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THE ORIENTATION NUMBER OF THREE COMPLETE GRAPHS WITH LINKAGES

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Abstract. For a graph G, let $\mathscr{D}(G)$ be the set of all strong orientations of G. The *orientation number* of G is $\vec{d}(G) = \min\{d(D)|D \in \mathscr{D}(G)\}$, where $d(D)$ denotes the diameter of the digraph D. In this paper, we consider the problem of determining the orientation number of three complete graphs with linkages.

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1. Introduction

Let G be a finite undirected simple graph with vertex set $V(G)$ and edge set $E(G)$. For a graph G and $x \in V(G)$, the degree of x in G is denoted by $d_G(x)$, and the maximum degree of G by $\Delta(G)$. For $v \in V(G)$, the *eccentricity of* v is $e_G(v) = \max\{d_G(v,x) \,|\, x \in V(G)\}$, where $d_G(v,x)$ denotes the length of a shortest (v,x)-path in G. The *diameter* of G is $d(G) = \max\{e_G(v) \,|\, v \in V(G)\}$.

Let D be a digraph with vertex set $V(D)$ and arc set $A(D)$ which has no loops and no two of its arcs have same tail and same head. The notions $e_D(v)$, for $v \in V(D)$, and $d(D)$ are defined as in the undirected graph.

An *orientation* of a graph G is a digraph D obtained from G by assigning a direction to each of its edge. A vertex v is *reachable* from a vertex u of a digraph D if there is a directed path in D from u to v. An orientation D of G is *strong* if any pair of vertices in D are mutually reachable in D. Robbins' one-way street theorem [7] states that a connected graph G has a strong orientation if and only if G is 2-edge-connected. For a 2-edge-connected graph G, let $\mathscr{D}(G)$ denote the set of all strong orientations of G. The *orientation number* of G is $\vec{d}(G) = \min\{d(D) \,|\, D \in \mathscr{D}(G)\}$. Any orientation D in $\mathscr{D}(G)$ with $d(D) = \vec{d}(G)$ is called an *optimal orientation* of G.

Given r fixed integers $n_1, n_2, \ldots, n_r$ with $n_r \geq n_{r-1} \geq \ldots \geq n_1 \geq 3$ and an integer m with $2 \leq r \leq m \leq \sum_{1 \leq i < j \leq r} n_i n_j$, the number of edges of the complete multipartite graph $K_{n_1,n_2,\ldots,n_r}$, let $\mathscr{G}(n_1, n_2 \ldots, n_r; m)$ denote the family of 2-edge connected graphs that are obtained from the disjoint union of r complete graphs $K_{n_1}, K_{n_2}, \ldots, K_{n_r}$ by adding

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m edges so that each edge links a vertex of K_{n_i} to a vertex of K_{n_j} for some i and j with $i \neq j$.

Define $\mathscr{G}_m^r = \{G : G \in \mathscr{G}(n_1, n_2, \ldots, n_r; m)$, where $n_1, n_2, \ldots, n_r$ are integers with $n_r \geq n_{r-1} \geq \cdots \geq n_1 \geq 3$ and $2 \leq r \leq m \leq \sum_{1 \leq i < j \leq r} n_i n_j\}$, $\mathscr{D}(\mathscr{G}_m^r) = \bigcup_{G \in \mathscr{G}_m^r} \mathscr{D}(G)$ and the parameter $\vec{d}(r; m) = \min\{\vec{d}(G) : G \in \mathscr{G}_m^r\}$. For a family of graphs $\mathscr{G}$, define $\vec{d}(\mathscr{G}) = \min\{\vec{d}(G) : G \in \mathscr{G}\}$. Hence, $\vec{d}(r; m) = \vec{d}(\mathscr{G}_m^r)$.

In [3], Koh and Ng considered the following problem: given a family of disjoint graphs, study the orientation number and design a corresponding optimal orientation for a resulting graph obtained by linking the given graphs with a set of additional edges.

For $r = 2$, Koh and Ng [3] proved the following:

- Let G_1 and G_2 be two bridgeless graphs of orders n_1 and n_2, respectively, and $\mathscr{G}_2^*$ be the family of graphs obtained by adding 2 edges to link G_1 and G_2. If $\Delta(G_1) = n_1 - 1$ and $\Delta(G_2) = n_2 - 1$, then $\vec{d}(\mathscr{G}_2^*) = 4$.
- $\min\{m : \vec{d}(2; m) = 3\} = 4$.
- For $p \geq 5$, $\vec{d}(\mathscr{G}(p, p; 2p)) = \vec{d}(\mathscr{G}(p, p+1; 2p)) = \vec{d}(\mathscr{G}(p, p+2; 2p+1)) = \vec{d}(\mathscr{G}(p, p+3; 2p+2)) = 2$.

Also, Ng [6] proved the following:

- $\vec{d}(\mathscr{G}(p, p+4; 2p+3)) = 2$.
- For $q \geq p + 5$, $\vec{d}(\mathscr{G}(p, q; 2p+4)) = 2$.

In this paper, we focus on the orientation number and designing a corresponding optimal orientation for three complete graphs with linkages.

Let D be a digraph. For $x, y \in V(D)$, write $x \rightarrow y$ or $y \leftarrow x$ if (x, y) is an arc in D. More generally, for $X, Y \subseteq V(D)$ with $X \cap Y = \phi$, write, $X \rightarrow Y$ if for every vertex x in X and for every vertex y in Y, we have $x \rightarrow y$. For simplicity, write $x \rightarrow Y$ for $\{x\} \rightarrow Y$ and $X \rightarrow y$ for $X \rightarrow \{y\}$. The *converse* of D, denoted by $\widetilde{D}$, is the digraph obtained from D by reversing each arc in D. It is clear that $d(D) = d(\widetilde{D})$. The subdigraph of D induced by $A \subseteq V(D)$ is denoted by $D[A]$.

We refer to [1] for notations and terminology not described here. For results on orientations of graphs, see a survey by Koh and Tay [4]. (Boesch and Tindell [2] and independently Maurer [5] proved that: $\vec{d}(K_n) = 2$ if $n \geq 3$ and $n \neq 4$, and $\vec{d}(K_4) = 3$. Soltés [8] proved that $\vec{d}(K_{p,q})$ is 3 if $2 \leq p \leq q \leq \binom{p}{\lfloor \frac{p}{2} \rfloor}$ and it is 4 if $q > \binom{p}{\lfloor \frac{p}{2} \rfloor}$, where $\lfloor x \rfloor$ denotes the greatest integer not exceeding the real x.)

2. Three complete graphs with linkages

In this section, we consider the orientation number for three complete graphs with linkages.

Theorem 2.1. *Let $i \in \{1, 2, 3\}$. Let G_i be a bridgeless graph of order $n_i \geq 3$ and let $\mathscr{G}(G_1, G_2, G_3; 3)$ be the family of 2-edge connected graphs obtained by adding 3 edges to link G_1, G_2 and G_3. If $\Delta(G_i) = n_i - 1$, then $\vec{d}(\mathscr{G}(G_1, G_2, G_3; 3)) = 6$.*

Proof: Let $x_i \in V(G_i)$ be a vertex such that $d_{G_i}(x_i) = n_i - 1$, A_i be a maximal independent subset of $G_i - x_i$, $G_i' = G_i - (A_i \cup \{x_i\})$ and $G = G_1 \cup G_2 \cup G_3 \cup \{x_1x_2, x_2x_3, x_1x_3\}$. Then $G \in \mathscr{G}(G_1, G_2, G_3; 3)$. Orient the edges of G as follows:

(i) $x_1 \rightarrow x_2 \rightarrow x_3 \rightarrow x_1$;

(ii) $A_i \rightarrow x_i \rightarrow V(G_i')$;

(iii) $u \rightarrow a$ if $u \in V(G_1')$, $a \in A_1$ and $ua \in E(G_1)$;

$v \rightarrow b$ if $v \in V(G_2')$, $b \in A_2$ and $vb \in E(G_2)$;

$w \to c$ if $w \in V(G_3')$, $c \in A_3$ and $wc \in E(G_3)$;
(iv) orient the remaining edges of G arbitrarily.
Let D be the resulting digraph. We claim that $d(D) \le 6$. By the nature of the orientation, we compute eccentricities only for vertices of G_1.

- Clearly, $x_1 \to V(G_1')$, $x_1 \to x_2 \to V(G_2')$, and $x_1 \to x_2 \to x_3 \to V(G_3')$. Let $a \in A_1$, $b \in A_2$, $c \in A_3$ be arbitrary. As each G_i is 2-edge-connected, there exist $u \in V(G_1')$, $v \in V(G_2')$, $w \in V(G_3')$ such that $ua \in E(G_1)$, $vb \in E(G_2)$, $wc \in E(G_3)$. Then $u \to a$, $v \to b$, $w \to c$. This shows that $e_D(x_1) \le 4$.
- Let $u \in V(G_1')$. By the choice of A_1, there exists $a \in A_1$ such that $ua \in E(G_1)$. Then $u \to a$. As $A_1 \to x_1$, $u \to a \to x_1$. This together with $e_D(x_1) \le 4$ implies that $e_D(u) \le 6$.
- Let $a \in A_1$. $A_1 \to x_1$ and $e_D(x_1) \le 4$ implies that $e_D(a) \le 5$.

