TWMS J. App. and Eng. Math. V.11, Special Issue, 2021, pp. 51-62

A NOTE ON INDICES OF PRIMEPOWER AND SEMIPRIME DIVISOR FUNCTION GRAPH

S. SHANMUGAVELAN[1], K. THANGA RAJESWARI[1], C. NATARAJAN[1], §

Abstract. The notion of using number theortic based graph seems to be one of the flourishing areas in Graph theory. One such concept is the divisor function graph $G_{D(n)}$ which is defined as: For any positive integer $n \geq 1$ with r divisors $d_1, d_2, d_3, ..., d_r$, divisor function graph $G_{D(n)}$ is a (V, E) graph with V as the set of all factors of n and E be defined in such a way that two vertices d_i and d_j are adjacent if and only if either $d_i \mid d_j$ or $d_j \mid d_i$, $i \neq j$. In this paper, we analyze the operation sum of two divisor function graphs and investigate several indices exclusively for prime powers and for semi primes. Also, we derive a result for an independent function.

Keywords: Divisor function graph, Harmonic index, Zagreb index, Energy.
AMS Subject Classification: 05C10

1. Introduction

Throughout the discussion $G = (V, E)$ is a non-trivial, simple, connected graph. The cardinality of neighbourhood set of vertex v in G is denoted by $deg(v)$ and the distance between any two verices u and v in G denoted by $d(u, v)$ is the length of the shortest $u - v$ path. For notation and graph theory terminologies not defined here we generally follow [2].

The idea of having a graph associated with divisor function $D(n)$ was introduced by Kannan, Narasimhan, Shanmugavelan [4] in 2015. Moreover they proved that $G_{D(n)}$ is always connected and a complete graph if and only if no two proper divisors in $D(n)$ are relatively prime. Also the chromatic number for $G_{D(n)}$ is at least 3 and it is Eulerian for a perfect square numbers.

Later in 2018, Narasimhan and Vignesh [5] introduced directed divisor function graph and proved it is unilaterally connected for prime powers and derived an algorithm for determining the size of it. Moreover, the extended study on colorability of $G_{D(n)}$ was made in 2018 by Narasimhan and Elamparithi [6] and they briefly discussed connectivity and its independence.

[1] Department of Mathematics, Srinivasa Ramanujan Centre, SASTRA Deemed to be University, Kumbakonam, India.
e-mail: shanmugavelan@src.sastra.edu; ORCID: https://orcid.org/0000-0002-9622-1283.
e-mail: shivamayam27@gmail.com; ORCID: https://orcid.org/0000-0002-2557-0486.
e-mail: natarajan_c@maths.sastra.edu; ORCID: https://orcid.org/0000-0002-1001-3556.

§ Manuscript received: October 10, 2019; accepted: April 2, 2020.

In this paper, several degree, distance based topological indices are analyzed for prime power divisor function graph, which is always regular and also we derive energy for $G_{D(p^n)}$ and a result on sum operation of Divisor fuction graph.

2. Indices of $G_{D(p^n)}$

Theorem 2.1. *For any prime power p^n, $G_{D(p^n)}$ has a harmonic index of $\frac{n+1}{2}$.*
Proof: *We know that, $G_{D(p^n)}$ is regular graph of degree n and it is also complete with $m = \frac{n(n+1)}{2}$ edges. By definition of Harmonic index,*

$$\begin{aligned} H(G_{D(p^n)}) &= \sum_{uv \in E(G_{D(p^n)})} \frac{2}{deg(u) + deg(v)} \\ &= \sum_{uv \in E(G_{D(p^n)})} \frac{2}{(n) + (n)} \\ &= \sum_{uv \in E(G_{D(p^n)})} \frac{1}{n} \\ &= m\frac{1}{n} \\ &= \frac{(n+1)}{2} \end{aligned}$$

Theorem 2.2. *The degree distance index of $G_{D(p^n)}$ is $n^2(n+1)$.*
Proof: *Clearly, $deg(u) = n, \forall u \in V(G_{D(p^n)})$. Since $G_{D(p^n)}$ is complete, each vertex is adjacent to remaining n vertices. Also,$d(u,v), \forall u, v \in V(G_{D(p^n)})$. By definition of degree distance index,*

$$\begin{aligned} Hence, DD(G_{D(p^n)}) &= \sum_{\{u,v\} \subseteq V(G_{D(p^n)})} [deg(u) + deg(v)]d(u,v) \\ &= \sum_{\{u,v\} \subseteq V(G_{D(p^n)})} [n+n](1) \\ &= 2nm, \; since\; m = \frac{n(n+1)}{2} \end{aligned}$$

$$Therefore, DD(G_{D(p^n)}) = n^2(n+1).$$

Definition 2.1. *The semiprime divisor function graph $G_{D(p \times q)} = (V, E)$, where $V = \{1, p, q, pq\}$ and the graph follows.*

