Research Article

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Decomposition and classification of length functions

Abstract: We study decompositions of length functions on integral domains as sums of length functions constructed from overrings. We find a standard representation when the integral domain admits a Jaffard family, when it is Noetherian and when it is a Prüfer domains such that every ideal has only finitely many minimal primes. We also show that there is a natural bijective correspondence between singular length functions and localizing systems.

Keywords: Length functions, Jaffard family, Prüfer domains, localizing systems

MSC 2010: 13C60, 13A18, 13F05, 13G05

1 Introduction

The concept of a (generalized) length function on the category $\mathrm{Mod}(R)$ of modules over a (commutative, unitary) ring R was introduced by Northcott and Reufel [12] as a generalization of the classical length of a module; more precisely, they defined a *length function* as a map from $\mathrm{Mod}(R)$ to the set of nonnegative real numbers (plus infinity) that is additive on exact sequences and such that the length of a module is the supremum of the length of its finitely generated submodules. In particular, they were interested in classifying all the possible length functions on a valuation domain; their results were later deepened and expressed in a different form by Zanardo [21]. Shortly after [12], Vámos [20] distinguished the two properties used to define a length functions (which he called *additivity* and *upper continuity*), showed that they were independent one from each other and classified all length functions on Noetherian rings. Ribenboim [13] subsequently considered length functions with values in an arbitrary ordered abelian group, though he needed to restrict the definition to a smaller class of modules (constructible modules) due to the possible non-completeness of the group (more precisely, due to the possible lack of suprema). More recently, length functions have been linked to the concept of *algebraic entropy* [2, 14, 15], the study of which also involves invariants satisfying a weaker form of additivity [16].

The purpose of this paper is to investigate two closely related problems: the first one is the possibility of "decomposing" a length function ℓ on an integral domain D as a sum of length functions defined on overrings of D (in particular, localizations of D); the second one is the possibility of expressing the set $\mathcal{L}(D)$ of the length functions on D (and/or some distinguished subset of $\mathcal{L}(D)$) as a product of the set of length functions on a family of overrings of D. Both problems can be seen, more generally, as asking for a way to find all length

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functions on D by reducing to "simpler" domains and cases. We pursue this problem first from a general point of view and then for two special classes of domains, namely Noetherian domains and Prüfer domains.

Section 3 can be seen as a generalization the case of Dedekind domains treated in [12, Section 7]. In it, we analyze the relationship between the length functions on D and the length functions on the overrings of D or, more generally, on D-algebras. We show that any length function on D induces, by restricting its domain, a length function on any D-algebra (Proposition 3.1) and, conversely, that a length function ℓ on a flat D-algebra T induces a length function on D by extension of scalars, i.e., defining $\ell^D(M) := \ell(M \otimes_D T)$ for every D-module M (Proposition 3.3); we also show that the latter construction can be adapted to using a whole family of flat algebras. Then we analyze how these two operations relate one to each other; in particular, we show that if we take a *Jaffard family* of D (that is, a family Θ of flat overrings of D that is complete, independent and locally finite; see Section 2 for a precise definition), then every length function ℓ is equal to the sum $\sum_{T \in \Theta} \ell \otimes T$ (Theorem 3.10) and that the set $\mathcal{L}_\infty(D)$ of length functions such that $\ell(D) = \infty$ is order-isomorphic to the product $\prod_{T \in \Theta} \mathcal{L}_\infty(T)$.

In the next two sections, we study how to obtain similar results when we do not have Jaffard families; in particular, we try to obtain an equality $\ell = \sum_{T \in \Theta} \ell \otimes T$ with the additional flexibility of allowing Θ to vary in function of ℓ.

In Section 4, using Vámos' results, we show that such a decomposition holds on Noetherian domains; more precisely, we show that any length function ℓ on a Noetherian domain D can be written as $\sum_{P \in \Sigma(\ell)} \ell \otimes D_P$, where $\Sigma(\ell) := \{P \in \mathrm{Spec}(D) \mid \ell(D/P) > 0\}$ (Proposition 4.5). This is accomplished by studying how length functions acts on D/I, where I is a primary ideal or an ideal admitting a primary decomposition. In the latter case, in particular, we show that if I is an ideal of a domain D with a primary decomposition, then $\ell(D/I)$ is equal to the sum of $\ell(D/Q)$, as Q ranges among the primary components of I (Proposition 4.7).

In Section 5, we study length functions on Prüfer domains; in this case, the main tool is the study of $\ell(D/I)$, where I is an ideal whose radical is prime. We show that, if every ideal of D has only finitely many minimal primes, then we can always write $\ell = \sum_{P \in \Sigma(\ell)} \ell \otimes D_P$, where $\Sigma(\ell)$ is defined in a manner similar but more complicated than the Noetherian case (Theorem 5.4). Since every D_P is a valuation domain and the length functions on valuation domains are classified [21], this representation allows a much deeper analysis on the length functions of D; we use it to partition $\Sigma(\ell)$ into four subsets (some of which can be empty), and then to characterize a length function ℓ through these subsets (Proposition 5.10). With this characterization, we can prove that, for this class of Prüfer domains, the set $\mathcal{L}(D)$ depends only on the topological structure of $\mathrm{Spec}(D)$ and on which prime ideals are idempotent (Theorem 5.12).

In the final section, Section 6, we connect singular length functions (i.e., the length functions such that $\ell(M)$ can only be 0 or ∞) to localizing systems and to stable semistar operations. More precisely, we show that, for any domain D, there is a natural bijection between the set of singular length functions and the set of localizing systems on D (Theorem 6.5); using the known correspondence between localizing systems and stable semistar operations, we obtain a bijection between singular length functions and stable semistar operations. We show that, for Prüfer domains, this correspondence connects also two natural "normalizations" of length functions and stable operations (Proposition 6.8), and we use it to give two examples of widely different behavior in how singular length functions can be decomposed.

2 Background and notation

All rings considered in the paper are commutative and unitary.

Let $\mathbb{R}^{\geq 0}$ denote the set of nonnegative real numbers, and let $\Gamma := \mathbb{R}^{\geq 0} \cup \{\infty\}$. Then Γ has a natural structure of (commutative) ordered additive semigroup, where, for every $r \in \Gamma$, $r + \infty = \infty + r = \infty$ and $r \leq \infty$.

If $\Lambda \subseteq \Gamma$ is a (not necessarily finite) set, we define the sum of Λ as the supremum of all the finite sums $\lambda_1 + \cdots + \lambda_n$, as $\{\lambda_1, \ldots, \lambda_n\}$ ranges among the finite subsets of Λ. Since all elements of Λ are nonnegative, this notion coincides with the usual sum if Λ is finite. If uncountably many elements of Λ are positive, then the sum of Λ is infinite.

Let R be a ring, and let $\mathrm{Mod}(R)$ be the category of R-modules. A map $\ell\colon \mathrm{Mod}(R) \to \Gamma$ is a *length function* on R if

- $\ell(0) = 0$,
- ℓ is *additive*: for every short exact sequence $0 \to M_1 \to M_2 \to M_3 \to 0$, we have $\ell(M_2) = \ell(M_1) + \ell(M_3)$,
- ℓ is *upper continuous*: for every R-module M,

$$\ell(M) = \sup\{\ell(N) \mid N \text{ is a finitely generated submodule of } M\}.$$

The first hypothesis is adopted by Northcott and Reufel [12] and by Vámos [20], but not by Zanardo [21]. The only function it excludes is the map ℓ_∞ sending every module to ∞. We call the length function ℓ_0 such that $\ell_0(M) = 0$ for every module M the *zero length function*; in [21, Section 3.1], ℓ_0 and ℓ_∞ are called *trivial* length functions, but we shall not use this terminology. If ℓ is different from the zero length function, then $\ell(R) > 0$.

If M_1 and M_2 are isomorphic modules, then $\ell(M_1) = \ell(M_2)$; indeed, the sequence $0 \to M_1 \to M_2 \to 0 \to 0$ is exact, and thus $\ell(M_2) = \ell(M_1) + \ell(0) = \ell(M_1)$ by additivity and since $\ell(0) = 0$. Moreover, if N is a submodule or a quotient of M then $\ell(N) \leq \ell(M)$.

Three examples of length functions on a ring R are

- the "usual" length function (i.e., the Jordan-Hölder length of a module): we denote it by length_R,
- the function ℓ such that $\ell(M) = 0$ if M is a torsion R-module, while $\ell(M) = \infty$ otherwise,
- if R is an integral domain, the rank function: $\mathrm{rank}(M) := \dim_K(M \otimes K)$, where K is the quotient field of R.

If R is an integral domain, then the rank function is, up to multiplication by a constant, the only length function ℓ such that $\ell(R) < \infty$ (see [12, Theorem 2]).

Let $\mathrm{Im}(\ell)$ denote the image of ℓ, i.e., the set of $\ell(M)$ as M ranges in $\mathrm{Mod}(R)$. We say that a length function ℓ is

- *singular* if $\mathrm{Im}(\ell) = \{0, \infty\}$,
- *discrete* if $\mathrm{Im}(\ell)$ is discrete in Γ.

We denote by

- $\mathcal{L}(R)$ the set of length functions on R,
- $\mathcal{L}_\infty(R)$ the set of length functions such that $\ell(R) = \infty$,
- $\mathcal{L}_{\mathrm{sing}}(R)$ the set of singular length functions,
- $\mathcal{L}_{\mathrm{disc}}(R)$ the set of discrete length functions.

The set $\mathcal{L}(R)$ has a natural order structure, where $\ell_1 \leq \ell_2$ if and only if $\ell_1(M) \leq \ell_2(M)$ for every R-module M. In this order, $\mathcal{L}(R)$ has both a minimum (ℓ_0) and a maximum (the function sending all nonzero modules to ∞).

We shall use often the following result.

Proposition 2.1 ([21, Proposition 3.3]). *Let ℓ_1, ℓ_2 be length functions of R. If $\ell_1(R/I) = \ell_2(R/I)$ for every ideal I of R, then $\ell_1 = \ell_2$.*

Let D be an integral domain with quotient field K; an *overring* of D is a ring comprised between D and K. A *Jaffard family* of D is a family Θ of overrings of D such that [18, Proposition 4.3]

- every $T \in \Theta$ is flat,
- $K \notin \Theta$,
- $I = \bigcap\{IT \mid T \in \Theta\}$ for every ideal I of D (i.e., Θ is *complete*),
- $TS = K$ for every $T \neq S$ belonging to Θ (i.e., Θ is *independent*),
- for every $x \in K$, there are only finitely many $T \in \Theta$ such that x is not a unit in T (i.e., Θ is *locally finite*).

Note that this is not the original definition; see [4, beginning of Section 6.3 and Theorem 6.3.5] for two other characterizations.

In particular, if Θ is a Jaffard family of D and $P \in \mathrm{Spec}(D)$ is nonzero, then there is exactly one $T \in \Theta$ such that $PT \neq T$ (see [4, Theorem 6.3.1 (1)]).

A *Prüfer domain* is an integral domain such that the localization at every prime ideal is a valuation domain. If V is a valuation domain and $P \in \mathrm{Spec}(V)$, we say that P is *branched* if P is minimal over a principal ideal; equivalently, if the union of all the prime ideals properly contained in P is different from P. A prime ideal that is not branched is called *unbranched*. If D is a Prüfer domain, we say that $P \in \mathrm{Spec}(D)$ is branched if PD_P is branched in D_P.

Unreferenced statements about Prüfer and valuation domains are standard; see, for example, [7] for a general reference.

3 Jaffard families

The following section is a generalization of [12, Section 7], of which we follow and generalize the method.

Let D be an integral domain, and let T be a D-algebra. Then every T-module is also (in a canonical way) a D-module; thus, given any $\ell \in \mathcal{L}(D)$, we can define a function ℓ_T by

$$\ell_T(M) := \ell(M) \quad \text{for every } M \in \mathrm{Mod}(T).$$

Proposition 3.1. *Let D, T, ℓ be as above.*
(a) *ℓ_T is a length function on T.*
(b) *If ℓ is singular, so is ℓ_T.*
(c) *If ℓ is discrete, so is ℓ_T.*

Proof. (a) Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be an exact sequence of T-modules. Then it is also an exact sequence of D-modules; hence, $\ell_T(M_2) = \ell(M_2) = \ell(M_1) + \ell(M_3) = \ell_T(M_1) + \ell_T(M_3)$. Thus, ℓ_T is additive. Suppose now $\ell_T(M) > x$ for some $x \in \mathbb{R}$. Then $\ell(M) > x$, and thus there is a finitely generated D-submodule N of M such that $\ell(N) > x$; let $N := e_1 D + \cdots + e_k D$. Then $NT = e_1 T + \cdots + e_n T$ is a submodule of M containing N, and thus $\ell_T(NT) = \ell(NT) \geq \ell(N) > x$. Hence, $\ell_T(M) = \sup\{\ell_T(N') \mid N' \subseteq M \text{ is finitely generated over } T\}$, and thus ℓ_T is upper continuous. Therefore, ℓ_T is a length function on T.

