n(n-1)\sum_{i=1}^{n}\left(\frac{4a}{b-a}\right)^i\frac{(2i-1)(n+i-2)!}{(2i)!(n-i)!}(1+bx)^i\right]$$

where $a - b + \alpha = 4an(n-1)$ and d_1 and d_2 are constants. The restriction on n arises because it is this value that ensures elementary functions (rather than infinite series) are admissible as solutions.

The classes of solutions found by Thirukkanesh and Maharaj (2008) are physically reasonable: the matter variables and metric functions are continuous and regular in the stellar interior; the interior line elements match to the Schwarzschild or Reissner-Nordstrom exterior line elements; the speed of sound is less than the speed of light; densities and pressures are consistent with observational results. Known solutions which are physically acceptable are contained in their general class. The Hansraj and Maharaj (2006) and Komathiraj and Maharaj (2007) solutions, which model charged relativistic spheres, are regained as special cases. The Durgapal and Bannerji (1983), Finch and Skea (1989) and Tikekar (1990) solutions, which model neutral relativistic spheres, are also special cases.

For completeness we present the explicit solutions mentioned above.

3.4.1 Case 1: Hansraj and Maharaj charged stars

If we set $a = 0$, $b = 1$ and $0 \leq \alpha \leq 1$ we get

$$\begin{aligned} y &= \left[d_1 - d_2\sqrt{(1-\alpha)(1+x)}\right]\cos\sqrt{(1-\alpha)(1+x)} \\ &\quad + \left[d_2 + d_1\sqrt{(1-\alpha)(1+x)}\right]\sin\sqrt{(1-\alpha)(1+x)} \end{aligned} \tag{3.4.2}$$

from the above class of solutions. The class of charged solutions (3.4.2) is the first category of models found by Hansraj and Maharaj (2006). If $\alpha = 1$ then

$$y = d_1 + d_2(1+x)^{\frac{3}{2}} \tag{3.4.3}$$

This is the second category of the Hansraj-Maharaj (2006) charged solutions. When $\alpha > 1$ then we obtain

$$\begin{aligned} y &= \left[d_2 + d_1\sqrt{(\alpha-1)(1+x)}\right]\sinh\sqrt{(\alpha-1)(1+x)} \\ &\quad + \left[d_1 - d_2\sqrt{(\alpha-1)(1+x)}\right]\cosh\sqrt{(\alpha-1)(1+x)} \end{aligned} \tag{3.4.4}$$

This is the third category of charged solutions found by Hansraj and Maharaj (2006). The exact solutions (3.4.2)-(3.4.4) model a charged relativistic sphere and satisfy the conditions for physical acceptability listed in §2.4.

3.4.2 Case 2: Maharaj and Komathiraj charged stars

For the case $a-1+\alpha = a(2n+1)(2n+3)$ we obtain

$$y =$$

$$d_1(1+ax)^{\frac{1}{2}}\left[1-(n+1)\sum_{i=1}^{n+1}\left(\frac{4a}{1-a}\right)^i\frac{(2i-1)(n+i)!}{(2i)!(n-i+1)!}(1+x)^i\right]$$

$$+d_2(1+x)^{\frac{3}{2}}\left[1+\frac{3}{(n+1)}\sum_{i=1}^{n}\left(\frac{4a}{1-a}\right)^i\frac{(2i+2)(n+i+1)!}{(n-i)!(2i+3)!}(1+x)^i\right] \quad (3.4.5)$$

In the case $a-1+\alpha = 4an(n-1)$ we find

$$y =$$

$$d_1(1+ax)^{\frac{1}{2}}(1+x)^{\frac{3}{2}}\left[1+\frac{3}{n(n-1)}\sum_{i=1}^{n-2}\left(\frac{4a}{1-a}\right)^i\frac{(2i+2)(n+i)!}{(2i+3)!(n-i-2)!}(1+x)^i\right]$$

$$+d_2\left[1-n(n-1)\sum_{i=1}^{n}\left(\frac{4a}{1-a}\right)^i\frac{(2i-1)(n+i-2)!}{(2i)!(n-i)!}(1+x)^i\right] \qquad (3.4.6)$$

The two categories of solutions (3.4.5) and (3.4.6) given above correspond to the Maharaj and Komathiraj (2007) model for a compact sphere in electric fields. The Maharaj and Komathiraj (2007) model for charged stars has a simple form written in terms of elementary functions. They are physically reasonable and contain the Durgapal and Bannerji (1983) model and other exact models corresponding to neutron stars as special cases.

3.4.3 Case 3: Finch and Skea neutron stars

When $\alpha = 0$, $a = 0$ and $b = 1$ we obtain

$$y = \left[d_1 - d_2\sqrt{1+x}\right]\cos\sqrt{1+x} + \left[d_2 + d_1\sqrt{1+x}\right]\sin\sqrt{1+x} \tag{3.4.7}$$

from the general solution. (Equivalently, we can set $\alpha = 0$ in (3.4.2).) Thus, we regain the Finch and Skea (1989) model for a neutron star when the electromagnetic field is absent. The Finch and Skea (1989) neutron star model has been proven to satisfy all the physical criteria for an isolated spherically symmetric stellar neutral matter distribution. This model has therefore been used in many investigations to study the interior of neutron stars in the context of general relativity.

3.4.4 Case 4: Durgapal and Bannerji neutron stars

If we take $\alpha = 0$ and $n = 0$ then we get

$$y = d_1(2-x)^{\frac{1}{2}}(5+2x) + d_2(1+x)^{\frac{3}{2}} \tag{3.4.8}$$

Here we have regained the neutron star model of Durgapal and Bannerji (1983). This model satisfies all criteria for being physically acceptable and has been used by many researchers to study neutral neutron stars.

3.4.5 Case 5: Tikekar superdense stars

If we take $\alpha = 0$ and $n = 2$ then we find

$$y = d_1 x(1 - \frac{7}{8}x^2)^{\frac{3}{2}} + d_2 \left[1 - \frac{7}{2}x^2 + \frac{49}{24}x^4\right] \tag{3.4.9}$$

Now we have regained the Tikekar (1990) model for superdense neutron stars. This model plays an important role in describing highly dense matter distributions, cold compact matter and core-envelope models for relativistic stars. It is interesting to note that the Tikekar (1990), superdense stars may be extended to include electric fields as demonstrated by Komathiraj and Maharaj (2007a).

Chapter 4

Shear-free models with heat flux

4.1 Introduction

In addition to radiating cosmological models, shear-free spacetimes are widely used to model relativistic stars which dissipate null radiation in the form of a radial heat flow. The heat flows from the hotter central regions to the stellar boundary. Various models involving gravitational collapse with radiative processes have been studied in the past. Deng and Mannheim (1990, 1991), Glass (1990), Santos *et al* (1985) and Stephani *et al* (2003) have discussed the physical features of shear-free solutions with heat flux. A necessary requirement for these models is that the interior spacetime must be matched at the boundary, where the radial pressure is nonzero, to the exterior Vaidya radiating spacetime. Studies of relativistic radiating stars are also useful in the investigation of the cosmic censorship hypothesis and radiative collapse with vanishing tidal forces (Herrera *et al* (2004), Maharaj and Govender (2005), Misthry *et al* (2008)). Wagh *et al* (2000) presented solutions to the Einstein field equations for a shear-free spherically symmetric spacetime, with radial heat flux, by choosing a barotropic equation of state. Herrera *et al* (2006) found analytical solutions to the field equations, for radiating collapsing spheres in the diffusion approximation. They demonstrated that the thermal evolution of the collapsing sphere can be modelled in causal thermodynamics. In this chapter we construct the model for shear-free spacetimes exhibiting heat flow. In §4.2, we discuss the spacetime geometry of shear-free spacetimes, and generate the Einstein field equations with heat flow in §4.3. From the field equations we deduce the condition of pressure isotropy which is written as a second order differential equation with variable coefficients. Some known solutions corresponding to the case of conformal flatness are reviewed in §4.4. We generate a number of new solutions in §4.5 by choosing a variety of particular forms for the gravitational potentials.

