COSMOLOGICAL MODELS
AND
THE DECELERATION PARAMETER

by

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Abstract

In this thesis we utilise a form for the Hubble constant first proposed by Berman (1983) to study a variety of cosmological models. In particular we investigate the Robertson–Walker spacetimes, the Bianchi I spacetime, and the scalar–tensor theory of gravitation of Lau and Prokhovnik (1986). The Einstein field equations with variable cosmological constant and gravitational constant are discussed and the Friedmann models are reviewed. The relationship between observation and the Friedmann models is reviewed. We present a number of new solutions to the Einstein field equations with variable cosmological constant and gravitational constant in the Robertson–Walker spacetimes for the assumed form of the Hubble parameter. We explicitly find forms for the scale factor, cosmological constant, gravitational constant, energy density and pressure in each case. Some of the models have an equation of state for an ideal gas. The gravitational constant may be increasing in certain regions of spacetime. The Bianchi I spacetime, which is homogeneous and anisotropic, is shown to be consistent with the Berman (1983) law by defining a function which reduces to the scale factor of Robertson–Walker. We illustrate that the scalar–tensor theory of Lau and Prokhovnik (1986) also admits solutions consistent with the Hubble variation proposed by Berman. This demonstrates that this approach is useful in seeking solutions to the Einstein field equations in general relativity and alternate theories of gravity.

To my parents and family

for the considerable amount of encouragement and patience

Preface

The study described in this thesis was carried out in the Department of Mathematics and Applied Mathematics, University of Natal, Durban, during the period January 1992 to December 1992. This thesis was completed under the excellent supervision of Dr. S. D. Maharaj.

This study represents original work by the author and it has not been submitted in any form to another University nor has it been previously published. Where use was made of the work of others it has been duly acknowledged in the text.

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Contents

0 Introduction

Cosmology is the study of the large scale evolution of the universe. Cosmologists have now produced a consistent picture of the history of the development of the universe from the big bang up to present times. The foundations of modern theoretical cosmology were laid with Einstein's publication of a paper indicating how the equations of general relativity could be applied to describe the behaviour of matter on a large scale. During the course of 1915, Einstein had published successive refinements of his basic field equations of general relativity. He had developed these as a covariant modification of the Newtonian theory of gravitation which was known to be consistent with observation to a high degree of accuracy. Hilbert (1915), however, obtained the field equations using the Lagrangian approach. For a consistent theory Einstein required that

(i) The form of the field equations are preserved under coordinate transformation, i.e. the field equations are tensor equations.

(ii) The theory of general relativity reduces to special relativity in the appropriate limit.

The theory of general relativity provides a model of the universe which is consistent with observational results.

The Einstein field equations couple the gravitational field to the matter content. Gravity is built into the theory through the field equations which link spacetime curvature to the matter distribution. The gravitational field in the Einstein field equations is contained in the Einstein tensor which is related to the curvature of spacetime via the Riemann tensor and the Ricci scalar. The matter content is represented by the isotropic energy– momentum tensor. The energy–momentum tensor is described by a relativistic fluid; for many applications in cosmology we consider a dust universe in which the pressure is zero. The Einstein field equations are a set of highly nonlinear partial differential equations subject to conservation laws, namely the Bianchi identities. In order to solve the highly nonlinear field equations of general relativity it is often assumed that spacetime admits symmetries in the hope that the field equations are simplified (Maharaj *et al* 1991).

Exact solutions to Einstein's field equations are necessary for applications in cosmology and astrophysics. Although a large number of solutions are known many of them do not satisfy a physical equation of state (Kramer *et al* 1980). Cosmological models are exact solutions of the Einstein field equations which should reproduce the physical properties of our universe. Every model is a great simplification of reality. Therefore we need to analyse many solutions so that we can infer which simplifications are valid and describe the physical universe. The standard cosmological models are the Robertson–Walker spacetimes which satisfy the cosmological principle which states that the universe is homogeneous and isotropic. The Friedmann models are the solutions of Einstein's field equations obeying the cosmological principle for a dust matter distribution. In the Friedmann models the universe originated at the initial singularity or big bang and subsequently expanded. The expansion of the universe was observationally verified by the observations of Hubble (1936). The Robertson–Walker models are also consistent with the observations of Penzias and Wilson (1965) that the universe is bathed in an isotropic microwave background radiation of approximately 3 degrees Kelvin. To investigate more general behaviour in cosmology we need to relax the assumptions of homogeneity and isotropy. A variety of such solutions is listed by Kramer *et al* (1980) amongst others.

At the time that Einstein proposed his field equations he wanted to find a solution describing a static closed universe. At that time the prevailing belief was that the universe was static. This belief was based on philosophical arguments rather than on mathematical grounds. In order to make the field equations fit in with the philosophical belief of the time, Einstein introduced an additional quantity, called the cosmological constant in the field equations. This was necessary to prevent the gravitational collapse of the universe. The cosmological constant acts as a repulsion mechanism and is possibly due to negative matter. However this addition is not necessary because the universe is expanding as discovered by Hubble (1936). In recent times the cosmological constant has resurfaced in theories which attempt to describe the mechanics of the early universe (Misner *et al* 1973). The classical Einstein field equations are easily adapted to cater for the variable cosmological constant and gravitational constant.

We outline briefly several different reasons for incorporating variable cosmological terms in the analysis of theories of gravity. It is believed that the cosmological constant corresponds to the vacuum energy density of the quantum field (Zel'dovich 1968), and that the cosmological constant was large during the early universe and has had an influence on its dynamics (DerSarkissian 1985; Kasper 1985; Villi 1985). The mass of the Higgs boson is believed to be related to the cosmological constant and the gravitational constant (Dreitlein 1974). There is a possibility of the cosmological constant being a function of temperature and that it is related to the process of broken symmetry (Bergmann 1968; Linde 1974; Wagoner 1970). Further interest in the cosmological constant arises within the contex of quantum gravity, supergravity theories, Kaluza–Klein theories, the inflationary universe scenario, particle physics and grand unified theories (Banerjee and Banerjee 1985; Lorenz–Petzold 1984; Singh and Singh 1983). The problems of singularity, horizon, flatness and monopole may be solved in cosmological models with variable cosmological constant (Ozer and Taha 1986, 1987). In some variable cosmological constant theories the problem of fine tuning can be explained (Canuto *et al* 1977).

In chapter I we consider some basic concepts of differentiable geometry and those elements of tensor analysis which are essential for this thesis. We cite appropriate literature for further details of results presented. Only results needed in later sections are discussed. We briefly introduce the mathematical structure of general relativity, namely a 4–dimensional differentiable manifold. Coordinate transformations are defined on the manifold and these lead to the transformation properties of tensor fields. The metric connection is defined and the covariant derivative is briefly discussed. The curvature tensor is defined and its properties are listed. Also defined are the Ricci tensor, Ricci scalar and the Einstein tensor. The energy–momentum tensor is introduced and the classical Einstein field equations with vanishing cosmological constant are motivated. Then the Einstein field equations with variable cosmological constant and gravitational constant are introduced. We note that the Einstein field equations may also be found using the variational principle of Hilbert (1915). The conservation laws for the classical Einstein field equations and the generalisation for variable cosmological constant and gravitational constant are given.

In chapter 2 we consider the geometry of the Robertson–Walker models. These are simple models of the universe satisfying the cosmological principle which states that the universe is homogeneous and isotropic. This model is described by a uniform perfect fluid energy–momentum tensor. The line element of this model and its various forms are given. Using the line element for the Robertson–Walker model we calculate and list the components of the connection coefficients, the Ricci tensor, curvature scalar, and the components of the Einstein tensor. The classical Einstein field equations for the Robertson–Walker models reduce to a system of two equations. We also present the field equations for the case of variable cosmological constant and gravitational constant. The Friedmann equation and the generalised Friedmann equations are obtained. In particular the dust Friedmann models are derived in detail. The properties of the Friedmann model are briefly discussed. Some elements of cosmology are discussed. The cosmological parameters such as the Hubble constant, the critical density and the deceleration parameter are defined and we present a brief discussion on their present day values.

In chapter 3 we assume the law of variation for Hubble's parameter which was proposed by Berman (1983) to solve the Einstein field equations. We present solutions to the classical Einstein field equations and relate these to Berman (1983) and Berman and Gomide (1988). The solution of Berman (1991) to the Einstein field equations with variable cosmological constant and gravitational constant are also investigated. We also find seven classes of new solutions to the Einstein field equations. These have the variable cosmological constant and gravitational constant and satisfy the Berman law for the Hubble parameter. In each case we explicitly present forms for the scale factor, the cosmological constant, the gravitational constant, the energy density and the pressure. The physical properties of the solutions are briefly discussed.

In chapter 4 we investigate the extension of the Berman (1983) law to the Bianchi I spacetime and the scalar–tensor theory of Lau and Prokhovnik (1986). The field equations of the Bianchi I spacetime are derived. By defining an analogue of the scale factor for the Bianchi I spacetime we investigate the Berman (1983) Hubble parameter. The field equations for the $k = 0$ Robertson–Walker spacetimes for the Lau and Prokhovnik (1986) theory are derived and we show that this scalar–tensor theory is consistent with our assumed form of the Hubble parameter.

In chapter 5 we briefly discuss the results obtained. We highlight the applicability of the form of the Hubble parameter used. We point out that this form may be useful in finding solutions to the Einstein field equations with variable cosmological constant and gravitational constant. In addition the Hubble law may be helpful in seeking solutions in scalar–tensor theories of gravity.

1 Tensors, Curvature and Field Equations

1.1 Introduction

In this chapter we provide a brief review and discussion of those essential features of differential geometry, manifolds and tensors that are necessary for this thesis. We begin by describing the 4–dimensional spacetime structure of a manifold which admits a Lorentzian metric at every point. A rigorous definition of a manifold is not given; we describe the fundamental characteristics heuristically for our applications in general relativity. The additional structure of an affine connection is introduced on the manifold as a consequence of the fundamental theorem of Riemannian geometry. For the purposes of general relativity we take spacetime to be a 4–dimensional pseudo–Riemannian manifold endowed with the metric connection. In §1.2 we also consider general coordinate transformations, tensor products and tensor fields as natural geometric objects on the manifold. For a more comprehensive treatment of a manifold and related concepts the reader is referred to Bishop and Goldberg (1968), Choquet–Bruhat *et al* (1977), Hawking and Ellis (1973), Misner *et al* (1973) and Wald (1984). The metric connection plays a significant role when considering curvature of spacetime in general relativity. The gravitational field is described by the nondegenerate symmetric metric tensor field which reduces to the Lorentzian metric of special relativity at a point. We consider the metric connection and the

Christoffel symbols in §1.3. The curvature tensor is derived and the various identities satisfied by the curvature tensor are given. Also the Ricci tensor, the Ricci scalar and the Einstein tensor are considered in this section. In §1.4 we introduce the symmetric energy–momentum tensor describing the matter content. Then we are in a position to introduce the classical Einstein field equations without the cosmological constant. We also briefly mention how the Einstein field equations may be derived from a variational principle. In addition we generalise the classical field equations to include the case of both variable gravitational and cosmological constants.

1.2 Manifolds and Tensor Fields

A differentiable manifold is essentially a space which has a structure that is locally similar to Euclidean space in that it may be covered by coordinate neighbourhoods. Even though the local structure of a manifold and Euclidean space are similar, it should be emphasised that their global structures may be very different. In general relativity we require a curved manifold which only locally resembles the flat spacetime of special relativity. The curvature, reflecting deviations from flatness, may be detected by means of geodesic deviation. An essential feature of a manifold is its dimension n. For the purposes of relativity it is sufficient to consider only the case $n = 4$. Points in the manifold may be labelled by the real coordinates

$$(x^a) = (x^0, x^1, x^2, x^3) = (ct, x^1, x^2, x^3)$$

where we adopt the convention that the speed of light $c = 1$. We require that the manifold supports a differentiable structure so that differentiation of functions, involving changes of coordinates, is permissible. For our purposes it is sufficient to assume that spacetime is a 4–dimensional, differentiable, connected, Hausdorff, oriented manifold M. A 4–dimensional differentiable manifold M is a set of points which is locally similar to $\Re^4$. Each point, contained in some coordinate neighbourhood of M, can be put into a bijective mapping with an open set of $\Re^4$ through a coordinate function. A collection of coordinate neighbourhoods (or atlas), satisfying certain well–defined conditions, with differentiably related coordinate functions transforms the set M into a differentiable manifold. The set M is connected so that any two points may be joined by a continuous curve. This ensures that the various parts of M interact and there are no disconnected regions. We require that M is Hausdorff to ensure that distinct events have disjoint neighbourhoods; in particular this means that geodesics have unique trajectories. The manifold M is orientable so that coordinate transformations have positive Jacobians

$$J \equiv \left| \frac{\partial x^{a'}}{\partial x^b} \right| > 0$$

in overlapping coordinate neighbourhoods. In the overlap the functional relationships given by

$$x^{a'} = x^{a'}(x^0, x^1, x^2, x^3)$$

and the inverse relationships

$$x^a = x^a(x^{0'}, x^{1'}, x^{2'}, x^{3'})$$

are both injective and differentiable. For further details on manifolds the reader is referred to Choquet–Bruhat *et al* (1977), Hawking and Ellis (1973), Misner *et al* (1973), Straumann (1984) and Wald (1984).