Hence, $d(D) \le 6$, and therefore $\vec{d}(G) \le 6$. Consequently, $\vec{d}(\mathscr{G}(G_1, G_2, G_3; 3)) \le 6$.

We next prove $\vec{d}(\mathscr{G}(G_1, G_2, G_3; 3)) \ge 6$ by the method of contradiction. Suppose there exists a graph G_0 in $\mathscr{G}(G_1, G_2, G_3; 3)$ and an orientation D_0 in $\mathscr{D}(G_0)$ such that $d(D_0) \le 5$. Since G_0 is 2-edge connected, the three edges added to $G_1 \cup G_2 \cup G_3$ to obtain G_0 must be $x'y'$, $y''z'$, $z''x''$ for some $x', x'' \in V(G_1)$, $y', y'' \in V(G_2)$, $z', z'' \in V(G_3)$. As $D_0 \in \mathscr{D}(G_0)$, in D_0, we have either $x' \to y'$, $y'' \to z'$, $z'' \to x''$ or $x' \leftarrow y'$, $y'' \leftarrow z'$, $z'' \leftarrow x''$. By symmetry, assume that $x' \to y'$, $y'' \to z'$, $z'' \to x''$. We consider three cases.

Case 1. Among the three pairs $\{x', x''\}$, $\{y', y''\}$, $\{z', z''\}$, at least two satisfy $x' = x''$, $y' = y''$, $z' = z''$, respectively.

Assume, by symmetry, that $x' = x''$ and $z' = z''$.

If there exists $x_0 \in V(G_1) \setminus \{x'\}$ such that $x' \to x_0$, then $y' = y''$. (Otherwise, $y' \ne y''$, and there is no directed path from x_0 to any vertex of $V(G_3) \setminus \{z'\}$, a contradiction.) For any $z_0 \in V(G_3) \setminus \{z'\}$, since $d_{D_0}(x_0, z_0) \le 5$, we have $x_0 \to x_0' \to x' \to y' \to z' \to z_0$ for some $x_0' \in V(G_1) \setminus \{x', x_0\}$. Hence, $z' \to (V(G_3) \setminus \{z'\})$. Consequently, there is no directed path from any vertex of $V(G_3) \setminus \{z'\}$ to z', a contradiction.

This contradiction shows that for any $x_0 \in V(G_1) \setminus \{x'\}$, we have $x' \leftarrow x_0$. Hence, $(V(G_1) \setminus \{x'\}) \to x'$. Then, there is no directed path from x' to any vertex of $V(G_1) \setminus \{x'\}$, once again a contradiction.

Case 2. Among the three pairs $\{x', x''\}$, $\{y', y''\}$, $\{z', z''\}$, exactly one satisfy $x' = x''$, $y' = y''$, $z' = z''$, respectively.

Assume, by symmetry, that $x' = x''$.

If $x_0 \in V(G_1) \setminus \{x'\}$ and $z_0 \in V(G_3) \setminus \{z', z''\}$, then since $d_{D_0}(x_0, z_0) \le 5$, $x_0 \to x' \to y' \to y'' \to z' \to z_0$. Hence, $(V(G_1) \setminus \{x'\}) \to x'$ and $z' \to (V(G_3) \setminus \{z', z''\})$. Then, there is no directed path from x' to any vertex in $V(G_1) \setminus \{x'\}$, a contradiction.

Case 3. $x' \ne x''$, $y' \ne y''$, $z' \ne z''$.

If $x_0 \in V(G_1) \setminus \{x', x''\}$ and $z_0 \in V(G_3) \setminus \{z', z''\}$, then since $d_{D_0}(x_0, z_0) \le 5$, $x_0 \to x' \to y' \to y'' \to z' \to z_0$. Hence, $(V(G_1) \setminus \{x', x''\}) \to x'$ and $z' \to (V(G_3) \setminus \{z', z''\})$. $d_{D_0}(z', y'') \le 5$ implies that $z' \to z'' \to x'' \to x' \to y' \to y''$. Now $d_{D_0}(z_0, z') \ge 6$, a contradiction. This contradiction shows that for any $x_0 \in V(G_1) \setminus \{x'\}$, we have $x' \leftarrow x_0$. Hence, $(V(G_1) \setminus \{x'\}) \to x'$. Then, there is no directed path from x' to any vertex of $V(G_1) \setminus \{x'\}$, once again a contradiction.

This completes the proof.

Theorem 2.2. *Let $i \in \{1, 2, 3\}$. Let G_i be a bridgeless graph of order $n_i \ge 3$ and let $\mathscr{G}(G_1, G_2, G_3; 4)$ be the family of 2-edge connected graphs obtained by adding 4 edges to link G_1, G_2 and G_3. If $K_{1,1,n_i-2} \subseteq G_i$, then $\vec{d}(\mathscr{G}(G_1, G_2, G_3; 4)) = 4$.*

Proof: Let $V(G_i) = \{x_j^i \,|\, j = 1, 2, \ldots, n_i\}$, $V_i = \{x_j^i \,|\, j = 3, 4, \ldots, n_i\}$, $d_{G_i}(x_1^i) = d_{G_i}(x_2^i) = n_i - 1$, and $G = G_1 \cup G_2 \cup G_3 \cup \{x_1^1x_1^2, x_2^1x_1^2, x_1^2x_1^3, x_1^2x_2^3\}$. Then $G \in \mathscr{G}(G_1, G_2, G_3; 4)$. Orient the edges of G as follows:
(i) $\{x_1^1, x_1^3\} \to x_1^2 \to \{x_2^1, x_2^3\}$;
(ii) $x_2^1 \to \{x_1^1\} \cup V_1$, $V_1 \to x_1^1$, $x_1^2 \to x_2^2 \to V_2 \to x_1^2$, $\{x_2^3\} \cup V_3 \to x_1^3$, $x_2^3 \to V_3$;
(iii) orient the remaining edges of G arbitrarily.
Let D be the resulting digraph. We claim that $d(D) \leq 4$.

The existence of the paths from: $x_1^2 \to x_2^2 \to V_2$, $x_1^2 \to x_2^1 \to V_1 \cup \{x_1^1\}$, and $x_1^2 \to x_2^3 \to V_3 \cup \{x_1^3\}$ shows that $e_D(x_1^2) \leq 2$. This together with: $x_2^1 \to x_1^1 \to x_1^2$ imply that $e_D(x_1^1) \leq 3$ and $e_D(x_2^1) \leq 4$; $x_2^2 \to x_3^2 \to x_1^2$ imply that $e_D(x_2^2) \leq 4$; for any $x_i^2 \in V_2$, $x_i^2 \to x_1^2$ imply that $e_D(x_i^2) \leq 3$. For any $x_i^1 \in V_1$, $x_i^1 \to x_1^1$ and $e_D(x_1^1) \leq 3$ implies that $e_D(x_i^1) \leq 4$. By the nature of the orientation, the bounds for the eccentricities of the vertices x_1^3, x_2^3, x_i^3, where $x_i^3 \in V_3$, are equal to the bounds of the eccentricities of the vertices x_1^1, x_2^1, x_i^1, where $x_i^1 \in V_1$.

This shows that $d(D) \leq 4$, and hence $\vec{d}(G) \leq 4$. Consequently, $\vec{d}(\mathscr{G}(G_1, G_2, G_3; 4)) \leq 4$.

We next prove $\vec{d}(\mathscr{G}(G_1, G_2, G_3; 4)) \geq 4$ by the method of contradiction. Suppose there is a G_0 in $\mathscr{G}(G_1, G_2, G_3; 4)$ and an orientation D_0 of G_0 such that $d(D_0) \leq 3$. We consider two cases.

Case 1. There is no edge with one end in G_r and other end in G_s for some $r, s \in \{1, 2, 3\}$ with $r \neq s$.

Since G_0 is 2-edge-connected, assume that the linked edges added to be $x_{r_1}^1x_{r_1}^2$, $x_{r_2}^1x_{r_2}^2$, $x_{r_3}^2x_{r_1}^3$ and $x_{r_4}^2x_{r_2}^3$. As $D_0 \in \mathscr{D}(G_0)$, without loss of generality, assume that, in D_0, we have $x_{r_1}^1 \to x_{r_1}^2$, $x_{r_2}^2 \to x_{r_2}^1$, $x_{r_3}^2 \to x_{r_1}^3$, $x_{r_2}^3 \to x_{r_4}^2$. Then, for any $x_p^1 \in V(G_1) \setminus \{x_{r_1}^1\}$ and for any $x_q^3 \in V(G_3) \setminus \{x_{r_1}^3\}$, $d_{D_0}(x_p^1, x_q^3) \geq 4$, a contradiction.

Case 2. For every $r, s \in \{1, 2, 3\}$ with $r \neq s$, there exists at least one edge with one end in G_r and other end in G_s.

Since G_0 is 2-edge-connected, assume that the linked edges added to be $x_{r_1}^1x_{r_1}^2$, $x_{r_2}^2x_{r_1}^3$, $x_{r_2}^1x_{r_2}^3$ and $x_{r_3}^1x_{r_3}^3$. As $D_0 \in \mathscr{D}(G_0)$, without loss of generality, assume that, in D_0, we have $x_{r_1}^1 \to x_{r_1}^2$, $x_{r_2}^2 \to x_{r_1}^3$, $x_{r_2}^3 \to x_{r_2}^1$ and either $x_{r_3}^1 \to x_{r_3}^3$ or $x_{r_3}^3 \to x_{r_3}^1$. Then, for any $x_p^3 \in V(G_3) \setminus \{x_{r_2}^3, x_{r_3}^3\}$ and for any $x_q^2 \in V(G_2) \setminus \{x_{r_1}^2\}$, $d_{D_0}(x_p^3, x_q^2) \geq 4$, a contradiction.