4.2 Spacetime geometry

Shear-free fluids are important in modelling inhomogeneous cosmological processes and radiating stellar models. Spherically symmetric spacetimes which are shear-free can be written as

$$ds^2 = -A^2dt^2 + B^2[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)] \tag{4.2.1}$$

in comoving coordinates $(x^a) = (t, r, \theta, \phi)$. The metric functions A and B depend on both the timelike coordinate t and the radial coordinate r.

The nonvanishing connection coefficients (2.2.2) are given by

$\Gamma^0{}_{00} = \frac{\dot{A}}{A}$ $\qquad$ $\Gamma^0{}_{01} = \frac{A'}{A}$

$\Gamma^0{}_{11} = \frac{B\dot{B}}{A^2}$ $\qquad$ $\Gamma^0{}_{22} = r^2\frac{B\dot{B}}{A^2}$

$\Gamma^0{}_{33} = r^2\sin^2\theta\frac{B\dot{B}}{A^2}$ $\qquad$ $\Gamma^1{}_{00} = \frac{AA'}{B^2}$

$$\Gamma^1{}_{11} = \frac{B'}{B} \qquad \Gamma^1{}_{22} = -r^2\left(\frac{B'}{B} + \frac{1}{r}\right)$$

$$\Gamma^1{}_{33} = -r^2 \sin^2\theta \left(\frac{B'}{B} + \frac{1}{r}\right) \qquad \Gamma^1{}_{01} = \frac{\dot{B}}{B}$$

$$\Gamma^2{}_{02} = \frac{\dot{B}}{B} \qquad \Gamma^3{}_{03} = \frac{\dot{B}}{B}$$

$$\Gamma^2{}_{12} = \frac{B'}{B} + \frac{1}{r} \qquad \Gamma^3{}_{13} = \frac{B'}{B} + \frac{1}{r}$$

$$\Gamma^2{}_{33} = -\sin\theta\cos\theta \qquad \Gamma^3{}_{23} = \cot\theta$$

for the metric (4.2.1). In the above, dots and primes denote differentiation with respect to t and r, respectively. Using the above connection coefficients and the definition for the Ricci tensor (2.2.4) we can write the nonvanishing Ricci tensor components as

$$R_{00} = \frac{AA''}{B^2} + AA'\frac{B'}{B^3} - 3\frac{\ddot{B}}{B} + 3\frac{\dot{A}}{A}\frac{\dot{B}}{B} + \frac{2}{r}\frac{AA'}{B^2} \tag{4.2.2a}$$

$$R_{01} = 2\left(\frac{\dot{B}B'}{B^2} - \frac{\dot{B}'}{B} + \frac{A'\dot{B}}{AB}\right) \tag{4.2.2b}$$

$$\begin{aligned} R_{11} = {} & 2\frac{\dot{B}^2}{A^2} + \frac{A'}{A}\frac{B'}{B} - \frac{2}{r}\frac{B'}{B} - B\dot{B}\frac{\dot{A}}{A^3} - \frac{A''}{A} \\ & + \frac{B\ddot{B}}{A^2} + 2\frac{B'^2}{B^2} - 2\frac{B''}{B} \end{aligned} \tag{4.2.2c}$$

$$\begin{aligned} R_{22} = {} & r^2\frac{B\ddot{B}}{A^2} - r^2 B\dot{B}\frac{\dot{A}}{A^3} + 2r^2\frac{\dot{B}^2}{A^2} - r^2\frac{A'}{A}\frac{B'}{B} - r\frac{A'}{A} \\ & - 3r\frac{B'}{B} - r^2\frac{B''}{B} \end{aligned} \tag{4.2.2d}$$

$$R_{33} = \sin^2\theta R_{22} \tag{4.2.2e}$$

Using the Ricci tensor components (4.2.2), and the definition (2.2.5), we obtain the Ricci scalar

$$\begin{aligned} R = {} & -2\frac{1}{B^2}\frac{A''}{A} - \frac{4}{r}\frac{1}{B^2}\frac{A'}{A} + \frac{6}{A^2}\frac{\dot{B}^2}{B^2} - \frac{8}{r}\frac{B'}{B^3} + 2\frac{B'^2}{B^4} \\ & - 2\frac{A'}{A}\frac{B'}{B^3} - 4\frac{B''}{B^3} - 6\frac{\dot{A}}{A^3}\frac{\dot{B}}{B} + 6\frac{\ddot{B}}{BA^2} \end{aligned} \tag{4.2.3}$$

for the metric (4.2.1). Now using the Ricci tensor components (4.2.2), and the Ricci scalar (4.2.3), we obtain the nonvanishing Einstein tensor components in the form

$$G_{00} = 3\frac{\dot{B}^2}{B^2} - \frac{A^2}{B^2}\left(2\frac{B''}{B} - \frac{B'^2}{B^2} + \frac{4}{r}\frac{B'}{B}\right) \tag{4.2.4a}$$

$$G_{01} = -\frac{2}{B^2}\left(B\dot{B}' - B'\dot{B} - B\dot{B}\frac{A'}{A}\right) \tag{4.2.4b}$$

$$\begin{aligned} G_{11} = {} & \frac{1}{A^2}\left(-2B\ddot{B} - \dot{B}^2 + 2B\dot{B}\frac{\dot{A}}{A}\right) + \\ & \frac{1}{B^2}\left(B'^2 + 2BB'\frac{A'}{A} + B^2\frac{2}{r}\frac{A'}{A} + BB'\frac{2}{r}\right) \end{aligned} \tag{4.2.4c}$$

$$\begin{aligned} G_{22} = {} & -2r^2\frac{B\ddot{B}}{A^2} + 2r^2B\dot{B}\frac{\dot{A}}{A^3} - r^2\frac{\dot{B}^2}{A^2} + \\ & r\frac{A'}{A} + r\frac{B'}{B} + r^2\frac{A''}{A} - r^2\frac{B'^2}{B^2} + r^2\frac{B''}{B} \end{aligned} \tag{4.2.4d}$$

$$G_{33} = \sin^2\theta G_{22} \tag{4.2.4e}$$

for the spacetime (4.2.1).

4.3 Field equations

For this particular model the nonvanishing components of the energy momentum tensor are written as

$$T_{00} = \rho A^2 \tag{4.3.1a}$$

$$T_{01} = -AB^2 q \tag{4.3.1b}$$

$$T_{11} = pB^2 \tag{4.3.1c}$$

$$T_{22} = pB^2 r^2 \tag{4.3.1d}$$

$$T_{33} = pB^2 r^2 \sin^2\theta \tag{4.3.1e}$$

Using (4.2.4) and the energy momentum tensor components (4.3.1) we obtain the Einstein field equations for this model

$$\rho = \frac{3\dot{B}^2}{A^2B^2} - \frac{1}{B^2}\left(\frac{2B''}{B} - \frac{B'^2}{B^2} + \frac{4B'}{rB}\right) \tag{4.3.2a}$$

$$\begin{aligned} p = {} & \frac{1}{A^2}\left(\frac{-2\ddot{B}}{B} - \frac{\dot{B}^2}{B^2} + \frac{2\dot{A}\dot{B}}{AB}\right) \\ & + \frac{1}{B^2}\left(\frac{B'^2}{B^2} + \frac{2A'B'}{AB} + \frac{2A'}{rA} + \frac{2B'}{rB}\right) \end{aligned} \tag{4.3.2b}$$

$$\begin{aligned} p = {} & \frac{-2\ddot{B}}{BA^2} + \frac{2\dot{A}\dot{B}}{BA^3} - \frac{\dot{B}^2}{A^2B^2} + \frac{A'}{rAB^2} \\ & + \frac{B'}{rB^3} + \frac{A''}{AB^2} - \frac{B'^2}{B^4} + \frac{B''}{B^3} \end{aligned} \tag{4.3.2c}$$

$$q = -\frac{2}{AB^2}\left(-\frac{\dot{B}'}{B} + \frac{B'\dot{B}}{B^2} + \frac{A'}{A}\frac{\dot{B}}{B}\right) \tag{4.3.2d}$$

The field equations above are a system of coupled partial differential equations and they model the evolution of the interior of a spherically symmetric radiating star or a radiating cosmological model.