(b) and (c) are obvious. □

In general, it is possible for ℓ_T to be the zero length function even if ℓ is not; for example, if $\ell(D/I) = 0$ and $T = D/I$, then $\ell_T(M)$ will be 0 for all T-modules M.

Proposition 3.2. *Let D be an integral domain, and let T be a D-algebra.*
(a) *If T is torsion-free over D, then $\ell(T) \neq 0$ for all nonzero $\ell \in \mathcal{L}(D)$.*
(b) *If T is torsion-free over D and $\ell \in \mathcal{L}_\infty(D)$, then $\ell_T \in \mathcal{L}_\infty(T)$.*
(c) *If T is torsion-free over D and $\mathrm{rank}_D(T) < \infty$, then $\ell \in \mathcal{L}_\infty(D)$ if and only if $\ell_T \in \mathcal{L}_\infty(T)$.*
(d) *If T is an overring of D, then $\ell \in \mathcal{L}_\infty(D)$ if and only if $\ell_T \in \mathcal{L}_\infty(T)$.*

Proof. (a) If T is torsion-free, then the canonical map $D \to T$ is injective; hence, if ℓ is an arbitrary nonzero length function, $\ell_T(T) = \ell(T) \geq \ell(D) > 0$.

(b) As above, for every $\ell \in \mathcal{L}_\infty(D)$, we have $\ell_T(T) = \ell(T) \geq \ell(D) = \infty$, and thus $\ell_T \in \mathcal{L}_\infty(T)$.

(c) Suppose T is torsion-free and $\mathrm{rank}_D(T) < \infty$, and let $\ell \in \mathcal{L}(D)$. If $\ell \in \mathcal{L}_\infty(D)$, then $\ell_T \in \mathcal{L}_\infty(T)$ by (b). On the other hand, if $\ell \notin \mathcal{L}_\infty(D)$, then by [12, Theorem 2], $\ell(M) = \alpha \cdot \mathrm{rank}_D(M)$ for every D-module M, where $\alpha := \ell(D)$; in particular, $\ell_T(T) = \ell(T) = \mathrm{rank}_D(T) < \infty$ by hypothesis, and thus $\ell_T \notin \mathcal{L}_\infty(T)$.

(d) follows from (c) since $T \otimes_D K \simeq K$ (where K is the quotient field of D) and so $\mathrm{rank}_D(T) = 1$. □

Hence, if T is a torsion-free D-algebra, then by the previous proposition we have a map

$$\widehat{\Lambda}_{D,T} : \mathcal{L}(D) \to \mathcal{L}(T), \; \ell \mapsto \ell_T,$$

that restricts to a map

$$\Lambda_{D,T} : \mathcal{L}_\infty(D) \to \mathcal{L}_\infty(T), \; \ell \mapsto \ell_T.$$

Clearly, both $\widehat{\Lambda}_{D,T}$ and $\Lambda_{D,T}$ are order-preserving.

A more interesting question is if (and how) we can construct a length function on D from a length function on a D-algebra T. We shall mainly be interested in the case when $T = D_P$ is a localization of D, but there is no harm in working more generally with a flat algebra.

Let thus T be a flat D-algebra, and let $\ell \in \mathcal{L}(T)$. We define ℓ^D as the map such that

$$\ell^D(M) := \ell(T \otimes_D M) \quad \text{for all } M \in \mathrm{Mod}(D).$$

This construction behaves similarly to the construction ℓ_T.

Proposition 3.3. *Let D, T, ℓ be as above.*
(a) $\ell^D \in \mathcal{L}(D)$.
(b) $\ell^D \in \mathcal{L}_\infty(D)$ *if and only if* $\ell \in \mathcal{L}_\infty(T)$.

Proof. (a) follows in the same way of [12, Proposition 2], using the flatness of T. (b) is immediate because $\ell^D(D) = \ell(T \otimes_D D) = \ell(T)$. □

Therefore, we can define a map $\widehat{\Psi}_{T,D}$ by setting

$$\widehat{\Psi}_{T,D}\colon \mathcal{L}(T) \to \mathcal{L}(D), \ \ell \mapsto \ell^D;$$

by part (b) of the previous proposition, also its restriction

$$\Psi_{T,D}\colon \mathcal{L}_\infty(T) \to \mathcal{L}_\infty(D), \ \ell \mapsto \ell^D$$

is well-defined.

Proposition 3.4. *Let D be an integral domain, and let T be a flat overring of D. Then $\widehat{\Lambda}_{D,T} \circ \widehat{\Psi}_{T,D}$ is the identity on $\mathcal{L}(T)$.*

Proof. Let $\ell \in \mathcal{L}(T)$. Then, for every $M \in \mathrm{Mod}(T)$, we have

$$(\widehat{\Lambda}_{D,T} \circ \widehat{\Psi}_{T,D})(\ell)(M) = (\ell^D)_T(M) = \ell^D(M) = \ell(M \otimes_D T).$$

Since T is a flat overring of D, the inclusion $D \hookrightarrow T$ is an epimorphism (being $D \hookrightarrow K$ an epimorphism; see [9, Chapitre IV, Corollaire 3.2] or [8, Proposition 4.5]); hence, for every T-module D, we have

$$T \otimes_D M \simeq T \otimes_D (T \otimes_T M) \simeq (T \otimes_D T) \otimes_T M \simeq T \otimes_T M \simeq M$$

as T-modules, with the second-to-last isomorphism coming from the fact that the inclusion is an epimorphism (see [9, Lemma 1.0] or [8, Lemma A.1]). Thus, $(\ell^D)_T(M) = \ell(M)$; since M was arbitrary, $(\ell^D)_T = \ell$, i.e., $\widehat{\Lambda}_{D,T} \circ \widehat{\Psi}_{T,D}$ is the identity. □

Remark 3.5. Proposition 3.4 does not work for arbitrary flat D-algebras. For example, if $T = M = D[x]$ is the polynomial ring over D, then $M \otimes_D T \simeq D[x,y] = T[y]$ as T-modules; hence, if ℓ is the rank function of T, we have $\ell(M) = 1$, while $(\ell^D)_T(M) = \infty$.

Let now ℓ be a length function on D, and let T be a flat D-algebra. We set

$$\ell \otimes T := (\widehat{\Psi}_{T,D} \circ \widehat{\Lambda}_{D,T})(\ell).$$

This notation comes from the fact that if M is a D-module, then

$$(\ell \otimes T)(M) = (\ell_T)^D(M) = \ell_T(M \otimes_D T) = \ell(M \otimes_D T).$$

The construction $\ell \otimes T$ is easily seen to satisfy an associative-like property; if T_1, T_2 are flat D-algebras, then $(\ell \otimes T_1) \otimes T_2 = \ell \otimes (T_1 \otimes_D T_2)$.

In general, $\ell \otimes T$ is different from ℓ; for example, if I is a proper ideal of D such that $IT = T$, then

$$(\ell \otimes T)(D/I) = \ell(D/I \otimes_D T) = \ell(T/IT) = \ell(0) = 0,$$

but clearly there may be length functions such that $\ell(D/I) \neq 0$.

Hence, if we want to construct an isomorphism from $\widehat{\Lambda}_{D,T}$ and $\widehat{\Psi}_{T,D}$, we need to consider more rings, and to do so, we must define the sum of a family of length functions.

Definition 3.6. Let $\Lambda := (\ell_\alpha)_{\alpha \in A}$ be a family of length functions on D. The *sum* of Λ, which we denote by $\sum_{\alpha \in A} \ell_\alpha$, is the map

$$\sum_{\alpha \in A} \ell_\alpha \colon \mathrm{Mod}(D) \to \Gamma, \ M \mapsto \sum_{\alpha \in A} \ell_\alpha(M).$$

Recall that the sum of an arbitrary family of elements of Γ is defined as the supremum of the set of the finite sums (see the beginning of Section 2).

Lemma 3.7. *The sum of a family Λ of length functions on D is a length function and the supremum of Λ in $\mathcal{L}(D)$.*

Proof. Just apply the definitions. □

Lemma 3.8. *Let $\{M_\alpha\}_{\alpha \in A}$ be a family of D-modules, and let ℓ be a length function on D. Then*

$$\ell\Big(\bigoplus_{\alpha \in A} M_\alpha\Big) = \sum_{\alpha \in A} \ell(M_\alpha).$$

Proof. If the family is finite, the claim follows immediately from additivity and induction on the cardinality of A.

If A is an arbitrary family, set N to be the direct sum of the M_α; then $\ell(N) \geq \ell(M_{\alpha_1}) + \cdots + \ell(M_{\alpha_n})$ for all finite subsets $\alpha_1, \ldots, \alpha_n$. Furthermore, every finitely generated submodule of N is contained in $M_{\alpha_1} \oplus \cdots \oplus M_{\alpha_n}$ for some $\alpha_1, \ldots, \alpha_n$; by upper continuity, it follows that $\ell(N)$ must be exactly the supremum. The claim is proved. □

The following lemma is a (partial) generalization of a property of h-local domains; see [11, Theorem 22].

Lemma 3.9. *Let D be an integral domain, and let Θ be a Jaffard family of D. If M is a torsion D-module, then*

$$M \simeq \bigoplus_{T \in \Theta} M \otimes_D T.$$

In particular, if $I \neq (0)$ is an ideal of D, then

$$\frac{D}{I} \simeq \bigoplus_{T \in \Theta} \frac{T}{IT} \simeq \bigoplus_{T \in \Theta}\Big(\frac{D}{I} \otimes_D T\Big).$$

Proof. We shall follow the proof of [11, Theorem 22, 3 ⇒ 4]; we start by showing that $K/D \simeq \bigoplus_{T \in \Theta} K/T$. Indeed, consider the natural map

$$\Phi\colon K \to \bigoplus_{T \in \Theta} K/T, \ d \mapsto (d + T)_{T \in \Theta}.$$

Note that Φ is well-defined since Θ is locally finite.

The kernel of Φ is $\bigcap_{T \in \Theta} T = D$. Hence, we need only to show that Φ is surjective, and to do so, it is enough to show that every element of the form $e(\alpha, U) := (0, \ldots, 0, \alpha + U, 0, \ldots, 0)$ is in the image of Φ; i.e., we need to show that, for every $U \in \Theta$ and every $\alpha \in K$, there is an $\alpha' \in K$ such that $\alpha' - \alpha \in U$, while $\alpha' \in T$ for every $T \in \Theta \setminus U$.

Let $U' := \bigcap_{T \in \Theta \setminus U} T$; by [18, Proposition 4.5 (b)], $U'U = K$. Hence, we have

$$U + U' = \bigcap_{T \in \Theta}(U + U')T = (U + UU') \cap \bigcap_{T \in \Theta \setminus U}(UT + U'T) = K.$$

In particular, $\alpha = \beta + \alpha'$ for some $\beta \in U$, $\alpha' \in U'$; since $\alpha' - \alpha = \beta \in U$, we have $e(\alpha, U) = \Phi(\alpha')$, as required. Hence, Φ is surjective and $K/D \simeq \bigoplus_{T \in \Theta} K/T$.

Let now M be a torsion D-module; then, since every T is flat, we have (see e.g. [11, p. 9–10])

$$M \simeq \mathrm{Tor}_1^D(K/D, M) \simeq \mathrm{Tor}_1^D\Big(\bigoplus_{T \in \Theta} K/T, M\Big) \simeq \bigoplus_{T \in \Theta} \mathrm{Tor}_1^T(K/T, M) \simeq \bigoplus_{T \in \Theta} M \otimes_D T,$$

as claimed. The final assertion follows from the fact that $D/I \otimes_D T \simeq T/IT$. □

Theorem 3.10. *Let D be an integral domain, and let Θ be a Jaffard family of D. For every length function $\ell \in \mathcal{L}_\infty(D)$, we have*

$$\ell = \sum_{T \in \Theta} \ell \otimes T.$$

Proof. Let $\ell^\sharp := \sum_{T \in \Theta} \ell \otimes T$; by Lemma 3.7, $\ell^\sharp$ is a length function on D. To show that $\ell = \ell^\sharp$, it is enough to show that $\ell(D/I) = \ell^\sharp(D/I)$ for every ideal I of D.