This completes the proof.

Recall that: $\mathscr{G}_m^3 = \{G : G \in \mathscr{G}(n_1, n_2, n_3; m)$, where n_1, n_2, n_3 are integers with $n_3 \geq n_2 \geq n_1 \geq 3$ and $3 \leq m \leq n_1n_2 + n_1n_3 + n_2n_3\}$. Set $\mathscr{G}_m^{3^*} = \{G : G \in \mathscr{G}(n_1, n_2, n_3; m)$, where n_1, n_2, n_3 are integers with $n_3 \geq n_2 \geq n_1 \geq 3$, $3 \leq m \leq n_1n_2 + n_1n_3 + n_2n_3$, $n_1 \neq 4$, $n_2 \neq 4$ and $n_3 \neq 4\}$.

Theorem 2.3. $\vec{d}(\mathscr{G}_9^{3^*}) \leq 3$.

Proof: Let $V(K_{n_1}) = \{x_1, x_2, \ldots, x_{n_1}\}$, $V(K_{n_2}) = \{y_1, y_2, \ldots, y_{n_2}\}$, $V(K_{n_3}) = \{z_1, z_2, \ldots, z_{n_3}\}$; $V_1 = \{x_3, x_4, \ldots, x_{n_1}\}$, $V_2 = \{y_3, y_4, \ldots, y_{n_2}\}$, $V_3 = \{z_3, z_4, \ldots, z_{n_3}\}$; G_1, G_2 and G_3 be the complete subgraphs of K_{n_1}, K_{n_2} and K_{n_3} induced by the sets V_1, V_2 and V_3, respectively; and $G = K_{n_1} \cup K_{n_2} \cup K_{n_3} \cup \{x_1y_2, x_1z_2, x_2y_1, x_2z_1, x_2y_2, x_2z_2, y_1z_2, y_2z_1, y_2z_2\}$. Then $G \in \mathscr{G}_9^{3^*}$. Orient the edges of G as follows:
(i) $x_1 \to V_1 \to x_2$, $x_1 \to x_2 \to \{y_1, y_2, z_1\}$;
(ii) $y_1 \to V_2 \to y_2$, $y_1 \to y_2 \to \{z_1, z_2, x_1\}$;
(iii) $z_1 \to V_3 \to z_2$, $z_1 \to z_2 \to \{x_1, x_2, y_1\}$; and
(iv) orient the edges of G_1, G_2 and G_3 such that $\vec{d}(G_1) \leq 3$, $\vec{d}(G_2) \leq 3$ and $\vec{d}(G_3) \leq 3$.

Let D be the resulting digraph. We claim that $d(D) \leq 3$. By the nature of the orientation, we compute eccentricity only for the vertices of K_{n_1}. The existence of the paths from: $x_1 \to V_1$, $x_1 \to x_2 \to y_2$, $x_1 \to x_2 \to y_1 \to V_2$, $x_1 \to x_2 \to z_1 \to \{z_2\} \cup V_3$, in D, shows that $e_D(x_1) \leq 3$; $x_2 \to y_2 \to x_1 \to V_1$, $x_2 \to y_1 \to V_2$, $x_2 \to z_1 \to \{z_2\} \cup V_3$, in D, shows that $e_D(x_2) \leq 3$; $V_1 \to x_2 \to y_2 \to \{x_1, z_2\}$, $V_1 \to x_2 \to y_1 \to V_2$, $V_1 \to x_2 \to z_1 \to V_3$, in D, and $\vec{d}(G_1) \leq 3$, shows that for every $x_i \in V_1$, $e_D(x_i) \leq 3$. Thus $d(D) \leq 3$, and hence $\vec{d}(G) \leq 3$. Consequently, $\vec{d}(\mathscr{G}_9^{3^*}) \leq 3$.

Theorem 2.4. $\vec{d}(\mathscr{G}(4, 4, 4; 12)) \leq 3$.

Proof: Let $\{x_1, x_2, x_3, x_4\}$, $\{y_1, y_2, y_3, y_4\}$, $\{z_1, z_2, z_3, z_4\}$ be the vertex sets of three disjoint copies of K_4 and from $3K_4$ obtain G by adding the 12 edges: $x_1y_1, y_1z_1, z_1x_1, x_1y_4, x_1z_4, y_1x_4, y_1z_4, z_1x_4, z_1y_4, x_4y_3, y_4z_3, z_4x_3$. Then $G \in \mathscr{G}(4, 4, 4; 12)$. Orient the edges of G as follows:
$x_4 \to \{x_1, x_2, x_3\}$, $x_3 \to \{x_1, x_2\}$, $x_2 \to x_1$,
$y_4 \to \{y_1, y_2, y_3\}$, $y_3 \to \{y_1, y_2\}$, $y_2 \to y_1$,
$z_4 \to \{z_1, z_2, z_3\}$, $z_3 \to \{z_1, z_2\}$, $z_2 \to z_1$,
$x_1 \to \{y_1, y_4, z_4\}$, $y_1 \to \{z_1, z_4, x_4\}$, $z_1 \to \{x_1, x_4, y_4\}$,
$x_4 \to y_3$, $y_4 \to z_3$, and $z_4 \to x_3$.
Let D be the resulting digraph. Direct verification shows that $d(D) = 3$.
This completes the proof.

Theorem 2.5. *Let* $n_3 \geq 5$ *or* $n_3 = 3$. *Then* $\vec{d}(\mathscr{G}(4, 4, n_3; 11)) \leq 3$.

Proof: Let $\{x_1, x_2, x_3, x_4\}$, $\{y_1, y_2, y_3, y_4\}$ and $\{z_1, z_2, \ldots, z_{n_3}\}$ be, respectively, the vertex sets of two disjoint copies of K_4 and K_{n_3}; let $V' = V(K_{n_3}) \setminus \{z_1, z_2\}$; and let $G = K_4 \cup K_4 \cup K_{n_3} \cup \{x_1y_1, x_1y_4, x_1z_1, x_1z_2, x_3z_2, x_4y_1, x_4y_3, x_4z_1, y_1z_1, y_1z_2, y_4z_1\}$. Then $G \in \mathscr{G}(4, 4, n_3; 11)$. Orient the edges of G as follows:
(*i*) $x_1 \to \{y_1, y_4, z_2\}$, $y_1 \to \{z_1, z_2, x_4\}$, $z_1 \to \{x_1, x_4, y_4\}$, $x_4 \to y_3$, $z_2 \to x_3$;
(*ii*) $x_4 \to \{x_3, x_2, x_1\}$, $\{x_3, x_2\} \to x_1$, $x_3 \to x_2$;
(*iii*) $y_4 \to \{y_3, y_2, y_1\}$, $\{y_3, y_2\} \to y_1$, $y_3 \to y_2$;
(*iv*) $z_2 \to z_1$, $z_2 \to V' \to z_1$; and
(*v*) orient the edges of $G[V']$ such that $\vec{d}(G[V']) \leq 3$.
Let D be the resulting digraph. We claim that $d(D) \leq 3$. We show this by computing upper bounds for eccentricities of the vertices.

Let $z_i \in V'$ be arbitrary. In D, the existence of the paths from: $x_1 \to y_4 \to \{y_2, y_3\}$, $x_1 \to z_2 \to \{z_1, z_i\}$, and $x_1 \to y_1 \to x_4 \to \{x_2, x_3\}$ shows that $e_D(x_1) \leq 3$; $x_2 \to x_1 \to z_2 \to \{x_3, z_i\}$, $x_2 \to x_1 \to y_1 \to \{z_1, x_4\}$, and $x_2 \to x_1 \to y_4 \to \{y_2, y_3\}$ shows that $e_D(x_2) \leq 3$; $x_3 \to x_2$, $x_3 \to x_1 \to y_1 \to x_4$, $x_3 \to x_1 \to y_4 \to \{y_2, y_3\}$, and $x_3 \to x_1 \to z_2 \to \{z_1, z_i\}$ shows that $e_D(x_3) \leq 3$; $x_4 \to \{x_2, x_3\}$, $x_4 \to x_1 \to y_1$, $x_4 \to x_1 \to y_4 \to \{y_2, y_3\}$, and $x_4 \to x_1 \to z_2 \to \{z_1, z_i\}$ shows that $e_D(x_4) \leq 3$; $y_1 \to x_4 \to \{x_1, x_2, x_3\}$, $y_1 \to z_1 \to y_4 \to \{y_2, y_3\}$, and $y_1 \to z_2 \to z_i$ shows that $e_D(y_1) \leq 3$; $y_2 \to y_1 \to x_4 \to \{x_1, x_2, x_3, y_3\}$, $y_2 \to y_1 \to z_1 \to y_4$, and $y_2 \to y_1 \to z_2 \to z_i$ shows that $e_D(y_2) \leq 3$; $y_3 \to y_2$, $y_3 \to y_1 \to z_1 \to y_4$, $y_3 \to y_1 \to x_4 \to \{x_1, x_2, x_3\}$, and $y_3 \to y_1 \to z_2 \to z_i$ shows that $e_D(y_3) \leq 3$; $y_4 \to y_2$, $y_4 \to y_1 \to z_1$, $y_4 \to y_1 \to x_4 \to \{x_1, x_2, x_3, y_3\}$, and $y_4 \to y_1 \to z_2 \to z_i$ shows that $e_D(y_4) \leq 3$; $z_1 \to x_4 \to \{x_1, x_2, x_3\}$, $z_1 \to y_4 \to \{y_1, y_2, y_3\}$, and $z_1 \to x_1 \to z_2 \to z_i$ shows that $e_D(z_1) \leq 3$; $z_2 \to z_i$, $z_2 \to z_1 \to x_4 \to \{x_1, x_2, x_3\}$, and $z_2 \to z_1 \to y_4 \to \{y_1, y_2, y_3\}$ shows that $e_D(z_2) \leq 3$; $z_i \to z_1 \to x_4 \to \{x_1, x_2, x_3\}$, $z_i \to z_1 \to y_4 \to \{y_1, y_2, y_3\}$, and $z_i \to z_1 \to x_1 \to z_2$ together with $\vec{d}(G[V']) \leq 3$

shows that $e_D(z_i) \leq 3$.