Equations (4.3.2b) and (4.3.2c) yield the consistency condition

$$\frac{A''}{A}\frac{1}{B^2} + \frac{B''}{B^3} - 2\frac{A'}{A}\frac{B'}{B^3} - 2\frac{B'^2}{B^4} - \frac{1}{B^2}\frac{1}{r}\left(\frac{A'}{A} + \frac{B'}{B}\right) \quad = \quad 0 \tag{4.3.3}$$

which is the condition of pressure isotropy. This equation governs the gravitational behaviour of the radiating model and must be solved to yield an exact solution to the system (4.3.2). In the present form it is difficult to solve, and we need to rewrite it in simpler form to make progress. We observe that (4.3.3) can be rewritten as

$$\frac{A_{rr}}{A} + \frac{B_{rr}}{B} = \left(2\frac{B_r}{B} + \frac{1}{r}\right)\left(\frac{A_r}{A} + \frac{B_r}{B}\right) \tag{4.3.4}$$

Then we introduce the new variable

$$x = r^2$$

so that the condition of pressure isotropy (4.3.4) can be written as

$$\left(\frac{A}{B}\right)_{xx} = 2A\left(\frac{1}{B}\right)_{xx} \tag{4.3.5}$$

where subscripts represent differentiation with respect to the new variable x. We present exact solutions to the governing equation (4.3.5) in the remainder of this chapter.

4.4 Known solutions

A number of solutions to (4.3.5), in closed form are known. Particular cases were considered by Bergmann (1981), Maiti (1982) and Modak (1984). These solutions are conformally flat which correspond to a vanishing Weyl tensor. The most general conformally flat solution with heat flux was found by Banerjee *et al* (1989). We can express the conformally flat solution in the form

$$\frac{A}{B} = 1 + C_1(t)r^2 \tag{4.4.1a}$$

$$B = \frac{1}{C_2(t)r^2 + C_3(t)} \tag{4.4.1b}$$

where $C_1(t), C_2(t)$ and $C_3(t)$ are functions of integration. The form of solution (4.4.1), was used by Di Prisco *et al* (2007), Herrera *et al* (2004), Maharaj and Govender (2005) and Misthry *et al* (2008) to study radiating relativistic spheres, and to generate temperature profiles in the causal theory of thermodynamics. Triginer and Pavon (1995) studied dissipative processes in inhomogeneous spacetimes for a particular case of (4.4.1).

Other particular solutions of (4.3.5) are known and these are listed by Krasinski (1997). It is interesting to note that a method of generating solutions to (4.3.5), was found by Deng and Mannheim (1990, 1991) which generates an infinite sequence of solutions.

4.5 New solutions

As pointed out in §4.4, particular solutions to (4.3.5), have been found previously. However these solutions are conformally flat or have a complicated form. We require solutions in simple form, preferably expressible in terms of elementary functions or special functions, to study the physical features of the model. In this section we demonstrate that it is possible to generate simple exact solutions. We first write (4.3.5), in the modified form

$$\left(\frac{1}{B}\right) A_{xx} + 2A_x \left(\frac{1}{B}\right)_x - A\left(\frac{1}{B}\right)_{xx} \quad = \quad 0 \tag{4.5.1}$$

Observe that (4.5.1), is linear in the function A if $\frac{1}{B}$ is specified; it is linear in terms of the function $\frac{1}{B}$ if A is a given quantity. We utilise this feature of (4.5.1), to generate several classes of new solutions. Observe that (4.5.1), is a partial differential equation. However, we can treat it as an ordinary differential equation in the integration process, because the variable t does not appear explicitly.

4.5.1 Solution I: $B^{-1} = (a + bx)^k$

It is possible to generate an Cauchy-Euler equation for a suitable choice of $\frac{1}{B}$. We assume the following functional form

$$\frac{1}{B} = (a + bx)^k \tag{4.5.2}$$

where a and b are functions of time and k is a real parameter. Then the condition (4.5.1) becomes

$$(a + bx)^2 A_{xx} + 2bk(a + bx)A_x - b^2k(k - 1)A = 0 \tag{4.5.3}$$

We can simplify (4.5.3), by introducing the new dependent variable

$$z = a + bx$$

Then (4.5.3) reduces to the Cauchy-Euler differential equation

$$z^2 b^2 \tilde{A}_{zz} + 2b^2 k z \tilde{A}_z - b^2 k(k-1)\tilde{A} = 0 \tag{4.5.4}$$

where $\tilde{A} = \tilde{A}(z, t)$. The characteristic equation corresponding to (4.5.4), is

$$m^2 + (2k-1)m - (k^2 - k) \quad = \quad 0$$

The roots of the characteristic equation are

$$\begin{aligned} m_1 &= \frac{(1-2k) + \sqrt{8k^2 - 8k + 1}}{2} \\ m_2 &= \frac{(1-2k) - \sqrt{8k^2 - 8k + 1}}{2} \end{aligned}$$

Three cases arise depending on the value of $8k^2 - 8k + 1$ which could be positive, negative or zero.

(i) Repeated roots

If $k = \frac{1}{2}\left(1 + \frac{1}{\sqrt{2}}\right)$ or $k = \frac{1}{2}\left(1 - \frac{1}{\sqrt{2}}\right)$ then the roots are repeated and $m_1 = m_2 = \frac{1}{2} - k$.

Then the solution of (4.5.4), is given by

$$\tilde{A} = [c + d \ln z]\, z^{(1-2k)/2}$$

In terms of the variable x we have

$$A(x,t) = [c + d \ln(a + bx)]\, (a + bx)^{(1-2k)/2} \tag{4.5.5}$$

where $c(t)$ and $d(t)$ are functions of integration.

(ii) Real distinct roots

If $\frac{1}{2}\left(1 - \frac{1}{\sqrt{2}}\right) < k < \frac{1}{2}\left(1 + \frac{1}{\sqrt{2}}\right)$ then the roots m_1 and m_2 are real and distinct and the solution of (4.5.4) is

$$\tilde{A} = cz^{\left[-(2k-1)+\sqrt{8k^2-8k+1}\right]/2} + dz^{\left[-(2k-1)-\sqrt{8k^2-8k+1}\right]/2} \tag{4.5.6}$$

where $c(t)$ and $d(t)$ result from integration. The closed form solution in terms of x is given by

$$\begin{aligned} A(x,t) &= c(a + bx)^{\left[(1-2k)+\sqrt{8k^2-8k+1}\right]/2} \\ &\quad + d(a + bx)^{\left[(1-2k)-\sqrt{8k^2-8k+1}\right]/2} \end{aligned} \tag{4.5.7}$$

for this case.