If $I = (0)$, then $\ell(D/I) = \infty = \ell^\sharp(D/I)$. Suppose $I \neq (0)$. By Lemmas 3.8 and 3.9, we have

$$\ell(D/I) = \ell\left(\bigoplus_{T \in \Theta} \frac{D}{I} \otimes T\right) = \sum_{T \in \Theta} \ell\left(\frac{D}{I} \otimes T\right) = \sum_{T \in \Theta} (\ell \otimes T)(D/I) = \ell^\sharp(D/I).$$

The claim is proved. □

Theorem 3.11. *Let Θ be a Jaffard family of D, and let Λ_Θ and Ψ_Θ be the maps*

$$\Lambda_\Theta : \mathcal{L}_\infty(D) \to \prod_{T \in \Theta} \mathcal{L}_\infty(T), \qquad \ell \mapsto (\ell_T)_{T \in \Theta},$$

$$\Psi_\Theta : \prod_{T \in \Theta} \mathcal{L}_\infty(T) \to \mathcal{L}_\infty(D), \qquad (\ell_{(T)})_{T \in \Theta} \mapsto \sum_{T \in \Theta} (\ell_{(T)})^D.$$

Then the following hold.
(a) *Λ_Θ and Ψ_Θ are order-preserving bijections between $\mathcal{L}_\infty(D)$ and $\prod_{T \in \Theta} \mathcal{L}_\infty(T)$, inverse one of each other.*
(b) *Λ_Θ restricts to a bijection from $\mathcal{L}_{\mathrm{sing}}(D)$ to $\prod_{T \in \Theta} \mathcal{L}_{\mathrm{sing}}(T)$.*
(c) *If Θ is finite, Λ_Θ restricts to a bijection from $\mathcal{L}_{\mathrm{disc}}(D)$ to $\prod_{T \in \Theta} \mathcal{L}_{\mathrm{disc}}(T)$.*

Proof. (a) By Propositions 3.1, 3.3 and Lemma 3.7, Λ_Θ and Ψ_Θ are well-defined.

By definition,

$$\Psi_\Theta \circ \Lambda_\Theta(\ell) = \sum_{T \in \Theta} (\ell_T)^D = \sum_{T \in \Theta} \ell \otimes T,$$

which is equal to ℓ by Theorem 3.10. Hence, $\Psi_\Theta \circ \Lambda_\Theta$ is the identity on $\mathcal{L}(D)$.

Take now

$$(\ell_{(T)})_{T \in \Theta} \in \prod_{T \in \Theta} \mathcal{L}_\infty(T).$$

Fix $U \in \Theta$, and let ℓ' be the component with respect to U of $(\Lambda_\Theta \circ \Psi_\Theta)(\ell^T)$. As in the proof of Theorem 3.10, we need to show that $\ell_{(U)}(U/J) = \ell'(U/J)$ for every ideal J of U. If $J = (0)$, then both sides are infinite; suppose $J \neq (0)$. Then

$$\ell'(U/J) = \ell_{(U)}\left(\sum_{T \in \Theta} (U/J) \otimes_D T\right) = \sum_{T \in \Theta} \ell_{(T)}((U/J) \otimes_D T).$$

If $U \neq T$, then

$$\frac{U}{J} \otimes_D T \simeq \frac{D}{J \cap D} \otimes_D U \otimes_D T \simeq \frac{T}{(J \cap D)T} \otimes U = (0)$$

since $(J \cap D)T = T$; hence, $\ell'(U/J)$ reduces to $\ell_{(U)}((U/J) \otimes_D U) = \ell_{(U)}(U/J)$. Therefore, $\ell' = \ell_{(U)}$, and so $\Lambda_\Theta \circ \Psi_\Theta$ is the identity, as claimed.

(b) If ℓ is singular, then so is every ℓ_T by Proposition 3.1 (b). Conversely, if every ℓ_T is singular, then $\sum_T \ell_T(M \otimes T)$ is always zero or infinite, and thus ℓ is singular.

(c) If ℓ is discrete, so is every ℓ_T by Proposition 3.1 (c). On the other hand, if Θ is finite, say

$$\Theta := \{T_1, \ldots, T_n\},$$

then $\mathrm{Im}(\ell) = \{a_1 + \cdots + a_n \mid a_i \in \mathrm{Im}(\ell_{T_i})\}$. Since each $\mathrm{Im}(\ell_{T_i})$ is discrete, so is $\mathrm{Im}(\ell)$. The claim is proved. □

The fact that every infinite length function ℓ can be "decomposed" as a sum of $\ell \otimes T$ is a rather special property, which puts some constraints on the family Θ.

Proposition 3.12. *Let D be an integral domain, and let Θ be a nonempty family of flat overrings of D.*
(a) *If, for every $\ell \in \mathcal{L}(D)$, we have $\ell = \sum_{T \in \Theta} \ell \otimes T$, then $\Theta = \{D\}$.*
(b) *If, for every $\ell \in \mathcal{L}_\infty(D)$, we have $\ell = \sum_{T \in \Theta} \ell \otimes T$, then for every nonzero prime P of D there is exactly one $T \in \Theta$ such that $PT \neq T$.*

Proof. Given a length function ℓ on D, let $\ell^\sharp := \sum_{T \in \Theta} \ell \otimes T$.

(a) Let ℓ be the rank function on D. Then $\ell(D) = \ell(T) = 1$ for every overring T of D, and in particular $(\ell \otimes T)(D) = 1$ for every flat overring T. Hence, $\ell^\sharp(D) = \sum_{T \in \Theta} 1 = |\Theta|$; since by hypothesis $\ell = \ell^\sharp$, we must have $|\Theta| = 1$; let $\Theta = \{T\}$.

Suppose $T \neq D$; then, since T is flat, there must be a prime ideal P of D such that $PT = T$. Let

$$\ell := (\mathrm{length}_{D_P})^D;$$

then

$$\ell(D/P) = \mathrm{length}_{D_P}(D/P \otimes_D D_P) = \mathrm{length}_{D_P}(D_P/PD_P) = 1.$$

On the other hand,

$$\ell^\sharp(D/P) = (\ell \otimes T)(D/P) = \ell(D/P \otimes T) = \ell(T/PT) = \ell(0) = 0,$$

against the hypothesis. Hence, T must be equal to D and $\Theta = \{D\}$, as claimed.

(b) Fix a nonzero prime ideal P of D, and let $\Theta' := \{P \in \Theta \mid PT \neq T\}$. Let $\ell := (\mathrm{length}_{D_P})^D$; then $\ell \in \mathcal{L}_\infty(D)$, and with the same calculation of the previous point, we see that $\ell(D/P) = 1$, while $\ell^\sharp(D/P) = |\Theta'|$, which means that $|\Theta'| = 1$. The claim is proved. $\qquad\square$

Remark 3.13. (i) Under the hypotheses of part (b) of the previous proposition, the only property needed to show that Θ is a Jaffard family is the local finiteness of Θ. We shall see in Example 6.9 that, at least for singular length functions, this is not actually necessary; on the other hand, we shall present in Example 6.10 the case of a one-dimensional domain having a length function ℓ that cannot be written as $\sum \ell \otimes D_M$ (with the sum ranging among the maximal ideals).

(ii) If Θ is an infinite Jaffard family, there could be elements in $\prod_T \mathcal{L}_{\mathrm{disc}}(T)$ that do not come from discrete length functions. For example, suppose $\Theta = \{T_1, \ldots, T_n, \ldots\}$ is countable and that, for every i, there is a discrete length function ℓ_i on T_i such that $\ell_i(M_i) \neq \{0, \infty\}$ for some torsion T_i-module M_i. By possibly multiplying for some constant, we can suppose $\ell_i(M_i) = 1/2^i$ for every i. Let $\ell := \Psi_\Theta(\ell_i)$. Then $\ell(M_i) = 1/2^i$ for every i; hence, we have $\ell(\bigoplus_{i \leq k} M_i) = 1 - 1/2^k$ and $\ell(\bigoplus_{i \in \mathbb{N}} M_i) = 1$. In particular, $\mathrm{Im}(\ell)$ is not discrete since 1 is not an isolated point of $\mathrm{Im}(\ell)$.

Example 3.14. Let D be a one-dimensional locally finite domain. Then $\{D_M \mid M \in \mathrm{Max}(D)\}$ is a Jaffard family of D. By Theorem 3.11, it follows that a length function ℓ on D is completely determined by the restrictions ℓ_{D_P}. In turn, this means that ℓ is completely determined by its values at $\ell(D_P/QD_P)$, where Q is a P-primary ideal; as we shall see in Proposition 4.4 below, this means that ℓ is determined by the values $\ell(D/Q)$, as Q ranges among the primary ideals of D.

Similarly, if D is an h-local domain (meaning that D is a locally finite domain such that every nonzero prime ideal is contained in only one maximal ideal), then $\{D_M \mid M \in \mathrm{Max}(D)\}$ is a Jaffard family, and thus every length function ℓ is determined at the local level.

4 Primary ideals

As shown by Proposition 3.12, the possibility of decomposing every length function ℓ as a sum of $\ell \otimes T$, as T ranges in a (fixed) family Θ, is a quite special property, since it is usually not easy to find Jaffard families of an integral domain D. In this section, and in the next one, we shall try to see when a decomposition $\ell = \sum_{T \in \Theta} \ell \otimes T$ can be reached if we allow Θ to be dependent on ℓ; to do so, we must treat length functions in a more intrinsic way.

Lemma 4.1. *Let ℓ be a length function of D, and let M be a D-module. Then we have $\ell(M) = 0$ if and only if $\ell(D/\mathrm{Ann}(x)) = 0$ for all $x \in M$.*

Proof. If $\ell(M) = 0$, then $\ell(N) = 0$ for all submodules N of M. In particular, this happens for $N = xD$ for every $x \in M$; however, $xD \simeq D/\mathrm{Ann}(x)$, and thus $\ell(D/\mathrm{Ann}(x)) = 0$.

Conversely, suppose $\ell(D/\mathrm{Ann}(x)) = 0$ for all $x \in M$. By upper continuity, it is enough to prove that $\ell(N) = 0$ for all finitely generated submodules N of M. Let thus $N = \langle x_1, \ldots, x_n \rangle$; then by [21, Proposition 2.2], $\ell(N) = \sum_i \ell(N_{i+1}/N_i)$, where $N_k := \langle x_1, \ldots, x_k \rangle$. However, N_{i+1}/N_i is a cyclic D-module, generated by $y := x_{i+1} + N_i$; in particular, it is isomorphic to $D/\mathrm{Ann}(y)$. However, $\mathrm{Ann}(y) \supseteq \mathrm{Ann}(x_{i+1})$, and thus $\ell(D/\mathrm{Ann}(y)) \leq \ell(D/\mathrm{Ann}(x_{i+1})) = 0$. Therefore, $\ell(N) = 0$, and the claim is proved. $\qquad\square$

The following is a slight generalization of [12, Lemma 2], of which we follow the proof.

Lemma 4.2. *Let R be a ring (not necessarily an integral domain); let $\alpha, \beta \in R$. Then the following hold.*
(a) $\ell(R/\alpha\beta R) \leq \ell(R/\alpha R) + \ell(R/\beta R)$.
(b) *If β is not a zero-divisor in R, then $\ell(R/\alpha\beta R) = \ell(R/\alpha R) + \ell(R/\beta R)$.*
(c) *If $\ell(R) < \infty$ and β is not a zero-divisor, then $\ell(R/\beta R) = 0$.*

Proof. Consider the exact sequence $0 \to \beta R/\alpha\beta R \to R/\alpha\beta R \to R/\beta R \to 0$. Then, by additivity,

$$\ell(R/\alpha\beta R) = \ell(R/\beta R) + \ell(\beta R/\alpha\beta R);$$

furthermore, $\ell(\beta R/\alpha\beta R) \leq \ell(R/\alpha R)$ since multiplication by β induces a surjection $R/\alpha R \twoheadrightarrow \beta R/\alpha\beta R$. (a) is proved.

If β is not a zero-divisor, then multiplication by β is an isomorphism, and thus (b) holds. In particular, if $\alpha = 0$, we have $\ell(R) = \ell(R/\beta R) + \ell(R)$, and if $\ell(R)$ is finite, then $\ell(R/\beta R)$ must be 0. Thus, (c) holds. $\square$

Lemma 4.3. *Let D be an integral domain, and let ℓ be a length function on D. Let Q be a P-primary ideal such that $\ell(D/Q) < \infty$, and let I be an ideal such that $Q \subsetneq I \nsubseteq P$. Then $\ell(D/I) = 0$.*

Proof. Let $x \in I \setminus P$; then $(Q : x) = Q$ since Q is P-primary [1, Lemma 4.4 (iii)], and in particular $\overline{x}$ is not a zero-divisor in $D/Q =: R$. By Lemma 4.2, it follows that $\ell_R(R/\overline{x}R) = 0$; however, by definition, we have $\ell_R(R/\overline{x}R) = \ell(D/(Q, x))$. Since $I \supseteq (Q, x)$, it follows that $\ell(D/I) \leq \ell(D/(Q, x)) = 0$, as claimed. $\square$

Proposition 4.4. *Let D be an integral domain, and let ℓ be a length function on D. Let $P' \subseteq P$ be two prime ideals, and let Q be a P'-primary ideal. Then $\ell(D/Q) = \ell(D_P/QD_P) = \ell_{D_P}(D_P/QD_P) = (\ell \otimes D_P)(D/Q)$.*

Proof. Since Q is P'-primary and $P' \subseteq P$, we have $Q = QD_{P'} \cap D = QD_P \cap D$, and thus there is an exact sequence
$$0 \to D/Q \xrightarrow{\iota} D_P/QD_P \xrightarrow{\pi} N \to 0$$
for some D-module N. If $\ell(D/Q) = \infty$, then also $\ell(D_P/QD_P) = \infty$, and we are done. Suppose $\ell(D/Q) < \infty$, and let $z = \pi(x) \in N$. Consider $I := \mathrm{Ann}(z)$; then I contains Q since $Q \subseteq \mathrm{Ann}(x)$. Furthermore, if

$$x = x' + QD_P \in D_P/QD_P,$$

there is an $s \in D \setminus P$ such that $sx' \in D$; hence, $sx \in \iota(D/Q)$, or equivalently $0 = \pi(sx) = sz$. Therefore, $I \nsubseteq P$, and thus $I \nsubseteq P'$; by Lemma 4.3, we have $\ell(D/I) = 0$. By Lemma 4.1, it follows that $\ell(N) = 0$, and thus $\ell(D/Q) = \ell(D_P/QD_P)$, as claimed.

