View metadata, citation and similar papers at core.ac.uk
brought to you by CORE
provided by Isik University Academic Open Access

TWMS J. App. and Eng. Math. V.11, Special Issue, 2021, pp. 25-29

STRONGER RECONSTRUCTION OF DISTANCE-HEREDITARY GRAPHS

P. DEVI PRIYA[1], S. MONIKANDAN[1], §

Abstract. A graph is said to be *set-reconstructible* if it is uniquely determined up to isomorphism from the set S of its non-isomorphic one-vertex deleted unlabeled subgraphs. Harary's conjecture asserts that every finite simple undirected graph on four or more vertices is set-reconstructible. A graph G is said to be *distance-hereditary* if for all connected induced subgraph F of G, $d_F(u,v) = d_G(u,v)$ for every pair of vertices $u, v \in V(F)$. In this paper, we have proved that the class of all 2-connected distance-hereditary graphs G with $\operatorname{diam}(G) = 2$ or $\operatorname{diam}(G) = \operatorname{diam}(\overline{G}) = 3$ are set-reconstructible.

Keywords: Set-reconstruction, connectivity, distance, distance-hereditary
AMS Subject Classification: 05C60, 05C40, 05C12

1. Introduction

The graphs dealt in this paper are finite, simple and undirected. Any definition and notation not given below are taken as in [1]. The *distance* $d(u,v)$ between two vertices u and v in a connected graph G is the minimum length of a path joining them. The *eccentricity* $e(v)$ of a vertex v is the distance to a vertex farthest from v. The *radius* $rad(G)$ is the minimum eccentricity of the vertices and the *diameter* $diam(G)$ is the maximum eccentricity. A graph G is *self-centered* if all vertices have the same eccentricity. Let H be an induced subgraph of a graph G. Then $N_H(u)$ is the set of all vertices adjacent to u in H. Also, $d_H(u,v)$ is the distance between the vertices u and v in H. The *connectivity* $\kappa(G)$ of a graph G is the minimum number of vertices whose removal results in a disconnected or trivial graph.

A *vertex-deleted subgraph* $G - v$ of a graph G is called a *card* of G. A graph H is a *set-reconstruction* of G if H has the same set S of (non-isomorphic) cards as G. A graph is *set-reconstructible* if it is isomorphic to all its set-reconstructions. A family $\mathscr{F}$ of graphs is *set-recognizable* if, for each $G \in \mathscr{F}$, every set-reconstruction of G is also in $\mathscr{F}$.
Harary's Conjecture [3]: All graphs with at least four vertices are set-reconstructible. This conjecture has been proved notoriously difficult, and has motivated a large amount

[1] Department of Mathematics, Manonmaniam Sundaranar University, Abishekapatti, Tirunelveli, Tamilnadu, INDIA.
e-mail: devipriya1409@gmail.com; ORCID: https://orcid.org/0000-0003-2077-6482.
e-mail: monikandans@gmail.com; ORCID: https://orcid.org/0000-0001-9805-2914.
§ Manuscript received: October 5, 2019; accepted: April 2, 2020.
TWMS Journal of Applied and Engineering Mathematics, Vol.11, Special Issue © Işık University, Department of Mathematics, 2021; all rights reserved.
The second author is supported by the University Grants Commission, Government of India. (F./2017-18/NFO-2017-18-OBC-TAM-53159)

.

of work in graph theory. Many parameters and several classes of graphs like disconnected graphs, trees and separable graphs without endvertices have been proved to be set-reconstructible by Manvel [5]. Also, it has been proved by Ramachandran and Monikandan [8] that all graphs are set-reconstructible if and only if all 2-connected graphs G such that $\text{diam}(G) = 2$ and all 2-connected graphs H such that $\text{diam}(H) = \text{diam}(\overline{H}) = 3$ are set-reconstructible. In this article, we prove that all distance-hereditary 2-connected graphs G such that $diam(G) = 2$ or $\text{diam}(G) = \text{diam}(\overline{G}) = 3$ are set-reconstructible.

The following well-known lemmas are used while proving our required result.