Let T_P represent the set of vectors tangent to a curve at a point P in M. It is easy to show that the set of tangent vectors T_P generates a vector space at P.

We generate the dual tangent space T_P^* at P by defining the real-valued function $T_P^* : T_P \longrightarrow \Re$. The dual space T_P^* satisfies the vector space axioms. We can then construct spaces $(T_s^r)_P$ of type (r, s) tensors at P by taking repeated tensor products of T_P and T_P^* (Bishop and Goldberg 1968, Misner *et al* 1973 and Schutz 1980). The space T_s^r is the set of multilinear functions mapped into $\Re$ at P. It is easily established that the space $(T_s^r)_P$ is also a vector space at P. A type (r, s) tensor field on M is an assignment to each point $P \in M$ a member of $(T_s^r)_P$. We represent the set of all type (r, s) tensor fields on M by T_s^r. The quantity $T^{a_1 a_2 \ldots a_r}{}_{b_1 b_2 \ldots b_s}$ represents the components of a (r, s) tensor field $\mathbf{T}$ in T_s^r, and under a change of coordinates transforms according to the rule

$$T^{a'_1 a'_2 \ldots a'_r}{}_{b'_1 b'_2 \ldots b'_s} = X^{a'_1}_{c_1} X^{a'_2}_{c_2} \ldots X^{a'_r}_{c_r} X^{d_1}_{b'_1} X^{d_2}_{b'_2} \ldots X^{d_s}_{b'_s} T^{c_1 c_2 \ldots c_r}{}_{d_1 d_2 \ldots d_s} \tag{1.1}$$

where the Jacobians of the matrices

$$X^{a'}_{b} = \frac{\partial x^{a'}}{\partial x^{b}}$$

and the inverse matrix

$$X^{b}_{a'} = \frac{\partial x^{b}}{\partial x^{a'}}$$

are nonzero in the overlapping coordinate neighbourhoods of the manifold.

In order to discuss metrical properties we need to introduce a differentiable metric tensor field $\mathbf{g}$ on the manifold. We require that the manifold M is endowed with an indefinite, symmetric metric tensor field $\mathbf{g}$ of rank two with signature $(- + + +)$. A manifold with an indefinite metric tensor field, as in general relativity, is called a pseudo-Riemannian manifold. A manifold with a positive definite metric tensor field is sometimes called a Riemannian manifold. The covariant components g_{ab} of the symmetric $(0, 2)$ metric tensor field $\mathbf{g}$ must satisfy (1.1) and are used to define invariantly the length of a curve in M. This length is defined by the following integral

$$s = \int_{u_1}^{u_2} |g_{ab}\dot{x}^a\dot{x}^b|^{\frac{1}{2}} du$$

where we have set $\dot{x}^a = dx^a/du$. This definition reduces to the classical expression for arc length in Euclidean space for a positive definite metric tensor. Equivalently we obtain the line element or fundamental metric form

$$ds^2 = g_{ab}dx^a dx^b \tag{1.2}$$

where we have omitted the modulus signs. The invariant relativistic quantity (1.2) is a measure of the infinitesimal interval between neighbouring points with coordinates given by x^a and $x^a + dx^a$. Spacetime M has the local property that at every point there exists inertial coordinate systems in which the metric tensor field takes the Lorentzian form

$$[g_{ab}] = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

In special relativity there exist global coordinate systems in which the metric tensor takes the above form. In the spacetime of general relativity we have only local Cartesian coordinate systems where g_{ab} takes the above form approximately. This is a reflection of the presence of curvature and the departure from flatness in the spacetime of general relativity.

1.3 Connections and Curvature

In Euclidean space it is easy to see that the inner product of vectors is constant if the vectors are parallel transported along a curve. It is possible to establish an analogue of this result in a Riemannian manifold. The fundamental theorem of Riemannian geometry states that there exists a unique symmetric connection which preserves inner products under parallel transport. As a consequence of this theorem the metric connection $\mathbf{\Gamma}$ may be expressed in terms of the metric tensor $\mathbf{g}$ and its derivatives:

$$\Gamma^a{}_{bc} = \tfrac{1}{2} g^{ad}(g_{cd,b} + g_{db,c} - g_{bc,d}) \qquad (1.3)$$

where commas denote partial differentiation. The connection coefficients $\Gamma^a{}_{bc}$ are sometimes called the Christoffel symbols of the second kind. The associated metric connection given by

$$\begin{aligned} \Gamma_{abc} &= g_{ad}\Gamma^d{}_{bc} \\ &= \tfrac{1}{2}(g_{ac,b} + g_{ba,c} - g_{bc,a}) \end{aligned}$$

is called the Christoffel symbols of the first kind.

The covariant derivative is the generalisation of the partial derivative in the manifold. It is a tensorial quantity and when applied to a type (r,s) tensor field $T^{a_1a_2\ldots a_r}{}_{b_1b_2\ldots b_s}$ it produces a type $(r,s+1)$ tensor field $T^{a_1a_2\ldots a_r}{}_{b_1b_2\ldots b_s;c}$ where semicolons denote covariant differentiation. For example, the covariant derivative of a covariant vector field V_a is given by

$$V_{a;b} = V_{a,b} - \Gamma^d{}_{ab}V_d$$

Unlike partial differentiation, the covariant derivative is not commutative in general.

As a result of this noncommutation a second covariant differentiation yields the Ricci identity

$$V_{a;bc} - V_{a;cb} = R^{d}{}_{abc} V_d$$

which is nonvanishing for a curved manifold. Here the quantity

$$R^{d}{}_{abc} = \Gamma^{d}{}_{ac,b} - \Gamma^{d}{}_{ab,c} + \Gamma^{e}{}_{ac}\Gamma^{d}{}_{eb} - \Gamma^{e}{}_{ab}\Gamma^{d}{}_{ec} \tag{1.4}$$

is called the Riemann tensor or curvature tensor which is defined in terms of the connection coefficients (1.3) and its derivatives.

We may establish the identity

$$R_{abcd} + R_{acdb} + R_{adbc} = 0$$

directly from the definition of the curvature tensor (1.4). The curvature tensor may also be expressed in terms of the second derivatives of the metric tensor and the Christoffel symbols of the first kind:

$$\begin{aligned} R_{abcd} &= g_{ae} R^{e}{}_{bcd} \\ &= \tfrac{1}{2}(g_{bc,ad} - g_{ac,bd} + g_{ad,bc} - g_{bd,ac}) - g^{ef}(\Gamma_{eac}\Gamma_{fbd} - \Gamma_{ead}\Gamma_{fbc}) \end{aligned}$$

Using this form for R_{abcd} it can be easily established that the curvature tensor (1.4) satisfies the following properties:

$$R_{abcd} = -R_{bacd}$$

$$R_{abcd} = -R_{abdc}$$

$$R_{abcd} = R_{cdab}$$

About any point $P \in M$ we can construct a coordinate system with $\Gamma^a{}_{bc}|_P = 0$ so that (1.4) yields

$$R^a{}_{bcd;e}|_P = \Gamma^a{}_{bd,ce}|_P - \Gamma^a{}_{bc,de}|_P$$

Cyclically permuting c, d and e and adding the resulting equations generates the Bianchi identity

$$R_{abcd;e} + R_{abde;c} + R_{abec;d} = 0$$

at the point P. As P is an arbitrary point in the manifold this result holds everywhere in the manifold M. As a consequence of the above properties we find that the curvature tensor has a maximum of twenty independent components. In general the curvature tensor has $n^2(n^2-1)/12$ independent components in a general n–dimensional manifold.

We can generate only one nonvanishing contraction of the curvature tensor, namely the symmetric Ricci tensor. The components of the Ricci tensor are defined by the contraction

$$\begin{aligned} R_{ab} &= R^c{}_{acb} \\ &= \Gamma^c{}_{ab,c} - \Gamma^c{}_{ac,b} + \Gamma^d{}_{ab}\Gamma^c{}_{cd} - \Gamma^d{}_{ac}\Gamma^c{}_{bd} \end{aligned} \qquad (1.5)$$

A contraction of the Ricci tensor (1.5) yields the scalar quantity

$$\begin{aligned} R &= g^{ab} R_{ab} \\ &= R^a{}_a \end{aligned} \qquad (1.6)$$

called the Ricci scalar or curvature scalar. The symmetric Einstein tensor $\mathbf{G}$ is defined, in terms of the Ricci tensor (1.5) and the Ricci scalar (1.6), by

$$G_{ab} = R_{ab} - \tfrac{1}{2}Rg_{ab} \tag{1.7}$$

The vanishing of the divergence of the Einstein tensor (1.7) necessarily follows by construction of **G**. This result is called the Bianchi identity

$$G^{ab}{}_{;b} = 0$$

and generates the conservation laws through the Einstein field equations.

1.4 Field Equations

The matter content of the universe may be described by a relativistic fluid which is uniquely represented in general for uncharged matter by the following decomposition of the symmetric energy–momentum tensor

$$T_{ab} = (\mu + p)u_a u_b + pg_{ab} + q_a u_b + q_b u_a + \pi_{ab} \tag{1.8}$$

where u^a is a 4–velocity and $u^a u_a = -1$. The quantity μ is the proper density, p is the isotropic pressure, q_a is the heat flow vector and π_{ab} is the anisotropic stress tensor. In general relativity the energy–momentum tensor (1.8) is taken as the source of the gravitational field. For many applications in cosmology the heat flow and shear are negligible. In this case the energy–momentum tensor takes the perfect fluid form and is given by the simple form

$$T_{ab} = (\mu + p)u_a u_b + pg_{ab} \tag{1.9}$$

From the viewpoint of thermodynamics, the coefficients of thermal conductivity and shear viscosity vanish for the perfect fluid (1.9).

The geometry of spacetime (represented by the Einstein tensor (1.7)) is related to the matter distribution (represented by the energy–momentum tensor (1.9)) via the Einstein field equations which take the form

$$R_{ab} - \tfrac{1}{2}Rg_{ab} = 8\pi G T_{ab} \tag{1.10}$$

where $8\pi G$ is the coupling constant (note that previously we set $c = 1$). All known physical equations describing classical fields of fundamental significance, including the Einstein field equations (1.10), can be derived from a variational principle. To establish this we suppose that the matter distribution T^{ab}, derivable from a covariant Lagrangian L_M, is given by

$$T^{ab} = \frac{1}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\; L_M)}{\delta g_{ab}}$$

where $g = |g_{ab}|$ and δ indicates variational differentiation. The Lagrangian density is then given by

$$\mathcal{L} = (R + 16\pi G L_M)\sqrt{-g}$$

which incorporates the pure gravitational field via the Ricci scalar R. Then it is possible to show that the Einstein field equations follow from the action integral given by

$$\begin{aligned} \delta \int \mathcal{L}\; d^4x &= 0 \\ \delta \int (R + 16\pi G L_M)\sqrt{-g}\; d^4x &= 0 \end{aligned}$$

The variational argument to obtain the Einstein field equations was first provided by Hilbert (1915).

The field equations (1.10) generate a system of ten coupled nonlinear partial differential equations that determine the behaviour of the gravitational field through the metric tensor field $\mathbf{g}$. As a consequence of the Bianchi identity $G^{ab}{}_{;b} = 0$ the ten field equations are not all independent. Also the Bianchi identity implies the conservation equations

$$T^{ab}{}_{;b} = 0 \tag{1.11}$$

A region of spacetime in which

$$T^{ab} = 0$$

is called empty; such a region is devoid not only of matter but of radiative energy and momentum. In the classical Newtonian limit, when the gravitational field is weak and varying slowly with time, the Einstein field equations (1.10) reduce to Poisson's equation

$$\nabla^2 V = 4\pi G\mu \quad .$$

where V is the Newtonian potential and μ is the matter density.

The Einstein field equations (1.10) do not admit a stable solution describing a static closed universe. To overcome this "problem" Einstein introduced the cosmological constant Λ by adding the term Λg_{ab} to the Einstein tensor (1.7). The constant Λ has to be small at present times so as not to interfere with the general relativity predictions for the solar system. With the addition of a cosmological constant Λ the Einstein field equations become

$$G_{ab} + \Lambda g_{ab} = 8\pi G T_{ab}$$

For a cosmological term Λ the above equations are implied by adapting the action integral as follows

$$\delta \int (R - 2\Lambda + 16\pi G L_M)\sqrt{-g}\, d^4x = 0$$

It was only later that Einstein realised that this addition was not necessary when Hubble discovered the expansion of the universe.