This completes the proof.

Theorem 2.6. *Let $n_2 \geq 5$ or $n_2 = 3$, and let $n_3 \geq 5$ or $n_3 = 3$. Then $\vec{d}(\mathscr{G}(4, n_2, n_3; 10) \leq 3$.*

Proof: Let $\{x_1, x_2, x_3, x_4\}, \{y_1, y_2, \ldots, y_{n_2}\}$, and $\{z_1, z_2, \ldots, z_{n_3}\}$ be, respectively, the vertex sets of K_4, K_{n_2} and K_{n_3}; let $V' = V(K_{n_2}) \setminus \{y_1, y_2\}$ and $V'' = V(K_{n_3}) \setminus \{z_1, z_2\}$; and let $G = K_4 \cup K_{n_2} \cup K_{n_3} \cup \{x_1y_1, x_1y_2, x_4y_1, x_1z_1, x_1z_2, x_4z_1, y_1z_1, y_1z_2, y_2z_1, x_3z_2\}$. Then $G \in \mathscr{G}(4, n_2, n_3; 10)$. Orient the edges of G as follows:
(i) $x_1 \to \{y_1, y_2, z_2\}$, $y_1 \to \{z_1, z_2, x_4\}$, $z_1 \to \{x_1, x_4, y_2\}$, $z_2 \to x_3$;
(ii) $x_4 \to \{x_3, x_2, x_1\}$, $x_3 \to \{x_2, x_1\}$, $x_2 \to x_1$;
(iii) $y_2 \to y_1$, $y_2 \to V' \to y_1$, $z_2 \to z_1$, $z_2 \to V'' \to z_1$; and
(iv) orient the edges of $G[V']$ and that of $G[V'']$ such that $\vec{d}(G[V']) \leq 3$ and $\vec{d}(G[V'']) \leq 3$.

Let D be the resulting digraph. We claim that $d(D) \leq 3$. We show this by computing upper bounds for eccentricities of the vertices.

Let $y_i \in V'$ and $z_j \in V''$ are arbitrary. In D, the existence of the paths from: $x_1 \to y_1 \to x_4 \to \{x_2, x_3\}$, $x_1 \to y_2 \to y_i$, and $x_1 \to z_2 \to \{z_1, z_j\}$ shows that $e_D(x_1) \leq 3$; $x_2 \to x_1 \to y_1 \to x_4$, $x_2 \to x_1 \to y_2 \to y_i$, and $x_2 \to x_1 \to z_2 \to \{x_3, z_1, z_j\}$ shows that $e_D(x_2) \leq 3$; $x_3 \to x_2$, $x_3 \to x_1 \to y_1 \to \{z_1, x_4\}$, $x_3 \to x_1 \to y_2 \to y_i$, and $x_3 \to x_1 \to z_2 \to z_j$ shows that $e_D(x_3) \leq 3$; $x_4 \to \{x_2, x_3\}$, $x_4 \to x_1 \to y_1 \to z_1$, $x_4 \to x_1 \to y_2 \to y_i$, and $x_4 \to x_1 \to z_2 \to z_j$ shows that $e_D(x_4) \leq 3$; $y_1 \to x_4 \to \{x_1, x_2, x_3\}$, $y_1 \to z_1 \to y_2 \to y_i$, and $y_1 \to z_2 \to z_j$ shows that $e_D(y_1) \leq 3$; $y_2 \to y_i$, $y_2 \to y_1 \to x_4 \to \{x_1, x_2, x_3\}$, $y_2 \to y_1 \to z_1$, and $y_2 \to y_1 \to z_2 \to z_j$ shows that $e_D(y_2) \leq 3$; $y_i \to y_1 \to x_4 \to \{x_1, x_2, x_3\}$, $y_i \to y_1 \to z_2 \to z_j$ and $y_i \to y_1 \to z_1 \to y_2$, together with $\vec{d}(G[V']) \leq 3$ shows that $e_D(y_i) \leq 3$; $z_1 \to x_4 \to \{x_1, x_2, x_3\}$, $z_1 \to y_2 \to \{y_1, y_i\}$, and $z_1 \to x_1 \to z_2 \to z_j$ shows that $e_D(z_1) \leq 3$; $z_2 \to z_j$, $z_2 \to z_1 \to x_4 \to \{x_1, x_2, x_3\}$, and $z_2 \to z_1 \to y_2 \to \{y_1, y_i\}$ shows that $e_D(z_2) \leq 3$; $z_j \to z_1 \to x_4 \to \{x_2, x_3\}$, $z_j \to z_1 \to y_2 \to \{y_1, y_i\}$, and $z_j \to z_1 \to x_1 \to z_2$, together with $\vec{d}(G[V'']) \leq 3$ shows that $e_D(z_j) \leq 3$.

This completes the proof.

Corollary 2.1.
(i) $\min\{m : \vec{d}(3; m) = 6\} = 3$.
(ii) $\min\{m : \vec{d}(3; m) = 4\} = 4$.
(iii) $\min\{m : \vec{d}(\mathscr{G}_m^{3^*}) = 3\} \leq 9$.
(iv) $\min\{m : \vec{d}(\mathscr{G}(4, 4, 4; m)) \leq 3\} \leq 12$.
(v) Let $n_3 \in \{3, 5, 6, 7, \ldots\}$. $\min\{m : \vec{d}(\mathscr{G}(4, 4, n_3; m)) \leq 3\} \leq 11$.
(vi) Let $n_2, n_3 \in \{3, 5, 6, 7, \ldots\}$. $\min\{m : \vec{d}(\mathscr{G}(4, n_2, n_3; m)) \leq 3\} \leq 10$.
(vii) $\min\{m : \vec{d}(3; m) = 3\} \leq 12$.

Proof: Proofs of (i), (ii), (iii), (iv), (v), and (vi) follows by Theorems 2.1, 2.2, 2.3, 2.4, 2.5, and 2.6. Proof of (vii) follows from (iii), (iv), (v) and (vi).

Problem 2.1. *Find* $\min\{m : \vec{d}(3; m) = 3\}$.

Theorem 2.7. *For $n \geq 5$ or $n = 3$, there exists a graph G in $\mathscr{G}(n, n, n; 6n)$ with $\vec{d}(G) = 2$.*

Proof: Let m be odd and let $V = \{v_0, v_1, \ldots, v_{m-1}\}$ be the vertex set of the complete graph K_m. Orient the edges of K_m as follows:
(i) $\{v_2, v_4, v_6, \ldots, v_{m-1}\} \to v_0 \to \{v_1, v_3, v_5, \ldots, v_{m-2}\}$;

(ii) $\{v_0\} \cup \{v_3, v_5, v_7, \ldots, v_{m-2}\} \to v_1 \to \{v_2, v_4, v_6, \ldots, v_{m-1}\}$;
(iii) $\{v_0, v_2, v_4, \ldots, v_{m-3}\} \to v_{m-2} \to \{v_1, v_3, v_5, \ldots, v_{m-4}\} \cup \{v_{m-1}\}$;
(iv) $\{v_1, v_3, v_5, \ldots, v_{m-2}\} \to v_{m-1} \to \{v_0, v_2, v_4, \ldots, v_{m-3}\}$;
(v) when $i \in \{2, 4, 6, \ldots, m-3\}$,
$(\{v_1, v_3, v_5, \ldots, v_{i-1}\} \cup \{v_{i+2}, v_{i+4}, v_{i+6}, \ldots, v_{m-1}\}) \to v_i \to$
$(\{v_0, v_2, v_4, \ldots, v_{i-2}\} \cup \{v_{i+1}, v_{i+3}, v_{i+5}, \ldots, v_{m-2}\})$;
(vi) when $i \in \{3, 5, 7, \ldots, m-4\}$,
$(\{v_0, v_2, v_4, \ldots, v_{i-1}\} \cup \{v_{i+2}, v_{i+4}, v_{i+6}, \ldots, v_{m-2}\}) \to v_i \to$
$(\{v_1, v_3, v_5, \ldots, v_{i-2}\} \cup \{v_{i+1}, v_{i+3}, v_{i+5}, \ldots, v_{m-1}\})$.

Let D be the resulting digraph. We claim that $d(D) = 2$. We show this by computing eccentricities for the vertices of D.