(iii) Complex roots

If $\frac{1}{2}\left(1+\frac{1}{\sqrt{2}}\right) < k < \frac{1}{2}\left(1-\frac{1}{\sqrt{2}}\right)$ then the roots m_1 and m_2 are complex and the solution of (4.5.4) is

$$\begin{aligned}\tilde{A} &= e^{(1-2k)(z-a)/2b} \\ &\times\left[c\cos\sqrt{8k^2-8k+1}\left(\frac{z-a}{b}\right)+d\sin\sqrt{8k^2-8k+1}\left(\frac{z-a}{b}\right)\right]\end{aligned} \tag{4.5.8}$$

where c and d are functions of integration. Then the closed form solution in terms of the original variable x is given by

$$A(x,t) = e^{(1-2k)x/2}\left[c\cos\sqrt{8k^2-8k+1}x + d\sin\sqrt{8k^2-8k+1}x\right] \tag{4.5.9}$$

for complex roots.

Hence we have generated a new class of solutions in terms of elementary functions, to the condition of pressure isotropy (4.5.1). We can present the solution in the compact form

$$A = \begin{cases} \left[c+d\ln(a+bx)\right](a+bx)^{(1-2k)/2}, & k=\frac{1}{2}\left(1+\frac{1}{\sqrt{2}}\right) \text{ or } \frac{1}{2}\left(1-\frac{1}{\sqrt{2}}\right) \\ \\ c(a+bx)^{\left[(1-2k)+\sqrt{8k^2-8k+1}\right]/2} & \\ +d(a+bx)^{\left[(1-2k)-\sqrt{8k^2-8k+1}\right]/2}, & \frac{1}{2}\left(1-\frac{1}{\sqrt{2}}\right) < k < \frac{1}{2}\left(1+\frac{1}{\sqrt{2}}\right) \\ \\ e^{(1-2k)x/2}\left[c\cos\sqrt{8k^2-8k+1}x\right. & \\ \left.+d\sin\sqrt{8k^2-8k+1}x\right], & \frac{1}{2}\left(1+\frac{1}{\sqrt{2}}\right) < k < \frac{1}{2}\left(1-\frac{1}{\sqrt{2}}\right) \end{cases} \tag{4.5.10}$$

$$\frac{1}{B} = (a + bx)^k$$

for the gravitational potentials A and B.

In the special case when $k = 1$, the roots are real and distinct. Then the above solution yields the particular case

$$\begin{aligned} A &= c + \frac{d}{a + bx} \\ &= \frac{(ca + d) + cbx}{a + bx} \qquad (4.5.11a) \\ \frac{1}{B} &= a + bx \qquad (4.5.11b) \end{aligned}$$

If we make the identification

$$\begin{aligned} ca + d &= 1 \\ cb &= C_1 \\ b &= C_2 \\ a &= C_3 \end{aligned}$$

then we observe that (4.5.11), is equivalent to the conformally flat solution (4.4.1). It is clear that the solutions found in this section reduce to the conformally flat case in the relevant limit. Hence we have found a new class of exact radiating models for shear-free fluids in terms of elementary functions which generalise the conformally flat case. These solutions will help in the construction of models where tidal effects are important, eg. in galaxy formation.

4.5.2 Solution II: $A = (a + bx)^k$

In an attempt to find other classes of solutions we observe that it is possible to choose power law forms for the potential A. This will generate solutions that differ from those found in §4.5.1. We now let

$$A = (a + bx)^k \tag{4.5.12}$$

so that (4.5.1) reduces to

$$(a + bx)^2 \left(\frac{1}{B}\right)_{xx} - 2kb(a + bx)\left(\frac{1}{B}\right)_x - b^2k(k-1)\left(\frac{1}{B}\right) = 0 \tag{4.5.13}$$

In a similar manner to the treatment above we obtain the second order Cauchy-Euler differential equation

$$z^2\tilde{v}'' - 2kz\tilde{v}' - k(k-1)\tilde{v} = 0 \tag{4.5.14}$$

where $v = 1/B$ and $\tilde{v} = \tilde{v}(z, t)$. The characteristic equation for this differential equation may be written as

$$m^2 - m(2k+1) - k(k-1) = 0 \tag{4.5.15}$$

For this case the roots of the characteristic equation work out to be

$$
\begin{aligned}
m_1 &= \frac{2k+1+\sqrt{8k^2+1}}{2} \\
m_2 &= \frac{2k+1-\sqrt{8k^2+1}}{2}
\end{aligned}
$$

It is immediately clear that $8k^2+1>0$. Consequently the roots of the characteristic equation are always real and distinct. Therefore the above choice for the potential A admits only one class of solutions corresponding to $8k^2+1>0$. The general closed form solution to (4.5.14), may be written as

$$
\tilde{v} = cz^{\left[2k+1+\sqrt{8k^2+1}\right]/2} + dz^{\left[2k+1-\sqrt{8k^2+1}\right]/2} \tag{4.5.17}
$$

Hence the general closed form solution to (4.5.13) is given by

$$
\begin{aligned}
\left(\frac{1}{B}\right)(x,t) = c(a+bx)^{\left[2k+1+\sqrt{8k^2+1}\right]/2} \\
+d(a+bx)^{\left[2k+1-\sqrt{8k^2+1}\right]/2}
\end{aligned} \tag{4.5.18}
$$

for the metric function $\frac{1}{B}$.

The exact models for this category of solution is given by (4.5.12) and (4.5.18). Note that the form of the potential A does not allow us to regain the conformally flat radiating limit. For this class of radiating models tidal forces are always present.

4.5.3 Solution III: $B^{-1} = e^{a+bx}$

Also, we observe that if an exponential form for the potential $\frac{1}{B}$ is chosen then we can find a new solution. In (4.5.1) we set

$$\frac{1}{B} = e^{a+bx} \tag{4.5.19}$$

so that we get

$$A'' + 2bA' - b^2 A = 0 \tag{4.5.20}$$

which is a second order ordinary differential equation with constant coefficients. The characteristic equation of (4.5.20) is

$$m^2 + 2bm - b^2 = 0$$

The roots are

$$m_1 = b(-1 + \sqrt{2})$$

$$m_2 = b(-1 - \sqrt{2})$$

which are real and distinct. This yields the general solution

$$A(x,t) = ce^{b(-1+\sqrt{2})x} + de^{b(-1-\sqrt{2})x} \tag{4.5.22}$$

Consequently we have generated another new exact solution given by (4.5.19) and (4.5.22) for a shear-free fluid with heat flux. This form of the solution is particularly suited to the asymptotic behaviour of the model because of the exponential dependence in the potentials.

4.5.4 Solution IV: $A = e^{a+bx}$

Conversely we can now choose the exponential form for the potential A and set

$$A = e^{a+bx} \tag{4.5.23}$$

In this case (4.5.1) reduces to

$$V'' - 2bV' - b^2 V = 0 \tag{4.5.24}$$

where $V = \frac{1}{B}$. This is a second order ordinary differential equation with constant coefficients. The corresponding characteristic equation is

$$m^2 - 2bm - b^2 = 0$$

for which the roots are

$$m_1 = b(1+\sqrt{2})$$

$$m_2 = b(1-\sqrt{2})$$

which are real and distinct. This admits the general solution

$$\left(\frac{1}{B}\right)(x,t) = ce^{b(1+\sqrt{2})x} + de^{b(1-\sqrt{2})x} \tag{4.5.26}$$

where once again $c(t)$ and $d(t)$ are functions of integration. In this section we have generated another exact solution which is given by (4.5.23) and (4.5.26).

4.5.5 Solution V: $B^{-1} = A^{\alpha}$

It is possible that we may express one of the potentials as a power of the second potential when choosing a particular form to generate an exact solution. With this in mind we made the particular choice

$$\frac{1}{B} = A^{\alpha} \tag{4.5.27}$$

for the potential $\frac{1}{B}$. Then (4.5.1) reduces to

$$(1-\alpha)AA_{xx} + (3\alpha - \alpha^2){A_x}^2 \;=\; 0 \tag{4.5.28}$$

in the potential A.