For certain applications we need to modify the Einstein field equations (1.10) for the more general case where the cosmological constant Λ and the gravitational constant G are dependent on the spacetime coordinates x^a. The modified field equations in this relativistic theory of gravitation are then given by

$$G_{ab} + \Lambda g_{ab} = 8\pi G T_{ab} \tag{1.12}$$

where in general

$$\Lambda = \Lambda(x^a) \qquad \text{and} \qquad G = G(x^a)$$

In fact for our applications in later chapters it is sufficient to suppose that the cosmological constant Λ and the gravitational constant G depend only on the timelike coordinate t. We may interpret the quantity $-\Lambda g_{ab}$ as that part of the energy–momentum tensor associated with the vacuum. It is speculated that Λ has a significant role in the early universe. Observations suggest that the present–day value of Λ is small (Misner *et al* 1973). Models in which the cosmological constant is variable include the scalar–tensor theories of Lau (1985), Lau and Prokhovnik (1986) and Maharaj and Beesham (1988). The first theory of gravity in which G decreases with time was suggested by Dirac (1937). Abdel–Rahman (1990), amongst others, suggests that the gravitational constant G may be increasing in time, at least in certain regions of spacetime. Other theories incorporating a variable gravitational constant G include the Hoyle–Narlikar theory (1971, 1972) and the Brans–Dicke theory (1961). However, the reader should note that, in addition to experimental limits from radar ranging of planets and lunar occultation studies, the helium synthesis analysis of Barrow (1978) strongly constrains the temporal evolution of the gravitational constant G. We obtain the analogue of the conservation equation (1.11) for the modified field equations (1.12)

$$8\pi (G T^{ab})_{;b} = \Lambda_{;b} g^{ab} \tag{1.13}$$

by utilising the Bianchi identity. For more information on the Einstein field equations the reader is referred to Misner *et al* (1973), Stephani (1990) and Wald (1984).

2 Robertson–Walker Models

2.1 Introduction

In this chapter the standard cosmology based on the Robertson–Walker models is reviewed. We present the observational information provided by astronomers and astrophysicists and relate these to the theory. The basic feature of the geometry of the standard model is that it is isotropic and spatially homogeneous. Thus all physical laws and geometrical properties are identical at all points of the spacetime manifold. The Robertson–Walker models accurately describe the dynamics of a homogeneous and isotropic universe. The dynamics of a Robertson–Walker universe are studied by applying Einstein's field equations to a matter content in the form of a homogeneous ideal fluid. The spacetime geometry is described and the various forms of the line element are given in §2.2 for the Robertson–Walker metric. We calculate the nonvanishing components of the connection coefficients, the Ricci tensor, the Ricci scalar and the Einstein tensor. In §2.3 we present the field equations for the classical Einstein field equations with $\Lambda = 0$. We also present the generalised Einstein field equations for variable gravitational constant G and cosmological constant Λ. In particular the general features of the Friedmann models are derived and discussed in §2.4. Also in this section we define the mass density of the present universe, the critical density, the Hubble constant and the deceleration parameter. We relate theoretical predictions for these quantities with observation. We also briefly discuss the implications of the microwave background radiation and the gravitational redshift for the study of modern cosmology. The age of the universe is analysed from the viewpoint of radioactive dating of meteorites. Some modern developments in research are briefly introduced.

2.2 Spacetime Geometry

The Robertson–Walker models are often used as a realistic description of the evolution of the universe in cosmology. In these models the matter distribution is spatially homogeneous and isotropic and has the perfect fluid form (1.9). In standard coordinates $(x^a) = (t, r, \theta, \phi)$ the Robertson–Walker line element has the form

$$ds^2 = -dt^2 + S^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right] \tag{2.1}$$

where $S(t)$ is the cosmic scale factor. Without loss of generality, the constant k takes on only three values: 0, 1 or -1. The constant k is related to the spatial geometry of a 3–dimensional manifold generated by $t =$ constant. For $k = 0$ the spatial geometry is flat, but for $k = 1$ or -1 it is curved. For constant positive curvature $k = 1$ the space is closed (it has finite volume). For constant negative curvature $k = -1$ the space is open (it has infinite volume). The scale factor $S(t)$ operates on the spatial part of (2.1) and determines the expansion or contraction of the universe. Note that $k = 0$ and $k = -1$ give spaces which continually expand, while $k = 1$ gives a space which expands to a maximum value, and then contracts. The Robertson–Walker spacetimes are the standard cosmological models and are consistent with observational results.

Sometimes other equivalent forms of the Robertson–Walker metric are used which we briefly describe here. As a consequence of the cosmological principle the 3–dimensional manifold generated by t = constant is a space of maximal symmetry. This implies that in isotropic coordinates we may transform the line element (2.1) to the expression

$$ds^2 = -dt^2 + \frac{S^2(t)}{\left(1 + \frac{1}{4}kr^2\right)^2}\left[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right]$$

Another possibility arises by introduction of a coordinate system that picks out a point in the universe as the origin of the coordinate system. Then the Robertson–Walker line element is given by

$$ds^2 = -dt^2 + S^2(t)\left[d\chi^2 + f^2(\chi)(d\theta^2 + \sin^2\theta d\phi^2)\right]$$

with the three possibilities

$$f(\chi) = \begin{cases} \chi & \text{for} \quad k = 0 \\ \sin\chi & \text{for} \quad k = 1 \\ \sinh\chi & \text{for} \quad k = -1 \end{cases}$$

for the function $f(\chi)$. Here the coordinate χ is related to the radial displacement of the preferred point.

For the line element (2.1) the nonvanishing connection coefficients (1.3) are given by

$$\Gamma^0{}_{11} = S\dot{S}/(1 - kr^2) \qquad \Gamma^0{}_{22} = S\dot{S}r^2$$

$$\Gamma^0{}_{33} = \sin^2\theta\,\Gamma^0{}_{22} \qquad \Gamma^1{}_{01} = \dot{S}/S$$

$$\begin{aligned}
\Gamma^1{}_{11} &= kr/(1-kr^2) & \Gamma^1{}_{22} &= -r(1-kr^2) \\
\Gamma^1{}_{33} &= \sin^2\theta\, \Gamma^1{}_{22} & \Gamma^2{}_{02} &= \dot{S}/S \\
\Gamma^2{}_{12} &= 1/r & \Gamma^2{}_{33} &= -\tfrac{1}{2}\sin 2\theta \\
\Gamma^3{}_{03} &= \dot{S}/S & \Gamma^3{}_{13} &= 1/r \\
\Gamma^3{}_{23} &= \cot\theta
\end{aligned} \tag{2.2}$$

In the above dots denote differentiation with respect to the time coordinate t. With the connection coefficients (2.2) we determine that the components of the Ricci tensor (1.5) for the metric (2.1) are

$$\begin{aligned}
R_{00} &= -3\ddot{S}/S \\
R_{11} &= (S\ddot{S} + 2\,\dot{S}^2 + 2k)/(1-kr^2) \\
R_{22} &= (S\ddot{S} + 2\,\dot{S}^2 + 2k)r^2 \\
R_{33} &= \sin^2\theta\, R_{22} \\
R_{ab} &= 0, \quad a \neq b
\end{aligned} \tag{2.3}$$

From the components (2.3) of the Ricci tensor R_{ab} we directly obtain the curvature scalar

$$R = 6(S\ddot{S} + \dot{S}^2 + k)/S^2 \tag{2.4}$$

We calculate the following components of the Einstein tensor (1.7) for the Robertson–Walker line element (2.1):

$$G_{00} = 3(\dot{S}^2 + k)/S^2$$

$$G_{11} = -(2S\ddot{S} + \dot{S}^2 + k)/(1 - kr^2)$$

$$G_{22} = -(2S\ddot{S} + \dot{S}^2 + k)r^2 \qquad (2.5)$$

$$G_{33} = \sin^2\theta \, G_{22}$$

$$G_{ab} = 0, \quad a \neq b$$

by utilising (2.3) and (2.4). Thus we completely describe the spacetime curvature by equations (2.2)–(2.5).

2.3 Field Equations

We first consider the classical Einstein field equations with cosmological constant $\Lambda = 0$ and a constant gravitational constant G, in which case the equations (1.10) are applicable. Consider the comoving fluid 4–velocity

$$u^a = \delta^a_0, \qquad u^a u_a = -1$$

and a perfect fluid energy–momentum tensor. Then from the matter tensor (1.9) and the Einstein tensor components (2.5), we find that the Einstein field equations

(1.10) reduce to the system

$$\frac{3}{S^2}(\dot{S}^2 + k) = 8\pi G\mu \tag{2.6}$$

$$2\frac{\ddot{S}}{S} + \frac{(\dot{S}^2 + k)}{S^2} = -8\pi G p \tag{2.7}$$

The field equations (2.6)–(2.7) are a system of two equations with the three unknowns S, μ and p. Equation (2.6) is called the Friedmann equation which does not contain the pressure p. Differentiating the Friedmann equation (2.6) with respect to the time coordinate t we obtain

$$6\frac{\dot{S}\ddot{S}}{S^2} - 16\pi G\frac{\dot{S}}{S}\mu = 8\pi G\dot{\mu}$$

Elimination of $\ddot{S}$ from this result and equation (2.7) yields the continuity equation

$$\dot{\mu} + 3\frac{\dot{S}}{S}(\mu + p) = 0 \tag{2.8}$$

which is a first order equation containing the pressure. We consider solutions to the classical Einstein field equations in §3.2 and §4.2 by assuming a form for the Hubble parameter.

Now we consider the more general case of variable cosmological constant Λ and gravitational constant G. As the Robertson–Walker spacetimes are homogeneous and isotropic we take

$$\Lambda = \Lambda(t) \quad \text{and} \quad G = G(t)$$

In this case the Einstein field equations (1.12) become

$$\frac{3}{S^2}(\dot{S}^2 + k) = 8\pi G\mu + \Lambda \tag{2.9}$$

$$2\frac{\ddot{S}}{S} + \frac{(\dot{S}^2 + k)}{S^2} = -8\pi G p + \Lambda \tag{2.10}$$

The system (2.9)–(2.10) is a system of two equations with five variables S, μ, p, Λ and G. This system of equations permits a wider class of solutions as both Λ and G are variable unlike the above case of the classical Einstein field equations. We present solutions to this class of equations with variable Λ and G in §3.3 and §3.4. In addition a Bianchi I spacetime and a scalar–tensor theory with variable Λ and G is analysed in chapter 4. Equation (2.9) is called the generalised Friedmann equation and reduces to (2.6) when $\Lambda = 0$ and G is constant. Differentiating the generalised Friedmann equation (2.9) with respect to the time coordinate t we obtain

$$6\frac{\dot{S}\ddot{S}}{S^2} - 16\pi G\frac{\dot{S}}{S}\mu - 2\Lambda\frac{\dot{S}}{S} = 8\pi\dot{G}\mu + 8\pi G\dot{\mu} + \dot{\Lambda}$$

Elimination of $\ddot{S}$ from this result and equation (2.10) gives the generalised continuity equation

$$\dot{\mu} + 3\frac{\dot{S}}{S}(\mu + p) + \frac{\dot{G}}{G}\mu + \frac{\dot{\Lambda}}{8\pi G} = 0 \tag{2.11}$$

This reduces to the conventional continuity equation (2.8) when $\Lambda = 0$ and G is constant. In an attempt to obtain solutions to the field equations we assume (as is often done) that the conservation law (1.11), $T^{ab}{}_{;b} = 0$, also holds. Then we have that equation (2.11) implies the two relationships

$$\dot{\mu} + 3\frac{\dot{S}}{S}(\mu + p) = 0 \tag{2.12}$$

$$8\pi\mu\dot{G} + \dot{\Lambda} = 0 \tag{2.13}$$

which facilitate the solution of the field equations. The result (2.12) is just the conventional continuity equation (2.8) and holds whenever (1.11) is applicable. Equation (2.13) simply relates G and Λ and does not explicitly contain the scale factor S.

2.4 Cosmology

There exist many solutions to the equations (2.6)-(2.7) in the literature for various equations of state. In this section we will analyse only the dust solutions with

$$\mu \neq 0, \qquad p = 0$$

For many applications in cosmology the pressure p is negligible. In this case the Einstein field equations may be completely integrated and the resulting solutions are called the Friedmann models. The continuity equation (2.8) with $p = 0$ is integrated to yield

$$\mu S^3 = \mu_0 {S_0}^3$$

where $S_0 = S(t_0)$ and $\mu_0 = \mu(t_0)$ denote present day values of S and μ, and t_0 is the present time. It is convenient to introduce the positive constant A given by

$$A^2 = \frac{8\pi G}{3}\mu_0 {S_0}^3$$

in terms of μ_0 and S_0. We list the three cases of the Friedmann models that arise.

