SUBGROUPS OF SOME (2,3,n) TRIANGLE GROUPS

by

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CONTENTS

PREFACE

This thesis is submitted in accordance with the degree of Doctor of Philosophy in the University of Glasgow. It presents the results of research undertaken by the author between October 1988 and October 1991.

All the results of the thesis are the original work of the author, except for the instances indicated within the text.

I would like to express my gratitude to my supervisor Dr W. W. Stothers for suggesting the subject of the thesis and for his assistance and encouragement throughout the research period.

I would also like to record my thanks to Professor R. W. Ogden for providing me with the opportunity to study at Glasgow, and to the Secretarial staff.

I should like to thank the Science and Engineering Research Council for financing the research through an SERC Studentship award.

Philip C. R. Stephenson

SUMMARY

As an abstract group, the (2,3,n) triangle group has the presentation

$$\Delta_n = < x, y : x^2 = y^3 = (yx)^n = 1 >$$

This thesis is concerned with subgroups of finite index in Δ_9, Δ_{11} and Δ_{13}.

With a subgroup of finite index, u, in the (2,3,11) triangle group, we associate a quintuple of non-negative integers (u,p,e,f,g), with $u \geq 1$ and $5u = 132(p-1) + 33e + 44f + 60g$.

We show in Theorem 1.4.6 that each quintuple, satisfying the conditions, corresponds to a subgroup of Δ_{11}.

With a subgroup of finite index, u, in the (2,3,13) triangle group, we associate a quintuple of non-negative integers (u,p,e,f,g), with $u \geq 1$ and $7u = 156(p-1) + 39e + 52f + 72g$.

We show in Theorem 3.3.6 that each quintuple, satisfying the conditions, corresponds to a subgroup of Δ_{13}.

With a subgroup of finite index, u, in the (2,3,9) triangle group, we associate a sextuple of non-negative integers (u,p,e,f,g_1,g_3), with $u \geq 1$, $u \equiv f \pmod 3$ and $u = 36(p-1) + 9e + 12f + 16g_1 + 12g_3$.

We show in Theorem 2.3.9 that each sextuple, satisfying the conditions, corresponds to a subgroup of Δ_9 with the following exceptions :

(a) (12n+9,0,1,0,0,n+3), $\forall\, n \geq 0$

(b) (24,0,0,0,0,5)

(c) (24,0,0,0,3,1)

(d) (24,0,0,3,0,2)

Coset diagrams are used extensively in the proofs, although to prove exception (a) for Δ_9, we make use of Hauptmodul equations (see [1] and [23]).

Computer programs were developed to generate all quintuples satisfying the relevant conditions for (2,3,11) subgroups for $u \leq 101$, all quintuples satisfying the relevant conditions for (2,3,13) subgroups for $u \leq 110$, and all

sextuples satisfying the relevant conditions for (2,3,9) subgroups for $u \leq 38$. These programs and their output are presented in the Appendices.

We show in Theorem 1.2.2 that quintuples, which satisfy the relevant (2,3,11) conditions, exist for each $u \geq 99$.

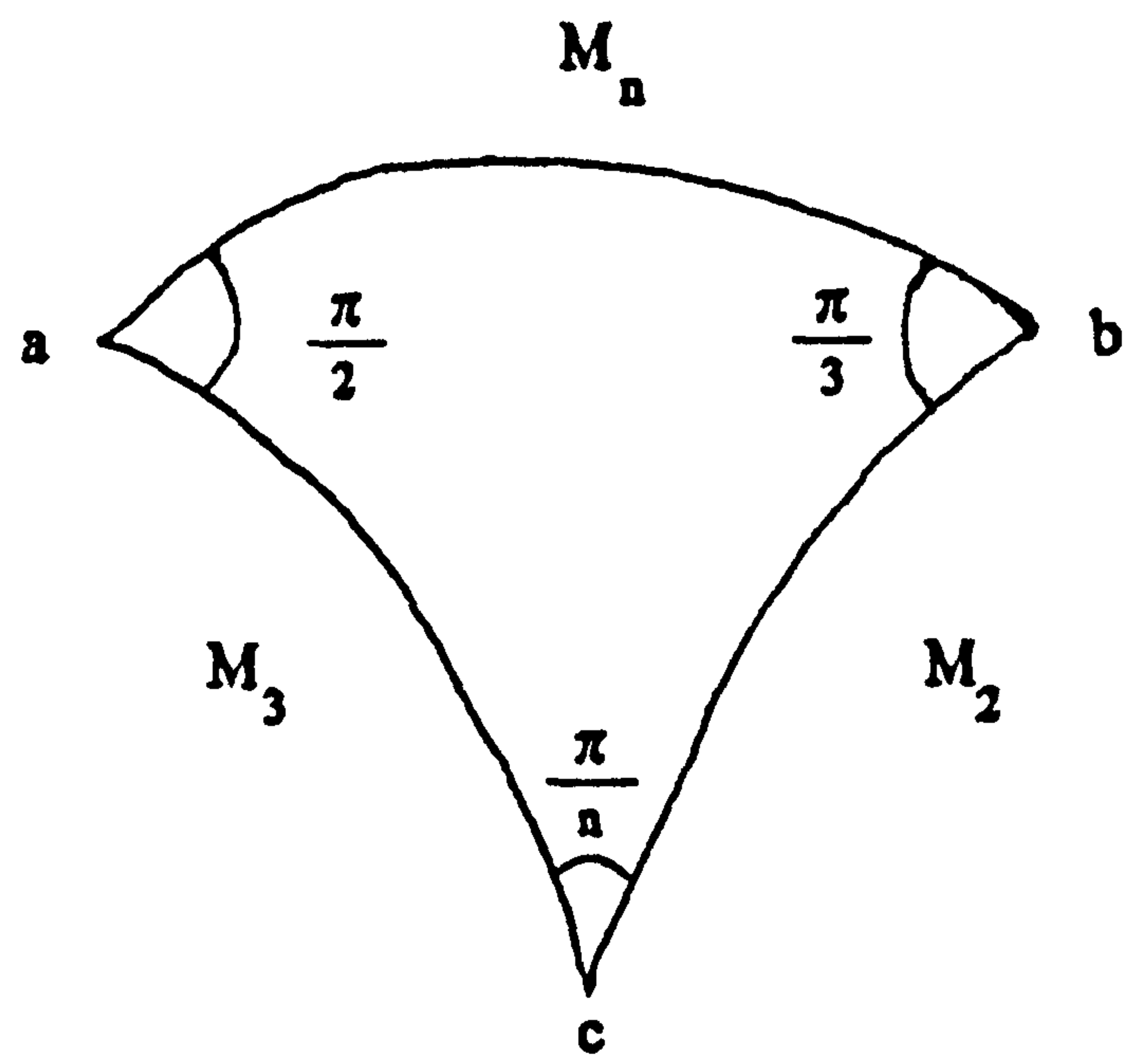
We show in Theorem 2.2.1 that sextuples, which satisfy the relevant (2,3,9) conditions, exist for each $u \geq 36$.

We show in Theorem 3.2.1 that quintuples, which satisfy the relevant (2,3,13) conditions, exist for each $u \geq 104$.

INTRODUCTION

An introduction to triangle groups can be found in [10], pp 65–106. For additional background material, refer to Appendix D.

The (2,3,n) triangle group has a fundamental domain consisting of two copies of a hyperbolic triangle with angles $\frac{\pi}{2}$, $\frac{\pi}{3}$ and $\frac{\pi}{n}$ (see [7], pg 236–238).



Let R_i represent hyperbolic-reflection in M_i $(i = 2, 3, n)$.

Let $x = R_nR_3$ and $y = R_2R_n$, so that $yx = R_2R_nR_nR_3 = R_2R_3$.

Then,

R_nR_3 is an anticlockwise hyperbolic-rotation of π about a,

R_2R_n is an anticlockwise hyperbolic-rotation of $\frac{2\pi}{3}$ about b,

R_2R_3 is an anticlockwise hyperbolic-rotation of $\frac{2\pi}{n}$ about c.

Hence, $x^2 = y^3 = (yx)^n = 1$, which agrees with the presentation for Δ_n given in the Summary. Note : the space Δ_n acts on is hyperbolic iff $n \geq 7$.

The results of the thesis can be regarded as results on the classical modular group. The modular group has presentation

$$\Gamma = < X, Y : X^2 = Y^3 = 1 >$$

and there exists an epimorphism $\theta_n : \Gamma \to \Delta_n$ defined by $X \to x$, $Y \to y$.

Equivalently, $\Delta_n \cong \Gamma/\mathrm{Ker}(\theta_n)$, where θ_n is the smallest normal subgroup of Γ containing $(yx)^n$. These associations between Δ_n and Γ can be found in [7], pg 300.

From [5], Δ_n is finite when $n < 6$, and infinite when $n \geq 6$. Δ_6 is soluble, but, for $n \geq 7$, Δ_n is insoluble, Fuchsian and SQ-universal (see [5], [13], [15]).

Δ_2, Δ_3, Δ_4 and Δ_5 are the rotation groups of a triangle, tetrahedron, octahedron and icosahedron, respectively (see [7], pg 300 and [9]).

Stothers ([20], [21], [22]) has studied subgroups of infinite index in the modular group. Subgroups of finite index in the modular group have been studied by Millington [11] and Stothers ([16], [17], [19]). Coset diagrams were used in most of these papers, and were also used in the study of the (2,3,n) groups and/or their subgroups of finite index by Conder ([3], [4], [5]), Mushtaq and Shaheen [12], and Stothers [18].

Defining a *specification* to be a list of non-negative integers (u,p,e,f,g) with $u \geq 1$ and $u = 84(p-1) + 21e + 28f + 36g$, Stothers [18] showed that each specification corresponds to a subgroup of finite index of the (2,3,7) group with the three exceptions (16,0,0,1,2), (21,1,1,0,0) and (36,1,0,0,1).

The symbol p in the specification of a subgroup G denotes the genus of the Riemann surface associated with G (see [7], [8] and [18]). We observe from the exceptional specifications for the (2,3,7) and (2,3,9) subgroups, that the genus p is either zero or one.

Using the genus formula for subgroups of finite index in the Modular group and results from [16], in particular Theorem 5.5 and its Corollary, it may be possible to show that any exception for (2,3,n) subgroups (with $n \geq 7$) must have $p \leq 1$.

The genus formula for subgroups of finite index in the Modular group is given by

$$g = 1 + \frac{1}{12}u - \frac{1}{4}e_2 - \frac{1}{3}e_3 - \frac{1}{2}h$$

using the notation in [16].

Note that there is a misprint in this formula in [16], although a correct version is given in [11]. Here, g, e_2, e_3 in [16] correspond to p, e, f in this thesis. Therefore, $g \leq 1 + \frac{1}{12}(u - 6h)$, since $e_2, e_3 \geq 0$.

So, for example, if $|u - 6h| < 12$, then $g \leq 1$.

Jones and Singerman discuss *map-subgroups* of (2,m,n) triangle groups in [6].

The correspondence between subgroups of index u in Δ_n and coset diagrams with u points, with the same specification, is a key result, given as Lemma 2.1 in [18]. We state this lemma here :

LEMMA (i) There is a correspondence between subgroups of index u in Δ_n and u point diagrams for Δ_n.

(ii) A subgroup with specification (u,p,e,f,g) corresponds to a diagram with e red points, f blue points and g green points. ∎

Compare this lemma with Theorem 1 in [11], Theorem 2.3 in [16] and Theorem 2.1 in [17].

Far more difficulties were experienced with the (2,3,9) group, dealt with in Chapter 2, than with the (2,3,11) group in Chapter 1 or the (2,3,13) group in Chapter 3. This was due to 11 and 13 being prime and 9 composite, and having to find proofs for the exceptions for the (2,3,9) group.

The methods used here could be applied to (2,3,n) groups for larger values of n. We would conjecture that, for prime $p \geq 11$, there are no exceptions for (2,3,p) groups.

It is worth noting Shimura's remark on pg 45 in [14], which implies that certain number theory techniques can be applied to (2,3,n) groups when n = 7, 9 or 11.

NOTATION

Δ_n	(2,3,n) triangle group
■	end of theorem
□	end of proof
$m \backslash n$	m divides n
∂p	degree of polynomial p
$m \to n$	vertices m and n are joined by a red line
n-cycle	a green cycle with n green lines
n^+-cycle	a green cycle with at least n green lines

CHAPTER 1

(2,3,11) TRIANGLE GROUP

§1.1 GENUS FORMULA

Reference [18], Pg 323, for first two paragraphs.

A subgroup of finite index u (≥ 1) in Δ_{11} has a fundamental domain consisting of u translates of that for Δ_{11}.

The domain has, say, e (resp. f, g) inequivalent elliptic vertices of order 2 (resp. 3, 11). Denote the genus of the corresponding Riemann surface by p.

Then, the genus formula can be derived from Theorem 2 in [15]. We get

$$2p - 2 + e(1 - \frac{1}{2}) + f(1 - \frac{1}{3}) + g(1 - \frac{1}{11}) = u(1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{11}),$$

which simplifies to

$$5u = 132(p - 1) + 33e + 44f + 60g \qquad (1.1.1)$$

This is the genus formula for subgroups of index u in Δ_{11}.

§1.2 SPECIFICATION

Adopting the same approach as in [18], Pg 324, we define a *specification* to be a list of non-negative integers (u,p,e,f,g), with $u \geq 1$, which satisfies (1.1.1).

To determine which values of u provide a solution for (1.1.1), we will make use of the following result from [2], Pg 299 :

RESULT 1.2.1

If a and b are coprime positive integers then for each $n \geq (a - 1)(b - 1)$, there exist integers $x, y \geq 0$ satisfying $ax + by = n$. ∎

THEOREM 1.2.2 The genus formula (1.1.1) has a solution for each $u \geq 99$, but not for $u = 98$.

Proof : $u \equiv e \pmod 4$, from (1.1.1).

Without loss of generality, let $p = 0$ and $e \leq 3$.

Then there are four cases :

(i) $u = 4v$, $e = 0$

(ii) $u = 4v + 1$, $e = 1$

(iii) $u = 4v + 2$, $e = 2$

(iv) $u = 4v + 3$, $e = 3$.

In each case, $v \geq 0$ and $f = 5k + f_0$, where $f_0 \in \{0, 1, 2, 3, 4\}$, $k \geq 0$.

case (i) : $u = 4v$, $e = 0$. Substitute values in (1.1.1).

$$20v = -132 + 220k + 44f_0 + 60g$$

$$\therefore \quad v + \frac{33 - 11f_0}{5} = 11k + 3g$$

$$\therefore \quad f_0 = 3 \quad \text{and} \quad v = 11k + 3g$$

By Result 1.2.1, this is solvable if $v \geq 20$, i.e. $u \geq 80$.

case (ii) : $u = 4v + 1$, $e = 1$. Substitute values in (1.1.1).

$$20v + 5 = -132 + 33 + 220k + 44f_0 + 60g$$

$$\therefore \quad v + \frac{26 - 11f_0}{5} = 11k + 3g$$

$$\therefore \quad f_0 = 1 \quad \text{and} \quad v + 3 = 11k + 3g$$

By Result 1.2.1, this is solvable if $v + 3 \geq 20$, i.e. $v \geq 17$, i.e. $u \geq 69$.

case (iii) : $u = 4v + 2$, $e = 2$. Substitute values in (1.1.1).

$$20v + 10 = -132 + 66 + 220k + 44f_0 + 60g$$

$$\therefore \quad v + \frac{19 - 11f_0}{5} = 11k + 3g$$

$\therefore\ f_0 = 4$ and $v - 5 = 11k + 3g$

By Result 1.2.1, this is solvable if $v - 5 \geq 20$, i.e. $v \geq 25$, i.e. $u \geq 102$.

case (iv) : $u = 4v + 3$, $e = 3$. Substitute values in (1.1.1).

$$20v + 15 = -132 + 99 + 220k + 44f_0 + 60g$$

$$v + \frac{12 - 11f_0}{5} = 11k + 3g$$

$\therefore\ f_0 = 2$ and $v - 2 = 11k + 3g$

By Result 1.2.1, this is solvable if $v - 2 \geq 20$, i.e. $v \geq 22$, i.e. $u \geq 91$.

From cases (i), (ii), (iii) and (iv), we deduce that (1.1.1) has a solution for each $u \geq 102$. Using (1.1.1), a computer program was developed to determine all the solutions for (1.1.1) for $u \leq 101$. This program and its output are shown in APPENDIX A. From the output, we see that solutions of (1.1.1) exist for $u = 99$, 100 and 101, but not for 98. These four values can also be checked by hand using (1.1.1).

The specifications listed in the program output for $u = 99, 100$ and 101 do satisfy (1.1.1).

For $u = 98$, we substitute this value into (1.1.1) and re-arrange to get

$$622 - 60g = 11(12p + 3e + 4f) \qquad (1.2.1)$$

Now we can conclude that $g \leq 10$ since $g, p, e, f \geq 0$. Next, we put each possible value of g $(0, 1, \ldots, 10)$ into (1.2.1). The RHS is divisible by 11, but the LHS is only divisible by 11 when $g = 10$, in which case we have

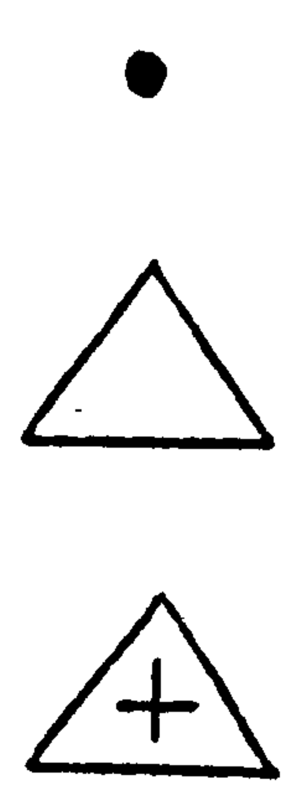
$$2 = 12p + 3e + 4f \qquad (1.2.2)$$

Now, p, e and f are non-negative integers, so clearly (1.2.2) has no solution. This implies that (1.1.1) has no solution for $u = 98$. □

§1.3 COSET DIAGRAMS

As described in [16] and [18], a coset diagram corresponding to a subgroup H of finite index in Δ_{11} is a directed edge-coloured graph with u vertices, corresponding to the cosets of H in Δ_{11}. A red (resp. blue, green) line indicates the effect of left-multiplication of a coset by x (resp. y, yx). Thus the coset diagram is equivalent to the representation of Δ_n on cosets of H. A coset diagram with specification (u,p,e,f,g) has u vertices, e red points, f blue points, g green points and genus p.

Colour code :



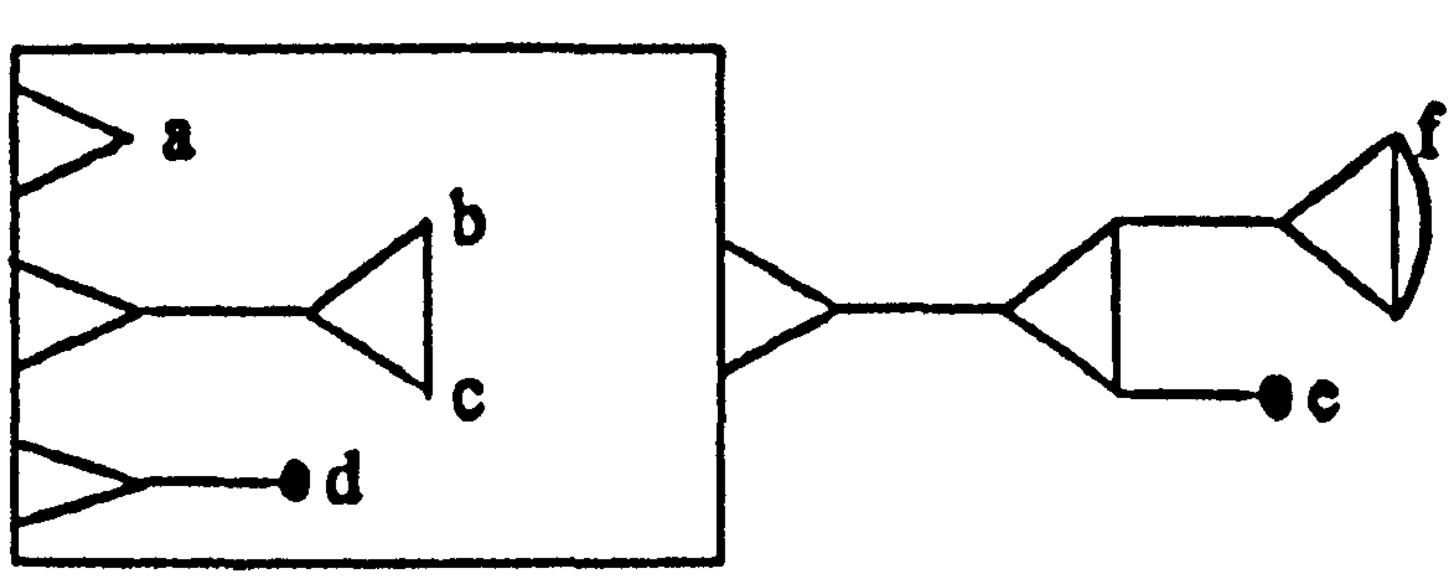
blue point

clockwise-orientated blue triangle

anticlockwise-orientated blue triangle

A vertex of a triangle, which has no other lines, indicates a red point.
A triangle with two vertices joined by another (red) line, indicates that one of these vertices is a green point.

For example, a coset diagram D with specification (23,0,3,2,1) is



a, b, c represent red points; d and e are blue points; and f is a green point.

The green lines have been omitted, but can be included by observing that a green line is determined by following a blue line and then a red line.

Composition of coset diagrams is described in Appendix E and in [18]. Lemma 2.2 in [18] will be used so often that its use will not be acknowledged, although we can assume the lemma will have been invoked when composition takes place. However, this lemma is stated below :

Lemma 2.2 in [18] Suppose that A and B are diagrams with specifications (u,p,e,f,g) and (u',p',e',f',g'), and are n(a) and n'(a) respectively.

(i) If $n, n' \geq 1$, then the (a)-composition of A and B has the specification (u+u',p+p',e+e'−4,f+f',g+g'), and is (n+n'−2)(a).

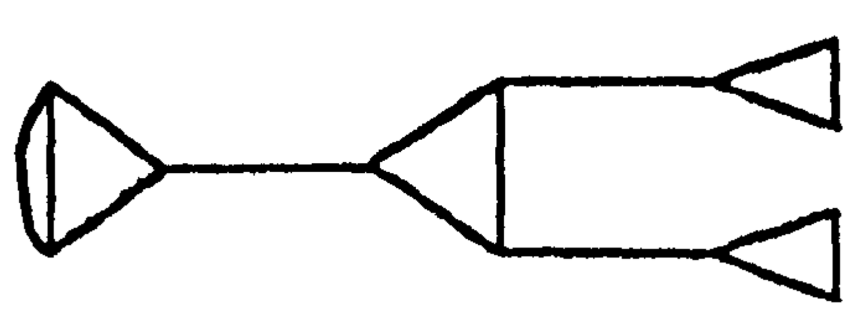
(ii) If $n \geq 2$, then the result of (a)-composition within A has specification (u,p+1,e−4,f,g) and is (n−2)(a). ∎

Using the notation in [18], we would describe the diagram D as 1(1) since there is one triangle with two adjacent red points (labelled b and c).

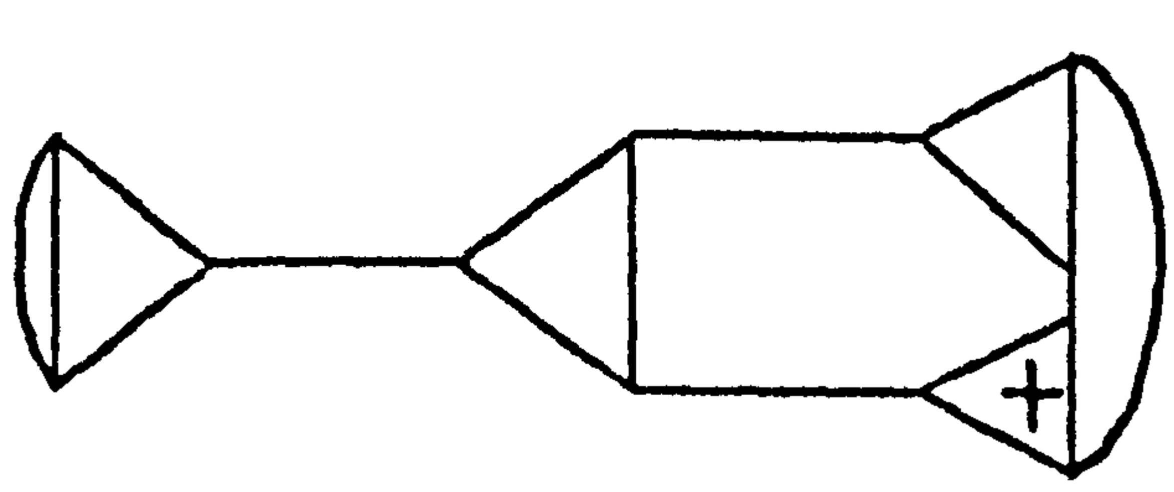
A diagram with n triangles of this sort would be described as n(1).

To demonstrate (1)-composition within the same diagram, we give diagrams for (12,0,4,0,1) 2(1) and (12,1,0,0,1), the second diagram being obtained from the first by undoing the four red points (recall : a red point represents a red loop) and then joining one pair with the other.

G_1 (12,0,4,0,1) 2(1)



(12,1,0,0,1)



§1.4 SUBGROUPS OF FINITE INDEX IN Δ_{11}

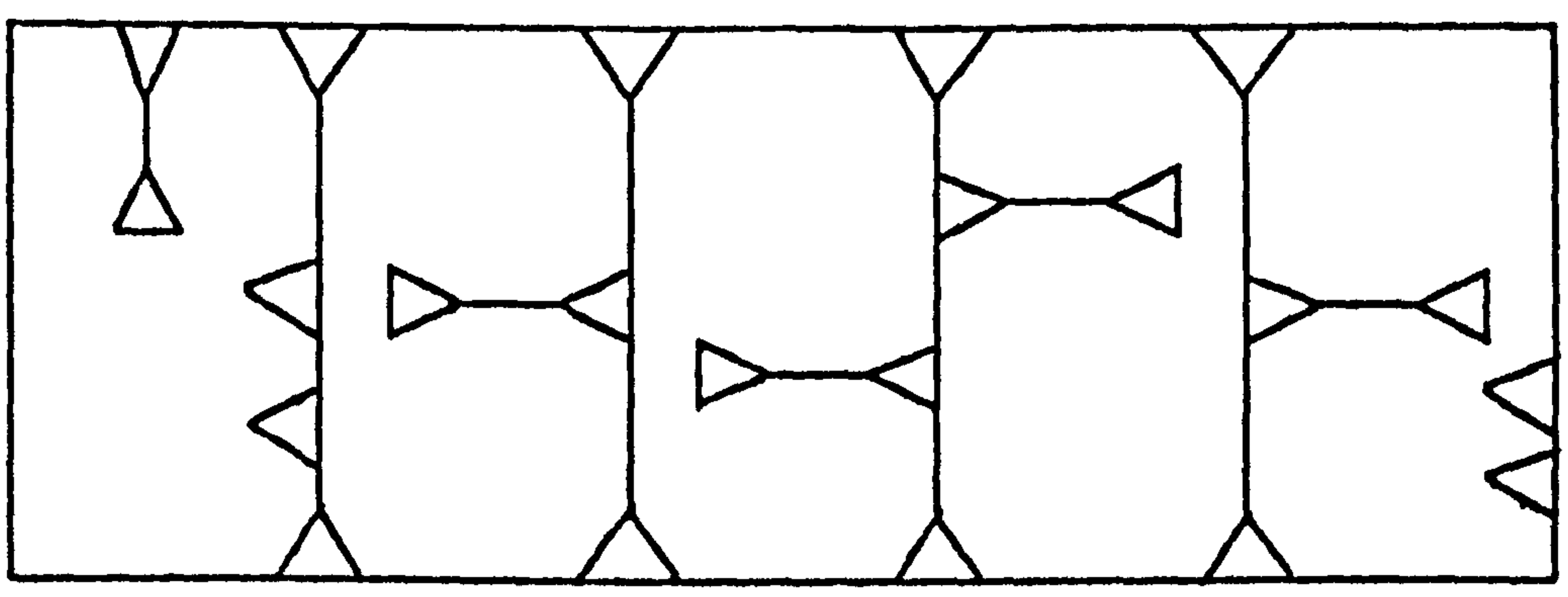
Before we can prove Theorem 1.4.6, we will need the following five lemmas.

LEMMA 1.4.1 If a specification $S = (u,p,e,f,g)$ satisfies (1.1.1) with $e \geq 4$, then there exists a coset diagram D with specification S, which is n(1) for some $n \geq 2$.

Proof : Let S be a counter-example with $p + e + f + g$ minimal. We want to show that no such S exists.

Consider the coset diagram

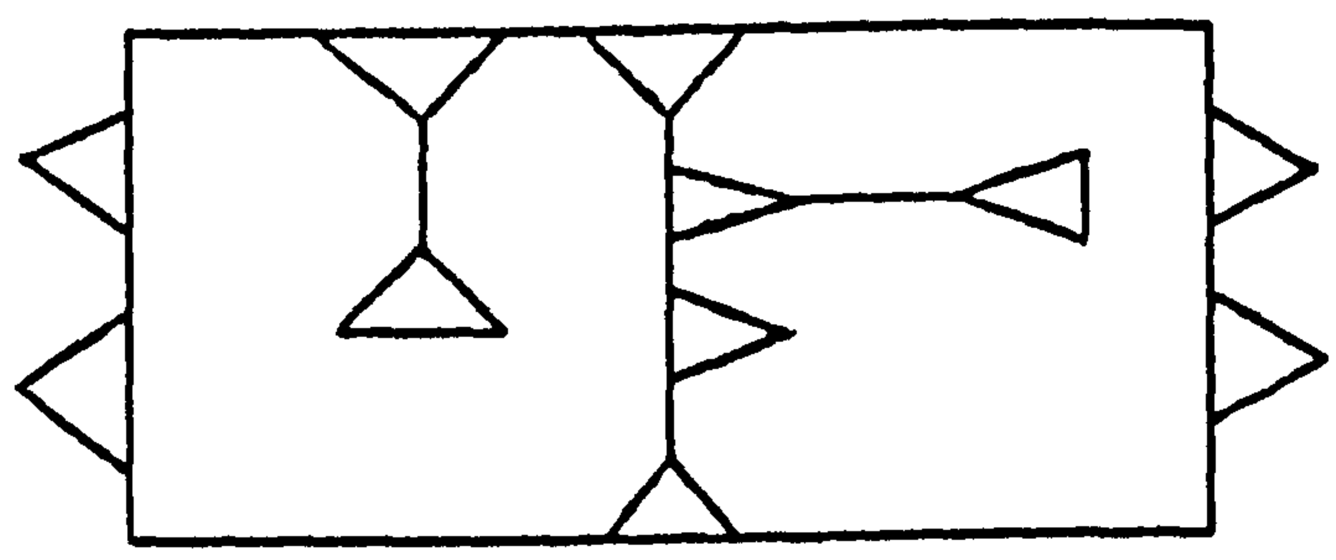
B =



This has specification (66,0,14,0,0) and is 5(1) 2(2). A 2-composition B(2)B of two copies of B has specification (132,0,24,0,0) and is 10(1) 2(2). By 2-composing this once and 1-composing it four times we get a diagram B_1 with specification (132,5,4,0,0) which is 2(1). We will use this to eliminate various possibilities for S.

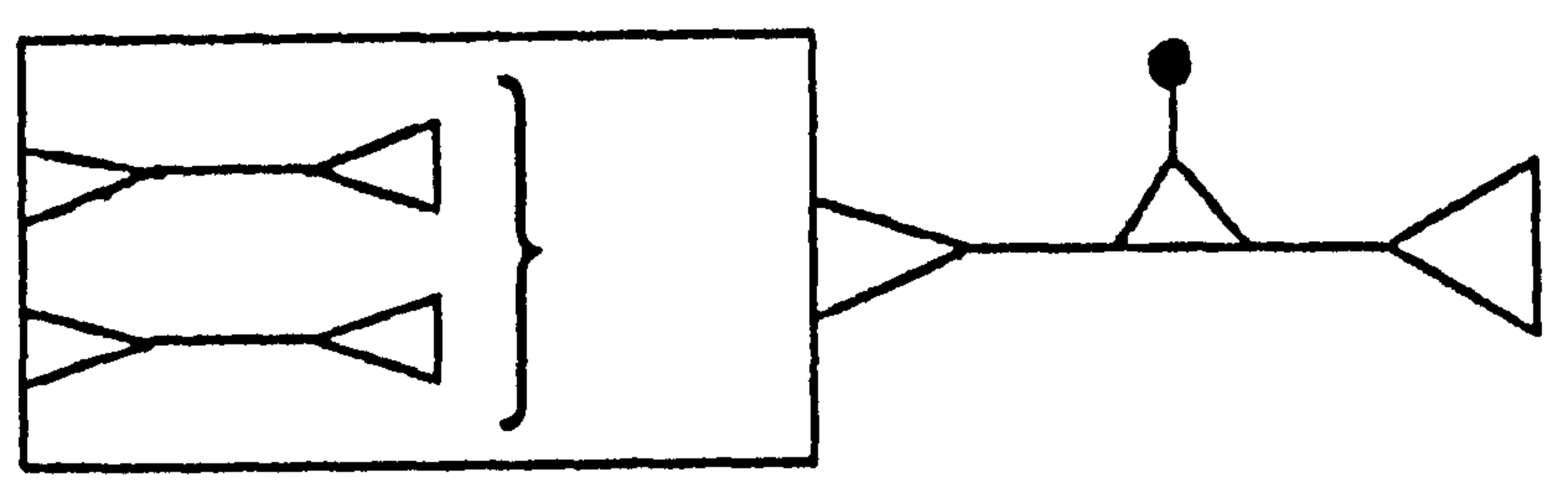
If $p \geq 5$ and D is a diagram with specification (u-132,p-5,e,f,g), which is n(1) for some $n \geq 2$, then a 1-composition $D + B_1$ has specification $S = (u,p,e,f,g)$ and is n(1) also. To put this another way, if our counter-example S has $p \geq 5$ then (u-132,p-5,e,f,g) is also a counter-example, contradicting the minimality of S. Thus, S has $p < 5$.

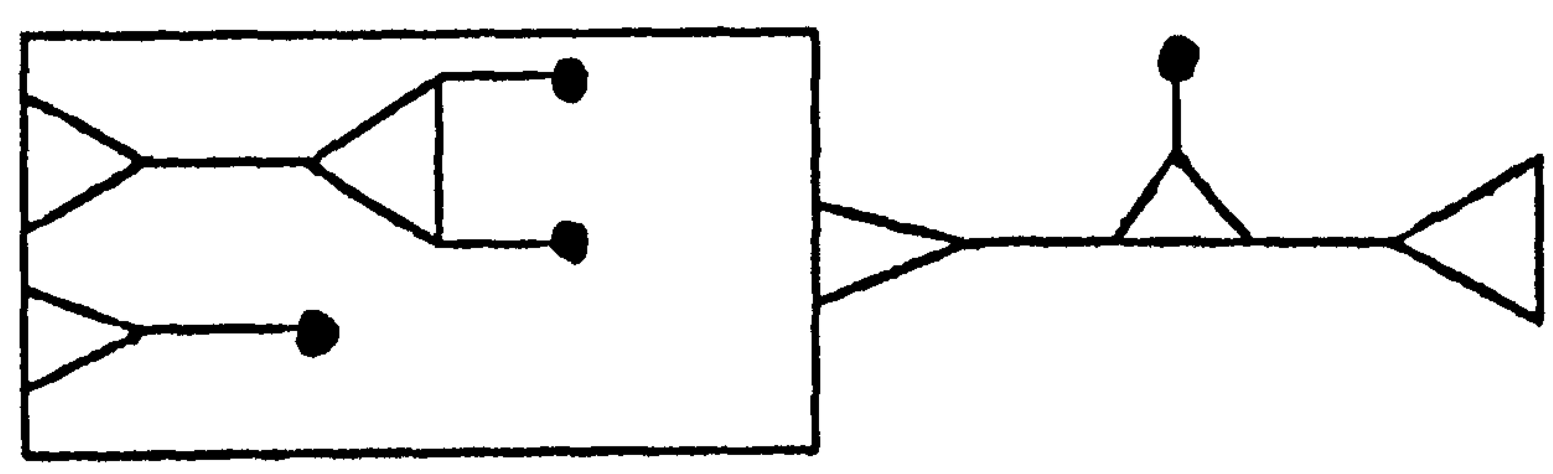
Now consider the diagram

E = 

This has specification (33,0,9,0,0) and is 2(1) 2(2). The 2-composition E_1 = E(2) has specification (33,1,5,0,0) and is 2(1).

If e ≥ 9 and D is a diagram with specification (u-33,p,e-5,f,g) which is n(1) for some n ≥ 2, then D + E has specification S and is n(1), n ≥ 2. Therefore, S has e < 9, by minimality.

A (22,0,6,1,0) 3(1) 

K_0 (22,0,2,4,0) 1(1) 

I_0 = A(1) = (22,1,2,1,0) 1(1). 1-composition of the bracketed triangles in A.

A_2 (44,1,4,2,0) 2(1) = A (22,0,6,1,0) 3(1) + I_0 (22,1,2,1,0) 1(1).

For p ≥ 1,

if f ≥ 2 and D (u-44,p-1,e,f-2,g) satisfies (1.1.1),

then D + A_2 has specification S which is n(1), n ≥ 2, and

if e ≥ 5 and D (u-33,p-1,e-1,f,g) satisfies (1.1.1),

then D + E_1 has specification S which is n(1), n ≥ 2.

Therefore, for p ≥ 1, S has f < 2 and e < 5 (i.e. e = 4).

F = A + K_0 = (44,0,4,5,0) 2(1).

If f ≥ 5 and D (u-44,p,e,f-5,g) satisfies (1.1.1), then D + F has specification S which is n(1), n ≥ 2. Therefore, S has f < 5.

A diagram for G_1 (12,0,4,0,1) 2(1) was exhibited in §1.3.

If $g \geq 1$ and $D(u-12,p,e,f,g-1)$ satisfies (1.1.1), then $D + G_1$ has specification S which is n(1), $n \geq 2$.

Therefore, S has $g < 1$. i.e. S has $g = 0$.

We now know that a minimal S would have one of the following two forms

(u,p,4,f,0)	:	$1 \leq p < 5$, $f < 2$	(1.4.1)
(u,0,e,f,0)	:	$4 \leq e < 9$, $f < 5$	(1.4.2)

Case (1.4.1) : Put $e = 4$ and $g = 0$ in (1.1.1) to get

$$5u = 132p + 44f$$

$$\therefore\ 0 \equiv 2p - 6f \pmod 5$$

$$\therefore\ p \equiv 3f \pmod 5$$

If $f = 0$, then $p \equiv 0 \pmod 5$, which does not have a solution for $1 \leq p < 5$.

If $f = 1$, then $p \equiv 3 \pmod 5$, so that $p = 3$, since $1 \leq p < 5$.

If $p = 3$, $e = 4$, $f = 1$ and $g = 0$ then $u = 88$.

(88,0,16,1,0) 6(1) 2(2) = A (22,0,6,1,0) 3(1) + B (66,0,14,0,0) 5(1) 2(2).

Now, 2-compose once and 1-compose twice to get

(88,3,4,1,0) 2(1).

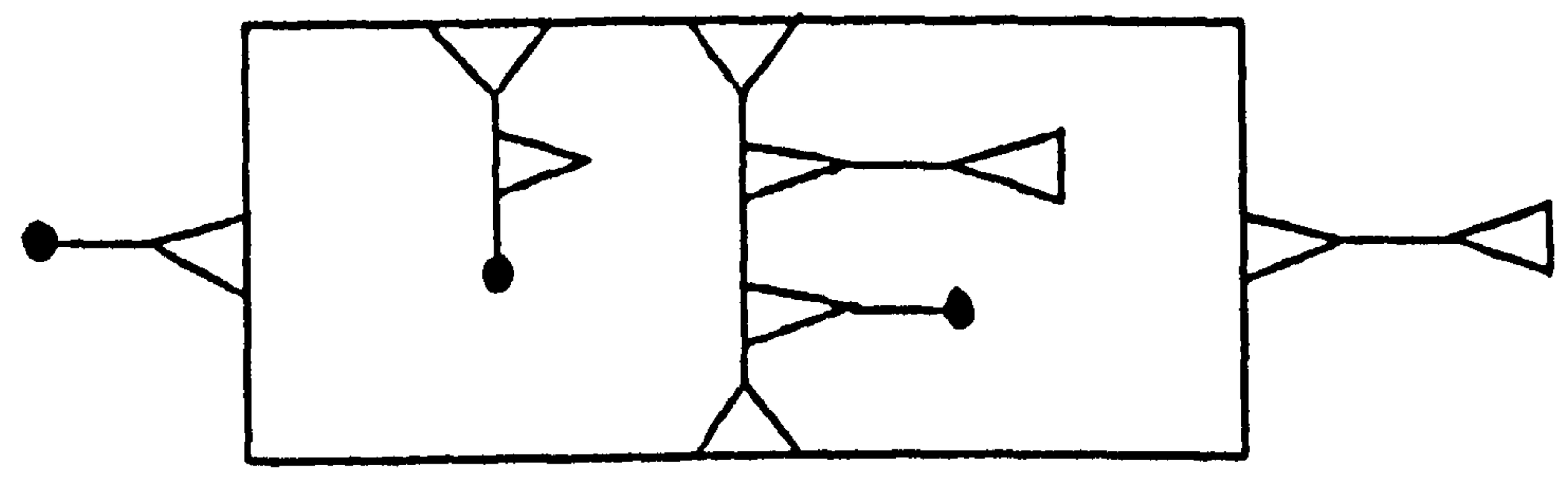
Case (1.4.2) : Put $p = 0$ and $g = 0$ in (1.1.1) to get

$$5u - 44f = 33(e - 4).$$

Replacing e by 4 (resp. 5, 6, 7, 8) in this equation and noting $f < 5$,

$e = 4$: $5u - 44f = 0$	$\therefore$	$f \equiv 0 \pmod 5$	$\therefore$	$f = 0$.
$e = 5$: $5u - 44f = 33$	$\therefore$	$f \equiv 3 \pmod 5$	$\therefore$	$f = 3$.
$e = 6$: $5u - 44f = 66$	$\therefore$	$f \equiv 1 \pmod 5$	$\therefore$	$f = 1$.
$e = 7$: $5u - 44f = 99$	$\therefore$	$f \equiv 4 \pmod 5$	$\therefore$	$f = 4$.
$e = 8$: $5u - 44f = 132$	$\therefore$	$f \equiv 2 \pmod 5$	$\therefore$	$f = 2$.

E_2 (33,0,5,3,0) 2(1)



The remaining five cases for a minimal S are therefore

(0,0,4,0,0)	:	null diagram (zero points/vertices)
(33,0,5,3,0) 2(1)	:	E_2
(22,0,6,1,0) 3(1)	:	A
(55,0,7,4,0) 3(1)	:	A + E_2
(44,0,8,2,0) 4(1)	:	A + A

Hence, no such S exists. □

LEMMA 1.4.2 If S (u,p,e,f,g) satisfies (1.1.1) and e = 3, then there exists a coset diagram with specification S which is 1(1).

Proof : Assume S is a counter-example with p + f + g minimal. We want to show that no such S exists.

Diagrams for E (33,0,9,0,0) 2(1) 2(2) and B (66,0,14,0,0) 5(1) 2(2) have already been exhibited.

E(2)B = (99,0,19,0,0) 7(1) 2(2).

Now 2-compose once and 1-compose three times to get

B_2 (99,4,3,0,0) 1(1).

If $p \geq 4$ and $D(u-99,p-4,4,f,g)$ satisfies (1.1.1), then D can be 2(1) by Lemma 1.4.1 so that $D + B_2$ has specification S which is 1(1).

Therefore, S has $p < 4$.

C (11,0,3,2,0) 1(1)

If $f \geq 2$ and $D(u-11,p,4,f-2,g)$ satisfies (1.1.1), then D can be 2(1) by Lemma 1.4.1 so that D + C has specification S which is 1(1).

Therefore, S has $f < 2$.

We now know that a minimal S would have one of the following two forms

$$(u,p,3,0,g) \quad : \quad p < 4 \tag{1.4.3}$$

$$(u,p,3,1,g) \quad : \quad p < 4 \tag{1.4.4}$$

Applying (1.1.1) to the specification in (1.4.3),

$5u = 132p - 33 + 60g$

$\therefore \; 0 \equiv 2p - 8 \pmod 5$

$\therefore \; p \equiv 4 \pmod 5$

But $0 \leq p < 4$, so there is no specification of the form in (1.4.3).

Applying (1.1.1) to the specification in (1.4.4),

$5u = 132p - 33 + 44 + 60g$

$\therefore \; 0 \equiv 2p - 4 \pmod 5$

$\therefore \; p \equiv 2 \pmod 5$

$\therefore \; p = 2$, since $0 \leq p < 4$.

We now know that a minimal S would have the following form

$$H_g(u,2,3,1,g) \quad : \quad u = 12g + 55 \tag{1.4.5}$$

The condition in (1.4.5) is obtained by substituting $p = 2$, $e = 3$ and $f = 1$ in (1.1.1).

Diagrams have already been exhibited for A (22,0,6,1,0) 3(1), E (33,0,9,0,0) 2(1) 2(2) and G_1 (12,0,4,0,1) 2(1) .

A(1)E = (55,0,11,1,0) 3(1) 2(2).

Now, 2-compose once and 1-compose once to get H_0 (55,2,3,1,0) 1(1).

For $g \geq 1$, $H_g = H_{g-1} + G_1 = (12g+55,2,3,1,g)$ 1(1).

Hence, no such S exists. □

<u>LEMMA 1.4.3</u> If S (u,p,e,f,g) satisfies (1.1.1) and $e = 2$, then there exists a coset diagram with specification S which is 1(1).

<u>Proof :</u> Assume S is a counter-example with $p + f + g$ minimal. We want to show that no such S exists.

A diagram has already been exhibited for B (66,0,14,0,0) 5(1) 2(2).

With B, 2-compose once and 1-compose twice to get B_3 (66,3,2,0,0) 1(1).

If $p \geq 3$ and D (u-66,p-3,4,f,g) satisfies (1.1.1), then D can be 2(1) by Lemma 1.4.1 so that $D + B_3$ has specification S which is 1(1).

Therefore, S has $p < 3$.

If $f \geq 4$ and D (u-22,p,4,f-4,g) satisfies (1.1.1), then D can be 2(1) by Lemma 1.4.1 so that $D + K_0$ has specification S which is 1(1).

Therefore, S has $f < 4$.

We now know that a minimal S would have one of the following three forms

(u,0,2,f,g)	:	$f < 4$	(1.4.6)
(u,1,2,f,g)	:	$f < 4$	(1.4.7)
(u,2,2,f,g)	:	$f < 4$	(1.4.8)

Applying (1.1.1) to the specifications in (1.4.6), (1.4.7) and (1.4.8),

(1.4.6) : $5u - 44f = -66 + 60g$

$\therefore \quad f \equiv 4 \pmod 5$

(1.4.7) : $5u - 44f = 66 + 60g$

$\therefore \quad f \equiv 1 \pmod 5$

(1.4.8) : $5u - 44f = 198 + 60g$

$\therefore \quad f \equiv 3 \pmod 5$

But $0 \leq f < 4$ in each case, so $f = 1$ in (1.4.7), $f = 3$ in (1.4.8), and there is no S of the form in (1.4.6).

Therefore, the remaining cases for a minimal S are

$I_g(u,1,2,1,g)$: $u = 12g + 22$ (1.4.9)

$J_g(u,2,2,3,g)$: $u = 12g + 66$ (1.4.10)

Diagrams have already been exhibited for I_0 1(1) and A_2 (2(1).

$J_0(66,2,2,3,0)\ 1(1) = I_0(22,1,2,1,0)\ 1(1) + A_2(44,1,4,2,0)\ 2(1)$.

(1.4.9) : For $g \geq 1$, $I_g = G_1 + I_{g-1} = (12g+22,1,2,1,g)\ 1(1)$

(1.4.10) : For $g \geq 1$, $J_g = G_1 + J_{g-1} = (12g+66,2,2,3,g)\ 1(1)$

Hence, no such S exists. □

LEMMA 1.4.4 If $S(u,p,e,f,g)$ satisfies (1.1.1) and $e = 1$, then there exists a coset diagram with specification S.

Proof : Assume S is a counter-example with $p + f + g$ minimal. We want to show that no such S exists.

If $p \geq 1$ and $D(u,p-1,5,f,g)$ satisfies (1.1.1), then D can be 2(1) by Lemma 1.4.1, so we can 1-compose D once to get a diagram with specification S. Therefore, S has $p < 1$. i.e. S has $p = 0$.

Putting $e = 1$ and $p = 0$ in (1.1.1) we have

$$5u = -132 + 33 + 44f + 60g$$

$$\therefore\ 5u - 44f = 60g - 99$$

$$\therefore\ f \equiv 1 \pmod 5$$

Therefore, a minimal S would be of the form

$$(u,0,1,f,g) \quad : \quad f \equiv 1 \pmod 5 \ \text{ and } \ u = 12g + \frac{11}{5}(4f - 9)$$

Therefore, the cases for a minimal S are

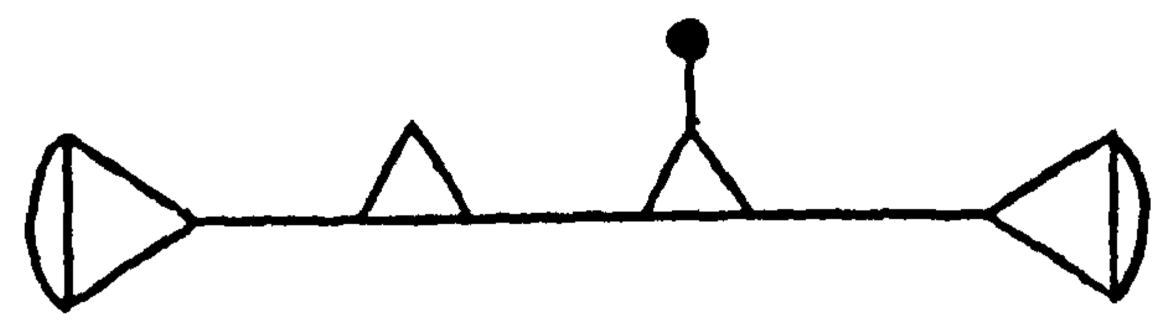
$(u,0,1,1,g)$	:	$u = 12g - 11$, $g \geq 1$	(1.4.11)
$(u,0,1,6,g)$	:	$u = 12g + 33$	
$(u,0,1,11,g)$	:	$u = 12g + 77$	
$(u,0,1,16,g)$	:	$u = 12g + 121$	

and so on, adding 5 to f, and adding 44 to u.

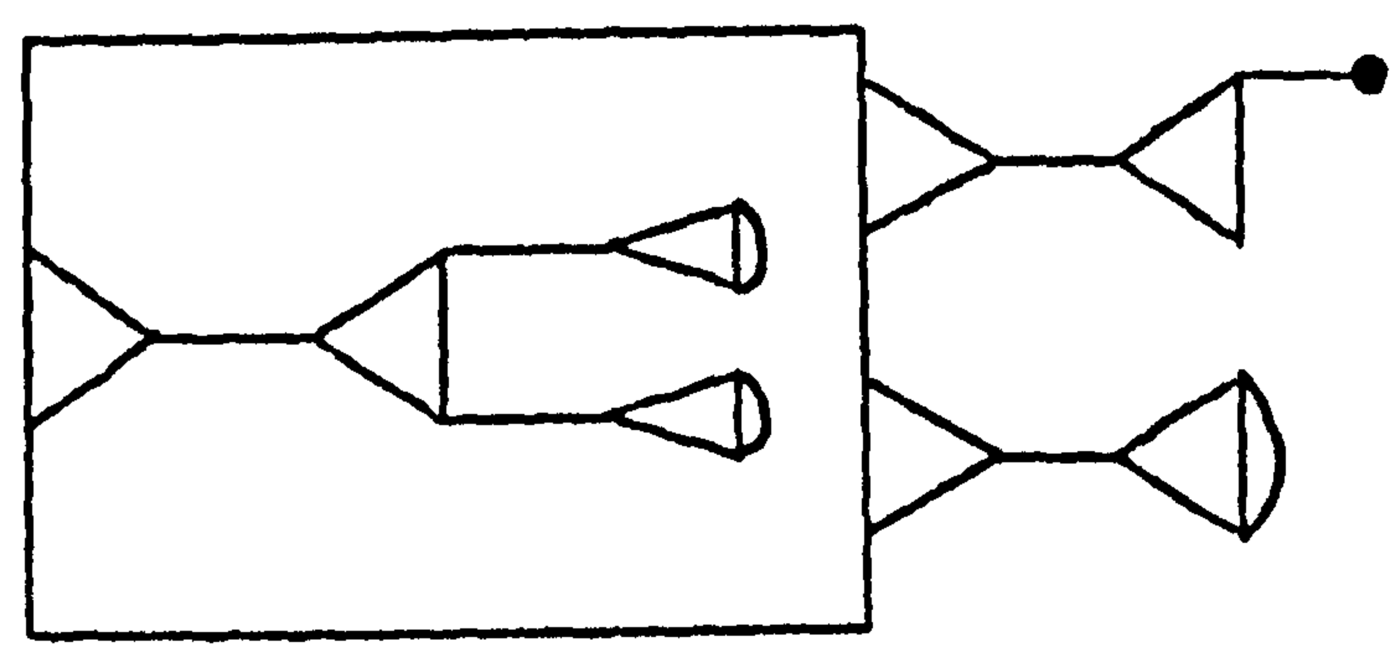
Consider (1.4.11) : (12g-11,0,1,1,g) , $g \geq 1$.

g = 1 (1,0,1,1,1) The diagram consists of a red point with a blue loop.