Lemma 1.1. *The connectivity of a graph G can be determined from the set of cards of G.*

Lemma 1.2. *Disconnected graphs are set-reconstructible.*

Lemma 1.3. *A graph G is set-reconstructible if and only its complement $\overline{G}$ is set- reconstructible.*

Lemma 1.4. *If $diam(G) \geq 3$, then $diam(\overline{G}) \leq 3$.*

Lemma 1.5. *If $rad(G) \geq 3$, then $rad(\overline{G}) \leq 2$.*

A graph G is said to be *distance-hereditary* if for all connected induced subgraph F of G, $d_F(u, v) = d_G(u, v)$ for every pair of vertices $u, v \in V(F)$.

Buckley and Harary [1] gave a characterization of distance-hereditary graphs as follows.

Theorem 1.1. [1] A graph G is distance-hereditary if and only if it contains no C_n, $n \geq 5$, $C_5 + e$, $C_6 + e$ nor $C_5 + \{e_1, e_2\}$, where the edges e_1 and e_2 have exactly one vertex in common, as an induced subgraph.

Graphs on at most 11 vertices [4] and disconnected graphs [5] are set-reconstructible and so we consider only connected graphs on at least 12 vertices.

In general, Kelly's Lemma cannot be applied for the set of cards. That is, one cannot determine the number of subgraphs in G isomorphic to a given graph F, where $|V(F)| < |V(G)|$, unless the multiplicity of each card in S is known. On the other hand, if F is a induced subgraph (or subgraph) of a card in S, then F must be a induced subgraph (or subgraph) of G. Using this remark, we prove the following lemma.

Lemma 1.6. *Distance-hereditary graphs are set-recognizable.*

Proof. If G itself is a cycle of order n, then G is a non-separable regular graph of degree two with n edges and hence it is set-reconstructible. Moreover, as $n \geq 12$, if G has one of the forbidden graphs given in Theorem 1.1, say F, as an induced subgraph, then F is an induced subgraph of each card $G - u$, where $u \in V(G) - V(F)$. Hence from the set S of one-vertex deleted subgraphs, we can determine whether G has F as an induced subgraph or not. □

Lemma 1.7. *The diameter of a distance-hereditary 2-connected graph G can be determined from the set of cards of G.*

Proof. Since G is distance-hereditary, the diameter of its maximal connected subgraph cannot exceed than that of G. Also, when $diam(G) = r$, there exist vertices u and v in $V(G)$ with $d(u, v) = r$. Thus, $d_{G-x}(u, v) = r$ for every x in $G - \{u, v\}$, as G is 2-connected. Hence, $diam(G) = max\{diam(H) : H\ is\ a\ maximal\ subgraph\ of\ G\}$.

□

2. Distance-hereditary graphs G with $\mathrm{diam}(G) = \mathrm{diam}(\overline{G}) = 3$

First we prove the following lemma which is used in proving our required class of graph to be set-reconstructible.

Lemma 2.1. *If G is self-centered with $rad(G) \geq 3$, then $\overline{G}$ is self-centered with $rad(\overline{G}) = 2$.*

Proof. Since $rad(G) \geq 3$, we have $diam(G) \geq 3$. By Lemmas 1.4 and 1.5, $diam(\overline{G}) \leq 3$ and $rad(\overline{G}) \leq 2$. Moreover, if there were a vertex with eccentricity 1 in $\overline{G}$, then that vertex would be an isolated vertex in G, giving a contradiction. Thus, $diam(\overline{G}) = 2$ or 3 and $rad(\overline{G}) = 2$.

Again, if $diam(\overline{G})$ were 3, then there exist two vertices u and v such that $d_{\overline{G}}(u, v) = 3$ and $N_{\overline{G}}(u) \cap N_{\overline{G}}(v) = \phi$. Hence, in G, the vertices u and v would be adjacent and each vertex in $V - \{u, v\}$ would be adjacent to at least one of the two vertices u and v. This implies that $rad(G) \leq 2$, giving a contradiction. □

Note that graphs with radius one has an $n - 1$ vertex and so is set-reconstructible. Thus, in view of Lemma 2.1, all graphs G with $\mathrm{diam}(G) = \mathrm{diam}(\overline{G}) = 3$ are set-reconstructible if all graphs G with $\mathrm{diam}(G) = 3$ and $rad(G) = 2$ are set-reconstructible.