(a) $k = 0$: In this simple case the Friedmann equation (2.6) gives the power–law solution

$$S = \left(\frac{3}{2}A\right)^{2/3} t^{2/3}$$

(b) $k = 1$: Here we set $S = A^2 \sin^2 \psi$ in equation (2.6) to obtain the parametric solution

$$\begin{aligned} S &= \frac{1}{2}A^2(1 - \cos 2\psi) \\ t &= \frac{1}{2}A^2(2\psi - \sin 2\psi) \end{aligned}$$

(c) $k = -1$: The Friedmann equation (2.6) is integrated by the substitution $S = A^2 \sinh^2 \psi$ to yield

$$
\begin{aligned}
S &= \frac{1}{2} A^2 (\cosh 2\psi - 1) \\
t &= \frac{1}{2} A^2 (\sinh 2\psi - 2\psi)
\end{aligned}
$$

We now briefly describe the behaviour of S for each of the above cases. For $k = 0$ we see that $S \propto t^{2/3}$ and the universe continually expands at a decreasing rate because $\dot{S} \to 0$ as $t \to \infty$. This spacetime is also called the Einstein–de Sitter model. For $k = 1$ the behaviour of S is determined parametrically. In this case the universe expands until a maximum value of S is attained and then the universe contracts. The graph of S is a cycloid so that this spacetime is both spatially and temporally closed. For $k = -1$ we observe that the behaviour of S is also parametrically determined. The universe continually expands and in the limit $\dot{S} \to 1$ as $t \to \infty$. The question that continues to attract attention is whether the real universe is in fact open or closed.

The Hubble constant $H(t)$ is defined by

$$
H(t) = \frac{\dot{S}(t)}{S(t)} \tag{2.14}
$$

and we denote its present day value by $H_0 \equiv H(t_0)$. We have previously set the present day value of the density parameter to be $\mu_0 \equiv \mu(t_0)$. Then we establish the following relationship

$$
\frac{k}{{S_0}^2} = \frac{8\pi G}{3} \left(\mu_0 - \frac{3}{8\pi G} {H_0}^2 \right)
$$

relating k to μ_0 and H_0. The critical density μ_c defined by

$$
\mu_c = \frac{3}{8\pi G} {H_0}^2
$$

depends on the present day value of H. Thus the present day density μ_0 and the critical density μ_c determine the value of k:

$$\begin{cases} k > 0 \\ k = 0 \\ k < 0 \end{cases} \quad \text{if} \quad \begin{cases} \mu_0 > \mu_c \\ \mu_0 = \mu_c \\ \mu_0 < \mu_c \end{cases}$$

The deceleration parameter $q(t)$ is given by the formula

$$q(t) = -\frac{S(t)\ddot{S}(t)}{\dot{S}^2(t)} \tag{2.15}$$

and its present day value is denoted by $q_0 \equiv q(t_0)$. From the above we can establish that

$$\begin{aligned} q_0 &= \frac{4\pi G}{3}\frac{\mu_0}{H_0{}^2} \\ &= \frac{1}{2}\left(\frac{\mu_0}{\mu_c}\right) \end{aligned}$$

relating the deceleration parameter q_0, the critical density μ_c and the present day density μ_0. We are now in a position to discuss the dynamics of the models using the observable parameters μ_0, μ_c and q_0 for the Friedmann models. For the flat model $k = 0$ we have the simplest universe with

$$\mu_0 = \mu_c \quad \Longrightarrow \quad q_0 = \frac{1}{2}$$

For the closed $k = 1$ model we obtain

$$\mu_0 > \mu_c \quad \Longrightarrow \quad q_0 > \frac{1}{2}$$

For the open $k = -1$ model we have that

$$\mu_0 < \mu_c \qquad \Longrightarrow \qquad 0 < q_0 < \frac{1}{2}$$

When $k = 0, -1$ the universe is continually expanding and for $k = 1$ the universe contracts after an initial expansion. In all three cases there is an initial singularity where the spacetime curvature diverges. This physically corresponds to a highly compact and hot phase of the universe. A full description of the physics at this stage of development requires a consistent theory of quantum gravity which would require major modifications in current theories of gravity and quantum mechanics. The formulation of such a quantum theory of gravity is a major unsolved problem. To determine which of the above models is an accurate description of the physical universe depends on the observed values of the cosmological parameters μ_0, μ_c and H_0.

Astronomical observations for H_0, q_0 and μ_0 have been given by Misner *et al* (1973), Stephani (1990), Weinberg (1972) and Will (1971). Recent estimates of H_0 give the approximate value

$$H_0{}^{-1} \approx (13 \times 10^9) \text{ yrs}$$

with possibly a wide margin of observational uncertainty. Substituting this value in $\mu_c = 3H_0{}^2/8\pi G$ gives the following value of the critical density

$$\mu_c \approx 1.3 \times 10^{-25} \ \text{kg m}^{-3}$$

Estimates for q_0 suggest that $q_0 \approx 1$ so that the universe is closed. The value of $q_0 \approx 1$ and the formula $q_0 = \mu_0/(2\mu_c)$ implies that

$$\mu_0 \approx 2.6 \times 10^{-25} \text{ kg m}^{-3}$$

This value is larger than the observed density of the universe which is based on the assumption that the matter content is mainly galactic. This leads to the problem of missing matter which is speculated to exist in the form of dark matter. On the other hand if we accepted the value of μ_0 based on masses of galaxies and their distribution we have $q_0 \approx 0.014$ and the universe is open. This conflicts with the above argument that $q_0 \approx 1$ and the closed universe. This does not mean that the Friedmann models are unacceptable: what is needed is more observational data and better interpretation techniques for analysing these results. It is hoped that recent advances in technology applied to experimental and observational aspects of cosmology will lead to an improvement of the estimates.

Two recent observational results that have had a major impact on modern cosmology are the gravitational redshift and the cosmic microwave background. The gravitational redshift was discovered by Hubble (1929). The spectral lines emitted from distant clusters of stars are systematically shifted towards the red of the spectrum. This effect is due to the expansion of the universe on a cosmological scale. The redshift is consistent with the cosmology of general relativity and in particular the concept of an expanding universe. Another fundamental discovery is the existence of the cosmic microwave background radiation made by Penzias and Wilson (1965). The universe is bathed in a microwave background radiation, not originating from stars of galaxies, with a temperature of approximately 3 degrees Kelvin. The existence of this radiation is consistent with the conclusion from the Robertson–Walker models that the universe has expanded from a much denser and hotter phase. The microwave background radiation exhibits a very high degree of isotropy which is consistent with the homogeneous and isotropic Robertson–Walker models. These observations suggest that any deviation from isotropy in the early universe was very small. The Hubble expansion of the universe and the microwave background radiation have been used to discard alternate models that do not exhibit this behaviour.

Observations on the age of the universe are consistent with the Robertson–Walker models. For the ages of meteorites we have that radioactive dating gives an age of approximately 4.5×10^9 years, and for terrestrial matter we obtain an age of 4.5×10^9 years. Other measurements of the lifetime of galactic material have been made using the observed relative abundances of pairs of related radioactive species. The measured current relative abundance implies a galactic lifetime of around 15×10^9 years. This sets a lower limit on the age of the universe. Models of stellar evolution used to make estimates of the lifetime of globular clusters in the galaxy also give an age about 15×10^9 years (Silk 1980).

As mentioned previously the Robertson–Walker models are based on the assumption that the universe is homogeneous and isotropic. This is consistent with the cosmological principle (Stephani 1990). On the largest scale this assumption certainly seems to be the case. Possible mechanisms for the evolution of large structures to yield the present observed distribution of matter are given by Silk (1980). Perhaps we should mention that with the improvement of observational techniques there has been a renewed interest in observational cosmology. Recent results provide a firmer basis for the comparision of observational results with theoretical predictions. We will not discuss the various arguments here except to mention briefly two debates. A problem with the standard Robertson–Walker models is the possibility that the universe may be inhomogeneous on a large scale (Ellis 1984) arising from the observed clustering of galaxies. The inflationary model proposed by Guth (1981) attempts to overcome other shortcomings in the standard model by postulating a period of rapid expansion in the early universe. Finally we should add that much of the present research is involved with bringing together general relativity and quantum mechanics to find a consistent theory of quantum gravity.

3 Robertson–Walker Cosmologies

3.1 Introduction

In this chapter we present various types of Robertson–Walker cosmological models by solving the Einstein field equations for both the cases of vanishing cosmological constant and variable cosmological constant. The scale factor is determined by the law of variation for Hubble's parameter proposed by Berman (1983). The variation of Hubble's law presented is not inconsistent with observations and helps in providing simple functional forms for the scale factor. It is interesting to note that this law yields a constant value for the deceleration parameter. Other forms for the deceleration parameter are being investigated by Beesham (1992). In §3.2 we obtain the general solution for the classical Einstein field equations with vanishing cosmological constant for the specified Hubble law. We briefly discuss some properties of the solutions presented. In earlier literature cosmological models with constant deceleration parameter have been presented by Berman (1983), Berman and Gomide (1988) and others. In §3.3 we consider the Einstein field equations with a perfect fluid and variable gravitational and cosmological constants for the Robertson–Walker metric. With the requirement that the normal conservation of energy momentum holds we find a solution to the field equations for $k = 0$. In this simple case we have the equation of state for a perfect gas. The solutions are characterised by the behaviour $\Lambda \propto t^{-2}$ for the cosmological constant which is consistent with observations which suggest that the cosmological constant has small values in the present universe. Berman (1991) has studied variable cosmological and gravitational theories with our form of the Hubble law. It is claimed that these solutions solve the monopole and horizon problems. In §3.4 we present a number of classes of new solutions for the variable gravitational and cosmological constants. These solutions allow for a wide range of behaviour for the gravitational constant. The ansatz utilised immediately leads to a form for the cosmological constant. Explicit forms for the gravitational constant, cosmological constant, scale factor, energy density and pressure are obtained in each case. The properties of these solutions are briefly discussed. Other forms of solution are possible. However we do not pursue these as they involve solutions in terms of special functions. In the most general case we would need to resort to numerical techniques.

3.2 Classical Einstein Field Equations

In this section we consider the classical Einstein field equations with cosmological constant $\Lambda = 0$ and gravitational constant G constant for the Robertson–Walker metric (2.1). As the two field equations (2.6)–(2.7) involve the three unknowns μ, p and S we need an additional assumption to find a solution. In this section we assume a form for the Hubble constant H, defined by equation (2.14), which is equivalent to supposing that the deceleration parameter q is a constant. This enables us to present the general solution to the field equations (2.6)–(2.7) for the Robertson–Walker line element (2.1). Berman (1983) proposed the following law of variation for Hubble's parameter

$$H = DS^{-m} \tag{3.1}$$

where D and m are constants. This law for H was also utilised by Berman and Gomide (1988). We explicitly provide the details of the arguments as it illustrates the technique utilised in later sections. In this simple case it is easy to interpret the results physically. The above form of H is the simplest possible form that facilitates the solution of the field equations. Also this form of the Hubble constant is not inconsistent with present day observations. From equations (2.14) and (3.1) we obtain the derivative for the scale factor S which takes the form

$$\dot{S} = DS^{-m+1}$$

Differentiating the above equation with respect to the time coordinate t we obtain the second derivative

$$\ddot{S} = -D^2(m-1)S^{-2m+1}$$

On substituting $\dot{S}$ and $\ddot{S}$ into equation (2.15) we find that the deceleration parameter has the simple form

$$q = m - 1 \tag{3.2}$$

which is a constant. We note that the case $m = 1$ is extremely restrictive as then $\dot{S}$ is a constant. As the functional form of the deceleration parameter is specified in equation (3.2) we can explicitly integrate the differential equation

$$\begin{aligned} q &= -\frac{\ddot{S}S}{\dot{S}^2} \\ &= m - 1 \end{aligned}$$

to obtain the scale factor

$$S = \begin{cases} [C + mDt]^{1/m} & \text{for} \quad m \neq 0 \\ Ee^{Dt} & \text{for} \quad m = 0 \end{cases} \tag{3.3}$$

where C and E are constants. Thus our Hubble variation law (3.1) leads to an explicit form for the scale factor. Substituting (3.3) in the Einstein field equation (2.6) we get the energy density

$$\mu = \frac{3}{8\pi G}\left[\frac{D^2}{S^{2m}} + \frac{k}{S^2}\right] \tag{3.4}$$

and from equation (2.7) the pressure

$$p = \frac{1}{8\pi G}\left[\frac{(2m-3)D^2}{S^{2m}} - \frac{k}{S^2}\right] \tag{3.5}$$

Equations (3.3)–(3.5) comprise the general solution to the classical Einstein field equations (2.6)–(2.7) for our Hubble law (3.1).

We do not pursue the physical properties of the above solutions in any detail except to make a few observations. It is possible to invert equation (3.3) to write the time coordinate t as a function of S. Then the present age t_0 of the universe is given by the result

$$t_0 = \begin{cases} {H_0}^{-1}/m - C/(mD) & \text{for} \quad m \neq 0 \\ \ln[S_0/E]^{1/D} & \text{for} \quad m = 0 \end{cases}$$

Also from (3.1) and (3.3) we have the following explicit functional form for Hubble's constant for the Robertson–Walker spacetimes:

$$H = \begin{cases} D/(C + mDt) & \text{for} \quad m \neq 0 \\ D & \text{for} \quad m = 0 \end{cases}$$

The case $m \neq 0$ is consistent with a decreasing Hubble constant H with increasing time t. From equations (3.4) and (3.5) we have the following relationship

$$p = \frac{1}{8\pi G}\left[\frac{2D^2(m-1)}{S^{2m}}\right] - \frac{1}{3}\mu$$

relating the energy density and the pressure. If $k = 0$ then we have the equation of state for an ideal gas

$$p = \alpha\mu$$

where

$$\alpha = \frac{1}{3}(2m-3)$$

is a constant. For a physical equation of state we require

$$0 \leq \alpha \leq 1$$

which restricts m. This ensures that properties of this model are physically reasonable, for example the speed of sound is less than the speed of light. Restrictions are imposed on the parameters in the above solutions if the energy conditions are to be satisfied. In fact under the restrictions imposed by Berman (1983) on his solutions the open Robertson–Walker models are excluded under the law (3.1). It is possible to perform a similar analysis presented above in alternate theories of gravity. For an application to the Pryce–Hoyle theory and the Brans–Dicke theory see Berman and Gomide (1988). A variety of models with interesting behaviour, for the scale factor specified above, is admitted for various values of the parameters. In addition to the scale factor note that the scalar fields in these alternate theories may be completely specified.