Two cases arise corresponding to $\alpha = 1$ and $\alpha \neq 1$. With $\alpha = 1$ we get from (4.5.28) that

$$A_x \;=\; 0$$

so that $A = A(t)$. Consequently $B = B(t)$ and the radial dependence of the model is lost. We therefore take $\alpha \neq 1$.

With $\alpha \neq 1$, (4.5.28) can be written as

$$\frac{d{A_x}^2}{{A_x}^2} = \frac{2\alpha(\alpha-3)}{1-\alpha}\frac{dA}{A} \tag{4.5.29}$$

We observe that this equation is easily integrable as it is separable. Upon integrating (4.5.29), we generate the first order differential equation

$$\frac{dA}{dx} = cA^{\frac{\alpha(\alpha-3)}{1-\alpha}} \tag{4.5.30}$$

Equation (4.5.30), is integrable and yields the general solution

$$A(x,t) = \left[\left(\frac{\alpha^2-2\alpha-1}{\alpha-1}\right)(cx+d)\right]^{\frac{\alpha-1}{\alpha^2-2\alpha-1}} \tag{4.5.31}$$

where c and d are constants of integration. Consequently (4.5.27) and (4.5.31) constitute another new exact solution to the Einstein field equations with heat flux.

4.5.6 Solution VI: $B^{-1} = A^\alpha + \beta$

In attempting to obtain a more general class of exact solutions to (4.5.1), we make another choice for $1/B$ which is more general, and which contains the choice of §4.5.5.

We make the assumption that

$$\frac{1}{B} = A^{\alpha} + \beta \tag{4.5.32}$$

where β is independent of x. Then (4.5.1) becomes

$$\left[A + \beta A^{1-\alpha} - \alpha A\right] A_{xx} + \left[2\alpha - \alpha(\alpha - 1)\right] {A_x}^2 \;\; = \;\; 0 \tag{4.5.33}$$

This can be reduced to the differential equation

$$\frac{d{A_x}^2}{{A_x}^2} = \frac{2\alpha(\alpha-3)A^{\alpha-1}}{\beta\left[\frac{1}{\beta}(1-\alpha)A^{\alpha}+1\right]} dA \tag{4.5.34}$$

Equation (4.5.34), can be integrated to generate the following differential equation

$$\frac{dA}{dx} = c(t)\left[\frac{1}{\beta}(1-\alpha)A^{\alpha}+1\right]^{\frac{\alpha-3}{1-\alpha}} \tag{4.5.35}$$

where $c(t)$ is a function of integration. We cannot integrate (4.5.35), in closed form for arbitrary α and β. However it is possible to plot the behaviour of A using software packages such as Mathematica. To regain the case considered in §4.5.5 we need to set $\beta = 0$ in (4.5.33).

Chapter 5

Shearing spacetimes

5.1 Introduction

In Chapter 4 we considered shear-free spacetimes. However it is important to include the effects of shear for many physical applications in cosmology and astrophysics. We observe that very few exact solutions are known with nonzero expansion, acceleration and shear as pointed out by Stephani *et al* (2003). The inclusion of nonvanishing shear leads to highly nonlinear equations with few new solutions although there have been a number of studies carried out by researchers. Naidu *et al* (2006) investigated the thermal evolution of a radiating anisotropic star with shear. They obtained rather simple, yet important solutions, to the field equations for which their model contains a Friedmann-like limit with vanishing heat flux. Maharaj and Misthry (2008), Misthry *et al* (2008) and Rajah and Maharaj (2008) also studied a collapsing star with non-vanishing shear and successfully found new classes of solutions in terms of elementary functions. They demonstrated that their solutions were regular at the stellar centre and that the solutions obtained by Naidu *et al* (2006) could be regained as a special case. In terms of cosmological models, Knutsen (1995) studied the properties of solutions obtained in noncomoving coordinates and which had shear, acceleration and expansion present. Kitamura (1994) obtained a class of exact solutions with shear corresponding to spherically symmetric perfect fluids; this class of models admit conformal transformations as demonstrated by Kitamura (1995). In this chapter we are concerned with spherically symmetric gravitational fields with $\dot{u}^a \neq 0$, $\Theta \neq 0$ and $\sigma \neq 0$. In §5.2, we study the spacetime geometry for the most general spherically symmetric metric. The quantities associated with the curvature are determined. The Einstein field equations, for a perfect fluid, are formulated in §5.3. In §5.4, we find a new class of expanding, accelerating and shearing spacetimes which are regular at the stellar centre. In §5.5, we demonstrate that a well known class of metrics do not satisfy the Einstein field equations.

5.2 Spacetime geometry

The most general spherically symmetric spacetime has nonvanishing acceleration, expansion and shear. These spacetimes are important in modelling astrophysical and cosmological processes. The line element for spherically symmetric spacetimes can be written as

$$ds^2 = -e^{2\nu(t,r)}dt^2 + e^{2\lambda(t,r)}dr^2 + Y^2\left(t,r\right)\left[d\theta^2 + \sin^2\theta d\phi^2\right] \tag{5.2.1}$$

where the functions ν, λ and Y are the gravitational potentials. We have utilised comoving coordinates $(x^a) = (t, r, \theta, \phi)$ related to the fluid four-velocity $u^a = e^{-\nu}\delta^a_0$. For the spherically symmetric metric (5.2.1), the kinematical quantities are given by

$$
\begin{aligned}
\omega_{ab} &= 0 \\
\dot{u}^a &= (0, \nu', 0, 0) \\
\Theta &= e^{-\nu}\left(\dot{\lambda} + \frac{2\dot{Y}}{Y}\right) \\
\sigma_1^1 &= \sigma_2^2 = -\frac{1}{2}\sigma_3^3 = \frac{1}{3}e^{-\nu}\left(\frac{\dot{Y}}{Y} - \dot{\lambda}\right)
\end{aligned}
$$

relative to the four-velocity $\boldsymbol{u}$, where dots and primes denote partial differentiation with respect to t and r respectively. In the above ω_{ab} is the vorticity tensor, $\dot{u}^a$ is the acceleration vector, Θ is the expansion scalar (or rate of expansion) and σ is the magnitude of the shear (or rate of shear). The vorticity vanishes since the spacetime is spherically symmetric. The acceleration, expansion and shear are nonzero in general. Since the work of this chapter is concerned with nonzero shear our solutions have to satisfy the condition

$$
\frac{\dot{Y}}{Y} - \dot{\lambda} \neq 0
$$

If the shear vanishes ($\sigma = 0$), then, after a suitable coordinate transformation, (5.2.1) assumes the form

$$
ds^2 = -e^{2\tilde{\nu}(t,r)}dt^2 + e^{2\tilde{\lambda}(t,r)}[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)]
$$

It is only in the case of vanishing shear that we can find coordinates which are simultaneously comoving and isotropic.