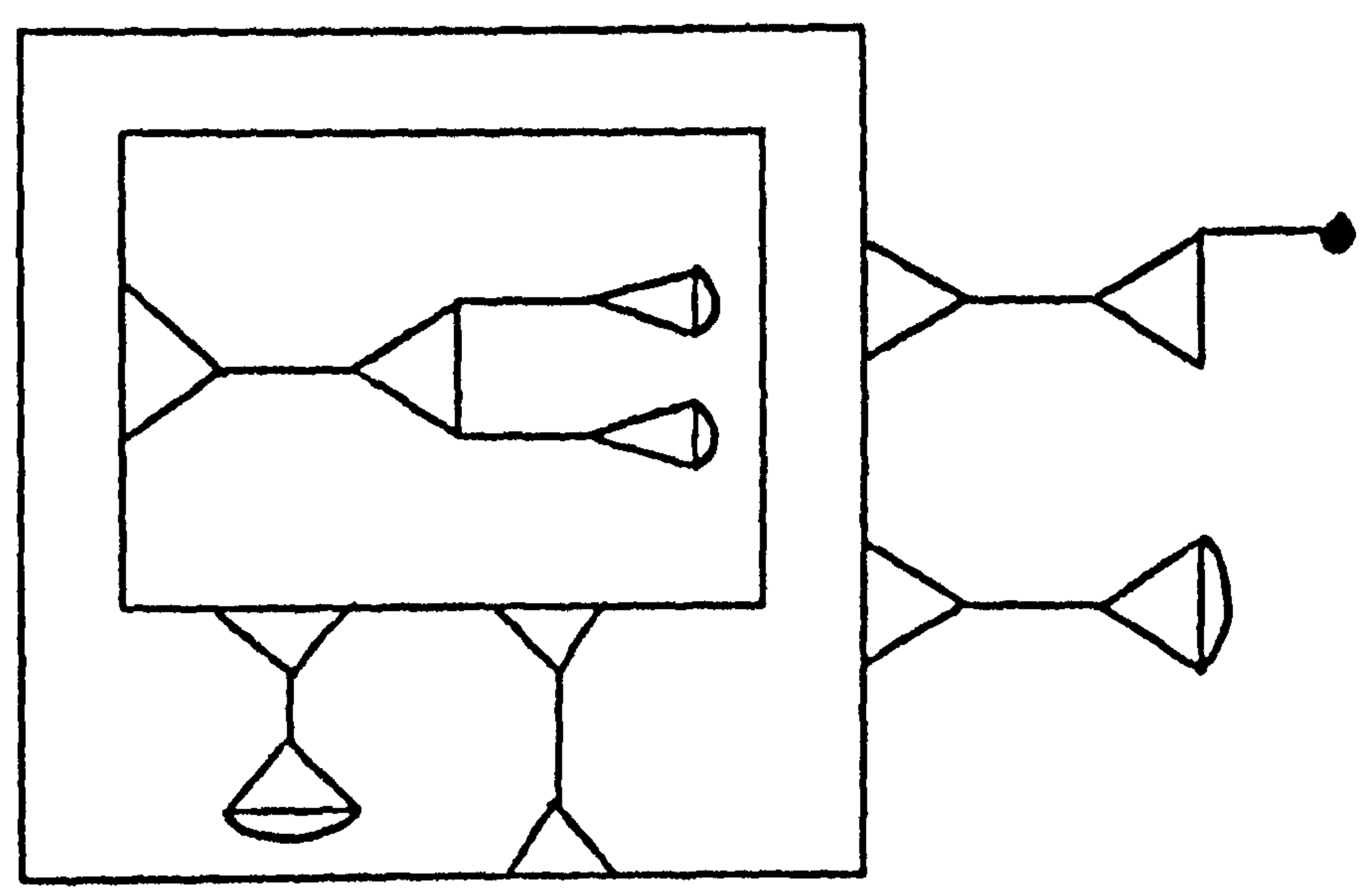
g = 2 (13,0,1,1,2)



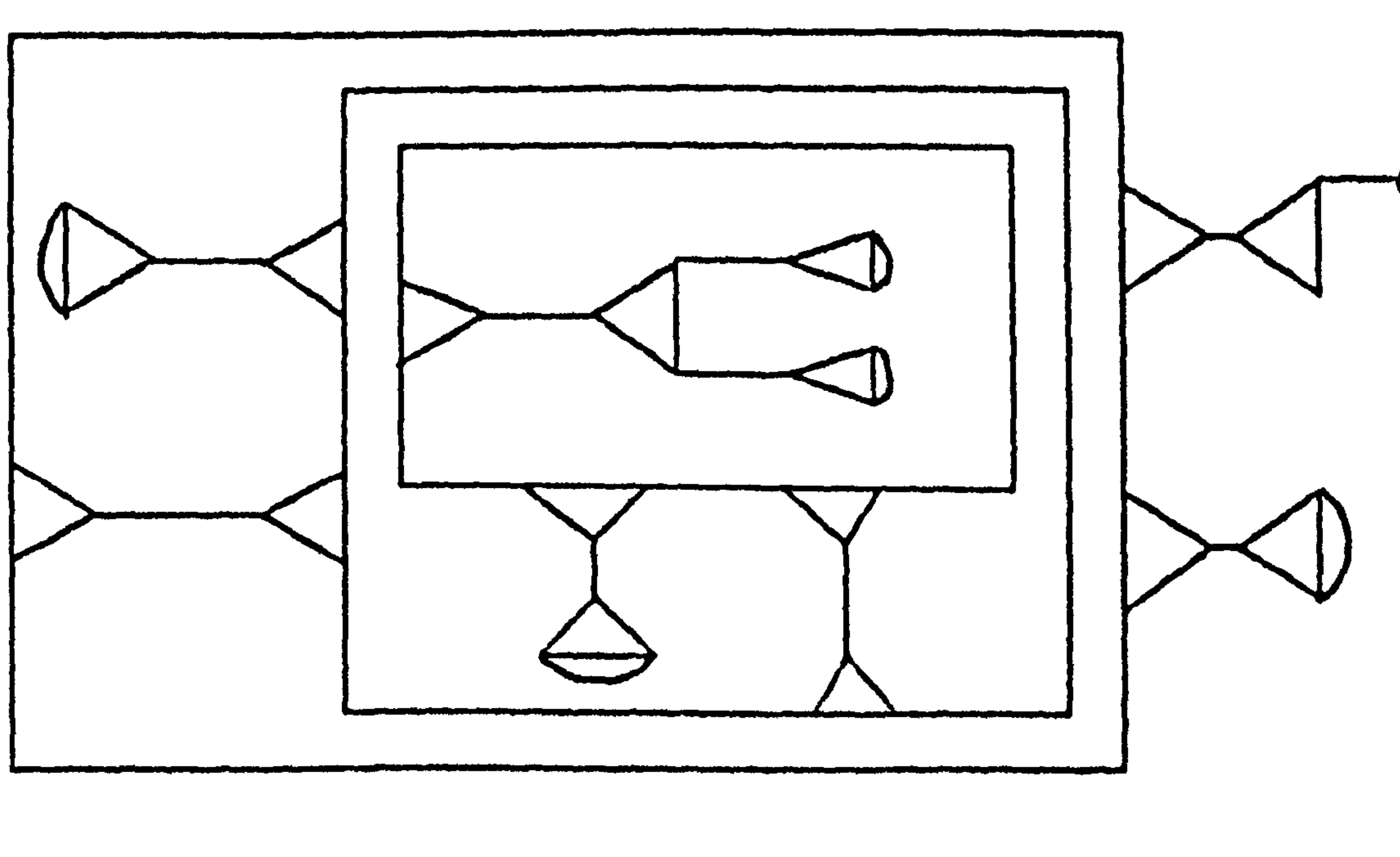
g = 3 (25,0,1,1,3)



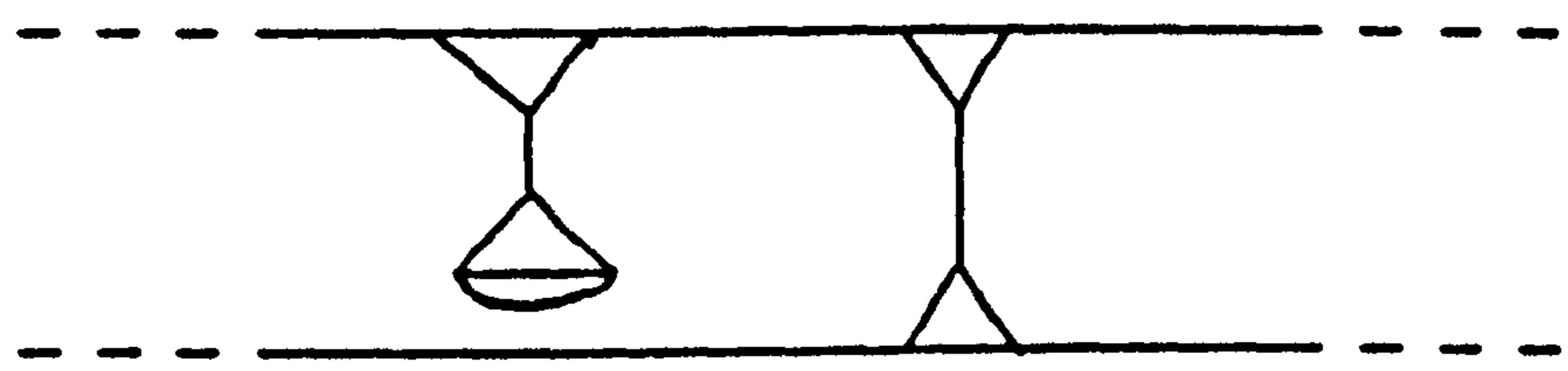
g = 4 (37,0,1,1,4)



g = 5 (49,0,1,1,5)



In general, for $g \geq 3$, there are $g-3$ sections of the form



and the remainder of the diagram is the same.

Let $G_n = (12n,0,4,0,n)$, $n \geq 1$, and $K_n = (44n+22,0,2,5n+4,0)$, $n \geq 0$.

Then, G_n can be 2(1) by Lemma 1.4.1, and K_n can be 1(1) by Lemma 1.4.3.

A diagram for C (11,0,3,2,0) 1(1) has already been exhibited.

The remaining cases for a minimal S can now be dealt with as follows.

$$(12g+33,0,1,6,g) = \begin{cases} C + K_0 , & g = 0, \\ (C + G_g) + K_0 , & g \geq 1 \end{cases}$$

$$(12g+77,0,1,11,g) = \begin{cases} C + K_1 , & g = 0, \\ (C + G_g) + K_1 , & g \geq 1 \end{cases}$$

... and so on. In general,

$$(12g+44n+33,0,1,5n+6,g) = \begin{cases} C + K_n , & g = 0, \\ (C + G_g) + K_n , & g \geq 1 \end{cases} , \quad n \geq 0$$

Hence, no such S exists. □

<u>LEMMA 1.4.5</u> If $S(u,p,e,f,g)$ satisfies (1.1.1) and $e = 0$, then there exists a coset diagram with specification S.

<u>Proof :</u> Assume S is a counter-example with $p + f + g$ minimal. We want to show that no such S exists.

If $p \geq 1$ and $D(u,p-1,4,f,g)$ satisfies (1.1.1), then D can be 2(1) by Lemma 1.4.1, so we can 1-compose D once to get a diagram with specification S. Therefore, S has $p < 1$. i.e. S has $p = 0$.

Putting $e = p = 0$ in (1.1.1) we have

$$5u = -132 + 44f + 60g$$

$$\therefore \quad f \equiv 3 \pmod 5$$

Therefore, a minimal S would be of the form

$$(u,0,0,f,g) \quad : \quad f \equiv 3 \pmod 5 \quad \text{and} \quad u = 12g + \frac{11}{5}(4f - 12)$$

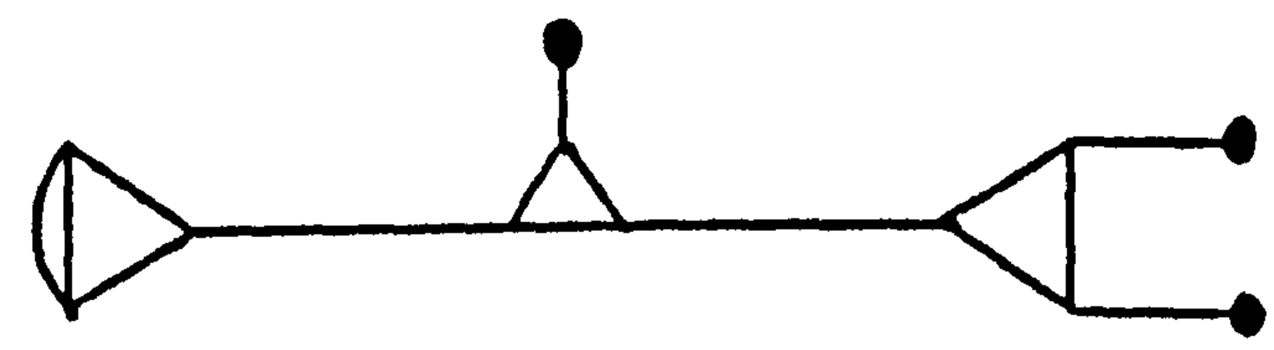
Therefore, the cases for a minimal S are

$$\begin{array}{lll} (u,0,0,3,g) & : & u = 12g\ ,\ g \geq 1 \\ (u,0,0,8,g) & : & u = 12g + 44 \\ (u,0,0,13,g) & : & u = 12g + 88 \\ (u,0,0,18,g) & : & u = 12g + 132 \end{array} \qquad (1.4.12)$$

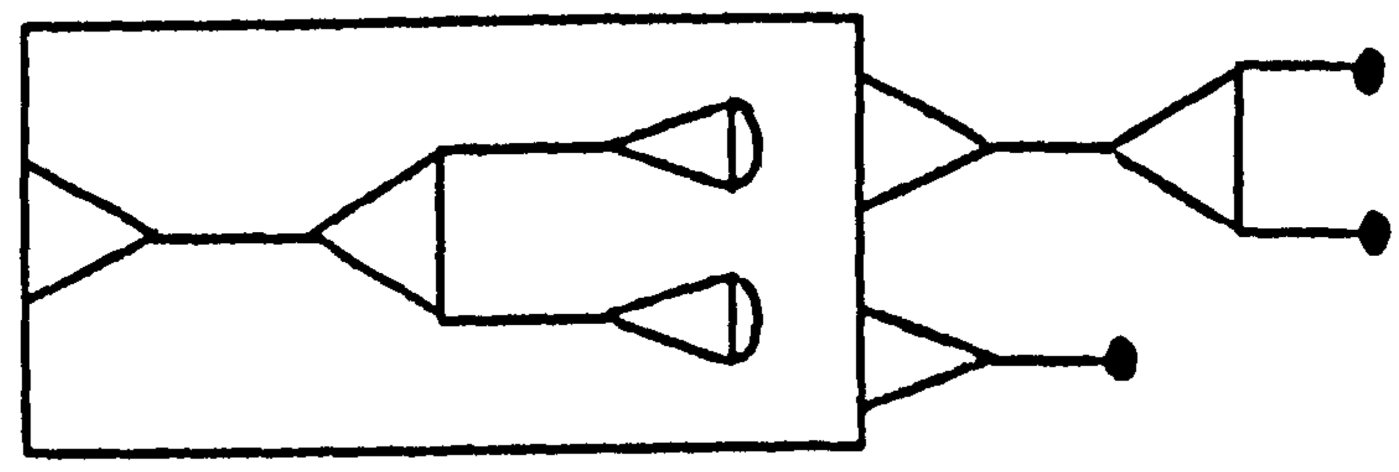
and so on, adding 5 to f, and adding 44 to u.

Consider (1.4.12) : (12g,0,0,3,g) , $g \geq 1$.

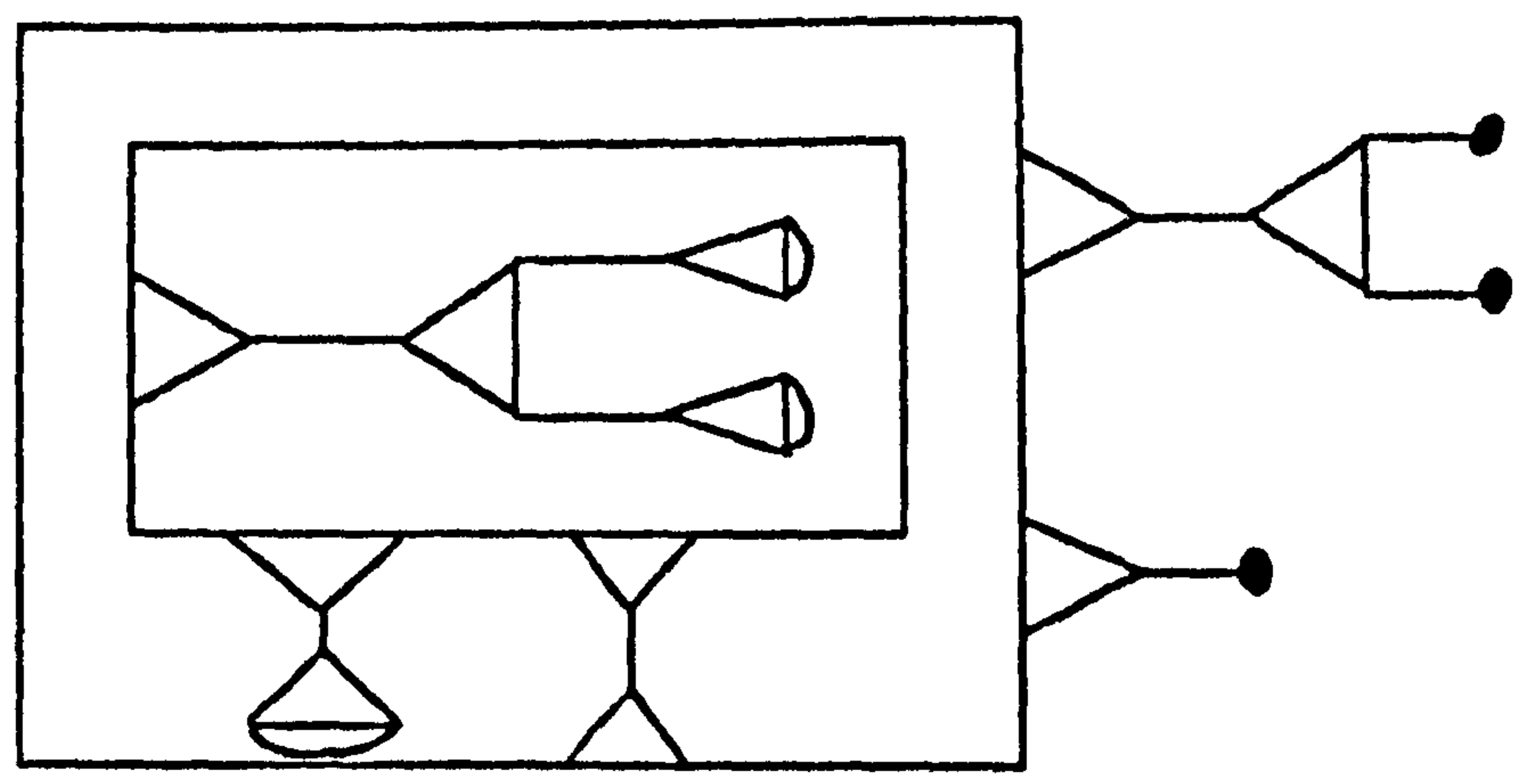
$g = 1$ (12,0,0,3,1)



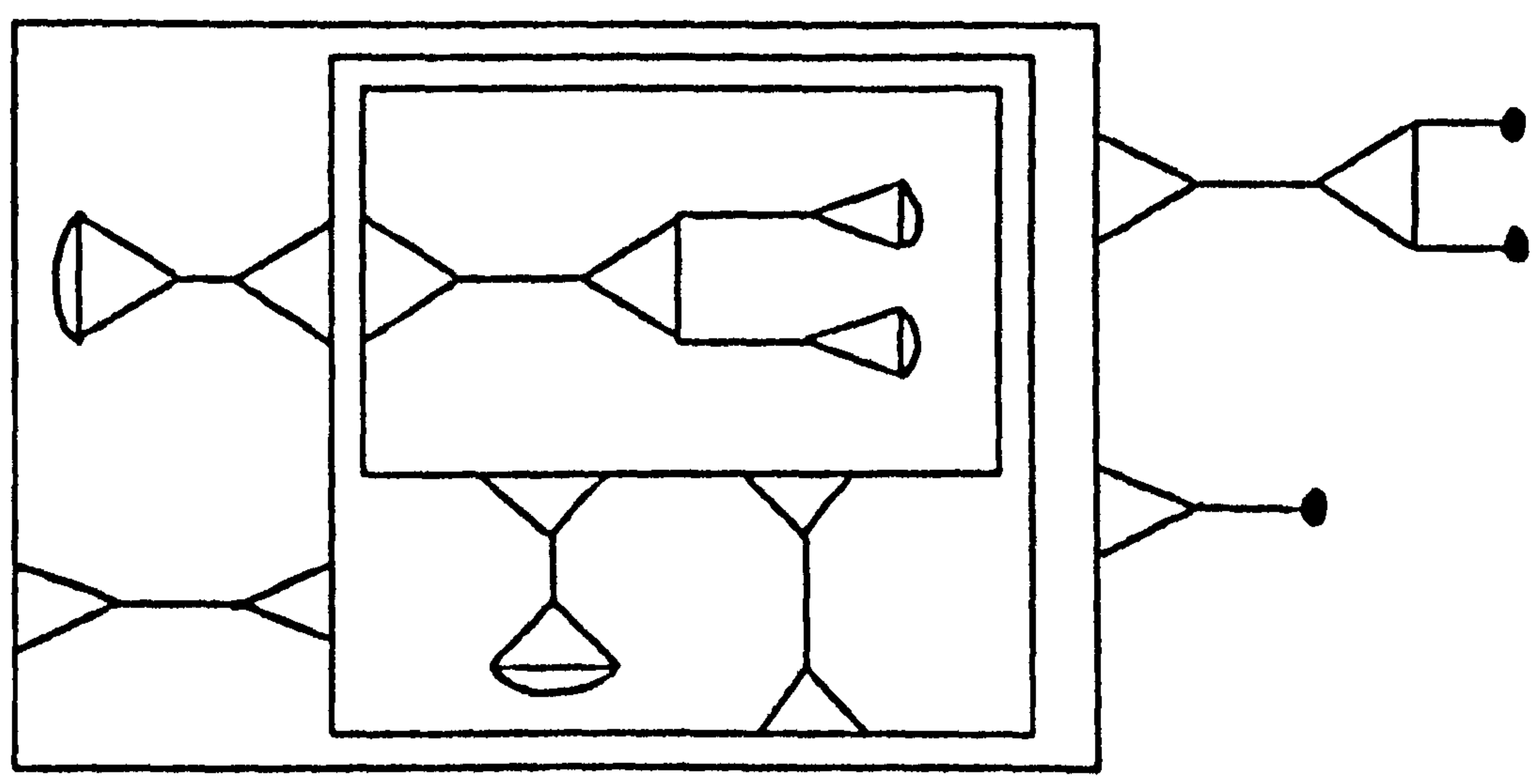
$g = 2$ (24,0,0,3,2)



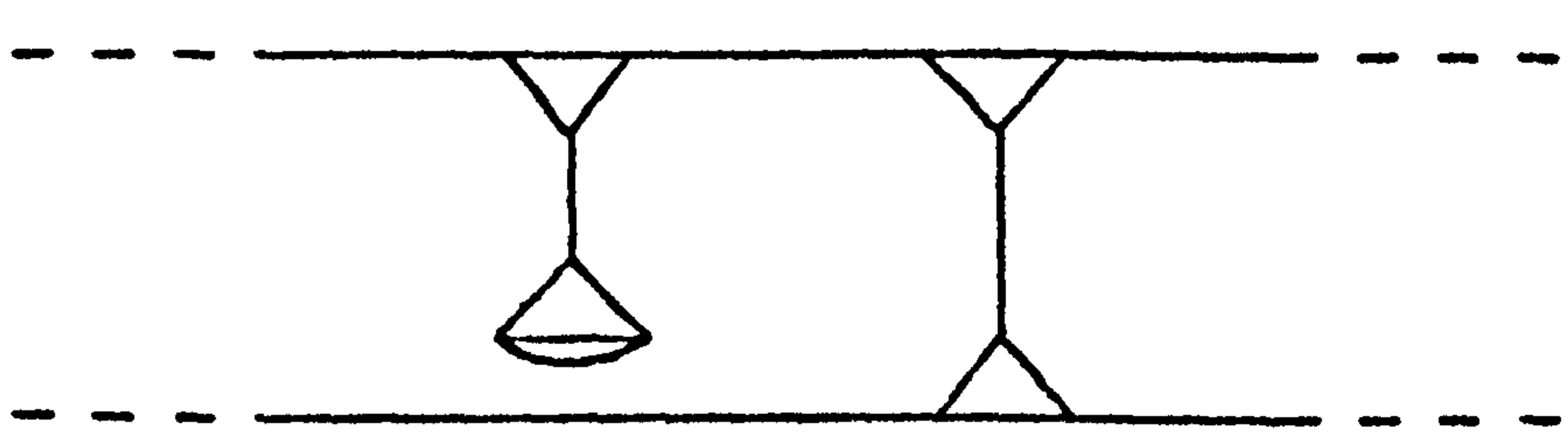
$g = 3$ (36,0,0,3,3)



$g = 4$ (48,0,0,3,4)



In general, for $g \geq 2$, there are $g - 2$ sections of the form



and the remainder of the diagram is the same.

Recall that we have

$G_n\ (12n,0,4,0,n)\ 2(1)$ and $K_n\ (44n+22,0,2,5n+4,0)\ 1(1)$.

The remaining cases for a minimal S can now be dealt with as follows.

$$(12g+44,0,0,8,g) = \left\{ \begin{array}{lll} K_0 + K_0 , & g = 0, \\ (K_0 + G_g) + K_0 , & g \geq 1 \end{array} \right\}$$

$$(12g+88,0,0,13,g) = \left\{ \begin{array}{lll} K_0 + K_1 , & g = 0, \\ (K_0 + G_g) + K_1 , & g \geq 1 \end{array} \right\}$$

... and so on. In general,

$$(12g+44n+44,0,0,5n+8,g) = \left\{ \begin{array}{lll} K_0 + K_n , & g = 0, \\ (K_0 + G_g) + K_n , & g \geq 1 \end{array} \right\} , n \geq 0$$

Hence, no such S exists. □

THEOREM 1.4.6 Every specification (u,p,e,f,g), satisfying the genus formula (1.1.1), corresponds to a subgroup of (finite) index u in Δ_{11}.

Proof : From Lemmas 1.4.1, 1.4.2, 1.4.3, 1.4.4 and 1.4.5, we know there exists a coset diagram for every specification (u,p,e,f,g) satisfying (1.1.1). From Lemma 2.1 in [18], there is a correspondence between subgroups of index u in Δ_{11} and u point coset diagrams for Δ_{11}. The theorem follows immediately. □

CHAPTER 2

(2,3,9) TRIANGLE GROUP

§2.1 GENUS FORMULA

The genus formula can be derived from Theorem 2 in [15]. We get

$$2p - 2 + e(1 - \tfrac{1}{2}) + f(1 - \tfrac{1}{3}) + g_1(1 - \tfrac{1}{9}) + g_3(1 - \tfrac{3}{9})$$

$$= \quad u(1 - \tfrac{1}{2} - \tfrac{1}{3} - \tfrac{1}{9}),$$

which simplifies to

$$u = 36(p - 1) + 9e + 12f + 16g_1 + 12g_3 \qquad (2.1.1)$$

Note that with Δ_{11} we had g, but with Δ_9 we have g_1 and g_3. This is because 9 is composite with divisor 3.

However, although the condition *u ≡ f (mod 3)* could be derived from (1.1.1), the genus formula for Δ_{11}, this is not the case for (2.1.1).

$$u = 36(p - 1) + 9e + 12f + 16g_1 + 12g_3 \quad : \quad u \equiv f \pmod 3 \qquad (2.1.2)$$

Therefore, the genus formula for subgroups of index u in Δ_9 is (2.1.2).

§2.2 SPECIFICATION

For Δ_9, we define a *specification* to be a list of non-negative integers (u,p,e,f,g_1,g_3), with $u \geq 1$, which satisfies the genus formula (2.1.2).

If a coset diagram with specification (u,p,e,f,g_1,g_3) exists, then there will be g_3 green 3-cycles and u vertices, including e red points, f blue points and g_1 green points.

THEOREM 2.2.1 The genus formula (2.1.2) has a solution for each $u \geq 36$, but not for $u = 35$.

Proof : $u \equiv e \pmod 4$, from (2.1.2).

Without loss of generality, let $p = 0$ and $e \leq 3$.

Then there are twelve cases, since $3\backslash(u - f)$ and $4\backslash(u - e)$:

(i)	$u = 12v$,	$e = 0$,	$f = 3k$	(vii)	$u = 12v + 6$,	$e = 2$,	$f = 3k$
(ii)	$u = 12v + 1$,	$e = 1$,	$f = 3k + 1$	(viii)	$u = 12v + 7$,	$e = 3$,	$f = 3k + 1$
(iii)	$u = 12v + 2$,	$e = 2$,	$f = 3k + 2$	(ix)	$u = 12v + 8$,	$e = 0$,	$f = 3k + 2$
(iv)	$u = 12v + 3$,	$e = 3$,	$f = 3k$	(x)	$u = 12v + 9$,	$e = 1$,	$f = 3k$
(v)	$u = 12v + 4$,	$e = 0$,	$f = 3k + 1$	(xi)	$u = 12v + 10$,	$e = 2$,	$f = 3k + 1$
(vi)	$u = 12v + 5$,	$e = 1$,	$f = 3k + 2$	(xii)	$u = 12v + 11$,	$e = 3$,	$f = 3k + 2$

case (i) : $u = 12v$, $e = 0$, $p = 0$, $f = 3k$. Substitute values in (2.1.2).

$$12v + 36 = 36k + 16g_1 + 12g_3$$

$$\therefore \quad 3v + 9 = 3(3k + g_3) + 4g_1$$

By Result 1.2.1, this is solvable if $3v + 9 \geq 6$, i.e. $3v \geq -3$, i.e. $u \geq -12$.

case (ii) : $u = 12v + 1$, $e = 1$, $p = 0$, $f = 3k + 1$. Put values in (2.1.2).

$$12v + 16 = 36k + 16g_1 + 12g_3$$

$$\therefore \quad 3v + 4 = 3(3k + g_3) + 4g_1$$

By Result 1.2.1, this is solvable if $3v + 4 \geq 6$, i.e. $3v \geq 2$, i.e. $u \geq 9$.

case (iii) : $u = 12v + 2$, $e = 2$, $p = 0$, $f = 3k + 2$. Put values in (2.1.2).

$$12v - 4 = 36k + 16g_1 + 12g_3$$

$$\therefore \quad 3v - 1 = 3(3k + g_3) + 4g_1$$

By Result 1.2.1, this is solvable if $3v - 1 \geq 6$, i.e. $3v \geq 7$, i.e. $u \geq 30$.

case (iv) : $u = 12v + 3$, $e = 3$, $p = 0$, $f = 3k$. Put values in (2.1.2).

$$12v + 12 = 36k + 16g_1 + 12g_3$$

$$\therefore \quad 3v + 3 = 3(3k + g_3) + 4g_1$$

By Result 1.2.1, this is solvable if $3v + 3 \geq 6$, i.e. $3v \geq 3$, i.e. $u \geq 15$.

case (v) : $u = 12v + 4$, $e = 0$, $p = 0$, $f = 3k + 1$. Put values in (2.1.2).

$$12v + 28 = 36k + 16g_1 + 12g_3$$

$$\therefore \quad 3v + 7 = 3(3k + g_3) + 4g_1$$

By Result 1.2.1, this is solvable if $3v + 7 \geq 6$, i.e. $3v \geq -1$, i.e. $u \geq 0$.

case (vi) : $u = 12v + 5$, $e = 1$, $p = 0$, $f = 3k + 2$. Put values in (2.1.2).

$$12v + 8 = 36k + 16g_1 + 12g_3$$

$$\therefore \quad 3v + 2 = 3(3k + g_3) + 4g_1$$

By Result 1.2.1, this is solvable if $3v + 2 \geq 6$, i.e. $3v \geq 4$, i.e. $u \geq 21$.

case (vii) : $u = 12v + 6$, $e = 2$, $p = 0$, $f = 3k$. Put values in (2.1.2).

$$12v + 24 = 36k + 16g_1 + 12g_3$$

$$\therefore \quad 3v + 6 = 3(3k + g_3) + 4g_1$$

By Result 1.2.1, this is solvable if $3v + 6 \geq 6$, i.e. $3v \geq 0$, i.e. $u \geq 6$.

case (viii) : $u = 12v + 7$, $e = 3$, $p = 0$, $f = 3k + 1$. Put values in (2.1.2).

$$12v + 4 = 36k + 16g_1 + 12g_3$$

$$\therefore \quad 3v + 1 = 3(3k + g_3) + 4g_1$$

By Result 1.2.1, this is solvable if $3v + 1 \geq 6$, i.e. $3v \geq 5$, i.e. $u \geq 27$.

case (ix) : $u = 12v + 8$, $e = 0$, $p = 0$, $f = 3k + 2$. Put values in (2.1.2).

$$12v + 20 = 36k + 16g_1 + 12g_3$$

$$\therefore \quad 3v + 5 = 3(3k + g_3) + 4g_1$$

By Result 1.2.1, this is solvable if $3v + 5 \geq 6$, i.e. $3v \geq 1$, i.e. $u \geq 12$.

case (x) : $u = 12v + 9$, $e = 1$, $p = 0$, $f = 3k$. Put values in (2.1.2).

$$12v + 36 = 36k + 16g_1 + 12g_3$$

$$\therefore \quad 3v + 9 = 3(3k + g_3) + 4g_1$$

By Result 1.2.1, this is solvable if $3v + 9 \geq 6$, i.e. $3v \geq -3$, i.e. $u \geq -3$.

case (xi) : $u = 12v + 10$, $e = 2$, $p = 0$, $f = 3k + 1$. Put values in (2.1.2).

$$12v + 16 = 36k + 16g_1 + 12g_3$$

$$\therefore \quad 3v + 4 = 3(3k + g_3) + 4g_1$$

By Result 1.2.1, this is solvable if $3v + 4 \geq 6$, i.e. $3v \geq 2$, i.e. $u \geq 18$.

case (xii) : $u = 12v + 11$, $e = 3$, $p = 0$, $f = 3k + 2$. Put values in (2.1.2).

$$12v - 4 = 36k + 16g_1 + 12g_3$$

$$\therefore \quad 3v - 1 = 3(3k + g_3) + 4g_1$$

By Result 1.2.1, this is solvable if $3v - 1 \geq 6$, i.e. $3v \geq 7$, i.e. $u \geq 39$.

From cases (i), (ii), ..., (xii), we deduce that (2.1.2) has a solution for each $u \geq 39$. Using (2.1.2), a computer program was developed to determine all the solutions for (2.1.2) for $u \leq 38$. This program and its output are shown in APPENDIX B. From the output, we see that solutions of (2.1.2) exist for $u = 36, 37$ and 38, but not for 35. These four values can also be checked by hand using (2.1.2).

The specifications listed in the program output for $u = 36$, 37 and 38 do satisfy (2.1.2).

For $u = 35$, we substitute this value into (2.1.2) and re-arrange to get

$$71 - (16g_1 + 12g_3) = 3(12p + 3e + 4f) \quad : \quad 35 \equiv f \pmod 3 \qquad (2.2.1)$$

Now we can conclude that $g_1 \leq 4$ and $g_3 \leq 5$, since $p, e, f, g_1, g_3 \geq 0$. Next, we put each possible arrangement of g_1 and g_3, such that $16g_1 + 12g_3 \leq 71$, into the equation in (2.2.1). The RHS is divisible by 3, but the LHS is only divisible by 3 when $g_1 = 2$ in which case $g_3 \leq 3$. This leaves us with four equations (from an initial eighteen equations) to check :

$$1 = 12p + 3e + 4f \quad : \quad 35 \equiv f \pmod 3 \qquad (2.2.2)$$

$$5 = 12p + 3e + 4f \quad : \quad 35 \equiv f \pmod 3 \qquad (2.2.3)$$

$$9 = 12p + 3e + 4f \quad : \quad 35 \equiv f \pmod 3 \qquad (2.2.4)$$

$$13 = 12p + 3e + 4f \quad : \quad 35 \equiv f \pmod 3 \qquad (2.2.5)$$

Noting that p, e and f are non-negative integers, it is clear that the equations in (2.2.2) and (2.2.3) do not have solutions, although the equations in (2.2.4) and (2.2.5) each have one solution.

The conditions $p = 0$, $e = 3$ and $f = 1$ satisfy the equation in (2.2.5), and the conditions $e = 3$, and $p = f = 0$ satisfy the equation in (2.2.4). However, the congruence is not satisfied for either $f = 0$ or $f = 1$, so that (2.2.2), (2.2.3), (2.2.4) and (2.2.5) do not have solutions.

This implies that (2.1.2) has no solution for $u = 35$. □

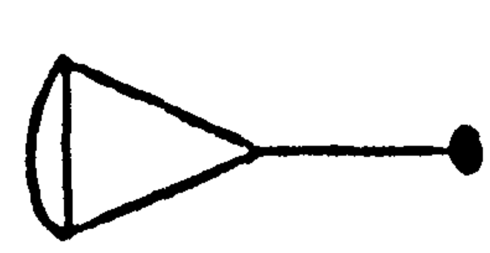
§2.3 SUBGROUPS OF FINITE INDEX IN Δ_9

Before we can prove Theorem 2.3.9, we will need the following eight lemmas. Note that some specifications will be used in more than one lemma.

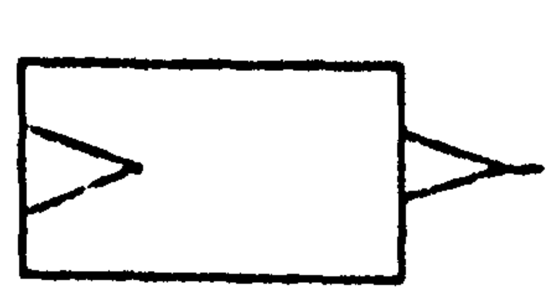
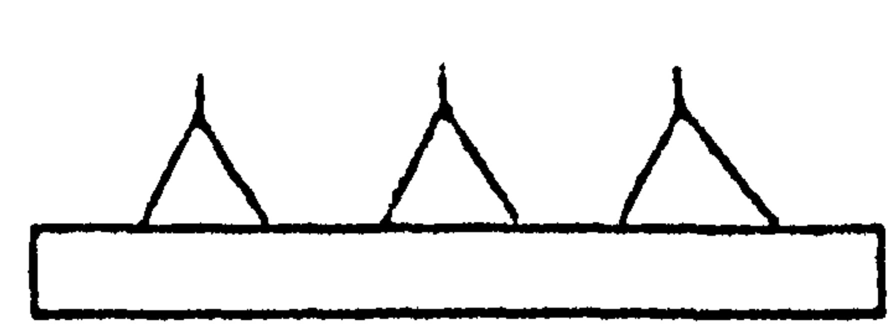
Composition of coset diagrams with specification (u,p,e,f,g_1,g_3) is done in exactly the same way as with specification (u,p,e,f,g).

<u>LEMMA 2.3.1</u> There are precisely four inequivalent types of green 3-cycles; two of them closed, two of them open.

The closed green 3-cycles are

(1)  and (2) 

The open green 3-cycles are

(3)  and (4) 

<u>Proof :</u> Recall that a green line is obtained by following a blue line and then a red line. We begin with a labelled blue triangle (labels 1, 2, 3 in clockwise order), and look at all possible green 3-cycles starting at label 1. For convenience, $3 \to 1$, for example, will represent a red line between vertex 3 and vertex 1.

There are five possible cases for a red line from vertex 2 :-

<u>Case (1)</u> : $2 \to 2$ (i.e. vertex 2 is a red point).

If $3 \to 1$, then we get a 2-cycle.

If $3 \to 3$, then we must have a red point at vertex 1 to complete a 3-cycle.

This gives a closed green 3-cycle of type (1).

If $3 \to$ blue point, then we get a cycle of length ≥ 4.

If $3 \to$ blue triangle, then we get an open green 3-cycle equivalent to type (3)

<u>Case (2)</u> : $2 \to 1$. This creates a green 1-cycle (green point) at vertex 1.

<u>Case (3)</u> : $2 \to 3$.

If $1 \to 1$, then we get a 2-cycle.

If $1 \to$ blue point, then we get a closed green 3-cycle of type (2).

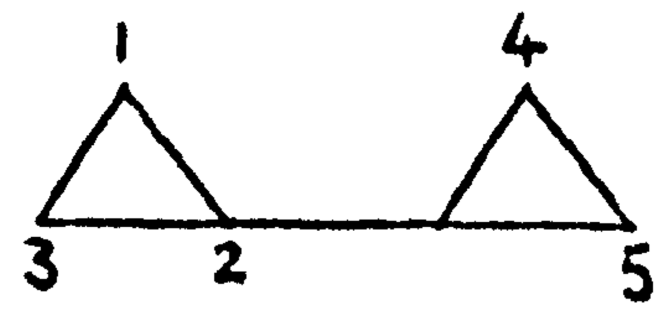
If $1 \to$ blue triangle, then we get a cycle of length ≥ 4.

<u>Case (4)</u> : $2 \to$ blue point.

We must have $3 \to 1$ to complete the green 3-cycle of type (2).

<u>Case (5)</u> : $2 \to$ blue triangle (with labels 4, 5).

If $4 \to 4$, then we must have $5 \to 1$ to complete the green 3-cycle, equivalent to type (3).



If $4 \to 5$, then we get a cycle of length ≥ 4.

If $4 \to 3$, then we must have $1 \to 1$ to complete the green 3-cycle, equivalent to type (3).

If $4 \to 1$, then we get a 2-cycle.

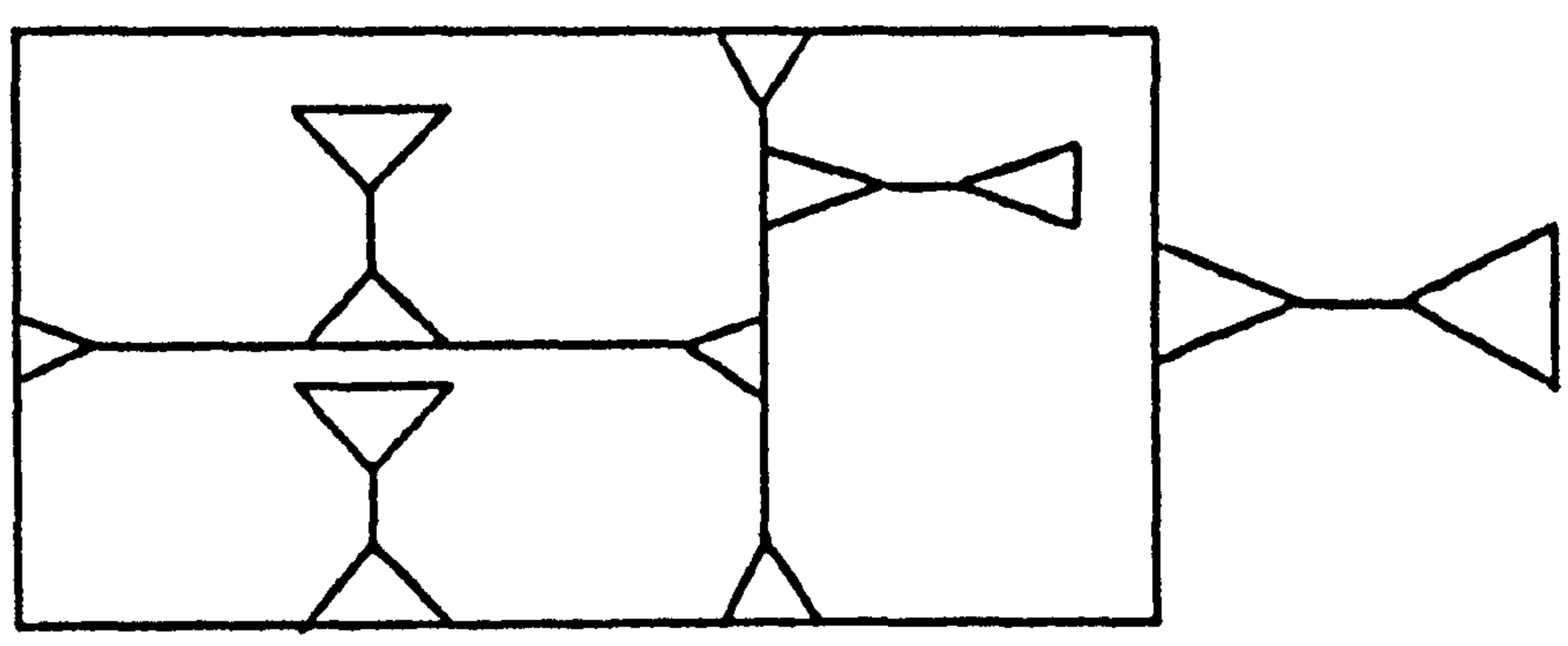
If $4 \to$ blue point, then we get a cycle of length ≥ 4.

If $4 \to$ blue triangle, then this new triangle must join to vertex 1 to give a green 3-cycle, equivalent to type (4). □

LEMMA 2.3.2 If $S(u,p,e,f,g_1,g_3)$ satisfies (2.1.2) and $e \geq 5$, then there exists a coset diagram with specification S which is n(1) where $n \geq 2$.

Proof : Assume S is a counter-example with $p + e + f + g_1 + g_3$ minimal. We want to show that no such S exists.

N_1 (36,0,8,0,0,0) 4(1)



$A_0 = N_1(1) = (36,1,4,0,0,0)$ 2(1).

If $p \geq 1$ and $D(u-36,p-1,e,f,g_1,g_3)$ satisfies (2.1.2), then $D + A_0$ has specification S which is n(1), $n \geq 2$.

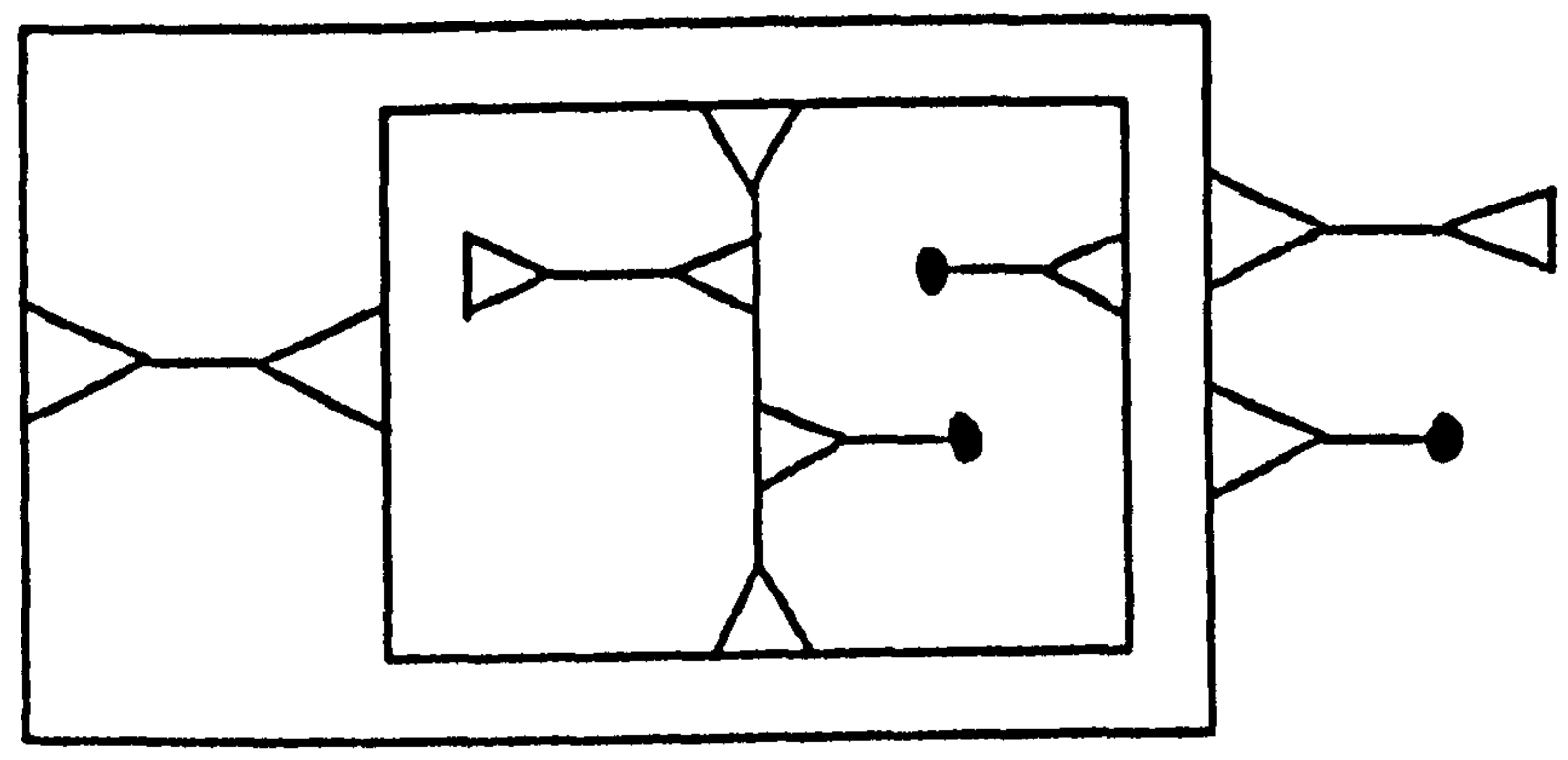
Therefore, S has $p < 1$. i.e. S has $p = 0$.

B_0 (9,0,5,0,0,0) 2(1)

If $e \geq 6$ and $D(u-9,p,e-1,f,g_1,g_3)$ satisfies (2.1.2), then $D + B_0$ has specification S which is n(1), $n \geq 2$.

Therefore, S has $e < 6$. i.e. S has $e = 5$.

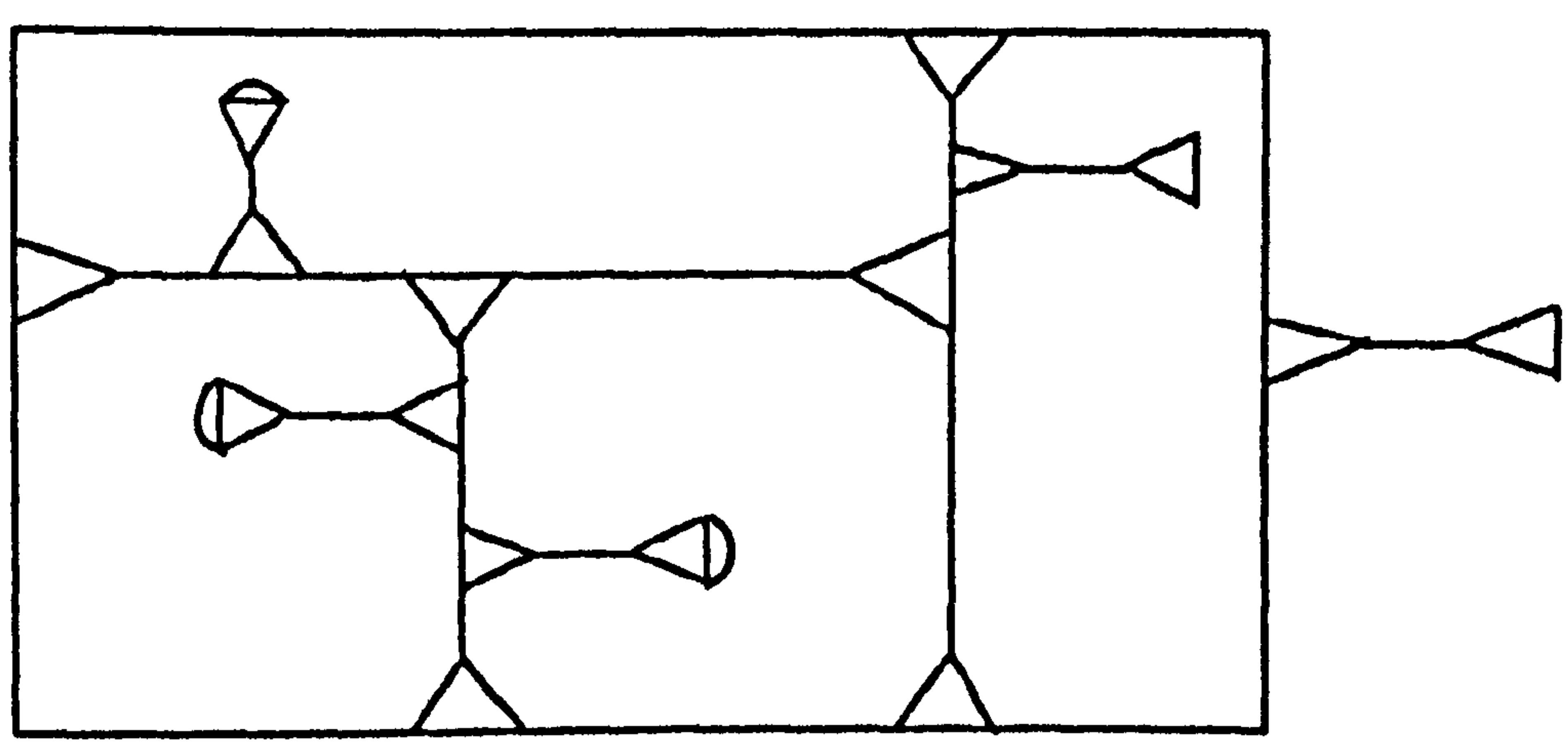
C_0 (36,0,4,3,0,0) 2(1)



If $f \geq 3$ and $D(u-36,p,e,f-3,g_1,g_3)$ satisfies (2.1.2), then $D + C_0$ has specification S which is n(1), $n \geq 2$.

Therefore, S has $f < 3$.

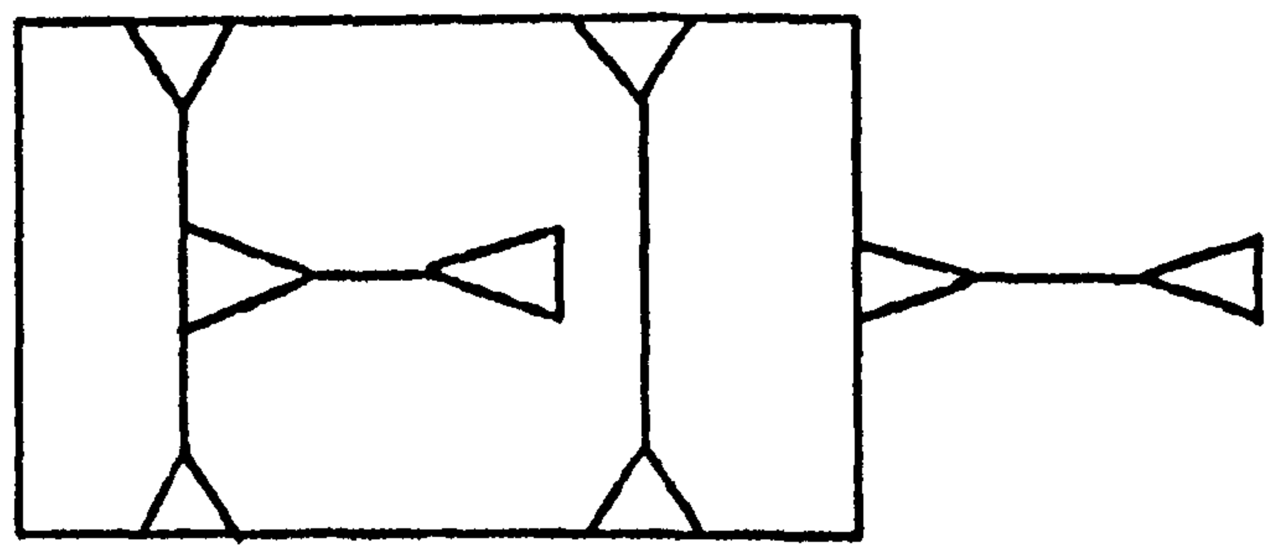
D_0 (48,0,4,0,3,0) 2(1)



If $g_1 \geq 3$ and $D(u-48,p,e,f,g_1-3,g_3)$ satisfies (2.1.2), then $D + D_0$ has specification S which is n(1), $n \geq 2$.

Therefore, S has $g_1 < 3$.

I_2 (24,0,4,0,0,2) 2(1)



If $g_3 \geq 2$ and $D(u-24,p,e,f,g_1,g_3-2)$ satisfies (2.1.2), then $D + I_2$ has specification S which is n(1), $n \geq 2$.

Therefore, S has $g_3 < 2$.

From (2.1.2), we have $u \equiv g_1 \pmod 3$ and $u \equiv f \pmod 3$.

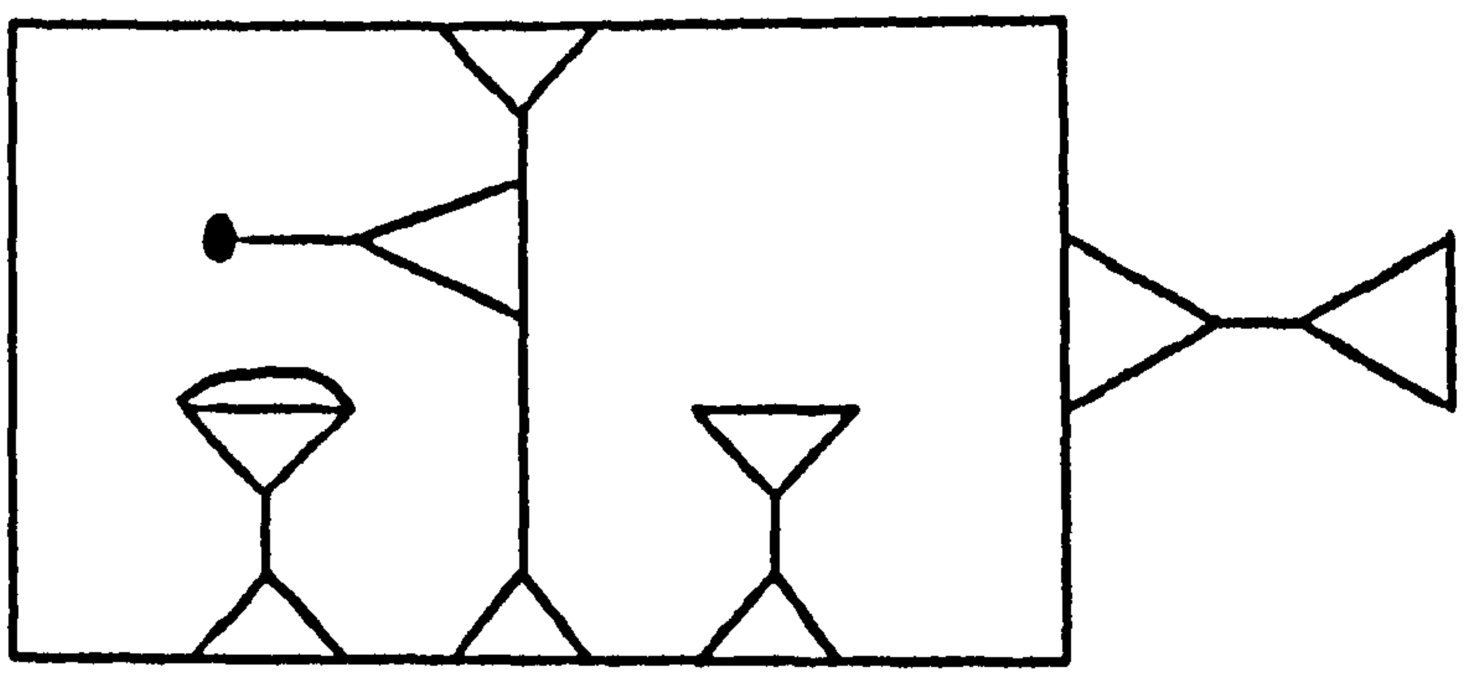
Therefore, $f \equiv g_1 \pmod 3$.