3.3 Variable Gravitational and Cosmological Constants

In this section we consider the generalised Einstein field equations (2.9)–(2.10) with variable gravitational constant $G(t)$ and variable cosmological constant $\Lambda(t)$ for the Robertson–Walker metric (2.1). An elementary solution found by Berman (1991) for the $k = 0$ Robertson–Walker spacetime is discussed in this section. Even though this solution is simple it displays interesting features that should characterise the class of solutions with variable cosmological constant and gravitational constant. A class of new solutions for variable $\Lambda(t)$ and $G(t)$ is presented in §3.4. We again assume that the variation of the Hubble parameter is given by equation (3.1). The Hubble parameter of the form

$$H = DS^{-m}$$

was assumed by Berman (1991). We consider only the case $m \neq 0$; consequently we have that

$$S = (C + mDt)^{1/m} \tag{3.6}$$

is the only form of the scale factor that need be considered. For $k = 0$ we see that the Einstein field equation (2.9) becomes

$$3\frac{\dot{S}^2}{S^2} = 8\pi G\mu + \Lambda$$

or equivalently we have

$$3D^2 = 8\pi G\mu S^{2m} + \Lambda S^{2m}$$

by using the Hubble law (3.1). By inspection of the above equation we observe that the following class of solutions is admitted

$$G\mu = AS^{-2m}$$

$$\Lambda = BS^{-2m} \tag{3.7}$$

where A and B are constants. The constants A, B and D are subject to the following condition

$$3D^2 = 8\pi A + B$$

This simple class of solutions is possible because $k = 0$. If $k \neq 0$ then the Einstein field equation (2.9) becomes

$$3D^2 + 3kS^{2m-2} = 8\pi G\mu S^{2m} + \Lambda S^{2m}$$

and the above class of solutions (3.7) has to be modified. This is done in the next section. On substituting

$$\mu = AS^{-2m}G^{-1}$$

into equation (2.13) we obtain the gravitational constant

$$G = \beta S^{mB/(4\pi A)} \tag{3.8}$$

where β is a positive constant of integration. Substitution of the value of G from equation (3.8) into $G\mu = AS^{-2m}$ yields the energy density

$$\mu = \frac{A}{\beta} S^{-2m-mB/(4\pi A)} \tag{3.9}$$

Then substituting the values of μ and S into the continuity equation (2.12) we obtain the pressure

$$p = \frac{A}{3\beta}\left[m\left(2 + \frac{B}{4\pi A}\right) - 3\right] S^{-2m-mB/(4\pi A)} \tag{3.10}$$

We note that the equation (16) given by Berman (1991) corresponding to our equation (3.10) has an incorrect coefficient on the right hand side. Equations (3.6)-(3.10) comprise the general solution to the generalised Einstein field equations (2.9)-(2.10) with variable cosmological and gravitational constants for the Hubble law (3.1).

It is interesting to observe that from equations (3.9) and (3.10) we obtain the equation of state for an ideal gas given by

$$p = \alpha\mu$$

where we have set the constant α to be

$$\alpha = \frac{1}{3}\left[m\left(2+\frac{B}{4\pi A}\right)-3\right]$$

This is similar to the situation in §3.2 for the classical Einstein field equations. If we suppose that $D > 0$ then we have the value

$$D = \left[\frac{8\pi A + B}{3}\right]^{1/2}$$

Furthermore if we impose the positive energy condition $\mu \geq 0$ then we have $A > 0$. It is possible to avoid the horizon and monopole problem with the above variable $G(t)$ and $\Lambda(t)$ solutions as suggested by Berman (1991). Other models considered which also have the relationship

$$\Lambda \propto \frac{1}{t^2}$$

include Berman (1990a), Berman and Som (1990), Berman *et al* (1989) and Bertolami (1986a,b). This form of Λ is physically reasonable as observations suggest that Λ is very small in the present universe. A decreasing functional form permits Λ to be large in the early universe. A partial list of cosmological models in which the gravitational constant G is a decreasing function of time are contained in Gron (1986), Hellings *et al* (1983), Rowan–Robinson (1981), Shapiro *et al* (1971) and Van Flandern (1981). The possibility of the G increasing with time, at least in some stages of the development of the universe, has been investigated by Abdel–Rahman (1990), Chow (1981), Levitt (1980) and Milne (1935).

3.4 New Variable Solutions

In §3.3 we analysed the details of the solution by Berman (1991) for $k = 0$. A different form of solution is required for $k \neq 0$ as the generalised Einstein field equations are more complicated. In this section we present a number of classes of new solutions for all cases of $k : 0, 1, -1$ for variable cosmological constant Λ and variable gravitational constant G. As far as we are aware the various classes of solutions presented in this section for the Hubble law (3.1) have not appeared in the literature. These solutions cover both the cases of $m = 0$ and $m \neq 0$ for the scale factor S:

$$S = \begin{cases} [C + mDt]^{1/m} & \text{for} \quad m \neq 0 \\ \\ Ee^{Dt} & \text{for} \quad m = 0 \end{cases}$$

Our new classes of solutions, extending those of Berman (1991), are found by assuming an ansatz that immediately leads to a solution of the Einstein field equation (2.9). The details of each class of solutions found are given together with the form of the scale factor S, the variable cosmological constant Λ, the variable gravitational constant G, the energy density μ and the pressure p.

We write the Einstein field equation (2.9) in the form

$$3\frac{\dot{S}^2}{S^2} = 8\pi G\mu + \Lambda - \frac{3k}{S^2} \tag{3.11}$$

or equivalently

$$3H^2 = 8\pi G\mu + \Lambda - \frac{3k}{S^2}$$

for $k = 0, 1, -1$. Again we utilise the variation of the Hubble parameter given by equation (3.1):

$$H = DS^{-m}$$

which is equivalent to assuming that the deceleration parameter

$$q = m - 1$$

is a constant. As the two field equations (2.9)–(2.10) involve five variables we need to adopt an ansatz to find a solution. The ansatz that we adopt essentially solves equation (3.11) and immediately provides a form for the cosmological constant Λ (the scale factor S is given by equation (3.3)). In an attempt to solve the Einstein field equations (2.9)–(2.10) we adopt the ansatz

$$\frac{3D^2}{S^{2m}} - \Lambda = K \tag{3.12}$$

$$8\pi G\mu - \frac{3k}{S^2} = K \tag{3.13}$$

where K is a constant. This ansatz has the advantage of providing many more classes of solutions than the elementary example given in §3.3. With the equations (3.12)–(3.13) we observe that the Einstein field equation (3.11) is identically satisfied. From equation (3.12) we have that the cosmological constant takes the following form for all classes of solution:

$$\Lambda = \frac{3D^2}{S^{2m}} - K \tag{3.14}$$

From equation (3.13) we can express the energy density μ in terms of the gravitational constant G and the scale factor S:

$$\mu = \frac{1}{8\pi G}\left[\frac{3k}{S^2} + K\right] \tag{3.15}$$

To obtain the pressure p we utilise the continuity equation

$$\dot{\mu} + 3\frac{\dot{S}}{S}(\mu + p) = 0$$

with the above forms of the scale factor S and the energy density μ. On substituting (3.15) and the derivative of equation (3.14) with respect to the time coordinate t into equation (2.13) we obtain the differential equation

$$\frac{\dot{G}}{G} = 6mD^2 \frac{\dot{S}}{S^{2m+1}} \frac{1}{3k/S^2 + K} \tag{3.16}$$

relating G to S. Thus as the scale factor S is specified by our assumed form for the Hubble parameter, the gravitational constant G is known in principle. The ansatz (3.12)–(3.13) enables us to integrate all the Einstein field equations for a number of values of m, k and K.

In the remainder of this section we present the various classes of solutions to the Einstein field equations for each of the cases considered. We consider seven cases in the sequence outlined below:

(a) $m = 0$

(b) $m \neq 0, K = 0, k \neq 0$

(c) $m \neq 0, K \neq 0, k = 0$

(d) $m = 2, K \neq 0, k \neq 0$

(e) $m = -2, K \neq 0, k \neq 0$

(f) $m = \frac{1}{2}, K \neq 0, k \neq 0$

(g) $m = \frac{2}{3}, K \neq 0, k \neq 0$

There are further classes of solution possible for other values of m. However the integration process becomes extremely complicated and here we present only the simple cases that follow easily from the integration process.

(a) $\underline{m = 0}$:

For $m = 0$ equation (3.3) gives the following form for the scale factor

$$S = Ee^{Dt} \tag{3.17}$$

The cosmological constant becomes

$$\Lambda = 3D^2 - K \tag{3.18}$$

by equation (3.14). With $m = 0$ we immediately note that equation (3.16) yields the gravitational constant

$$G = A \tag{3.19}$$

where A is a constant. With the help of equation (3.19) we find that equation (3.15) gives the density

$$\mu = \frac{1}{8\pi A}\left[\frac{3k}{S^2} + K\right] \tag{3.20}$$

Substituting (3.17) and (3.20) into the continuity equation (2.12) we obtain the pressure

$$p = -\frac{1}{8\pi A}\left[\frac{k}{S^2} + K\right] \tag{3.21}$$

The solutions of the Einstein field equations (2.9)-(2.10) for $m = 0$ are given by the set of equations (3.17)-(3.21).

The essential characteristic of these solutions is that Λ and G are strictly constants because of the restriction $m = 0$. The cosmological constant Λ vanishes when

$$K = 3D^2$$

and is positive for

$$K < 3D^2$$

The other cases considered in this section have Λ and G variable. The scale factor S is exponential in t, so that if $D > 0$ then the universe is exponentially expanding always. Such a model is not a physical description of our present universe but could be applicable in the early universe in the inflationary scenario. On substituting $m = 0$ into equation (3.2) we get the deceleration parameter $q = -1$ for this class of solutions. By comparing equations (3.20) and (3.21) we find that the equation of state is given by

$$p = -\frac{1}{3}\mu - \frac{K}{12\pi A}$$

so that the pressure may be negative for appropriate values of the parameters. For the $k = 0$ Robertson–Walker model we have

$$\begin{aligned} p &= -\frac{K}{8\pi A} \\ &= -\mu \end{aligned}$$

so that both μ and p are constants and have opposite signs to each other.

(b) $\underline{m \neq 0, K = 0, k \neq 0}$:

For $m \neq 0$ equation (3.3) gives the scale factor

$$S = [C + mDt]^{1/m} \tag{3.22}$$

With $K = 0$ the cosmological constant given by equation (3.14) is modified to the form

$$\Lambda = \frac{3D^2}{S^{2m}} \tag{3.23}$$

Equation (3.16) together with $K = 0$ gives the differential equation

$$\frac{\dot{G}}{G} = \frac{2mD^2}{k} \frac{\dot{S}}{S^{2m-1}}$$

which upon integration yields

$$G = \alpha \exp\left\{ \frac{mD^2}{k(1-m)} S^{2-2m} \right\} \tag{3.24}$$

where α is a positive constant of integration. From equation (3.15) the density is given by

$$\mu = \frac{3k}{8\pi\alpha} S^{-2} \exp\left\{ \frac{mD^2}{k(m-1)} S^{2-2m} \right\} \tag{3.25}$$

where we have utilised equation (3.24). Substituting (3.22) and (3.25) into the continuity equation (2.12) yields the pressure

$$p = -\frac{1}{8\pi\alpha} \left[\frac{4mD^3}{S^{3m}} + \frac{3k}{S^2} \right] \exp\left\{ \frac{mD^2}{k(m-1)} S^{2-2m} \right\} \tag{3.26}$$

The solutions of the Einstein field equations (2.9)–(2.10) for $m \neq 0, K = 0, k \neq 0$ are given by the class of equations (3.22)–(3.26).

Both Λ and G depend on the time coordinate t unlike the previous case (a). In contrast to case (a) the scale factor is of the power law type and not exponential. If $m = 1$ then μ, G and p are not defined. Thus we require $m \neq 1$. This case shares the common feature with the remaining cases that G may be increasing in time in certain regions of spacetime. This is similar to the model proposed by Abdel–Rahman (1990). Note that if $m = 1$ then $q = 0$ and this severely restricts the form of the scale factor. The energy density μ is negative if $k = -1$. The relationship between the energy density and the pressure is given by

$$p = -\left[1 + \frac{4mD^3}{3kS^{3m-2}}\right]\mu$$

With $m = \frac{2}{3}$ this relationship becomes

$$p = -\left(1 + \frac{8D^3}{9k}\right)\mu$$

which is the equation of state of an ideal gas. The positivity of the pressure is dependent on the values of D and k.