For the line element (5.2.1), the of nonvanishing connection coefficients are given below

$\Gamma^0{}_{00} = \dot{\nu}$ $\quad\quad$ $\Gamma^0{}_{01} = \nu'$

$\Gamma^0{}_{11} = e^{2(\lambda-\nu)}\dot{\lambda}$ $\quad\quad$ $\Gamma^0{}_{22} = e^{-2\nu}Y\dot{Y}$

$\Gamma^0{}_{33} = \sin^2\theta e^{-2\nu}Y\dot{Y}$ $\quad\quad$ $\Gamma^1{}_{00} = e^{2(\nu-\lambda)}\nu'$

$\Gamma^1{}_{01} = \dot{\lambda}$ $\quad\quad$ $\Gamma^1{}_{11} = \lambda'$

$\Gamma^1{}_{22} = -e^{-2\lambda}YY'$ $\quad\quad$ $\Gamma^1{}_{33} = -\sin^2\theta e^{-2\lambda}YY'$

$\Gamma^2{}_{02} = \frac{\dot{Y}}{Y}$ $\quad\quad$ $\Gamma^2{}_{12} = \frac{Y'}{Y}$

$\Gamma^2{}_{33} = -\sin\theta\cos\theta$ $\quad\quad$ $\Gamma^3{}_{03} = \frac{\dot{Y}}{Y}$

$\Gamma^3{}_{13} = \frac{Y'}{Y}$ $\quad\quad$ $\Gamma^3{}_{23} = \cot\theta$

Then utilizing the above nonzero connection coefficients, and the definition for the Ricci tensor (2.2.4), we obtain the nonzero Ricci tensor components in the form

$$
\begin{aligned}
R_{00} &= -\ddot{\lambda} - \dot{\lambda}^2 + \dot{\lambda}\dot{\nu} + 2\dot{\nu}\frac{\dot{Y}}{Y} - 2\frac{\ddot{Y}}{Y} \\
&\quad + e^{2(\nu-\lambda)}\left(\nu'' + \nu'^2 - \nu'\lambda' + 2\nu'\frac{Y'}{Y}\right) \qquad (5.2.2a)
\end{aligned}
$$

$$
R_{01} = 2\left(\dot{\lambda}\frac{Y'}{Y} + \nu'\frac{\dot{Y}}{Y} - \frac{\dot{Y}'}{Y}\right) \qquad (5.2.2b)
$$

$$
\begin{aligned}
R_{11} &= -\nu'' - \nu'^2 + \lambda'\nu' + 2\lambda'\frac{Y'}{Y} - 2\frac{Y''}{Y} \\
&\quad + e^{2(\lambda-\nu)}\left(\ddot{\lambda} + \dot{\lambda}^2 - \dot{\lambda}\dot{\nu} + 2\dot{\lambda}\frac{\dot{Y}}{Y}\right) \qquad (5.2.2c)
\end{aligned}
$$

$$
\begin{aligned}
R_{22} &= e^{-2\nu}Y\dot{Y}\left(\dot{\lambda} - \dot{\nu} + \frac{\dot{Y}}{Y} + \frac{\ddot{Y}}{\dot{Y}}\right) \\
&\quad + e^{-2\lambda}YY'\left(\lambda' - \nu' - \frac{Y'}{Y} - \frac{Y''}{Y'}\right) + 1 \qquad (5.2.2d)
\end{aligned}
$$

$$
R_{33} = \sin^2\theta R_{22} \qquad (5.2.2e)
$$

The Ricci tensor components (5.2.2), may be used to generate the following expression for the Ricci scalar

$$
\begin{aligned}
R &= 2e^{-2\nu}\left(\ddot{\lambda} + \dot{\lambda}^2 - \dot{\lambda}\dot{\nu} + 2\dot{\lambda}\frac{\dot{Y}}{Y} - 2\dot{\nu}\frac{\dot{Y}}{Y} + \frac{\dot{Y}^2}{Y^2} + 2\frac{\ddot{Y}}{Y}\right) - \\
&\quad 2e^{-2\lambda}\left(\nu'' + \nu'^2 - \nu'\lambda' - 2\lambda'\frac{Y'}{Y} + 2\nu'\frac{Y'}{Y} + \frac{Y'^2}{Y^2} + 2\frac{Y''}{Y}\right) + \frac{2}{Y^2} \qquad (5.2.3)
\end{aligned}
$$

For the line element (5.2.1), the corresponding Einstein tensor components are given as

$$G_{00} = 2\dot{\lambda}\frac{\dot{Y}}{Y} + \frac{\dot{Y}^2}{Y^2} - e^{2(\nu-\lambda)}\left(-2\lambda'\frac{Y'}{Y} + \frac{Y'^2}{Y^2} + 2\frac{Y''}{Y}\right) + \frac{e^{2\nu}}{Y^2} \qquad (5.2.4a)$$

$$G_{01} = 2\dot{\lambda}\frac{Y'}{Y} + 2\nu'\frac{\dot{Y}}{Y} - 2\frac{\dot{Y}'}{Y} \qquad (5.2.4b)$$

$$G_{11} = 2\nu'\frac{Y'}{Y} + \frac{Y'^2}{Y^2} + e^{2(\lambda-\nu)}\left(2\dot{\nu}\frac{\dot{Y}}{Y} - \frac{\dot{Y}}{Y^2} - 2\frac{\ddot{Y}}{Y}\right) - \frac{e^{2\lambda}}{Y^2} \qquad (5.2.4c)$$

$$G_{22} = -e^{-2\nu}\left[\left(\ddot{\lambda} + \dot{\lambda}^2 - \dot{\lambda}\dot{\nu}\right)Y^2 + \left(\dot{\lambda}\dot{Y} - \dot{\nu}\dot{Y} + \ddot{Y}\right)Y\right]$$

$$+e^{-2\lambda}\left[\left(\nu'' + \nu'^2 - \nu'\lambda'\right)Y^2 + \left(\nu'Y' - \lambda'Y' + Y''\right)Y\right] \qquad (5.2.4d)$$

$$G_{33} = \sin^2\theta G_{22} \qquad (5.2.4e)$$

which follow from (5.2.2) and (5.2.3).

5.3 The field equations

For the line element (5.2.1), the energy momentum tensor has the following nonzero components

$$T_{00} = \rho e^{2\nu} \tag{5.3.1a}$$

$$T_{11} = p e^{2\lambda} \tag{5.3.1b}$$

$$T_{22} = p Y^2 \tag{5.3.1c}$$

$$T_{33} = p \sin^2\theta Y^2 \tag{5.3.1d}$$

for a perfect fluid with vanishing heat flux.

Equating (5.2.4) and (5.3.1) leads to the Einstein field equations

$$\rho = \frac{1}{Y^2} - \frac{2}{Y}e^{-2\lambda}\left(Y'' - \lambda' Y' + \frac{Y'^2}{2Y}\right) + \frac{2}{Y}e^{-2\nu}\left(\dot{\lambda}\dot{Y} + \frac{\dot{Y}^2}{2Y}\right) \tag{5.3.2a}$$

$$p = -\frac{1}{Y^2} + \frac{2}{Y}e^{-2\lambda}\left(\nu' Y' + \frac{Y'^2}{2Y}\right) - \frac{2}{Y}e^{-2\nu}\left(\ddot{Y} - \dot{\nu}\dot{Y} + \frac{\dot{Y}^2}{2Y}\right) \tag{5.3.2b}$$

$$\begin{aligned} p &= e^{-2\lambda}\left[\nu'' + \nu'^2 - \nu'\lambda' + \frac{1}{Y}\left(\nu' Y' - \lambda' Y' + Y''\right)\right] \\ &\quad - e^{-2\nu}\left[\ddot{\lambda} + \dot{\lambda}^2 - \dot{\lambda}\dot{\nu} + \frac{1}{Y}\left(\dot{\lambda}\dot{Y} - \dot{\nu}\dot{Y} + \ddot{Y}\right)\right] \end{aligned} \tag{5.3.2c}$$

$$0 = \dot{Y}' - \dot{Y}\nu' - Y'\dot{\lambda} \tag{5.3.2d}$$

which are highly nonlinear. From the conservation of energy momentum (2.3.5), we generate the differential equations

$$p' = -(\rho + p)\,\nu' \tag{5.3.3a}$$

$$\dot{\rho} = -(\rho + p)\left(\dot{\lambda} + 2\frac{\dot{Y}}{Y}\right) \tag{5.3.3b}$$

which are conservation equations. The result (5.3.3), may be obtained directly from the field equations (5.3.2). The conservation equations are sometimes used in conjunction with the field equations to obtain a solution. The system (5.3.2), comprises four equations in the five unknowns ρ, p, ν, λ and Y. Actually there are only three independent equations; (5.3.2b) and (5.3.2c) generate the condition of pressure isotropy. To obtain a solution it is necessary to impose additional restrictions. There are few solutions, in comoving coordinates, to the system (5.3.2) which are known with nonzero expansion, acceleration and shear as pointed out by Stephani *et al* (2003). In fact, only the two general classes of Marklund and Bradley (1999) and Maharaj *et al* (1993) in comoving coordinates have been published in the literature.