The second and the third equalities come from the definitions. $\square$

In particular, we can obtain a version of Vámos' results in the vein of Theorem 3.10.

Proposition 4.5. *Let D be a Noetherian domain, let ℓ be a length function on D, and let*

$$\Sigma(\ell) := \{P \in \mathrm{Spec}(D) \mid \ell(D/P) > 0\}.$$

Then

$$\ell = \sum_{P \in \Sigma(\ell)} \ell \otimes D_P.$$

Proof. Let $\ell^\sharp := \sum_{P \in \Sigma(\ell)} \ell \otimes D_P$; then $\ell^\sharp$ is a length function. By [20, Corollary to Lemma 2], to show that $\ell = \ell^\sharp$ it is enough to show that $\ell(D/Q) = \ell^\sharp(D/Q)$ for every prime ideal Q of D.

If $Q \notin \Sigma(\ell)$, then $\ell(D/Q) = 0$ (by definition of $\Sigma(\ell)$). Furthermore, any ideal I properly containing Q satisfies $\ell(D/I) = 0$ (by [20, Lemma 3] or Lemma 4.3), and thus no prime ideal containing Q belongs to $\Sigma(\ell)$. However, if $Q \nsubseteq P$, then

$$(\ell \otimes D_P)(D/Q) = \ell(D/Q \otimes D_P) = \ell(D_P/QD_P) = \ell(0) = 0,$$

and thus also $\ell^\sharp(D/Q) = 0$.

Suppose now that $Q \in \Sigma(\ell)$. Then, using the previous reasoning, we have

$$\ell^\sharp(D/Q) = \sum_{\substack{P \in \Sigma(\ell) \\ P \supseteq Q}} (\ell \otimes D_P)(D/Q) = \sum_{\substack{P \in \Sigma(\ell) \\ P \supseteq Q}} \ell(D_P/QD_P) = \sum_{\substack{P \in \Sigma(\ell) \\ P \supseteq Q}} \ell(D/Q),$$

with the last equality coming from Proposition 4.4.

If Q is a maximal element of $\Sigma(\ell)$, then $\ell^\sharp(D/Q) = \ell(D/Q)$, as claimed. Suppose Q is not maximal in $\Sigma(\ell)$; then $\ell(D/Q) = \infty$ since $\ell(D/Q) > 0$ (being $Q \in \Sigma(\ell)$) and $\ell(D/Q)$ cannot be finite (using again [20, Lemma 3]/ Lemma 4.3). Hence, both $\ell(D/Q)$ and $\ell^\sharp(D/Q)$ are infinite.

Therefore, $\ell(D/Q) = \ell^\sharp(D/Q)$ for every prime ideal Q, and thus $\ell = \ell^\sharp$, as claimed. $\square$

More generally, Proposition 4.4 shows that, for a P-primary ideal Q, the value of $\ell(D/Q)$ depends only on ℓ_{D_P}, which is a length function over D_P. We now want to extend this result to ideals having a primary decomposition; we premise a lemma, which already appeared, without proof, in [20].

Lemma 4.6. *Let I, J be ideals of a ring R (not necessarily a domain), and let ℓ be a length function on R. Then*

$$\ell(R/I) + \ell(R/J) = \ell(R/(I + J)) + \ell(R/(I \cap J)).$$

Proof. To simplify the notation, let $\tau(A) := \ell(R/A)$ for every ideal A. From the two exact sequences

$$0 \to \frac{I}{I \cap J} \to \frac{D}{I \cap J} \to \frac{D}{I} \to 0 \quad \text{and} \quad 0 \to \frac{I + J}{J} \to \frac{D}{J} \to \frac{D}{I + J} \to 0$$

and additivity, we have

$$\begin{cases} \tau(I \cap J) = \tau(I) + \ell(I/(I \cap J)), \\ \tau(J) = \tau(I + J) + \ell((I + J)/J). \end{cases}$$

Hence,

$$\tau(I) + \tau(J) + \ell(I/(I \cap J)) = \tau(I \cap J) + \tau(I + J) + \ell((I + J)/J).$$

Since $I/(I \cap J) \simeq (I + J)/J$, the claim follows if $\ell(I/(I \cap J)) < \infty$. If not, then $\tau(J) \geq \ell((I + J)/J) = \infty$, and thus $\tau(I \cap J) \geq \tau(J) = \infty$; hence, both sides are infinite, and the claim again follows. $\square$

Proposition 4.7. *Let D be an integral domain, and let ℓ be a length function on D. Let I be an ideal of D having a primary decomposition $Q_1 \cap \cdots \cap Q_n$. Then*

$$\ell(D/I) = \sum_{i=1}^n \ell(D/Q_i).$$

Proof. Set $\tau(A) := \ell(D/A)$ for every ideal A of D.

If $\tau(Q_i) = \infty$ for some i, then $\tau(I) = \infty$, and we are done. Suppose $\tau(Q_i) < \infty$ for every i; we proceed by induction on the number n of components of I. If $n = 1$, the claim is obvious. Suppose the claim holds up to $n - 1$; without loss of generality, $P_1 := \mathrm{rad}(Q_1)$ is a minimal prime of I. Then $I = Q_1 \cap J$, where $J := Q_2 \cap \cdots \cap Q_n$. By Lemma 4.6, we have

$$\tau(I) + \tau(Q_1 + J) = \tau(Q_1) + \tau(J).$$

The ideal $Q_1 + J$ is not contained in P_1, for otherwise $J \subseteq P_1$, which is impossible since no primary component of J is contained in P_1. By Lemma 4.3, this implies that $\tau(Q_1 + J) = 0$, and thus $\tau(I) = \tau(Q_1) + \tau(J)$; the claim now follows by induction. $\square$

5 Prüfer domains

In this section, we obtain a standard representation for length functions on Prüfer domains where every ideal has only finitely many minimal primes.

The starting point for this section is the following extension of Proposition 4.4.

Proposition 5.1. *Let D be a Prüfer domain, and let ℓ be a length function on D. Let I be an ideal of D such that $P := \mathrm{rad}(I)$ is prime. Then $\ell(D/I) = \ell(D_P/ID_P) = (\ell \otimes D_P)(D/I)$.*

Proof. Let $J := ID_P \cap D$; then J is a P-primary ideal, and, by Proposition 4.4, $\ell(D/J) = \ell(D_P/JD_P) = \ell(D_P/ID_P)$, with the last equality coming from the fact that $JD_P = ID_P$. Thus, we must prove that $\ell(D/I) = \ell(D/J)$.

If $\ell(D/J) = \infty$, then $\ell(D/I) \geq \ell(D/J) = \infty$, and we are done. Suppose $\ell(D/J) < \infty$, and consider the exact sequence $0 \to J/I \to D/I \to D/J \to 0$. Then $\ell(D/I) = \ell(D/J) + \ell(J/I)$, and thus we need to show that $\ell(J/I) = 0$.

Let $x \in J$, and let $A := (I : x)$; we claim that $P \subsetneq A$. Indeed, let $M \in \mathrm{Max}(D)$; if $I \nsubseteq M$ (equivalently, if $J \nsubseteq M$), then $AD_M = (ID_M : x) = D_M$ contains $PD_M = D_M$. Suppose $I \subseteq M$; then $AD_M = (ID_M : xD_M)$. If $AD_M \nsupseteq PD_M$, then $AD_M \subsetneq PD_M$; localizing further at D_P, we have $AD_P \subseteq PD_P$. However, $AD_P = (ID_P : x) = D_P$, a contradiction. Hence, $P \subseteq A$; but since $AD_P \neq PD_P$, we must also $P \neq A$ and so $P \subsetneq A$.

Since $\ell(D/J) < \infty$, we can now apply Lemma 4.3, obtaining $\ell(D/A) = 0$; by Lemma 4.1, we have $\ell(J/I) = 0$ (since $(I : x)$ is equal to the annihilator of $x + I$ in J/I), and thus $\ell(D/I) = \ell(D/J)$, as claimed. □

Definition 5.2. Let D be a Prüfer domain, and let ℓ be a length function on D. The *total spectrum* of ℓ is

$$\Sigma(\ell) := \{P \in \mathrm{Spec}(D) \mid \ell(D/Q) > 0 \text{ for some } P\text{-primary ideal } Q\}.$$

Lemma 5.3. *Let D be a Prüfer domain, ℓ a length function on D, and let $P' \subsetneq P$ be prime ideals such that $P \in \Sigma(\ell)$. Then $\ell(D/P') = \infty$, and in particular $P' \in \Sigma(\ell)$.*

Proof. Let L be a P-primary ideal. Then $P' = P'D_P \cap D \subseteq LD_P \cap D = L$. If $\ell(D/P') < \infty$, then by Lemma 4.3 (applied with $Q = P'$), we would have $\ell(D/L) = 0$. Since L was an arbitrary P-primary ideal, it would follow that $P \notin \Sigma(\ell)$, against the hypothesis. Hence, $\ell(D/P') = \infty$. □

The total spectrum of ℓ is exactly the set we are looking for.

Theorem 5.4. *Let D be a Prüfer domain such that every ideal of D has only finitely many minimal primes. For every length function ℓ on D, we have*

$$\ell = \sum_{P \in \Sigma(\ell)} \ell \otimes D_P.$$

Proof. Let $\ell^\sharp := \sum_{P \in \Sigma(\ell)} \ell \otimes D_P$; then $\ell^\sharp$ is a length function by Lemma 3.7. To show that $\ell = \ell^\sharp$, it is enough to show that $\ell(D/I) = \ell^\sharp(D/I)$ for every ideal I of D. Let thus I be an ideal of D, and let $\{P_1, \ldots, P_n\}$ be the minimal primes of I.

For each i, let $T_i := \bigcap\{D_Q \mid Q \in \mathcal{V}(P_i)\}$, where $\mathcal{V}(A) := \{P \in \mathrm{Spec}(D) \mid A \subseteq P\}$; then T_i is a Prüfer domain whose prime ideals are the extension of the prime ideals comparable with P_i. Let $J_i := IT_i \cap D$; then $\mathrm{rad}(J_i) = P_i$, and, since every maximal ideal containing I survives in some T_i, we have $I = J_1 \cap \cdots \cap J_n$. Fix i, and let $L_i := \bigcap_{k \neq i} J_k$; then the minimal primes of L_i are $P_1, \ldots, P_{i-1}, P_{i+1}, \ldots, P_n$. In particular, since $\mathrm{rad}(J_i) = P_i$ and $\mathrm{Spec}(D)$ is a tree, there are no prime ideals containing both J_i and L_i; thus, $J_i + L_i = D$. By Lemma 4.6, it follows that $\ell'(D/I) = \ell'(D/(J_i \cap L_i)) = \ell'(D/J_i) + \ell'(D/L_i)$ for every length function ℓ'; by induction, it follows that $\ell'(D/I) = \sum_i \ell'(D/J_i)$ for every ℓ'. In particular, it holds for $\ell' = \ell$ and for $\ell' = \ell^\sharp$; hence, we need only to prove that $\ell(D/J_i) = \ell^\sharp(D/J_i)$ for every J_i, or equivalently that $\ell(D/J) = \ell^\sharp(D/J)$ for every J such that $\mathrm{rad}(J) = P \in \mathrm{Spec}(D)$.