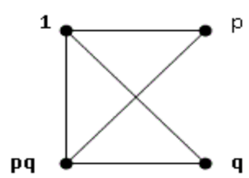


Figure 1. $G_{D(p \times q)}$

Theorem 2.3. *The degree distance index of semiprime Divisor function graph* $G_{D(p\times q)}$ *is* 34.
Proof: *Clearly, using figure 1,* $deg(1) = 3; deg(p) = 2; deg(q) = 2; deg(pq) = 3$

$$\begin{aligned} DD(G) &= \sum_{\{u,v\}\subseteq V(G)} [deg(u) + deg(v)]d(u, v) \\ &= [(3 + 2)(1) + (3 + 3)(1) + (3 + 2)(1) \\ &+ (2 + 3)(1) + (2 + 2)(2) + (2 + 3)(1)] \\ Therefore, DD(G_{D(p\times q)}) &= 34. \end{aligned}$$

Theorem 2.4. *The Wiener index of* $G_{D(p^n)}$ *is* $\frac{n(n+1)}{2}$.
Proof: *Clearly for any fixed vertex* p^i, $0 \leq i \leq n + 1$, *as origin there are* n *distinct terminus vertices each with distance 1. By the definition of Wiener index,*

$$\begin{aligned} W(G) &= \sum_{\{u,v\}\subseteq V(G)} d(u, v) \\ W(G_{D(p^n)}) &= \sum_{\{u,v\}\subseteq V(G)} (1) \end{aligned}$$

For the vertex $p^0 = 1$, *its possible distances are* $d(1, p), d(1, p^2), ..., d(1, p^n)$, *where each distance is exactly one.*
Therefore, The distances whose origin is 1 is accounted for n.
For the next vertex p^1, *its possible distances are* $d(p^1, p^2), d(p^1, p^3), ..., d(p^1, p^n)$.
Therefore, the distances whose origin is p^1 *is accounted for* $n - 1$.
For any arbitrary vertex p^i, *its possible distances are* $d(p^i, p^{(i+1)}), d(p^i, p^{(i+2)}), ..., d(p^i, p^n)$.
Hence, the distances whose origin is p^i *is accounted for* $n - i$.
On adding these, we have

$$\begin{aligned} W(G_{D(p^n)}) &= n + (n - 1) + ... + (n - i) + ... + 2 + 1 \\ &= \frac{n(n + 1)}{2}. \end{aligned}$$

Theorem 2.5. *The Wiener index of semiprime Divisor function graph* $G_{D(p\times q)}$ *is* 7.
Proof: *Clearly, using the figure 1,* $d(p, q) = 2$ *and all the other pairs of vertices has distance equal to* 1.

$$\begin{aligned} Hence, W(G) &= \sum_{\{u,v\}\subseteq V(G)} d(u, v) \\ W(G_{D(p\times q)}) &= d(1, p) + d(1, q) + d(1, pq) + d(p, q) + d(p, pq) + d(q, pq) \\ &= 1 + 1 + 1 + 2 + 1 + 1 \\ Therefore, W(G_{D(p\times q)}) &= 7. \end{aligned}$$

Theorem 2.6. *The hyper Wiener index of* $G_{D(p^n)}$ *is* $\frac{n(n+1)}{2}$.

Proof: *Clearly, there is a edge between each pair of vertices.*
Hence $d(u,v)=1, \forall u,v \in V(G_{D(p^n)})$. *By the definition of hyper Wiener index,*

$$\begin{aligned}
WW(G) &= \frac{1}{2}\sum_{\{u,v\}\subseteq V(G)} (d(u,v)+d(u,v)^2) \\
WW(G_{D(p^n)}) &= \frac{1}{2}\sum_{\{u,v\}\subseteq V(G)} (1+1^2) \\
&= \sum_{\{u,v\}\subseteq V(G)} 1
\end{aligned}$$

Similar to the proof of theorem 2.4, we have

$$WW(G_{D(p^n)}) = \frac{n(n+1)}{2}.$$

Theorem 2.7. *The hyper Wiener index of semiprime Divisor function graph* $G_{D(p\times q)}$ *is* 8.
Proof:

$$\begin{aligned}
Hence, WW(G_{D(p\times q)}) &= \frac{1}{2}\sum_{\{u,v\}\subseteq V(G)} (d(u,v)+d(u,v)^2) \\
&= \frac{1}{2}([d(1,p)+d(1,p)^2]+[d(1,q)+d(1,q)^2]+[d(1,pq)+d(1,pq)^2] \\
&+[d(p,q)+d(p,q)^2]+[d(p,pq)+d(p,pq)^2]+[d(q,pq)+d(q,pq)^2]) \\
&= \frac{1}{2}[(1+1^2)+(1+1^2)+(1+1^2)+(2+2^2)+(1+1^2)+(1+1^2)
\end{aligned}$$