The existence of the paths, in D, from: $v_0 \to \{v_1, v_3, v_5, \ldots, v_{m-2}\}$ and $v_0 \to v_1 \to \{v_2, v_4, v_6, \ldots, v_{m-1}\}$ shows that $e_D(v_0) \leq 2$; $v_1 \to \{v_2, v_4, v_6, \ldots, v_{m-1}\}$ and $v_1 \to v_2 \to \{v_0\} \cup \{v_3, v_5, v_7, \ldots, v_{m-2}\}$ shows that $e_D(v_1) \leq 2$; for $i \in \{2, 4, 6, \ldots, m-5\}$, $v_i \to \{v_0, v_2, v_4, \ldots, v_{i-2}\} \cup \{v_{i+1}, v_{i+3}, v_{i+5}, \ldots, v_{m-2}\}$, $v_i \to v_{i+1} \to \{v_1, v_3, v_5, \ldots, v_{i-1}\} \cup \{v_{i+2}, v_{i+4}, v_{i+6}, \ldots, v_{m-1}$ shows that $e_D(v_i) \leq 2$; for $i \in \{3, 5, 7, \ldots, m-4\}$, $v_i \to \{v_1, v_3, v_5, \ldots, v_{i-2}\} \cup \{v_{i+1}, v_{i+3}, v_{i+5}, \ldots, v_{m-1}\}$, $v_i \to v_{i+1} \to \{v_0, v_2, v_4, \ldots, v_{i-1}\} \cup \{v_{i+2}, v_{i+4}, v_{i+6}, \ldots, v_{m-2}\}$ shows that $e_D(v_i) \leq 2$; $v_{m-3} \to \{v_0, v_2, v_4, \ldots, v_{m-5}\} \cup \{v_{m-2}\}$ and $v_{m-3} \to v_{m-2} \to \{v_1, v_3, v_5, \ldots, v_{m-4}\} \cup \{v_{m-1}\}$ shows that $e_D(v_{m-3}) \leq 2$; $v_{m-2} \to \{v_1, v_3, v_5, \ldots, v_{m-4}\}$ and $v_{m-2} \to v_{m-1} \to \{v_0, v_2, v_4, \ldots, v_{m-3}\}$ shows that $e_D(v_{m-2}) \leq 2$; $v_{m-1} \to \{v_0, v_2, v_4, \ldots, v_{m-3}\}$ and $v_{m-1} \to v_0 \to \{v_1, v_3, v_5, \ldots, v_{m-2}\}$ shows that $e_D(v_{m-1}) \leq 2$.

We consider two cases.

Case 1. $n = m$ is odd.

Let $V_1 = \{x_0, x_1, \ldots, x_{m-1}\}$, $V_2 = \{y_0, y_1, \ldots, y_{m-1}\}$, and $V_3 = \{z_0, z_1, \ldots, z_{m-1}\}$ be the vertex sets of three disjoint complete graphs K_m.

Let $G = 3K_m \cup \{x_i y_i, x_{i+1} y_i, y_i z_i, y_i z_{i+1}, x_i z_{m-1-i}, x_{m-i} z_i : i \in \{0, 1, 2, \ldots, m-1\}\}$, where suffixes are reduced modulo m. Then $G \in \mathscr{G}(m, m, m; 6m)$. Orient the edges of G as follows:

(i) if $v_i \to v_j$, then $x_i \to x_j$, $y_i \leftarrow y_j$ and $z_i \to z_j$;
(ii) $x_i \to \{y_i, z_{m-1-i}\}$, $y_i \to \{x_{i+1}, z_{i+1}\}$, and $z_i \to \{y_i, x_{m-i}\}$.

Let D' be the resulting digraph. We claim that $d(D') = 2$. We show this by computing eccentricities for the vertices of D'. Let $D'_i = D'[V_i]$, $i \in \{1, 2, 3\}$. As $D'_1 \cong \widetilde{D'_2} \cong D'_3 \cong D$, $d(D'_i) = 2$.

The existence of the paths: $x_0 \to y_0$, $x_0 \to y_0 \to y_j$ for $j \in \{2, 4, 6, \ldots, m-1\}$, $x_0 \to x_j \to y_j$ for $j \in \{1, 3, 5, \ldots, m-2\}$, $x_0 \to z_{m-1} \to z_j$ for $j \in \{0, 2, 4, \ldots, m-3\}$, $x_0 \to x_j \to z_{m-1-j}$ for $j \in \{1, 3, 5, \ldots, m-2\}$, and $x_0 \to z_{m-1}$, in D', together with $e_{D'_1}(x_0) \leq 2$ shows that $e_{D'}(x_0) \leq 2$.

The existence of the paths: $x_1 \to y_1 \to y_j$ for $j \in \{0\} \cup \{3, 5, 7, \ldots, m-2\}$, $x_1 \to y_1$, $x_1 \to x_j \to y_j$ for $j \in \{2, 4, 6, \ldots, m-1\}$, $x_1 \to z_{m-2} \to z_j$ for $j \in \{1, 3, 5, \ldots, m-4\} \cup \{m-1\}$, $x_1 \to x_j \to z_{m-1-j}$ for $j \in \{2, 4, 6, \ldots, m-1\}$, and $x_1 \to z_{m-2}$, in D', together with $e_{D'_1}(x_1) \leq 2$ shows that $e_{D'}(x_1) \leq 2$.

Let $i \in \{2, 4, 6, \ldots, m-3\}$. The existence of the paths from: $x_i \to y_i$, $x_i \to y_i \to \{y_1, y_3, y_5, \ldots, y_{i-1}\} \cup \{y_{i+2}, y_{i+4}, y_{i+6}, \ldots, y_{m-1}\}$, $x_i \to x_j \to y_j$ for $j \in \{0, 2, 4, \ldots, i-2\} \cup \{i+1, i+3, i+5, \ldots, m-2\}$, $x_i \to z_{m-1-i} \to \{z_0, z_2, z_4, \ldots, z_{m-i-3}\} \cup \{z_{m-i}, z_{m-i+2}, z_{m-i+4}, \ldots, z_{m-2}\}$, $x_i \to x_j \to z_{m-1-j}$ for $j \in \{0, 2, 4, \ldots, i-2\} \cup \{i+1, i+3, i+5, \ldots, m-2\}$, and $x_i \to z_{m-1-i}$, in D', together with $e_{D'_1}(x_i) \leq 2$ shows that $e_{D'}(x_i) \leq 2$.

Let $i \in \{3, 5, 7, \ldots, m-4\}$. The existence of the paths from: $x_i \to y_i$, $x_i \to y_i \to \{y_0, y_2, y_4, \ldots, y_{i-1}\} \cup \{y_{i+2}, y_{i+4}, y_{i+6}, \ldots, y_{m-2}\}$, $x_i \to x_j \to y_j$ for $j \in \{1, 3, 5, \ldots, i-$

$2\}\cup\{i+1,i+3,i+5,\ldots,m-1\}$, $x_i \to z_{m-i-1} \to \{z_1,z_3,z_5,\ldots,z_{m-i-3}\}\cup\{z_{m-i},z_{m-i+2},z_{m-i+4},\ldots,z_{m-1}\}$, $x_i \to x_j \to z_{m-j-1}$ for $j \in \{1,3,5,\ldots,i-2\}\cup\{i+1,i+3,i+5,\ldots,m-1\}$, and $x_i \to z_{m-i-1}$, in D', together with $e_{D'_1}(x_i) \le 2$ shows that $e_{D'}(x_i) \le 2$.

The existence of the paths from: $x_{m-2} \to y_{m-2}, x_{m-2} \to y_{m-2} \to \{y_0,y_2,y_4,\ldots,y_{m-3}\}$, $x_{m-2} \to x_j \to y_j$ for $j \in \{1,3,5,\ldots,m-4\}\cup\{m-1\}$, $x_{m-2} \to z_1$, $x_{m-2} \to z_1 \to \{z_2,z_4,z_6,\ldots,z_{m-1}\}$, and $x_{m-2} \to x_j \to z_{m-1-j}$ for $j \in \{1,3,5,\ldots,m-4\}\cup\{m-1\}$, in D', together with $e_{D'_1}(x_{m-2}) \le 2$ shows that $e_{D'}(x_{m-2}) \le 2$.

The existence of the paths from: $x_{m-1} \to y_{m-1}, x_{m-1} \to y_{m-1} \to \{y_1,y_3,y_5,\ldots,y_{m-2}\}$, $x_{m-1} \to x_j \to y_j$ for $j \in \{0,2,4,\ldots,m-3\}$, $x_{m-1} \to z_0, x_{m-1} \to z_0 \to \{z_1,z_3,z_5,\ldots,z_{m-2}\}$, and $x_{m-1} \to x_j \to z_{m-1-j}$ for $j \in \{0,2,4,\ldots,m-3\}$, in D', together with $e_{D'_1}(x_{m-1}) \le 2$ shows that $e_{D'}(x_{m-1}) \le 2$.

The existence of the paths from: $y_0 \to x_1$, $y_0 \to x_1 \to \{x_2,x_4,x_6,\ldots,x_{m-1}\}$, $y_0 \to y_j \to x_{j+1}$ for $j \in \{2,4,6,\ldots,m-1\}$, $y_0 \to z_1$, $y_0 \to z_1 \to \{z_2,z_4,z_6,\ldots,z_{m-1}\}$, and $y_0 \to y_j \to z_{j+1}$ for $j \in \{2,4,6,\ldots,m-1\}$, in D', together with $e_{D'_2}(y_0) \le 2$ shows that $e_{D'}(y_0) \le 2$.