Lemma 2.2. *Distance-hereditary 2-connected graphs G with $diam(G) = 3$ and $rad(G) = 2$ are set-recognizable.*

Proof. Using Lemma 1.6, 1.7, the class of all distance-hereditary 2-connected graphs with diameter two are set-recognizable.

Since G is a graph of radius 2, there exists u in $V(G)$ with $e(u) = 2$. In addition, since G is a 2-connected distance-hereditary graph, we have $e(u) = 2$ in $G - x$ for all x in $N_1(u)$. Suppose that F is a 2-connected distance-hereditary graph with $diam(F) = rad(F) = 3$ and contains a vertex u_1 with $e(u_1) = 2$ in $F - u_2$ for some vertex u_2 in F. Then the only vertex at distance 3 from u_1 is u_2. Let Q be the set of neighbours of u_2 in F. Trivially, $F - \{Q - q\}$ is connected for each q in Q. Since F is distance-hereditary of diameter 3, it follows that $d(x, u_2) \leq 3$ for all x in F and hence $d(x, q) \leq 2$ for all x. Thus, $e(q) = 2$ in F, giving a contradiction to $rad(F) = 3$. Hence, the graph F has no such maximal subgraphs. Therefore, the graph G is set-recognizable. □

Let u be a vertex in G with $e(u) = 2$. We define $N_1(u) = \{v \in V(G) : d(u, v) = 1\}$, $Z = \{v \in V(G) : d(u, v) = 2\}$, $X = \{v \in V(G) : d(v, z) \geq 2, \text{for any } z \in N_2(u)\}$ and $Y = N_1(u) - X$ and we take $\kappa(G)$ to be k.

Theorem 2.1.[2] *If W is a minimum vertex cut of G, then either u belongs to W with $W - \{u\} \subseteq X$ or $W \subseteq Y$, and the following hold.*

(i) If $u \in W$ with $W - \{u\} \subseteq X$, then $deg(u) = |N_{G-W}(x_i)| + (k - 1)$, where $x_i \in X$.

(ii) If $W \subseteq Y$, then $N_{G-W}(y_i) = N_{G-W}(y_j)$ for $1 \leq i, j \leq k$ and $i \neq j$, where $y_i, y_j \in Y$. Also any two such distinct k-vertex cuts are disjoint. Moreover, any minimal $(k+1)$-vertex cut and k-vertex cut are disjoint.

Theorem 2.2. *All distance-hereditary 2-connected graphs G with diameter 3 and radius 2 are set-reconstructible.*

Proof. Set-recognization of G follows by Lemmas 1.6 and 1.7.

We now set-reconstruct G as follows. Draw a card $G - v$, from S, with a $(k - 1)$-vertex cut $W_{k-1} = \{w_1, w_2, \ldots, w_{k-1}\}$ such that $d(v)$ is maximum. Clearly, $W = \{v\} \cup W_{k-1}$ is a k-vertex cut of G. Thus, $W \subseteq X \cup \{u\}$ or $W \subseteq Y$. Suppose $d(v) = |N_{G-W_{k-1}}(w_i)| + (k-1)$, for $1 \leq i \leq k-1$, then all graphs, obtained by adding a new vertex to $G-v$ and making it adjacent to the common neighbours of the vertices in W_{k-1} and to each vertex in W_{k-1}, are isomorphic and they are G.

Suppose that $S(G)$ has no such cards. Then, by Theorem 2.1, each k-vertex cut of G is contained in Y. Now, we can identify the number of k-vertex cuts of G as follows. Draw a card $G-y$ containing the minimum number of k-vertex cuts and no $(k-1)$-vertex cut (existence of such a card is guaranteed by $G-u$). Then the only k-vertex cut of G are those of $G-y$ and so the number, say m, of k-vertex cuts of G can be determined. Let $W_1, W_2, \ldots, W_m$ be the m vertex cuts of G of size k and $G[W_i]$ be the subgraph of G induced by W_i.

Now G is set-reconstructed as follows. Consider a card $G-y$ with a $(k-1)$-vertex cut. By Theorem 2.1, the card $G-y$ must be obtained by deleting the vertex y from a k-vertex cut of G and so, it contains precisely $m-1$ vertex cuts of size k. All graphs obtained by replacing the $(k-1)$-vertex cut of $G-y$ by $G[W_i]$, where $G[W_i]$ is the graph not induced by any of the remaining $m-1$ vertex cuts of size k of $G-y$, and making each vertex in W_i adjacent to the common neighbours of those vertices in the $(k-1)$-vertex cut of $G-y$ are isomorphic and they are G. $\square$

Thus we have the following theorem.