We now know that a minimal S would have one of the following two forms

$$(u,0,5,f,g_1,0) \quad : \quad f = g_1 \in \{0, 1, 2\} \qquad (2.3.1)$$

$$(u,0,5,f,g_1,1) \quad : \quad f = g_1 \in \{0, 1, 2\} \qquad (2.3.2)$$

F_0 (28,0,4,1,1,0) 2(1)



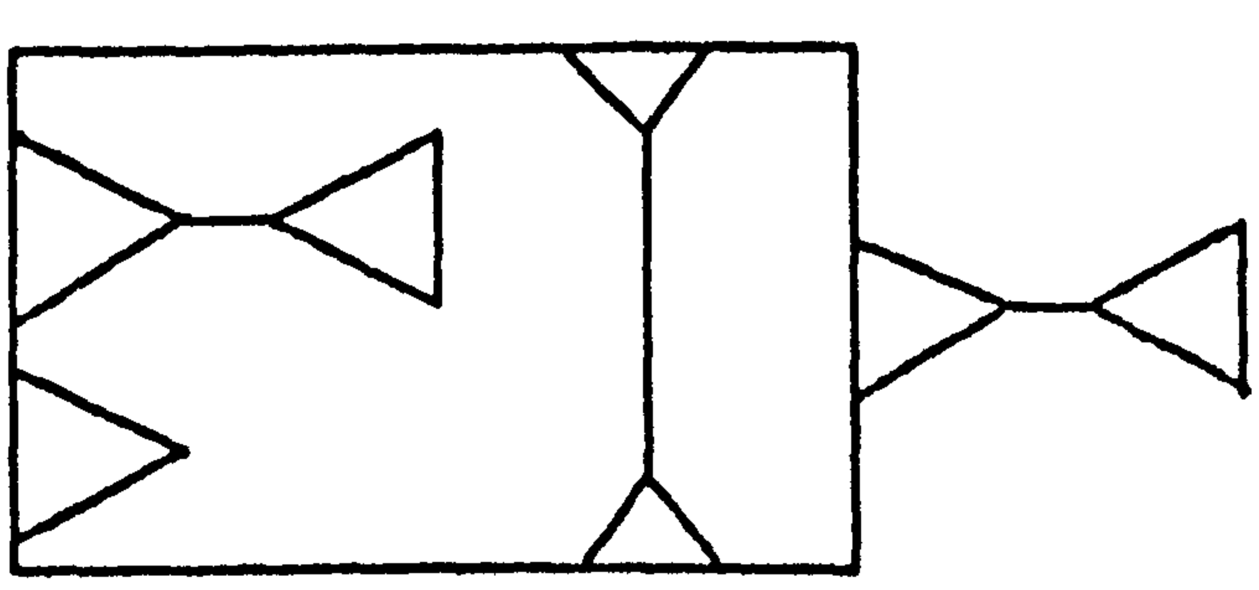
Consider (2.3.1).

$f = g_1 = 0$: (9,0,5,0,0,0) 2(1) = B_0

$f = g_1 = 1$: (37,0,5,1,1,0) 2(1) = $B_0 + F_0$

$f = g_1 = 2$: (65,0,5,2,2,0) 2(1) = $(B_0 + F_0) + F_0$

B_1 (21,0,5,0,0,1) 2(1)



Consider (2.3.2).

$f = g_1 = 0$: (21,0,5,0,0,1) 2(1) = B_1

$f = g_1 = 1$: (49,0,5,1,1,1) 2(1) = $B_1 + F_0$

$f = g_1 = 2$: (77,0,5,2,2,1) 2(1) = $(B_1 + F_0) + F_0$

Hence, no such S exists. □

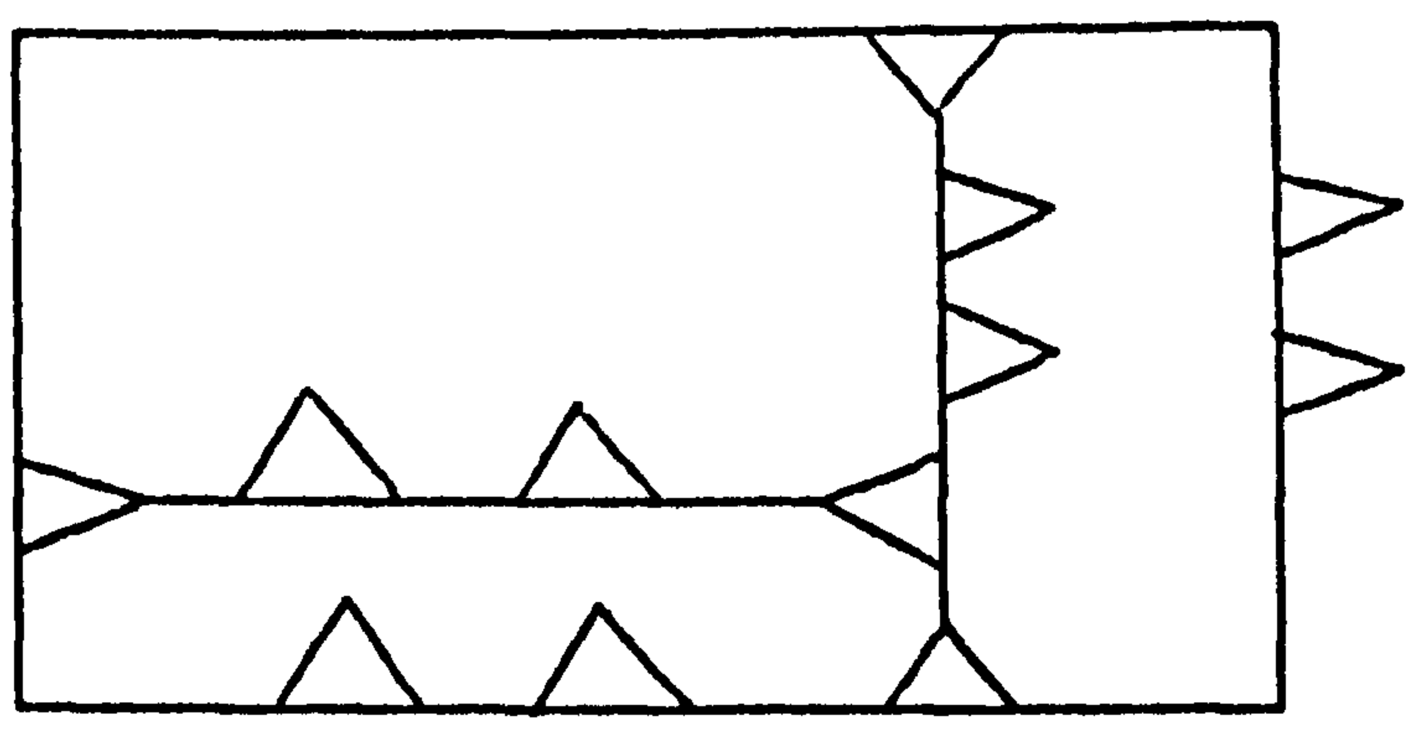
<u>LEMMA 2.3.3</u> If $S(u,p,e,f,g_1,g_3)$ satisfies (2.1.2) and $e = 4$, then there exist two coset diagrams with specification S, one of which is 2(1) and the other 2(2), with the following exceptions :

There exist coset diagrams, which are 1(1) 1(2), with specifications

(12,0,4,0,0,1) and (36,0,4,0,0,3).

<u>Proof :</u> Assume S is a counter-example with $p + f + g_1 + g_3$ minimal. We want to show that no such S exists.

N_2 (36,0,8,0,0,0) 4(2)

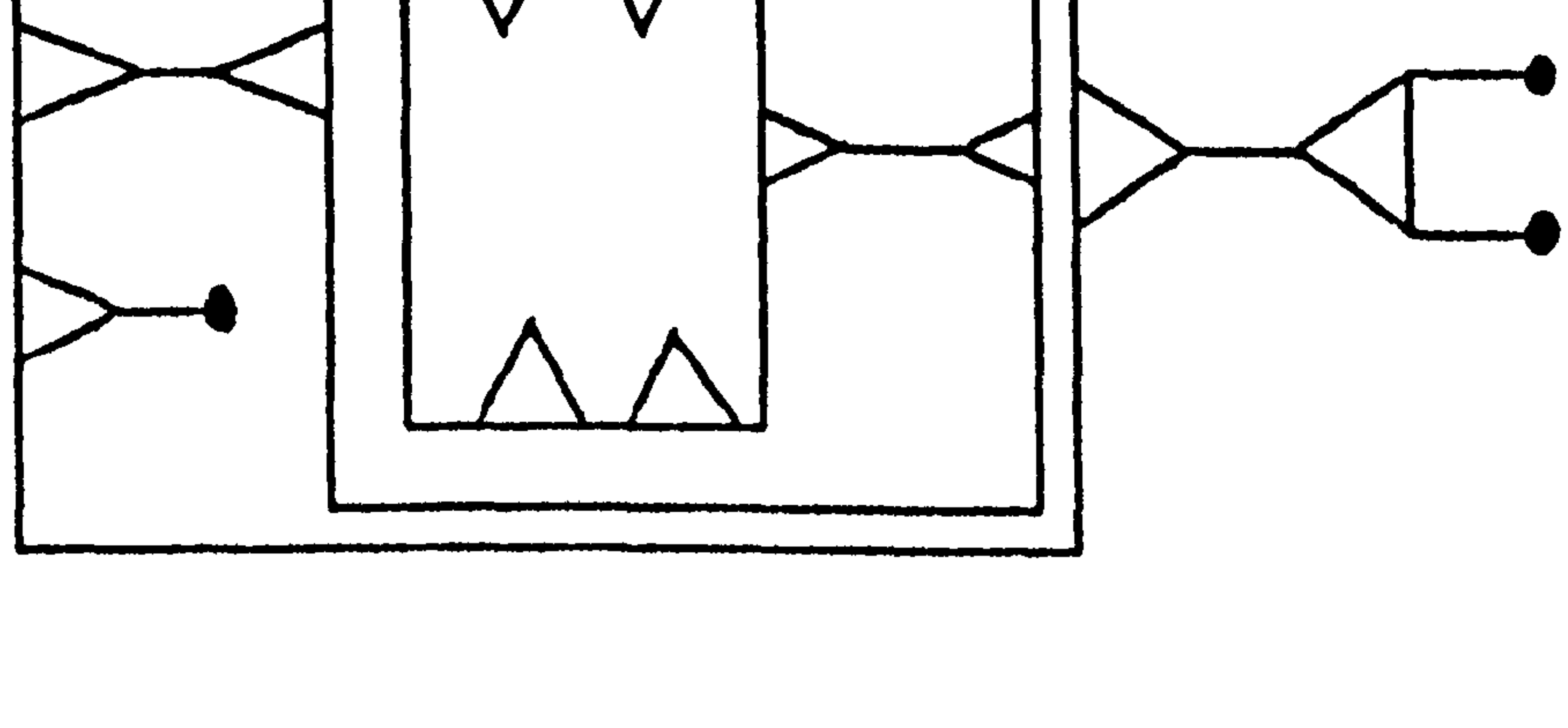


$A_0^2 = N_2(2) = (36,1,4,0,0,0)$ 2(2). The superscript in A_0^2 indicates that the diagram is 2(2).

A diagram has already been exhibited for A_0 (36,1,4,0,0,0) 2(1).

If $p \geq 2$ and $D(u-36,p-1,4,f,g_1,g_3)$ satisfies (2.1.2), then A_0 2(1) + D 2(1) and A_0^2 2(2) + D 2(2) each have specification S which are respectively 2(1) and 2(2). Therefore, S has $p < 2$. Note that we had to take $p \geq 2$ rather than 1, to ensure that D could not equal an exception.

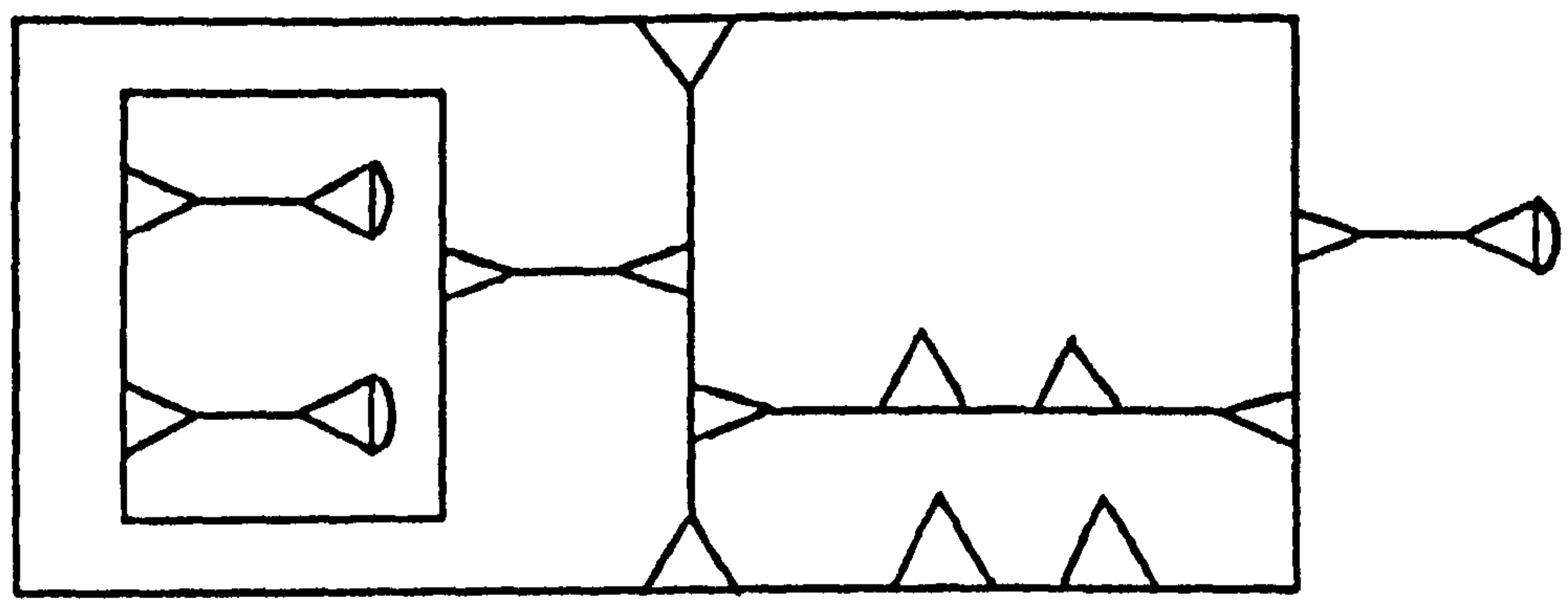
C_0^2 (36,0,4,3,0,0) 2(2)



A diagram has already been exhibited for C_0 (36,0,4,3,0,0) 2(1).

If $f \geq 4$ and $D(u-36,p,4,f-3,g_1,g_3)$ satisfies (2.1.2), then C_0 2(1) + D 2(1) and C_0^2 2(2) + D 2(2) each have specification S which are respectively 2(1) and 2(2). Therefore, S has $f < 4$.

D_0^2 (48,0,4,0,3,0) 2(2)



A diagram has already been exhibited for D_0 (48,0,4,0,3,0) 2(1).

If $g_1 \geq 4$ and $D(u-48,p,4,f,g_1-3,g_3)$ satisfies (2.1.2), then D_0 2(1) + D 2(1) and D_0^2 2(2) + D 2(2) each have specification S which are respectively 2(1) and 2(2). Therefore, S has $g_1 < 4$.

As shown in Lemma 2.3.2, we have $f \equiv g_1 \pmod 3$.

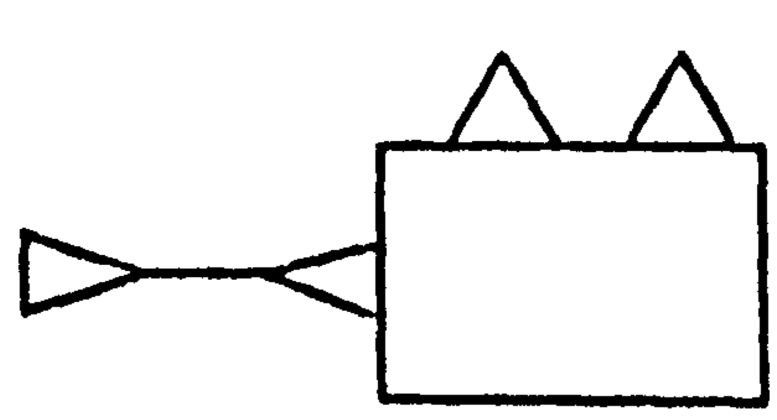
We now know that a minimal S would have the following form

$$(u,p,4,f,g_1,g_3) \quad : \quad p < 2 \ , \ f < 4 \ , \ g_1 < 4 \ , \ f \equiv g_1 \pmod 3.$$

This gives us twelve cases to consider :

<u>Case (1)</u> : I_n (12n,0,4,0,0,n) , $n \geq 1$.

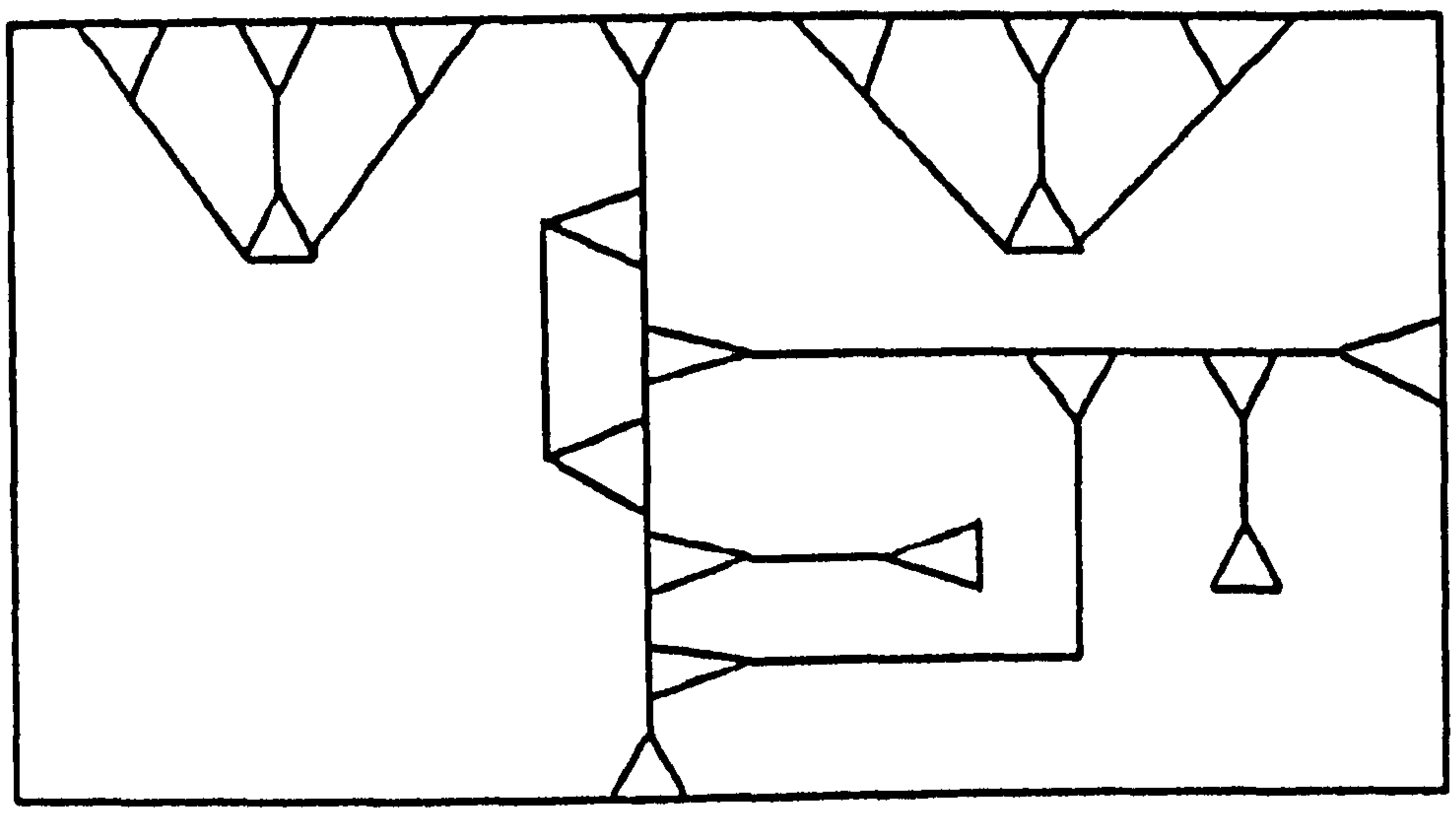
I_1 (12,0,4,0,0,1) 1(1) 1(2)



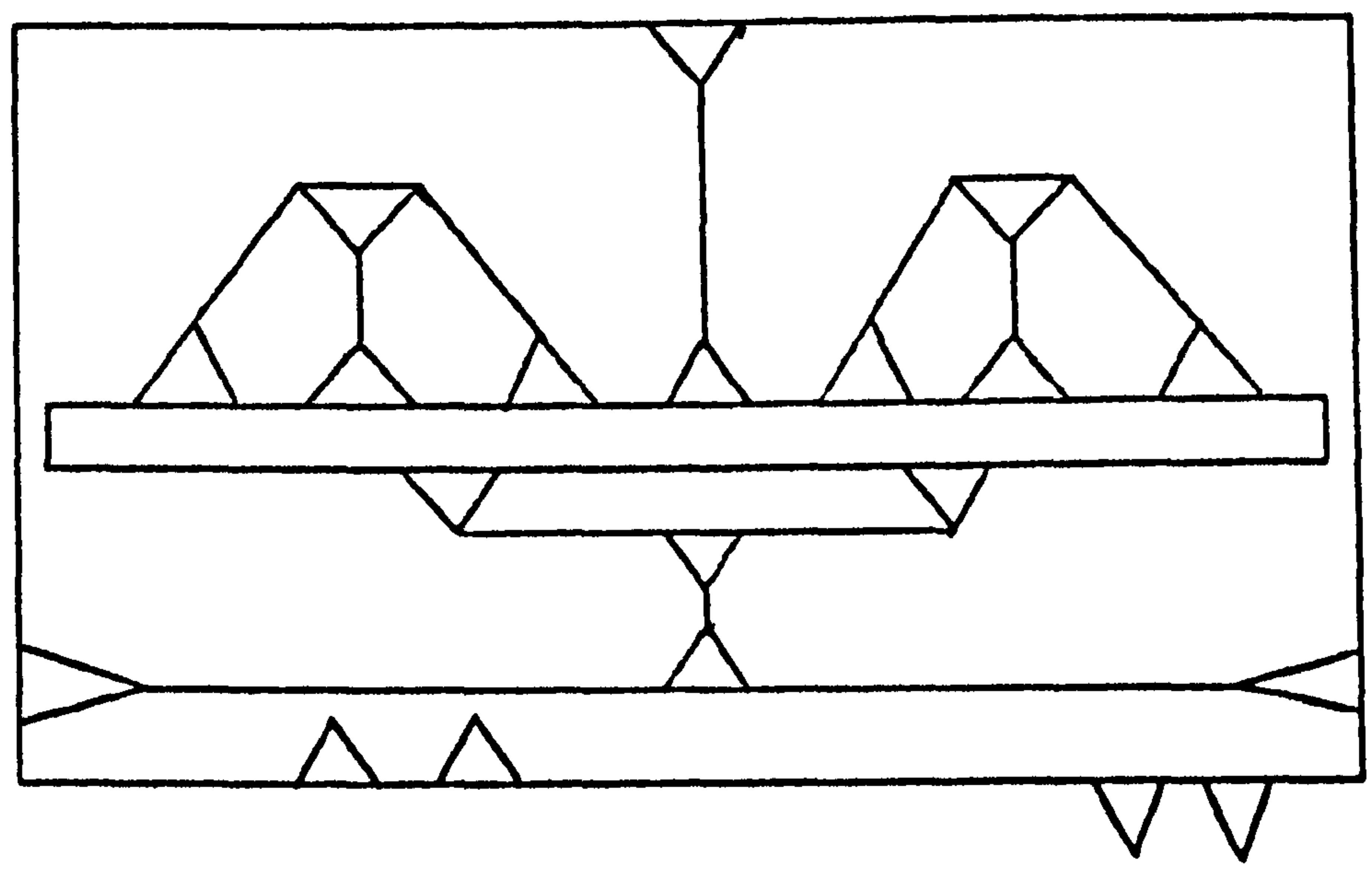
A diagram has already been exhibited for I_2 (24,0,4,0,0,2) 2(1).

I_3 (36,0,4,0,0,3) 1(1) 1(2) = I_1 1(1) 1(2) + I_2 2(1)

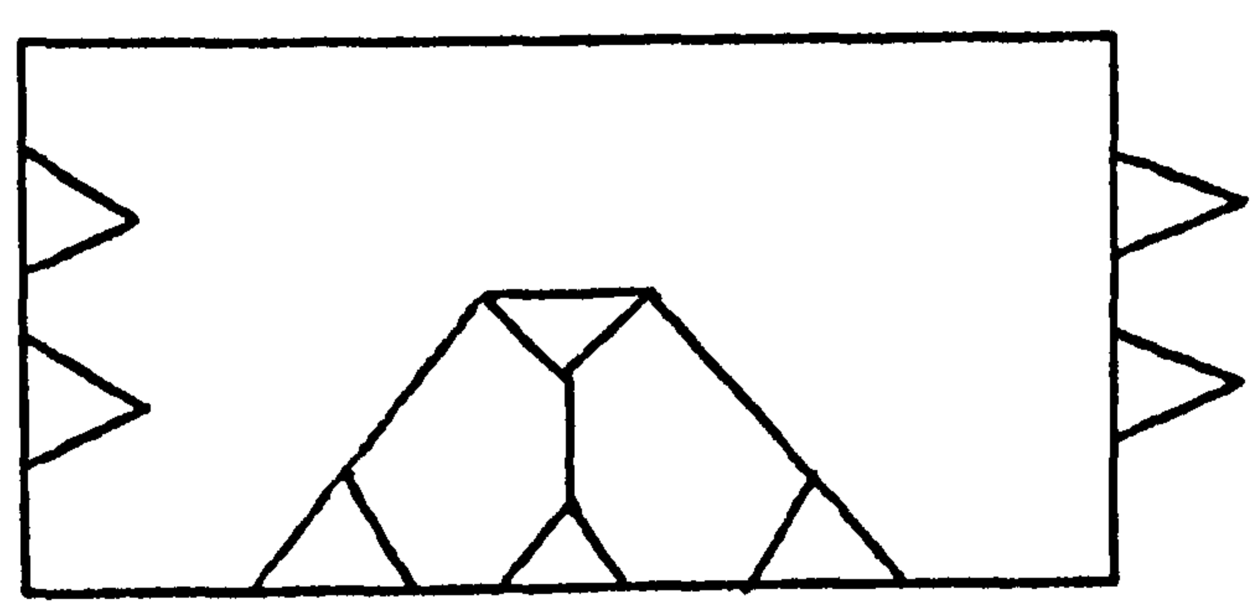
I_5 (60,0,4,0,0,5) 2(1)



I_5^2 (60,0,4,0,0,5) 2(2)



I_2^2 (24,0,4,0,0,2) 2(2)



$$\forall \text{ even } n \geq 4, \quad \left\{ \begin{array}{l} I_n\,(12n,0,4,0,0,n)\,2(1) = I_{n-2}\,2(1) + I_2\,2(1) \\ I_n^2\,(12n,0,4,0,0,n)\,2(2) = I_{n-2}^2\,2(2) + I_2^2\,2(2) \end{array} \right\}$$

$$\forall \text{ odd } n \geq 7, \quad \left\{ \begin{array}{l} I_n\,(12n,0,4,0,0,n)\,2(1) = I_{n-2}\,2(1) + I_2\,2(1) \\ I_n^2\,(12n,0,4,0,0,n)\,2(2) = I_{n-2}^2\,2(2) + I_2^2\,2(2) \end{array} \right\}$$

Therefore, 2(1) and 2(2) diagrams for I_n exist $\forall n$, with the exceptions of I_1 1(1) 1(2) and I_3 1(1) 1(2). Proofs for these exceptions follow.

Case (1A) : Assume (12,0,4,0,0,1) has a coset diagram D which is 2(1) or 2(2)

D has 4 Δ, with four red points, one green 3-cycle and one green 9-cycle. By Lemma 2.3.1, the 3-cycle must be of type (4), as otherwise the diagram could not possibly become 2(1) or 2(2). Let us start with a 3-cycle of type (4), and then build it up. The fourth triangle must join to one of the other three, and, by symmetry, it does not matter which one. Each of the four free vertices must be a red point. The resulting diagram, shown at the beginning of Case (1), is 1(1) 1(2). Contradiction. Hence, no diagram exists for I_1 which is 2(1) or 2(2).

Case (1B) : Assume (36,0,4,0,0,3) has a coset diagram D which is 2(1)

D has 12 Δ, with four red points, three green 3-cycles and three green 9-cycles. We will use the same notation as in Lemma 2.3.1.

A cycle with at least n green lines will be denoted by an n^+-cycle.

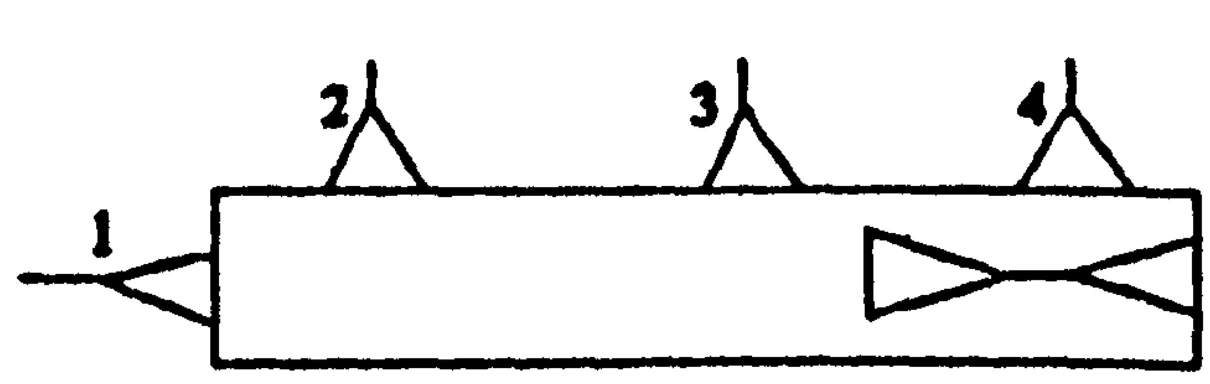
If there is only one incomplete 9-cycle in the construction of a diagram, and we have an m^+-cycle and an n^+-cycle where $m > 3$ and $n > 3$, then both cycles must belong to the same 9-cycle.

If $m + n > 9$, then we must have a contradiction. This might be expressed as

"If $a \to b$, then (7^+-cycle through c , 4^+-cycle through d) $\twoheadrightarrow$ 11^+-cycle",

where a, b, c and d refer to vertices.

D is 2(1), so we can start with a 9-cycle as follows, and then build it up, noting that all 3-cycles are of type (4) by Lemma 2.3.1.

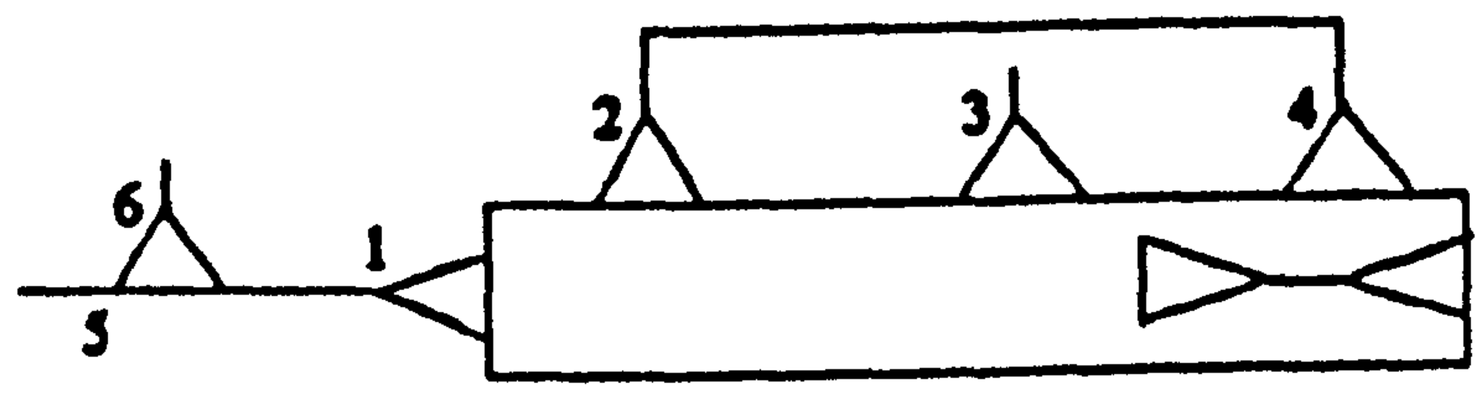


If $2 \to 1$ or $2 \to 3$, then 2-cycle. If $2 \to 2$, then D cannot be 2(1).

Case (1Ba) : Assume 2 → 4.

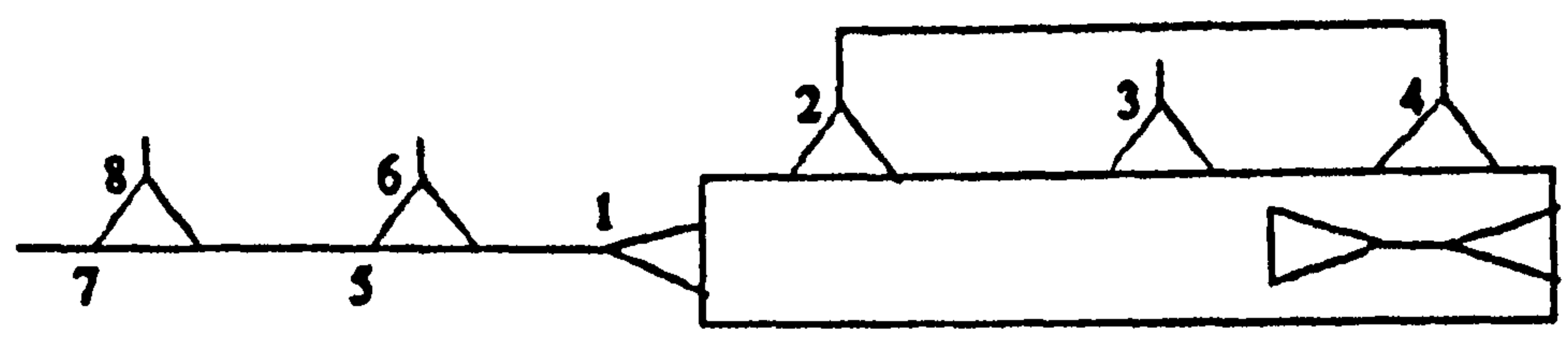
If 1 → 3, then D complete with only 6 Δ. If 1 → 1 (red point), then 5-cycle.

∴ 1 → Δ.



If 5 → 6, then 1-cycle. If 5 → 3, then 12^+-cycle.

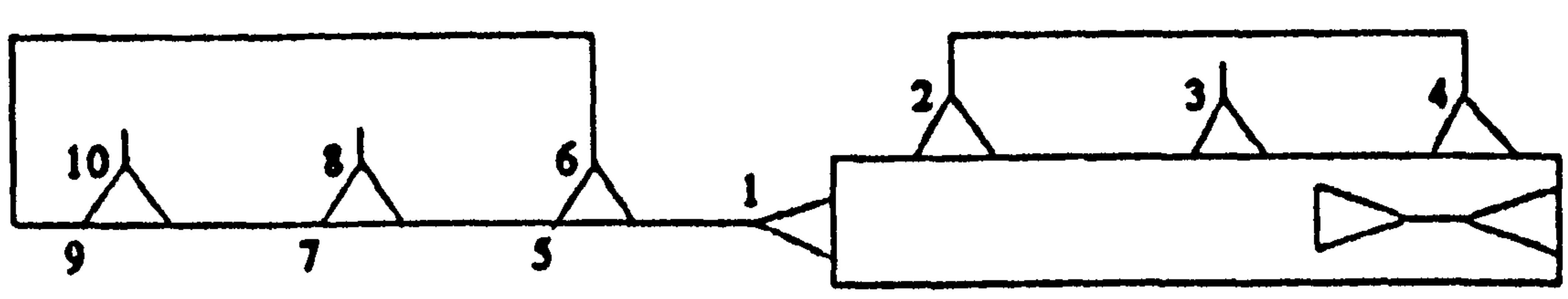
If 5 → 5, then 6 → 6 ⇒ 8-cycle. ∴ 5 → Δ.



If 7 → 8, then 1-cycle. If 7 → 6, then 8-cycle. If 7 → 3, then 13^+-cycle.

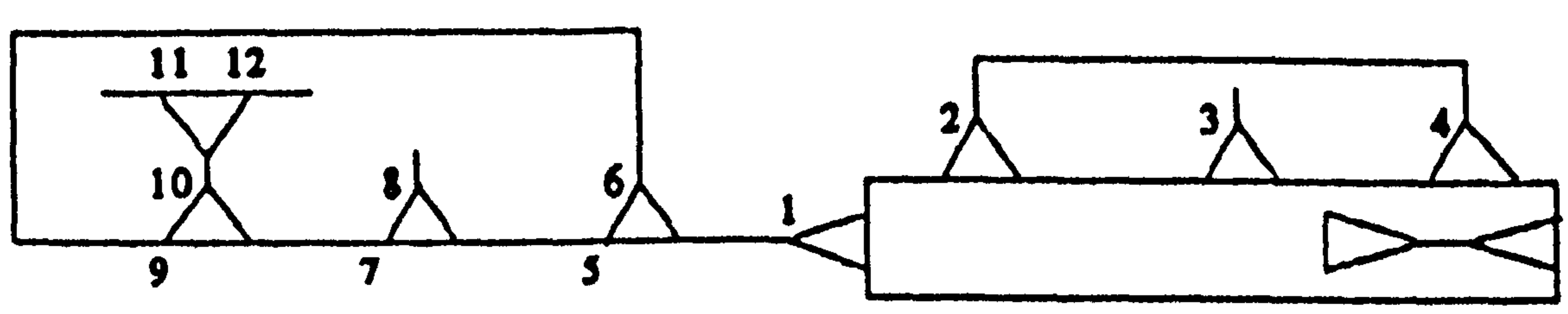
If 7 → 7, then 8 → 6 for 9-cycle, but 2-cycle created.

∴ 7 → Δ → 6 to complete 9-cycle.



If 10 → 3, then 8 → Δ and 11^+-cycle. If 10 → 8, then 2-cycle.

If 10 → 10, then D cannot be 2(1). ∴ 10 → Δ.



If 11 → 3, then 8 → 12 to complete 9-cycle, but D completed with only 10 Δ.

If 11 → 8, then 4-cycle. If 11 → 12, then 1-cycle.

If 11 → 11, then 12 → 12 ⇒ 8^+-cycle through 8, 4^+-cycle through 3 ⇒ 12^+-cycle.

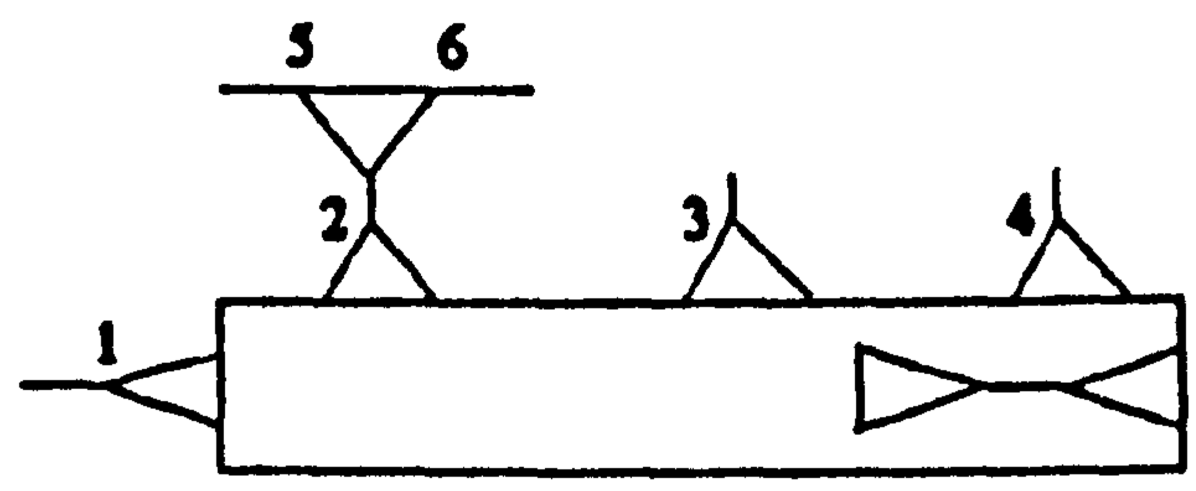
∴ $11 \to \Delta$.

two 9-cycles, 3^+-cycle through 12, 4^+-cycle through 3, 5^+-cycle through 8

$\Rightarrow$ $8 \to 12$ $\Rightarrow$ 7^+-cycle through 8, 4^+-cycle through 3 $\Rightarrow$ 11^+-cycle.

Contradiction. ∴ Case (1Ba) not possible.

Hence, $2 \to \Delta$.

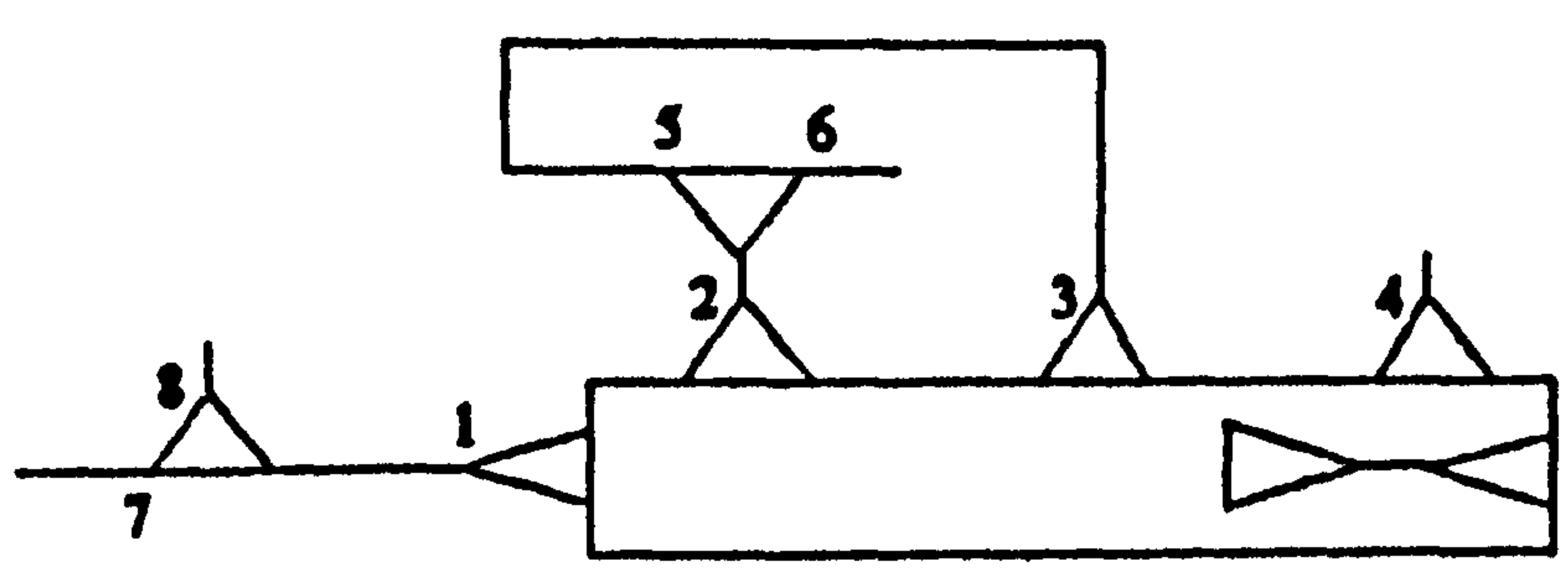


If $3 \to 4$, then 2-cycle. If $3 \to 3$, then D cannot be 2(1).

<u>Case (1Bb) : Assume $3 \to 5$.</u>

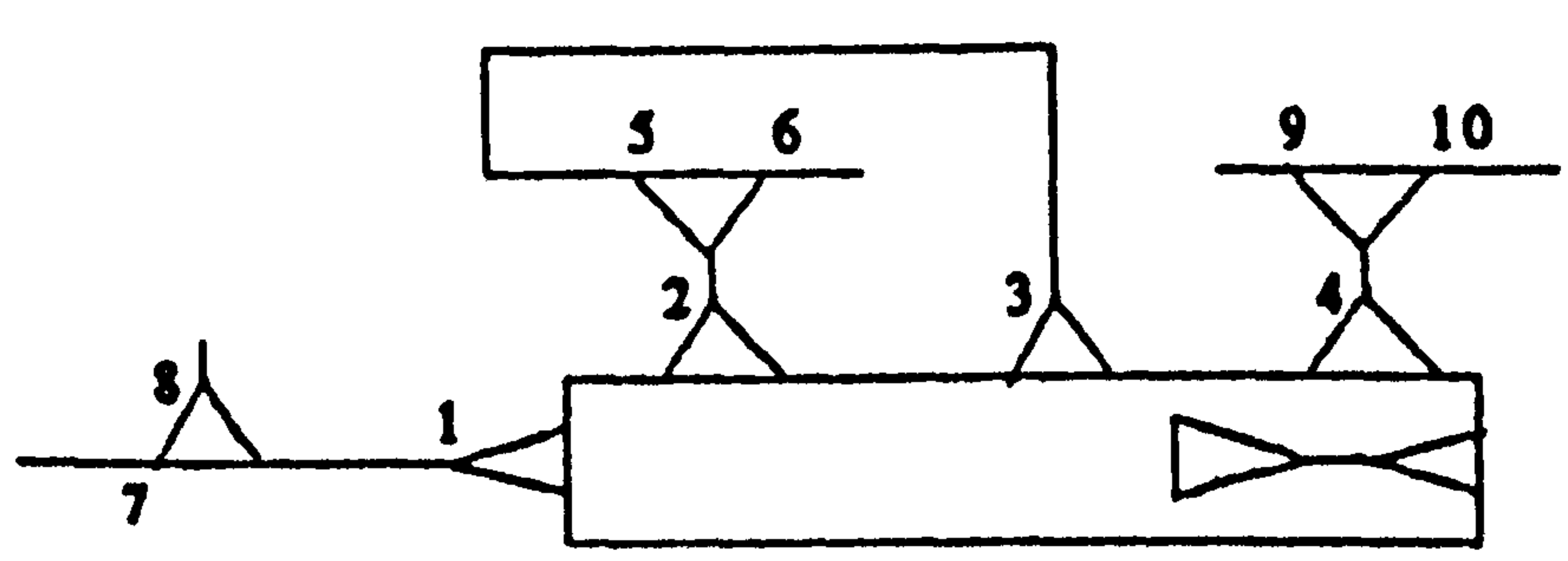
If $1 \to 4$, then 5-cycle. If $1 \to 6$, then 12^+-cycle.

If $1 \to 1$, then D cannot be 2(1). ∴ $1 \to \Delta$.



If $4 \to 4$, then D cannot be 2(1). If $4 \to 6$, then 14^+-cycle.

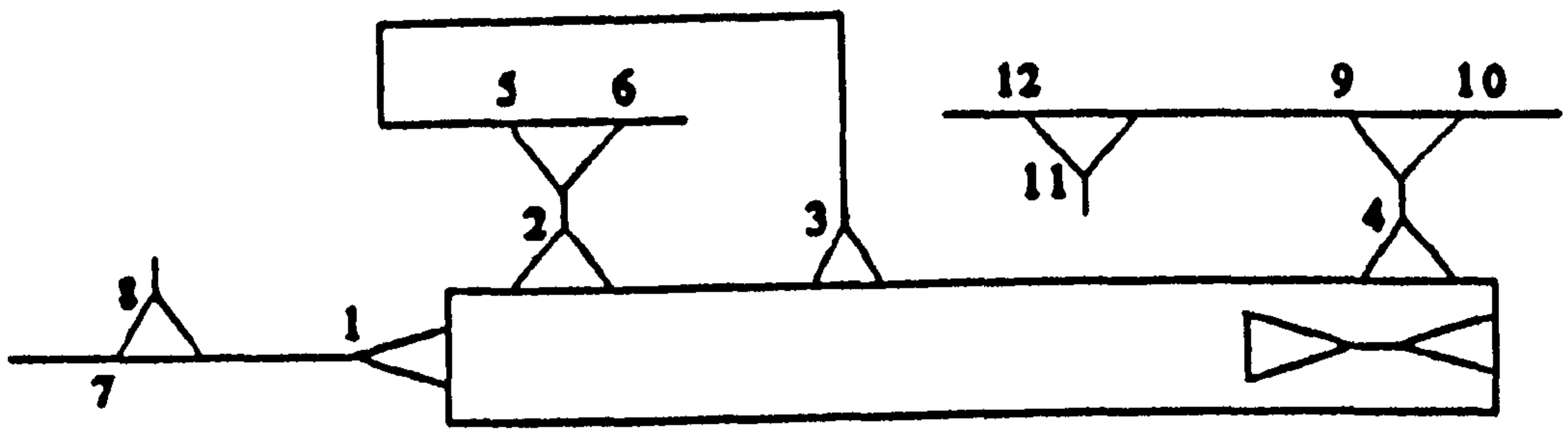
If $4 \to 7$, then 4-cycle. If $4 \to 8$, then 6-cycle. ∴ $4 \to \Delta$.



If $9 \to 8$, then 7-cycle. If $9 \to 10$, then 1-cycle. If $9 \to 6$, then 12^+-cycle.

If $9 \to 9$, then $10 \to 10$ $\Rightarrow$ 13^+-cycle. If $9 \to 7$, then $8 \to \Delta$ $\Rightarrow$ 10^+-cycle.

∴ $9 \to \Delta$.



If $11 \to 11$, then $12 \to 12 \Rightarrow 11^{+}$-cycle. If $11 \to 12$, then 1-cycle.

If $11 \to 8$, then 8-cycle. If $11 \to 6$ or $11 \to 10$, then 13^{+}-cycle.

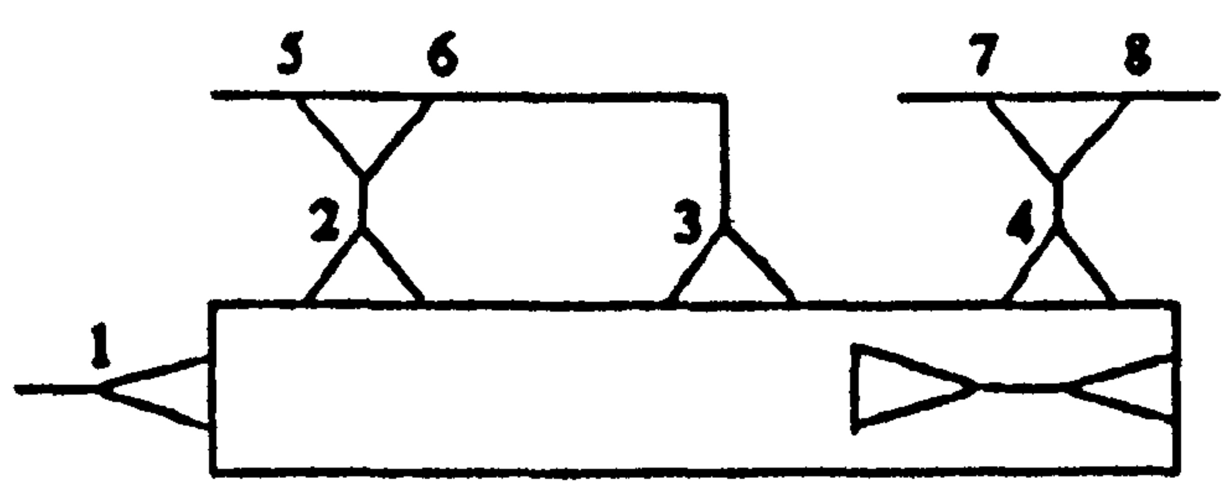
If $11 \to 7$, then $8 \to 8 \Rightarrow$ D is not 2(1).

$\therefore$ $11 \to \Delta \to 8$. $\therefore$ two 9-cycles, 4^{+}-cycle, 5^{+}-cycle $\Rightarrow 6 \to 7 \Rightarrow 11^{+}$-cycle.

Contradiction. $\therefore$ Case (1Bb) not possible.

<u>Case (1Bc) : Assume $3 \to 6$.</u>

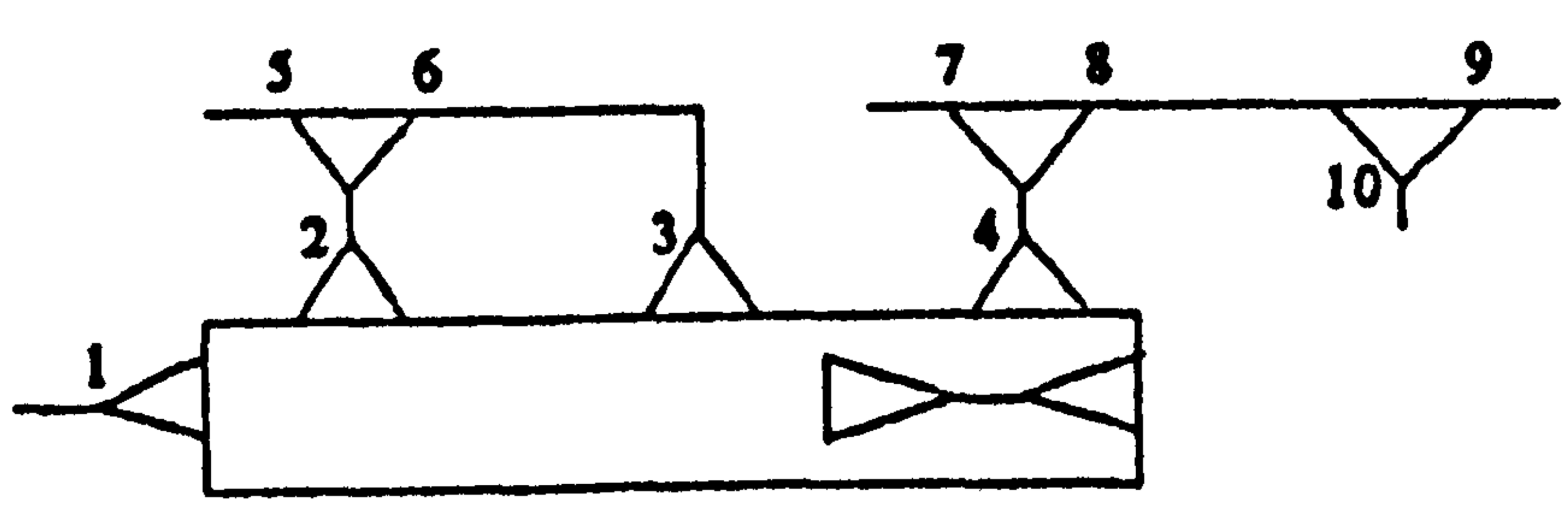
If $4 \to 4$, then D cannot be 2(1). If $4 \to 1$ or $4 \to 5$, then 2-cycle. $\therefore$ $4 \to \Delta$.



If $8 \to 1$, then 4-cycle. If $8 \to 7$, then 1-cycle.

If $8 \to 8$, then $7 \to 7 \Rightarrow$ D 2(1), $1 \to 5 \Rightarrow$ 9-cycle $\Rightarrow$ D completed with only 8 Δ.

If $8 \to 5$, then $1 \to \Delta \Rightarrow$ 1-cycle. $\therefore$ $8 \to \Delta$.

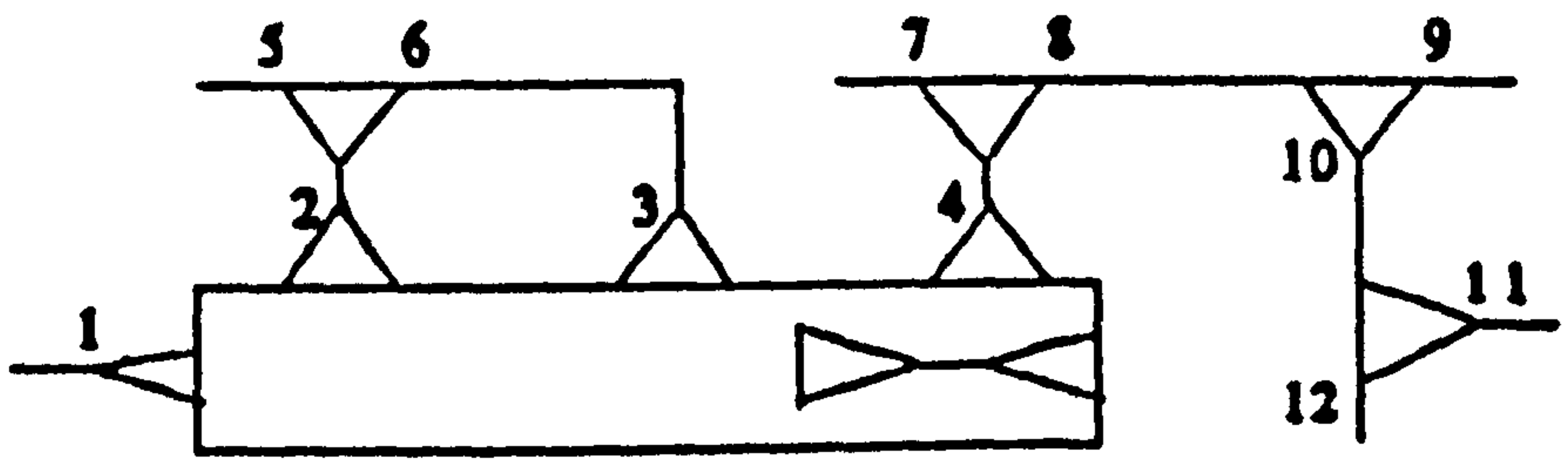


If $10 \to 9$, then 1-cycle. If $10 \to 1$, then 5-cycle.