$Hence, WW(G_{D(p\times q)}) = 8.$

Theorem 2.8. *The Randic index of* $G_{D(p^n)}$ *is* $\frac{(n+1)}{2}$

Proof: Clearly, $|V(G_{D(p^n)})| = n+1$ and $deg(u)=n, \forall u \in V(G_{D(p^n)})$. By the definition of Randic index,

$$\begin{aligned}
R(G) &= \sum_{uv\in E(G)} \frac{1}{\sqrt{d(u)d(v)}} \\
R(G_{D(p^n)}) &= \sum_{uv\in E(G)} \frac{1}{\sqrt{n\times n}} \\
&= \sum_{uv\in E(G)} \frac{1}{n} \\
&= \frac{n(n+1)}{2\times n}, \ since\ m = \frac{n(n+1)}{2} \\
R(G_{D(p^n)}) &= \frac{(n+1)}{2}
\end{aligned}$$

Theorem 2.9. *The Randic index of semiprime Divisor function graph* $G_{D(p\times q)}$ *is* $\frac{2\sqrt{6}+1}{3}$.

Proof: *Clearly, using the figure 1,* $d(p,q) = 2$ *and all the other pairs of vertices has distance equal to* 1.

$$R(G) = \sum_{uv \in E(G)} \frac{1}{\sqrt{deg(u)deg(v)}}$$
$$R(G_{D(p\times q)}) = \frac{1}{\sqrt{3\times 2}} + \frac{1}{\sqrt{3\times 2}} + \frac{1}{\sqrt{3\times 3}} + \frac{1}{\sqrt{2\times 3}} + \frac{1}{\sqrt{2\times 3}}$$
$$R(G_{D(p\times q)}) = \frac{2\sqrt{6}+1}{3}.$$

Theorem 2.10. *The first Zagreb index of* $G_{D(p^n)}$ *is* $(n+1)n^2$.
Proof: *Clearly,* $V(G_{D(p^n)}) = \{1, p, p^2, ..., p^n\}$.
Therefore $|V(G_{D(p^n)})| = n+1$ *and* $deg(u) = n, \forall u \in V(G_{D(p^n)})$. *By the definition of first Zagreb index,*

$$M_1(G) = \sum_{v_i \in V(G)} deg(v_i)^2$$
$$M_1(G_{D(p^n)}) = \sum_{v_i \in V(G)} n^2$$
$$M_1(G_{D(p^n)}) = (n+1)n^2.$$

Theorem 2.11. *The first Zagreb index of semiprime Divisor function graph* $G_{D(p\times q)}$ *is* 26.
Proof:

$$M_1(G) = \sum_{v_i \in V(G)} deg(v_i)^2$$
$$M_1(G_{D(p\times q)}) = deg(1)^2 + deg(p)^2 + deg(q)^2 + deg(pq)^2$$
$$= 3^2 + 3^2 + 2^2 + 2^2$$
$$Hence, M_1(G_{D(p\times q)}) = 26.$$

Theorem 2.12. *The second Zagreb index of* $G_{D(p^n)}$ *is* $\frac{n^3(n+1)}{2}$.
Proof: *Clearly,* $|V(G_{D(p^n)})| = n+1$ *and* $deg(u) = n, \forall u \in V(G_{D(p^n)})$. *By the definition of second Zagreb index,*

$$M_2(G) = \sum_{v_i v_j \in E(G)} deg(v_i)deg(v_j)$$
$$M_2(G_{D(p^n)}) = \sum_{v_i v_j \in E(G)} (n)(n)$$
$$= n^2(\frac{n(n+1)}{2}),\ since\ m = \frac{n(n+1)}{2}$$
$$M_2(G_{D(p^n)}) = \frac{n^3(n+1)}{2}.$$

Theorem 2.13. *The second Zagreb index of semiprime divisor function graph* $G_{D(p\times q)}$ *is* 33.