By Proposition 5.1, $\ell(D/J) = (\ell \otimes D_P)(D/J)$. On the other hand,

$$\ell^\sharp(D/J) = \sum_{Q \in \Sigma(\ell)} (\ell \otimes D_Q)(D/J) = \sum_{Q \in \Sigma(\ell)} \ell(D_Q/JD_Q).$$

If $Q \nsupseteq P$, then $JD_Q = D_Q$, and so $\ell(D_Q/JD_Q) = 0$. Hence,

$$\ell^\sharp(D/J) = \sum_{\substack{Q \in \Sigma(\ell) \\ Q \supseteq P}} \ell(D_Q/JD_Q) = \sum_{\substack{Q \in \Sigma(\ell) \\ Q \supseteq P}} (\ell \otimes D_Q)(D/J). \tag{5.1}$$

If P is maximal in $\Sigma(\ell)$, then (5.1) reduces to $\ell^\sharp(D/J) = (\ell \otimes D_P)(D/J)$, and so is equal to $\ell(D/J)$. If P is not maximal, then by Lemma 5.3 $\ell(D/P) = \infty$, and so both $\ell(D/J)$ and $\ell^\sharp(D/J)$ are infinite; in particular, they are equal. The claim is proved. □

This theorem does not hold for general Prüfer domain. We shall see an example at the end of the paper (Example 6.10).

Since $\ell \otimes D_P = (\ell')^D$, where ℓ' is a length function on D_P, the previous theorem effectively reduces the study of the length functions on D to the local case. If V is a valuation domain, the length functions on V have been studied in [12, 21]; they can be divided into the following four classes.

- Torsion singular length functions: if $P \in \mathrm{Spec}(V)$, we define t_P as

$$t_P(M) := \begin{cases} 0 & \text{if } M \text{ is a torsion } V/P\text{-module,} \\ \infty & \text{otherwise.} \end{cases}$$

- Idempotent singular length functions: if $P \in \mathrm{Spec}(V)$ is idempotent, we define i_P as

$$i_P(M) := \begin{cases} 0 & \text{if } M \text{ is a } V/P\text{-module,} \\ \infty & \text{otherwise.} \end{cases}$$

- L-rank functions: if $P \in \mathrm{Spec}(V)$ is idempotent, and $\alpha \in \mathbb{R}^+$, then $\ell = \alpha \cdot \mathrm{rk}_P$ for some $\alpha \in \mathbb{R}^+$, where

$$\mathrm{rk}_P(M) := \begin{cases} \mathrm{rank}_{V/P}(M) & \text{if } M \text{ is a } V/P\text{-module,} \\ \infty & \text{otherwise.} \end{cases}$$

- Valuative length functions: let $P \in \mathrm{Spec}(D)$ be a branched prime ideal. Let Q be the largest prime ideal contained in P, and let v be a valuation on D_P/Q. We define L_v as the function

$$L_v(M) := \sup \sum_{i=1}^{s} \inf\{v(\alpha) \mid \alpha \in \mathrm{Ann}(E_i/E_{i-1})\},$$

where the supremum is taken over all finite chains of submodules $(0) = E_0 \subsetneq E_1 \subsetneq \cdots \subsetneq E_s = M$.

Remark 5.5. (i) The four classes of length functions are pairwise disjoint; however, the classes of idempotent singular length functions and of L-rank functions could be merged by considering the functions of type $\alpha \cdot \mathrm{rk}_P$ for $\alpha \in \mathbb{R}^{\geq 0}$, i.e., by allowing $\alpha = 0$ in the definition of L-rank function (and using the convention $0 \cdot \infty = \infty$). However, in view of the study of singular length functions (see Corollary 5.8 and Section 6 from below Proposition 6.7 onwards), it is more useful to consider them separately.

(ii) $i_{(0)}$ is the zero length function.

(iii) The rank function on V is $\mathrm{rk}_{(0)}$; on the other hand, if M is the maximal ideal of V, then the "usual" length is rk_M if M is idempotent, while if $M = mV$ is principal (and thus, in particular, branched), it is equal to L_v, where v is a rank-one valuation on V/Q normalized in such a way that $v(\overline{m}) = 1$.

Suppose now that D is a Prüfer domain, $P \in \mathrm{Spec}(D)$, and let ℓ be a length function on the valuation domain D_P. Calculating $\ell^D(D/I)$ (where I is an ideal of D) corresponds to calculating the values of $\ell(D_P/ID_P)$, which can be done easily by considering the various classes of length function of valuation domains; the results are the following:

$$(t_{PD_P})^D(D/I) = \begin{cases} 0 & \text{if } PD_P \subsetneq ID_P, \\ \infty & \text{otherwise;} \end{cases}$$

$$(i_{PD_P})^D(D/I) = \begin{cases} 0 & \text{if } PD_P \subseteq ID_P, \\ \infty & \text{otherwise;} \end{cases}$$

$$(\mathrm{rk}_{PD_P})^D(D/I) = \begin{cases} \mathrm{rank}_{D/P}(D_P/ID_P) & \text{if } PD_P \subseteq ID_P, \\ \infty & \text{otherwise} \end{cases} = \begin{cases} 0 & \text{if } PD_P \subsetneq ID_P, \\ 1 & \text{if } PD_P = ID_P, \\ \infty & \text{otherwise;} \end{cases}$$

$$(L_v)^D(D/I) = \begin{cases} 0 & \text{if } I \not\subseteq P, \\ \inf v(ID_P/QD_P) & \text{if } P \text{ is minimal over } I, \\ \infty & \text{otherwise.} \end{cases}$$

Let now ℓ be a length function on the Prüfer domain D (not necessarily satisfying the hypothesis of Theorem 5.4). To each class of length functions on valuation domains, we can associate a subset of the spectrum of D:

- $\Sigma_t(\ell) := \{P \in \mathrm{Spec}(D) \mid \ell_{D_P} = t_{PD_P}\}$,
- $\Sigma_i(\ell) := \{P \in \mathrm{Spec}(D) \mid \ell_{D_P} = i_{PD_P}\}$,
- $\Sigma_r(\ell) := \{P \in \mathrm{Spec}(D) \mid \ell_{D_P} = \alpha \mathrm{rk}_{PD_P} \text{ for some } \alpha \in \mathbb{R}^+\}$,
- $\Sigma_v(\ell) := \{P \in \mathrm{Spec}(D) \mid \ell_{D_P} = L_v \text{ for some rank-one valuation } v \text{ on } D_P/QD_P\}$.

Lemma 5.6. *Let D be a Prüfer domain, ℓ a length function.*
(a) *The four sets $\Sigma_t(\ell), \Sigma_i(\ell), \Sigma_r(\ell)$ and $\Sigma_v(\ell)$ are pairwise disjoint.*
(b) *$\Sigma(\ell) = \Sigma_t(\ell) \cup \Sigma_i(\ell) \cup \Sigma_r(\ell) \cup \Sigma_v(\ell)$.*
(c) *If $P, Q \in \mathrm{Spec}(D)$ are such that $P \in \Sigma(\ell)$ and $Q \subsetneq P$, then $Q \in \Sigma_t(\ell)$.*

Proof. (a) follows from the fact that the four classes of length functions on valuation domains are disjoint; (b) follows from the calculations of $\ell^D(D/I)$. (c) is another version of Lemma 5.3 since $\ell(D/Q) = \infty$ if and only if $Q \in \Sigma_t(\ell)$. □

With this terminology, we get a restatement of Theorem 5.4 and a way to characterize singular and discrete length functions.

Corollary 5.7. *Let D be a Prüfer domain such that every ideal of D has only finitely many minimal primes, and let ℓ be a length function on D. Then*

$$\ell = \sum_{P \in \Sigma_t(\ell)} (t_{PD_P})^D + \sum_{P \in \Sigma_i(\ell)} (i_{PD_P})^D + \sum_{P \in \Sigma_r(\ell)} \ell(D/P)(\mathrm{rk}_{PD_P})^D + \sum_{P \in \Sigma_v(\ell)} (L_{v_P})^D,$$

where v_P is a rank-one valuation on D_P/QD_P (and Q is the prime ideal directly below P).

Corollary 5.8. *Let D be a Prüfer domain such that every ideal of D has only finitely many minimal primes, and let ℓ be a length function on D.*
(a) *ℓ is singular if and only if $\Sigma_r(\ell) = \Sigma_v(\ell) = \emptyset$.*
(b) *ℓ is discrete if and only if $\Sigma_v(\ell)$ does not contain any idempotent prime and the family*

$$\{\ell(D/P) \mid P \in \Sigma_r(\ell) \cup \Sigma_v(\ell)\}$$

is discrete.

The representation of Corollary 5.7 can also be seen as a way to define a length function: given

$$\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4 \subseteq \mathrm{Spec}(D),$$

real numbers $\alpha_P \in \mathbb{R}^+$ and valuations v_P on D_P/LD_P, we can define a length function ℓ by

$$\ell := \sum_{P \in \Sigma_1} (t_{PD_P})^D + \sum_{P \in \Sigma_2} (i_{PD_P})^D + \sum_{P \in \Sigma_3} \alpha_P(\mathrm{rk}_{PD_P})^D + \sum_{P \in \Sigma_4} (L_{v_P})^D.$$

In general, there is no guarantee that this representation is the same as the one obtained in the corollary, i.e., the conditions $\Sigma_1 = \Sigma_t(\ell)$, $\Sigma_2 = \Sigma_i(\ell)$, etc. need not to be satisfied; indeed, being arbitrary, the sets Σ_j usually do not satisfy conditions (a) and (c) of Lemma 5.6. For example, if $P \nsubseteq Q$ are two prime ideals, the families $\Sigma_1 = \{Q\}$, $\Sigma_2 = \Sigma_3 = \Sigma_4 = \emptyset$ and $\Sigma_1' = \{Q, P\}$, $\Sigma_2' = \Sigma_3' = \Sigma_4' = \emptyset$ give rise to the same ℓ.

However, we can obtain uniqueness just by excluding the more obvious problems. To express it in a slightly less unwieldy way, we introduce the following definition.

Definition 5.9. Let $(\mathcal{P}, \leq)$ be a partially ordered set. A family $\{X_1, \dots, X_n\}$ of subsets of $\mathcal{P}$ is a *layered family with core X_k* if

- $X_i \cap X_j = \emptyset$ if $i \neq j$,
- if $x \in \bigcup_i X_i$ and $y < x$, then $y \in X_k$.

Under this terminology, Lemma 5.6 (a) and (c) can be reparaphrased by saying that $\{\Sigma_t(\ell), \Sigma_i(\ell), \Sigma_r(\ell), \Sigma_v(\ell)\}$ is a layered family with core $\Sigma_t(\ell)$.

Proposition 5.10. *Let D be a Prüfer domain. Let $\{\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4\}$ and $\{\Sigma_1', \Sigma_2', \Sigma_3', \Sigma_4'\}$ be layered families of* Spec(D) *with core Σ_1 and Σ_1', respectively, and suppose that*

- *each prime in $\Sigma_2, \Sigma_3, \Sigma_2'$ and Σ_3' is idempotent,*
- *each prime in Σ_4 and Σ_4' is branched,*
- *if P is unbranched and every prime properly contained in P belongs to Σ_1 (resp., Σ_1'), then $P \in \Sigma_1 \cup \Sigma_2 \cup \Sigma_3$ (resp., $P \in \Sigma_1' \cup \Sigma_2' \cup \Sigma_3'$).*

Furthermore, for every $P \in \Sigma_3$ (resp., $P \in \Sigma_3'$), let $\alpha_P \in \mathbb{R}^+$ (resp., $\alpha_P' \in \mathbb{R}^+$), and for every $P \in \Sigma_4$ (resp., $P \in \Sigma_4'$), let v_P (resp., v_P') be a valuation relative to D_P/QD_P, where Q is the largest prime properly contained in P. Let ℓ and ℓ' be the length functions

$$\ell = \sum_{P \in \Sigma_1} (t_{PD_P})^D + \sum_{P \in \Sigma_2} (i_{PD_P})^D + \sum_{P \in \Sigma_3} \alpha_P (\mathrm{rk}_{PD_P})^D + \sum_{P \in \Sigma_4} (L_{v_P})^D,$$

$$\ell' = \sum_{P \in \Sigma_1'} (t_{PD_P})^D + \sum_{P \in \Sigma_2'} (i_{PD_P})^D + \sum_{P \in \Sigma_3'} \alpha_P' (\mathrm{rk}_{PD_P})^D + \sum_{P \in \Sigma_4'} (L_{v_P'})^D.$$

Then $\ell = \ell'$ if and only if the following hold:

- $\Sigma_j = \Sigma_j'$ *for $j = 1, 2, 3, 4$;*
- $\alpha_P = \alpha_P'$ *for all $P \in \Sigma_3$;*
- $v_P = v_P'$ *for all $P \in \Sigma_4$.*

Proof. If the three properties hold, then clearly $\ell = \ell'$. Suppose now that $\ell = \ell'$. Let $\Sigma := \bigcup_j \Sigma_j$ and $\Sigma' := \bigcup_j \Sigma_j'$.