5.4 New shearing solutions

In this section we present a new class of exact solutions, in terms of elementary functions, which are expanding, accelerating and shearing. As a starting point we choose the simple form

$$ds^2 = -e^{2\nu(r)}dt^2 + e^{2\lambda(r)}dr^2 + (r+\alpha)^2T^2(t)(d\theta^2 + \sin^2\theta d\phi^2) \tag{5.4.1}$$

for the line element where α is a real constant. This is the simplest form that allows for $\dot{u}^a \neq 0, \Theta \neq 0$ and $\sigma \neq 0$. Particular models associated with (5.4.1) have been studied by Hajj-Boutros (1985), Maharaj *et al* (1993) and Wesson (1978), and other models as given in Stephani *et al* (2003).

The field equations (5.3.2) simplify, because of the reduced metric (5.4.1), and we obtain

$$\rho = \frac{1}{(r+\alpha)^2 T^2} + \frac{2}{(r+\alpha)} e^{-2\lambda}\left[\lambda' - \frac{1}{2(r+\alpha)}\right] + e^{-2\nu}\left(\frac{\dot{T}^2}{T^2}\right) \tag{5.4.2a}$$

$$\begin{aligned} p = {} & \frac{1}{(r+\alpha)T}\left[-\frac{1}{(r+\alpha)T} + T\left(2\nu' + \frac{1}{(r+\alpha)}\right)e^{-2\lambda}\right. \\ & \left. -2(r+\alpha)\left(\ddot{T} + \frac{\dot{T}^2}{2T}\right)e^{-2\nu}\right] \end{aligned} \tag{5.4.2b}$$

$$p = e^{-2\lambda}\left[(\nu'' + \nu'^2 - \nu'\lambda') + \frac{1}{(r+\alpha)}(\nu' - \lambda')\right] - \frac{\ddot{T}}{T}e^{-2\nu} \tag{5.4.2c}$$

$$0 = 1 - (r+\alpha)\nu' \tag{5.4.2d}$$

Equation (5.4.2d) can be integrated to give

$$e^{2\nu} = a^2 (r+\alpha)^2 \tag{5.4.3}$$

where a is a constant of integration.

Equating (5.4.2b) and (5.4.2c) leads to

$$\frac{1}{T^2} + \frac{(r+\alpha)^2}{e^{2\nu}}\left[\frac{\ddot{T}}{T} + \frac{\dot{T}^2}{T^2}\right] =$$

$$\frac{(r+\alpha)}{e^{2\lambda}}\left[\frac{1}{r+\alpha} + \nu' + \lambda' + (r+\alpha)(\nu'\lambda' - \nu'^2 - \nu'')\right] \tag{5.4.4}$$

which is the condition of pressure isotropy. We can eliminate $e^{2\nu}$ from (5.4.4), with the help of (5.4.3), to get

$$\frac{1}{T^2} + \frac{1}{a^2}\left[\frac{\ddot{T}}{T} + \frac{\dot{T}^2}{T^2}\right] = \frac{2(r+\alpha)}{e^{2\lambda}}\left(\frac{1}{r+\alpha} + \lambda'\right) \tag{5.4.5}$$

Observe that in (5.4.5) the left hand side is a function of the coordinate t and the right hand side is a function of the coordinate r. This implies that

$$\frac{1}{T^2} + \frac{1}{a^2}\left[\frac{\ddot{T}}{T} + \frac{\dot{T}^2}{T^2}\right] = 2k \tag{5.4.6a}$$

$$\frac{(r+\alpha)}{e^{2\lambda}}\left(\frac{1}{r+\alpha} + \lambda'\right) = k \tag{5.4.6b}$$

where k is an arbitrary constant. We can write (5.4.6a) in the form

$$(T^2)^{\cdot\cdot} - 4a^2kT^2 + 2a^2 = 0 \tag{5.4.7}$$

which is linear in T^2. Three classes of solutions are possible depending on the roots of the characteristic equation, and we obtain

$$T^2 = \begin{cases} -a^2t^2 + ct + d, & k = 0 \\ \\ c\sin(2ant) + d\cos(2ant) - \frac{1}{2n^2}, & k = -n^2 < 0 \\ \\ ce^{2ant} + de^{-2ant} + \frac{1}{2n^2}, & k = n^2 > 0 \end{cases} \tag{5.4.8}$$

Therefore the general solution of (5.4.6a) is known. If we let $e^{2\lambda} = y$ then (5.4.6b) is transformed to

$$y' + \left(\frac{2}{r+\alpha}\right) y - \left(\frac{k}{r+\alpha}\right) y^2 \;\; = \;\; 0$$

This is a Riccati equation which is integrable. The general solution of (5.4.6b) can then be written as

$$e^{2\lambda} = \frac{1}{k + b(r+\alpha)^2} \tag{5.4.9}$$

where b is a constant.

The solutions to (5.4.6) may be given for the three cases $k = 0, k < 0$ and $k > 0$. For these cases, the metric (5.4.1) may be written as

$\underline{k = 0}$:

$$ds^2 = -a^2(r+\alpha)^2dt^2 + \left(\frac{1}{b(r+\alpha)^2}\right) dr^2 + (r+\alpha)^2$$

$$\times(-a^2t^2 + ct + d)(d\theta^2 + \sin^2\theta d\phi^2) \tag{5.4.10}$$

$\underline{k = -n^2 < 0}$:

$$ds^2 = -a^2(r+\alpha)^2 dt^2 + \left(\frac{1}{-n^2 + b(r+\alpha)^2}\right) dr^2 + (r+\alpha)^2$$

$$\times \left(c\sin(2ant) + d\cos(2ant) - \frac{1}{2n^2}\right)(d\theta^2 + \sin^2\theta d\phi^2) \qquad (5.4.11)$$

$\underline{k = n^2 > 0}$:

$$ds^2 = -a^2(r+\alpha)^2 dt^2 + \left(\frac{1}{n^2 + b(r+\alpha)^2}\right) dr^2 + (r+\alpha)^2$$

$$\times \left(ce^{2ant} + de^{-2ant} + \frac{1}{2n^2}\right)(d\theta^2 + \sin^2\theta d\phi^2) \qquad (5.4.12)$$

where c and d are constants of integration. By (5.4.2), using (5.4.3), (5.4.8) and (5.4.9), the energy density and pressure may be written as

$$\rho = -3b + \frac{k}{(r+\alpha)^2} - \frac{1}{a^2(r+\alpha)^2}\frac{\ddot{T}}{T} \qquad (5.4.13a)$$

$$p = 3b + \frac{k}{(r+\alpha)^2} - \frac{1}{a^2(r+\alpha)^2}\frac{\ddot{T}}{T} \qquad (5.4.13b)$$

respectively. Combining (5.4.13b) and (5.4.13b), we obtain the following equation of state

$$p = \rho + 6b \qquad (5.4.14)$$

which is of the barotropic form $p = p(\rho)$. This equation of state is the same as obtained by Maharaj *et al* (1993) although in the results above it was obtained from a different nonsingular metric solution. This is a linear equation of state which generalises the stiff equation of state ($p = \rho$).