Proof:

$$\begin{aligned}
Hence, M_2(G) &= \sum_{v_iv_j \in E(G)} deg(v_i)deg(v_j)\\
M_2(G_{D(p\times q)}) &= [deg(1)deg(p)] + [deg(1)deg(q)] + [deg(1)deg(pq)]\\
&+ [deg(p)deg(pq)] + [deg(q)deg(pq)]\\
&= [3.2] + [3.2] + [3.3] + [2.3] + [2.3]\\
M_2(G_{D(p\times q)}) &= 33.
\end{aligned}$$

Theorem 2.14. *The Gutman index of $G_{D(p^n)}$ is $\frac{n^3(n+1)}{2}$.*
Proof: *By the definition of Gutman index,*

$$\begin{aligned}
Gut(G) &= \sum_{u\neq v} deg(u)deg(v)d(u,v)\\
Clearly,\ Gut(G_{D(p^n)}) &= \sum_{p^i\neq p^j} deg(p^i)deg(p^j)d(p^i,p^j)\\
&= \sum_{p^i\neq p^j} n^2\\
&= n^2 \times \frac{n(n+1)}{2}\\
Gut(G_{D(p^n)}) &= \frac{n^3(n+1)}{2}.
\end{aligned}$$

Theorem 2.15. *The Gutman index of semiprime divisor function graph $G_{D(p\times q)}$ is 41.*
Proof: *Clearly, using the figure 1, $d(p,q) = 2$ and all the other pairs of vertices has distance equal to 1.*

$$\begin{aligned}
Gut(G) &= \sum_{u\neq v} deg(u)deg(v)d(u,v)\\
Gut(G_{D(p\times q)}) &= [deg(1)deg(p)d(1,p)] + [deg(1)deg(q)d(1,q)] + [deg(1)deg(pq)d(1,pq)]\\
&+ [deg(p)deg(q)d(p,q)] + [deg(p)deg(pq)d(p,pq)] + [deg(q)deg(pq)d(q,pq)]\\
&= [3.2.1] + [3.2.1] + [3.3.1] + [2.2.2] + [2.3.1] + [2.3.1]\\
Gut(G_{D(p\times q)}) &= 41
\end{aligned}$$

Theorem 2.16. *The Detour Gutman index of $G_{D(p^n)}$ is $\frac{n^4(n+1)}{2}$.*
Proof: *Clearly, $|V(G_{D(p^n)})| = n+1$ and $deg(u) = n, \forall u \in V(G_{D(p^n)})$.*
Also, the longest path $D(u,v) = n, \forall u \in V(G_{D(p^n)})$. By the definition of detour Gutman index,

$$\begin{aligned}
DGut(G) &= \sum_{u\neq v} deg(u)deg(v)D(u,v)\\
DGut(G_{D(p^n)}) &= \sum_{u\neq v} n.n.n\\
&= n^3(\frac{n(n+1)}{2})\\
DGut(G_{D(p^n)}) &= \frac{n^4(n+1)}{2}.
\end{aligned}$$

Theorem 2.17. *The Detour Gutman index of* $G_{D(p\times q)}$ *is* 90.
Proof: *Clearly, D(1,p)=2; D(1,q)=3; D(1,pq)=2; D(p,q)=3; D(p,pq)=3; D(q,pq)=2.*

$$DGut(G) = \sum_{u\neq v} deg(u)deg(v)D(u,v)$$

$$\begin{aligned} DGut(G_{D(p\times q)}) &= [deg(1)deg(p)D(1,p)] + [deg(1)deg(q)D(1,q)] + [deg(1)deg(pq)D(1,pq)] \\ &+ [deg(p)deg(q)D(p,q)] + [deg(p)deg(pq)D(p,pq)] + [deg(q)deg(pq)D(q,pq)] \\ &= [3.2.2] + [3.2.3] + [3.3.2] + [2.2.3] + [2.3.3] + [2.3.2] \\ DGut(G_{D(p\times q)}) &= 90. \end{aligned}$$

Definition 2.2. *For a vertex* v*, the sum degree of* v *is defined as* $S_v = \sum_{u\in N(v)} deg(u)$ *and for a vertex* v*, the multiplication degree of* v *is defined as* $M_v = \prod_{u\in N(v)} deg(u)$. *The R degree of a vertex* v *of a simple connected graph* G *is defined as* $r(v) = S_v + M_v$.
The first R index of a simple connected graph G *defined as* $R^1(G) = \sum_{v\in G} (r(v))^2$.
The second R index of a simple connected graph G *defined as* $R^2(G) = \sum_{uv\in E} r(u)r(v)$.
The third R index of a simple connected graph G *defined as* $R^3(G) = \sum_{uv\in E} [r(u) + r(v)]$.