Let P be a prime ideal of D. Then a direct calculation shows that

$$\ell(D/P) = \begin{cases} 0 & \text{if } P \in \Sigma_2 \cup \Sigma_4 \text{ or } P \notin \Sigma, \\ \alpha_P & \text{if } P \in \Sigma_3, \\ \infty & \text{if } P \in \Sigma_1, \end{cases}$$

and analogously (*mutatis mutandis*) for ℓ'. In particular, $\ell(D/P) = \infty$ if and only if $P \in \Sigma_1$, and $\ell'(D/P) = \infty$ if and only if $P \in \Sigma_1'$; since $\ell = \ell'$, it follows that $\Sigma_1 = \Sigma_1'$. Likewise, $\ell(D/P) \notin \{0, \infty\}$ if and only if $P \in \Sigma_3$, and analogously for ℓ'; hence, $\Sigma_3 = \Sigma_3'$ and $\alpha_P = \alpha_P'$ for all $P \in \Sigma_3$.

Let now $\mathcal{B}$ and $\mathcal{U}$, respectively, be the set of branched and unbranched prime ideals of D. Let P be a branched prime ideal, and consider a P-primary ideal L with $L \subsetneq P$. Another calculation shows that

$$\ell(D/L) = \begin{cases} 0 & \text{if } P \notin \Sigma, \\ \inf v_P(LD_P/QD_P) & \text{if } P \in \Sigma_4, \\ \infty & \text{if } P \in \Sigma_1 \cup \Sigma_2 \cup \Sigma_3, \end{cases}$$

where Q is the biggest prime ideal properly contained in P. In particular, $P \in \Sigma$ if and only if $\ell(D/L) > 0$ (since $L \subsetneq P$); hence, $\Sigma \cap \mathcal{B} = \Sigma' \cap \mathcal{B}$. Furthermore, $\ell(D/L) = \infty$ if and only if $P \in \Sigma_1 \cup \Sigma_2 \cup \Sigma_3$, and similarly for ℓ'; since $\Sigma_1 = \Sigma_1'$ and $\Sigma_3 = \Sigma_3'$ by the previous reasoning, and the Σ_i and Σ_i' are disjoint, it follows that $\Sigma_2 \cap \mathcal{B} = \Sigma_2' \cap \mathcal{B}$.

Let P be an unbranched prime ideal. If $P \notin \Sigma$, then by hypothesis, there is a $Q \subsetneq P$ not contained in Σ; in particular (eventually passing to the minimal prime of an $x \in P \setminus Q$), we can suppose that Q is branched. By the previous reasoning, $Q \notin \Sigma'$, and thus also $P \notin \Sigma'$. By the same reasoning, if $P \notin \Sigma'$, then $P \notin \Sigma$; hence, $\Sigma \cap \mathcal{U} = \Sigma' \cap \mathcal{U}$. Furthermore, $\Sigma_4 \cap \mathcal{U} = \emptyset = \Sigma_4' \cap \mathcal{U}$; hence, $\Sigma_2 \cap \mathcal{U} = \Sigma_2' \cap \mathcal{U}$.

Putting together the two cases, we see that $\Sigma = \Sigma'$ and $\Sigma_2 = \Sigma_2'$; hence, $\Sigma_4 = \Sigma_4'$. Moreover,

$$\inf v_P(LD_P/QD_P) = \inf v_P'(LD_P/QD_P)$$

implies that $v_P = v_P'$. The claim is proved. □

Remark 5.11. If we drop the hypothesis on the unbranched prime ideals, the proposition above does not hold. For example, let P be an unbranched prime ideal of D, and let Δ be the set of prime ideals properly

contained in P. Then

$$\left(\sum_{Q\in\Delta}(t_{QD_Q})^D\right)(D/I) = \begin{cases} 0 & \text{if } QD_Q \subsetneqq ID_Q \text{ for every } Q \in \Delta, \\ \infty & \text{otherwise.} \end{cases}$$

The first condition holds if and only if $PD_P \subseteq ID_P$; hence, $\sum_{Q\in\Delta}(t_{QD_Q})^D = (i_{PD_P})^D$. In the notation of Proposition 5.10, this means the length functions induced by the families $\Sigma_1 = \Delta$, $\Sigma_2 = \Sigma_3 = \Sigma_4 = \emptyset$ and $\Sigma_1' = \Delta$, $\Sigma_2' = \{P\}$, $\Sigma_3' = \Sigma_4' = \emptyset$ are the same, and thus uniqueness does not hold.

As a consequence, we can prove that (under the hypotheses of Theorem 5.4) the set $\mathcal{L}(D)$ depends only on $\mathrm{Spec}(D)$ and the idempotence of the primes of D.

Theorem 5.12. *Let A, B be Prüfer domains such that every ideal of A and B has only finitely many minimal primes. Suppose that there is a homeomorphism $\phi\colon \mathrm{Spec}(A) \to \mathrm{Spec}(B)$ such that a prime ideal P is idempotent if and only if $\phi(P)$ is idempotent. Then there is an order isomorphism $\overline{\phi}\colon \mathcal{L}(A) \to \mathcal{L}(B)$ that respects the classes of $\ell \otimes D_P$, i.e., such that $\phi(\Lambda(\ell)) = \Lambda(\overline{\phi}(\ell))$ for each $\Lambda \in \{\Sigma_t, \Sigma_i, \Sigma_r, \Sigma_v\}$.*

Proof. We first note that, if $P \in \mathrm{Spec}(A)$, then P is branched if and only if $\phi(P)$ is branched; indeed, the homeomorphism ϕ induces homeomorphisms $\phi_P\colon \mathrm{Spec}(A_P) \to \mathrm{Spec}(B_{\phi(P)})$, and the maximal ideal M of a valuation ring V is branched if and only if $\mathrm{Spec}(V) \setminus \{M\}$ is compact. Since $\mathrm{Spec}(A_P) \setminus \{PA_P\}$ corresponds to $\mathrm{Spec}(B_{\phi(P)}) \setminus \{\phi(P)B_{\phi(P)}\}$, we have that P is branched if and only if $\phi(P)$ is branched.

For every branched prime ideal P of A, fix a valuation v_P on A_P/LA_P (viewed as a map to $\mathbb{R}$), where L is the largest prime properly contained in P; then all valuations on A_P/LA_P are in the form λv_P for some $\lambda \in \mathbb{R}^+$. Similarly, if P is a branched prime ideal of B, fix a valuation w_P on B_P/LB_P (with the same notation for L).

Let $\ell \in \mathcal{L}(A)$. By Corollary 5.7, and with the notation as above, we can write

$$\ell = \sum_{P\in\Sigma_t(\ell)}(t_{PA_P})^A + \sum_{P\in\Sigma_i(\ell)}(i_{PA_P})^A + \sum_{P\in\Sigma_r(\ell)}\ell(A/P)(\mathrm{rk}_{PA_P})^A + \sum_{P\in\Sigma_v(\ell)}(L_{\lambda_P v_P})^A;$$

hence, we define

$$\overline{\phi}(\ell) := \sum_{Q\in\phi(\Sigma_t(\ell))}(t_{QB_Q})^B + \sum_{Q\in\phi(\Sigma_i(\ell))}(i_{QB_Q})^B + \sum_{Q\in\phi(\Sigma_r(\ell))}\ell(A/\phi^{-1}(Q))(\mathrm{rk}_{QB_Q})^B + \sum_{Q\in\phi(\Sigma_v(\ell))}(L_{\lambda_{\phi^{-1}(Q)} w_Q})^B.$$

By the previous reasoning, every prime of $\phi(\Sigma_i(\ell))$ and $\phi(\Sigma_r(\ell))$ is idempotent and every prime of $\phi(\Sigma_v(\ell))$ is branched; hence, $\overline{\phi}$ is a well-defined map from $\mathcal{L}(A)$ to $\mathcal{L}(B)$. Furthermore, since ϕ is a homeomorphism, it is straightforward to see that $\{\phi(\Sigma_t(\ell)), \phi(\Sigma_i(\ell)), \phi(\Sigma_r(\ell)), \phi(\Sigma_v(\ell))\}$ is a layered family with core $\phi(\Sigma_t(\ell))$, and that if every prime contained in the branched prime Q of B is in $\phi(\Sigma_t(\ell))$, then Q is in $\phi(\Sigma_t(\ell)) \cup \phi(\Sigma_i(\ell)) \cup \phi(\Sigma_r(\ell))$. By Proposition 5.10, thus, $\overline{\phi}$ is injective, and it respects the classes of $\ell \otimes D_P$.

With the same reasoning, we can build an injective map $\overline{\phi^{-1}}\colon \mathcal{L}(B) \to \mathcal{L}(A)$, and it is an easy verification that $\overline{\phi^{-1}}$ is the inverse of $\overline{\phi}$. Hence, $\overline{\phi}$ and $\overline{\phi^{-1}}$ are bijections between $\mathcal{L}(A)$ and $\mathcal{L}(B)$.

Suppose now $\ell_1 \le \ell_2$ are length functions on A: we claim that $\overline{\phi}(\ell_1) \le \overline{\phi}(\ell_2)$. It is enough to verify the inequality at B/J, where J is an ideal of B; furthermore, with the same reasoning of the proof of Theorem 5.4, we can reduce this verification to the case where $\mathrm{rad}(J) =: Q$ is a prime ideal, and by Proposition 5.1, we can further suppose that J is a primary ideal. Let $P := \phi^{-1}(Q)$.

If $P \notin \Sigma(\ell_1)$, then $Q \notin \phi(\Sigma(\ell_1)) = \Sigma(\overline{\phi}(\ell_1))$, and thus, by Proposition 5.1, we have $\ell_1(B/J) = 0$. Hence, $\overline{\phi}(\ell_1)(B/J) \le \overline{\phi}(\ell_2)(B/J)$. Suppose thus that $P \in \Sigma(\ell_1)$, i.e., that $Q \in \Sigma(\overline{\phi}(\ell_1))$.

Then $\overline{\phi}(\ell_1)(B/J) = (\overline{\phi}(\ell_1) \otimes B_Q)(B/J)$. Furthermore, if Q is a prime ideal such that $Q \notin \phi(\Sigma_v(\ell))$ (equivalently, if $P \notin \Sigma_v(\ell)$), then $\ell(A/J) = \overline{\phi}(\ell)(B/Q)$ for every length function ℓ; in particular,

$$\overline{\phi}(\ell_1)(B/Q) = \ell_1(A/P) \le \ell_2(A/P) = \overline{\phi}(\ell_2)(B/Q).$$

Thus, if $J = Q$ we are done; suppose now that $J \subsetneqq Q$.

If $P \notin \Sigma_v(\ell)$ and L_1, L_2 are P-primary ideals such that $L_1, L_2 \subsetneqq P$, then $\ell(A/L_1) = \ell(A/L_2)$ (equal to 0 if $P \notin \Sigma(\ell)$ and to ∞ if $P \in \Sigma(\ell)$); hence, if $P \in \Sigma(\ell_1) \setminus \Sigma_v(\ell)$, then $\ell_1(A/L) = \infty$ for every P-primary ideal $L \subsetneqq P$, which means that $\ell_2(A/L) = \infty$; it follows that also $\overline{\phi}(\ell_1)(B/J) = \infty = \overline{\phi}(\ell_2)(B/J)$, and in particular $\overline{\phi}(\ell_1)(B/J) \le \overline{\phi}(\ell_2)(B/J)$.

Hence, we only need to consider that case $P \in \Sigma_v(\ell_1)$; in particular, P is branched, and so Q is as well. If $P \notin \Sigma_v(\ell_2)$, then $\ell_2(A/L) = \infty$ for all P-primary ideals $L \subsetneq P$; hence $\overline{\phi}(\ell_1)(B/J) \le \overline{\phi}(\ell_2)(B/J)$ since the latter is equal to ∞.

Suppose thus that $P \in \Sigma_v(\ell_1) \cap \Sigma_v(\ell_2)$; let P' be the largest prime ideal of A properly contained in P, and let Q' be the largest prime ideal of B properly contained in Q. Let $\pi_Q : B_Q \to B_Q/Q'B_Q$ be the canonical quotient map. If QB_Q is principal, then $JB_Q = (QB_Q)^n$ for some n, and we define $I := (PA_P)^n \cap A$. If QB_Q is not principal, then let $\delta := \inf\{w_Q(\pi_Q(x)) \mid x \in J\} \in \mathbb{R}$. If $\pi_P : A_P \to A_P/P'A_P$ is the quotient, then we define $I := \{y \in A \mid v_P(\pi_P(y)) \ge \delta\}$. In both cases, I is an ideal of A whose radical is P, and by construction $\ell(A/I) = \delta = \overline{\phi}(\ell)(B/J)$ for every length function ℓ such that $P \in \Sigma_v(\ell)$; as above, it follows that $\overline{\phi}(\ell_1)(B/J) \le \overline{\phi}(\ell_2)(B/J)$.