The existence of the paths from: $y_1 \to x_2 \to \{x_0\}\cup\{x_3,x_5,x_7,\ldots,x_{m-2}\}$, $y_1 \to y_j \to x_{j+1}$ for $j \in \{0\}\cup\{3,5,7,\ldots,m-2\}$, $y_1 \to x_2$, $y_1 \to z_2 \to \{z_0\}\cup\{z_3,z_5,z_7,\ldots,z_{m-2}\}$, $y_1 \to y_j \to z_{j+1}$ for $j \in \{0\}\cup\{3,5,7,\ldots,m-2\}$, and $y_1 \to z_2$, in D', together with $e_{D'_2}(y_1) \le 2$ shows that $e_{D'}(y_1) \le 2$.

Let $i \in \{2,4,6,\ldots,m-5\}$. The existence of the paths from: $y_i \to x_{i+1}$, $y_i \to x_{i+1} \to \{x_1,x_3,x_5,\ldots,x_{i-1}\}\cup\{x_{i+2},x_{i+4},x_{i+6},\ldots,x_{m-1}\}$, $y_i \to y_j \to x_{j+1}$ for $j \in \{1,3,5,\ldots,i-1\}\cup\{i+2,i+4,i+6,\ldots,m-1\}$, $y_i \to z_{i+1}$, $y_i \to z_{i+1} \to \{z_1,z_3,z_5,\ldots,z_{i-1}\}\cup\{z_{i+2},z_{i+4},z_{i+6},\ldots,z_{m-1}\}$, and $y_i \to y_j \to z_{j+1}$ for $j \in \{1,3,5,\ldots,i-1\}\cup\{i+2,i+4,i+6,\ldots,m-1\}$, in D', together with $e_{D'_2}(y_i) \le 2$ shows that $e_{D'}(y_i) \le 2$.

Let $i \in \{3,5,7,\ldots,m-4\}$. The existence of the paths from: $y_i \to x_{i+1}$, $y_i \to x_{i+1} \to \{x_0,x_2,x_4,\ldots,x_{i-1}\}\cup\{x_{i+2},x_{i+4},x_{i+6},\ldots,x_{m-2}\}$, $y_i \to y_j \to x_{j+1}$ for $j \in \{0,2,4,\ldots,i-1\}\cup\{i+2,i+4,i+6,\ldots,m-2\}$, $y_i \to z_{i+1}$, $y_i \to z_{i+1} \to \{z_0,z_2,z_4,\ldots,z_{i-1}\}\cup\{z_{i+2},z_{i+4},z_{i+6},\ldots,z_{m-2}\}$, and $y_i \to y_j \to z_{j+1}$ for $j \in \{0,2,4,\ldots,i-1\}\cup\{i+2,i+4,i+6,\ldots,m-2\}$, in D', together with $e_{D'_2}(y_i) \le 2$ shows that $e_{D'}(y_i) \le 2$.

The existence of the paths from: $y_{m-3} \to x_{m-2}, y_{m-3} \to x_{m-2} \to \{x_1,x_3,x_5,\ldots,x_{m-4}\}\cup\{x_{m-1}\}$, $y_{m-3} \to y_j \to x_{j+1}$ for $j \in \{1,3,5,\ldots,m-4\}\cup\{m-1\}$, $y_{m-3} \to z_{m-2}$, $y_{m-3} \to z_{m-2} \to \{z_1,z_3,z_5,\ldots,z_{m-4}\}\cup\{z_{m-1}\}$, and $y_{m-3} \to y_j \to z_{j+1}$ for $j \in \{1,3,5,\ldots,m-4\}\cup\{m-1\}$, in D', together with $e_{D'_2}(y_{m-3}) \le 2$ shows that $e_{D'}(y_{m-3}) \le 2$.

The existence of the paths from: $y_{m-2} \to x_{m-1}, y_{m-2} \to x_{m-1} \to \{x_0,x_2,x_4,\ldots,x_{m-3}\}$, $y_{m-2} \to y_j \to x_{j+1}$ for $j \in \{0,2,4,\ldots,m-3\}$, $y_{m-2} \to z_{m-1}$, $y_{m-2} \to z_{m-1} \to \{z_0,z_2,z_4,\ldots,z_{m-3}\}$, and $y_{m-2} \to y_j \to z_{j+1}$ for $j \in \{0,2,4,\ldots,m-3\}$, in D', together with $e_{D'_2}(y_{m-2}) \le 2$ shows that $e_{D'}(y_{m-2}) \le 2$.

The existence of the paths from: $y_{m-1} \to x_0, y_{m-1} \to x_0 \to \{x_1,x_3,x_5,\ldots,x_{m-2}\}$, $y_{m-1} \to y_j \to x_{j+1}$ for $j \in \{1,3,5,\ldots,m-2\}$, $y_{m-1} \to z_0$, $y_{m-1} \to z_0 \to \{z_1,z_3,z_5,\ldots,z_{m-2}\}$, and $y_{m-1} \to y_j \to z_{j+1}$ for $j \in \{1,3,5,\ldots,m-2\}$, in D', together with $e_{D'_2}(y_{m-1}) \le 2$ shows that $e_{D'}(y_{m-1}) \le 2$.

The existence of the paths from: $z_0 \to x_0$, $z_0 \to x_0 \to \{x_1,x_3,x_5,\ldots,x_{m-2}\}$, $z_0 \to z_j \to x_{m-j}$ for $j \in \{1,3,5,\ldots,m-2\}$, $z_0 \to y_0, z_0 \to y_0 \to \{y_2,y_4,y_6,\ldots,y_{m-1}\}$, $z_0 \to z_j \to y_j$ for $j \in \{1,3,5,\ldots,m-2\}$, in D', together with $e_{D'_3}(z_0) \le 2$ shows that $e_{D'}(z_0) \le 2$.

The existence of the paths from: $z_1 \to x_{m-1} \to \{x_0,x_2,x_4,\ldots,x_{m-3}\}, z_1 \to z_j \to x_{m-j}$ for $j \in \{2,4,6,\ldots,m-1\}$, $z_1 \to x_{m-1}$, $z_1 \to y_1$, $z_1 \to y_1 \to \{y_0\}\cup$

$\{y_3, y_5, y_7, \ldots, y_{m-2}\}$, and $z_1 \to z_j \to y_j$ for $j \in \{2, 4, 6, \ldots, m-1\}$, in D', together with $e_{D'_3}(z_1) \leq 2$ shows that $e_{D'}(z_1) \leq 2$.

The existence of the paths from: $z_2 \to x_{m-2}$, $z_2 \to x_{m-2} \to \{x_1, x_3, x_5, \ldots, x_{m-4}\} \cup \{x_{m-1}\}$, $z_2 \to z_j \to x_{m-j}$ for $j \in \{0\} \cup \{3, 5, 7, \ldots, m-2\}$, $z_2 \to y_2$, $z_2 \to y_2 \to \{y_1\} \cup \{y_4, y_6, y_8, \ldots, y_{m-1}\}$, and $z_2 \to z_j \to y_j$ for $j \in \{0\} \cup \{3, 5, 7, \ldots, m-2\}$, in D', together with $e_{D'_3}(z_2) \leq 2$ shows that $e_{D'}(z_2) \leq 2$.

Let $i \in \{4, 6, 8, \ldots, m-3\}$. The existence of the paths from: $z_i \to x_{m-i}$, $z_i \to x_{m-i} \to \{x_1, x_3, x_5, \ldots, x_{m-i-2}\} \cup \{x_{m-i+1}, x_{m-i+3}, x_{m-i+5}, \ldots, x_{m-1}\}$, $z_i \to z_j \to x_{m-j}$ for $j \in \{0, 2, 4, \ldots, i-2\} \cup \{i+1, i+3, i+5, \ldots, m-2\}$, $z_i \to y_i$, $z_i \to y_i \to \{y_1, y_3, y_5, \ldots, y_{i-1}\} \cup \{y_{i+2}, y_{i+4}, y_{i+6}, \ldots, y_{m-1}\}$, and $z_i \to z_j \to y_j$ for $j \in \{0, 2, 4, \ldots, i-2\} \cup \{i+1, i+3, i+5, \ldots, m-2\}$, in D', together with $e_{D'_3}(z_i) \leq 2$ shows that $e_{D'}(z_i) \leq 2$.

Let $i \in \{3, 5, 7, \ldots, m-4\}$. The existence of the paths from: $z_i \to x_{m-i}$, $z_i \to x_{m-i} \to \{x_0, x_2, x_4, \ldots, x_{m-i-2}\} \cup \{x_{m-i+1}, x_{m-i+3}, x_{m-i+5}, \ldots, x_{m-2}\}$, $z_i \to z_j \to x_{m-j}$ for $j \in \{1, 3, 5, \ldots, i-2\} \cup \{i+1, i+3, i+5, \ldots, m-1\}$, $z_i \to y_i$, $z_i \to y_i \to \{y_0, y_2, y_4, \ldots, y_{i-1}\} \cup \{y_{i+2}, y_{i+4}, y_{i+6}, \ldots, y_{m-2}\}$, $z_i \to z_j \to y_j$ for $j \in \{1, 3, 5, \ldots, i-2\} \cup \{i+1, i+3, i+5, \ldots, m-1\}$, in D', together with $e_{D'_3}(z_i) \leq 2$ shows that $e_{D'}(z_i) \leq 2$.