Theorem 2.18. *The first R index of* $G_{D(p^n)}$ *is* $n^2(n+1)(n^2+n^{2n-2}+2n^n)$.
Proof: *Here* $V(G_{D(p^n)}) = \{1, p, p^2, ..., p^n\}$.
Therefore $|V(G_{D(p^n)})| = n+1$ *and* $deg(u) = n, \forall u \in V(G_{D(p^n)})$ *since* $G_{D(p^n)}$ *is complete, each vertex is adjacent to remaining* n *vertices. For an arbitrary vertex* p^i,

$$\begin{aligned} S_{p^i} &= \sum_{p^j\in N(p^i)} deg(p^j) \\ &= n + n + ... + n \\ Hence, S_{p^i} &= n^2, \forall p^i \in V(G_{D(p^n)}). \\ Also, M_{p^i} &= \prod_{p^j\in N(p^i)} deg(p^j) \\ &= \prod_{p^j\in N(p^i)} n \\ M_{p^i} &= n^n, \forall p^i \in V(G_{D(p^n)}) \\ Then, r(p^i) &= S_{p^i} + M_{p^i} = n^2 + n^n, \forall p^i \in V(G_{D(p^n)}). \\ Hence, R^1(G_{D(p^n)}) &= \sum_{p^i\in G_{D(p^n)}} (r(p^i))^2 \\ R^1(G_{D(p^n)}) &= \sum_{p^i\in G_{D(p^n)}} (n^2+n^n)^2 \\ &= \sum_{p^i\in G_D(p^n)} n^4 + n^{2n} + 2n^{n+2} \\ &= (n+1)(n^4 + n^{2n} + 2n^{n+2}) \\ Hence, R^1(G_{D(p^n)}) &= n^2(n+1)(n^2 + n^{2n-2} + 2n^n). \end{aligned}$$

Theorem 2.19. *The second R index of $G_{D(p^n)}$ is $\frac{n^3(n+1)(n^2+n^{2n-2}+2n^n)}{2}$.*
Proof: *Clearly, $S_v = n^2$, $M_v = n^n$ and so $r(v) = S_v + M_v = n^2 + n^n, \forall v \in V(G_{D(p^n)})$.*

$$Hence, R^2(G_{D(p^n)}) = \sum_{p^ip^j \in E(G_{D(p^n)})} r(p^i)r(p^j)$$

$$R^2(G_{D(p^n)}) = \sum_{p^ip^j \in E(G_{D(p^n)})} (n^2+n^n)(n^2+n^n)$$

$$= mn^2(n^2+n^{2n-2}+2n^n)$$

$$Therefore, R^2(G_{D(p^n)}) = \frac{n^3(n+1)(n^2+n^{2n-2}+2n^n)}{2}.$$

Theorem 2.20. *The third R index of $G_{D(p^n)}$ is $n(n+1)(n^2+n^n)$.*
Proof:

$$R^3(G_{D(p^n)}) = \sum_{p^ip^j \in E(G_{D(p^n)})} [r(p^i)+r(p^j)]$$

$$R^3(G_{D(p^n)}) = \sum_{p^ip^j \in E(G_{D(p^n)})} (n^2+n^n+n^2+n^n)$$

$$= 2m(n^2+n^n)$$

$$= n(n+1)(n^2+n^n)$$

$$R^3(G_{D(p^n)}) = n^3(n+1)(n^{n-2}+1).$$

Theorem 2.21. *The Balaban index of $G_{D(p^n)}$ is $\frac{n(n+1)^2}{2(n^2-n+4)}$.*
Proof: *Clearly, $deg(p^i) = n, 0 \leq i \leq n+1$ and $\mu = m-n+c = m-n+1$, since $c = 1$. By the definition of Balaban index,*

$$J = \frac{m}{\mu+1} \sum_{(p^i,p^j) \in E(G)} [deg(p^i)deg(p^j)]^{\frac{-1}{2}}$$

$$= \frac{m}{m-n+2} \sum_{(p^i,p^j) \in E(G)} \frac{1}{n}$$

$$= \frac{m}{m-n+2} \times m \times \frac{1}{n}$$

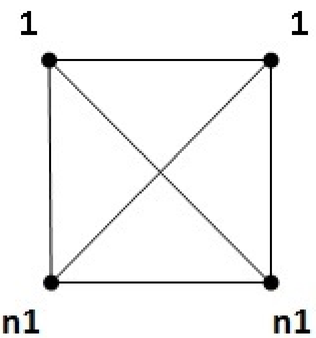
$$= \frac{n^2(n+1)^2}{2^2 n(\frac{n(n+1)}{2}-n+2)}, \ since\ m = \frac{n(n+1)}{2}$$

$$= \frac{n(n+1)^2}{2(n^2-n+4)}.$$

3. Sum of two divisor function graphs

Theorem 3.1. *If n_1 or n_2 or both are prime then the sum of divisor function graphs $G_{D(n_1)}$ and $G_{D(n_2)}$ is a simple graph, $(G_{D(n_1)} + G_{D(n_2)}) = (p_1+p_2, q)$. The number of edges in sum of two divisor function graphs (say) $q = q_1+q_2+p_2+|S|+1$ where $S = V(G_{D(n_2)}) \cap P$ and $P = \{x/x = n_1.n, n \in N, n \leq n_2\}$*
Proof: *Without loss of generality, let $n_1 \leq n_2$.*
Suppose $n_1|n_2$. Assume n_1 is prime.
case-1: n_2 is prime