Therefore, $\overline{\phi}(\ell_1) \le \overline{\phi}(\ell_2)$, and so $\overline{\phi}$ is order-preserving. By symmetry, the same happens for $\overline{\phi^{-1}} = \overline{\phi}^{-1}$; hence, $\overline{\phi}$ is an order isomorphism, as claimed. $\qquad\square$

Remark 5.13. A different way to reduce the study of length functions over Prüfer domains to valuation rings is to apply the theory developed in Section 3 on Jaffard families, together with some results involving quotients and a recursive algorithm, in a manner similar to what has been done in [19] for the study of star and semistar operations. However, this method yields results that are at the same time weaker and more complex than the ones obtained in this section; for the sake of completeness, we summarize here briefly the main ideas of this alternative argument.

Let D be a Prüfer domain. Under some hypothesis on D (namely that D is semilocal or that $\mathrm{Spec}(D)$ is a Noetherian topological space), it is possible to construct a canonical Jaffard family Θ of D (the "standard decomposition") that is the finest possible, in the sense that each $T \in \Theta$ does not possess any nontrivial Jaffard family (see [18, Section 6]). For example, if D is also finite-dimensional, any member of Θ is obtained by an intersection $\bigcap\{D_M \mid M \in \mathrm{Max}(D), P \subseteq M\}$, where P is a prime ideal of D of height 1.

Take a $T \in \Theta$. If T is semilocal, we can find a nonzero prime ideal $Q \subseteq \mathrm{Jac}(T)$ such that $\mathrm{Jac}(T/Q)$ does not contain any nonzero prime. (Here $\mathrm{Jac}(A)$ denotes the Jacobson radical of A, that is, the intersection of its maximal ideals.) Extending the results of Section 3 to quotients, it is possible to express $\mathcal{L}_\infty(T)$ as the union of $\mathcal{L}_\infty(T/Q)$ and a distinguished subset of $\mathcal{L}_\infty(T_Q)$; since T_Q is a valuation domain, the latter is known, while by construction T/Q admits a nontrivial standard decomposition $\Theta_{T/Q}$. Hence, it is possible to repeat the construction starting from T/Q instead of D.

Suppose that, at some point, this procedure stops by reaching valuation domains; then, since the sets $\mathcal{L}_\infty(V)$ of these rings are known, by climbing back quotients and localizations we can obtain a description of the set $\mathcal{L}_\infty(D)$. This works, for example, if D is semilocal or if it is locally finite and finite-dimensional.

However, this algorithm involves alternating two types of procedures (finding a Jaffard family and taking quotients), whose effect on the set of length functions is quite different. Therefore, this description becomes extremely complex, especially when an element T of the Jaffard family Θ is quotiented more than once. Finally, the hypotheses that are needed to be put on D for this algorithm to work are less general than the one used in this section; indeed, if D is semilocal or if it is locally finite and finite-dimensional, then every nonzero ideal I is contained in only finitely many maximal ideals, and thus (since $\mathrm{Spec}(D)$ is a tree), I has only finitely many minimal primes.

6 Singular length functions

In this section, we characterize singular length functions through purely ideal-theoretic means, by using the concept of localizing system (see [5] or [6, Section 5.1]). We denote by $\mathcal{I}(D)$ the set of ideals of D.

Definition 6.1. Let D be an integral domain. A *localizing system* on D is a set $\mathcal{F} \subseteq \mathcal{I}(D)$ such that,

- if $I \in \mathcal{F}$ and $I \subseteq J$, then $J \in \mathcal{F}$,
- if $I \in \mathcal{F}$ and $(J : iD) \in \mathcal{F}$ for all $i \in I$, then $J \in \mathcal{F}$.

We denote by $\mathrm{LocSist}(D)$ the set of localizing systems on D.

Our next aim is to prove that to every length function can be associated a localizing system, and conversely.

Definition 6.2. Let ℓ be a length function. The *zero locus* of ℓ is $Z(\ell) := \{I \in \mathcal{I}(D) \mid \ell(D/I) = 0\}$.

Proposition 6.3. *The zero locus of a length function ℓ is a localizing system.*

Proof. If $I \in Z(\ell)$ and $I \subseteq J$, then $\ell(D/J) \leq \ell(D/I) = 0$, and so $J \in Z(\ell)$. Suppose $I \in Z(\ell)$, and let J be an ideal such that $(J : iD) \in Z(\ell)$ for every $i \in I$. From the exact sequence $0 \to (I + J)/J \to D/J \to D/(I + J) \to 0$, and from the fact that $I + J \in Z(\ell)$ (since $I + J \supseteq I \in Z(\ell)$), we have $\ell(D/J) = \ell((I + J)/J)$. For every $i + J \in (I + J)/J$, we have $\mathrm{Ann}(i + J) = (J : iD) \in Z(\ell)$ by hypothesis; by Lemma 4.1, it follows that $\ell((I + J)/J) = 0$, and thus $\ell(D/J) = 0$, i.e., $J \in Z(\ell)$. Thus, $Z(\ell)$ is a localizing system. $\square$

Conversely, let $\mathcal{F}$ be a localizing system on D. The *length function associated to $\mathcal{F}$* is

$$\ell_{\mathcal{F}}(M) := \begin{cases} 0 & \text{if } \mathrm{Ann}(x) \in \mathcal{F} \text{ for all } x \in M, \\ \infty & \text{otherwise.} \end{cases}$$

Proposition 6.4. *For any localizing system $\mathcal{F}$ on D, $\ell_{\mathcal{F}}$ is a length function.*

Proof. Clearly, $\ell_{\mathcal{F}}$ is upper continuous. Let $0 \to M_1 \to M_2 \xrightarrow{\pi} M_3 \to 0$ be an exact sequence of D-modules. If $\ell_{\mathcal{F}}(M_2) = 0$, then $\mathrm{Ann}(x) \in \mathcal{F}$ for all $x \in M_2$; in particular, $\mathrm{Ann}(y) \in \mathcal{F}$ for all $y \in M_1$ (and so $\ell_{\mathcal{F}}(M_1) = 0$) and $\mathrm{Ann}(z) = \mathrm{Ann}(\pi(x)) \supseteq \mathrm{Ann}(x)$ for all $z = \pi(x) \in M_3$ (and so $\ell_{\mathcal{F}}(M_3) = 0$). In particular,

$$\ell_{\mathcal{F}}(M_2) = \ell_{\mathcal{F}}(M_1) + \ell_{\mathcal{F}}(M_3).$$

Suppose now $\ell_{\mathcal{F}}(M_2) = \infty$; then there is an $x \in M_2$ such that $\mathrm{Ann}(x) \notin \mathcal{F}$. Suppose $\ell_{\mathcal{F}}(M_1) = \ell_{\mathcal{F}}(M_3) = 0$, let $z := \pi(x)$, and consider the exact sequence $0 \to xD \cap M_1 \to xD \to zD \to 0$. Since we supposed $\ell_{\mathcal{F}}(M_3) = 0$, we must have $I := \mathrm{Ann}(z) \in \mathcal{F}$; furthermore, for every $i \in I$, we have $ix \in xD \cap M_1$, and thus $\mathrm{Ann}(ix) \in \mathcal{F}$. However, $\mathrm{Ann}(ix) = (\mathrm{Ann}(x) : iD)$; since $I \in \mathcal{F}$, this would mean that $\mathrm{Ann}(x) \in \mathcal{F}$, against the hypothesis on x. Therefore, one between $\ell_{\mathcal{F}}(M_1)$ and $\ell_{\mathcal{F}}(M_3)$ is infinite, and thus $\ell_{\mathcal{F}}(M_2) = \ell_{\mathcal{F}}(M_1) + \ell_{\mathcal{F}}(M_3)$. Hence, $\ell_{\mathcal{F}}$ is a length function. $\square$

Theorem 6.5. *Let D be an integral domain. The two maps*

$$\mathcal{L}_{\mathrm{sing}}(D) \to \mathrm{LocSist}(D), \ \ell \mapsto Z(\ell),$$
$$\mathrm{LocSist}(D) \to \mathcal{L}_{\mathrm{sing}}(D), \quad \mathcal{F} \mapsto \ell_{\mathcal{F}}$$

are bijections, one inverse of the other. Furthermore, if $\mathrm{LocSist}(D)$ is endowed with the containment order, they are order-reversing isomorphisms.

Proof. Since a singular length function is characterized by its zero locus, we need to show that $Z(\ell_{\mathcal{F}}) = \mathcal{F}$ and that $\ell_{Z(\ell)} = \ell$. Indeed,
$$Z(\ell_{\mathcal{F}}) = \{I \in \mathcal{I}(D) \mid \ell_{\mathcal{F}}(D/I) = 0\}.$$

However, for every $x \in D/I$, we have $\mathrm{Ann}(x) \supseteq \mathrm{Ann}(1 + I) = I$, and thus $\ell_{\mathcal{F}}(D/I) = 0$ if and only if $I \in Z(\ell)$. Therefore, $Z(\ell_{\mathcal{F}}) = \mathcal{F}$.

On the other hand,

$$\ell_{Z(\ell)}(M) = \begin{cases} 0 & \text{if } \mathrm{Ann}(x) \in Z(\ell) \text{ for all } x \in M, \\ \infty & \text{otherwise.} \end{cases}$$

If $\mathrm{Ann}(x) \in Z(\ell)$ for all $x \in M$, then $\ell(M) = 0$ by Lemma 4.1, while if $\mathrm{Ann}(x) \notin Z(\ell)$ for some $x \in M$, then $xD \simeq D/\mathrm{Ann}(x)$, and thus, since ℓ is singular, $\ell(M) \geq \ell(xD) = \infty$. Thus, $\ell_{Z(\ell)} = \ell$, as claimed.

The last claim follows from the fact that, for singular length functions, one has $\ell_1 \leq \ell_2$ if and only if $Z(\ell_1) \supseteq Z(\ell_2)$. $\square$

Localizing systems are also closely related to the concept of stable semistar operations. Let $\mathbf{F}(D)$ be the set of D-submodules of the quotient field K; a *stable semistar operation* on D is a map $\star : \mathbf{F}(D) \to \mathbf{F}(D)$ such that,

for every $I, J \in \mathbf{F}(D)$ and every $x \in K$,

- $I \subseteq I^\star$,
- if $I \subseteq J$, then $I^\star \subseteq J^\star$,
- $(I^\star)^\star = I^\star$,
- $(xI)^\star = x \cdot I^\star$,
- $(I \cap J)^\star = I^\star \cap J^\star$.

(A map that satisfies the first four properties is called a *semistar operation*.) We denote by $\mathrm{SStar}_{\mathrm{st}}(D)$ the set of stable semistar operations.

There is a natural bijection between stable semistar operations and localizing systems; if $\star$ is a stable semistar operation, then the set $\mathcal{F}^\star := \{I \in \mathcal{I}(D) \mid 1 \in I^\star\}$ is a localizing system, while if $\mathcal{F}$ is a localizing system, then $\star_{\mathcal{F}} : I \mapsto \bigcup\{(I : E) \mid E \in \mathcal{F}\}$ is a stable semistar operation; these two correspondences are inverse one of each other [5, Theorem 2.10]. By composing them with the maps considered in Theorem 6.5, we obtain two bijections

$$\Phi : \mathrm{SStar}_{\mathrm{st}}(D) \to \mathcal{L}_{\mathrm{sing}}(D), \quad \star \mapsto \ell_{\mathcal{F}^\star},$$

$$\Phi^{-1} : \mathcal{L}_{\mathrm{sing}} \to \mathrm{SStar}_{\mathrm{st}}(D), \quad \ell \mapsto \star_{Z(\ell)}$$

which are order-reversing isomorphisms if $\mathrm{SStar}_{\mathrm{st}}(D)$ is endowed with the order such that $\star_1 \leq \star_2$ if $I^{\star_1} \subseteq I^{\star_2}$ for every $I \in \mathbf{F}(D)$. Note that, in this order, the infimum of a family Δ is the map sending I to $\bigcap_{\star \in \Delta} I^\star$.

Proposition 6.6. *Let Λ be a nonempty set of stable semistar operations on the integral domain D. Then*

$$\Phi(\inf \Lambda) = \sum_{\star \in \Lambda} \Phi(\star).$$

Proof. Since Φ is an order-reversing isomorphism, we have $\Phi(\inf \Lambda) = \sup \Phi(\Lambda)$; the claim now follows from Lemma 3.7. $\square$

A special subset of stable semistar operations are *spectral semistar operations*, i.e., the closures in the form

$$s_\Delta : I \mapsto \bigcap_{P \in \Delta} ID_P,$$

where $\Delta \subseteq \mathrm{Spec}(D)$; furthermore, we can suppose that Δ is *closed by generizations*, i.e., it is such that if $P \in \Delta$ and $Q \subseteq P$, then also $Q \in \Delta$. The corresponding localizing system is

$$\mathcal{F}_\Delta := \{I \in \mathcal{I}(D) \mid I \nsubseteq P \text{ for every } P \in \Delta\},$$

while the associated length function ℓ_Δ is such that

$$\ell_\Delta(D/I) = \begin{cases} 0 & \text{if } I \nsubseteq P \text{ for every } P \in \Delta, \\ \infty & \text{if } I \subseteq P \text{ for some } P \in \Delta, \end{cases}$$

or, more generally,

$$\ell_\Delta(M) = \begin{cases} 0 & \text{if } \mathrm{Ann}(x) \nsubseteq P \text{ for every } P \in \Delta \text{ and } x \in M, \\ \infty & \text{if } \mathrm{Ann}(x) \subseteq P \text{ for some } P \in \Delta \text{ and } x \in M \end{cases}$$

for every D-module M. In particular, if $\Delta = \mathrm{Spec}(D)$, then $\ell_\Delta(M) = 0$ if and only if $M = 0$, while if $\Delta = \{(0)\}$, then $\ell_\Delta(M) = 0$ if and only if M is torsion.