Theorem 2.3. *All distance-hereditary 2-connected graphs G with $diam(G) = diam(\overline{G}) = 3$ are set-reconstructible.*

3. Distance-hereditary graphs G with diameter two

In this section, by G, we mean a distance-hereditary graph of diameter two and connectivity k. As graphs of radius one are set-reconstructible, we can assume that $rad(G) = 2$. So, we consider the vertex u and the sets X, Y, and Z as in previous section with an additional condition that each vertex in X has a neighbour in Y.

Lemma 3.1.[2] If W is a minimum vertex cut of G, then $W \subseteq Y$. Moreover, if W_1 and W_2 are two distinct k-vertex cuts, then $W_1 \cap W_2 = \phi$.

Theorem 3.1. *All distance-hereditary 2-connected graphs G with diameter two are set-reconstructible.*

Proof. Set-recognizability follows by Lemmas 1.6 and 1.7. Set-reconstruction follows by proceeding as in Theorem 2.2. $\square$

4. Conclusion

It is not known whether graphs G with $diam(G) = 2$ are set-recognizable or not, even though they are proved to be recognizable form the full collection of cards. However, existence of an induced path of length k in a card of a graph does not imply that the diameter of the graph is at least k (example C_5). Graphs with diameter one are precisely complete graphs, which are set-recognizable. Hence, if graphs with diameter two are set recognizable, then graphs with diameter at least three are set recognizable. Also " $diam(G) \geq 3$ and $diam(\overline{G}) \geq 3$" if and only if " $diam(G) = diam(\overline{G}) = 3$ ". Hence, if graphs with diameter two are set recognizable, then graphs G with $diam(G) = diam(\overline{G}) = 3$ are also set-recognizable. In Lemma 1.4, we have proved a weaker result that all distance-hereditary 2-connected graphs G with $diam(G) = 2$ are set-recognizable.

References

[1] Buckley, F., and Harary, F. (1990), Distance in Graphs, Addison-Wesley.
[2] Devi Priya, P., and Monikandan, S. (2018), Reconstruction of distance hereditary 2-connected graphs, Discrete Math., 341, (8), 2326-2331.
[3] Harary, F. (1964), On the reconstruction of a graph from a collection of subgraphs, Theory of Graphs and Its Applications (M. Fiedler, ed.), Prague , Reprinted Academic Press, New York, 47-52.
[4] McKay, B. D. (1997), Small graphs are reconstructible, Australas J. Combin., 15, 123-126.
[5] Manvel, B. (1976), On Reconstructing graphs from their sets of subgraphs, J. Combin. Theory Ser. B, 21, 156-165.
[6] Ramachandran, S., and Monikandan, S. (2007), All graphs are set reconstructible if all 2-connected graphs are set reconstructible, Ars Combin., 83, 341-352.
[7] Ramachandran, S., and Monikandan, S. (2009), Graph reconstruction conjecture: reductions using complement, connectivity and distance, Bull. Inst. Combin. Appl., 56, 103-108.
[8] Ramachandran, S., and Monikandan, S. (2009), A sufficient condition for Set Reconstruction, Ars Combin., 91, 439-446.

P. Devi Priya is doing Ph.D. in the Department of Mathematics, Manonmaniam Sundaranar University since 2016 under UGC-NFOBC fellowship. She has secured University First Rank in M.Sc. Integrated programme in Mathematics during 2010-2015. She has been awarded INSPIRE FELLOWSHIP by DST, New Delhi and YOUNG SCIENTIST AWARD of citation 2018-2019 for her outstanding achievements in Mathematics by Dr. K. V. Rao Scientific Society, Hyderabad.

S. Monikandan has got his Ph.D. from Manonmaniam Sundaranar University in 2006 for his thesis on Ulam's Conjecture in Graph Theory. He is working as an Associate Professor of this University since 2005. He has guided four Ph.D. students and thirty M.Phil. students. He has written three books and completed successfully two major research projects on this problem sponsored by DST-SERB, New Delhi. He has 32 research publications of which 15 were published in SCI journals.