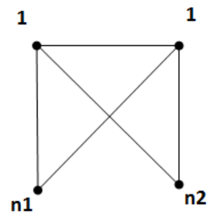
since $n_1|n_2$ *and both are prime which implies* $n_1 = n_2$.
By definition of sum of graphs, $V(G_{D(n_1)} + G_{D(n_2)}) = p_1 + p_2$.
By the definition of sum of graphs there are q_1 *edges of* $G_{D(n_1)}$ *and* q_2 *edges of* $G_{D(n_2)}$ *and since* 1 *is the divisor of* n_1 *in* $G_{D(n_1)}$, 1 *is adjacent with* p_2 *vertices.*
Also, here $P = \{n_1, 2n_1, 3n_1, ..., n_2n_1\}$ *(By definition of* P *given above).*
$V(G_{D(n_2)}) = \{1, n_2\}$ *and* $S = \{1, n_2\} \cap \{n_1, 2n_1, 3n_1, ..., n_2n_1\}$
since $n_1 = n_2$, $S = \{1, n_2\} \cap \{n_2, 2n_2, 3n_2, ..., n_2n_2\}$
Therefore, $S = \{n_2\}$ *which implies* $|S| = 1$.
Suppose S *is empty.* $P = \{n_1, 2n_1, ..., n_2n_1\}$ *is the set of multiples of* n_1.
$V(G_{D(n_2)}) = \{d1 = 1, d2, d3, ..., dn = n_2\}$ *is the set of divisors of* n_2. *Since* $n_1|n_2$, n_1 *must be in the set* $V(G_{D(n_2)})$. *Therefore, there exist at least one element in* $S = V(G_{D(n_2)})$ *which is a contradiction to the fact that* S *is empty. Hence,* P *and* $V(G_{D(n_2)})$ *has at least one element in common.*
Also since, 1 *is the divisor of* n_2 *in* $G_{D(n_2)}$, 1 *is adjacent to* n_1 *in* $G_{D(n_1)}$.
Hence, $q = q_1 + q_2 + p_2 + |S| + 1$.
Using the graph, $V(G_{D(n_1)}) = p_1 = 2; V(G_{D(n_2)}) = p_2 = 2;$
$E(G_{D(n_1)}) = q_1 = 1; E(G_{D(n_2)}) = q_2 = 1$ *and* $|S| = 1$.
Therefore $V(G_{D(n_1)} + G_{D(n_2)}) = p = p_1 + p_2 = 2 + 2 = 4$ *and*
$E(G_{D(n_1)} + G_{D(n_2)}) = q = 1 + 1 + 2 + 1 + 1 = 6$.
Therefore the result is true in this case.



case-2: n_2 **is composite.**
Given n_1 *is prime. Since* $n_1|n_2$, n_2 *is a multiple of* n_1. *By definition of* $G_{D(n_1)} + G_{D(n_2)}$, $V(G_{D(n_1)} + G_{D(n_2)}) = p_1 + p_2$. *To find q: By definition of sum of graphs there are* q_1 *edges of* $G_{D(n_1)}$ *and* q_2 *edges of* $G_{D(n_2)}$ *in* $G_{D(n_1)} + G_{D(n_2)}$ *and since* 1 *is the divisor of* n_1, 1 *is adjacent with* p_2 *vertices.*
Now,$P = \{n_1, 2n_1, ..., n_2n_1\}$ *and* $V(G_{D(n_2)}) = \{1, d_2, d_3, ..., n_2\}$ *which implies*
$S = V(G_{D(n_2)}) \cap P \geq 1$. *If* $S = \phi$ *for* $1 \leq j \leq n$, $d_j \notin S = P \cap V(G_{D(n_2)})$ *which is a contradiction since* P *and* $V(G_{D(n_2)})$ *has* n_1 *as common element as* $n_1|n_2$.
Therefore $|S| \geq 1$. *As* 1 *is the divisor of* n_2 *in* $G_{D(n_2)}$, 1 *is adjacent to* n_1 *in* $G_{D(n_1)}$.
Hence, $q = q_1 + q_2 + p_2 + |S| + 1$.