Such length functions have a decomposition like the ones found in Theorems 3.10 and 5.4.

Proposition 6.7. *Let D be an integral domain, and let $\Delta \subseteq \mathrm{Spec}(D)$ be closed by generizations; let $\ell := \Phi(s_\Delta)$, and let $\Sigma(\ell) := \{P \in \mathrm{Spec}(D) \mid P \notin Z(\ell)\}$. Then*
(a) $\Delta = \Sigma(\ell)$,
(b) $\ell = \sum_{P \in \Sigma(\ell)} \ell \otimes D_P$.

Proof. From the bijections between $\mathrm{SStar}_{\mathrm{st}}(D)$, $\mathrm{LocSist}(D)$ and $\mathcal{L}_{\mathrm{sing}}(D)$, we see that prime P is in $Z(\ell)$ if and only if $1 \in P^{s_\Delta}$; since Δ is closed by generizations, it follows that $P \in Z(\ell)$ if and only if $P \notin \Delta$, and thus $\Delta = \Sigma(\ell)$.

The semistar operation s_Δ is the infimum of the family $s_{\{P\}}$, as P ranges in Δ; by Proposition 6.6, it follows that

$$\ell = \Phi(s_\Delta) = \sum_{P \in \Delta} \Phi(s_{\{P\}}),$$

and thus we only need to show that $\Phi(s_{\{P\}}) = \ell \otimes D_P$; to do so, it is enough to show that their zero loci are equal. We have

$$Z(\Phi(s_{\{P\}})) = \mathcal{F}^{s_{\{P\}}} = \{I \in \mathcal{I}(D) \mid I \nsubseteq P\}.$$

If $I \nsubseteq P$, then $D_P/ID_P = 0$, and thus

$$(\ell \otimes D_P)(D/I) = \ell(D/I \otimes_D D_P) = \ell(D_P/ID_P) = \ell(0) = 0,$$

i.e., $I \in Z(\ell \otimes D_P)$; on the other hand, if $I \subseteq P$ then, using Proposition 4.4,

$$(\ell \otimes D_P)(D/I) \geq (\ell \otimes D_P)(D/P) = \ell(D/P) = \infty$$

since $P^{s_{\{P\}}} = PD_P$ does not contain 1. Hence, $I \notin Z(\ell \otimes D_P)$, and thus $Z(\Phi(s_{\{P\}})) = Z(\ell \otimes D_P)$. The claim is proved. $\square$

Let now D be a Prüfer domain, and let $\star \in \mathrm{SStar}_{st}(D)$. The *normalized stable version* of $\star$ is (see [17, Section 4])

$$\widehat{\star} : I \mapsto \bigcap_{P \in \Sigma_1(\star)} ID_P \cap \bigcap_{P \in \Sigma_2(\star)} (ID_P)^{v_{D_P}},$$

where v_{D_P} is the v-operation on D_P (i.e., if J is an ideal of D_P, then $J^{v_{D_P}} = \bigcap\{yD_P \mid J \subseteq yD_P\}$), and

$$\Sigma_1(\star) := \{P \in \mathrm{Spec}(D) \mid 1 \notin P^\star\},$$
$$\Sigma_2(\star) := \{P \in \mathrm{Spec}(D) \mid 1 \in P^\star,\ 1 \notin Q^\star \text{ for some } P\text{-primary ideal } Q\}.$$

(In the terminology of [17], $\Sigma_1(\star) =: \mathrm{QSpec}^\star(D)$ is the *quasi-spectrum* of $\star$, while $\Sigma_2(\star) =: \mathrm{PsSpec}^\star(D)$ is the *pseudo-spectrum*.) By [17, Proposition 3.4], and in the terminology introduced in Definition 5.9, furthermore, $\{\Sigma_1(\star), \Sigma_2(\star)\}$ is a layered family with core $\Sigma_1(\star)$.

This construction is analogous to the passage from a length function ℓ to

$$\ell^\sharp := \sum_{P \in \Sigma_t(\ell)} (t_{PD_P})^D + \sum_{P \in \Sigma_i(\ell)} (i_{PD_P})^D = \sum_{P \in \Sigma(\ell)} \ell \otimes D_P,$$

as the next proposition shows.

Proposition 6.8. *Let D be a Prüfer domain. Then, for every stable semistar operation $\star$, we have $\Phi(\widehat{\star}) = \Phi(\star)^\sharp$.*

Proof. Let Δ be the set formed by the functions $d_P : I \mapsto ID_P$, for $P \in \Sigma_1(\star)$, and $v_P := I \mapsto (ID_P)^{v_{D_P}}$, as $P \in \Sigma_2(\star)$. By Proposition 6.6, it follows that

$$\Phi(\widehat{\star}) = \sum_{P \in \Sigma_1(\star)} \Phi(d_P) + \sum_{P \in \Sigma_2(\star)} \Phi(v_P).$$

On the other hand, by unpacking the definitions, we have $\Sigma_t(\Phi(\star)) = \Sigma_1(\star)$ and $\Sigma_i(\Phi(\star)) = \Sigma_2(\star)$; hence, it is enough to show that $\Phi(d_P) = (t_{PD_P})^D$ and $\Phi(v_P) = (i_{PD_P})^D$ for every $P \in \mathrm{Spec}(D)$, and to do so, it is enough to consider their zero loci.

By a direct calculation,

$$Z(\Phi(d_P)) = \{I \in \mathcal{I}(D) \mid I \nsubseteq P\} = \{I \in \mathcal{I}(D) \mid PD_P \subsetneq ID_P\} = Z((t_{PD_P})^D);$$

analogously,

$$Z(\Phi(v_P)) = \{I \in \mathcal{I}(D) \mid PD_P \subseteq ID_P\} = Z((i_{PD_P})^D)$$

since $(ID_P)^{v_{D_P}} = D_P$ if and only if ID_P is equal to D_P or to PD_P. The claim is proved. $\square$

Suppose now that every ideal of D has only finitely many minimal primes. Then Theorem 5.4 (and Corollary 5.8) can be seen as a version of [17, Theorem 4.5 and Corollary 4.6]; like any singular length function can be written as $\sum_{P \in \Sigma_1} \ell \otimes D_P + \sum_{P \in \Sigma_2} \ell \otimes D_P$, a stable semistar operation can be written as the infimum of the semistar operations d_P (as P ranges in some Σ_1) and v_P (as P ranges in Σ_2).

More generally, suppose that $\star$ is a semistar operation which is equal to its normalized stable version, i.e., suppose that there are $\Sigma_1, \Sigma_2 \subseteq \mathrm{Spec}(D)$ such that $\{\Sigma_1, \Sigma_2\}$ is a layered family with core Σ_1 and

$$\star : I \mapsto \bigcap_{P \in \Sigma_1} ID_P \cap \bigcap_{P \in \Sigma_2} (ID_P)^{v_{D_P}}.$$

Then, by Proposition 6.8, we see that the corresponding length function $\ell = \ell^\sharp$ can be decomposed as $\ell = \sum_{P \in \Sigma(\ell)} \ell \otimes D_P$. Thus, the fact that a stable semistar operation is determined at the local level (through the closures d_P and v_P) corresponds to the fact that the corresponding length function depends exclusively on length functions on the localizations of D.

We end the paper with two examples of Prüfer domains of dimension 1 that are not locally finite and whose behavior with respect to decomposition is very different; more precisely, in Example 6.9, we present an example where every singular length function can be decomposed (despite the domain not satisfying the hypothesis of Theorem 5.4), while in Example 6.10, we give a singular length function that cannot be decomposed.

Example 6.9. Let D be an almost Dedekind domain (i.e., an integral domain such that D_M is a discrete valuation ring for every $M \in \mathrm{Max}(D)$), and suppose that there is only a finite (nonzero) number of maximal ideals of D that are not finitely generated. (See [10] for explicit examples of domains with this property.) In particular, D is one-dimensional and $\mathrm{Spec}(D)$ is not Noetherian, and so there are ideals with infinitely many minimal primes.

We claim that every singular length function ℓ can be written as $\ell = \sum_{M \in \mathrm{Max}(D)} \ell \otimes D_M$, and to do so, we want to show that every stable semistar operation $\star$ is equal to $\star = s_\Delta$ for some $\Delta \subseteq \mathrm{Spec}(D)$. If not, then by [5, Theorem 4.12(3)], there is a proper ideal I of D such that $I = I^\star \cap D$ but $P \neq P^\star \cap D$ for every prime ideal P containing I; since D is one-dimensional, it follows that $1 \in P^\star$ for every P containing I, or equivalently that $P^\star = D^\star$.

Suppose that $P = pD$ contains I and is principal; then $1 \in P^\star = (pD)^\star = pD^\star$, and thus $1/p \in T := D^\star$. Hence, $I^\star = (ID)^\star = (ID^\star)^\star \supseteq ID[1/p]$. The ideal $ID[1/p] \cap D$ is not contained in pD since

$$(ID[1/p] \cap D)D_P = ID[1/p]D_P \cap DD_P = D_P;$$

thus, $ID[1/p] \cap D \neq I$, a contradiction. Hence, I is not contained in any principal prime ideal; however, this means that I has a primary decomposition, namely $I = \bigcap_i (ID_{M_i} \cap D)$, where $\{M_1, \dots, M_n\}$ are the maximal ideal of D containing I. By Proposition 4.7, it follows that $\ell(D/I) = \sum_i \ell(D/(ID_{M_i} \cap D))$; however, by Proposition 4.4,

$$\ell(D/(ID_M \cap D)) = \ell(D_M/ID_M) = \ell(D_M/(MD_M)^k) = 0$$

since D_M is a DVR (and so $ID_M = (MD_M)^k$ for some k) and $\ell(D_M/MD_M) = \ell(D/M) = 0$. It follows that $\ell(D/I) = 0$, and thus that $I \in Z(\ell) = \{I \in \mathfrak{I}(D) \mid 1 \in I^\star\}$, against the fact that $I = I^\star \cap D$. Hence, $\star$ is spectral, and so $\star = s_\Delta$; by Proposition 6.7, we have

$$\ell = \sum_{P \in \Sigma(\ell)} \ell \otimes D_P = \sum_{M \in \mathrm{Max}(D)} \ell \otimes D_M,$$

with the last equality coming from the fact that if $M \notin \Sigma(\ell)$, then $\ell \otimes D_M$ sends every proper quotient D/I to 0.

Example 6.10. Let $D := \mathbb{A}$ be the ring of all algebraic integers. By [3, Example 4.5] and [17, Example 4.2], we can build a stable semistar operation $\star$ such that $Q^\star = D$ for every primary ideal Q, while $D^\star = D$ (and so $(xD)^\star = xD$ for every $x \in D$). The corresponding localizing system contains every ideal contained in only finitely many maximal ideals, but it does not contain any proper principal ideal; hence, the associated length function ℓ is such that $\ell(D/I) = 0$ if I is contained in only finitely many maximal ideals, while $\ell(D/xD) = \infty$ for all nonunits $x \in D$.

By Proposition 4.4, if Q is P-primary, then $\ell_{D_P}(D_P/QD_P) = \ell(D/Q) = 0$, while $\ell_{D_P}(D_P) = \infty$; hence, $\ell_{D_P} = t_{(0)}$ for every $P \in \mathrm{Max}(D)$. It follows that $\Sigma_t(\ell) = \{(0)\}$, while $\Sigma_i(\ell) = \Sigma_r(\ell) = \Sigma_v(\ell) = \emptyset$; therefore, setting $\ell^\sharp := \sum_{P \in \Sigma(\ell)} \ell \otimes D_P$, we have

$$\ell^\sharp(M) = (t_{(0)})^D(M) = \begin{cases} 0 & \text{if } M \text{ is a torsion } D\text{-module,} \\ \infty & \text{otherwise,} \end{cases}$$

and thus $\ell^\sharp(D/I) = 0$ for every proper ideal I of D. In particular, $\ell \neq \ell^\sharp$.

Furthermore, $(\ell \otimes D_M)(D/I) = 0$ for every nonzero ideal I and every maximal ideal M; hence, we also have

$$\ell \neq \sum_{P \in \Delta} \ell \otimes D_P$$

for every family $\Delta \subseteq \mathrm{Spec}(D)$.

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