(c) $\underline{m \neq 0, K \neq 0, k = 0}$:

As for case (b) the condition for $m \neq 0$ implies that the scale factor is given by

$$S = [C + mDt]^{1/m} \tag{3.27}$$

The cosmological constant is of the form

$$\Lambda = \frac{3D^2}{S^{2m}} - K \tag{3.28}$$

given by (3.14). With $k = 0$, equation (3.16) with $K \neq 0$ becomes the ordinary differential equation

$$\frac{\dot{G}}{G} = \frac{6mD^2}{K}\frac{\dot{S}}{S^{2m+1}}$$

which gives upon integration

$$G = \alpha \exp\left\{\frac{3D^2}{KS^{2m}}\right\} \tag{3.29}$$

where α is a positive constant of integration. On substituting (3.29) into equation (3.15) gives the density

$$\mu = \frac{K}{8\pi\alpha}\exp\left\{-\frac{3D^2}{KS^{2m}}\right\} \tag{3.30}$$

On substituting (3.27) and (3.30) into the continuity equation (2.12) we obtain the pressure

$$p = -\frac{1}{8\pi\alpha}\left[\frac{2mD^2}{S^{2m}} + K\right]\exp\left\{-\frac{3D^2}{KS^{2m}}\right\} \tag{3.31}$$

The solutions of the Einstein field equations (2.9)–(2.10) for $m \neq 0, K \neq 0, k = 0$ are given by the class of equations (3.27)–(3.31).

This case differs from the previous case in that the constant K in Λ is nonvanishing. For appropriate values of the parameters the gravitational constant may be an increasing function with time. The relationship between μ and p is given by

$$p = \left[-\frac{2mD^2}{KS^{2m}} - 1\right]\mu$$

In this case it is not possible to have an equation of state for an ideal gas as $m \neq 0$ by assumption. However if $m = \frac{1}{2}$ we obtain a simple relationship relating the energy density to the pressure

$$p = \left[-\frac{D^2}{K}S^{-1} - 1\right]\mu$$

from the above. This has the asymptotic behaviour that as t increases

$$p \approx -\mu$$

so that the pressure becomes negative.

(d) $\underline{m = 2, K \neq 0, k \neq 0}$:

For $m = 2$ the scale factor given by equation (3.3) takes the form

$$S = [C + 2Dt]^{1/2} \tag{3.32}$$

With $m = 2$ the cosmological constant in equation (3.14) is given by

$$\Lambda = \frac{3D^2}{S^4} - K \tag{3.33}$$

For $m = 2$ the differential equation (3.16) takes the form

$$\frac{\dot{G}}{G} = \frac{12D^2\dot{S}/S^5}{3k/S^2 + K}$$

To integrate this equation we make the substitution

$$u = C + 2Dt$$

so that we get

$$\frac{dG}{G} = \frac{6D^2}{3ku^2 + Ku^3}\,du$$

$$= 6D^2\left[\frac{1}{3ku^2} - \frac{K}{3k^2u} + \frac{K^2}{3k^2(3k + Ku)}\right]du$$

This is integrated to yield

$$\ln G \propto \ln\left[\frac{(3ku^{-1} + K)^{K/k}}{\exp\{u^{-1}\}}\right]^{2D^2/k}$$

or equivalently we have for the gravitational constant

$$G = \alpha\left[\frac{(3kS^{-2} + K)^{K/k}}{\exp\{S^{-2}\}}\right]^{2D^2/k} \tag{3.34}$$

where α is a positive constant of integration. Then from equation (3.15) the density is given by the formula

$$\mu = \frac{1}{8\pi\alpha}\left[3kS^{-2} + K\right]\left[\frac{\exp\{S^{-2}\}}{(3kS^{-2} + K)^{K/k}}\right]^{2D^2/k} \tag{3.35}$$

where we have used equation (3.34). Substitution of (3.32) and (3.35) into the continuity equation (2.12) yields the pressure

$$p = -\frac{1}{8\pi\alpha}\left[3kS^{-2}+K\right]\left[\frac{\exp\{S^{-2}\}}{(3kS^{-2}+K)^{K/k}}\right]^{2D^2/k} \times$$

$$\left[\frac{4D^2S^{-4}(2K-3kS^{-2})}{k(3kS^{-2}+K)}+1\right]+\frac{kS^{-2}}{4\pi\alpha}\left[\frac{\exp\{S^{-2}\}}{(3kS^{-2}+K)^{K/k}}\right]^{2D^2/k} \tag{3.36}$$

The solutions of the Einstein field equations (2.9)–(2.10) for $m = 2, K \neq 0, k \neq 0$ are given by the class of equations (3.32)–(3.36).

Unlike the cases considered thus far we have a specific value for m. This gives a value $q = 1$ for the deceleration parameter. The form of the solutions presented are more complicated than the previous cases. Depending on the parameters in equations (3.35) and (3.36) we may have both $\mu \geq 0$ and $p \geq 0$. A wide range of behaviour is possible for the gravitational constant. The relationship between the energy density and the pressure is given by

$$p = -\frac{1}{k\left(3kS^{-2}+K\right)}\left[4D^2S^{-4}\left(2K-3kS^{-2}\right)+k\left(kS^{-2}+K\right)\right]\mu$$

which differs substantially from the equation of state for an ideal gas.

(e) $\underline{m = -2, K \neq 0, k \neq 0}$:

For $m = -2$ equation (3.3) gives the scale factor

$$S = \frac{1}{\sqrt{C-2Dt}} \tag{3.37}$$

With $m = -2$ the cosmological constant which is given by equation (3.14) assumes the form

$$\Lambda = \frac{3D^2}{S^{-4}} - K \tag{3.38}$$

For $m = -2$ the differential equation (3.16) takes the form

$$\frac{\dot{G}}{G} = -\frac{12D^2\dot{S}/S^{-3}}{3k/S^2 + K}$$

To integrate this equation we make the substitution

$$u = C - 2Dt$$

so that we get

$$\begin{aligned}\frac{dG}{G} &= \frac{6D^2}{3ku^4 + Ku^3}\,du \\ &= 6D^2\left[\frac{1}{Ku^3} - \frac{3k}{K^2u^2} - \frac{9k^2}{K^3u} + \frac{27k^3}{K^3\,(3ku + K)}\right]du\end{aligned}$$

This equation is integrated to yield

$$\ln G \propto \ln\left[\frac{\exp\{3k/(K^2u) - 1/(2Ku^2)\}}{[u\,(3ku + K)]^{9k^2/K^3}}\right]$$

or equivalently we have for the gravitational constant

$$G = \alpha\,\frac{\exp\{(3k/K^2)S^2 - (1/2K)S^4\}}{[S^{-2}\,(3kS^{-2} + K)]^{9k^2/K^3}} \tag{3.39}$$

where α is a positive constant of integration. From equation (3.15) we find that the energy density is given by the expression

$$\mu = \frac{1}{8\pi\alpha}\left[3kS^{-2} + K\right]\left[S^{-2}(3kS^{-2} + K)\right]^{9k^2/K^3}\exp\left\{\frac{1}{2K}S^4 - \frac{3k}{K^2}S^2\right\} \tag{3.40}$$

where we have used equation (3.39). On substituting (3.37) and (3.40) into the continuity equation (2.12) we obtain the pressure

$$p = \frac{1}{4\pi\alpha}\left[3kS^{-2}+K\right]^{9k^2/K^3+1}\left[\frac{6k^2}{K^3}+\frac{3k-S^6}{3S^2(3kS^{-2}+K)}-\frac{1}{2}\right]\times$$

$$S^{-18k^2/K^3}\exp\left\{\frac{1}{2K}S^4-\frac{3k}{K^2}S^2\right\} \tag{3.41}$$

The solutions of the Einstein field equations (2.9)–(2.10) for $m=-2, K\neq 0, k\neq 0$ are given by the class of equations (3.37)–(3.41).

The value of m is chosen so that the integration process is simplified. The value of $q=-3$ for the deceleration parameter differs from that in case (d). As for (e) the form of the solutions are complicated. Depending on the parameters in equations (3.40) and (3.41) we may have both $\mu\geq 0$ and $p\geq 0$. It is possible to investigate the behaviour of the gravitational constant and the cosmological constant for appropriate choices of the parameters in certain intervals of spacetime. The relationship between the energy density and the pressure is given by

$$p = 2\left[\frac{6k^2}{K^3}+\frac{3k-S^6}{3S^2\left(3kS^{-2}+K\right)}\right]\mu$$

As for case (d) we do not obtain the equation of state for an ideal gas.

(f) $m=\frac{1}{2}, K\neq 0, k\neq 0$:

For $m=\frac{1}{2}$ equation (3.3) gives the scale factor

$$S = \left[C+\frac{1}{2}Dt\right]^2 \tag{3.42}$$

With $m=\frac{1}{2}$ equation (3.14) gives the cosmological constant

$$\Lambda = \frac{3D^2}{S}-K \tag{3.43}$$

For $m = \frac{1}{2}$ the differential equation (3.16) becomes

$$\frac{\dot{G}}{G} = \frac{3D^2\dot{S}/S^{-2}}{3k/S^2 + K}$$

To integrate this equation we use the substitution

$$u = C + \frac{1}{2}Dt$$

which gives the differential equation

$$\frac{dG}{G} = \frac{6D^2u}{3k + Ku^4}\, du$$

This equation is integrated to yield

$$\ln G \propto \sqrt{\frac{3}{kK}}\, D^2 \arctan\left(\sqrt{\frac{K}{3k}}\, u^2\right)$$

We restrict our attention only to the case

$$kK > 0$$

for simplicity. It is also possible to perform the integration process for $kK < 0$; the form of the solution is similar to that presented here. Equivalently we may write for the gravitational constant

$$G = \alpha \exp\left\{\sqrt{\frac{3}{kK}}\, D^2 \arctan\left(\sqrt{\frac{K}{3k}}\, S\right)\right\} \tag{3.44}$$

where α is a positive constant of integration. From equation (3.15) we find that the energy density is given by

$$\mu = \frac{1}{8\pi\alpha}\left[3kS^{-2} + K\right] \exp\left\{-\sqrt{\frac{3}{kK}}\, D^2 \arctan\left(\sqrt{\frac{K}{3k}}\, S\right)\right\} \tag{3.45}$$

where we have used equation (3.44). On substituting equations (3.42) and (3.45) into the continuity equation (2.12) we obtain the pressure

$$p = -\frac{1}{8\pi\alpha}\left[\left(k - D^2 S\right) S^{-2} + K\right] \exp\left\{-\sqrt{\frac{3}{kK}} D^2 \arctan\left(\sqrt{\frac{K}{3k}} S\right)\right\} \qquad (3.46)$$

The solutions of the Einstein field equations (2.9)-(2.10) for $m = \frac{1}{2}, K \neq 0, k \neq 0$ are given by the set of equations (3.42)-(3.46).

The value of m is chosen to show that the integration process may yield solutions that are more complicated than those given previously. For this class of solutions the deceleration parameter has the value $q = -\frac{1}{2}$ as $m = \frac{1}{2}$. Depending on the parameters in equations (3.45) and (3.46) we may have both $\mu \geq 0$ and $p \geq 0$. The gravitational constant has a more complicated behaviour than the cases studied previously. Note that we need to impose the restriction that $kK > 0$ for these solutions to be applicable. The relationship between the energy density and the pressure is given by

$$p = \left[\frac{2k + D^2 S}{3k + KS^2} - 1\right] \mu$$

In this case it is not possible to have an equation of state for an ideal gas. As in case (c) as t increases we have the asymptotic relationship

$$p \approx -\mu$$

so that the pressures may be again negative.