Suppose n_1 *does not divide* n_2, *where* n_1 *is prime.*
case-3: n_2 **is prime**
By definition of sum of graphs, $V(G_{D(n_1)} + G_{D(n_2)}) = p_1 + p_2$.
To find the number of edges q, By definition of sum of graphs, q_1 *edges of* $G_{D(n_1)}$ *and* q_2 *edges of* $G_{D(n_2)}$ *are in* $G_{D(n_1)} + G_{D(n_2)}$ *and since* 1 *is the divisor of* n_1 *in* $G_{D(n_1)}$, 1 *is adjacent to* p_2 *vertices. Since* n_1 *does not divides* n_2, $S = P \cap V(G_{D(n_2)}) = \phi$. *Therefore,*

$|S| = 0$. As 1 is the vertex of $G_{D(n_2)}$, it is adjacent to n_1 in $G_{D(n_1)}$.
Hence $q = q_1 + q_2 + p_2 + |S| + 1$.
Also, $q_1 = 1, q_2 = 1, p_1 = 2, p_2 = 2$ and $|S| = 0$
Therefore $q = 1 + 1 + 2 + 0 + 1 = 5$. The graph also has 5 edges.
The result is true in this case.



case 4: n_2 is composite
By definition of sum of graphs, $V(G_{D(n_1)} + G_{D(n_2)}) = p_1 + p_2$. To find q: By definition of sum of graphs we have q_1 edges of $G_{D(n_1)}$ and q_2 edges of $G_{D(n_2)}$. Since 1 is the divisor of n_1 in $G_{D(n_1)}$, 1 is adjacent to p_2 vertices and as 1 is the vertex of $G_{D(n_2)}$, it is adjacent to n_1 in $G_{D(n_1)}$. Also, $S = V(G_{D(n_2)}) \cap P = \phi$ (since n_1 does not divide n_2) which implies $|S| = 0$. Hence $q = q_1 + q_2 + p_2 + |S| + 1$.
Therefore the result is true in all cases.
Hence the theorem.

Remark 3.1. *Note that in case-1 of above theorem two edges need to delete for obtaining a divisor function graph, one edge is from 1 in $G_{D(n_1)}$ to 1 in $G_{D(n_2)}$ and the other edge from n_1 in $G_{D(n_1)}$ to n_2 in $G_{D(n_2)}$ are deleted since $n_1 = n_2$ in case-1. In all other cases, the edge from 1 in $G_{D(n_1)}$ to 1 in $G_{D(n_2)}$ is deleted to get the simple graph as a divisor function graph.*

4. Energy and Independent function of $G_{D(p^n)}$, $G_{D(pq)}$

Theorem 4.1. *If p is prime and n is any positive integer then Energy of $G_{D(p^n)}$ is $2n$.*
Proof: *We know that $|V(G_{D(p^n)})| = n + 1$. The adjacency matrix for $G_{D(p^n)}$ is,*

$$A[G_{D(p^n)}] = \begin{bmatrix} 0 & 1 & \dots & 1 \\ 1 & 0 & \dots & 1 \\ . & . & . & . \\ 1 & 1 & . & 0 \end{bmatrix}_{(n+1)\times(n+1)}.$$

That is, all the $(i,j)^{th}$ entries are units except the leading diagonal.

The characteristic equation of the $A[G_{D(p^n)}] = f(\lambda) = det(\lambda I_n - A[G_{D(p^n)}]) = 0$ is

$$\Rightarrow \det \begin{bmatrix} \lambda & -1 & \dots & -1 \\ -1 & \lambda & \dots & -1 \\ . & . & . & . \\ -1 & -1 & . & \lambda \end{bmatrix}_{(n+1)\times(n+1)} = 0.$$

$\Rightarrow (\lambda - 1)^n \times (\lambda + n) = 0.$

Therefore, the eigen values are $1, 1, .., 1$(n times). That is, Algebraic Multiplicity of 1 is n and $(-n)$ is also an eigen value. As the energy, $E(G)$, of a simple graph G is defined to be the sum of the absolute values of the eigen values of G.

Therefore, Energy of $G_{D(p^n)} = \sum_{i=1}^{n+1} |\lambda_i|$.
$= 1 + 1 + ... + 1 + |-n|$
Energy of $G_{D(p^n)} = 2n$.

Theorem 4.2. *Energy of a semi prime $G_{D(pq)}$ is $2+\sqrt{17}$ where $p \neq q$ be distict primes.*
Proof: *Clearly, $G_{D(pq)} \cong K_n - e$, for any edge $e \in E(G_{D(q)})$ and its adjacency matrix is given by $A(G_{D(pq)})$=*

$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$ *which is a singular matrix of order 4×4.*

Then its characteristic equation of $A[G_{D(p^n)}] = f(\lambda) = det(\lambda I_n - A[G_{D(pq)}]) = 0$ is

$$\Rightarrow \det \begin{bmatrix} \lambda+1 & 0 & 0 & 1 \\ 1 & -\lambda & 0 & 1 \\ 1 & 0 & -\lambda & 1 \\ 1 & 1 & 1 & -\lambda \end{bmatrix} = 0.$$

On further simplification, the latent roots are 0,-1 and $\frac{1 \pm \sqrt{17}}{2}$. Its energy is $2+\sqrt{17}$.