When $\alpha = 0$, our results (5.4.10)-(5.4.12) reduce to the result of Maharaj *et al* (1993). Observe that at the point corresponding to $r = 0$, the Maharaj *et al* (1993) result is singular and cannot be used to model the stellar centre. This is an undesirable feature in a relativistic stellar model. In our models (5.4.10)-(5.4.12), the point corresponding to $r = 0$ does not produce a singularity in the line element (5.4.1) since $\alpha \neq 0$ in general. Hence, the class of new solutions (5.4.10)-(5.4.12) are physically reasonable and may be used to model stellar centres. They also have the advantage of being given in terms of elementary functions which facilitates the investigation of the gravitational behaviour. We emphasize that the class of solutions presented in this section are expanding, accelerating and shearing. It is interesting to observe that the solutions (5.4.10)-(5.4.12) admit a conformal Killing vector of the form

$$\boldsymbol{X} = \frac{\partial}{\partial r}$$

which is orthogonal to the spacelike hypersurfaces. This was first observed by Maharaj and Maharaj (1994). The influence of the electromagnetic field on the metric (5.4.1) was considered by Moodley *et al* (2003).

5.5 Correction: Marklund and Bradley solution

The second class of expanding, accelerating and shearing solutions that has been reported is due to Marklund and Bradley (1999). The line element in this class has the form

$$ds^2 = -\frac{r^2}{4t^2(ct^2 - t + a)^2}dt^2 + \frac{1}{a - br^2}dr^2 + \frac{t}{r^2}(d\theta^2 + \sin^2\theta d\phi^2) \qquad (5.5.1)$$

We deduce from this line element that the metric functions must be

$$e^{2\nu} = \frac{r^2}{4t^2(ct^2 - t + a)^2}$$

$$e^{2\lambda} = \frac{1}{a - br^2}$$

$$Y^2 = \frac{t}{r^2}$$

If we substitute these functional forms in (5.3.2d) then we obtain

$$\frac{1}{r^2 t^{1/2}} = 0 \qquad (5.5.2)$$

which is an inconsistency. Hence the Marklund and Bradley (1999) result is not a solution of the Einstein field equations. This is contrary to the claims in the literature.

We can demonstrate in principle why the Marklund and Bradley (1999) result does not work. Note that the Marklund and Bradley (1999) "solution" is of the form

$$ds^2 = -e^{2\nu_1(t)}e^{2\nu_2(r)}dt^2 + e^{2\lambda(r)}dr^2 + {Y_1}^2(t){Y_2}^2(r)(d\theta^2 + \sin^2\theta d\phi^2) \qquad (5.5.3)$$

With the form (5.5.3) we find that (5.3.2d) becomes

$$\dot{Y}_1({Y_2}' - Y_2{\nu_2}') = 0$$

So that when $\dot{Y}_1 \neq 0$ we must have

$$\frac{{Y_2}'}{Y_2} = {\nu_2}'$$

Hence we have the relationship

$$e^{\nu_2} \propto Y_2 \tag{5.5.4}$$

Consequently, we have established that for metrics of the form (5.5.3), as chosen by Marklund and Bradley (1999), the condition (5.5.4) must hold. This means that the radial dependence of $e^{\nu(t,r)}$ and $Y(t,r)$ must be the same. This is clearly not the case in the Marklund and Bradley (1999) model. Our argument given here indicates why the solution of Marklund and Bradley (1999) fails. Note that the condition (5.5.4) necessarily follows because the metric coefficients $\nu(t,r)$ and $Y(t,r)$ are separable functions.

Chapter 6

Conclusion

The main objective of this dissertation was to study the spherically symmetric spacetimes, and associated relativistic models used to describe stars and cosmological processes. It was our aim to find new exact solutions to the Einstein field equations for relativistic stars which are static and nonstatic models with vanishing shear in the presence of heat flux. Nonstatic models with nonvanishing shear in the absence of heat flux were also considered. Solutions to the highly nonlinear system of coupled differential field equations were sought by solving the condition of pressure isotropy. Our assumptions effectively reduced the pressure isotropy condition to a simple second order differential equation with variable coefficients. We solved this master equation by choosing particular forms for the gravitational potentials, and obtained several new classes of exact solutions in terms of elementary functions. A new solution which is appropriate in describing the centre of a star was also presented for the shearing model. This solution contains the result obtained by Maharaj *et al* (1993) as a special case. We also made a number of general remarks on the shearing solution found by Marklund and Bradley (1999) and showed that it is inconsistent.

We now provide an overview of the main results obtained during the course of our investigations:

- In Chapter 2, we introduced the relevant definitions and formalisms of differential geometry that were necessary for later chapters. We generated the Einstein field equations for a neutral fluid matter distribution, and also the Einstein-Maxwell system of equations for charged matter.

- In Chapter 3, we constructed the basic model for static spherically symmetric spacetimes containing neutral as well as charged perfect fluids. For the case of the charged perfect fluid model we showed that the Einstein-Maxwell system can be rewritten as a simpler system by using the transformation of Durgapal and Bannerji (1983). A number of exact solutions are known to the field equations which could model the interior of a dense static star. The two general classes of exact solutions for charged relativistic stars, obtained by Thirukkanesh and Maharaj (2008), were presented. We demonstrated that this general class contains well known solutions for neutral and charged static stars. The explicit solutions, found previously, that model charged compact spheres and neutral neutron stars were explicitly regained.

- Chapter 4 formed a major part of this study. We constructed the model for a shear-free spacetime with nonvanishing radial heat flow. It is well known that such a model is effective in describing radiative processes in both astrophysics and cosmology. We produced a second order differential equation with variable coefficients representing the condition of pressure isotropy. It was our main purpose to solve this master equation to generate new exact solutions. The pressure isotropy equation contains two dependant variables, namely the gravitational potentials A and B, and can be written as

$$\left(\frac{1}{B}\right) A_{xx} + 2A_x \left(\frac{1}{B}\right)_x - A \left(\frac{1}{B}\right)_{xx} = 0$$

The known solutions for this equation, corresponding to conformal flatness, were presented. We generated several new classes of exact solutions corresponding to the following choices

- (a) $\frac{1}{B} = (a + bx)^k$

- (b) $A = (a + bx)^k$

- (c) $\frac{1}{B} = e^{a+bx}$

- (d) $A = e^{a+bx}$

- (e) $\frac{1}{B} = A^\alpha$

- (f) $\frac{1}{B} = A^\alpha + \beta$

for the potentials. In each case we were able to solve the condition of pressure isotropy and present exact solutions in terms of elementary functions. These solutions are new and have not been published previously. It is remarkable that our simple ansatz allows for such a wide variety of simple models. It is important to note that the conformally flat solution

$$\begin{aligned} \frac{A}{B} &= 1 + C_1(t)r^2 \\ B &= \frac{1}{C_2(t)r^2 + C_3(t)} \end{aligned}$$

is contained in our general class of solutions in the relevant limit.

- In Chapter 5, we considered the model for a spacetime with nonzero shear, acceleration and expansion. We presented three new classes of solutions to the field equations which are generalizations of those found by Maharaj *et al* (1993). These new solutions have the barotropic equation of state

$$p = \rho + 6b$$

  which is a generalization of the stiff equation of state $p = \rho$. Note that this new class of shearing models is appropriate for describing the centre of relativistic stars as the metric functions remain regular unlike the models of Maharaj *et al* (1993). Also, we showed that the Marklund and Bradley (1999) model is inconsistent and does not satisfy the Einstein field equations.

In the above we have highlighted only those items of particular interest to spherically symmetric gravitational fields. The primary aim of this dissertation was to study the appropriate models for relativistic stars and cosmological processes, as well as to find new exact solutions if possible. We have produced a number of new solutions in terms of simple elementary functions which generalise earlier treatments. We have not carried out any qualitative analysis of the behaviour of our new solutions or used them to predict the overall evolution of the systems which we have studied. This is outside the scope of this dissertation. In future work we aim to find other physically relevant solutions and use them, in conjunction with other physical theories, to predict the behaviour of the gravitating systems.

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