(g) $m = \frac{2}{3}, K \neq 0, k \neq 0$:

For $m = \frac{2}{3}$ equation (3.3) gives the scale factor

$$S = \left[C + \frac{2}{3} D t\right]^{3/2} \qquad (3.47)$$

With $m = \frac{2}{3}$ equation (3.14) gives the cosmological constant

$$\Lambda = \frac{3D^2}{S^{4/3}} - K \tag{3.48}$$

For $m = \frac{2}{3}$ the differential equation (3.16) becomes

$$\frac{\dot{G}}{G} = \frac{4D^2\dot{S}/S^{-7/3}}{3k/S^2 + K}$$

To integrate this equation we make the substitution

$$u = C + \frac{2}{3}Dt$$

From this we obtain the differential equation

$$\frac{dG}{G} = \frac{6D^2}{3k + Ku^3}\,du$$

$$= \frac{6D^2}{K}\left[\frac{1}{3a^2(a+u)} - \frac{u-2a}{3a^2(u^2 - au + a^2)}\right] du$$

where for simplicity we set

$$a^3 = \frac{3k}{K}$$

On integration we obtain

$$\ln G \propto \ln\left[\frac{(u+a)^2}{u^2 - au + a^2}\exp\left\{6\arctan\left(\frac{2u-a}{\sqrt{3}a}\right)\right\}\right]^{D^2/Ka^2}$$

or equivalently we have for the gravitational constant

$$G = \alpha\left[\frac{(S^{2/3}+a)^2}{S^{4/3} - aS^{2/3} + a^2}\exp\left\{6\arctan\left(\frac{2S^{2/3}-a}{\sqrt{3}a}\right)\right\}\right]^{D^2/Ka^2} \tag{3.49}$$

where α is a positive constant of integration. From equation (3.15) we find that the energy density is given by

$$\mu = \frac{1}{8\pi\alpha}\left[3kS^{-2} + K\right]\left[\frac{S^{4/3} - aS^{2/3} + a^2}{(S^{2/3}+a)^2}\right] \times$$

$$\exp\left\{-6\arctan\left(\frac{2S^{2/3}-a}{\sqrt{3}a}\right)\right\}\Bigg]^{D^2/Ka^2} \tag{3.50}$$

where we have used equation (3.49). On substituting equations (3.47) and (3.50) into the continuity equation (2.12) we obtain the pressure

$$p = -\frac{1}{8\pi\alpha}\left[\frac{S^{4/3}-aS^{2/3}+a^2}{\left(S^{2/3}+a\right)^2}\exp\left\{-6\arctan\left(\frac{2S^{2/3}-a}{\sqrt{3}a}\right)\right\}\right]^{D^2/Ka^2}\times$$

$$\left[kS^{-2}+K+\frac{2D^2}{3Ka}\left[3kS^{-2}+K\right]\left[\frac{\left(S^{2/3}+a\right)^2}{S^{4/3}-aS^{2/3}+a^2}\right]\right]\times$$

$$\left[\frac{S^{2/3}-a}{\left(S^{2/3}+a\right)^3}-\frac{4\sqrt{3}\left(S^{4/3}-aS^{2/3}+a^2\right)}{\left(S^{2/3}+a\right)^2\left(3a^2+\left(2S^{2/3}-a\right)^2\right)}\right] \tag{3.51}$$

The solutions of the Einstein field equations (2.9)–(2.10) for $m=\frac{2}{3}, K\neq 0, k\neq 0$ are given by the set of equations (3.47)–(3.51).

The value of m chosen shows that the integration process becomes even more complicated than the previous cases. It is clear that other choices of m will lead to complicated integrals which will have solutions in terms of special functions. The value of the deceleration parameter is $q=-\frac{1}{3}$ with $m=\frac{2}{3}$. The form of the solutions are the most complicated in this case. To plot explicity the behaviour of the gravitational constant against time will be extremely complicated for $m=\frac{2}{3}$. Note that the value of a chosen for the integration process is nonzero since $K\neq 0$ and $k\neq 0$. Depending on the values of the parameters in equations (3.50) and (3.51) we may have both $\mu\geq 0$ and $p\geq 0$ at least for some regions of spacetime. The relationship between the energy density and the pressure is given by

$$p = -\left[\frac{kS^{-2}+K}{3kS^{-2}+K} + \frac{2D^2\left(S^{2/3}+a\right)^2}{3Ka\left(S^{4/3}-aS^{2/3}+a^2\right)} \times \right.$$

$$\left.\left[\frac{S^{2/3}-a}{\left(S^{2/3}+a\right)^3} - \frac{4\sqrt{3}\left(S^{4/3}-aS^{2/3}+a^2\right)}{\left(S^{2/3}+a\right)^2\left(3a^2+\left(2S^{2/3}-a\right)^2\right)}\right]\right]\mu$$

Clearly in this case we cannot obtain the equation of state for an ideal gas from the above equation.

In §3.3 we have presented a number of new solutions to the Einstein field equations with variable cosmological constant and gravitational constant which satisfy the Hubble variation law given by equation (3.1). It is remarkable that this simple law leads to a wide class of solutions. A generalisation of this Hubble law is presently being considered by Beesham (1992). We have obtained explicit solutions for the scale factor S in the following cases:

$$m = 0; \qquad S = Ee^{Dt}$$

$$m \neq 0; \qquad S = [C + mDt]^{1/m}$$

$$m = 2; \qquad S = [C + 2Dt]^{1/2}$$

$$m = -2; \qquad S = [C - 2Dt]^{-1/2}$$

$$m = \frac{1}{2}; \qquad S = \left[C + \frac{1}{2}Dt\right]^2$$

$$m = \frac{2}{3}; \qquad S = \left[C + \frac{2}{3}Dt\right]^{3/2}$$

We have also explicitly obtained the functional forms for the energy density μ, pressure p, the cosmological constant Λ and the gravitational constant G in each case. We have not pursued the physical properties of these solutions in any detail. This is the subject of further investigation. It is interesting to observe that solutions are admitted in which the gravitational constant may be increasing with time (see Abdel–Rahman (1990) and also §3.3). The ansatz utilised to solve the Einstein field equation (2.9) is very simple. It might be worthwhile to investigate other possibilities that lead to solutions to the Einstein field equations with interesting behaviour for the gravitational constant and cosmological constant.

4 Other Cosmologies

4.1 Introduction

As observed in §3.2 the Berman law (3.1) for the Hubble parameter has also been analysed in the Pryce–Hoyle and Brans–Dicke theories by Berman and Gomide (1988). More recently Berman (1990b,c,d) and Berman and Som (1990) have exhaustively investigated this Hubble law in the Brans–Dicke theory with the objective of analysing density perturbations in cosmological models. In the previous two chapters we have studied spacetimes with the Robertson–Walker line element for a homogeneous and isotropic universe with a perfect fluid energy momentum tensor. Our intention in this chapter is to show that the Hubble law used previously may be extended to other spacetimes and even alternate theories of gravity. In §4.2 we consider a gravitational field with less symmetry than the Robertson–Walker models, namely the Bianchi I spacetime with three Killing vectors, which is used to describe a homogeneous and anisotropic universe. The metric used is a generalisation of the $k = 0$ Robertson–Walker spacetime. We list the components of the Einstein tensor and the Einstein field equations for variable cosmological and gravitational constants. The solution to the vacuum classical Einstein field equations is presented and showed to be consistent with the Hubble law used in chapter 3. A simple solution corresponding to variable cosmological and gravitational constants is also found.

This suggests the possibility of seeking more solutions in Bianchi I and other Bianchi models. In §4.3 we consider the scalar–tensor theory of Lau and Prokhovnik (1986) which is consistent with the Dirac Large Numbers Hypothesis (Dirac 1938, 1979). We investigate the solution of Maharaj and Beesham (1988) to the field equations of Lau and Prokhovnik (1986) for the $k = 0$ Robertson–Walker spacetime. We find that the theory of Lau and Prokhovnik (1986) is consistent with the Hubble's variation law of chapter 3. This illustrates that the Hubble variation law also extends to alternate theories of gravity involving scalar fields.

4.2 Anisotropic Bianchi I model

The nine Bianchi cosmologies, each characterised by three Killing vector symmetries, may be used to describe anisotropies in the universe. The simplest Bianchi spacetime is of type I. We consider the spatially homogeneous and anisotropic spacetime described by the line element

$$ds^2 = -dt^2 + A^2(t)dx^2 + B^2(t)dy^2 + C^2(t)dz^2 \tag{4.1}$$

This spacetime is a generalisation of the $k = 0$ Robertson–Walker spacetime and is often utilised in the study of anisotropic models. It differs from the $k = 0$ Robertson–Walker spacetime in that it is not isotropic. The line element (4.1) is a Bianchi I spacetime with an Abelian 3–dimensional Lie algebra of motions (Kramer *et al* 1980). The Killing vectors of the Bianchi I spacetime is given by

$$\begin{aligned}
\mathbf{X}_1 &= \frac{\partial}{\partial x} \\
\mathbf{X}_2 &= \frac{\partial}{\partial y}
\end{aligned}$$

$$\mathbf{X}_3 = \frac{\partial}{\partial z}$$

A detailed analysis of the group structure and classification scheme of all the Bianchi cosmologies is provided in the treatments by Ellis and MacCallum (1969) and Ryan and Shepley (1975).

The components of the Einstein tensor (1.7) for the line element (4.1) are given by the following system

$$G_{00} = \frac{\dot{A}\dot{B}}{AB} + \frac{\dot{A}\dot{C}}{AC} + \frac{\dot{B}\dot{C}}{BC} \tag{4.2}$$

$$G_{11} = -A^2 \left[\frac{\ddot{B}}{B} + \frac{\dot{B}\dot{C}}{BC} + \frac{\ddot{C}}{C}\right] \tag{4.3}$$

$$G_{22} = -B^2 \left[\frac{\ddot{A}}{A} + \frac{\dot{A}\dot{C}}{AC} + \frac{\ddot{C}}{C}\right] \tag{4.4}$$

$$G_{33} = -C^2 \left[\frac{\ddot{A}}{A} + \frac{\dot{A}\dot{B}}{AB} + \frac{\ddot{B}}{B}\right] \tag{4.5}$$

$$G_{ab} = 0, \qquad a \neq b \tag{4.6}$$

With the help of the perfect fluid energy momentum tensor (1.9) and the Einstein tensor components (4.2)–(4.6) we have that the Einstein field equations (1.12) with variable cosmological constant and gravitational constant can be written as the coupled system of differential equations

$$\frac{\dot{A}\dot{B}}{AB} + \frac{\dot{A}\dot{C}}{AC} + \frac{\dot{B}\dot{C}}{BC} - \Lambda = 8\pi G\mu \tag{4.7}$$

$$\frac{\ddot{B}}{B} + \frac{\dot{B}\dot{C}}{BC} + \frac{\ddot{C}}{C} - \Lambda = -8\pi G p \tag{4.8}$$

$$\frac{\ddot{A}}{A} + \frac{\dot{A}\dot{C}}{AC} + \frac{\ddot{C}}{C} - \Lambda = -8\pi G p \tag{4.9}$$

$$\frac{\ddot{A}}{A} + \frac{\dot{A}\dot{B}}{AB} + \frac{\ddot{B}}{B} - \Lambda = -8\pi G p \tag{4.10}$$

With $\Lambda = 0$ and G a constant we regain the classical Einstein field equations from the above equations. In the case $\mu = 0 = p$ and $\Lambda = 0$ we obtain the vacuum classical Einstein field equations given by

$$\frac{\dot{A}\dot{B}}{AB} + \frac{\dot{B}\dot{C}}{BC} + \frac{\dot{A}\dot{C}}{AC} = 0 \tag{4.11}$$

$$\frac{\ddot{B}}{B} + \frac{\dot{B}\dot{C}}{BC} + \frac{\ddot{C}}{C} = 0 \tag{4.12}$$

$$\frac{\ddot{A}}{A} + \frac{\dot{A}\dot{C}}{AC} + \frac{\ddot{C}}{C} = 0 \tag{4.13}$$

$$\frac{\ddot{A}}{A} + \frac{\dot{A}\dot{B}}{AB} + \frac{\ddot{B}}{B} = 0 \tag{4.14}$$

from equations (4.7)–(4.10).

The Bianchi I spacetime has received much attention as it is simple but does cater for anisotropy. The solution of the vacuum field equations (4.11)–(4.14) is well known. This solution is called the Kasner solution. We express the Kasner solution in a form which is more appropriate for our purposes:

$$A = [\alpha + \beta t]^{p_1}$$

$$B = [\alpha + \beta t]^{p_2}$$

$$C = [\alpha + \beta t]^{p_3}$$

where α and β are constants and

$$p_1 + p_2 + p_3 = 1$$

$${p_1}^2 + {p_2}^2 + {p_3}^2 = 1$$

must be satisfied for a consistent solution. The constants α and β are not essential to the solution and may be eliminated using the transformation

$$\tilde{t} \rightarrow \alpha + \beta t$$

We note that the general Bianchi I solution for dust

$$\mu \neq 0, \qquad p = 0$$

is also known and is listed by Stephani (1990). The form of solution for dust is similar to the Kasner solution given above. Other special cases of solution are listed by Kramer *et al* (1980).