The existence of the paths from: $z_{m-2} \to x_2$, $z_{m-2} \to x_2 \to \{x_0\} \cup \{x_3, x_5, x_7, \ldots, x_{m-2}\}$, $z_{m-2} \to z_j \to x_{m-j}$ for $j \in \{1, 3, 5, \ldots, m-4\} \cup \{m-1\}$, $z_{m-2} \to y_{m-2}$, $z_{m-2} \to y_{m-2} \to \{y_0, y_2, y_4, \ldots, y_{m-3}\}$, and $z_{m-2} \to z_j \to y_j$ for $j \in \{1, 3, 5, \ldots, m-4\} \cup \{m-1\}$, in D', together with $e_{D'_3}(z_{m-2}) \leq 2$ shows that $e_{D'}(z_{m-2}) \leq 2$.

The existence of the paths from: $z_{m-1} \to x_1$, $z_{m-1} \to x_1 \to \{x_2, x_4, x_6, \ldots, x_{m-1}\}$, $z_{m-1} \to z_j \to x_{m-j}$ for $j \in \{0, 2, 4, \ldots, m-3\}$, $z_{m-1} \to y_{m-1} \to \{y_1, y_3, y_5, \ldots, y_{m-2}\}$, $z_{m-1} \to z_j \to y_j$ for $j \in \{0, 2, 4, \ldots, m-3\}$, and $z_{m-1} \to y_{m-1}$, in D', together with $e_{D'_3}(z_{m-1}) \leq 2$, shows that $e_{D'}(z_{m-1}) \leq 2$.

This completes the proof of the claim $d(D') = 2$.

Case 2. $n = m+1$ is even.

Let $V'_1 = V_1 \cup \{x\}$, $V'_2 = V_2 \cup \{y\}$, and $V'_3 = V_3 \cup \{z\}$, where V_1, V_2, V_3 are as in Case 1; let $G = 3K_n \cup \{xy, yz, zx, xz_{m-1}, yz_{m-1}, yx_{m-1}\} \cup \{xy_i, zx_i, zy_i, x_iy_i, y_iz_i, x_iz_{m-i-1} : i \in \{0, 1, 2, \ldots, m-1\}\}$, where suffixes are reduced modulo m. Then $G \in \mathscr{G}(n, n, n; 6n)$. Orient the edges of G as follows:

(*i*) if $v_i \to v_j$, then $x_i \to x_j$, $y_i \leftarrow y_j$ and $z_i \to z_j$;

(*ii*) $x \to V_1$, $\{y_0, y_1, y_2, \ldots, y_{m-3}\} \to y \to \{y_{m-2}, y_{m-1}\}$, and $z \to V_3$;

(*iii'*) $y \to x$, $x \to z$, $y \to z$,

$z_{m-1} \to x$, $z_{m-1} \to y$, $x_{m-1} \to y$,

$\{y_i \to x, x_i \to z, y_i \to z, x_i \to y_i, z_i \to y_i, z_{m-i-1} \to x_i : i \in \{0, 1, 2, \ldots, m-1\}\}$.

Let D' be the resulting digraph. We claim that $d(D') = 2$. We show this by computing eccentricities for the vertices of D'.

The existence of the paths from: $x \to V_1$, $x \to x_i \to y_i$ for $i \in \{0, 1, 2, \ldots, m-1\}$, $x \to x_{m-1} \to y$, $x \to z \to V_3$, and $x \to z$, in D', shows that $e_{D'}(x) \leq 2$.

The existence of the paths from: $y \to x \to V_1$, $y \to x$, $y \to y_{m-2} \to \{y_0, y_2, y_4, \ldots, y_{m-3}\}$, $y \to y_{m-1} \to \{y_1, y_3, y_5, \ldots, y_{m-2}\}$, $y \to y_{m-1}$, $y \to z \to V_3$, and $y \to z$, in D', shows that $e_{D'}(y) \leq 2$.

The existence of the paths from: $z \to z_{m-i-1} \to x_i$ for $i \in \{0, 1, 2, \ldots, m-1\}$, $z \to z_{m-1} \to x$, $z \to z_i \to y_i$ for $i \in \{0, 1, 2, \ldots, m-1\}$, $z \to z_{m-1} \to y$, and $z \to V_3$, in D', shows that $e_{D'}(z) \leq 2$.

For $i \in \{0, 1, 2, \ldots, m-1\}$, $x_i \to y_i \to x$ shows that $d_{D'}(x_i, x) \leq 2$, $y_i \to x$ shows that $d_{D'}(y_i, x) = 1$, and $z_i \to y_i \to x$ shows that $d_{D'}(z_i, x) \leq 2$. $y \to x$ shows that $d_{D'}(y, x) = 1$. $z \to z_{m-1} \to x$ shows that $d_{D'}(z, x) \leq 2$.

For $i \in \{0, 1, 2, \ldots, m-3\}$, $x_i \to y_i \to y$ shows that $d_{D'}(x_i, y) \leq 2$, $y_i \to y$ shows that $d_{D'}(y_i, y) = 1$, and $z_i \to y_i \to y$ shows that $d_{D'}(z_i, y) \leq 2$. $x_{m-2} \to x_{m-1} \to y$ shows that $d_{D'}(x_{m-2}, y) \leq 2$. $x_{m-1} \to y$ shows that $d_{D'}(x_{m-1}, y) = 1$. $y_{m-2} \to y_0 \to y$ shows that $d_{D'}(y_{m-2}, y) \leq 2$. $y_{m-1} \to y_1 \to y$ shows that $d_{D'}(y_{m-1}, y) \leq 2$. $z_{m-2} \to z_{m-1} \to y$ shows that $d_{D'}(z_{m-2}, y) \leq 2$. $z_{m-1} \to y$ shows that $d_{D'}(z_{m-1}, y) = 1$. $x \to x_{m-1} \to y$ shows that $d_{D'}(x, y) \leq 2$. $z \to z_{m-1} \to y$ shows that $d_{D'}(z, y) \leq 2$.

For $i \in \{0, 1, 2, \ldots, m-1\}$, $x_i \to z$ shows that $d_{D'}(x_i, z) = 1$, $y_i \to z$ shows that $d_{D'}(y_i, z) = 1$, and $z_i \to y_i \to z$ shows that $d_{D'}(z_i, z) \leq 2$. $x \to z$ shows that $d_{D'}(x, z) = 1$. $y \to z$ shows that $d_{D'}(y, z) = 1$.

For $i \in \{0, 1, 2, \ldots, m-1\}$, $d_{D'}(x_0, y_i) \leq 2$ follows from the existence of the paths: $x_0 \to y_0$, $x_0 \to y_0 \to y_j$ for $j \in \{2, 4, 6, \ldots, m-1\}$, and $x_0 \to x_j \to y_j$ for $j \in \{1, 3, 5, \ldots, m-2\}$, in D'.

For $i \in \{0, 1, 2, \ldots, m-1\}$, $d_{D'}(x_1, y_i) \leq 2$ follows from the existence of the paths: $x_1 \to y_1$, $x_1 \to y_1 \to y_j$ for $j \in \{0\} \cup \{3, 5, 7, \ldots, m-2\}$, $x_1 \to x_j \to y_j$ for $j \in \{2, 4, 6, \ldots, m-1\}$, in D'.

For $i \in \{2, 4, 6, \ldots, m-3\}$ and $j \in \{0, 1, 2, \ldots, m-1\}$, $d_{D'}(x_i, y_j) \leq 2$ follows from the existence of the paths from: $x_i \to y_i$, $x_i \to y_i \to \{y_1, y_3, y_5, \ldots, y_{i-1}\} \cup \{y_{i+2}, y_{i+4}, y_{i+6}, \ldots, y_{m-1}\}$, and $x_i \to x_k \to y_k$ for $k \in \{0, 2, 4, \ldots, i-2\} \cup \{i+1, i+3, i+5, \ldots, m-2\}$, in D'.

For $i \in \{3, 5, 7, \ldots, m-4\}$ and $j \in \{0, 1, 2, \ldots, m-1\}$, $d_{D'}(x_i, y_j) \leq 2$ follows from the existence of the paths from: $x_i \to y_i$, $x_i \to y_i \to \{y_0, y_2, y_4, \ldots, y_{i-1}\} \cup \{y_{i+2}, y_{i+4}, y_{i+6}, \ldots, y_{m-2}\}$, and $x_i \to x_k \to y_k$ for $k \in \{1, 3, 5, \ldots, i-2\} \cup \{i+1, i+3, i+5, \ldots, m-1\}$, in D'.

For $i \in \{0, 1, 2, \ldots, m-1\}$, $d_{D'}(x_{m-2}, y_i) \leq 2$ follows from the existence of the paths from: $x_{m-2} \to y_{m-2}$, $x_{m-2} \to y_{m-2} \to \{y_0, y_2, y_4, \ldots, y_{m-3}\}$, and $x_{m-2} \to x_j \to y_j$ for $j \in \{1, 3, 5, \ldots, m-4\} \cup \{m-1\}$, in D'.