If $10 \to 10$, then $9 \to 9$, $1 \to \Delta \to 7 \Rightarrow 5 \to \Delta \Rightarrow 5^{+}$-cycle, 6^{+}-cycle $\Rightarrow 11^{+}$-cycle.

If $10 \to 5$, then $1 \to \Delta \Rightarrow 10^{+}$-cycle.

If $10 \to 7$, then $1 \to 5 \Rightarrow$ 9-cycle $\Rightarrow$ D completed with only 9 Δ. $\therefore$ $10 \to \Delta$.

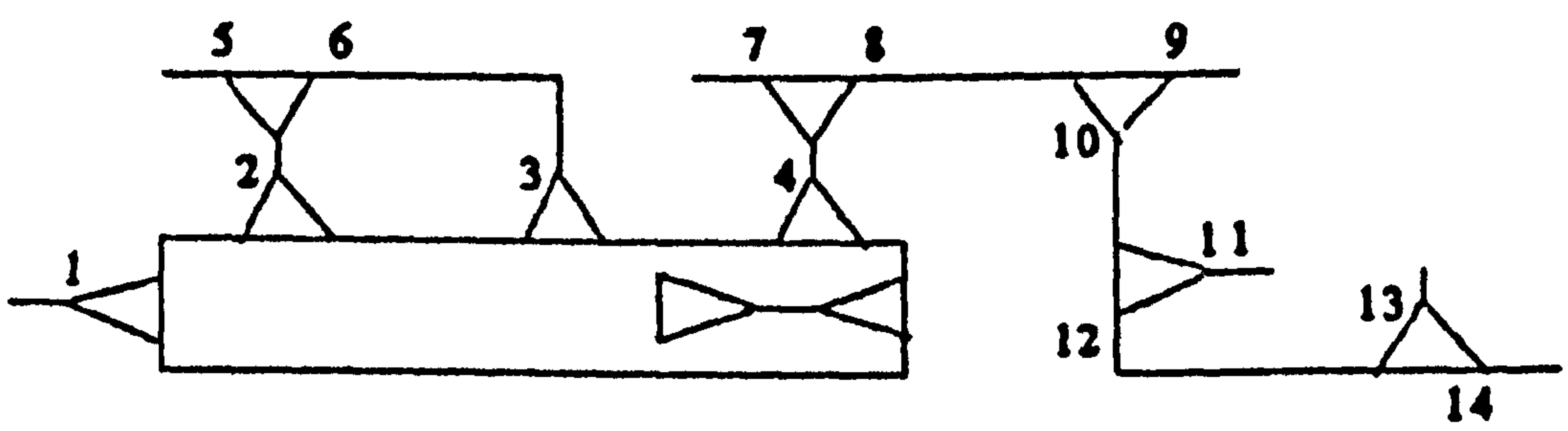


If 12 → 1, then 6-cycle. If 12 → 11, then 1-cycle.

If 12 → 5, then 1 → 1 ⇒ D cannot be 2(1). If 12 → 7, then 10^+-cycle.

If 12 → 9, then 1 → Δ → 7 ⇒ 5 → Δ ⇒ 5^+-cycle, 6^+-cycle ⇒ 11^+-cycle.

If 12 → 12, then 11 → 11 ⇒ D 2(1) ⇒ 1 → 9 ⇒ 5 → Δ ⇒ 11^+-cycle. ∴ 12 → Δ.



If 14 → 1, then 7-cycle. If 14 → 5, then 10^+-cycle.

If 14 → 7, then 11^+-cycle. If 14 → 13, then 1-cycle.

If 14 → 9, then 1 → 7 ⇒ 5 → Δ ⇒ 2-cycle.

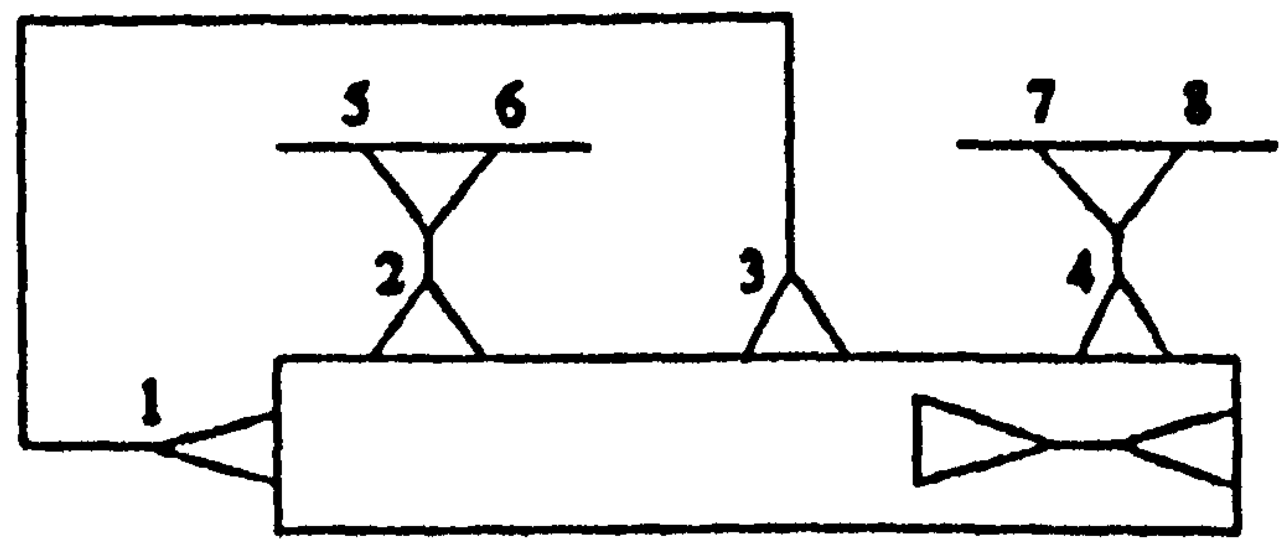
If 14 → 11, then 1 → 9 ⇒ 5 → Δ ⇒ 5^+-cycle, 6^+-cycle ⇒ 11^+-cycle.

If 14 → 14, then 13 → 13 ⇒ D 2(1) ⇒ 10^+-cycle.

∴ 14 → Δ. This new triangle must contain the other two red points to make D 2(1). This creates 11^+-cycle. Contradiction. ∴ Case (1Bc) not possible.

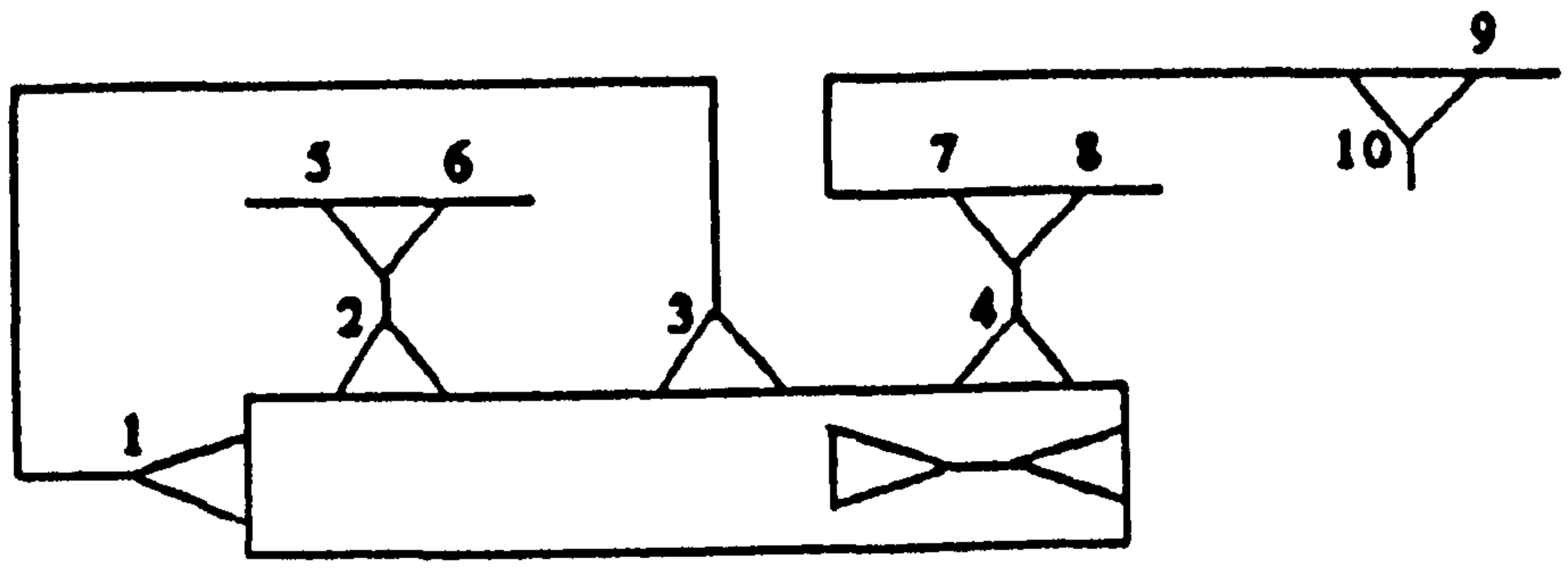
Case (1Bd) : Assume 3 → 1.

If 4 → 4 then D cannot be 2(1). If 4 → 5 or 4 → 6, then 12^+-cycle. ∴ 4 → Δ.



If 7 → 5, then 8 → Δ → 6 ⇒ 10^+-cycle. If 7 → 6, then 13^+-cycle.

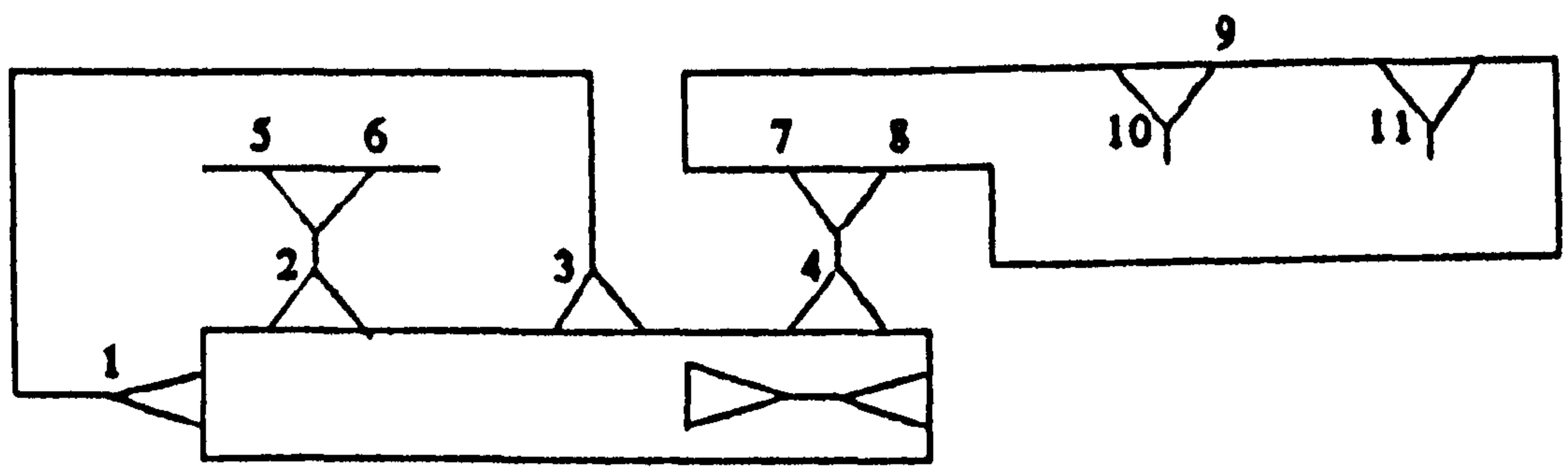
If 7→7, then 8→8 ⇒ D 2(1) ⇒ 8-cycle. If 7→8, then 1-cycle. ∴ 7→Δ.



If 9→5, then 6→8 ⇒ 10→Δ ⇒ 11^+-cycle. If 9→6, then 14^+-cycle.

If 9→8, then 8-cycle. If 9→9, then 10→10 ⇒ D 2(1) ⇒ 11^+-cycle.

If 9→10, then 1-cycle. ∴ 9→Δ→8, to complete 9-cycle.



If 10→5, then 6→11 ⇒ D completed with only 10Δ.

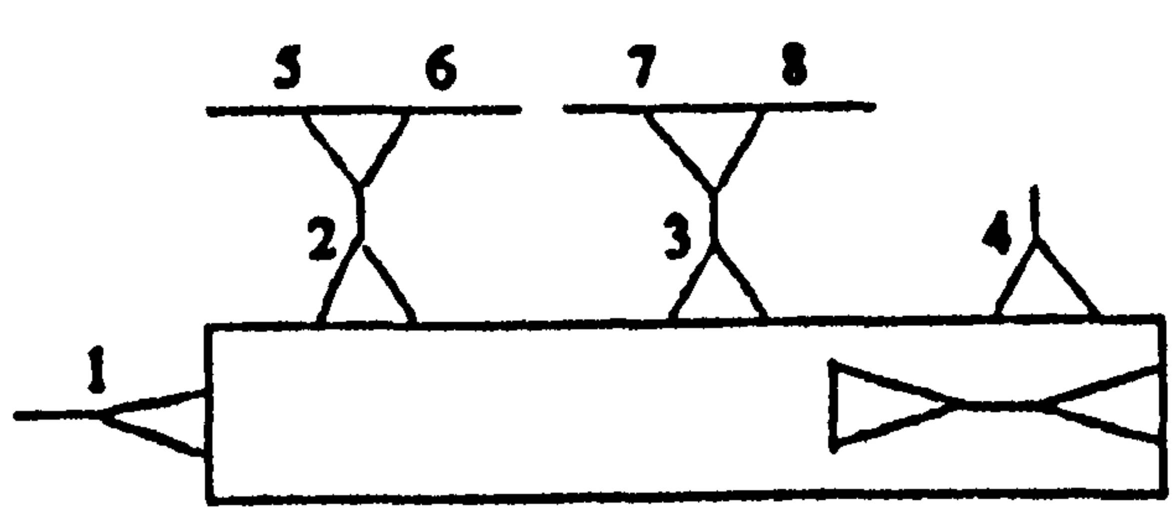
If 10→6, then 4^+-cycle, 8^+-cycle ⇒ 12^+-cycle.

If 10→10 then D cannot be 2(1). If 10→11, then 2-cycle.

∴ 10→Δ. ∴ 4^+-cycle, 6^+-cycle ⇒ 10^+-cycle.

Contradiction. ∴ Case (1Bd) not possible.

Hence, 3→Δ.



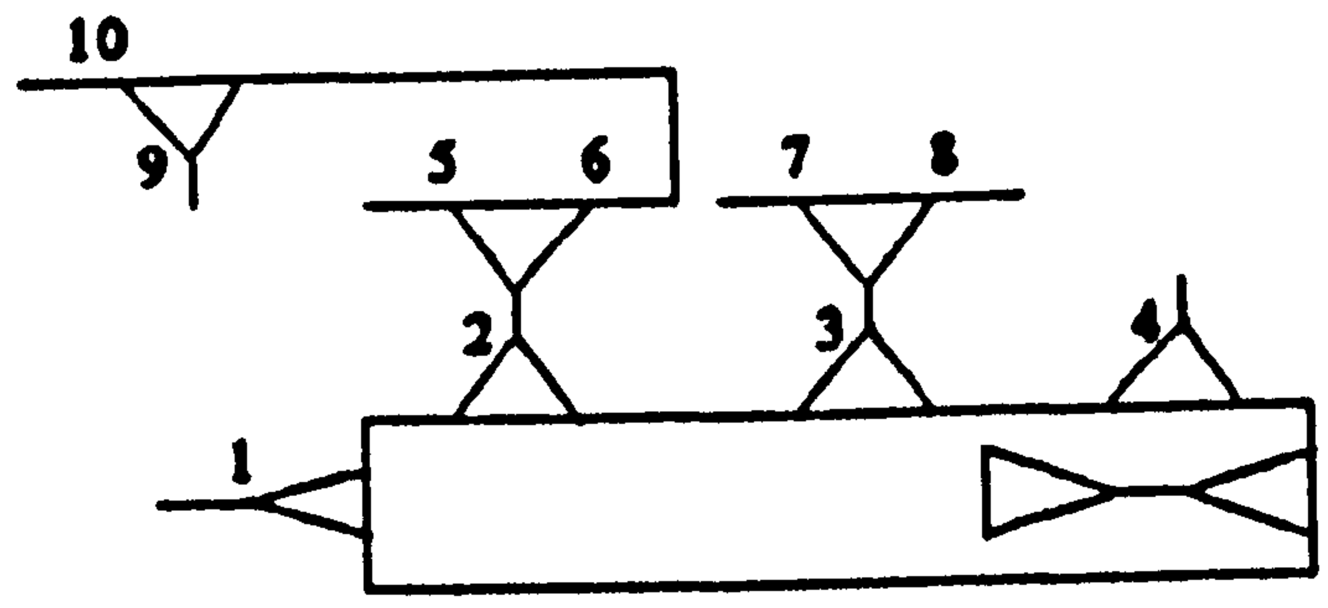
If 6→1, then 8→Δ ⇒ 4→Δ→Δ→7 ⇒ 5→Δ ⇒ 5^+-cycle, 6^+-cycle ⇒ 11^+-cycle.

If 6→6, then 5→5 ⇒ 7→Δ→1 ⇒ 8→Δ ⇒ 4→Δ ⇒ two 5^+-cycles ⇒ 10^+-cycle.

If 6→8, then 7→Δ ⇒ 5→Δ ⇒ 4→Δ ⇒ 4^+-cycle, 6^+-cycle ⇒ 10^+-cycle.

If 6→4, then 7→Δ→Δ→8 ⇒ 1→Δ ⇒ 5→Δ ⇒ 5^+-cycle, 6^+-cycle ⇒ 11^+-cycle.

If 6→7, then 4-cycle. If 6→5, then 1-cycle. ∴ 6→Δ.



If $10 \to 1$, then $7 \to \Delta \to 4 \Rightarrow 5 \to \Delta \Rightarrow 8 \to \Delta \Rightarrow$ two 5^+-cycles $\Rightarrow 10^+$-cycle.

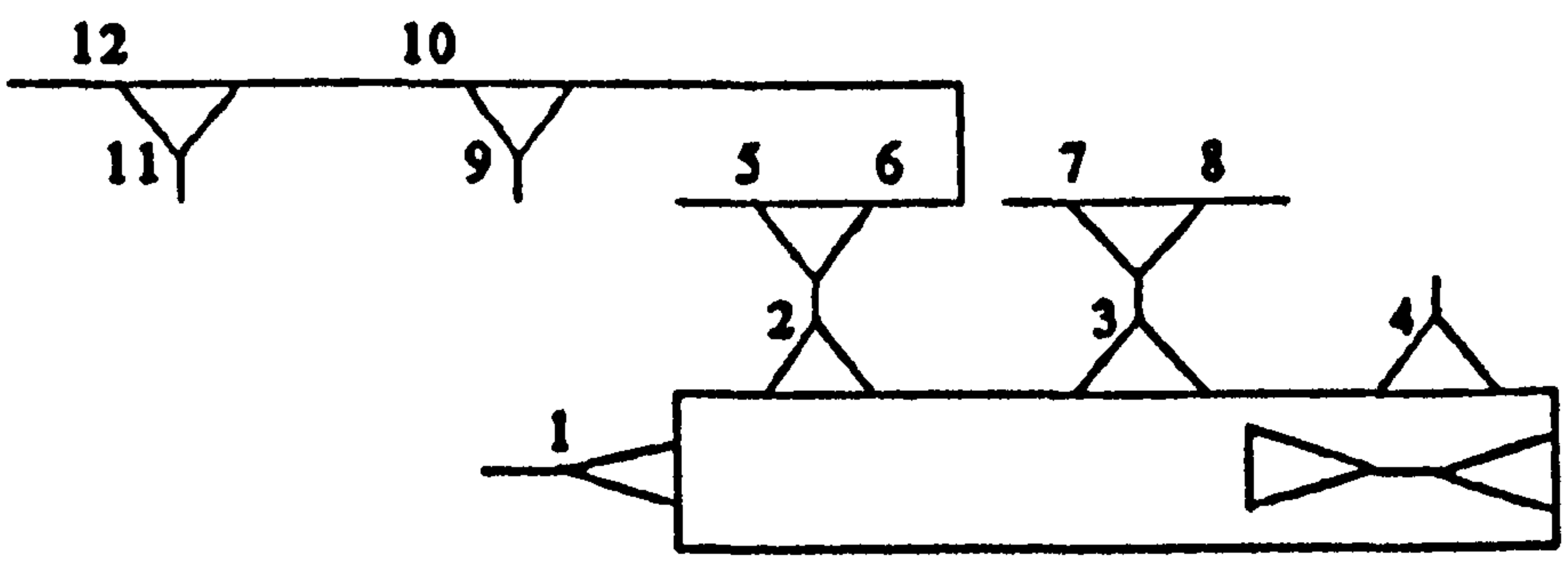
If $10 \to 4$, then $7 \to \Delta \to 8 \Rightarrow 9 \to \Delta \Rightarrow 1 \to \Delta \Rightarrow 4^+$-cycle, 6^+-cycle $\Rightarrow 10^+$-cycle.

If $10 \to 5$, then $7 \to \Delta \to 1 \Rightarrow 4 \to \Delta \Rightarrow 4^+$-cycle, 5^+-cycle $\Rightarrow$ 9-cycle $\Rightarrow$ 2-cycle.

If $10 \to 8$, then $9 \to \Delta \Rightarrow 4 \to \Delta \Rightarrow 4^+$-cycle, 6^+-cycle $\Rightarrow 10^+$-cycle.

If $10 \to 10$, then $9 \to 9$, $7 \to \Delta \to 5 \Rightarrow 1 \to \Delta \Rightarrow 4 \to \Delta \Rightarrow$ two 5^+-cycles $\Rightarrow 10^+$-cycle.

If $10 \to 7$, then 5-cycle. If $10 \to 9$, then 1-cycle. $\therefore$ $10 \to \Delta$.



If $7 \to 1$, then $5 \to 12 \Rightarrow 9 \to \Delta \Rightarrow 4 \to \Delta \Rightarrow$ two 4^+-cycles, 5^+-cycle $\Rightarrow 13^+$-cycle.

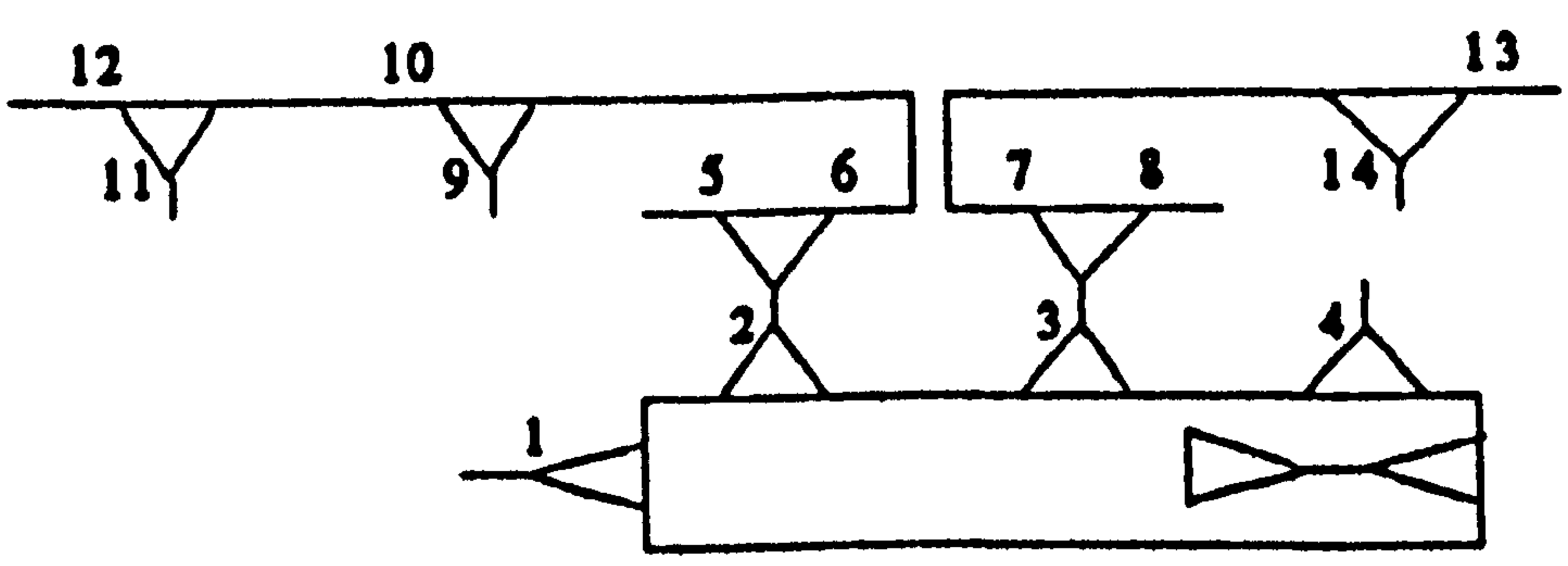
If $7 \to 4$, then $1 \to 12 \Rightarrow 5 \to \Delta \Rightarrow 11 \to \Delta \Rightarrow 4^+$-cycle, 6^+-cycle $\Rightarrow 10^+$-cycle.

If $7 \to 5$, then $9 \to \Delta \to 12 \Rightarrow 1 \to \Delta$ with two red points $\Rightarrow 10^+$-cycle.

If $7 \to 7$, then $8 \to 8 \Rightarrow 10^+$-cycle. If $7 \to 8$, then 1-cycle.

If $7 \to 9$, then $11 \to \Delta \to 12 \Rightarrow 1 \to \Delta$ with two red points $\Rightarrow 4 \to 5 \Rightarrow$ 6-cycle.

If $7 \to 11$, then $12 \to 9 \Rightarrow 12^+$-cycle. If $7 \to 12$, then 6-cycle. $\therefore$ $7 \to \Delta$.

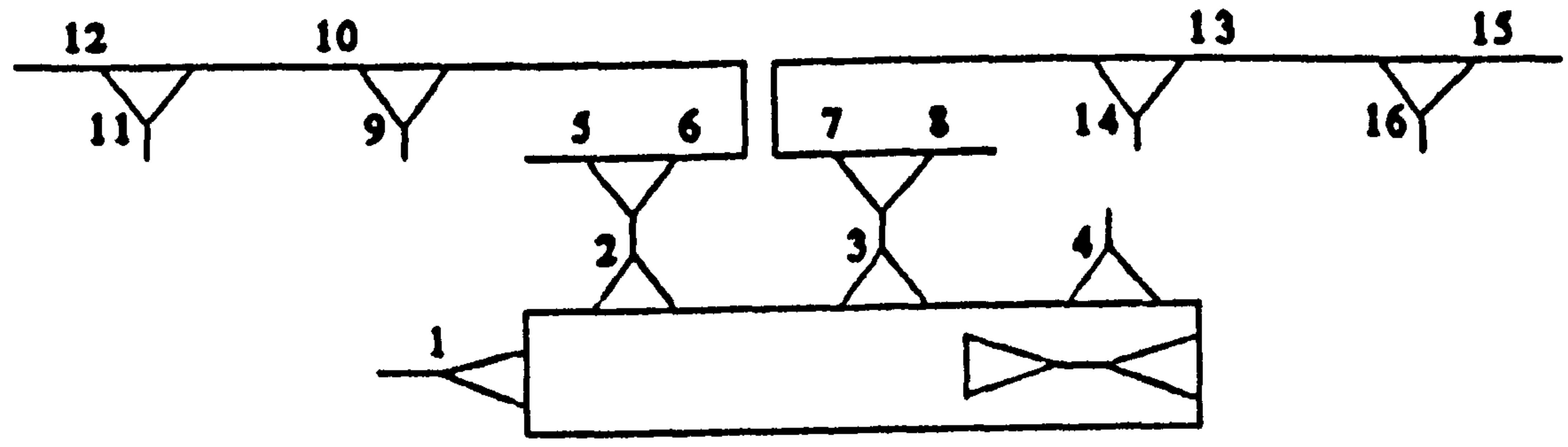


If $13 \to 1$ or $13 \to 4$ or $13 \to 8$ or $13 \to 11$, then 10^+-cycle.

If $13 \to 5$, then $9 \to 12 \Rightarrow 14 \to \Delta$,2 red points $\Rightarrow 1 \to 8 \Rightarrow 4 \to 11 \Rightarrow$ four 9-cycles.

If 13 → 9, then 11 → 12 ⇒ 1-cycle. If 13 → 12, then 7-cycle.

If 13 → 13, then 14 → 14 ⇒ 10^+-cycle. If 13 → 14, then 1-cycle. ∴ 13 → Δ.



To make D 2(1) we must have 11 → 11 or 15 → 15.

If 11 → 11, then 12 → 12 ⇒ 11^+-cycle. If 15 → 15, then 16 → 16 ⇒ 11^+-cycle.

Contradiction. Hence (36,0,4,0,0,3) does not have a diagram which is 2(1).

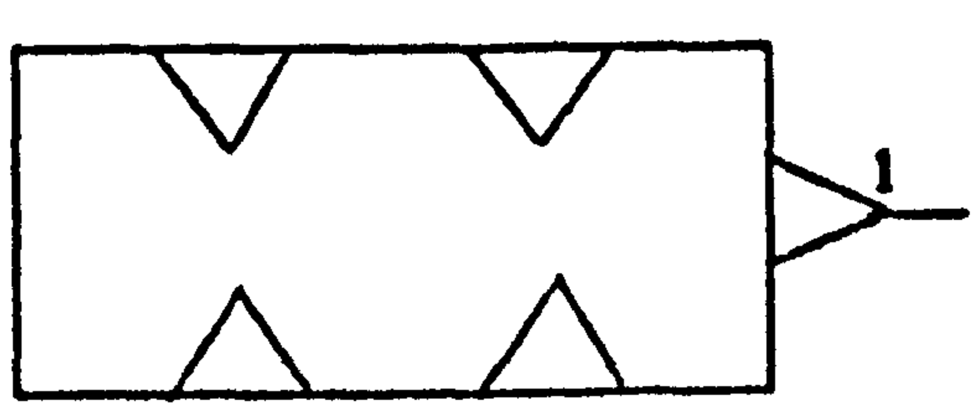
<u>Case (1C) : Assume (36,0,4,0,0,3) has a coset diagram D which is 2(2)</u>

D has 12Δ, with four red points, three green 3-cycles and three green 9-cycles. We will use the same notation as in Case (1B).

D is 2(2), so either all four red points will be in the same 9-cycle or two 9-cycles will each contain two red points.

<u>Case (1C1) : Assume one 9-cycle contains all four red points</u>

We can start with a 9-cycle as follows, and then build it up noting that all 3-cycles are of type (4) by Lemma 2.3.1.



To save space, only one more diagram will be drawn. This next diagram can be referred to for vertex labels mentioned up until then.

1 → Δ. If 2 → 3, then 1-cycle. ∴ 2 → Δ. ∴ 3 → 4. ∴ 5 → Δ.

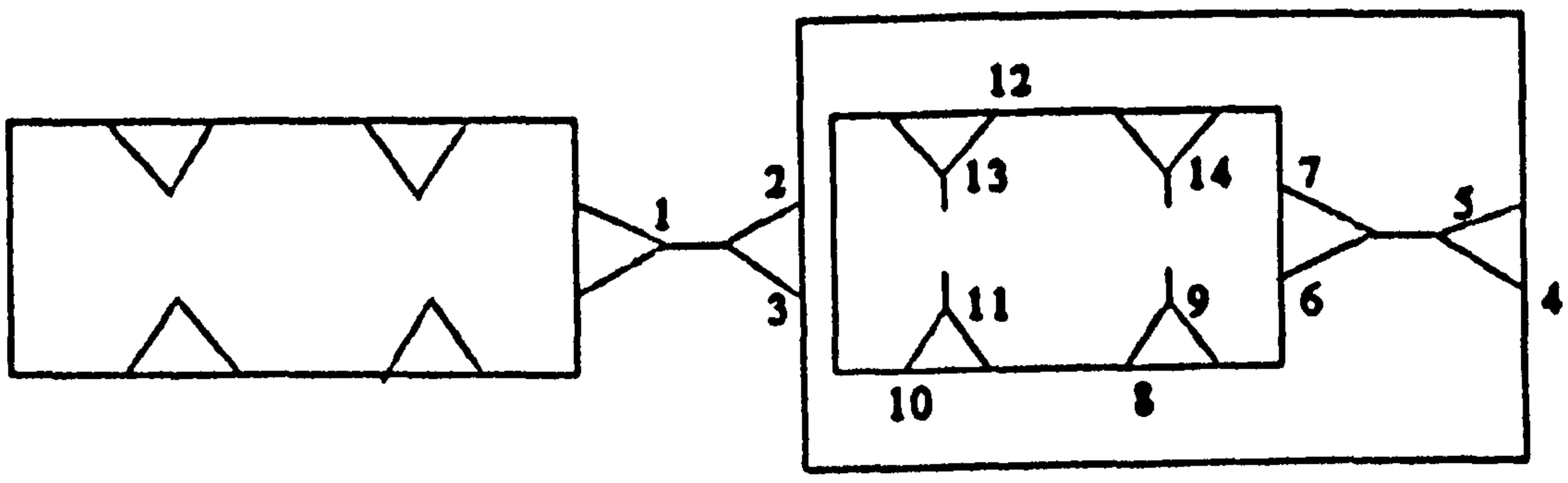
If 6 → 7, then 1-cycle. ∴ 6 → Δ. If 8 → 7, then 6-cycle.

If 8 → 9, then 1-cycle. ∴ 8 → Δ. If 10 → 7, then 7-cycle.

If $10 \to 9$, then $7 \to 7$ ⇒ five red points. If $10 \to 11$, then 1-cycle. ∴ $10 \to \Delta$.

If $12 \to 7$, then 8-cycle. If $12 \to 9$, then 10^+-cycle.

If $12 \to 11$, then 10^+-cycle. If $12 \to 13$, then 1-cycle. ∴ $12 \to \Delta \to 7$.



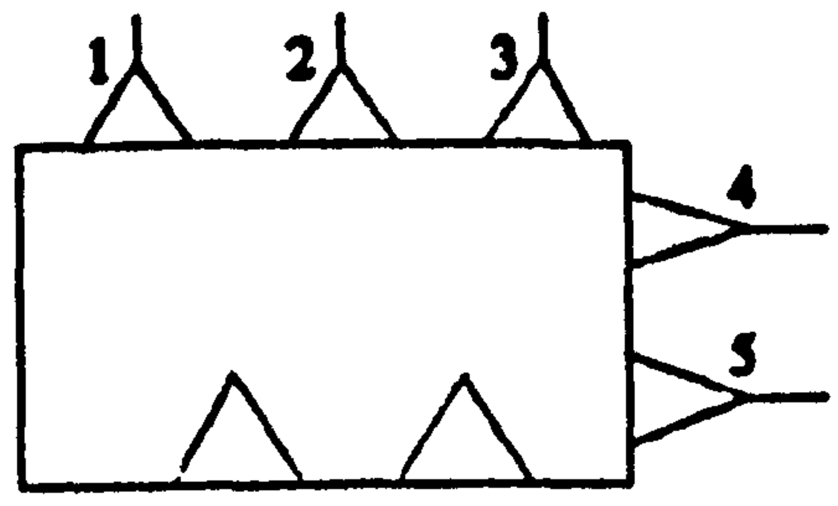
If $9 \to 11$, then 2-cycle. If $9 \to 13$, then $11 \to 14$ ⇒ four 9-cycles.

If $9 \to 14$, then $11 \to 13$ ⇒ 2-cycle.

Contradiction. Hence, Case (1C1) not possible.

Case (1C2) : Assume one 9-cycle contains exactly two red points

We can start with a 9-cycle as follows, and then build it up noting that all 3-cycles are of type (4) by Lemma 2.3.1.

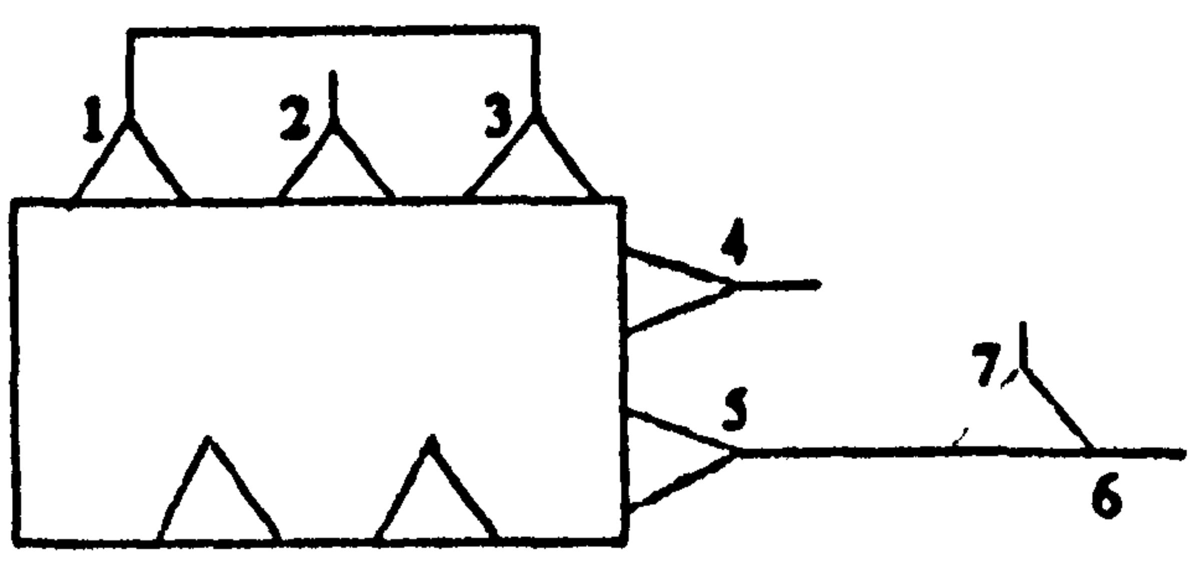


If $1 \to 2$, then 2-cycle. If $1 \to 5$, then 4-cycle.

Case (1C2a) : Assume $1 \to 3$.

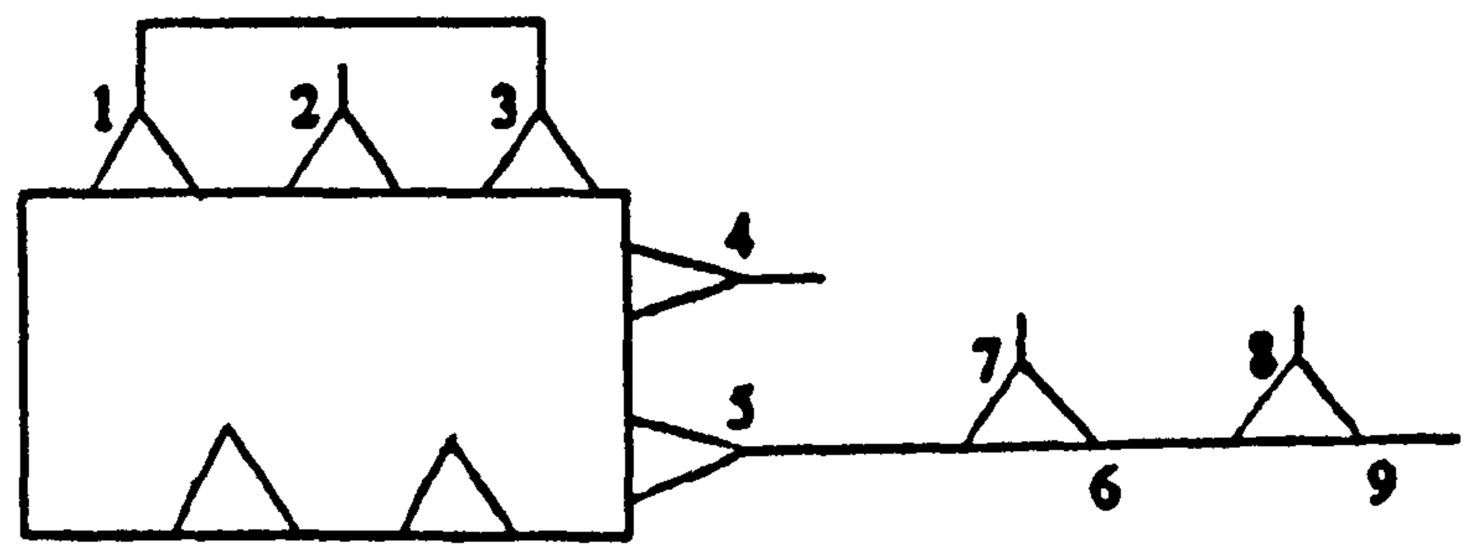
If $5 \to 2$, then 12^+-cycle. If $5 \to 4$, then 2-cycle.

If $5 \to 5$, then $4 \to 4$ ⇒ 8-cycle. ∴ $5 \to \Delta$.



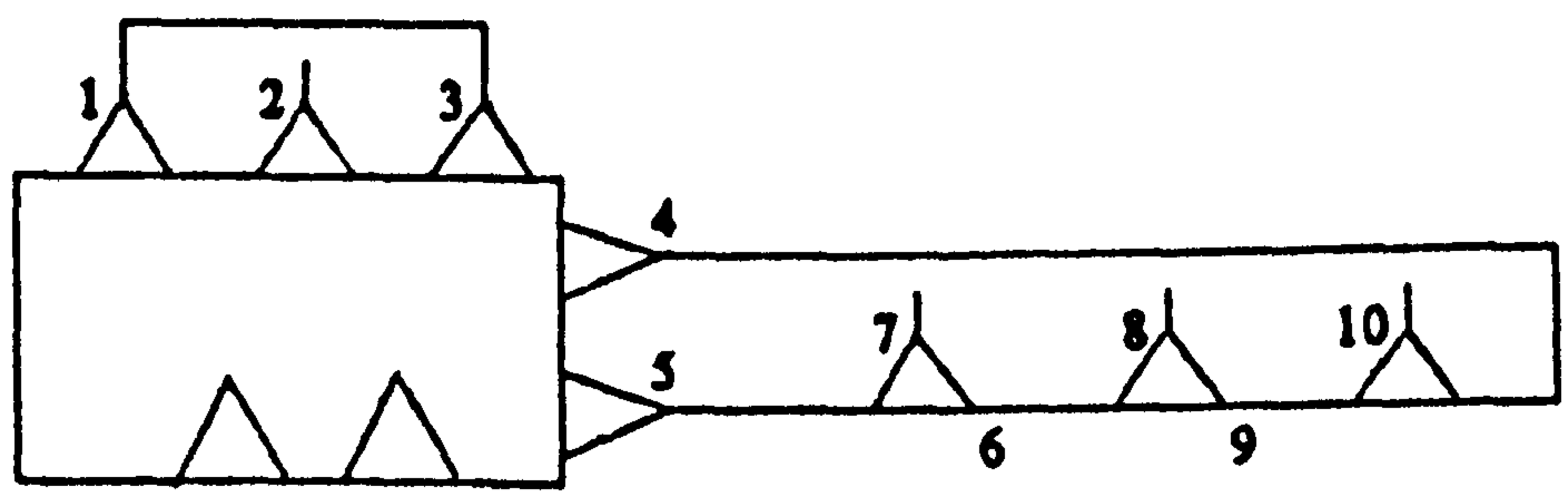
If $6 \to 2$, then 12^+-cycle. If $6 \to 4$, then 7-cycle. If $6 \to 7$, then 1-cycle.

If $6 \to 6$, then D cannot be 2(2). $\therefore$ $6 \to \Delta$.



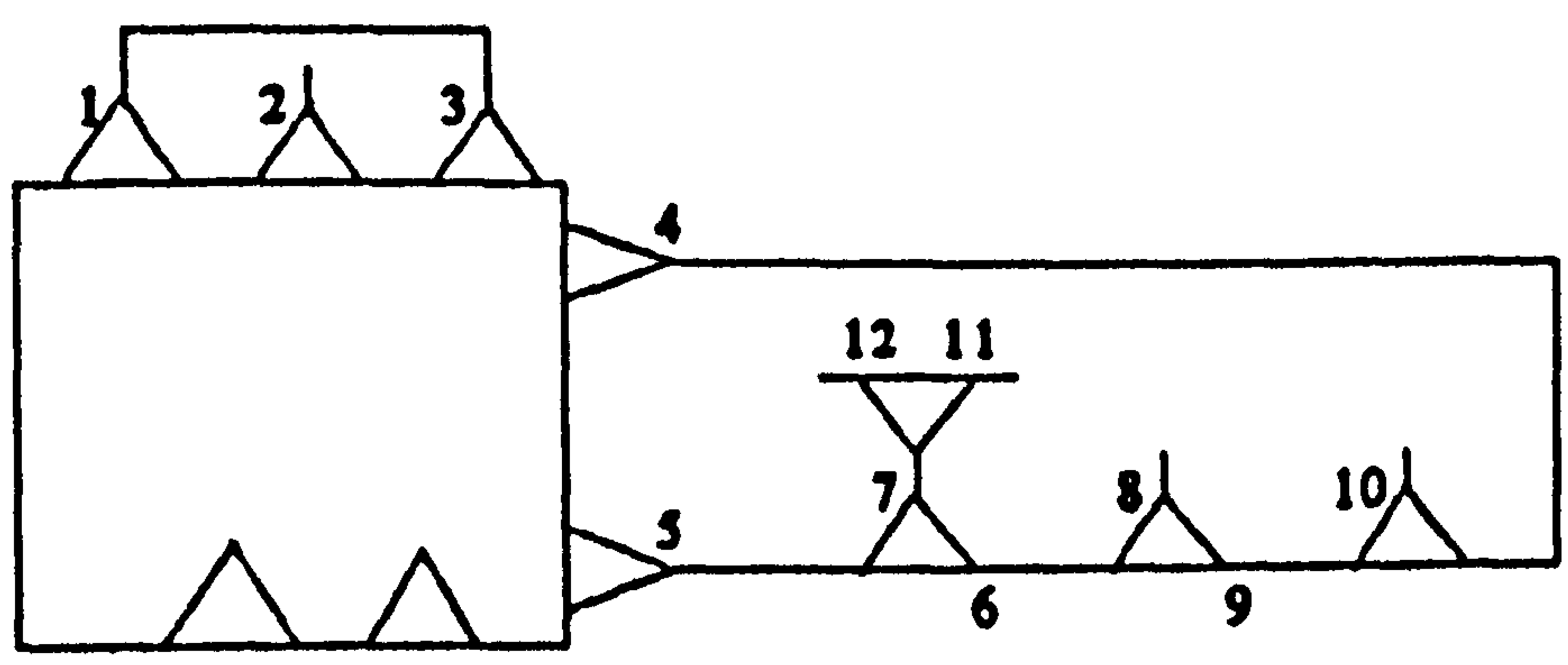
If $9 \to 2$, then 13^+-cycle. If $9 \to 4$, then 8-cycle. If $9 \to 7$, then 11^+-cycle.

If $9 \to 8$, then 1-cycle. If $9 \to 9$, then D cannot be 2(2). $\therefore$ $9 \to \Delta \to 4$.



If $7 \to 2$, then 10^+-cycle. If $7 \to 8$, then 2-cycle. If $7 \to 10$, then 4-cycle.

If $7 \to 7$, then $8 \to 8 \Rightarrow 10 \to \Delta \Rightarrow 10^+$-cycle. $\therefore$ $7 \to \Delta$.



Two 9-cycles, 5^+-cycle, 4^+-cycle $\Rightarrow 2 \to 12 \Rightarrow 10^+$-cycle.

Contradiction. $\therefore$ Case (1C2a) not possible.

<u>Case (1C2b) : Assume $1 \to 1$.</u>

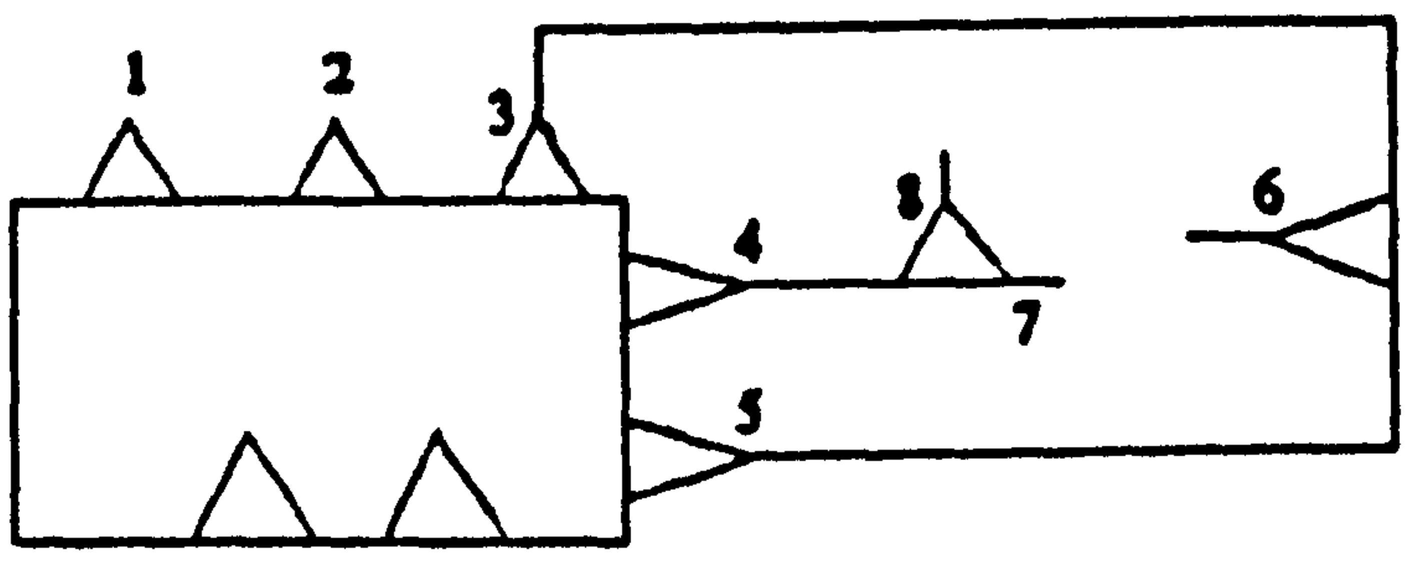
$2 \to 2$, to make D 2(2). If $3 \to 4$, then 2-cycle. If $3 \to 5$, then 8-cycle.

$\therefore$ $3 \to \Delta \to 5$. If $4 \to 6$, then D complete with only 8 Δ. $\therefore$ $4 \to \Delta$.

(See next diagram). If $6 \to 7$ or $6 \to 8$, then 4-cycle.

$\therefore$ $6 \to \Delta$. $\therefore$ two 5^+-cycles $\Rightarrow 10^+$-cycle.

Contradiction. $\therefore$ Case (1C2b) not possible.



<u>Case (1C2c) : Assume 1 → 4.</u>

If 5 → 2 or 5 → 3, then 12^+-cycle. If 5 → 5, then D cannot be 2(2). ∴ 5 → Δ.

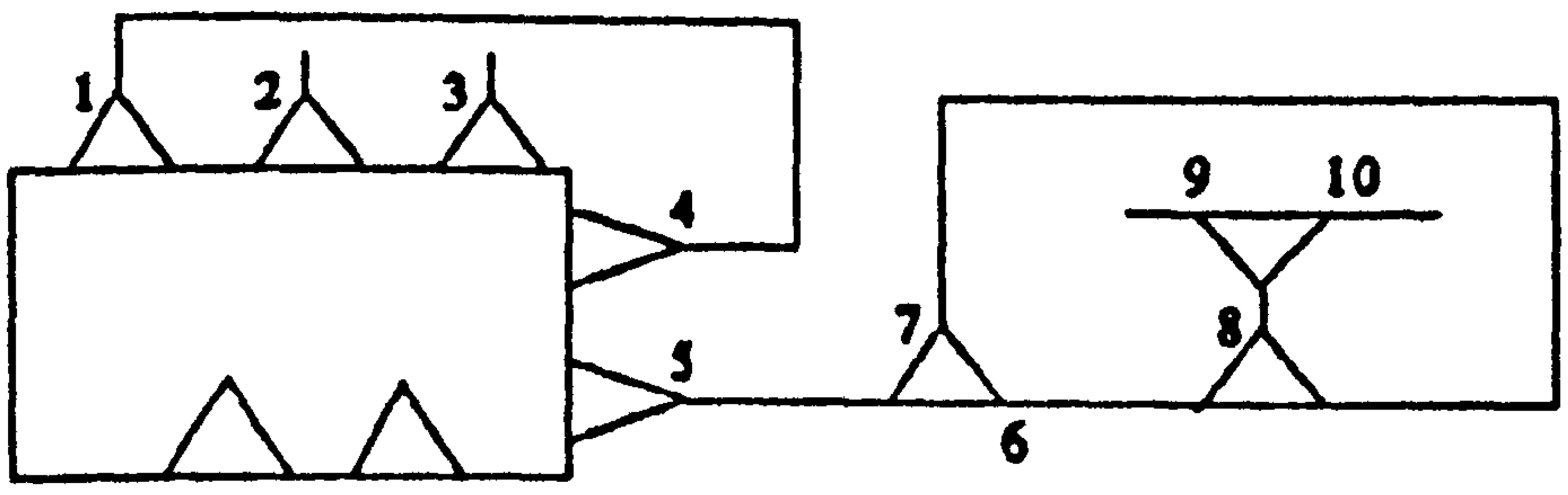
If 6 → 2, then 12^+-cycle. If 6 → 3, then 10^+-cycle. (See next diagram).

If 6 → 6, then D cannot be 2(2). If 6 → 7, then 1-cycle. ∴ 6 → Δ → 7.

If 8 → 2, then 3 → 3 ⇒ D complete with only 9 Δ.

If 8 → 3, then 2 → 2 ⇒ D complete with only 9 Δ.

If 8 → 8, then D cannot be 2(2). ∴ 8 → Δ.

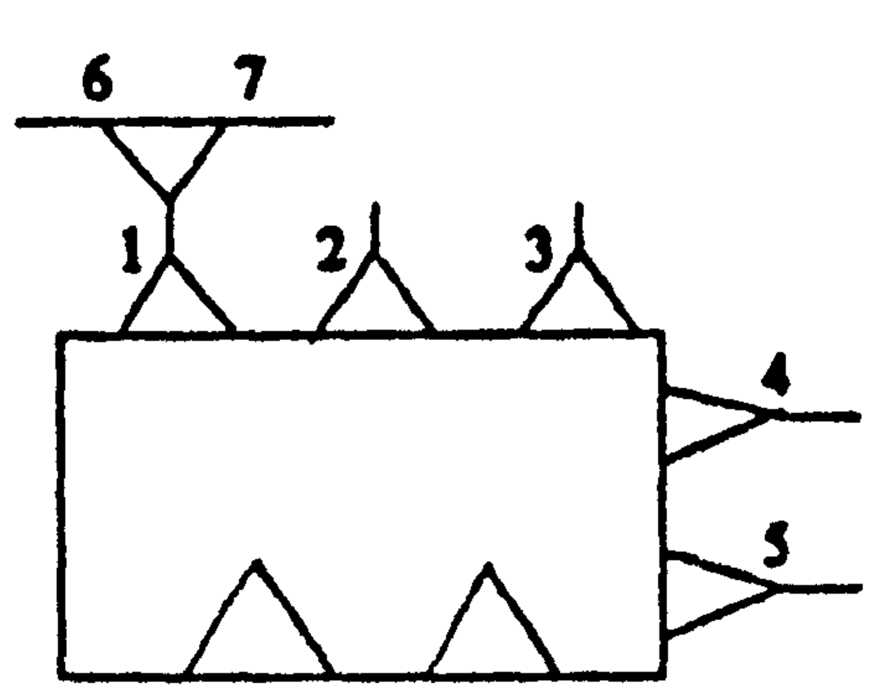


∴ two 9-cycles, 5^+-cycle through 10, 4^+-cycle through 3.

∴ 2 → 10 and 3 → 9 ⇒ 9-cycle. ∴ D complete with only 10 Δ.

Contradiction. ∴ Case (1C2c) not possible.

Hence, 1 → Δ.



If 6 → 7, then 1-cycle. If 6 → 5, then 5-cycle.

Case (1C2d) : Assume $6 \to 2$.