Definition 4.1. *[7] Let $G(V,E)$ be a graph. A function $f : V \to [0,1]$ is called an Independent function(IF), if for every vertex $v \in V$ with $f(v) > 0$, we have $\sum\limits_{u \in N[v]} f(u) = 1$*

Theorem 4.3. *If $f : V \to [0,1]$ such that $f(v) = 1, \forall v \in I$, then f becomes an Independent function of $G_{D(n)}$ for some independent set I.*
Proof: *Let us prove the theorem for two cases.*
case-1: n is prime
Clearly, $V(G_{deg(p)}) = \{1, p\}$. So, pendant vertices in the graph forms an Independent set. Also, $N[1] = \{p\}$ and $N[p] = \{1\}$.
$\Rightarrow f(1) = 1$ *and* $f(p) = 1$
Hence, $\sum\limits_{u \in N[v]} f(u) = 1$
case-2: n is composite
Let $n = p_1^{\alpha_1} p_2^{\alpha_2} ... p_n^{\alpha_n}$ and take I as the set of all prime divisors of n which clearly an independent set such that d in number.
Let the number of divisors of n be $\tau(n) = (\alpha_1+1)(\alpha_2+1)...(\alpha_n+1) = n_1$ (say). And so $\tau(n) - d$ non-prime divisors which doesnot forms an independent set . Clearly $|I| > 1, \forall v \in I$ such that $f(v) > 0$, that is, there exist at least one Independent function f. Since each vertex in I is prime, by repeatedly applying case-1 we obtain the result.

5. Conclusion and Future Scope

In this article, we estimated several indices for special case of a divisor function graph such as prime powers and semiprime.
Also, We estimated the energy for prime power and semiprime divisor function graphs to be $2n$ and $2+\sqrt{17}$.

Our future works are to evaluate the indices of $G_{D(n)}$ other than semiprime and prime powers and to determine the size of other operations on the divisor function graph.

Acknowledgement. The authors thank the Management of SASTRA Deemed University for providing this opportunity. Also, the authors would like to record their gratitude to the organizers of ICGTA2019.

References

[1] Balakrishnan, R., (2004), The energy of a graph, Linear Algebra and its Applications, Volume-387, pp.287-295.

[2] Bondy, J. A., USR Murty(2017), Graph Theory with Application, New York.

[3] Fajtlowicz, S., (1987) On conjectures of Graffti-II, Congr. Num. (60) pp.187-197.

[4] Kannan, K., Narasimhan, D. and Shanmugavelan, S. (2015), The Graph of Divisor function D(n), IJPAM, 102 (3) pp. 483-494.

[5] Narasimhan, D., Vignesh, R. and Elamparithi, A., (2018), Directed Divisor Function Graph $G_{Dij}(n)$", International Journal of Engineering and Advanced Technology (IJEAT), Volume-8, Issue-2S, pp. 214-218.

[6] Narasimhan, D. Elamparithi, A. and Vignesh, R.,(2018), Connectivity, Independency and Colorability of Divisor Function Graph $G_{D(n)}$, International Journal of Engineering and Advanced Technology (IJEAT), Volume-8, Issue-2S, pp. 209-213.

[7] Sangeetha, K. J., Maheswari, B., (2015), Independent functions of Euler Totient Cayley Graph, IJCER, ISSN(e):2250-3005, Volume-05, Issue-2.

S. Shanmugavelan works as an Assistant professor in Srinivasa Ramanujan Centre, SASTRA Deemed University, Kumbakonam, India. He received his MSc. degree from SASTRA Deemed University in 2015 and awarded M.Phil from Bharathidasan University in 2016. His research areas of interest are Domination in graphs, Number theoretic graphs and Topological indicies.

K. Thanga Rajeswari is a post graduate student in Department of Mathematics, Srinivasa Ramanujan Centre, SASTRA Deemed University. She graduated in 2018 from SASTRA. Her research interests focus mainly in Graph theoretical problems.

C. Natarajan received his Master's degree from IIT Madras, Chennai in the year 2004 and Ph. D degree in 2016 from SASTRA Deemed University, Thanjavur, India. His research areas include domination, graph colourings and Topological indices. He has 15 years of teaching experience and he is currently working as Assistant Professor in the Department of Mathematics, Srinivasa Ramanujan Centre, SASTRA Deemed University, Kumbakonam. He published more than 20 research articles in peer reviewed journals of national and international repute.