We will show that the Hubble variation (3.1)

$$H = DS^{-m}$$

used previously is consistent with the Bianchi I spacetime (4.1) for the vacuum field equations (4.11)–(4.14). We note that it is possible in principle to perform a similar analysis for the Einstein field equations (4.7)–(4.10) with variable cosmological constant and gravitational constant. To perform an analogous discussion to the previous chapter we need to define the function

$$S = (ABC)^{1/3}$$

as an "average" of the anisotropy. Clearly this definition for the Bianchi I spacetime reduces to the scale factor of the flat $k = 0$ Robertson–Walker spacetime when we have $A = B = C$. Then the above definition gives the following form for Hubble's constant

$$H = \frac{\dot{S}}{S}$$

$$= \frac{1}{3}(\ln\ ABC)^{\cdot}$$

This form of the Hubble parameter was utilised by Misner *et al* (1973) in studying adiabatic cooling of anisotropy in the early universe. For the vacuum Kasner solution the Hubble law is of the form

$$H = \frac{1}{3}\left[\ln(\alpha + \beta t)^{p_1+p_2+p_3}\right]^{\cdot}$$

$$= \frac{1}{3}\left[\frac{\beta}{\alpha + \beta t}\right] \quad (4.15)$$

from definition. Is this form of solution consistent with the Berman variation law? To answer this question we must compare this result with the Hubble law obtained from (3.1). Using the scale factor S defined above for the anisotropic Bianchi I spacetime we obtain the following form

$$H = DS^{-m}$$

$$= \frac{D}{(\alpha + \beta t)^{m/3}} \quad (4.16)$$

On comparing equations (4.15) and (4.16) we have

$$m = 3$$

$$D = \frac{1}{3}\beta$$

Thus we have verified that the vacuum Kasner solution is consistent with the Hubble variation law

$$H = \frac{D}{C + mDt}$$

with $m = 3$. In fact the vacuum Kasner solution remains unchanged with this variation of the Hubble law as the only modification involves a rescaling of the arbitrary constant β.

The above solution for the Bianchi I spacetime is interesting as it suggests that the class of solutions presented in chapter 3 for Robertson–Walker spacetimes may be extended to other spacetimes with less symmetry. It is possible that this approach may lead to new solutions of the Einstein field equations. We may extend the arguments given above in the Bianchi I spacetime to include the case of variable cosmological constant and gravitational constant. We illustrate this possibility with an elementary solution of the Einstein field equations (4.7)–(4.10). It is interesting to note that the vacuum Kasner solution is given by

$$A = [\alpha + \beta t]^{p_1}$$

$$B = [\alpha + \beta t]^{p_2}$$

$$C = [\alpha + \beta t]^{p_3}$$

with the conditions

$$p_1 + p_2 + p_3 = 1$$

$$p_1{}^2 + p_2{}^2 + p_3{}^2 = 1$$

consistent with the Berman law

$$H = DS^{-m}$$

extends to the case of variable cosmological constant and gravitational constant. It is clear by simple inspection that this solution is admitted by the field equations (4.7)–(4.10) if the cosmological constant

$$\Lambda = 8\pi G p$$

and the pressure

$$p = -\mu$$

Thus the pressures are negative for a Kasner–type solution with variable cosmological constant and gravitational constant. We note that there is freedom in the solution as we can arbitrarily specify the behaviour of the cosmological constant or the gravitational constant. Even though this solution is very simple it illustrates that there are solutions to the Einstein field equations with variable cosmological constant Λ and gravitational constant G consistent with the Berman law (3.1). This is an area for future research. The simplest starting point would be to choose the form of Λ and G so that the metric functions generate a behaviour which is similar to that of the Kasner solution.

4.3 Theory of Lau and Prokhovnik

The law for the variation of Hubble's parameter (3.1) is also consistent with scalar–tensor theories of gravity that reduce to Einstein's general relativity. We illustrate this with the scalar–tensor theory of Lau and Prokhovnik (1986). This is a theory with variable cosmological constant and gravitational constant but, in addition, it has a scalar field ψ. The theory was structured so that it is consistent with the Dirac Large Numbers Hypothesis (Dirac 1938, 1979). This theory was also investigated by Maharaj and Beesham (1988) who presented solutions to the field equations of Lau and Prokhovnik (1986) for the flat $k = 0$ Robertson–Walker spacetime. The generalised field equations in the scalar–tensor theory of Lau and Prokhovnik (1986) are given by

$$R_{ab} - \frac{1}{2}Rg_{ab} + \Lambda g_{ab} = 8\pi G T_{ab} + \psi_{,a}\psi_{,b} \qquad (4.17)$$

$$\dot{\psi}\Box\psi + \dot{\Lambda} + \frac{1}{2}\dot{g}^{00}\dot{\psi}^2 + g^{00}\dot{\psi}\ddot{\psi} + 8\pi\dot{G}L_m = 0 \qquad (4.18)$$

where

$$\Box\psi = g^{ab}\psi_{;ab}$$

Here L_m is the matter Lagrangian density including all non–gravitational fields. The quantity

$$\Lambda = \lambda(t) - \frac{1}{2}g^{00}\dot{\psi}^2$$

is a generalisation of the normal cosmological constant but is equivalent to the cosmological constant used before in Robertson–Walker spacetimes. The field equation (4.17) is a generalisation of the classical Einstein field equation (1.10) to incorporate variable cosmological constant Λ, gravitational constant G and scalar fields ψ. The other field equation (4.18) governs the behaviour of the scalar field ψ. For details of the derivation of (4.17)–(4.18) see Lau and Prokhovnik (1986).

In this section we follow the notation of Lau and Prokhovnik (1986). This enables us easily to compare our results with those of Maharaj and Beesham (1988). The $k = 0$ Robertson–Walker spacetimes with flat spatial sections are characterised by the metric

$$ds^2 \;=\; -\,dt^2 + S^2(t)\left[dx^2 + dy^2 + dz^2\right] \tag{4.19}$$

which was given previously in polar coordinates by equation (2.1) in §2.2. We use the energy–momentum tensor

$$T^{ab} \;=\; \mu\, u^a u^b$$

which is a special case of equation (1.9) in chapter 1 with the pressure $p = 0$. Thus in this case the matter Lagrangian density is given by

$$L_m \;=\; \mu$$

for a dust energy–momentum tensor. For compatibility with the Dirac Large Numbers Hypothesis we must have (Dirac 1938, 1979; Lau 1985)

$$S^2(t) \;=\; \beta_1\,(\alpha + \beta t)^{2/3} \tag{4.20}$$

$$G(t) \;=\; \beta_2\,(\alpha + \beta t)^{-1} \tag{4.21}$$

$$(L_m =)\mu(t) \;=\; \beta_3\,(\alpha + \beta t)^{-1} \tag{4.22}$$

where $\beta_1, \beta_2, \beta_3$ are constants in the above. To integrate completely the field equations (4.17)–(4.18) it remains to obtain forms for the cosmological constant Λ and the scalar field ψ.

Using the metric (4.19) it is easy to show that the field equation (4.18) reduces to

$$2\dot{\psi}\ddot{\psi} + \dot{\Lambda} + 3\frac{\dot{S}}{S}\dot{\psi}^2 + 8\pi\dot{G}\mu = 0 \tag{4.23}$$

Substituting equations (4.20)–(4.22) in equation (4.23) we obtain

$$(\dot{\psi}^2)^{\cdot} + \dot{\Lambda} + \beta(\alpha+\beta t)^{-1}\dot{\psi}^2 = 8\pi\beta\beta_2\beta_3(\alpha+\beta t)^{-3} \tag{4.24}$$

The (0, 0) component of equation (4.17) is given by

$$3\frac{\dot{S}^2}{S^2} - \Lambda = \dot{\psi}^2 + 8\pi G\mu \tag{4.25}$$

Substituting equations (4.20)–(4.22) in equation (4.25) we obtain

$$\frac{1}{3}\beta^2(\alpha+\beta t)^{-2} - \Lambda - \dot{\psi}^2 = 8\pi\beta_2\beta_3(\alpha+\beta t)^{-2} \tag{4.26}$$

On differentiating equation (4.26) with respect to time we have

$$-\frac{2}{3}\beta^3(\alpha+\beta t)^{-3} - \dot{\Lambda} - (\dot{\psi}^2)^{\cdot} = -16\pi\beta\beta_2\beta_3(\alpha+\beta t)^{-3} \tag{4.27}$$

Adding equations (4.24) and (4.27) we obtain

$$\dot{\psi}^2 = \left(\frac{2}{3}\beta^2 - 8\pi\beta_2\beta_3\right)(\alpha+\beta t)^{-2} \tag{4.28}$$

Equation (4.28) has the general solution for the scalar field

$$\psi = \frac{1}{\beta}\left(\frac{2}{3}\beta^2 - 8\pi\beta_2\beta_3\right)^{1/2}\ln(\alpha+\beta t) + A \tag{4.29}$$

where A is constant. Substitution of equation (4.28) in equation (4.26) yields

$$\Lambda = -\frac{1}{3}\beta^2(\alpha+\beta t)^{-2} \tag{4.30}$$

for the cosmological constant.

The solution to the field equations in the theory of Lau and Prokhovnik (1986) is given by equations (4.20)–(4.22), (4.29) and (4.30). The solutions presented are analogous to those of Maharaj and Beesham (1988). It is interesting to observe that the cosmological constant has the behaviour

$$\Lambda \propto \frac{1}{t^2}$$

This is the same form as the solutions presented in chapter 3 for variable cosmological constant and gravitational constant without a scalar field ψ. This form of the cosmological constant is consistent with observations of present day values for the cosmological constant which are small.

Using equation (4.20) we have that

$$\begin{aligned} H &= \frac{\dot{S}}{S} \\ &= \frac{\frac{1}{3}\beta}{\alpha + \beta t} \end{aligned}$$

However from chapter 3 for $m \neq 0$ we have that

$$H = \frac{D}{C + mDt}$$

which follows from the Berman hypothesis that the deceleration parameter is constant. Thus we have established that if

$$\begin{aligned} m &= 3 \\ C &= 3\alpha \\ D &= \beta \end{aligned}$$

then the dust solutions, for the $k = 0$ Robertson–Walker spacetime, in the theory of Lau and Prokhovnik (1986) are consistent with the Hubble variation law

$$H = DS^{-m}$$

Thus the above Hubble variation may be useful in studying solutions of the field equations in scalar–tensor theories. It has the advantage of immediately specifying the scale factor. This is helpful in alternate theories of gravity as the normal variables are supplemented with the cosmological constant, gravitational constant and scalar fields. The Berman (1983) ansatz provides a mechanism to reduce the number of variables in an undetermined system of differential equations.

5 Conclusion

Our main objective was to consider the relevance of a Hubble law proposed by Berman (1983) to the study of cosmological models. In this thesis we consider a number of exact solutions of Einstein's field equations to the Robertson–Walker spacetimes, the Bianchi I spacetime and scalar–tensor theories. The Einstein field equations are simplified by assuming the variation law for Hubble's parameter used by Berman (1983). Explicit solutions are presented in the Robertson–Walker spacetimes, the Bianchi I spacetime and the scalar–tensor theory of Lau and Prokhovnik (1986). This illustrates that the Hubble variation used is consistent and is relevant in the study of cosmological models.

In chapter 1 of this thesis we provide only those aspects of general relativity and differential geometry necessary for later chapters. The curvature tensor and associated quantities are defined. The Einstein field equations are motivated and the matter distribution is described by the energy–momentum tensor. We also introduce the Einstein field equations with variable cosmological constant and gravitational constant.

In chapter 2 we consider the Robertson–Walker spacetimes satisfying the cosmological principle. The energy–momentum tensor is a perfect fluid. We derive the classical Einstein field equations and the field equations with variable cosmological constant and gravitational constant. The properties of the Friedmann model are reviewed in general. The Hubble constant, critical density and the deceleration parameter are important cosmological parameters. The age of the universe, the Hubble expansion and the microwave background radiation are investigated and related to observation. Some recent developments in modern cosmology are pointed out.

In chapter 3 we comprehensively investigate solutions to the Robertson–Walker spacetimes. We utilise the Berman (1983) Hubble law to find solutions to the classical Einstein field equations. The solution of Berman (1991) for variable cosmological constant and gravitational constant are discussed. We also present a large class of new solutions to the Einstein field equations with variable cosmological constant and gravitational constant. In each case we list the functional forms for the scale factor, cosmological constant, gravitational constant, energy density and pressure. The properties of the solutions are briefly investigated. In many cases the equation of state of an ideal gas is possible. Many of the solutions have the gravitational constant increasing in certain regions of spacetime. This is similar to the behaviour of the model of Abdel–Rahman (1990).

In chapter 4 we consider the Bianchi I spacetime and scalar–tensor theory of Lau and Prokhovnik (1986). For the Bianchi I spacetime we show that the Berman (1983) law is consistent with the vacuum Kasner solution. It is possible to extend this result in Bianchi I spacetime to variable cosmological constant and gravitational constant. We also show that the $k = 0$ Robertson–Walker solution of Maharaj and Beesham (1988) for the scalar–tensor theory of Lau and Prokhovnik (1986) is consistent with our assumed form for the Hubble parameter.

We should point out that the idea of cosmological dependence on S^{-m} has been discussed in another context by Bishop (1976) and Landsberg and Bishop

(1975). These papers are essentially concerned with seeking a cosmological explaination for the decrease in the gravitational constant G with time. The scalar field is assumed to vary as a power of the cosmological expansion factor (i.e. $\psi \propto S^{-m}$) in the scalar– tensor theory of Nordtvedt (1970). A set of models may be obtained which is compatible with observation. The models found by Bishop (1976) is similar to those of Newtonian cosmology obtained from an impotence principle.

We have demonstrated that the Hubble law used in this thesis provides consistent cosmological models. It is applicable both to general relativity and alternate theories of gravity. It is clear that our ansatz should lead to other solutions to the Einstein field equations with interesting behaviour. It might be interesting to investigate the form of the solutions permitted in other cosmological models.

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