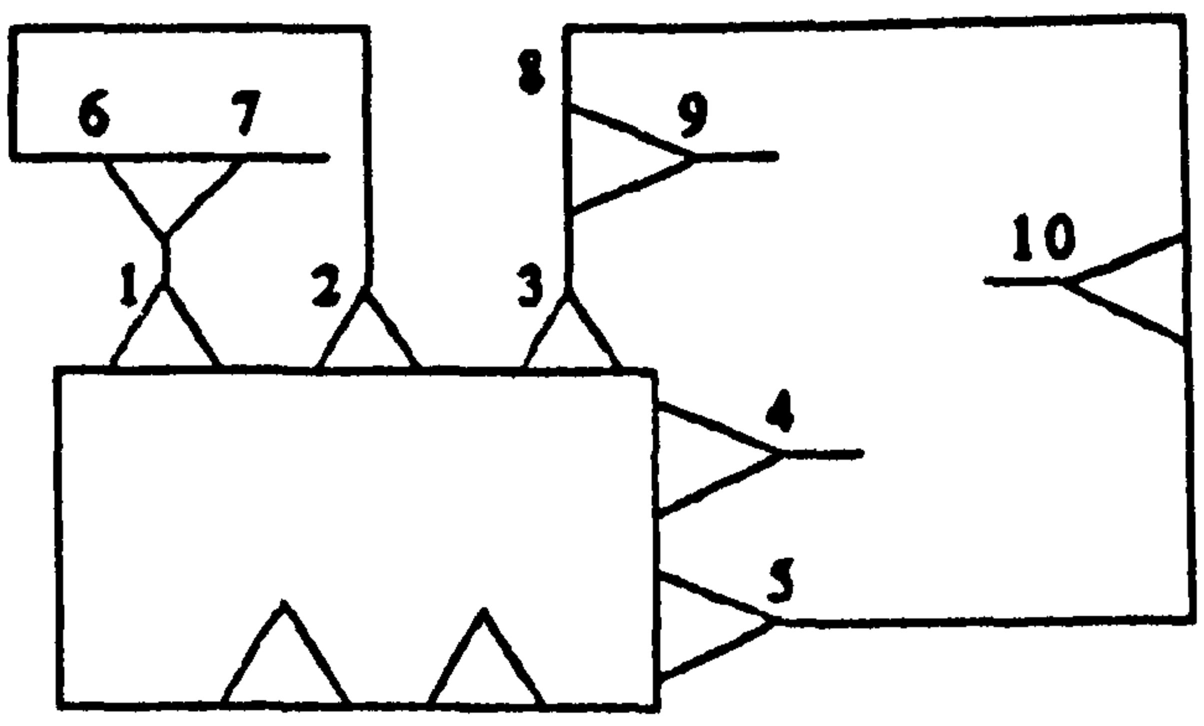
If $3 \to 4$, then 2-cycle. If $3 \to 5$, then 7-cycle. If $3 \to 7$, then 13^+-cycle.

If $3 \to 3$, then $4 \to 4$ ⇒ 11^+-cycle. ∴ $3 \to \Delta$. (See next diagram).

If $8 \to 8$, then D cannot be 2(2). If $8 \to 9$, then 1-cycle.

If $8 \to 7$, then 13^+-cycle. If $8 \to 4$, then 10^+-cycle.

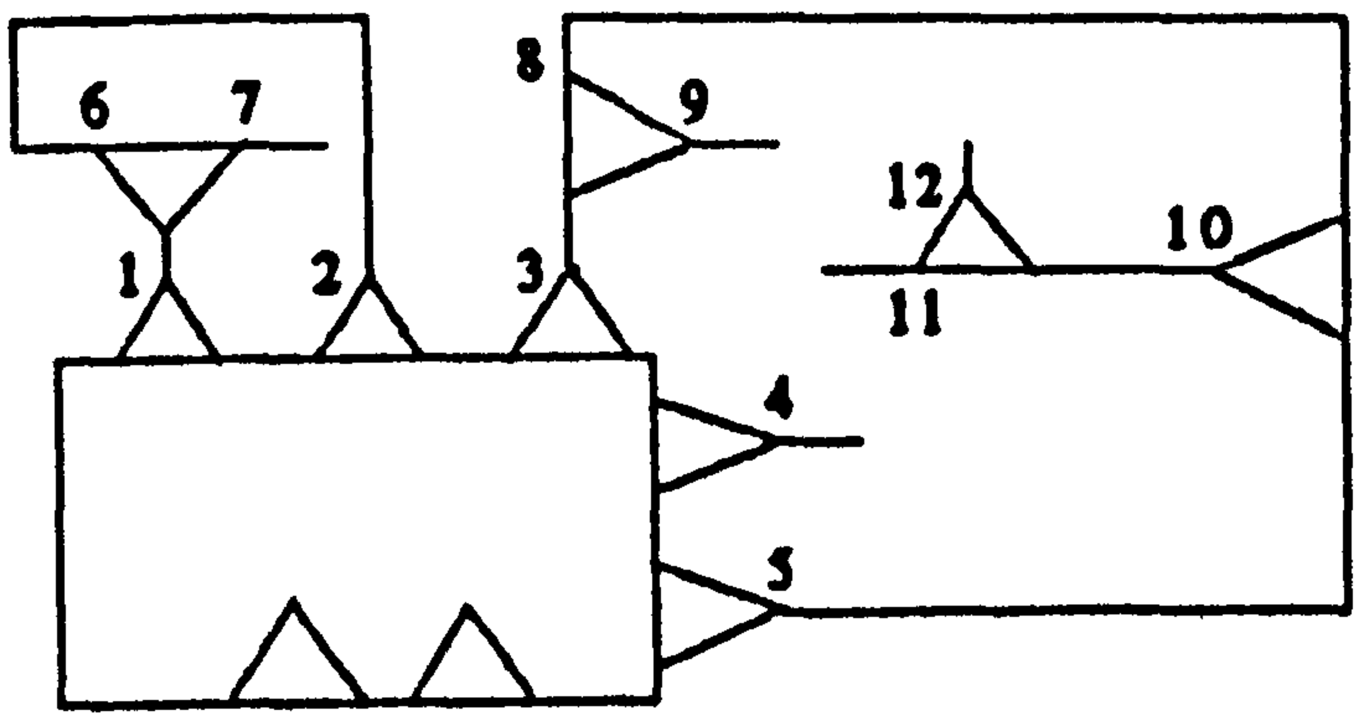
If $8 \to 5$, then 8-cycle. ∴ $8 \to \Delta \to 5$.



If $10 \to 4$, then $9 \to \Delta$ ⇒ 4^+-cycle, 7^+-cycle ⇒ 11^+-cycle.

If $10 \to 7$, then $4 \to 9$ ⇒ D complete with only $10\,\Delta$. If $10 \to 9$, then 2-cycle.

If $10 \to 10$, then $9 \to 9$ ⇒ 4^+-cycle, 8^+-cycle ⇒ 12^+-cycle. ∴ $10 \to \Delta$.



If $7 \to 7$, then 4-cycle. If $7 \to 4$, then 11^+-cycle. If $7 \to 9$, then 10^+-cycle.

If $7 \to 11$, then $4 \to 12$ ⇒ $9 \to \Delta$ ⇒ four 9-cycles.

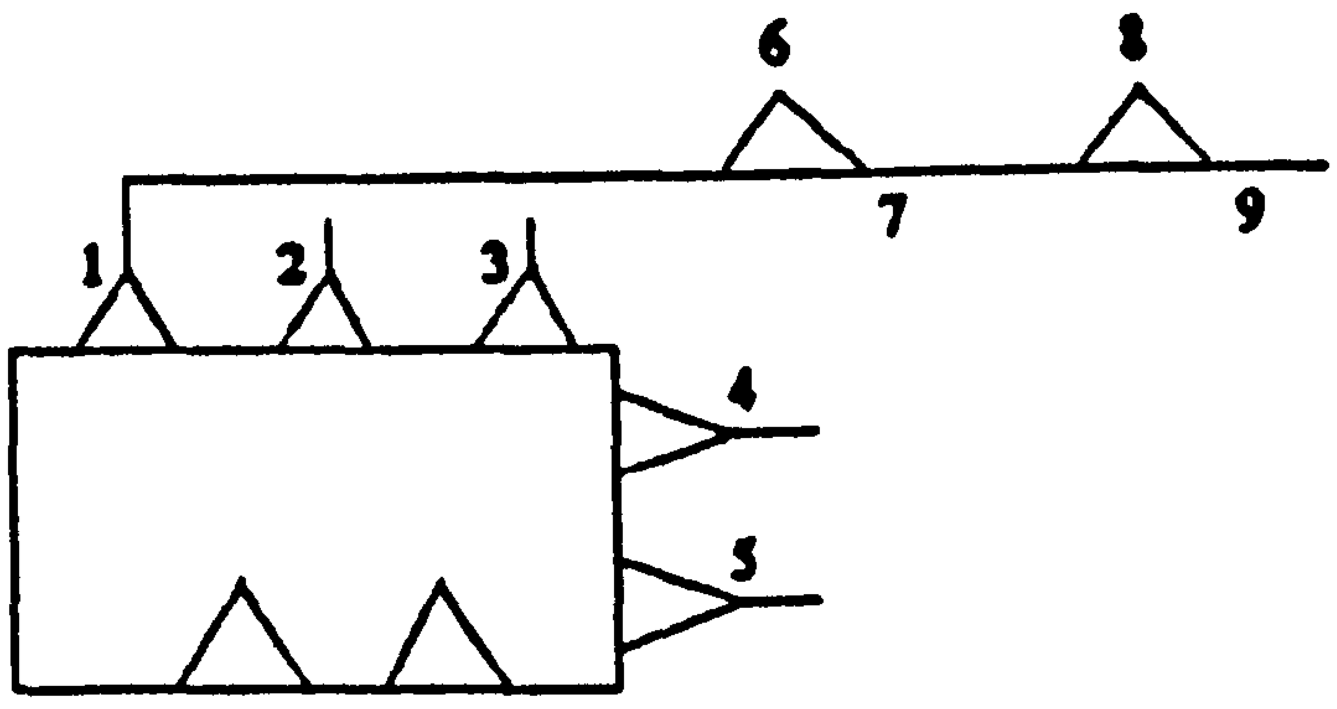
If $7 \to 12$, then 4^+-cycle, 8^+-cycle ⇒ 12^+-cycle. ∴ $7 \to \Delta$.

∴ 4^+-cycle, 6^+-cycle ⇒ 10^+-cycle.

Contradiction. ∴ Case (1C2d) not possible.

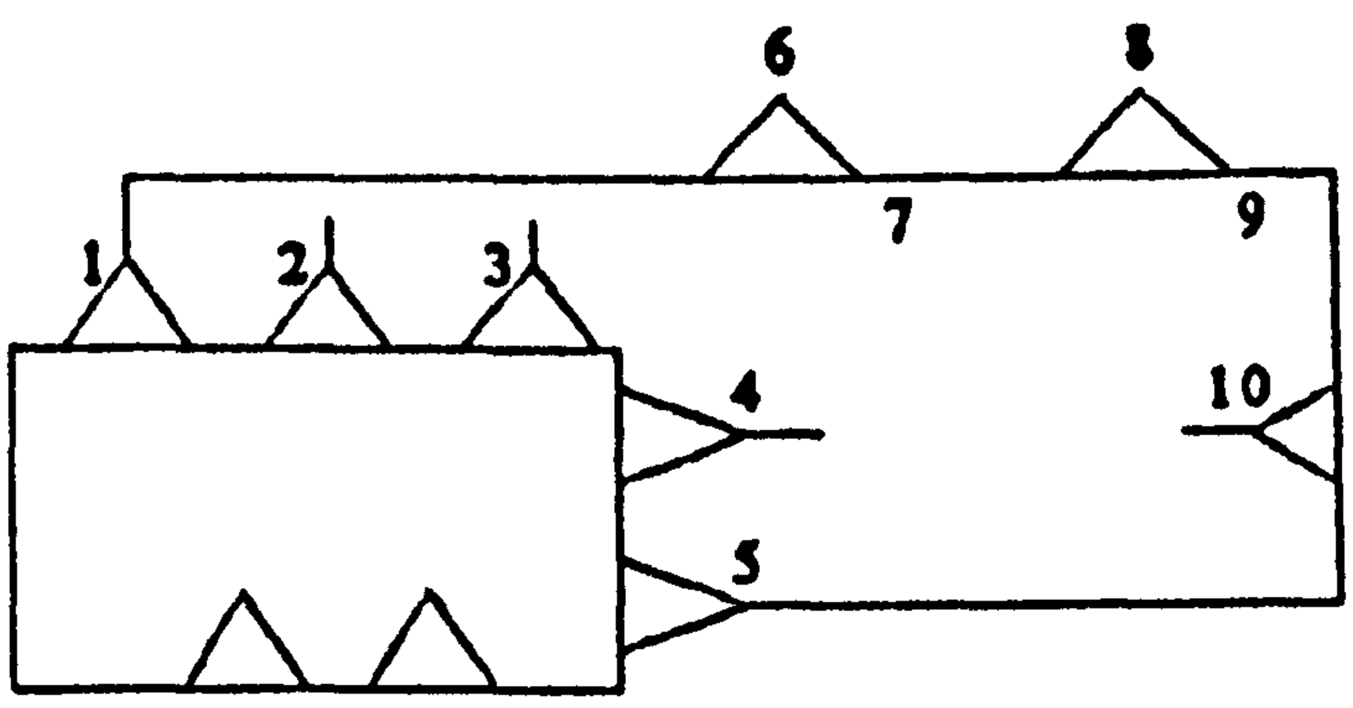
Case (1C2e) : Assume $6 \to 6$.

Then $7 \to \Delta$ and $8 \to 8$, to make D 2(2). (See next diagram).



If $9 \to 2$ or $9 \to 3$ or $9 \to 4$, then 10^+-cycle. If $9 \to 5$, then 8-cycle.

$\therefore\ 9 \to \Delta \to 5$.



If $10 \to 2$, then 5-cycle. If $10 \to 3$, then 5^+-cycle, 7^+-cycle $\Rightarrow$ 12^+-cycle.

If $10 \to 4$, then $3 \to \Delta \to 2$ $\Rightarrow$ 1-cycle. $\therefore\ 10 \to \Delta$.

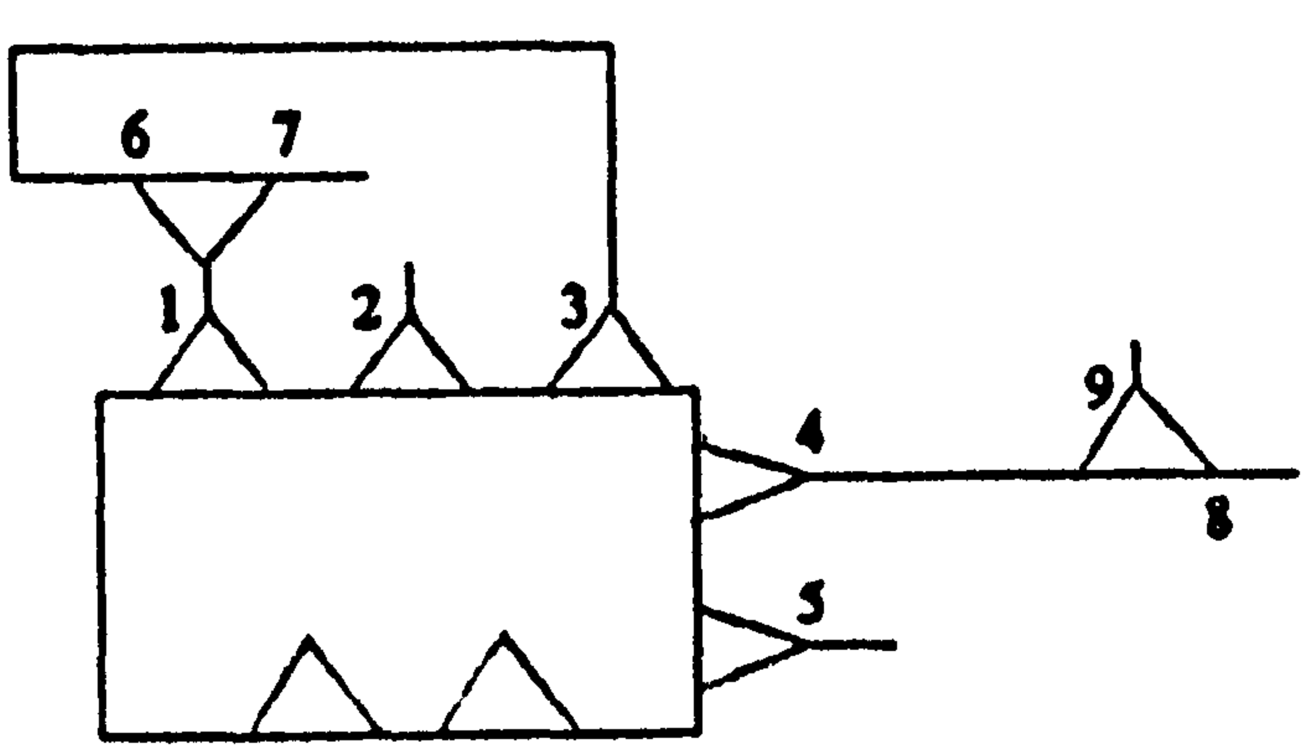
$\therefore\ 4^+$-cycle, 6^+-cycle $\Rightarrow$ 10^+-cycle.

Contradiction. $\therefore$ Case (1C2e) not possible.

<u>Case (1C2f) : Assume $6 \to 3$.</u>

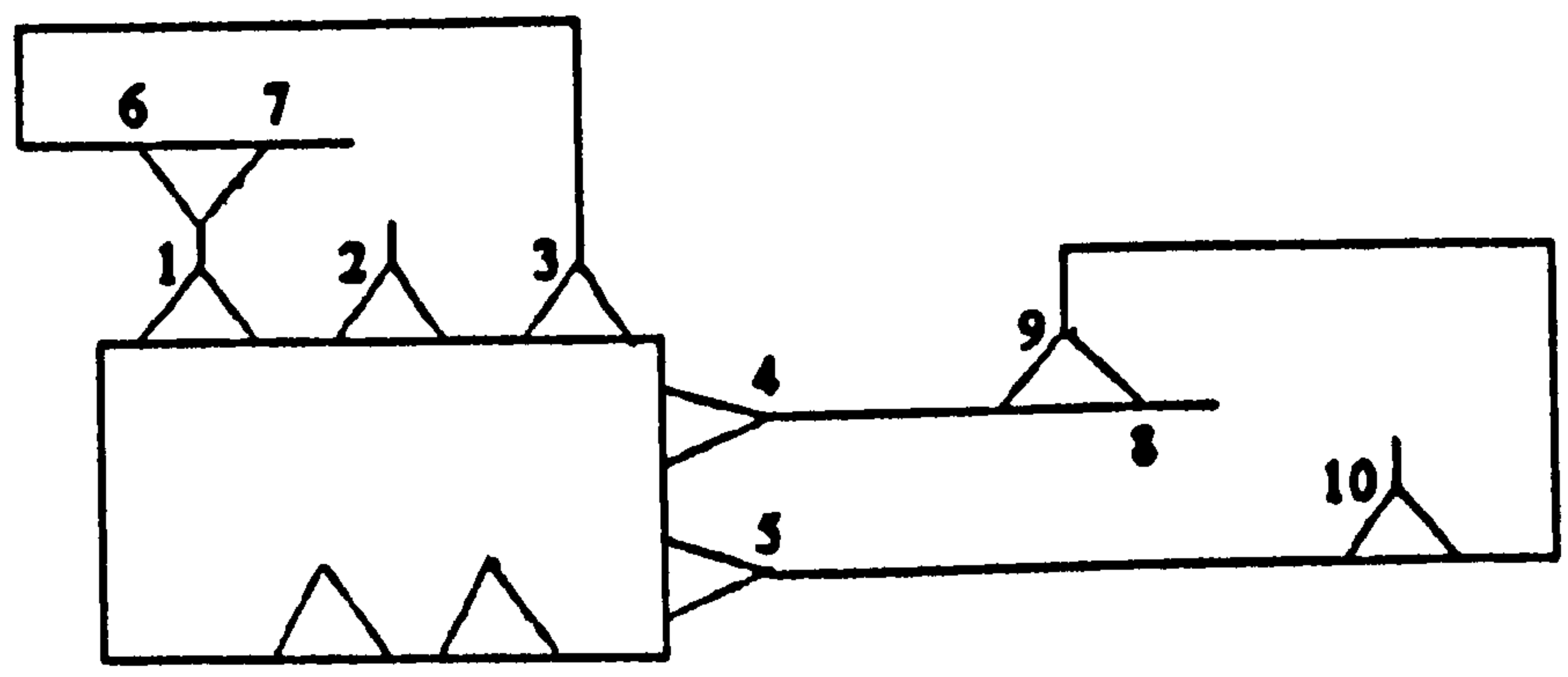
If $4 \to 4$, then $5 \to 5$ $\Rightarrow$ $7 \to \Delta$ $\Rightarrow$ $2 \to \Delta$ $\Rightarrow$ two 5^+-cycles $\Rightarrow$ 10^+-cycle.

If $4 \to 5$, then 2-cycle. If $4 \to 2$ or $4 \to 7$, then 10^+-cycle. $\therefore\ 4 \to \Delta$.



If $5 \to 2$ or $5 \to 5$ or $5 \to 7$, then 11^+-cycle. If $5 \to 9$, then 8-cycle.

If $5 \to 8$, then $9 \to 9$ $\Rightarrow$ D cannot be 2(2). $\therefore\ 5 \to \Delta \to 9$.



If $2 \to 2$, then D cannot be 2(2).

If $2 \to 7$, then $8 \to \Delta \Rightarrow 10 \to \Delta \Rightarrow 4^+$-cycle, 6^+-cycle $\Rightarrow 10^+$-cycle.

If $2 \to 8$ or $2 \to 10$, then 5^+-cycle, 7^+-cycle $\Rightarrow 12^+$-cycle.

$\therefore$ $2 \to \Delta$. $\therefore$ two 9-cycles, three 4^+-cycles $\Rightarrow 12^+$-cycle.

Contradiction. $\therefore$ Case (1C2f) not possible.

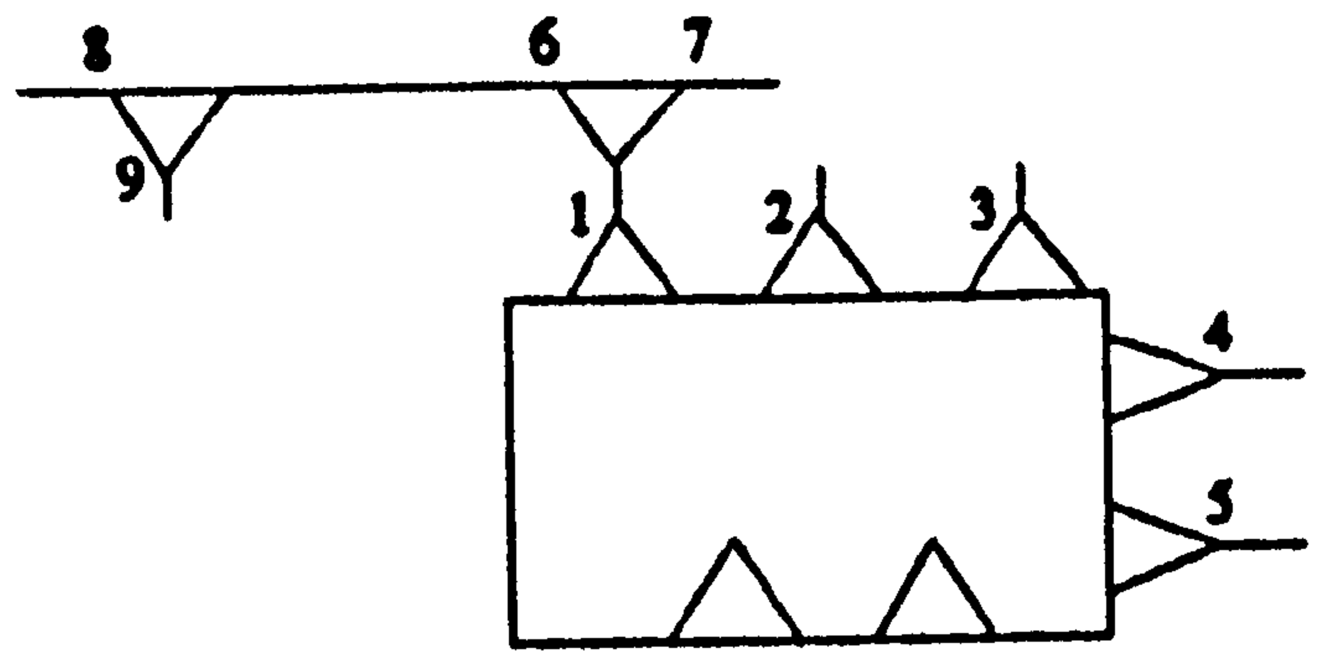
Case (1C2g) : Assume $6 \to 4$.

If $5 \to 2$ or $5 \to 3$, then 12^+-cycle. If $5 \to 7$, then 13^+-cycle.

If $5 \to 5$, then 7-cycle. If $5 \to \Delta$, then 1-cycle.

Contradiction. $\therefore$ Case (1C2g) not possible.

Hence, $6 \to \Delta$.



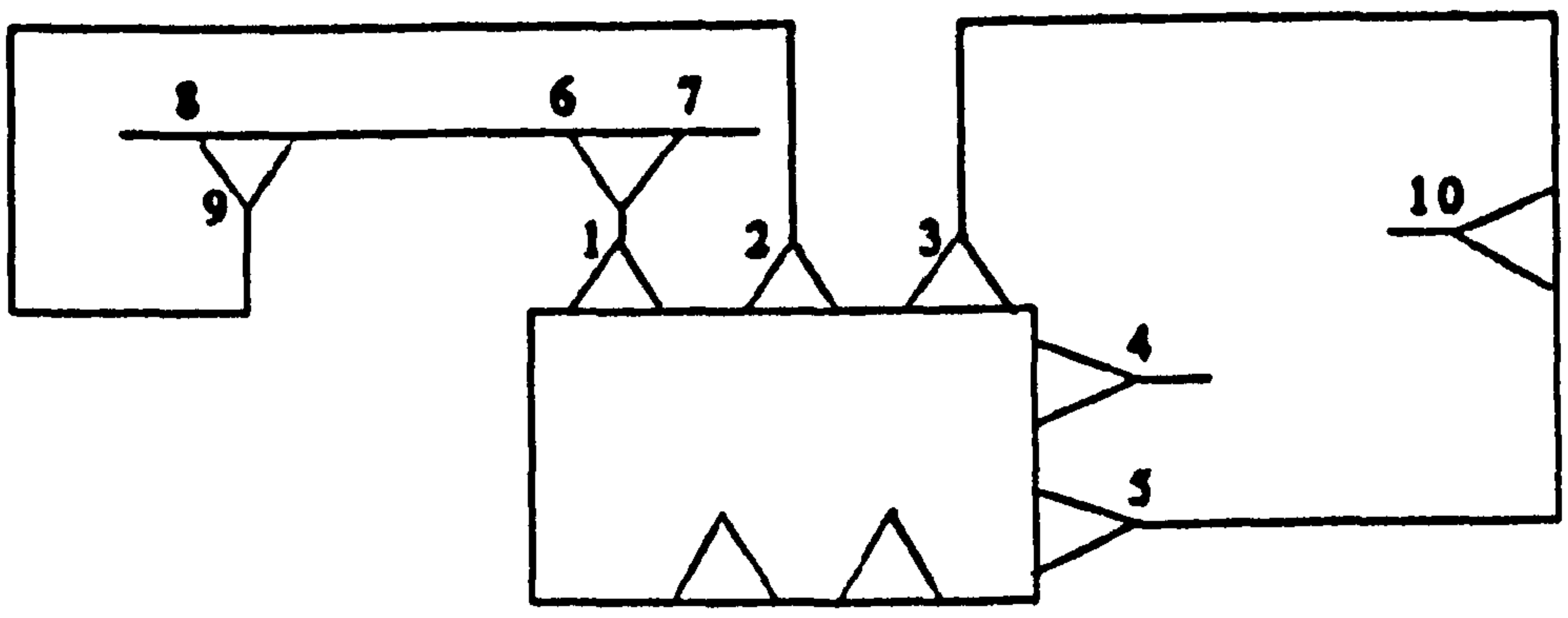
If $9 \to 4$, then $5 \to \Delta \Rightarrow 10^+$-cycle. If $9 \to 8$, then 1-cycle.

If $9 \to 5$, then 6-cycle.

Case (1C2h) : Assume $9 \to 2$.

If $3 \to 5$, then 8-cycle. If $3 \to 7$, then 12^+-cycle. If $3 \to 8$, then 10^+-cycle.

If $3 \to 3$, then $4 \to 4 \Rightarrow 12^+$-cycle. If $3 \to 4$, then 2-cycle. $\therefore$ $3 \to \Delta \to 5$.



If $4 \to 7$ or $4 \to 8$, then 5^+-cycle, 7^+-cycle $\Rightarrow$ 12^+-cycle.

If $4 \to 4$, then D cannot be 2(2).

If $4 \to 10$, then $7 \to \Delta$ $\Rightarrow$ $8 \to \Delta$ $\Rightarrow$ 4^+-cycle, 6^+-cycle $\Rightarrow$ 10^+-cycle.

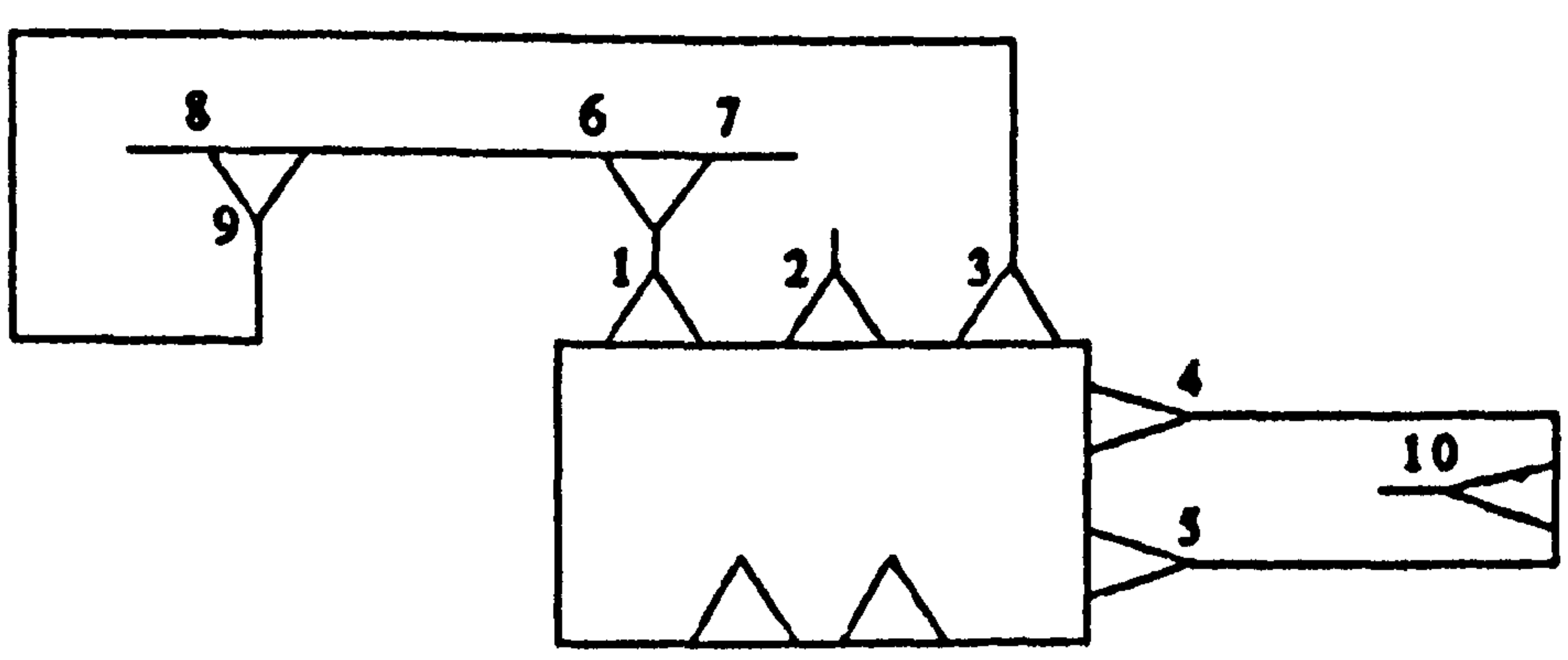
If $4 \to \Delta$, then three 4^+-cycles $\Rightarrow$ 12^+-cycle.

Contradiction. $\therefore$ Case (1C2h) not possible.

<u>Case (1C2i) : Assume $9 \to 3$.</u>

If $4 \to 2$ or $4 \to 7$, then 11^+-cycle. If $4 \to 4$, then $5 \to 5$ $\Rightarrow$ 10-cycle.

If $4 \to 5$, then 2-cycle. If $4 \to 8$, then 10^+-cycle. $\therefore$ $4 \to \Delta \to 5$.



If $2 \to 2$, then D cannot be 2(2). If $2 \to 10$, then 10^+-cycle.

If $2 \to 7$ or $2 \to 8$, then $10 \to \Delta$ $\Rightarrow$ 5^+-cycle, 6^+-cycle $\Rightarrow$ 11^+-cycle.

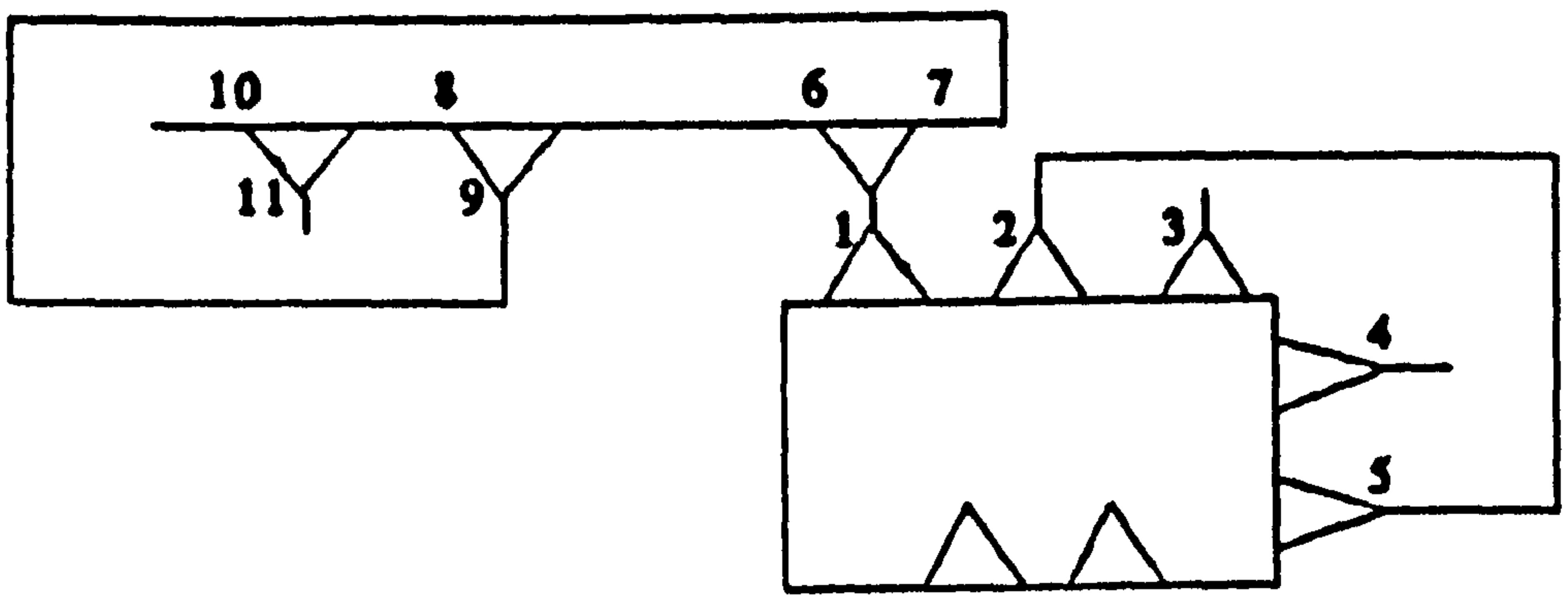
If $2 \to \Delta$, then three 4^+-cycles $\Rightarrow$ 12^+-cycle.

Contradiction. $\therefore$ Case (1C2i) not possible.

<u>Case (1C2j) : Assume $9 \to 7$.</u>

Then $2 \to 5$ to complete 9-cycle. If $8 \to 8$, then D cannot be 2(2).

If $8 \to 3$, then $4 \to \Delta$ $\Rightarrow$ 11^+-cycle. If $8 \to 4$, then $3 \to \Delta$ $\Rightarrow$ 11^+-cycle. $\therefore$ $8 \to \Delta$.



4^+-cycle, 5^+-cycle ➡ 9^+-cycle ➡ $3 \to 10$ and $11 \to 4$ ➡ D complete with only 10Δ.

Contradiction. ∴ Case (1C2j) not possible.

<u>Case (1C2k) : Assume $9 \to 9$.</u>

Then, $8 \to \Delta \to 5$, where Δ has a red point.

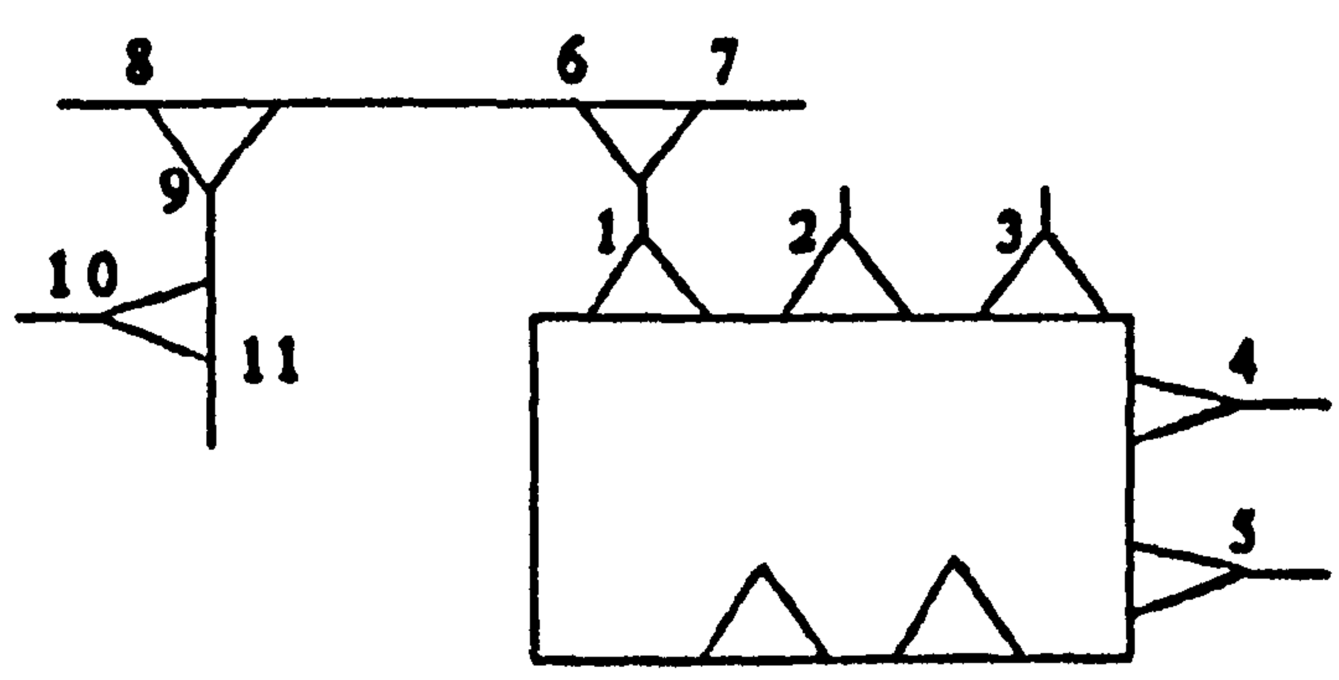
If $7 \to 2$, then $3 \to \Delta \to \Delta \to 4$ ➡ 2-cycle. If $7 \to 4$, then 5-cycle.

If $7 \to 3$, then 5^+-cycle, 7^+-cycle ➡ 12^+-cycle.

If $7 \to \Delta$, then two 9-cycles, 4^+-cycle, 6^+-cycle ➡ 10^+-cycle.

Contradiction. ∴ Case (1C2k) not possible.

Hence, $9 \to \Delta$.



If $11 \to 10$, then 1-cycle. If $11 \to 5$, then 7-cycle.

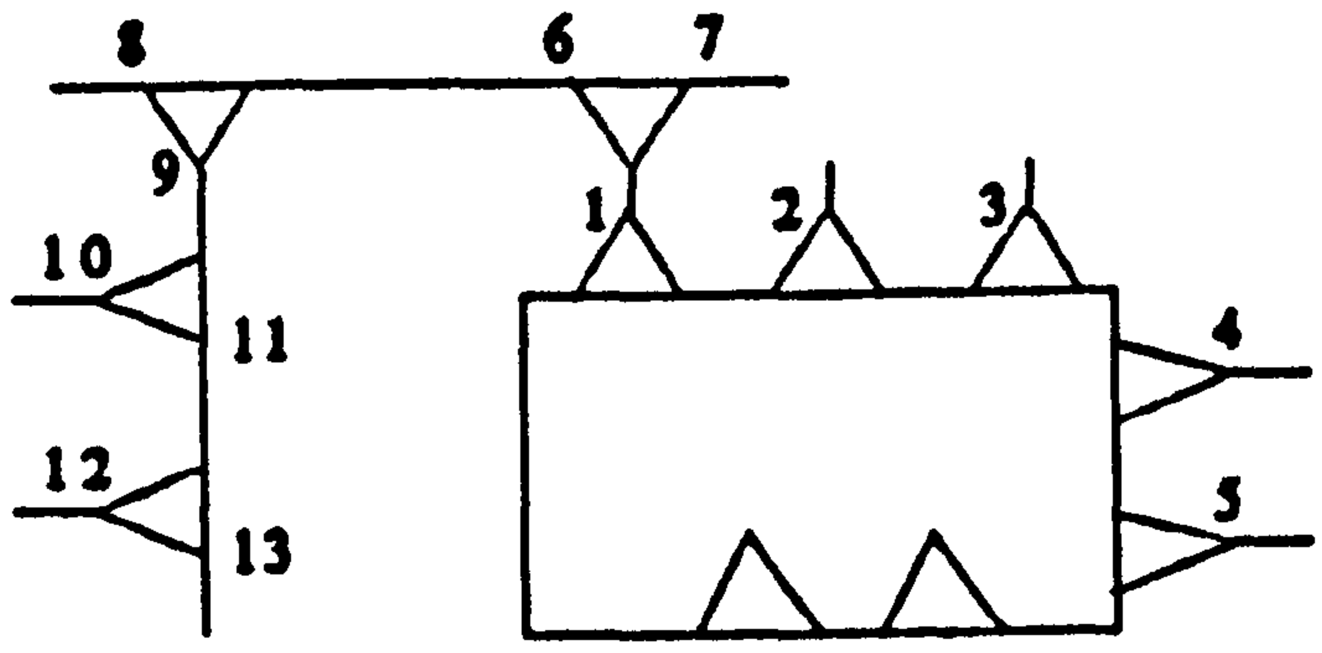
If $11 \to 7$ or $11 \to 11$, then 10^+-cycle. If $11 \to 3$, then $4 \to 5$ ➡ 2-cycle.

If $11 \to 2$, then $3 \to 5$ ➡ $4 \to \Delta$ ➡ 4^+-cycle, 6^+-cycle ➡ 10^+-cycle.

If $11 \to 4$, then $5 \to 5$ ➡ D cannot be 2(2).

If $11 \to 8$, then $5 \to 7$ ➡ $10 \to \Delta$ ➡ two 5^+-cycles ➡ 10^+-cycle.

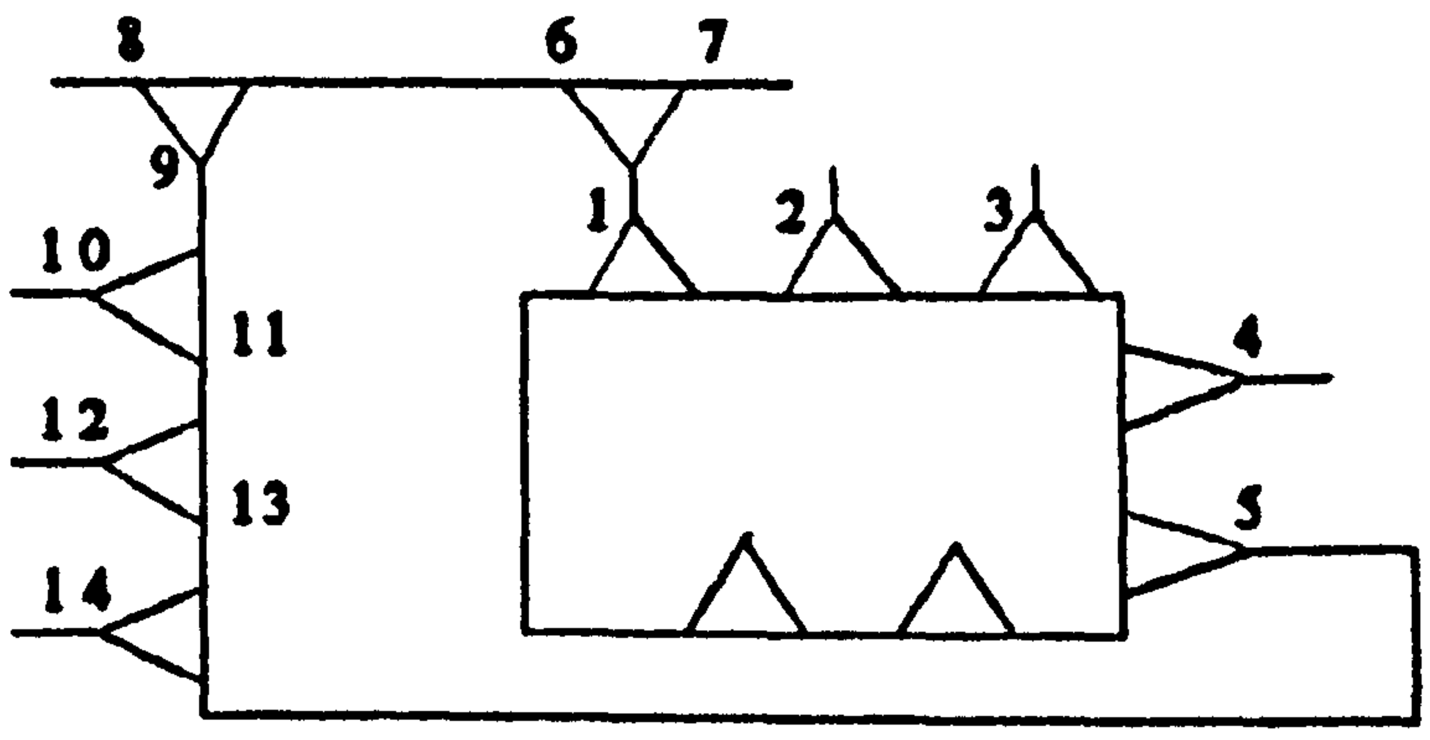
∴ $11 \to \Delta$.



If 13 → 2 or 13 → 3 or 13 → 4 or 13 → 8 or 13 → 10, then 10^+-cycle.

If 13 → 5, then 8-cycle. If 13 → 7, then 11^+-cycle. If 13 → 12, then 1-cycle.

If 13 → 13, then 5 → 12 ⇒ D cannot be 2(2). Hence, 13 → Δ → 5.



If 2 → 2, then 3 → 3 ⇒ 4 → 14 ⇒ 7 → 12 ⇒ 8 → 10 ⇒ 2-cycle.

If 3 → 3, then 4 → 4 ⇒ 2 → 7 ⇒ 8 → 14 ⇒ 10 → 12 ⇒ 2-cycle.

If 4 → 4, then 3 → 3 ⇒ 2-cycle.

If 7 → 7, then 8 → 8 ⇒ 4 → 14 ⇒ 4^+-cycle, 7^+-cycle ⇒ 11^+-cycle.

If 14 → 14, then 12 → 12 ⇒ 2 → 7 ⇒ 4^+-cycle, 7^+-cycle ⇒ 11^+-cycle.

If 12 → 12, then 10 → 10 ⇒ 7 → 14 ⇒ 6^+-cycle, 8^+-cycle ⇒ 14^+-cycle.

∴ 8 → 8 and 10 → 10. ∴ 7 → 2. ∴ 4 → 14. ∴ 3 → 12. ∴ 4-cycle.

Contradiction. Hence, Case (1C2) not possible.

Therefore, Case (1C) is not possible.

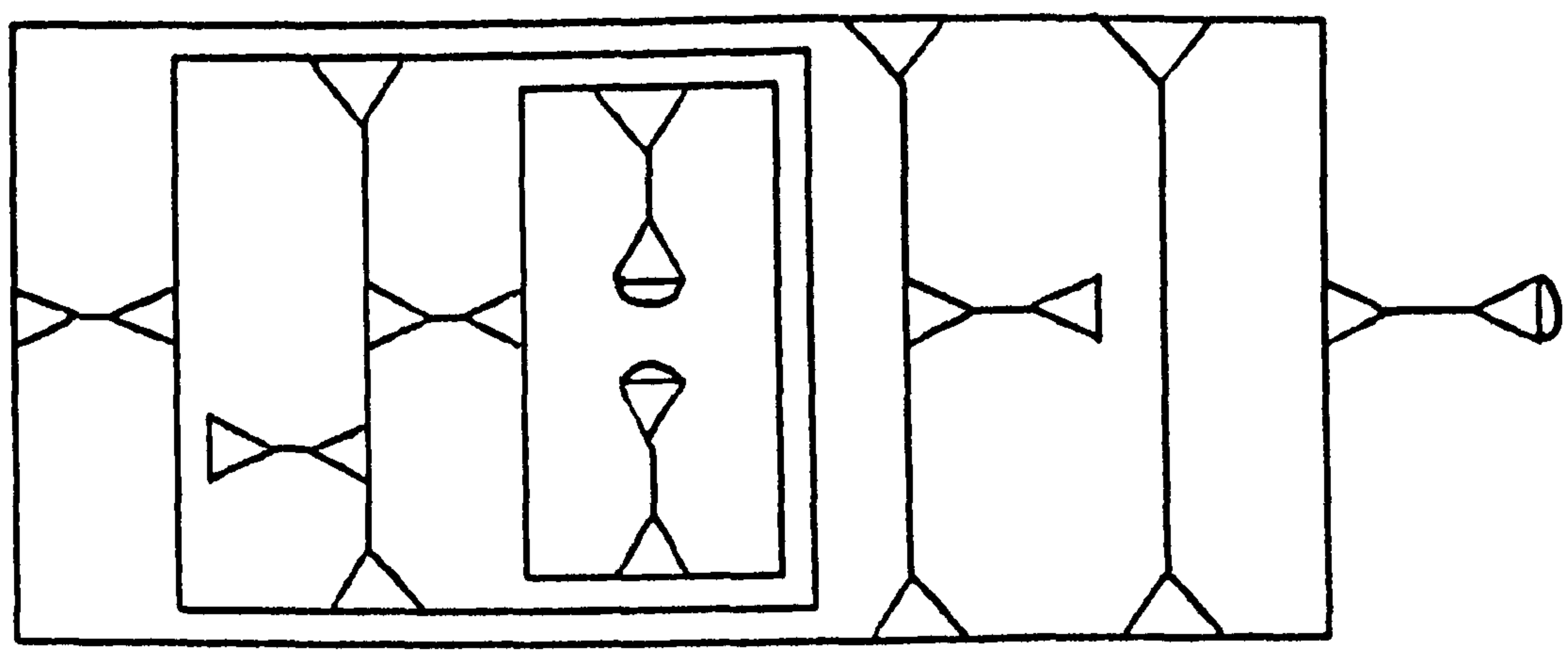
Hence, (36,0,4,0,0,3) does not have a diagram which is 2(2).

Hence, no diagram exists for I_3 which is 2(1) or 2(2).

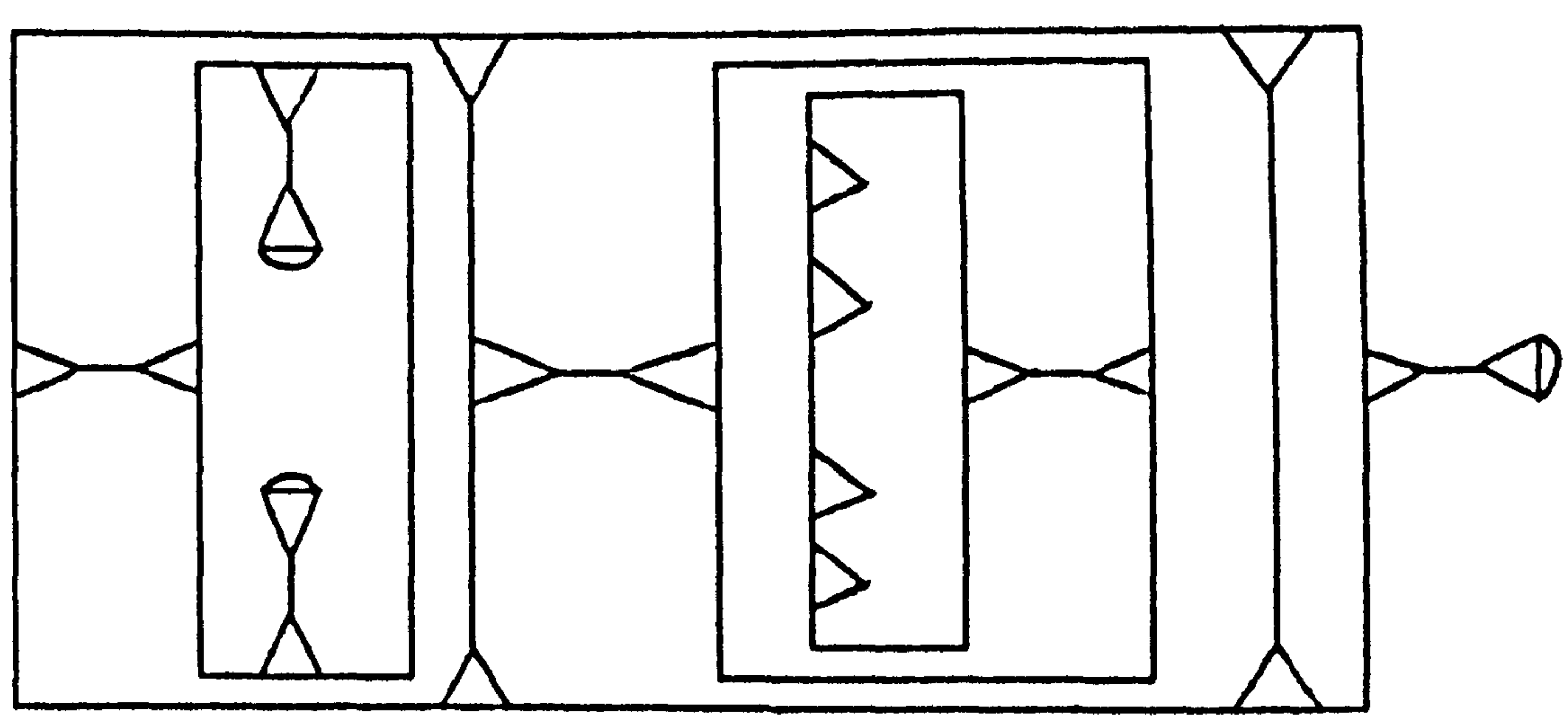
<u>Case (2)</u> : $D_n\,(12n+48,0,4,0,3,n)$, $n \geq 0$.

Diagrams have already been exhibited for $D_0\,2(1)$ and $D_0^2\,2(2)$.