For $i \in \{0, 1, 2, \ldots, m-1\}$, $d_{D'}(x_{m-1}, y_i) \leq 2$ follows from the existence of the paths from: $x_{m-1} \to y_{m-1}$, $x_{m-1} \to y_{m-1} \to \{y_1, y_3, y_5, \ldots, y_{m-2}\}$, and $x_{m-1} \to x_j \to y_j$ for $j \in \{0, 2, 4, \ldots, m-3\}$, in D'.

For $i, j \in \{0, 1, 2, \ldots, m-1\}$, $d_{D'}(x_i, z_j) \leq 2$ follows from the existence of the path: $x_i \to z \to z_j$, in D'.

For $i, j \in \{0, 1, 2, \ldots, m-1\}$, $d_{D'}(y_i, x_j) \leq 2$ follows from the existence of the path: $y_i \to x \to x_j$, in D'.

For $i, j \in \{0, 1, 2, \ldots, m-1\}$, $d_{D'}(y_i, z_j) \leq 2$ follows from the existence of the path: $y_i \to z \to z_j$, in D'.

For $i \in \{0, 1, 2, \ldots, m-1\}$, $d_{D'}(z_0, x_i) \leq 2$ follows from the existence of the paths: $z_0 \to x_{m-1} \to x_j$ for $j \in \{0, 2, 4, \ldots, m-3\}$, $z_0 \to z_j \to x_{m-1-j}$ for $j \in \{1, 3, 5, \ldots, m-2\}$, and $z_0 \to x_{m-1}$, in D'.

For $i \in \{0, 1, 2, \ldots, m-1\}$, $d_{D'}(z_1, x_i) \leq 2$ follows from the existence of the paths: $z_1 \to x_{m-2} \to x_j$ for $j \in \{1, 3, 5, \ldots, m-4\} \cup \{m-1\}$, $z_1 \to z_j \to x_{m-1-j}$ for $j \in \{2, 4, 6, \ldots, m-1\}$, and $z_1 \to x_{m-2}$, in D'.

For $i \in \{2, 4, 6, \ldots, m-3\}$ and $j \in \{0, 1, 2, \ldots, m-1\}$, $d_{D'}(z_i, x_j) \leq 2$ follows from the existence of the paths from: $z_i \to x_{m-1-i} \to \{x_0, x_2, x_4, \ldots, x_{m-i-3}\} \cup \{x_{m-i}, x_{m-i+2}, x_{m-i+4}, \ldots, x_{m-2}\}$, $z_i \to z_k \to x_{m-1-k}$ for $k \in \{0, 2, 4, \ldots, i-2\} \cup \{i+1, i+3, i+5, \ldots, m-2\}$, and $z_i \to x_{m-1-i}$, in D'.

For $i \in \{3, 5, 7, \ldots, m-4\}$ and $j \in \{0, 1, 2, \ldots, m-1\}$, $d_{D'}(z_i, x_j) \leq 2$ follows from the existence of the paths from: $z_i \to x_{m-i-1} \to \{x_1, x_3, x_5, \ldots, x_{m-i-3}\} \cup \{x_{m-i}, x_{m-i+2},$

$x_{m-i+4}, \ldots, x_{m-1}\}$, $z_i \to z_k \to x_{m-k-1}$ for $k \in \{1, 3, 5, \ldots, i-2\} \cup \{i+1, i+3, i+5, \ldots, m-1\}$, and $z_i \to x_{m-i-1}$, in D'.

For $i \in \{0, 1, 2, \ldots, m-1\}$, $d_{D'}(z_{m-2}, x_i) \leq 2$ follows from the existence of the paths from: $z_{m-2} \to x_1$, $z_{m-2} \to x_1 \to \{x_2, x_4, x_6, \ldots, x_{m-1}\}$, and $z_{m-2} \to z_j \to x_{m-1-j}$ for $j \in \{1, 3, 5, \ldots, m-4\} \cup \{m-1\}$, in D'.

For $i \in \{0, 1, 2, \ldots, m-1\}$, $d_{D'}(z_{m-1}, x_i) \leq 2$ follows from the existence of the paths from: $z_{m-1} \to x_0$, $z_{m-1} \to x_0 \to \{x_1, x_3, x_5, \ldots, x_{m-2}\}$, and $z_{m-1} \to z_j \to x_{m-1-j}$ for $j \in \{0, 2, 4, \ldots, m-3\}$, in D'.

For $i \in \{0, 1, 2, \ldots, m-1\}$, $d_{D'}(z_0, y_i) \leq 2$ follows from the existence of the paths from: $z_0 \to y_0$, $z_0 \to y_0 \to \{y_2, y_4, y_6, \ldots, y_{m-1}\}$, and $z_0 \to z_j \to y_j$ for $j \in \{1, 3, 5, \ldots, m-2\}$, in D'.

For $i \in \{0, 1, 2, \ldots, m-1\}$, $d_{D'}(z_1, y_i) \leq 2$ follows from the existence of the paths from: $z_1 \to y_1$, $z_1 \to y_1 \to \{y_0\} \cup \{y_3, y_5, y_7, \ldots, y_{m-2}\}$, and $z_1 \to z_j \to y_j$ for $j \in \{2, 4, 6, \ldots, m-1\}$, in D'.

For $i \in \{0, 1, 2, \ldots, m-1\}$, $d_{D'}(z_2, y_i) \leq 2$ follows from the existence of the paths from: $z_2 \to y_2$, $z_2 \to y_2 \to \{y_1\} \cup \{y_4, y_6, y_8, \ldots, y_{m-1}\}$, and $z_2 \to z_j \to y_j$ for $j \in \{0\} \cup \{3, 5, 7, \ldots, m-2\}$, in D'.

For $i \in \{4, 6, 8, \ldots, m-3\}$ and $j \in \{0, 1, 2, \ldots, m-1\}$, $d_{D'}(z_i, y_j) \leq 2$ follows from the existence of the paths from: $z_i \to y_i$, $z_i \to y_i \to \{y_1, y_3, y_5, \ldots, y_{i-1}\} \cup \{y_{i+2}, y_{i+4}, y_{i+6}, \ldots, y_{m-1}\}$, and $z_i \to z_k \to y_k$ for $k \in \{0, 2, 4, \ldots, i-2\} \cup \{i+1, i+3, i+5, \ldots, m-2\}$, in D'.

For $i \in \{3, 5, 7, \ldots, m-4\}$ and $j \in \{0, 1, 2, \ldots, m-1\}$, $d_{D'}(z_i, y_j) \leq 2$ follows from the existence of the paths from: $z_i \to y_i$, $z_i \to y_i \to \{y_0, y_2, y_4, \ldots, y_{i-1}\} \cup \{y_{i+2}, y_{i+4}, y_{i+6}, \ldots, y_{m-2}\}$, $z_i \to z_k \to y_k$ for $k \in \{1, 3, 5, \ldots, i-2\} \cup \{i+1, i+3, i+5, \ldots, m-1\}$, in D.

For $i \in \{0, 1, 2, \ldots, m-1\}$, $d_{D'}(z_{m-2}, y_i) \leq 2$ follows from the existence of the paths from: $z_{m-2} \to y_{m-2}$, $z_{m-2} \to y_{m-2} \to \{y_0, y_2, y_4, \ldots, y_{m-3}\}$, and $z_{m-2} \to z_j \to y_j$ for $j \in \{1, 3, 5, \ldots, m-4\} \cup \{m-1\}$, in D'.

For $i \in \{0, 1, 2, \ldots, m-1\}$, $d_{D'}(z_{m-1}, y_i) \leq 2$ follows from the existence of the paths from: $z_{m-1} \to y_{m-1} \to \{y_1, y_3, y_5, \ldots, y_{m-2}\}$, $z_{m-1} \to z_j \to y_j$ for $j \in \{0, 2, 4, \ldots, m-3\}$, and $z_{m-1} \to y_{m-1}$, in D'.

This completes the proof of the claim $d(D') = 2$.

Corollary 2.2. *If* $n \geq 5$ *or* $n = 3$, $\min\{m : \vec{d}(\mathscr{G}(n, n, n; m)) = 2\} \leq 6n$.

Problem 2.2. *Find* $\min\{m : \vec{d}(\mathscr{G}(n, n, n; m)) = 2\}$.

Problem 2.3. *Find* $\min\{m : \vec{d}(3; m) = 2\}$.

References

[1] Bang-Jensen, J., and Gutin, G. (2009), Digraphs: Theory, Algorithms and Applications, Second Edition, Springer, London.

[2] Boesch. F., and Tindell, R. (1980), Robbin's theorem for mixed multigraphs, Amer. Math. Monthly, 87, 716-719.

[3] Koh, K. M., and Ng, K. L. (2005), The orientation number of two complete graphs with linkages, Discrete Math., 295, 91-106.

[4] Koh, K. M., and Tay, E. G. (2002), Optimal orientations of graphs and digraphs: A survey, Graphs and Combin., 18, 745-756.

[5] S.B. Maurer, S. B. (1980), The king chicken theorems, Math. Mag., 53, 67-80.

[6] Ng, K. L. (2009), On a conjecture concerning the orientation number of a graph, Discrete Math., 309, 1603-1610.

[7] Robbins, H. (1939), A theorem on graphs with an application to a problem of traffic control, Amer. Math. Monthly, 46, 281-283.
[8] Soltés, L. (1986), Orientations of graphs minimizing the radius or the diameter, Math. Slovaca, 36, 289-296.

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