$D_1\,(60,0,4,0,3,1)\,2(1)$



$D_1^2\,(60,0,4,0,3,1)\,2(2)$

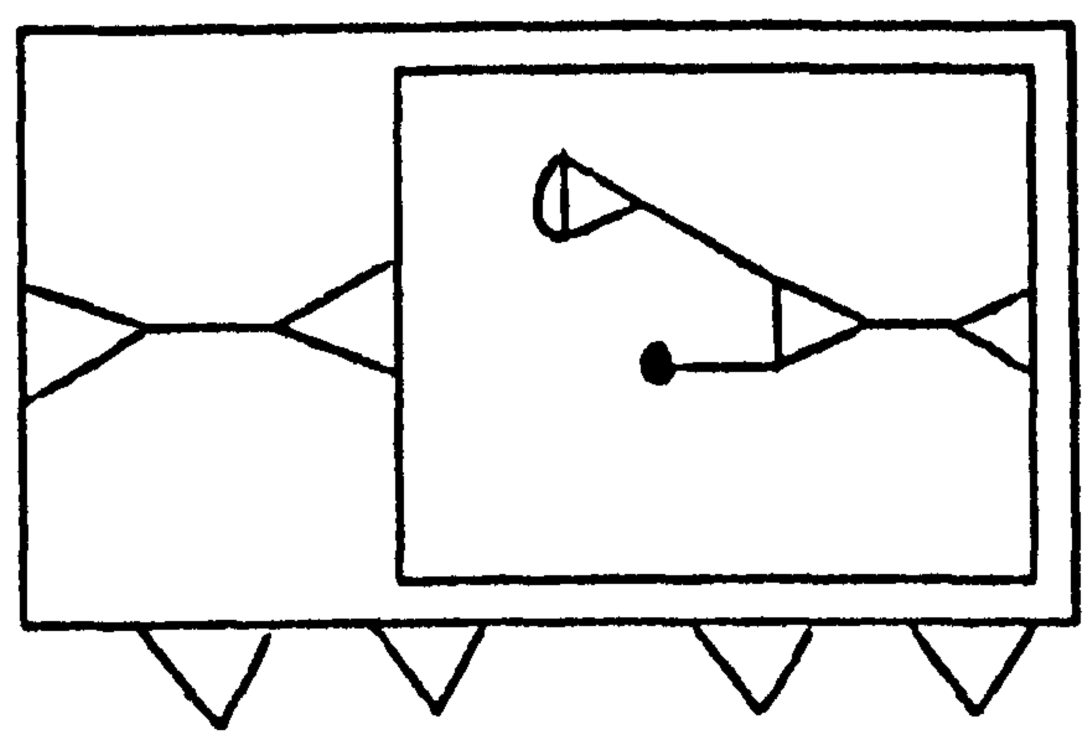


$$\forall n \geq 2, \quad \left\{ \begin{array}{l} D_n\,(12n+48,0,4,0,3,n)\,2(1) = D_{n-2} + I_2\,(24,0,4,0,0,2)\,2(1) \\ D_n^2\,(12n+48,0,4,0,3,n)\,2(2) = D_{n-2}^2 + I_2^2\,(24,0,4,0,0,2)\,2(2) \end{array} \right\}$$

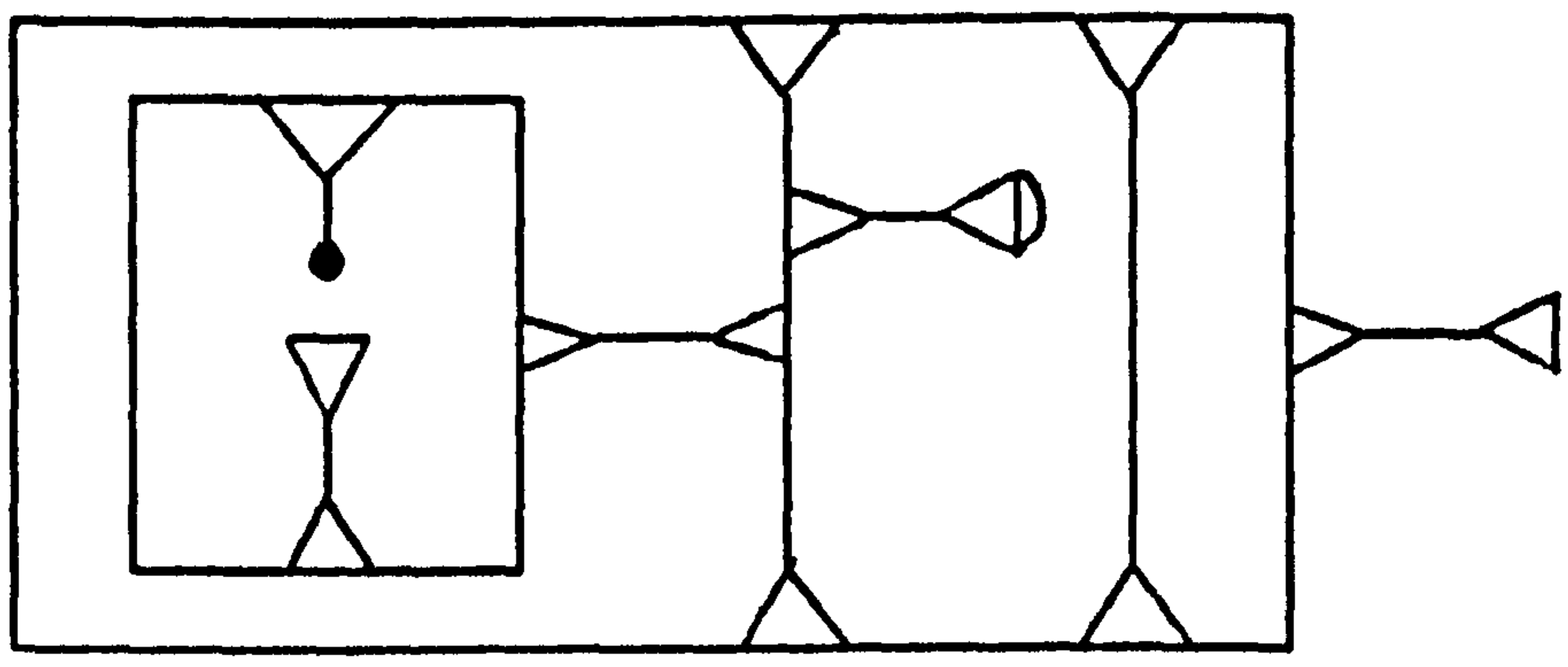
<u>Case (3)</u> : $F_n\,(12n+28,0,4,1,1,n)$, $n \geq 0$.

A diagram has already been exhibited for $F_0\,2(1)$.

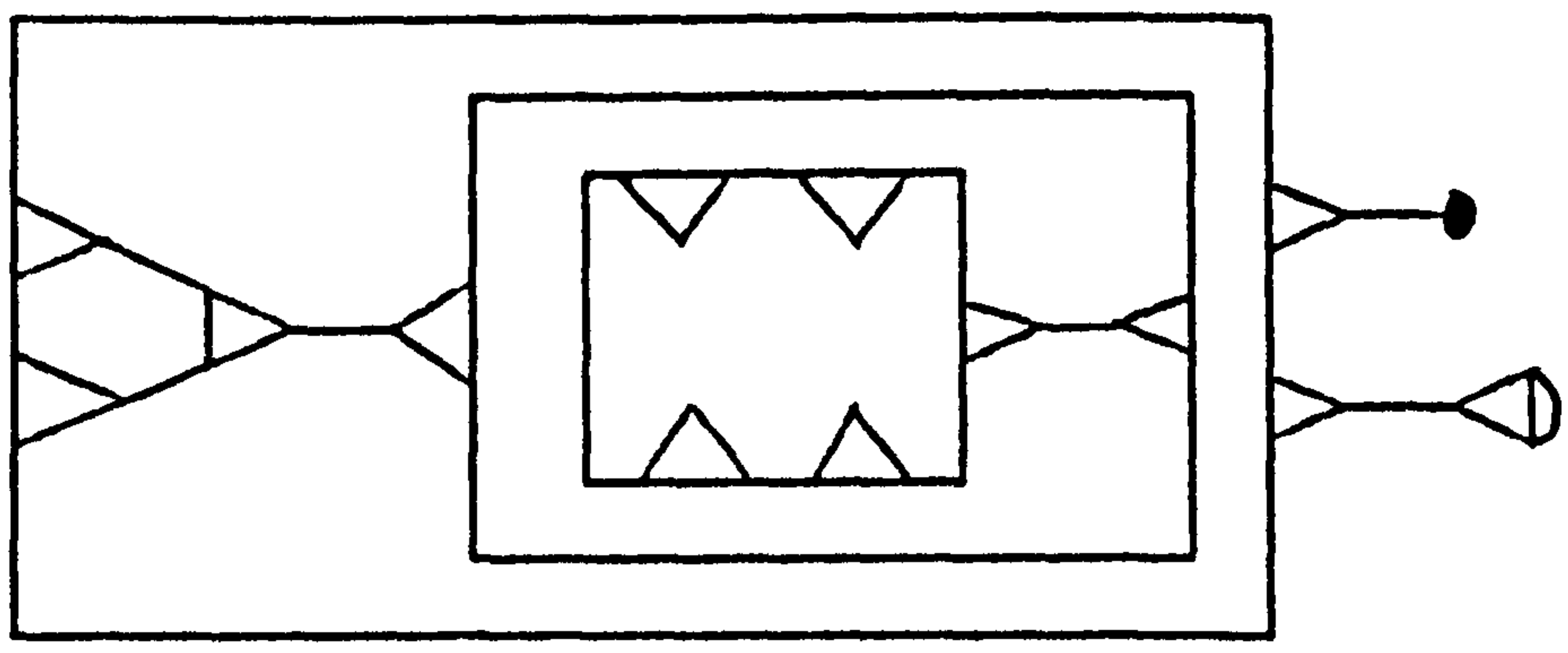
$F_0^2\,(28,0,4,1,1,0)\,2(2)$



F_1 (40,0,4,1,1,1) 2(1)



F_1^2 (40,0,4,1,1,1) 2(2)



$$\forall n \geq 2, \left\{ \begin{array}{l} F_n (12n+28,0,4,1,1,n)\ 2(1) = F_{n-2}\ 2(1) + I_2 (24,0,4,0,0,2)\ 2(1) \\ F_n^2 (12n+28,0,4,1,1,n)\ 2(2) = F_{n-2}^2\ 2(2) + I_2^2 (24,0,4,0,0,2)\ 2(2) \end{array} \right\}$$

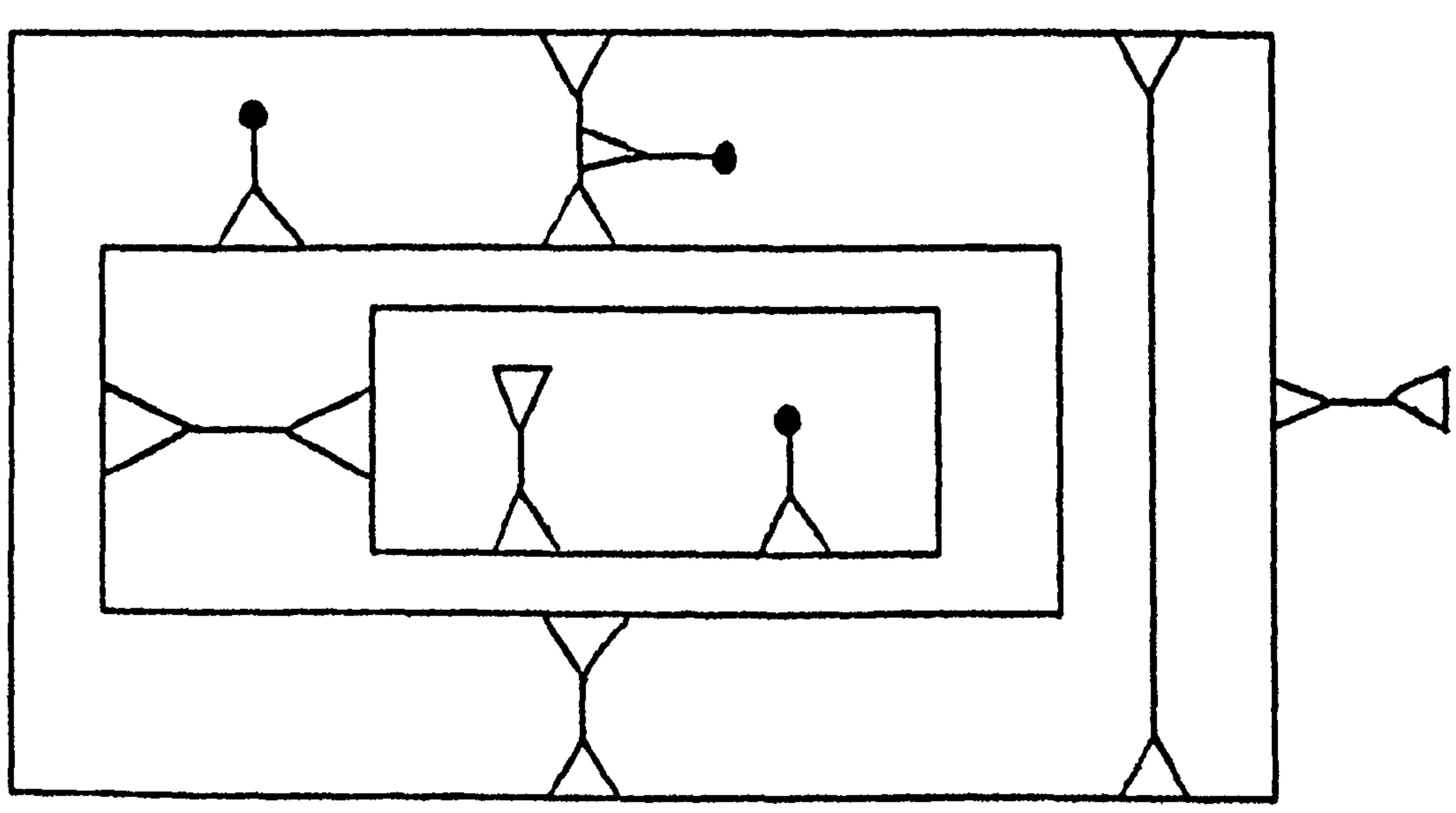
<u>Case (4)</u> : K_n (12n+56,0,4,2,2,n) , $n \geq 0$.

$$\forall n \geq 0, \left\{ \begin{array}{l} K_n (12n+56,0,4,2,2,n)\ 2(1) = F_0\ 2(1) + F_n (12n+28,0,4,1,1,n)\ 2(1) \\ K_n^2 (12n+56,0,4,2,2,n)\ 2(2) = F_0^2\ 2(2) + F_n^2 (12n+28,0,4,1,1,n)\ 2(2) \end{array} \right\}$$

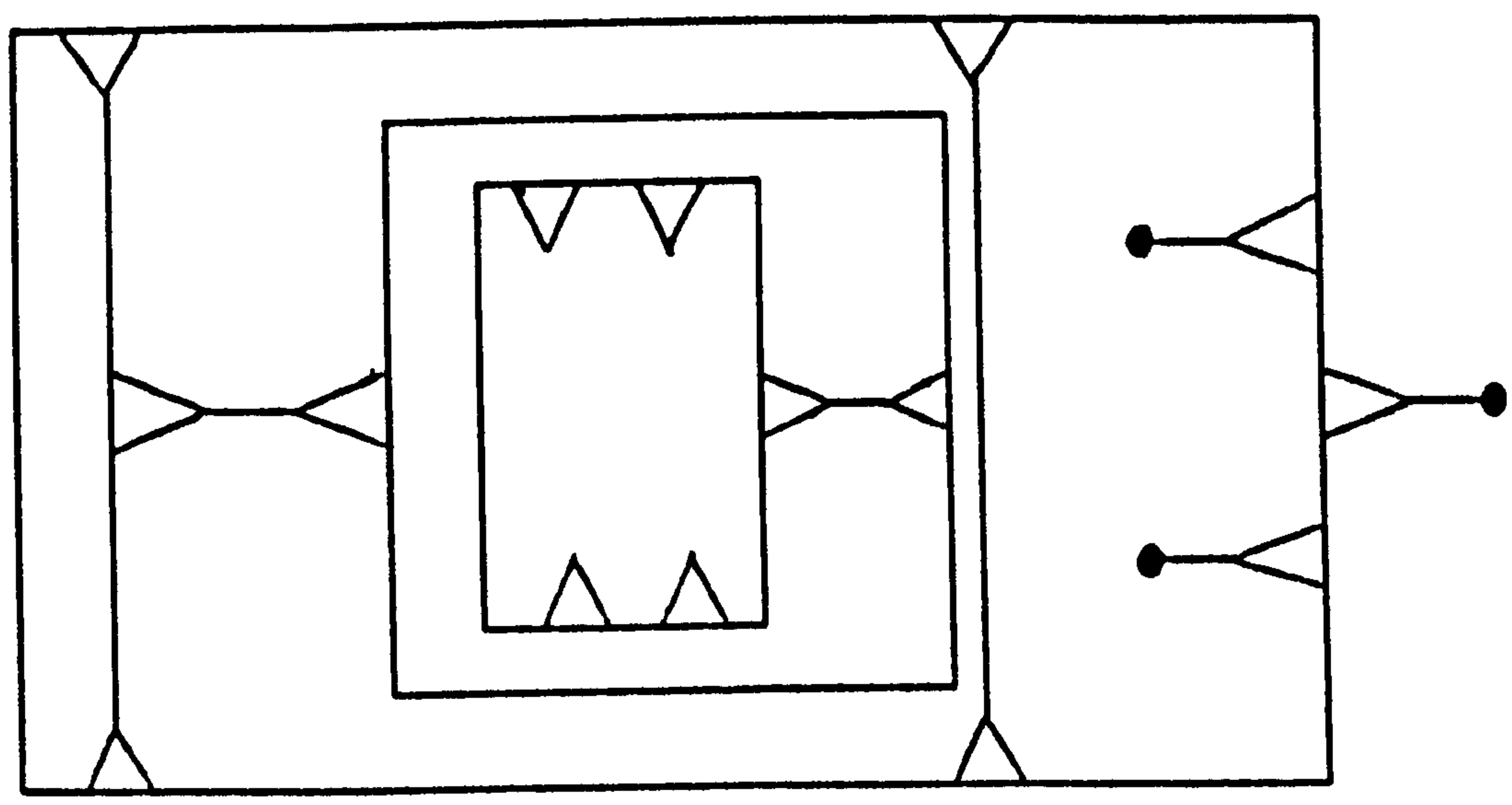
<u>Case (5)</u> : C_n (12n+36,0,4,3,0,n) , $n \geq 0$.

Diagrams have already been exhibited for C_0 2(1) and C_0^2 2(2).

C_1 (48,0,4,3,0,1) 2(1)



$C_1^2\,(48,0,4,3,0,1)\,2(2)$



$$\forall n \geq 2, \left\{ \begin{array}{l} C_n\,(12n+36,0,4,3,0,n)\,2(1) = C_{n-2}\,2(1) + I_2\,(24,0,4,0,0,2)\,2(1) \\ C_n^2\,(12n+36,0,4,3,0,n)\,2(2) = C_{n-2}^2\,2(2) + I_2^2\,(24,0,4,0,0,2)\,2(2) \end{array} \right\}$$

<u>Case (6)</u> : $L_n\,(12n+84,0,4,3,3,n)$, $n \geq 0$.

$$\forall n \geq 0, \left\{ \begin{array}{l} L_n\,(12n+84,0,4,3,3,n)\,2(1) = K_n\,(12n+56,0,4,2,2,n)\,2(1) + F_0\,2(1) \\ L_n^2\,(12n+84,0,4,3,3,n)\,2(2) = K_n^2\,(12n+56,0,4,2,2,n)\,2(2) + F_0^2\,2(2) \end{array} \right\}$$

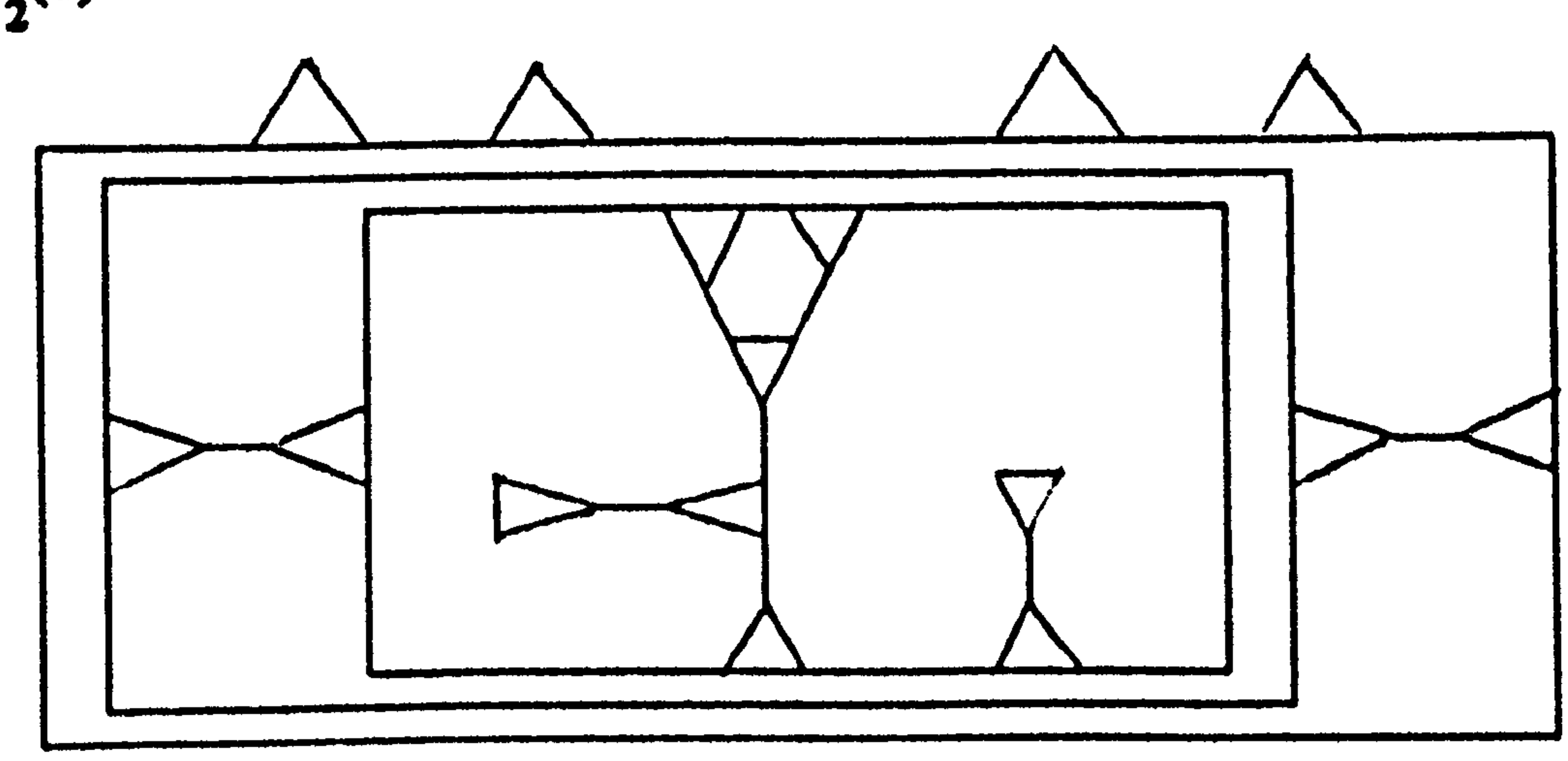
<u>Case (7)</u> : $A_n\,(12n+36,1,4,0,0,n)$, $n \geq 0$.

Diagrams for $N_1\,(36,0,8,0,0,0)\,4(1)$ and $N_2\,(36,0,8,0,0,0)\,4(2)$ have already been exhibited.

$A_0\,(36,1,4,0,0,0)\,2(1) = N_1(1)$

$A_0^2\,(36,1,4,0,0,0)\,2(2) = N_2(2)$

$N_3\,(48,0,8,0,0,1)\,2(1)\,2(2)$



$A_1\,(48,1,4,0,0,1)\,2(1) = N_3(2)$

$A_1^2\,(48,1,4,0,0,1)\,2(2) = N_3(1)$

$$\forall n \geq 2, \left\{ \begin{array}{l} A_n(12n+36,1,4,0,0,n)\, 2(1) = A_{n-2}\, 2(1) + I_2(24,0,4,0,0,2)\, 2(1) \\ A_n^2(12n+36,1,4,0,0,n)\, 2(2) = A_{n-2}^2\, 2(2) + I_2^2(24,0,4,0,0,2)\, 2(2) \end{array} \right\}$$

Case (8) : $E_n(12n+84,1,4,0,3,n)$, $n \geq 0$.

$$\forall n \geq 0, \left\{ \begin{array}{l} E_n(12n+84,1,4,0,3,n)\, 2(1) = D_n(12n+48,0,4,0,3,n)\, 2(1) + A_0\, 2(1) \\ E_n^2(12n+84,1,4,0,3,n)\, 2(2) = D_n^2(12n+48,0,4,0,3,n)\, 2(2) + A_0^2\, 2(2) \end{array} \right\}$$

Case (9) : $G_n(12n+64,1,4,1,1,n)$, $n \geq 0$.

$$\forall n \geq 0, \left\{ \begin{array}{l} G_n(12n+64,1,4,1,1,n)\, 2(1) = F_n(12n+28,0,4,1,1,n)\, 2(1) + A_0\, 2(1) \\ G_n^2(12n+64,1,4,1,1,n)\, 2(2) = F_n^2(12n+28,0,4,1,1,n)\, 2(2) + A_0^2\, 2(2) \end{array} \right\}$$

Case (10) : $H_n(12n+92,1,4,2,2,n)$, $n \geq 0$.

$$\forall n \geq 0, \left\{ \begin{array}{l} H_n(12n+92,1,4,2,2,n)\, 2(1) = K_n(12n+56,0,4,2,2,n)\, 2(1) + A_0\, 2(1) \\ H_n^2(12n+92,1,4,2,2,n)\, 2(2) = K_n^2(12n+56,0,4,2,2,n)\, 2(2) + A_0^2\, 2(2) \end{array} \right\}$$

Case (11) : $J_n(12n+72,1,4,3,0,n)$, $n \geq 0$.

$$\forall n \geq 0, \left\{ \begin{array}{l} J_n(12n+72,1,4,3,0,n)\, 2(1) = C_n(12n+36,0,4,3,0,n)\, 2(1) + A_0\, 2(1) \\ J_n^2(12n+72,1,4,3,0,n)\, 2(2) = C_n^2(12n+36,0,4,3,0,n)\, 2(2) + A_0^2\, 2(2) \end{array} \right\}$$

Case (12) : $M_n(12n+120,1,4,3,3,n)$, $n \geq 0$.

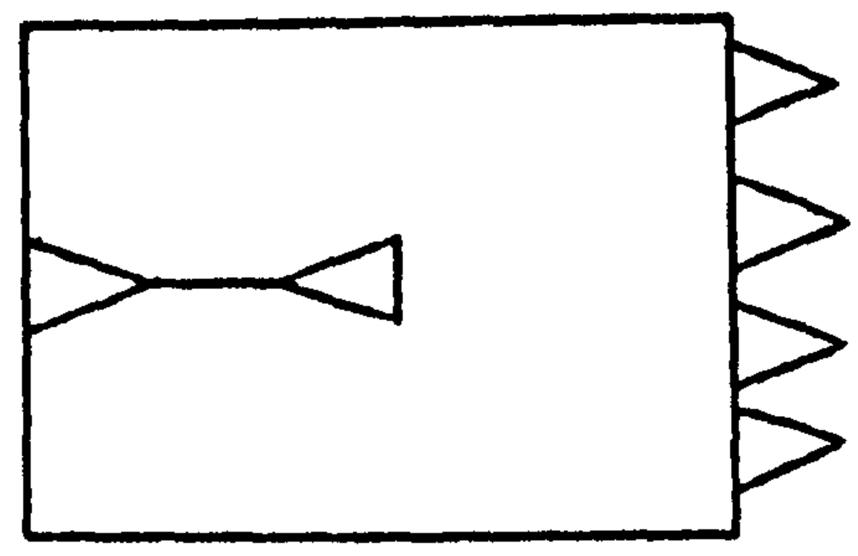
$$\forall n \geq 0, \left\{ \begin{array}{l} M_n(12n+120,1,4,3,3,n)\, 2(1) = L_n(12n+84,0,4,3,3,n)\, 2(1) + A_0\, 2(1) \\ M_n^2(12n+120,1,4,3,3,n)\, 2(2) = L_n^2(12n+84,0,4,3,3,n)\, 2(2) + A_0^2\, 2(2) \end{array} \right\}$$

Hence, no such S exists. □

<u>LEMMA 2.3.4</u> If $S(u,p,e,f,g_1,g_3)$ satisfies (2.1.2) and $e = 3$, then there exists a coset diagram with specification S which is 1(1).

<u>Proof :</u> Assume S is a counter-example with $p + f + g_1 + g_3$ minimal. We want to show that no such S exists.

N_4 (18,0,6,0,0,0) 1(1) 2(2)

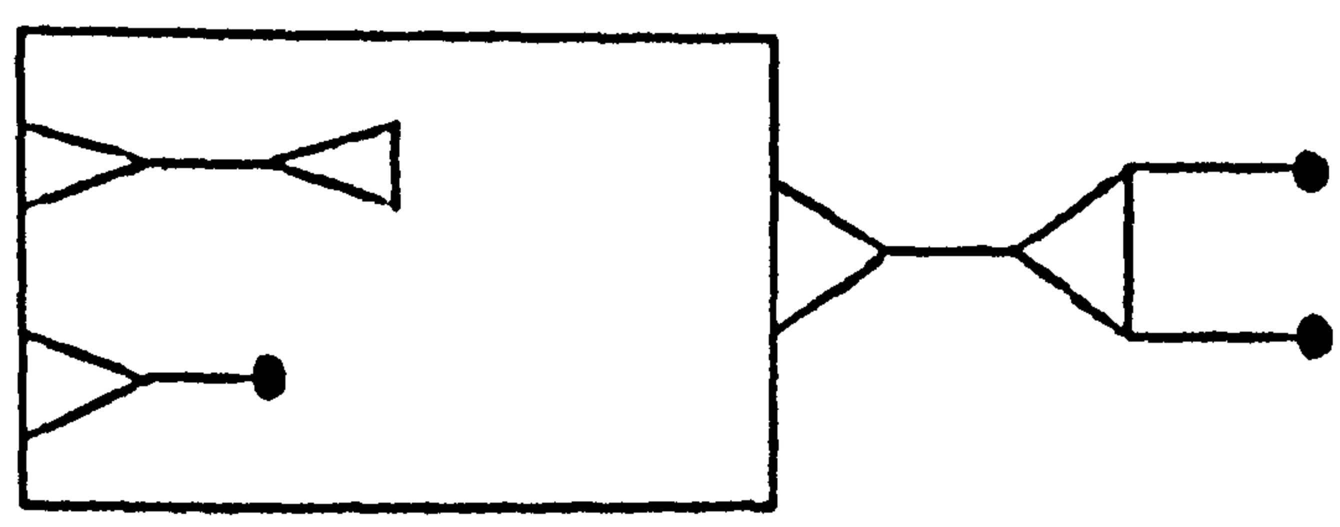


Let $P_0 = N_4(2) = (18,1,2,0,0,0)$ 1(1).

If $p \geq 1$ and $D(u-18,p-1,5,f,g_1,g_3)$ satisfies (2.1.2), then D can be 2(1) by Lemma 2.3.2, so that $D + P_0$ has specification S which is 1(1).

Therefore, S has $p < 1$. i.e. S has $p = 0$.

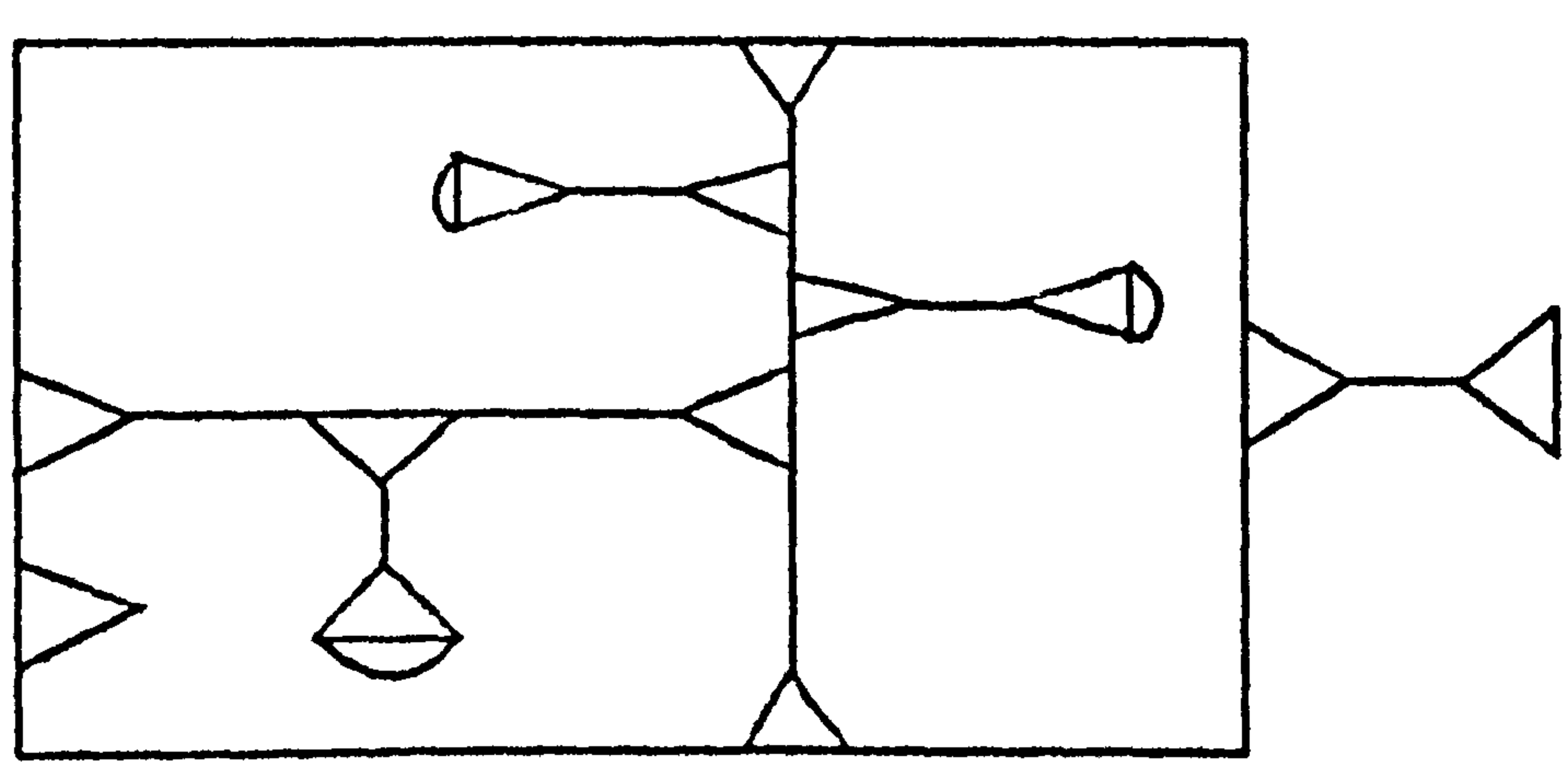
Q (18,0,2,3,0,0) 1(1)



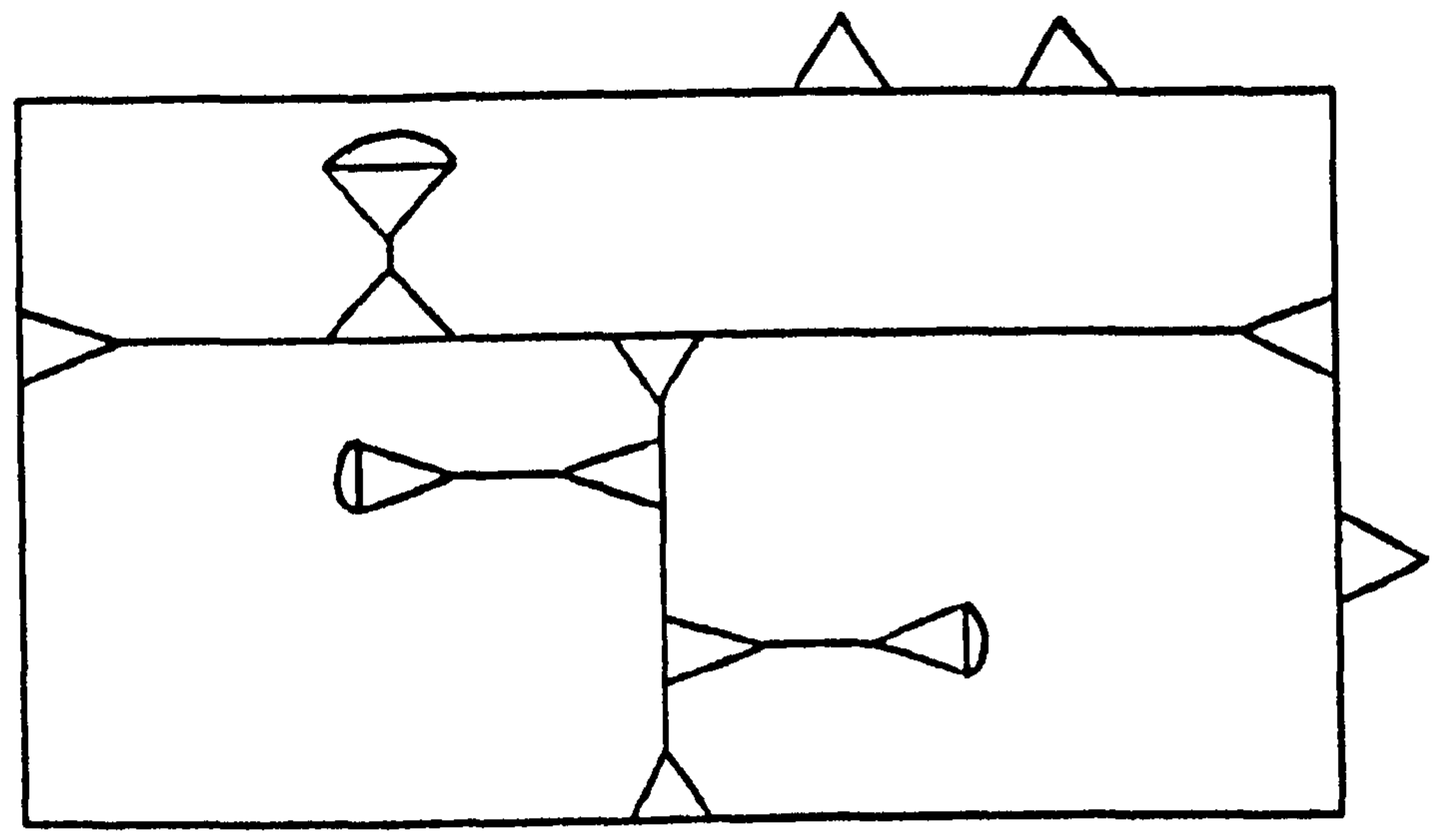
If $f \geq 3$ and $D(u-18,p,5,f-3,g_1,g_3)$ satisfies (2.1.2), then D can be 2(1) by Lemma 2.3.2, so that D + Q has specification S which is 1(1).

Therefore, S has $f < 3$.

R (39,0,3,0,3,0) 1(1)



R^2 (39,0,3,0,3,0) 1(2)



If $g_1 \geq 3$ and D$(u-39,p,4,f,g_1-3,g_3)$ satisfies (2.1.2), then D can be either 2(1) or 1(1) 1(2) by Lemma 2.3.3, so that both D 2(1) + R 1(1) and D 1(1) 1(2) + R^2 1(2) have specification S, which is 1(1).

Therefore, S has $g_1 < 3$.

T_0(3,0,3,0,0,1) can be regarded as 1(1) or 1(2), since the diagram consists of one blue triangle with a red point on each vertex.

If $g_3 \geq 1$ and D$(u-3,p,4,f,g_1,g_3-1)$ satisfies (2.1.2), then D can be either 2(1) or 1(1) 1(2) by Lemma 2.3.3, so that both D 2(1) + T_0 1(1) and D 1(1) 1(2) + T_0 1(2) have specification S, which is 1(1).

Therefore, S has $g_3 < 1$. i.e. S has $g_3 = 0$.

As shown in Lemma 2.3.2, we have $f \equiv g_1 \pmod 3$.

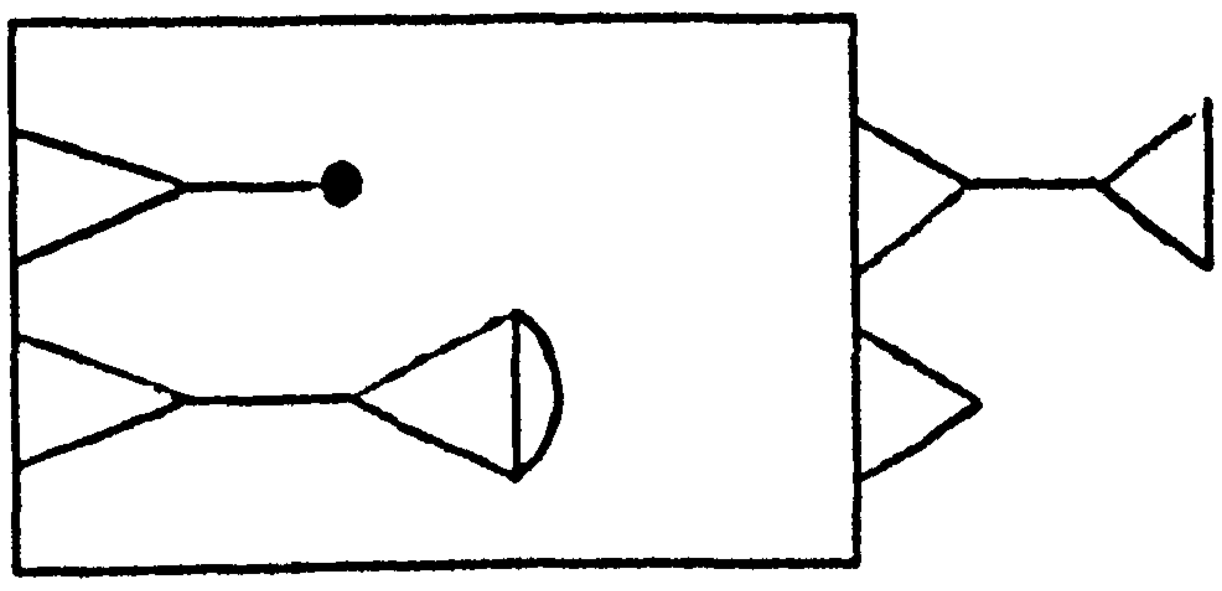
We now know that a minimal S would have the form

$(u,0,3,f,g_1,0) \quad : \quad f = g_1 \in \{0, 1, 2\}$.

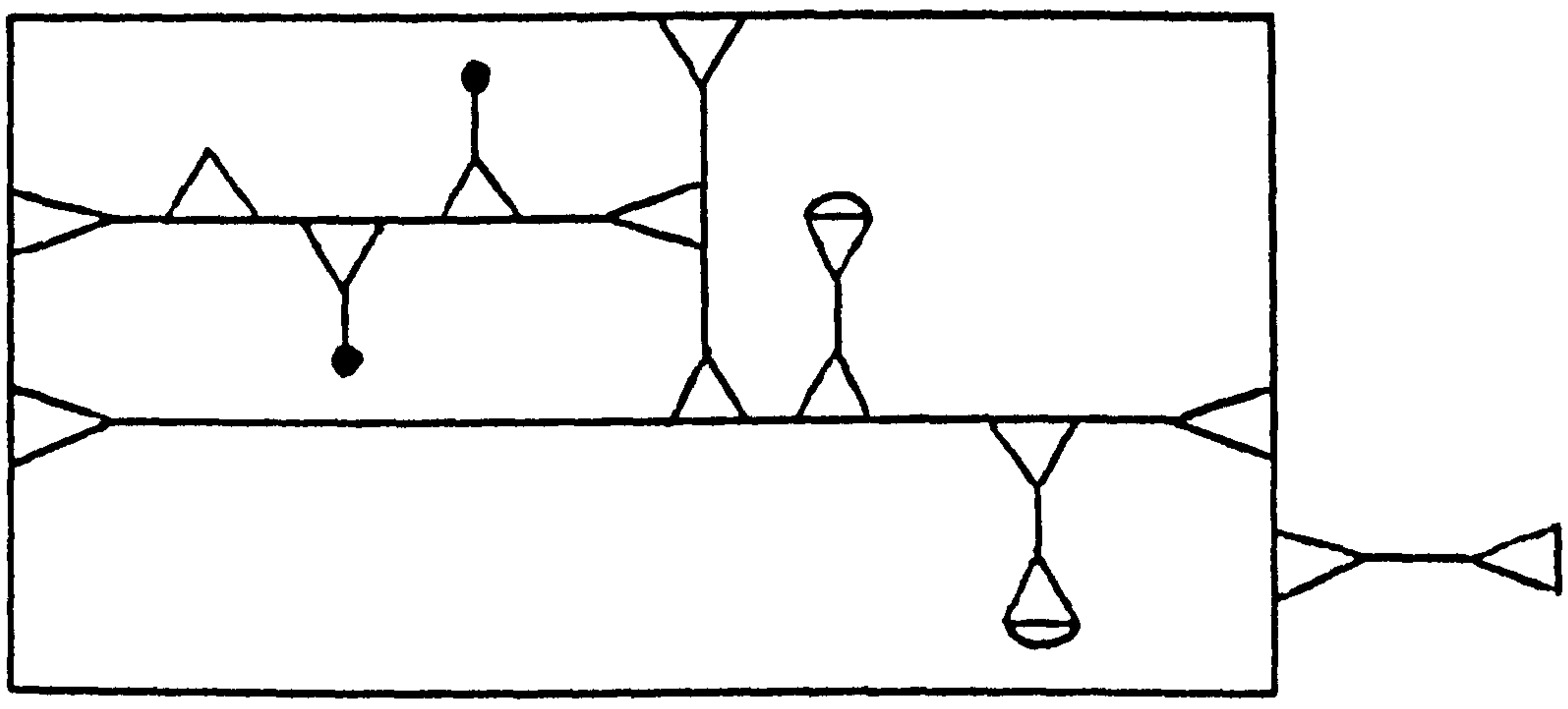
This gives us three cases to consider :

$f = g_1 = 0 \Rightarrow u = -9 < 0$, so that we can ignore this case.

(19,0,3,1,1,0) 1(1)



(47,0,3,2,2,0) 1(1)



Hence, no such S exists. □

<u>LEMMA 2.3.5</u> If $S(u,p,e,f,g_1,g_3)$ satisfies (2.1.2) and $e = 2$, then there exists a coset diagram with specification S which is 1(2), with the following exceptions :

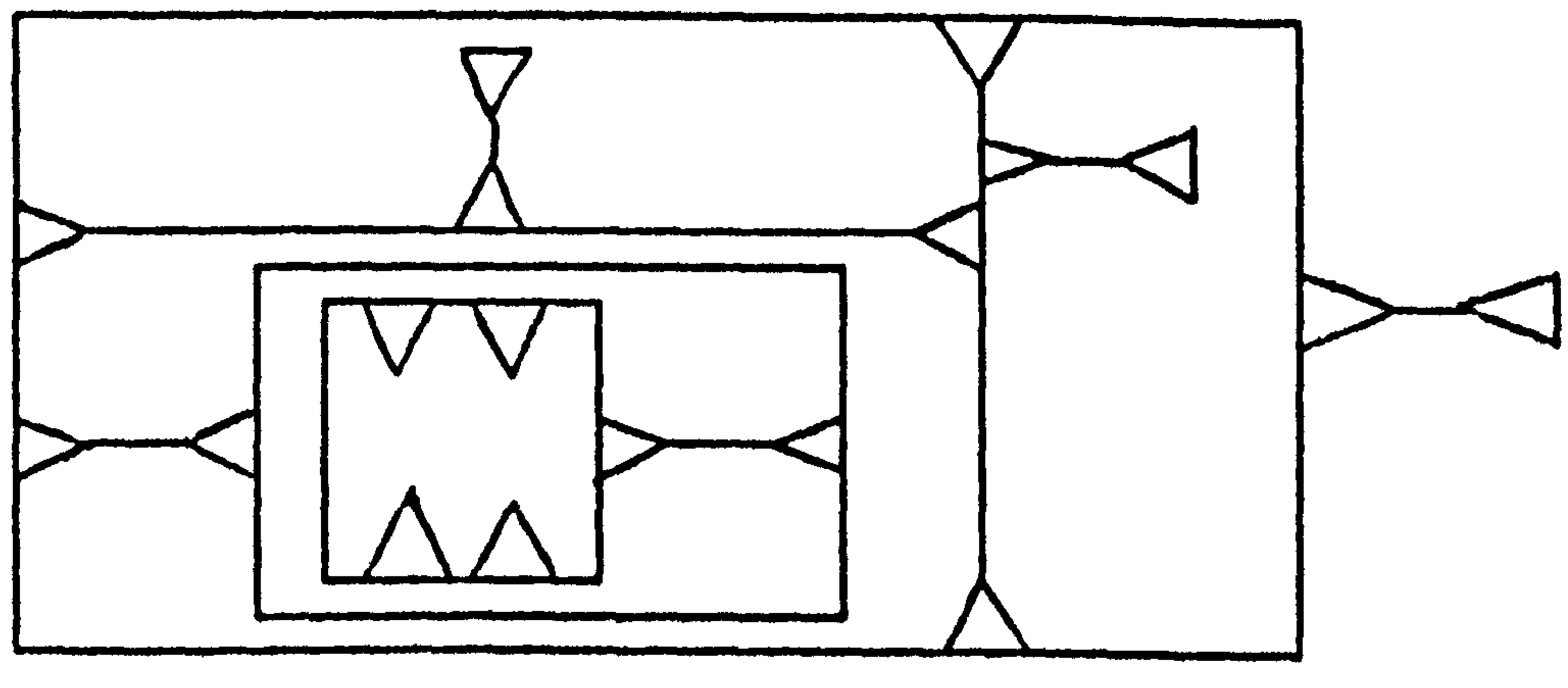
There exist coset diagrams with specifications

(a) (18,1,2,0,0,0), which is 1(1)

(b) (12n+6,0,2,0,0,n+2), $n \geq 0$.

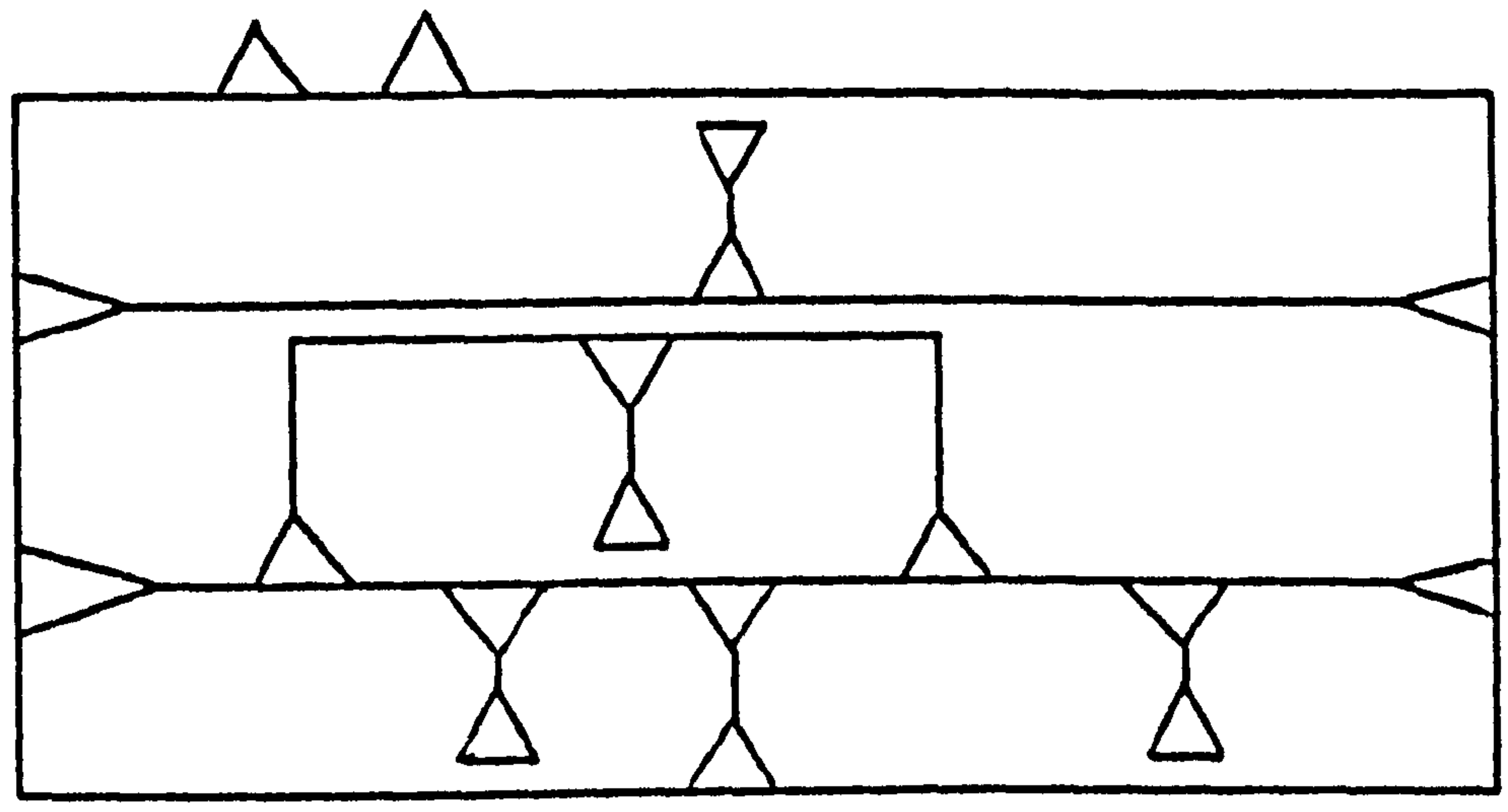
<u>Proof :</u> Assume S is a counter-example with $p + f + g_1 + g_3$ minimal. We want to show that no such S exists.

N_5 (54,0,10,0,0,0) 3(1) 2(2)



1-compose once and 2-compose once to get U (54,2,2,0,0,0) 1(1).

N_6 (54,0,10,0,0,0) 4(1) 1(2)

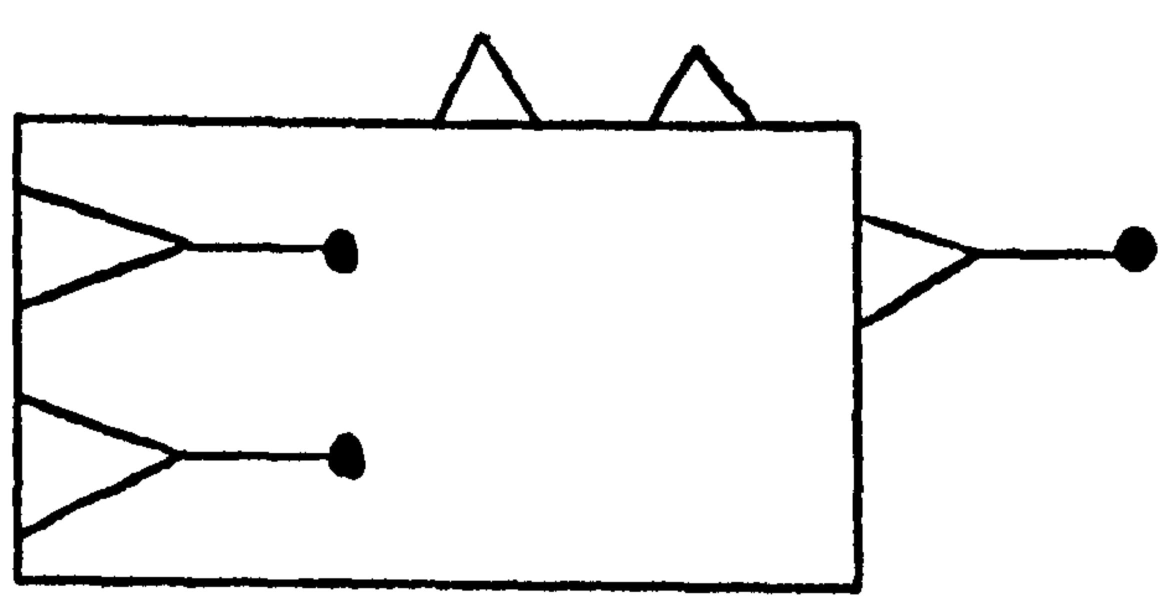


1-compose twice to get U^2 (54,2,2,0,0,0) 1(2).

If $p \geq 2$ and D (u-54,p-2,4,f,g_1,g_3) satisfies (2.1.2), then D can be either 2(2) or 1(1) 1(2) by Lemma 2.3.3, so that both D 2(2) + U^2 1(2) and D 1(1) 1(2) + U 1(1) have specification S, which is 1(2).

Therefore, S has $p < 2$.

Q^2 (18,0,2,3,0,0) 1(2)

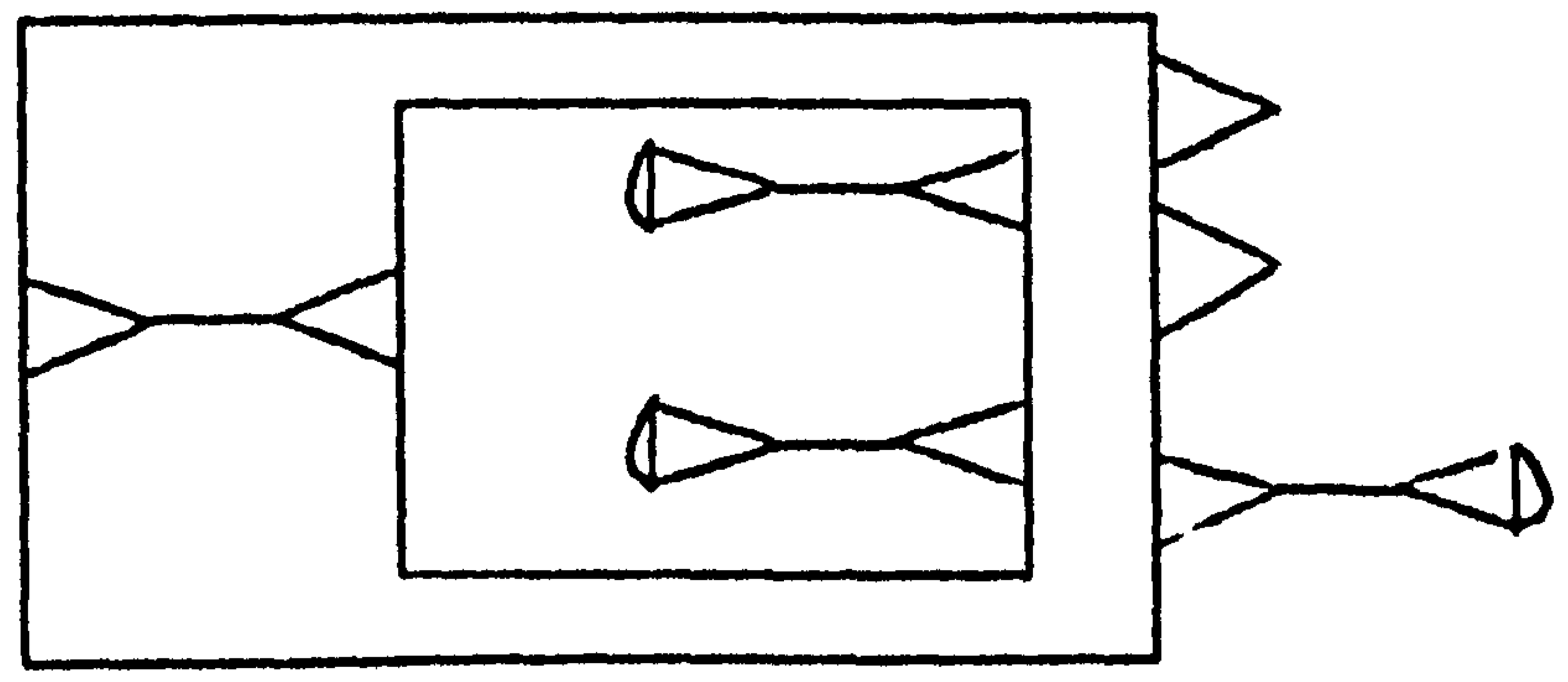


A diagram has already been exhibited for Q (18,0,2,3,0,0) 1(1).

If $f \geq 3$ and D (u-18,p,4,f-3,g_1,g_3) satisfies (2.1.2), then D can be either 2(2) or 1(1) 1(2) by Lemma 2.3.3, so that both D 2(2) + Q^2 1(2) and D 1(1) 1(2) + Q 1(1) have specification S, which is 1(2).

Therefore, S has $f < 3$.

V_0 (30,0,2,0,3,0) 1(2)



If $g_1 \geq 4$ and $D(u-30,p,4,f,g_1-3,g_3)$ satisfies (2.1.2), then D can be 2(2) by Lemma 2.3.3, so that D 2(2) + V_0 1(2) has specification S, which is 1(2).

Therefore, S has $g_1 < 4$.

As shown in Lemma 2.3.2, we have $f \equiv g_1 \pmod 3$.

We now know that a minimal S would have the form

$$(u,p,2,f,g_1,g_3) \quad : \quad p < 2,\ f < 3,\ g_1 < 4,\ f \equiv g_1 \pmod 3$$

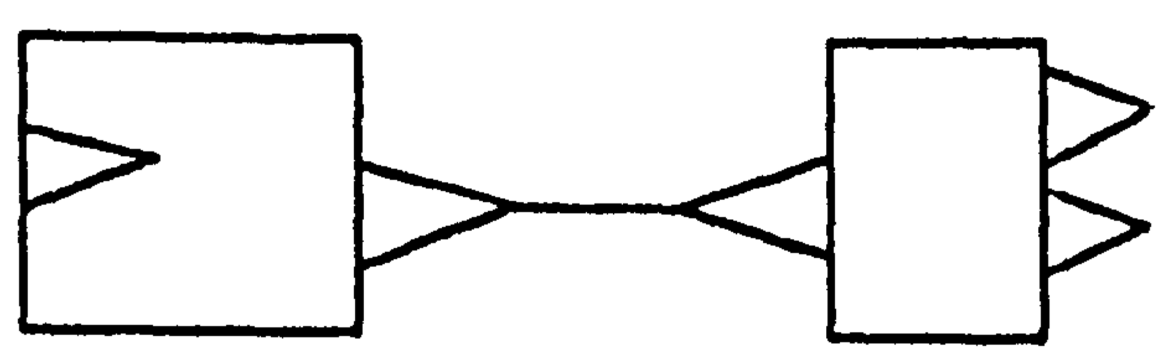
This gives us eight cases to consider :

<u>Case (1)</u> : $W_n(12n+6,0,2,0,0,n+2)$, $n \geq 0$.

Let $T_n = (12n+3,0,3,0,0,n+1)$.

T_0 (3,0,3,0,0,1) 1(2)

T_1 (15,0,3,0,0,2) 1(2)



$\forall\, n \geq 2$, T_n 1(2) = T_{n-2} 1(2) + Γ_2^2(24,0,4,0,0,2) 2(2).

$\therefore$ $W_n = T_0(3,0,3,0,0,1)$ 1(2) + $T_n(12n+3,0,3,0,0,n+1)$ 1(2), $\forall\, n \geq 0$.

Therefore, a coset diagram for W_n exists for each $n \geq 0$.

However, no diagram exists for W_n which is 1(2) $\forall\, n \geq 0$, as then we could have

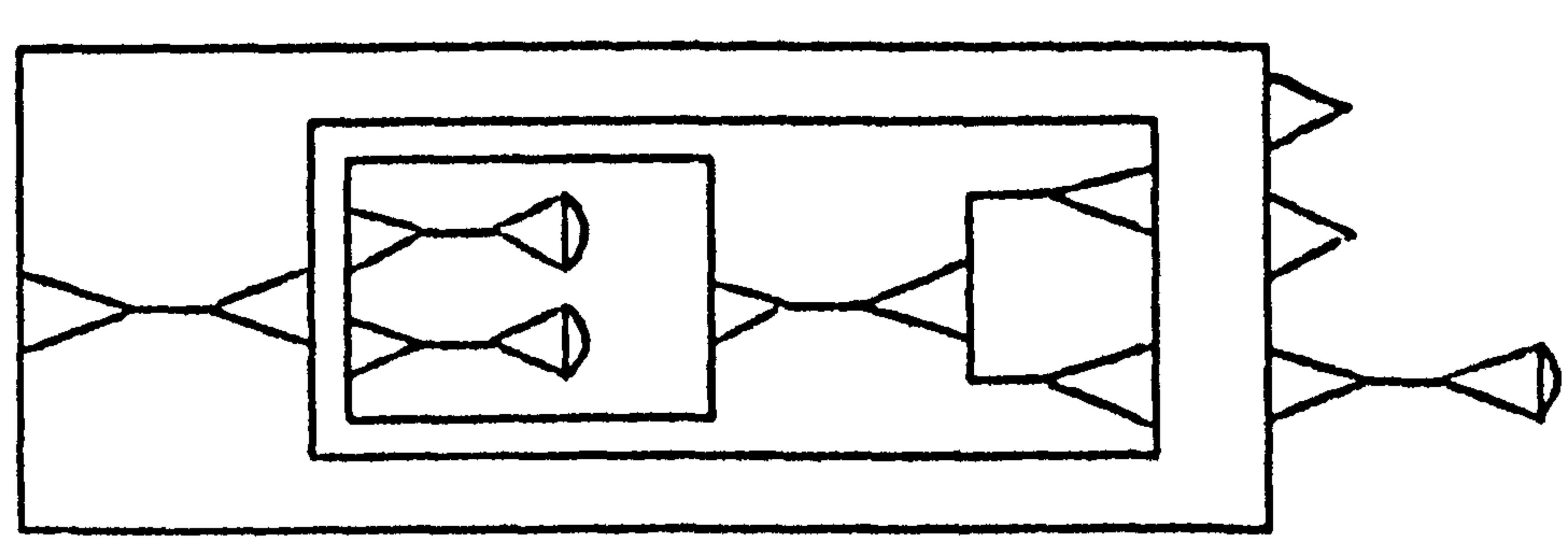
$$W_n\ 1(2) + T_0(3,0,3,0,0,1)\ 1(2) = (12n+9,0,1,0,0,n+3),\ \forall\, n \geq 0.$$

But this would contradict Lemma 2.3.7. This is exception (b).

<u>Case (2)</u> : $V_n(12n+30,0,2,0,3,n)$, $n \geq 0$.

A diagram has already been exhibited for V_0 (30,0,2,0,3,0) 1(2).

V_1 (42,0,2,0,3,1) 1(2)

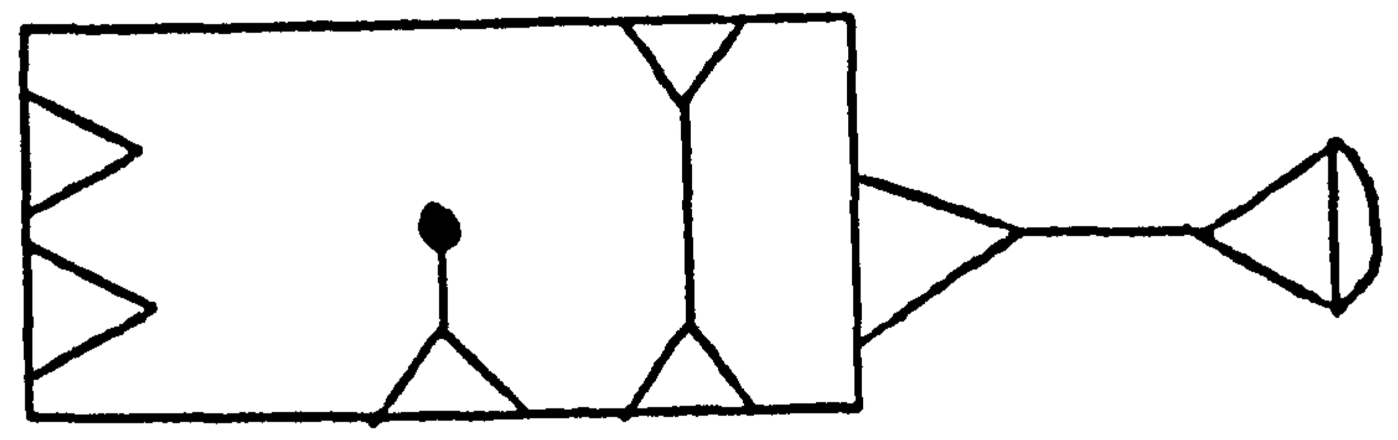


$V_n\, 1(2) = V_{n-2}\, 1(2) + I_2^2\,(24,0,4,0,0,2)\, 2(2),\ \forall\, n \geq 2.$

<u>Case (3)</u> : $X_n\,(12n+10,0,2,1,1,n)$, $n \geq 0$.

$X_0\,(10,0,2,1,1,0)\ 1(2)$

$X_1\,(22,0,2,1,1,1)\ 1(2)$



$X_n\, 1(2) = X_{n-2}\, 1(2) + I_2^2\,(24,0,4,0,0,2)\, 2(2),\ \forall\, n \geq 2.$

<u>Case (4)</u> : $Y_n\,(12n+38,0,2,2,2,n)\ 1(2)$

$= X_n\,(12n+10,0,2,1,1,n)\ 1(2) + F_0^2\,(28,0,4,1,1,0)\ 2(2),\ \forall\, n \geq 0.$

<u>Case (5)</u> : $P_n\,(12n+18,1,2,0,0,n)$, $n \geq 0$.

A diagram has already been exhibited for $N_4\,(18,0,6,0,0,0)\ 1(1)\ 2(2)$.

$P_0\,(18,1,2,0,0,0)\ 1(1) = N_4\,(2)$. This is exception (a), and its proof follows.

<u>Case (5A) : Assume (18,1,2,0,0,0) has a coset diagram which is 1(2)</u>

Then either there exists a diagram for X (18,0,6,0,0,0) 3(2), or there exists a diagram for Y (18,0,6,0,0,0) 2(1) 1(2), both of which could then be composed to give a diagram for (18,1,2,0,0,0) 1(2).

<u>Case (5A1) : Assume (18,0,6,0,0,0) has a coset diagram X which is 3(2)</u>

X has 6Δ, six red points and two green 9-cycles. We can start with a 1(2) part of the diagram as follows and then build this up noting that there are no blue points, 1-cycles or 3-cycles.

If $2 \to 1$, then 2-cycle. If $2 \to 2$, then $1 \to \Delta \Rightarrow$ X cannot be 3(2). $\therefore$ $2 \to \Delta$.

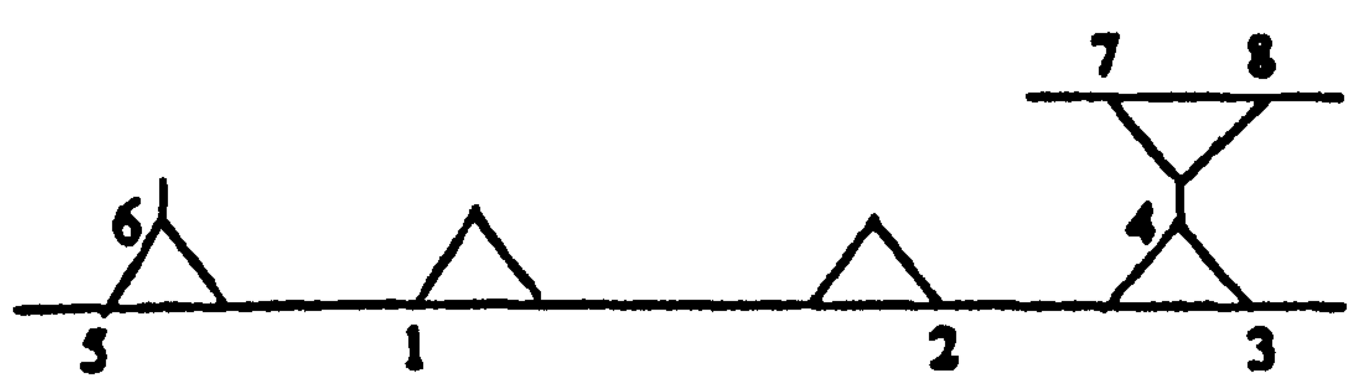
If $1 \to 1$, then X cannot be 3(2). If $1 \to 3$, then 3-cycle.

If $1 \to 4$, then 5-cycle. $\therefore$ $1 \to \Delta$.

If $4 \to 3$, then 1-cycle. If $4 \to 6$, then 6-cycle.

If $4 \to 4$, then $3 \to \Delta \Rightarrow 5 \to \Delta \Rightarrow 6 \to 6 \Rightarrow 10^+$-cycle.

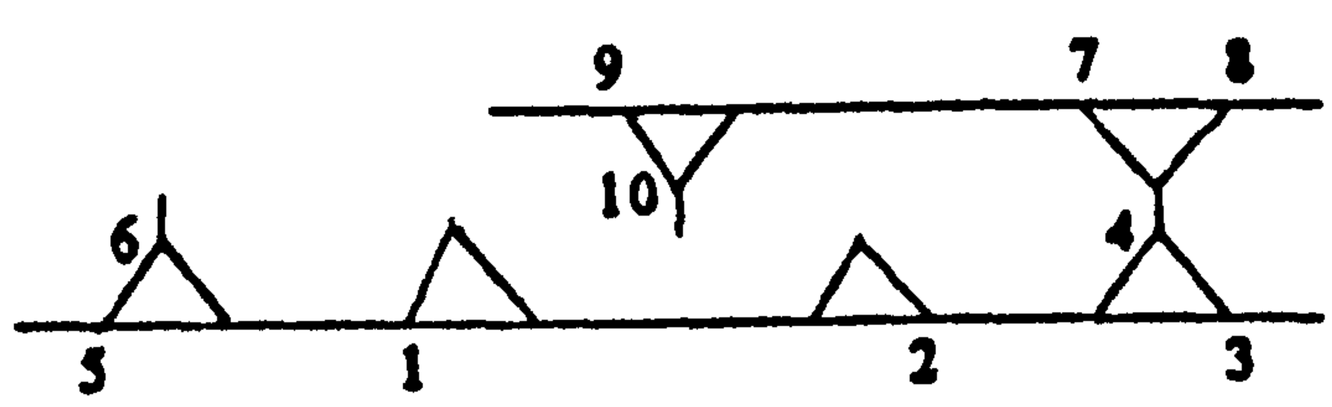
If $4 \to 5$, then $6 \to \Delta \Rightarrow$ 1-cycle. $\therefore$ $4 \to \Delta$.



If $7 \to 3$, then 11^+-cycle. If $7 \to 6$, then 7-cycle. If $7 \to 8$, then 1-cycle.

If $7 \to 7$, then $8 \to \Delta$ with a red point $\Rightarrow$ X cannot be 3(2).

If $7 \to 5$, then $6 \to \Delta \Rightarrow 10^+$-cycle. $\therefore$ $7 \to \Delta$.



If $5 \to 3$, then 4-cycle. If $5 \to 6$, then 1-cycle.

If $5 \to 5$ then X cannot be 3(2).

If $5 \to 8$, then $3 \to 3$, $6 \to 6$, $9 \to 9$, $10 \to 10 \Rightarrow$ 6-cycle.

If $5 \to 9$, then $3 \to 3$, $6 \to 6$, $8 \to 8$, $10 \to 10 \Rightarrow$ 8-cycle.

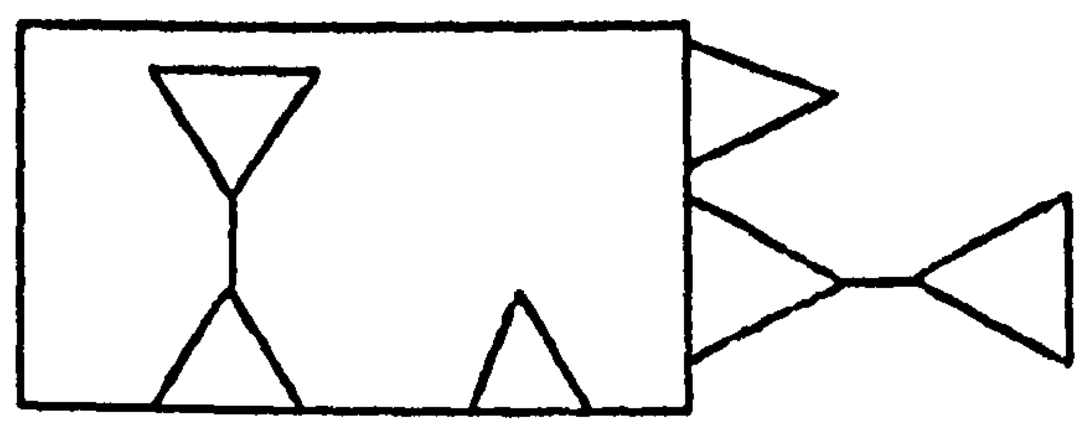
If $5 \to 10$, then $3 \to 3$, $6 \to 6$, $8 \to 8$, $9 \to 9 \Rightarrow$ X is not 3(2).

Contradiction. $\therefore$ Case(5A1) is not possible.

Case (5A2) : Assume (18,0,6,0,0,0) has a coset diagram Y which is 2(1) 1(2)

Y has 6Δ, six red points and two green 9-cycles. Any 1(1) part of a diagram contributes five green lines. Therefore, the two 1(1) parts of Y must belong to different 9-cycles. This takes care of 4Δ. The other 2Δ must each have a red point.

However, to obtain two 9-cycles, we must have the following diagram :-

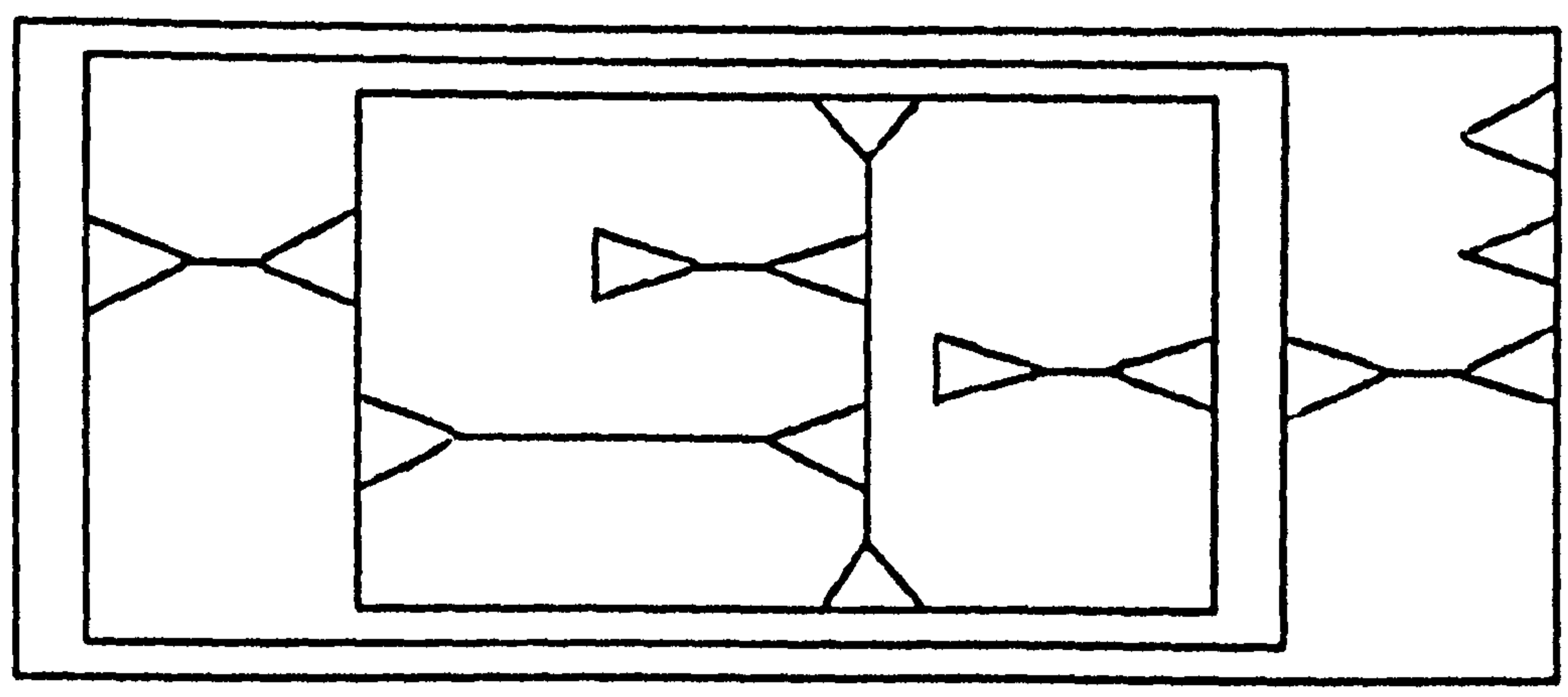


This diagram is 2(1), not 2(1) 1(1).

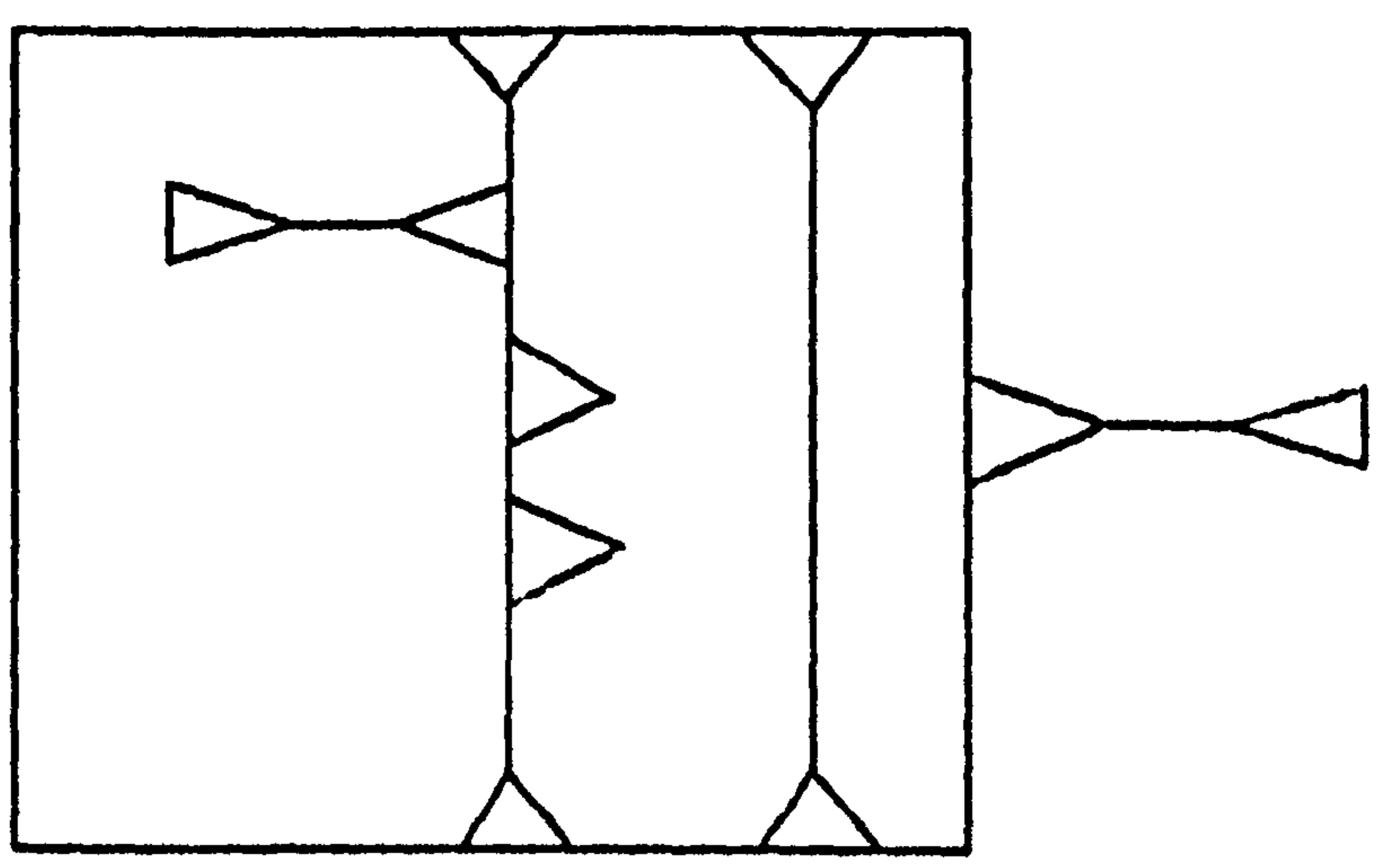
Contradiction. ∴ Case(5A2) is not possible.

Therefore, Case (5A) is not possible. Hence, P_0 does not have a 1(2) diagram.

N_7 (42,0,6,0,0,2) 2(1) 1(2)



N_8 (30,0,6,0,0,1) 2(1) 1(2)



P_2 (42,1,2,0,0,2) 1(2) = N_7 (1).

P_1 (30,1,2,0,0,1) 1(2) = N_8 (1).

For n ≥ 3, P_n 1(2) = P_{n-2} 1(2) + Γ_2^2 (24,0,4,0,0,2) 2(2).

Case (6) : Z_n (12n+66,1,2,0,3,n) 1(2)

$= A_0^2$ (36,1,4,0,0,0) 2(2) + V_n (12n+30,0,2,0,3,n) 1(2), $\forall n \geq 0$.

Case (7) : (12n+46,1,2,1,1,n) 1(2)

$= A_0^2$ (36,1,4,0,0,0) 2(2) + X_n (12n+10,0,2,1,1,n) 1(2), $\forall n \geq 0$.

Case (8) : (12n+74,1,2,2,2,n) 1(2)

$= A_0^2$ (36,1,4,0,0,0) 2(2) + Y_n (12n+38,0,2,2,2,n) 1(2), $\forall n \geq 0$.

Hence, no such S exists. □

LEMMA 2.3.6 If S (u,p,e,f,g_1,g_3) satisfies (2.1.2) and $e = 2$, then there exists a coset diagram with specification S which is 1(1), with the following exceptions :

There exist coset diagrams with specifications

(a) (30,0,2,0,3,0), which is 1(2)

(b) (12n+6,0,2,0,0,n+2), $n \geq 0$.

Proof : Assume S is a counter-example with $p + f + g_1 + g_3$ minimal. We want to show that no such S exists.

In Lemma 2.3.5, we obtained

U (54,2,2,0,0,0) 1(1) and U^2 (54,2,2,0,0,0) 1(2).

If $p \geq 2$ and D $(u-54,p-2,4,f,g_1,g_3)$ satisfies (2.1.2), then D can be either 2(1) or 1(1) 1(2) by Lemma 2.3.3, so that both D 2(1) + U 1(1) and D 1(1) 1(2) + U^2 1(2) have specification S, which is 1(1).

Therefore, S has $p < 2$.

Diagrams have already been exhibited for

$Q\,(18,0,2,3,0,0)\;1(1)$ and $Q^2\,(18,0,2,3,0,0)\;1(2)$.

If $f \geq 3$ and $D\,(u-18,p,4,f-3,g_1,g_3)$ satisfies (2.1.2), then D can be either 2(1) or 1(1) 1(2) by Lemma 2.3.3, so that both $D\,2(1) + Q\,1(1)$ and $D\,1(1)\,1(2) + Q^2\,1(2)$ have specification S, which is 1(1).

Therefore, S has $f < 3$.

Diagrams have already been exhibited for

$N_4\,(18,0,6,0,0,0)\;1(1)\,2(2)$ and $V_0\,(30,0,2,0,3,0)\;1(2)$.

$\therefore \quad D_0^{1/2}\,(48,0,4,0,3,0)\;1(1)\,1(2) = N_4\,1(1)\,2(2) + V_0\,1(2)$.

If $g_1 \geq 4$ and $D\,(u-48,p,2,f,g_1-3,g_3)$ satisfies (2.1.2), then D can be 1(2) by Lemma 2.3.5, so that $D\,1(2) + D_0^{1/2}\,1(1)\,1(2)$ has specification S, which is 1(1). Therefore, S has $g_1 < 4$.

As shown in Lemma 2.3.2, we have $f \equiv g_1 \pmod 3$.

We now know that a minimal S would have the form

$$(u,p,2,f,g_1,g_3) \quad : \quad p < 2,\ f < 3,\ g_1 < 4,\ f \equiv g_1 \pmod 3$$

This gives us eight cases to consider :

Case (1) : $W_n\,(12n+6,0,2,0,0,n+2)$, $n \geq 0$.

In Lemma 2.3.5, Case (1), we showed that a diagram exists for each $n \geq 0$.

However, no diagram exists for W_n which is 1(1) $\forall\, n \geq 0$, as then we could have

$$W_n\,1(1) + T_0\,(3,0,3,0,0,1)\;1(1) = (12n+9,0,1,0,0,n+3),\ \forall\, n \geq 0.$$

But this would contradict Lemma 2.3.7. This is exception (b).

Note that the diagram for T_0 can be interpreted as either 1(1) or 1(2).

Case (2) : $V_n\,(12n+30,0,2,0,3,n)$, $n \geq 0$.

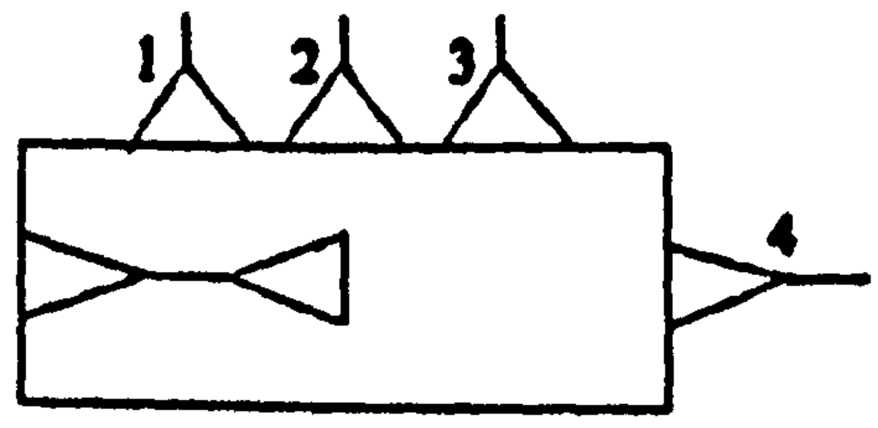
A diagram has already been exhibited for $V_0\,(30,0,2,0,3,0)\;1(2)$.

This is exception (a), and its proof follows.

Case (2A) : Assume (30,0,2,0,3,0) has a coset diagram D which is 1(1)

D has 10 Δ, two red points on the same triangle, three green 1-cycles and three green 9-cycles.

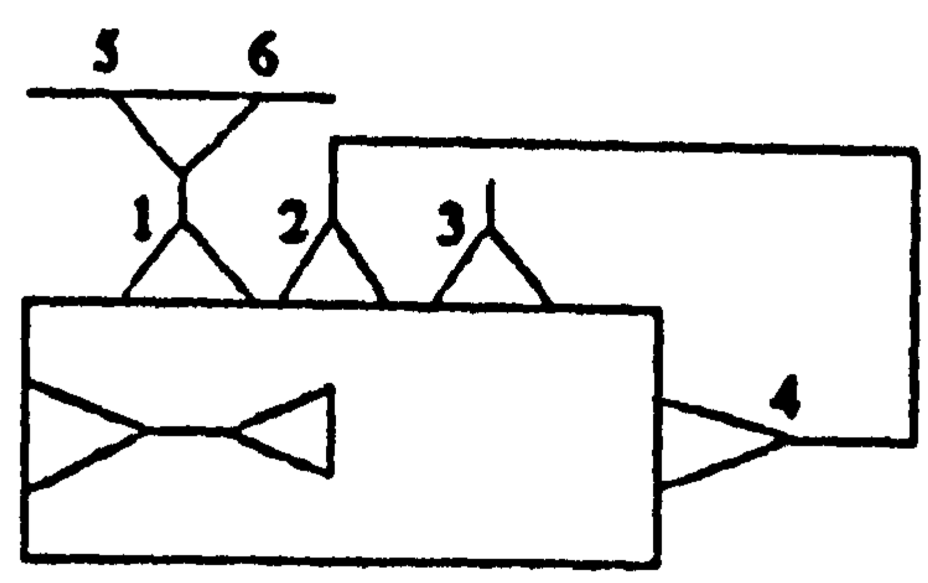
D is 1(1), so we can start with a 9-cycle as follows, and then build this up noting that there are no blue points, 3-cycles or any more red points.



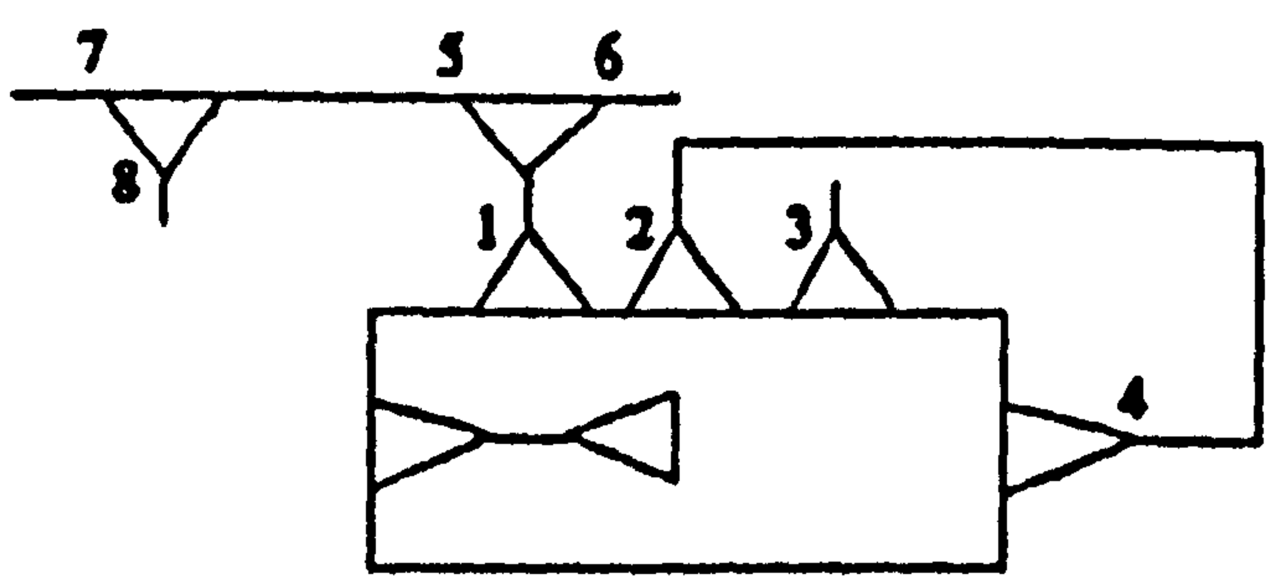
If $2 \to 1$ or $2 \to 3$, then 2-cycle.

Case (2A1) : Assume $2 \to 4$

If $1 \to 3$, then D has only 6Δ. $\therefore$ $1 \to \Delta$.

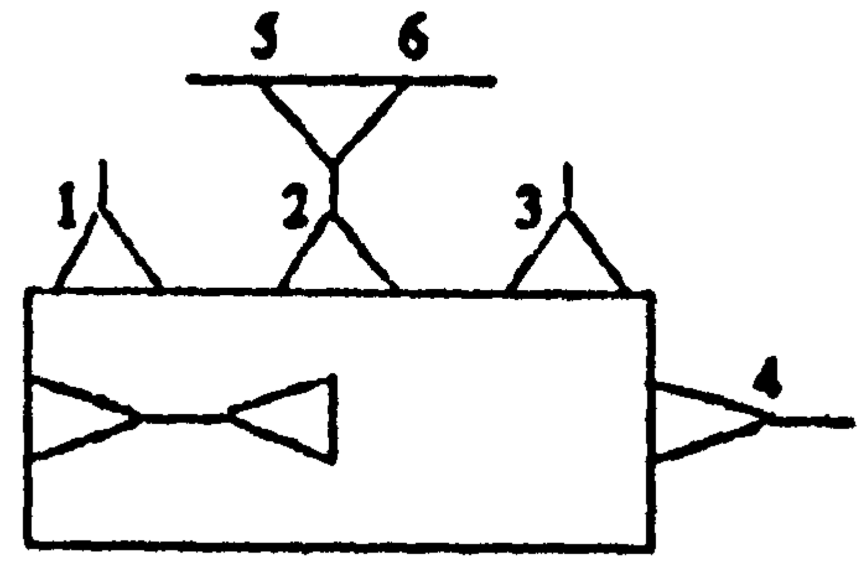


If $5 \to 3$, then 12^+-cycle. If $5 \to 6$, then 7-cycle. $\therefore$ $5 \to \Delta$.



To obtain three 1-cycles, we must have $3 \to \Delta$, $6 \to \Delta$ $\Rightarrow$ 6-cycle.

Contradiction. $\therefore$ Case (2A1) is not possible. Hence, $2 \to \Delta$.

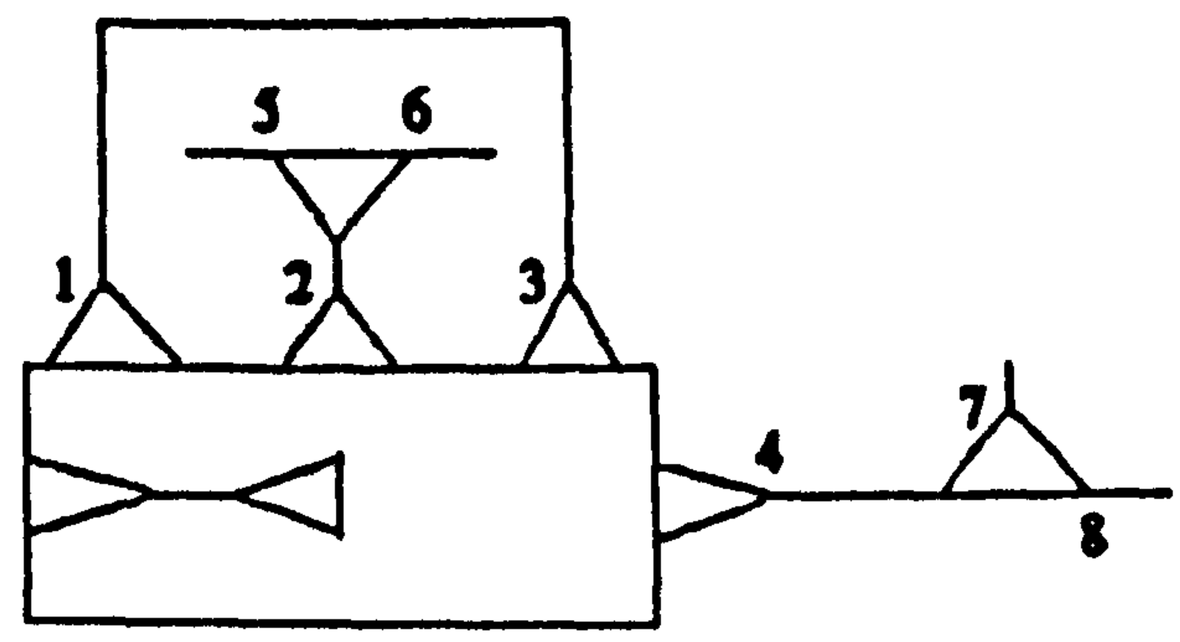


If 3 → 4, then 2-cycle. If 3 → 6, then 3-cycle.

If 3 → 5, then 1 → Δ, 4 → Δ, 6 → Δ for three 1-cycles ⇒ 6-cycle.

Case (2A2) : Assume 3 → 1

If 4 → 5 or 4 → 6, then 12^+-cycle. ∴ 4 → Δ.

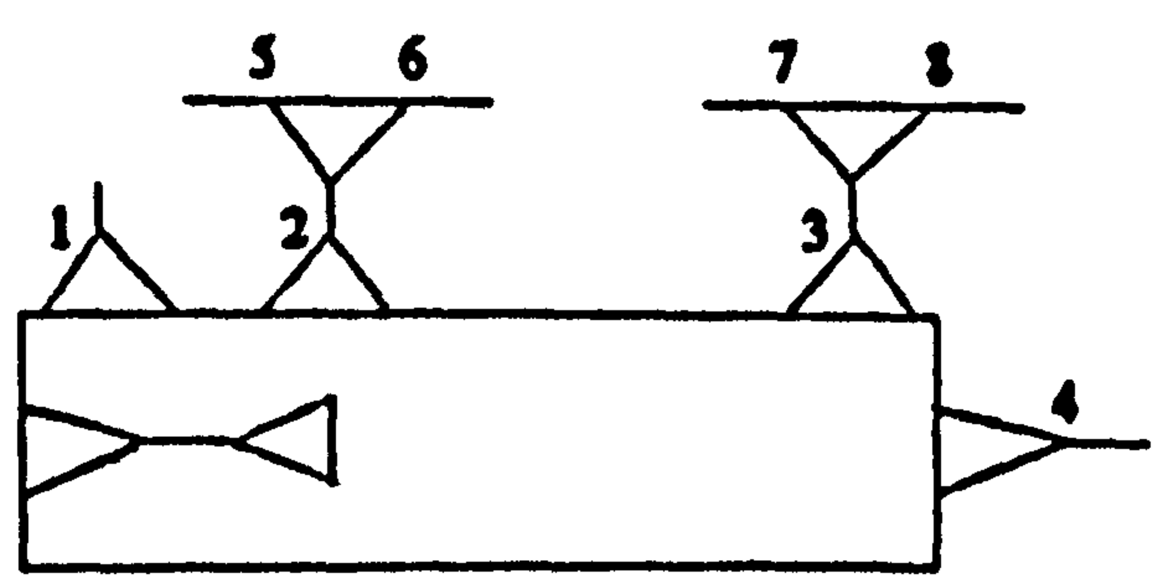


If 7 → 5 or 7 → 6, then three 1-cycles not possible.

If 7 → 8, then 7-cycle. ∴ 7 → Δ and 8 → Δ. But this creates a 6-cycle.

Contradiction. ∴ Case (2A2) is not possible.

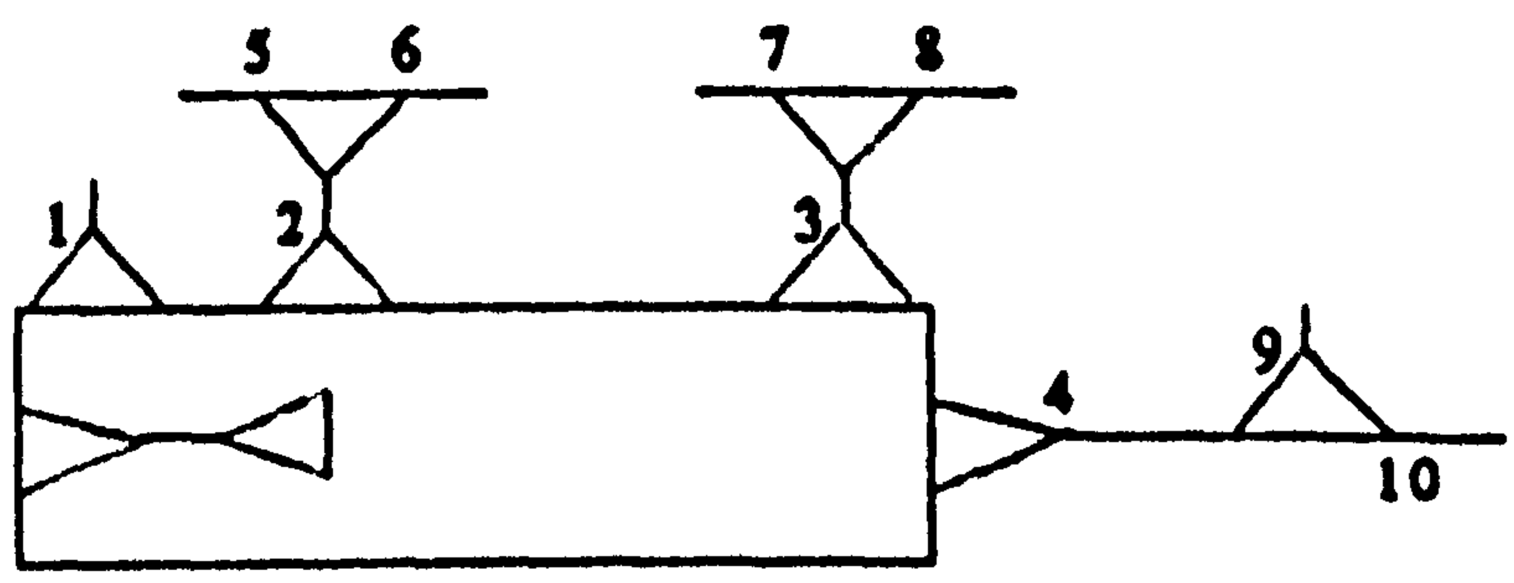
Hence, 3 → Δ.



If 4 → 5, then 1 → Δ and 6 → Δ ⇒ 8-cycle. If 4 → 1 or 4 → 8, then 3-cycle.

If 4 → 6, then 1 → Δ and 5 → Δ ⇒ 7-cycle.

If 4 → 7, then 1 → Δ and 8 → Δ ⇒ 6-cycle. ∴ 4 → Δ.



If 1 → 5, then 3-cycle. If 1 → 10, then 4-cycle.

If 1 → 6, then 5 → Δ ⇒ 6-cycle. If 1 → 7, then 8 → Δ ⇒ 7-cycle.

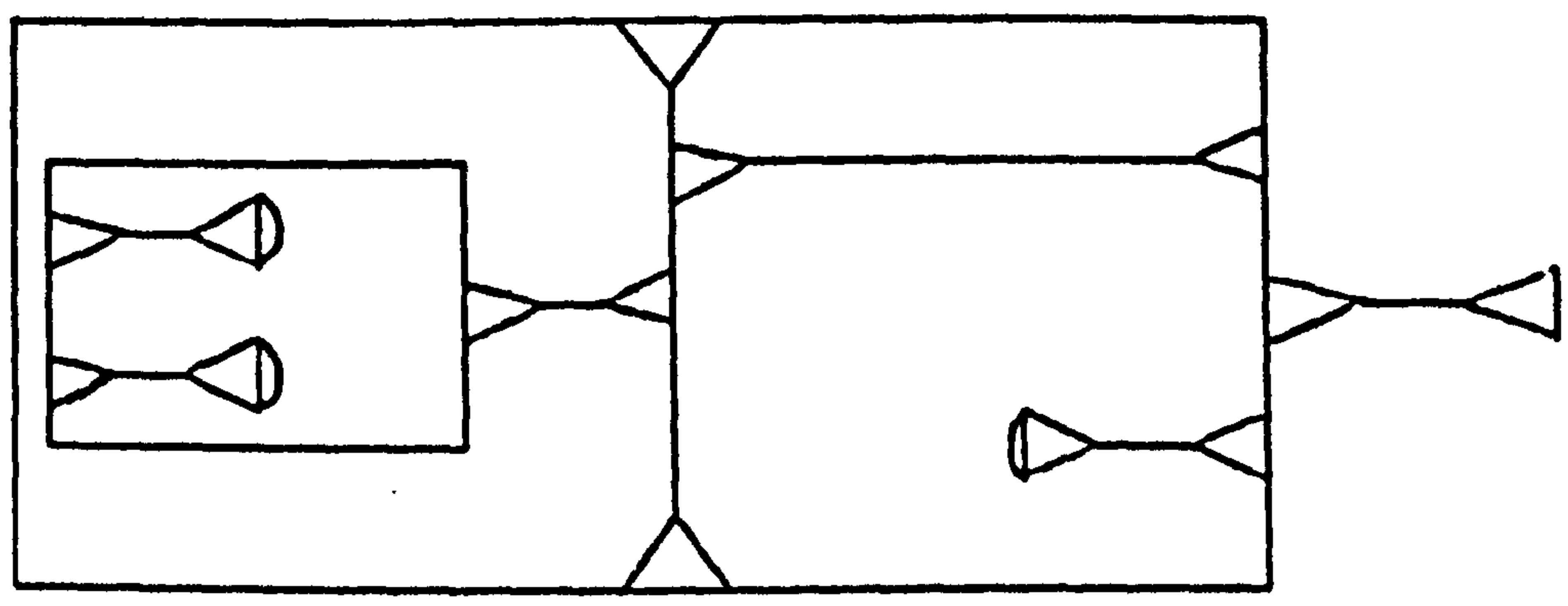
If $1 \to 8$, then $7 \to \Delta \Rightarrow$ 8-cycle. If $1 \to 9$, then $10 \to \Delta \Rightarrow$ 7-cycle.

If $1 \to \Delta$, then D has four 1-cycles.

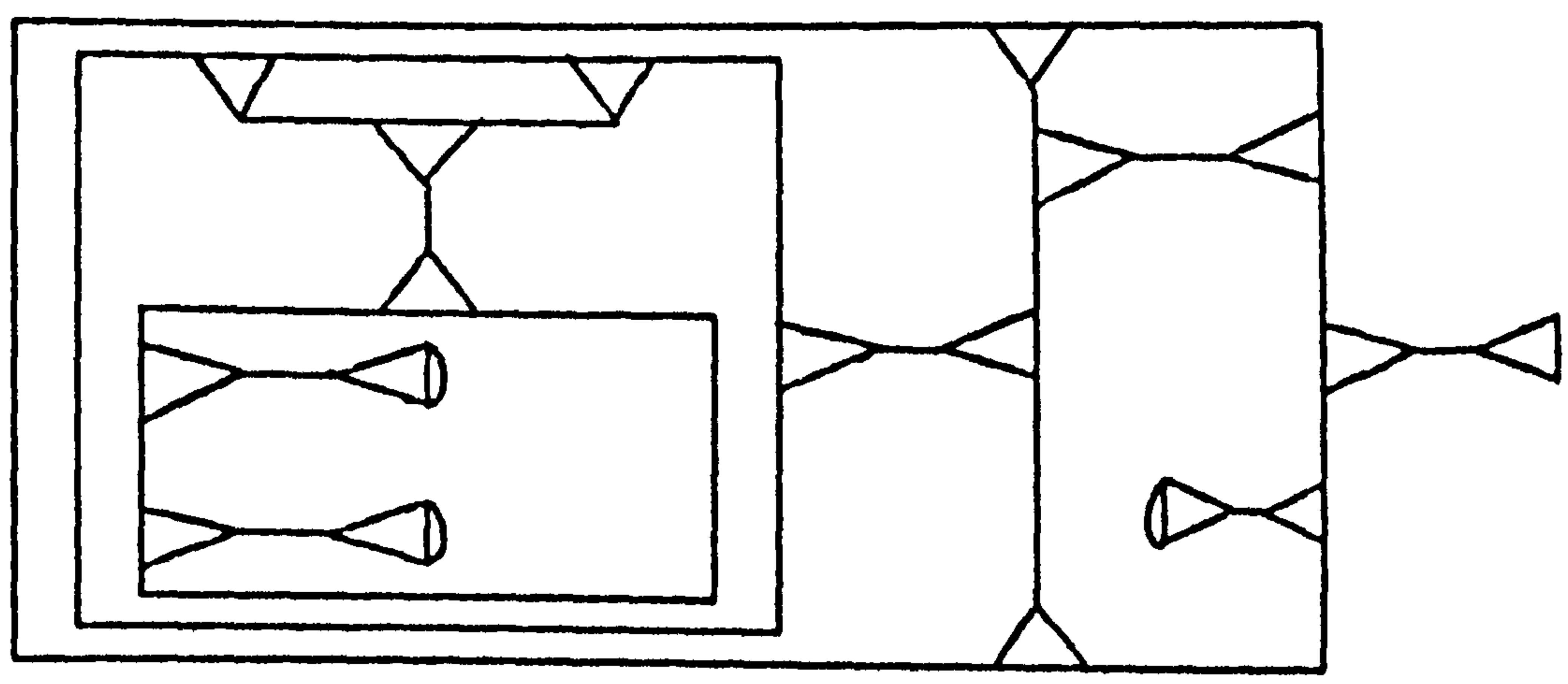
Contradiction. Therefore, Case (2A) not possible.

Hence, no diagram exists for V_0 which is 1(1).

V_1 (42,0,2,0,3,1) 1(1)



V_2 (54,0,2,0,3,2) 1(1)

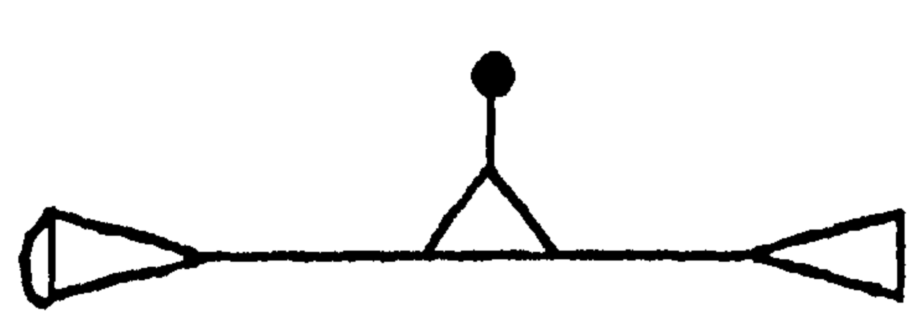


$\forall n \geq 3$, V_n 1(1) = V_{n-2} 1(1) + I_2 (24,0,4,0,0,2) 2(1).

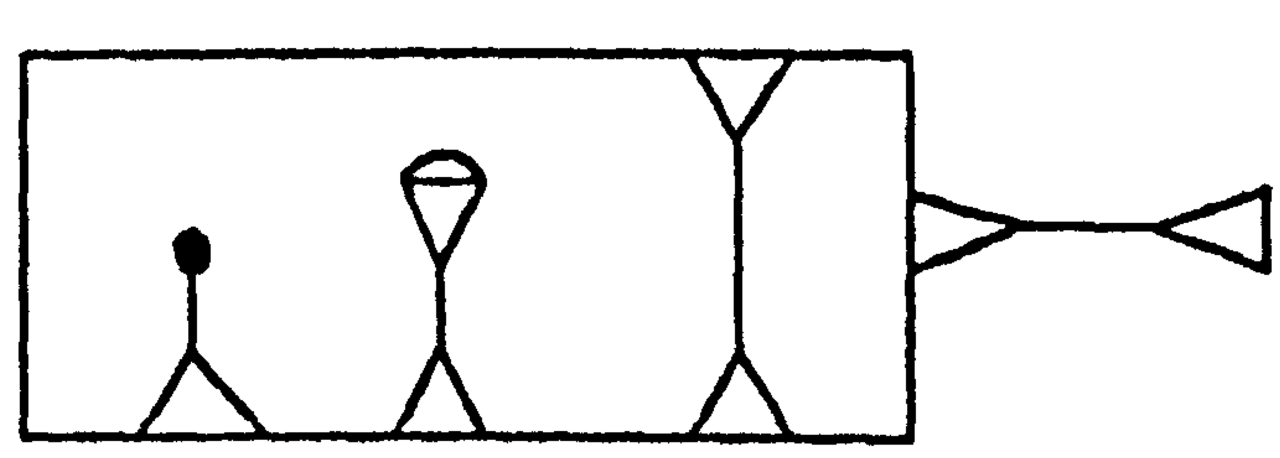
Hence, there exists a diagram for V_n which is 1(1), $\forall n \geq 1$.

<u>Case (3)</u> : X_n (12n+10,0,2,1,1,n) , $n \geq 0$.

X_0 (10,0,2,1,1,0) 1(1)



X_1 (22,0,2,1,1,1) 1(1)



$\forall n \geq 2$, X_n 1(1) = X_{n-2} 1(1) + I_2 (24,0,4,0,0,2) 2(1).

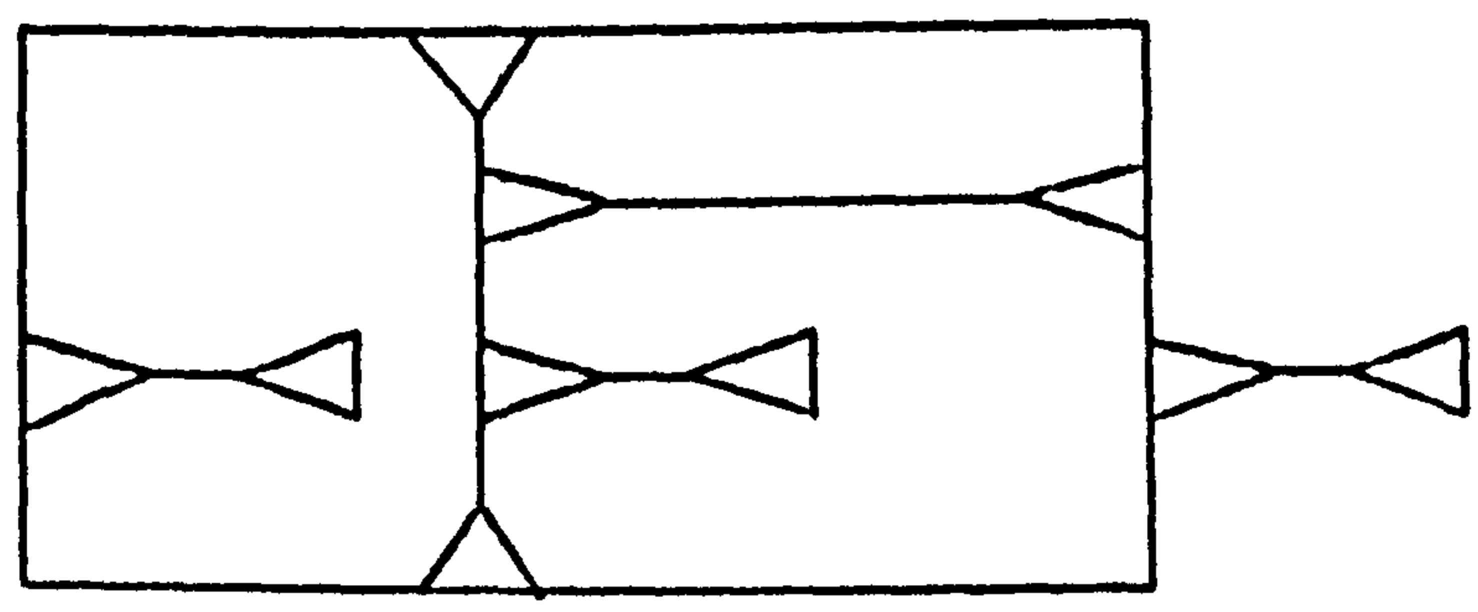
Case (4) : Y_n (12n+38,0,2,2,2,n) 1(1)

$= X_n$ (12n+10,0,2,1,1,n) 1(1) + F_0 (28,0,4,1,1,0) 2(1), $\forall n \geq 0$.

Case (5) : P_n (12n+18,1,2,0,0,n), $n \geq 0$.

A diagram has already been exhibited for N_4 (18,0,6,0,0,0) 1(1) 2(2).

N_9 (30,0,6,0,0,1) 3(1)

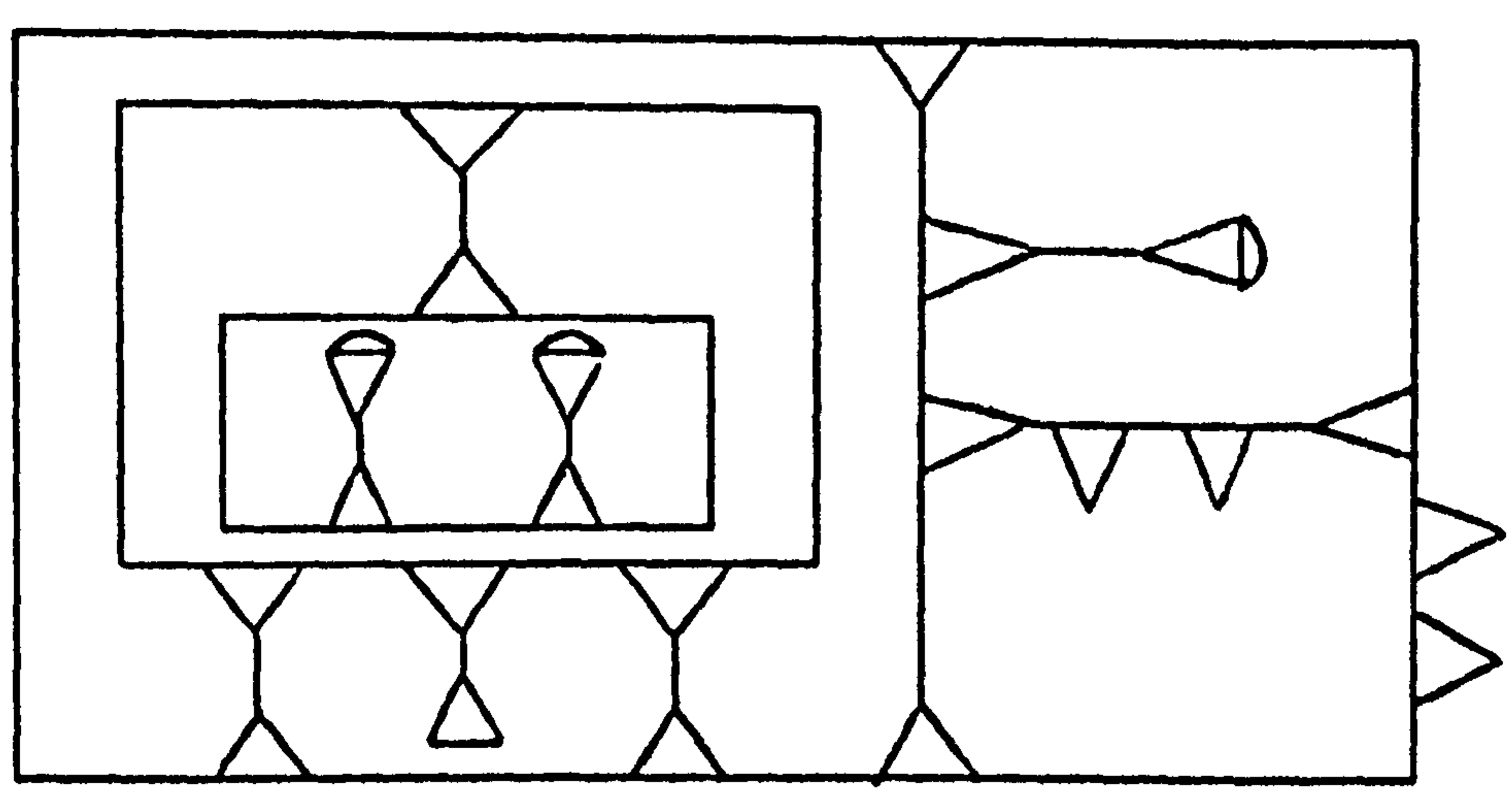


P_0 (18,1,2,0,0,0) 1(1) = N_4 (2). P_1 (30,1,2,0,0,1) 1(1) = N_9 (1).

$\forall n \geq 2$, P_n 1(1) = P_{n-2} 1(1) + I_2 (24,0,4,0,0,2) 2(1).

Case (6) : Z_n (12n+66,1,2,0,3,n), $n \geq 0$.

N_{10} (66,0,6,0,3,0) 1(1) 2(2)



Z_0 (66,1,2,0,3,0) 1(1) = N_{10} (2).

$\forall n \geq 1$, Z_n 1(1) = A_0 (36,1,4,0,0,0) 2(1) + V_n (12n+30,0,2,0,3,n) 1(1).

Case (7) : (12n+46,1,2,1,1,n) 1(1)

$= A_0$ (36,1,4,0,0,0) 2(1) + X_n (12n+10,0,2,1,1,n) 1(1), $\forall n \geq 0$.

Case (8) : (12n+74,1,2,2,2,n) 1(1)

$= A_0$ (36,1,4,0,0,0) 2(1) + Y_n (12n+38,0,2,2,2,n) 1(1), $\forall n \geq 0$.

Hence, no such S exists. □

LEMMA 2.3.7 If S (u,p,e,f,g_1,g_3) satisfies (2.1.2) and e = 1, then there exists a coset diagram with specification S, with the following exception :

(12n+9,0,1,0,0,n+3), $n \geq 0$.

Proof : Assume S is a counter-example with p + f + g_1 + g_3 minimal. We want to show that no such S exists.

If $p \geq 1$ and D (u,p−1,5,f,g_1,g_3) satisfies (2.1.2), then D can be 2(1) by Lemma 2.3.2, so we can 1-compose D once to get a diagram with specification S. Therefore, S has p < 1. i.e. S has p = 0.

A diagram has already been exhibited for Q (18,0,2,3,0,0) 1(1).

If $f \geq 3$ and D (u−18,p,3,f−3,g_1,g_3) satisfies (2.1.2), then D can be 1(1) by Lemma 2.3.4, so that D 1(1) + Q 1(1) has a diagram with specification S. Therefore, S has f < 3.

A diagram has already been exhibited for R^2 (39,0,3,0,3,0) 1(2).

If $g_1 \geq 4$ and D (u−39,p,2,f,g_1−3,g_3) satisfies (2.1.2), then D can be 1(2) by Lemma 2.3.5, so that D 1(2) + R^2 1(2) has a diagram with specification S. Therefore, S has $g_1 < 4$.

As shown in Lemma 2.3.2, we have $f \equiv g_1$ (mod 3).

We now know that a minimal S would have the form

(u,0,1,f,g_1,g_3) : $f < 3$, $g_1 < 4$, $f \equiv g_1$ (mod 3)

This gives us four cases to consider :

Case (1) : (12n+1,0,1,1,1,n) , n ≥ 0.

n = 0 (1,0,1,1,1,0) : The diagram consists of a blue point with a red loop.

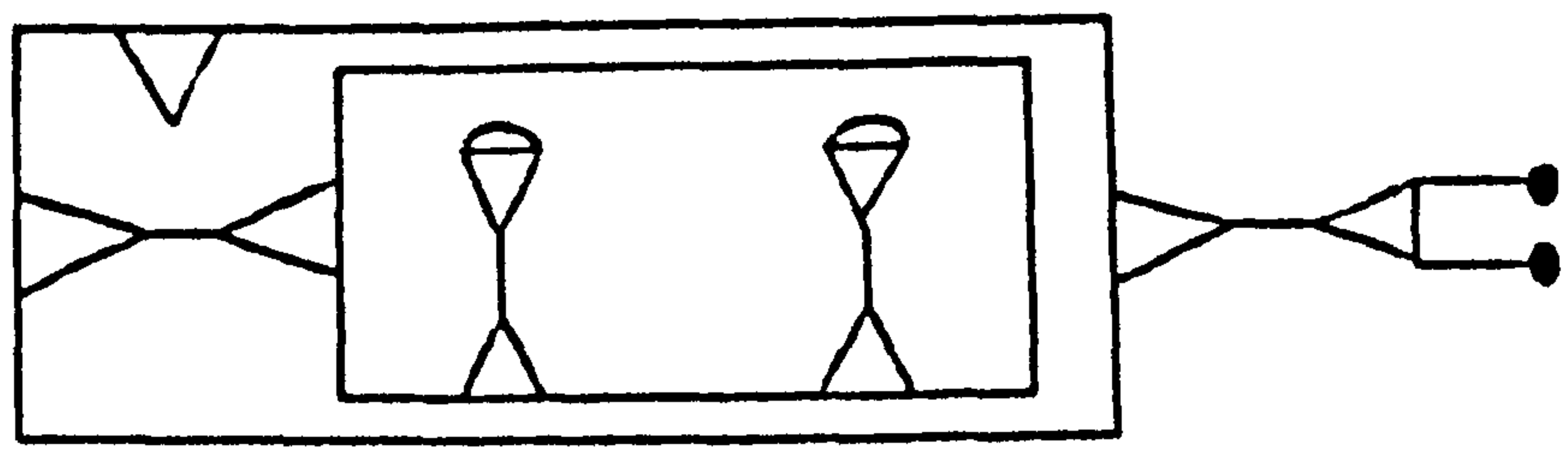
From Lemma 2.3.6, we can have X_n (12n+10,0,2,1,1,n) 1(1), ∀ n ≥ 0.

∴ (12n+13,0,1,1,1,n+1) = X_n 1(1) + T_0 (3,0,3,0,0,1) 1(1), ∀ n ≥ 0.

Hence, a diagram exists ∀ n ≥ 0.

Case (2) : (12n+29,0,1,2,2,n) , n ≥ 0.

n = 0 (29,0,1,2,2,0)



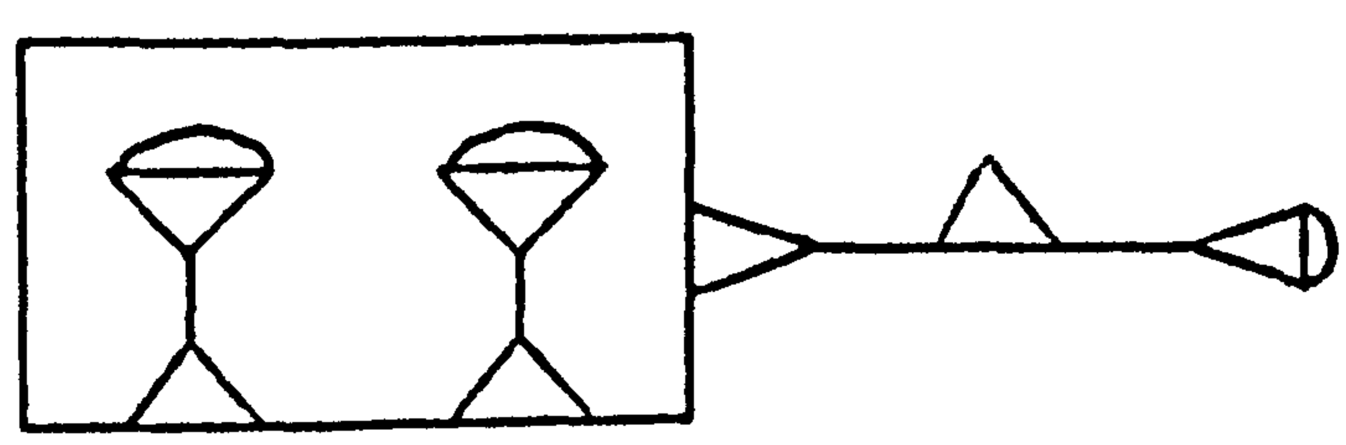
From Lemma 2.3.6, we can have Y_n (12n+38,0,2,2,2,n) 1(1), ∀ n ≥ 0.

∴ (12n+41,0,1,2,2,n+1) = Y_n 1(1) + T_0 (3,0,3,0,0,1) 1(1), ∀ n ≥ 0.

Hence, a diagram exists ∀ n ≥ 0.

Case (3) : (12n+21,0,1,0,3,n) , n ≥ 0.

n = 0 (21,0,1,0,3,0)



From Lemma 2.3.5, we can have V_n (12n+30,0,2,0,3,n) 1(2), ∀ n ≥ 0.

∴ (12n+33,0,1,0,3,n+1) = V_n 1(2) + T_0 (3,0,3,0,0,1) 1(2), ∀ n ≥ 0.

Hence, a diagram exists ∀ n ≥ 0.

Case (4) : (12n+9,0,1,0,0,n+3) , n ≥ 0.

This is the exception. Assume, with a view to obtaining a contradiction, that (12n+9,0,1,0,0,n+3) has a coset diagram for some n ≥ 0.

$u = 12n + 9$, $p = f = g_1 = 0$, $e = 1$, $g_3 = n + 3$, $g_9 = \frac{1}{9}(u - 3g_3 - g_1) = n$.

The diagram has $\frac{u-e}{2} = 6n + 4$ red lines, and $\frac{u-f}{3} = 4n + 3$ blue triangles.

By Theorem 2 in [1], the following two equations are satisfied :

$$q_n^9 q_{n+3}^3 j = p_{4n+3}^3 \tag{2.3.3}$$

$$q_n^9 q_{n+3}^3 (j - 1728) = r_{6n+4}^2 s_1 \tag{2.3.4}$$

p_i, q_i, r_i, s_i are coprime polynomials of order i, and j is a Hauptmodul.

Subtract equation (2.3.4) from equation (2.3.3) to get

$$1728 q_n^9 q_{n+3}^3 = p_{4n+3}^3 - r_{6n+4}^2 s_1$$

$$\therefore\ r_{6n+4}^2 s_1 = p_{4n+3}^3 - (12 q_n^3 q_{n+3})^3 \tag{2.3.5}$$

Shift s_1 to become x, and then multiply equation (2.3.5) by $1/(x^{12n+9})$ to get

$$\frac{r_{6n+4}^2}{x^{12n+8}} = \frac{p_{4n+3}^3}{x^{12n+9}} - \frac{(12 q_n^3 q_{n+3})^3}{x^{12n+9}}$$

$$\therefore \left[\frac{r_{6n+4}}{x^{6n+4}}\right]^2 = \left[\frac{p_{4n+3}}{x^{4n+3}}\right]^3 - \left[\frac{12 q_n^3 q_{n+3}}{x^{4n+3}}\right]^3$$

Let $y = \frac{1}{x}$, $r = \frac{r_{6n+4}}{x^{6n+4}}$, $p = \frac{p_{4n+3}}{x^{4n+3}}$, $q = \frac{12 q_n^3 q_{n+3}}{x^{4n+3}}$.

Then, $r^2 = p^3 - q^3$,

where $\partial r = 6n + 4$ and $\partial p = \partial q = 4n + 3$ when expressed in terms of y.

$$\therefore\ r^2 = (p - q)(p - \omega q)(p - \omega^2 q) \quad , \text{ where } \omega = \exp(2\pi i/3) \tag{2.3.6}$$

Scale p and q so that the leading coefficients become 1. Then,

$p = y^{4n+3} +$ (polynomial in y of degree $\le 4n + 2$). Similarly for q.

Now, $p - \omega q$ and $p - \omega^2 q$ must both be of degree $4n + 3$, and $p - q$ must be at most of degree $4n + 2$. However, r^2 is of degree $12n + 8$, so that $p - q$ must be of degee $4n + 2$.

Let f be a polynomial in y of degree $\in \{1, 2, \ldots, 4n+2\}$.

Then, for any distinct non-zero constants a and b,

$$\left\{\begin{array}{l} f \setminus (p - aq) \\ f \setminus (p - bq) \end{array}\right\} \Rightarrow \left\{\begin{array}{l} f \setminus (b - a)q \\ f \setminus 2p - (a + b)q \end{array}\right\} \Rightarrow \left\{\begin{array}{l} f \setminus q \\ f \setminus 2p - (a + b)q \end{array}\right\} \Rightarrow \left\{\begin{array}{l} f \setminus q \\ f \setminus p \end{array}\right\}$$

But p and q are coprime (since p_i and q_i are coprime), so we can conclude that $p - q$, $p - \omega q$ and $p - \omega^2 q$ must be mutually coprime.

The LHS of equation (2.3.6) is a perfect square, and, therefore, so is the RHS. As the three terms in the RHS are mutually coprime, each of these terms must be a perfect square. However, both $p - \omega q$ and $p - \omega^2 q$ are of odd degree.

This contradiction implies that the original assumption is false. Hence, the specification (12n+9,0,1,0,0,n+3) does not have a coset diagram $\forall\, n \geq 0$. Hence, no such S exists. □

LEMMA 2.3.8 If $S(u,p,e,f,g_1,g_3)$ satisfies (2.1.2) and $e = 0$, then there exists a coset diagram with specification S, with the following exceptions:

(a) (24,0,0,0,0,5)

(b) (24,0,0,0,3,1)

(c) (24,0,0,3,0,2)

Proof : Assume S is a counter-example with $p + f + g_1 + g_3$ minimal. We want to show that no such S exists.

If $p \geq 1$ and $D(u,p-1,4,f,g_1,g_3)$ satisfies (2.1.2), then D can be either 2(1) or 1(1) 1(2) by Lemma 2.3.3, so we can either 1-compose or 1,2-compose D once to get a diagram with specification S.

Therefore, S has $p < 1$. i.e. S has $p = 0$.

A diagram has already been exhibited for Q (18,0,2,3,0,0) 1(1).

If $f \geq 4$ and $D(u-18,p,2,f-3,g_1,g_3)$ satisfies (2.1.2), then D can be 1(1) by Lemma 2.3.6, so that D 1(1) + Q 1(1) has a diagram with specification S.

Therefore, S has $f < 4$.

A diagram has already been exhibited for V_0 (30,0,2,0,3,0) 1(2).

If $g_1 \geq 4$ and $D(u-30,p,2,f,g_1-3,g_3)$ satisfies (2.1.2), then D can be 1(2) by Lemma 2.3.5, so that D 1(2) + V_0 1(2) has a diagram with specification S.

Therefore, S has $g_1 < 4$.

As shown in Lemma 2.3.2, we have $f \equiv g_1 \pmod 3$.

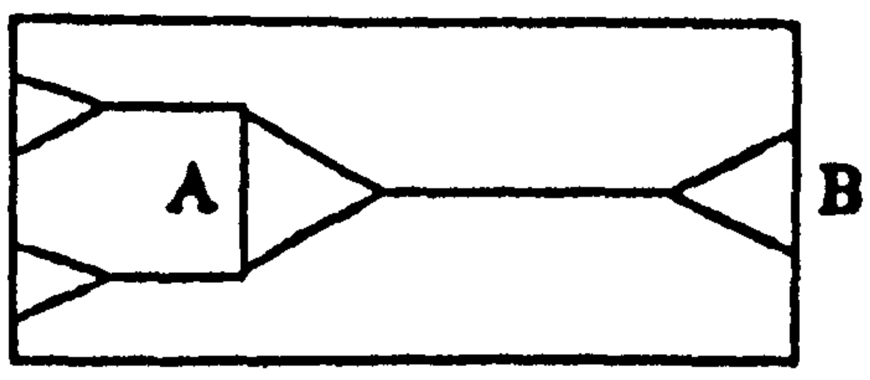
We now know that a minimal S would have the form

$$(u,0,0,f,g_1,g_3) \quad : \quad f < 4,\ g_1 < 4,\ f \equiv g_1 \pmod 3$$

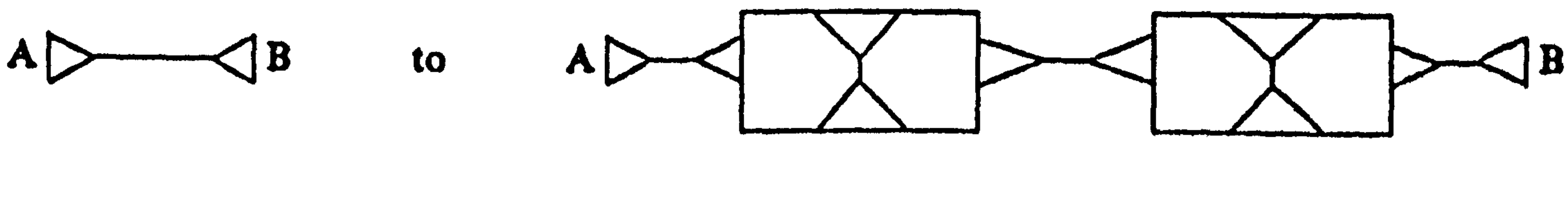
This gives us six cases to consider :

Case (1) : (12n,0,0,0,0,n+3) , $n \geq 1$.

n = 1 (12,0,0,0,0,4)

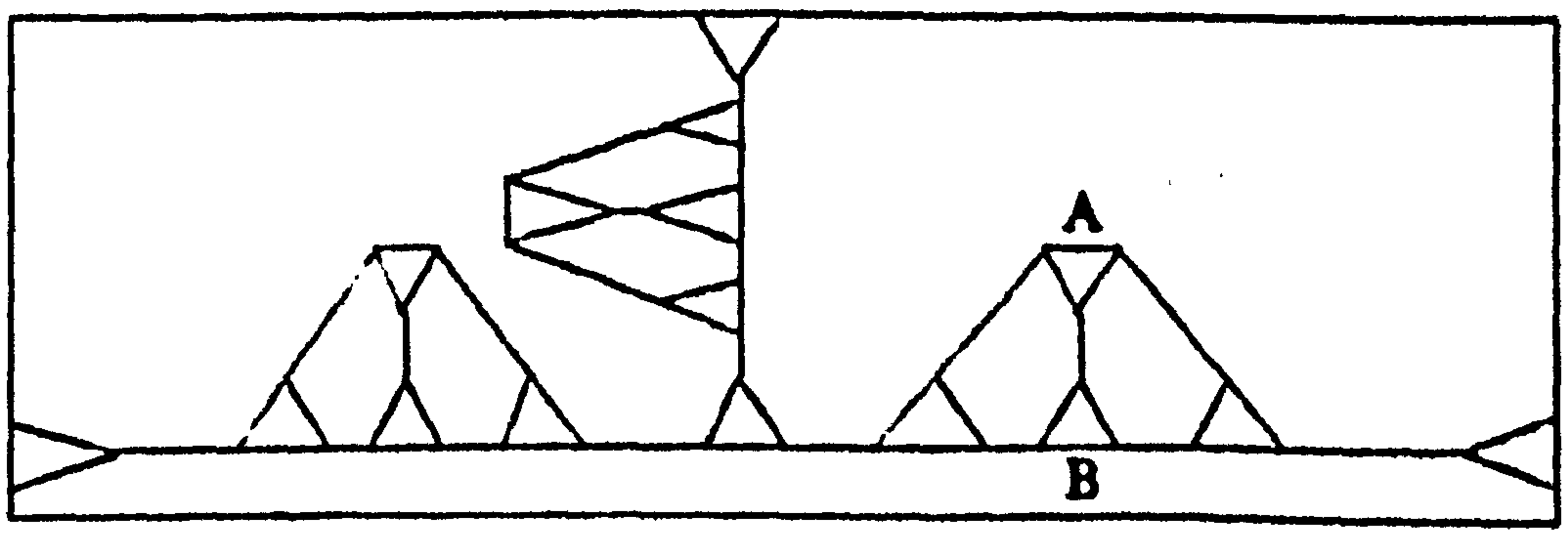


For n = 3, we can expand the section between triangles A and B in the diagram for n = 1 to change from



Similarly for n = 5, 7, 9, ...

n = 4 (48,0,0,0,0,7)



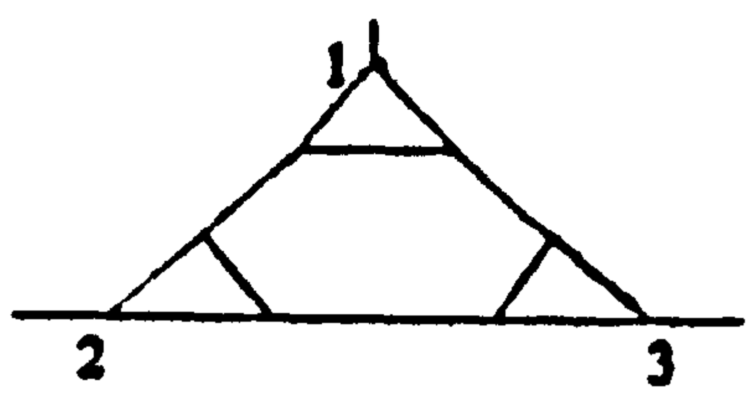
For $n = 6$, we can expand the section between triangles A and B in the diagram for $n = 4$, as we did with $n = 3$. Similarly for $n = 8, 10, 12, \ldots$

Hence, a coset diagram exists for all $n \geq 3$ and for $n = 1$.

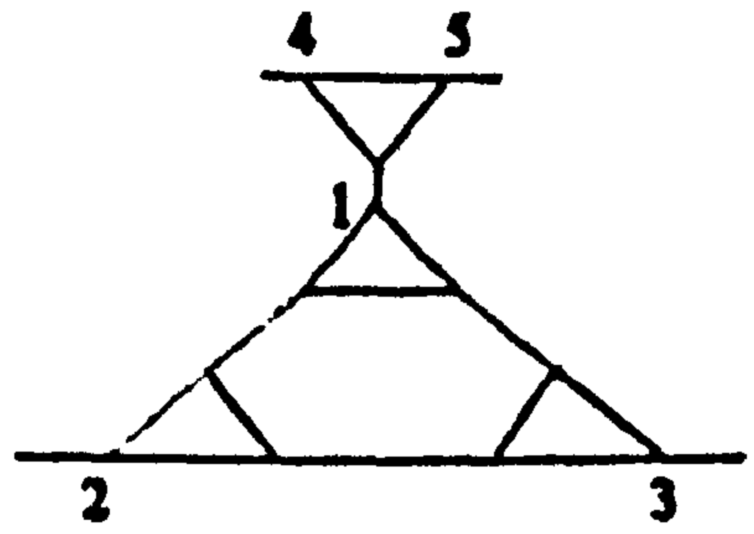
However, no coset diagram exists for $n = 2$, as we now show.

Assume, with a view to obtaining a contradiction, that there exists a coset diagram D for (24,0,0,0,0,5).

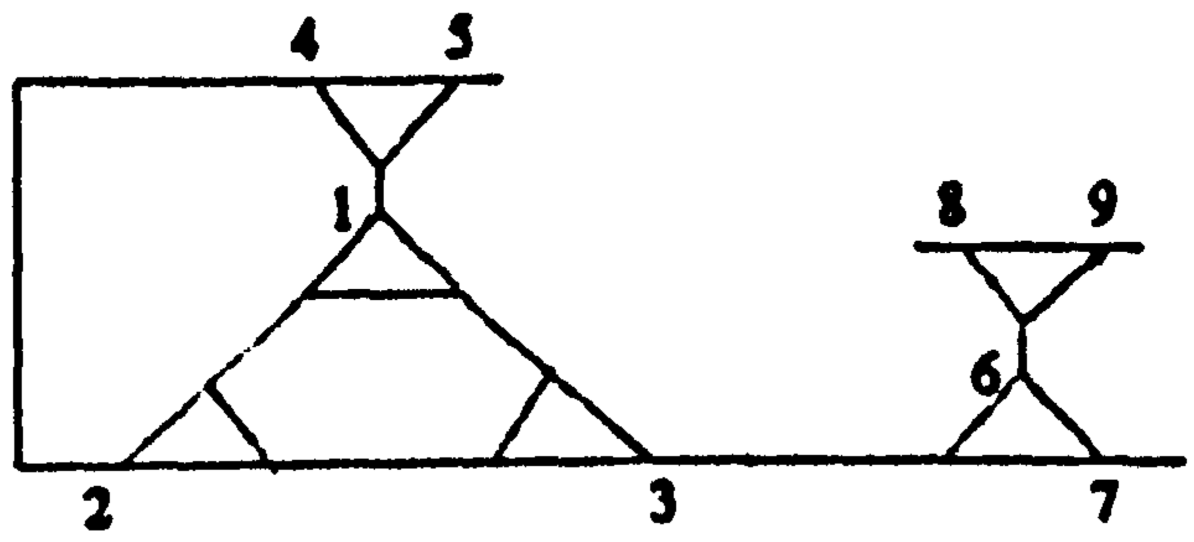
Then, D has eight triangles, with one green 9-cycle and five green 3-cycles. By Lemma 2.3.1, all of these 3-cycles must be of type (4). Therefore, we can start with a 3-cycle of type (4), and then build this up, noting that there are no red, blue or green points. We will use the same notation as used in Lemmas 2.3.1 and 2.3.3.



If $1 \to 2$ or $1 \to 3$, then 2-cycle. $\therefore$ We must have $1 \to \Delta$.



<u>Case (1a) : Assume $2 \to 4$.</u>



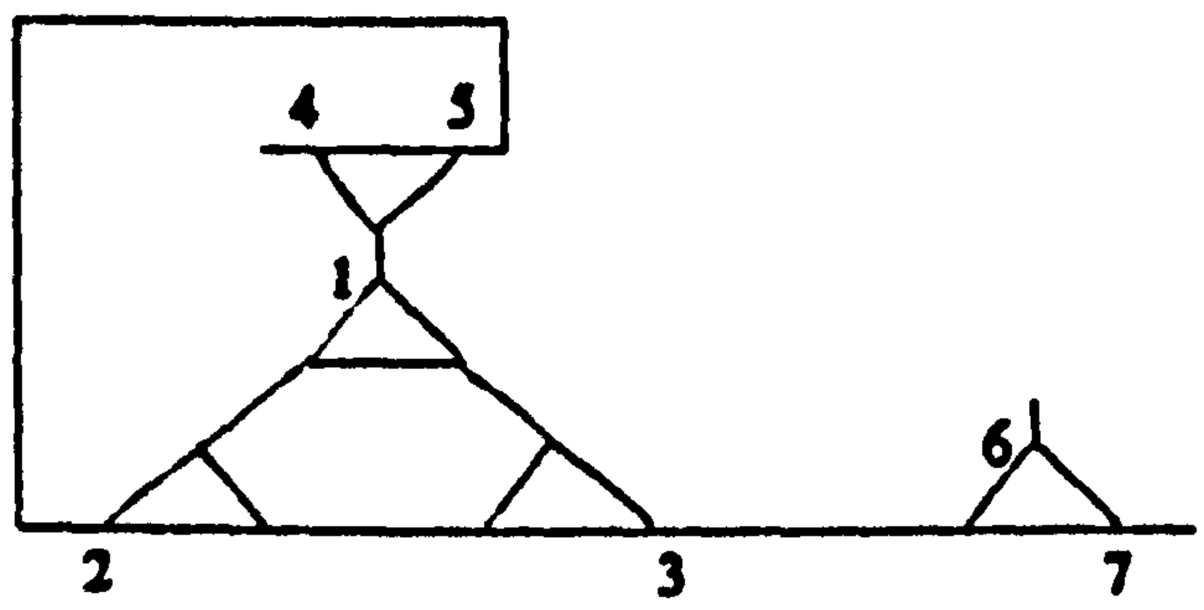
If $3 \to 5$, then D would be complete with only four triangles. $\therefore$ $3 \to \Delta$.

If $6 \to 5$, then 4-cycle. If $6 \to 7$, then 1-cycle. $\therefore$ $6 \to \Delta$.

We have 4^+-cycle through 7, and 5^+-cycle through 5. To combine these cycles, vertex 5 would have to be a red point.

Contradiction. ∴ Case (1a) not possible.

Case (1b) : Assume $2 \rightarrow 5$.

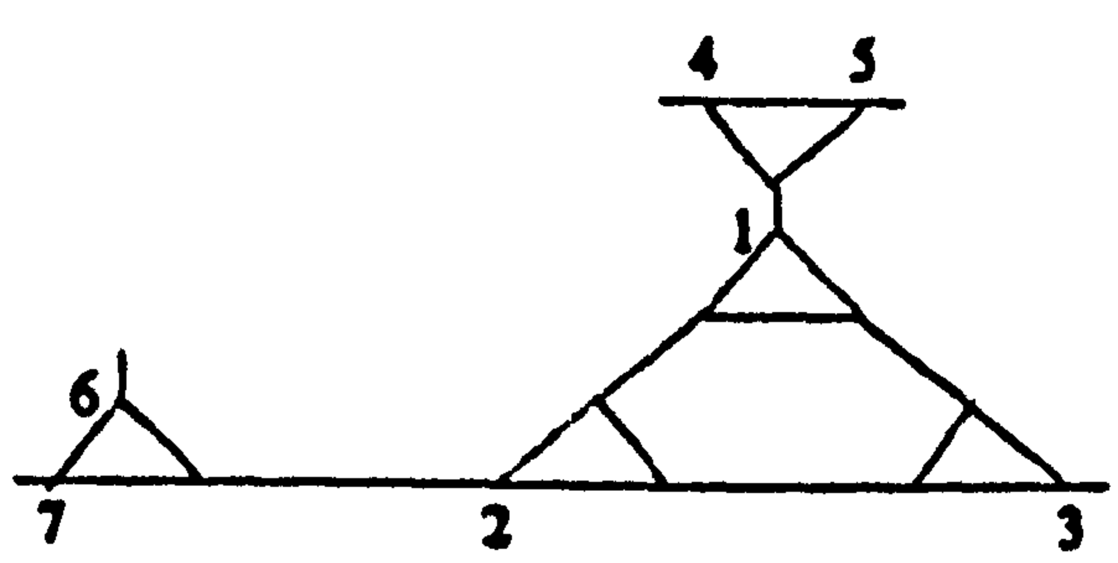


If $3 \rightarrow 4$, then D would be complete with only four triangles. ∴ $3 \rightarrow \Delta$.

7^+-cycle through 7 , 4^+-cycle through 4 $\Rightarrow$ 11^+-cycle.

Contradiction. ∴ Case (1b) not possible.

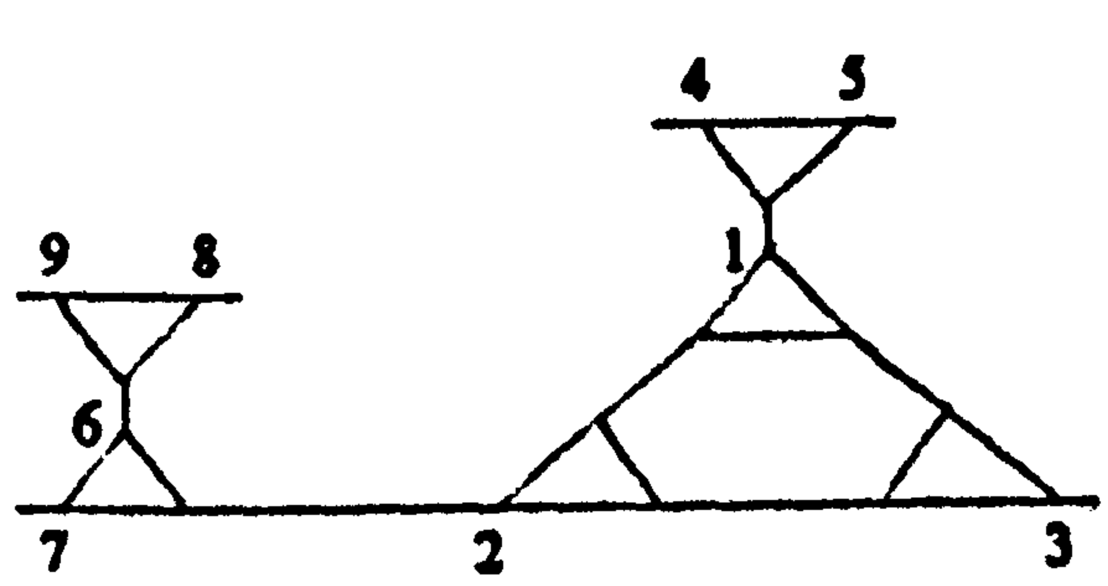
If $2 \rightarrow 3$, then 2-cycle. ∴ $2 \rightarrow \Delta$.



If $6 \rightarrow 7$, then 1-cycle. If $6 \rightarrow 4$, then 4-cycle.

If $6 \rightarrow 3$, then 7^+-cycle through 5 , 4^+-cycle through 7 $\Rightarrow$ 11^+-cycle.

If $6 \rightarrow 5$, then $4 \rightarrow 7$ and $4 \rightarrow 3$ both create 12^+-cycles so that $4 \rightarrow \Delta$, in which case we have a 7^+-cycle through 4 and a 4^+-cycle through 7 which would create an 11^+-cycle. ∴ $6 \rightarrow \Delta$.

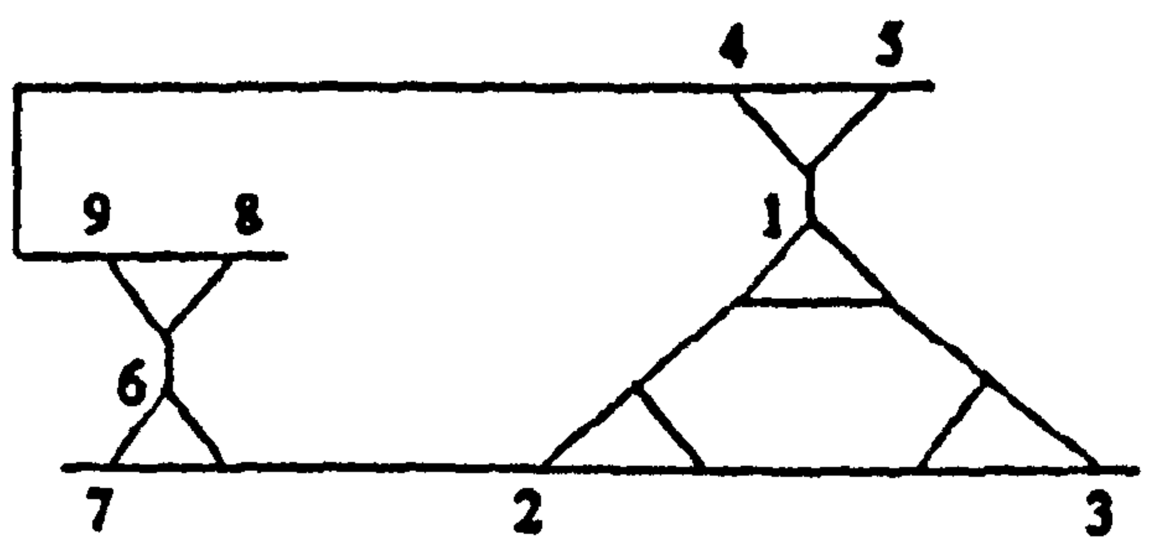


If $4 \to 5$, then 1-cycle. If $4 \to 8$, then 5-cycle.

If $4 \to 3$, then 8^+-cycle through 8 , 4^+-cycle through 5 $\Rightarrow$ 12^+-cycle.

If $4 \to 7$, then 7^+-cycle through 8 , 4^+-cycle through 3 $\Rightarrow$ 11^+-cycle.

<u>Case (1c) : Assume $4 \to 9$.</u>

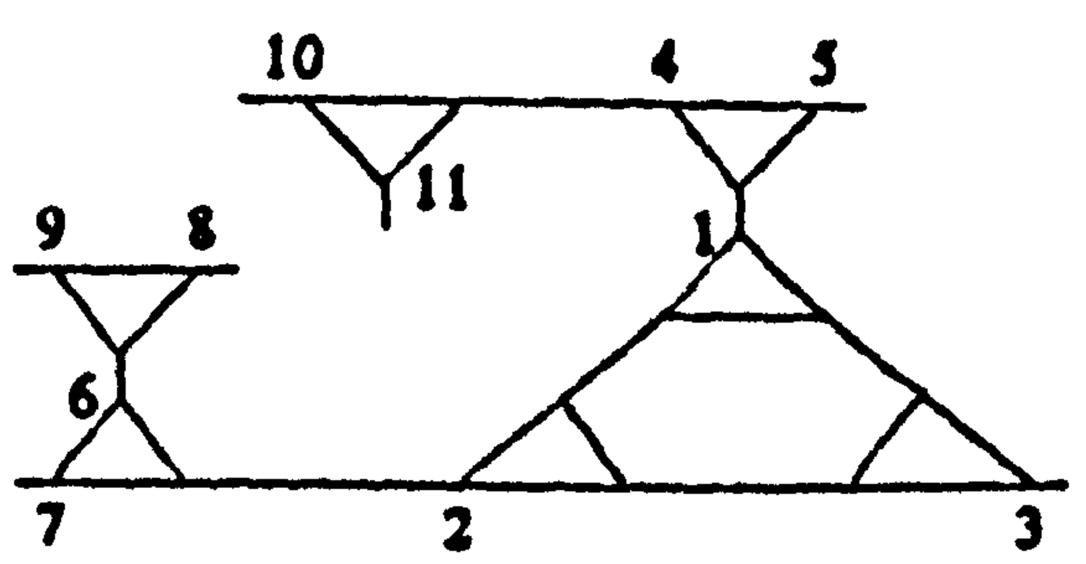


If $3 \to 5$, then 6^+-cycle through 7 , 6^+-cycle through 8 $\Rightarrow$ 12^+-cycle.

If $3 \to 7$, then 6^+-cycle through 5 , 6^+-cycle through 8 $\Rightarrow$ 12^+-cycle.

If $3 \to 8$, then 12^+-cycle through 5. $\quad\therefore$ $3 \to \Delta$.

Now, we have a 6^+-cycle through 8 and a 4^+-cycle through 5 which would create a 10^+-cycle. Contradiction. $\therefore$ Case (1c) not possible. $\therefore$ $4 \to \Delta$.



If $3 \to 5$, then 5^+-cycle through 10 , 6^+-cycle through 8 $\Rightarrow$ 11^+-cycle.

If $3 \to 7$ or $3 \to 10$, then 5^+-cycle through 5 , 6^+-cycle through 8 $\Rightarrow$ 11^+-cycle.

If $3 \to 8$, then $5 \to 11$ to complete 9-cycle $\Rightarrow$ 4^+-cycle through 9.

If $3 \to 9$, then 4^+-cycle through 5 , 6^+-cycle through 8 $\Rightarrow$ 10^+-cycle.

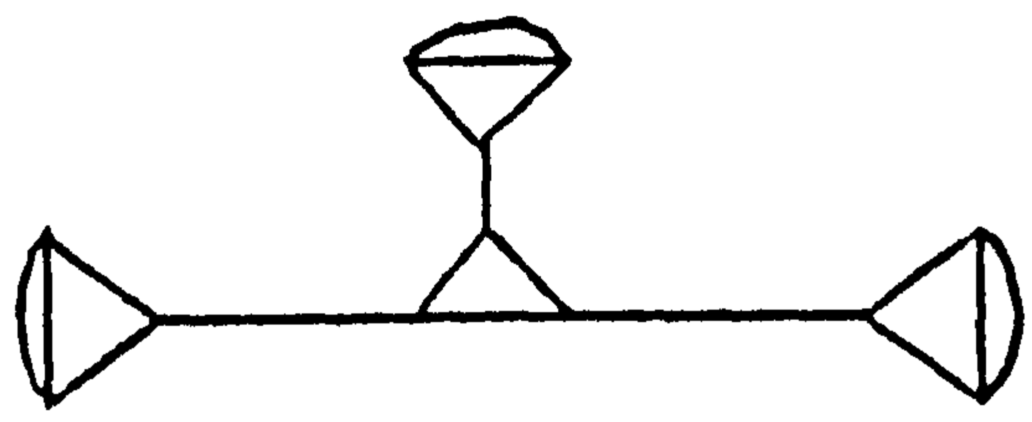
If $3 \to 11$, then $7 \to 8$ to complete 9 cycle $\Rightarrow$ 4^+-cycle through 5.

$\therefore$ $3 \to \Delta$. Now, we have a 6^+-cycle through 8 and a 4^+-cycle through 5 which would create a 10^+-cycle. Contradiction.

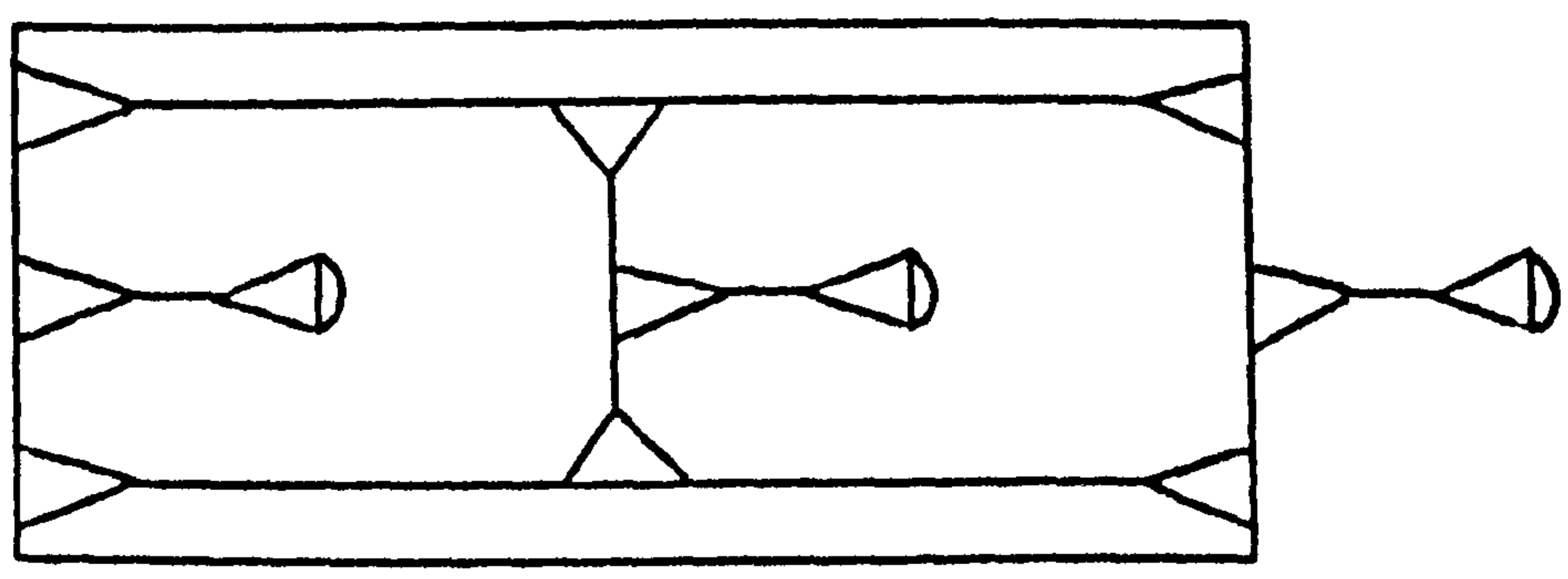
Therefore, there does not exist a coset diagram for (24,0,0,0,0,5).

<u>Case (2)</u> : (12n,0,0,0,3,n-1) , $n \geq 1$.

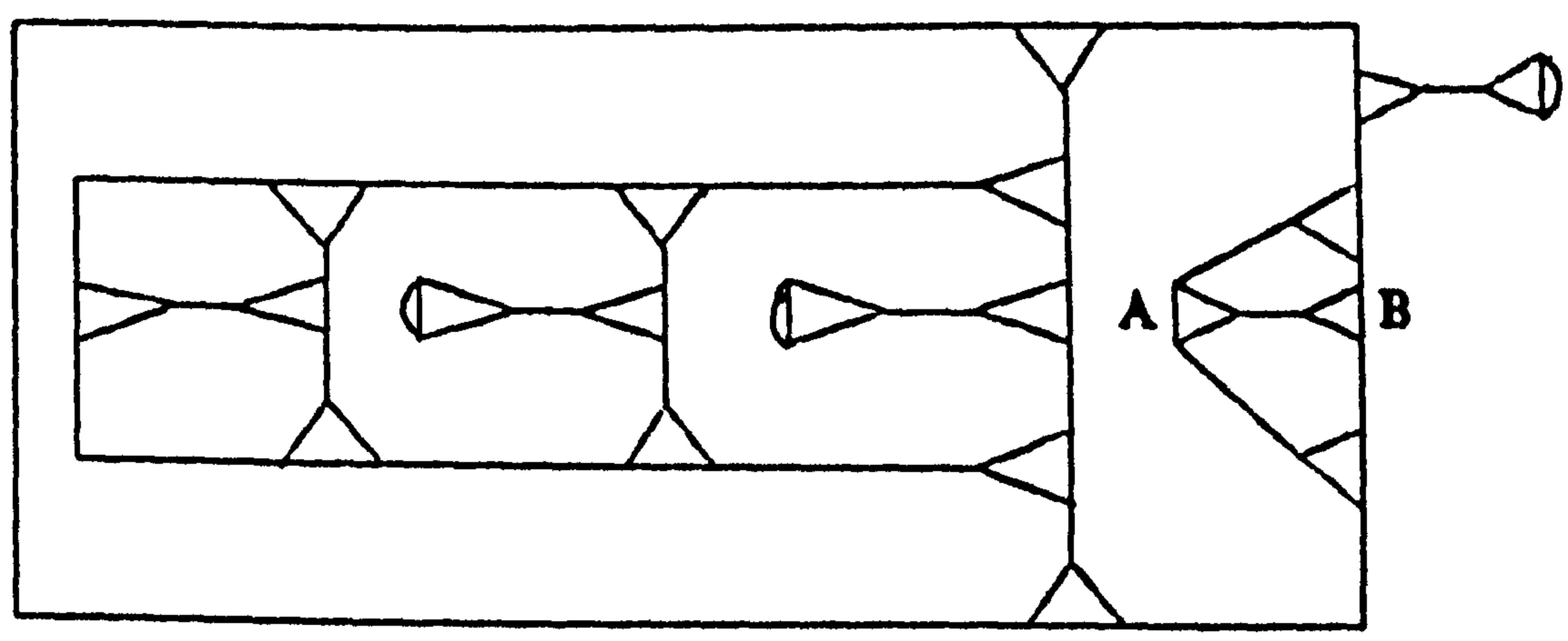
n = 1 (12,0,0,0,3,0)



n = 3 (36,0,0,0,3,2)

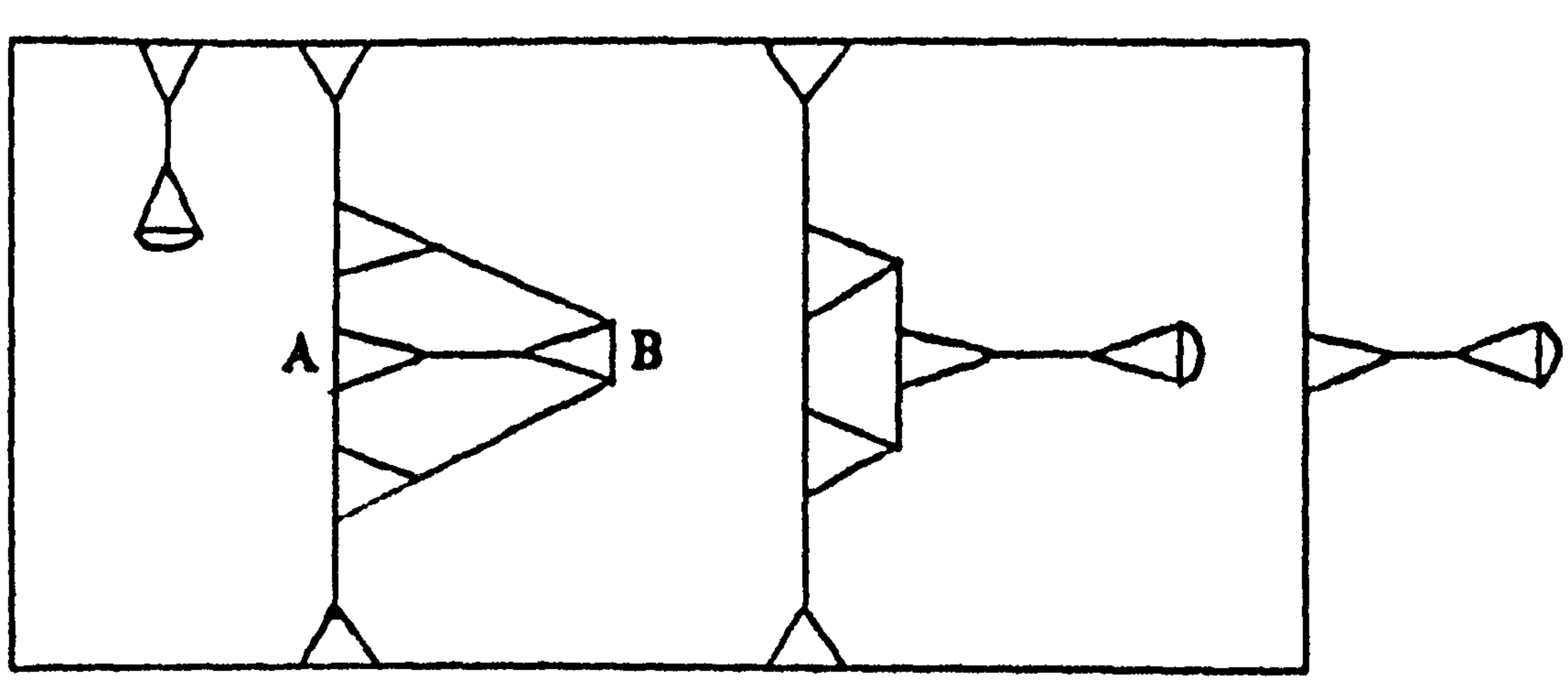


n = 5 (60,0,0,0,3,4)



For n = 7, we can expand the section between triangles A and B in the diagram for n = 5, as we did with Case (1). Similarly for n = 9, 11, 13, ...

n = 4 (48,0,0,0,3,3)



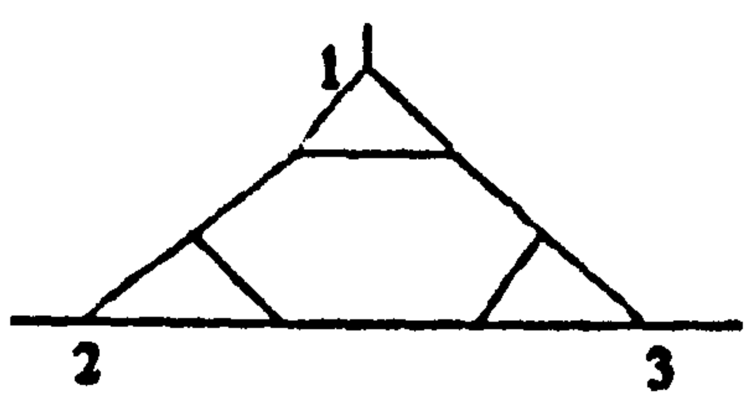
For n = 6, we can expand the section between triangles A and B in the diagram for n = 4, as we did with Case (1). Similarly for n = 8, 10, 12, ...

Hence, a coset diagram exists for all $n \geq 3$ and for n = 1.

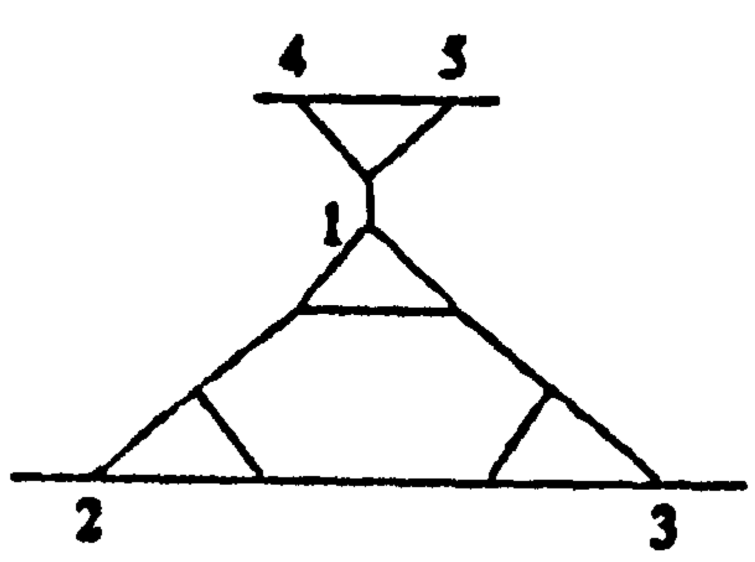
However, no coset diagram exists for n = 2, as we now show.

Assume, with a view to obtaining a contradiction, that there exists a coset diagram D for (24,0,0,0,3,1).

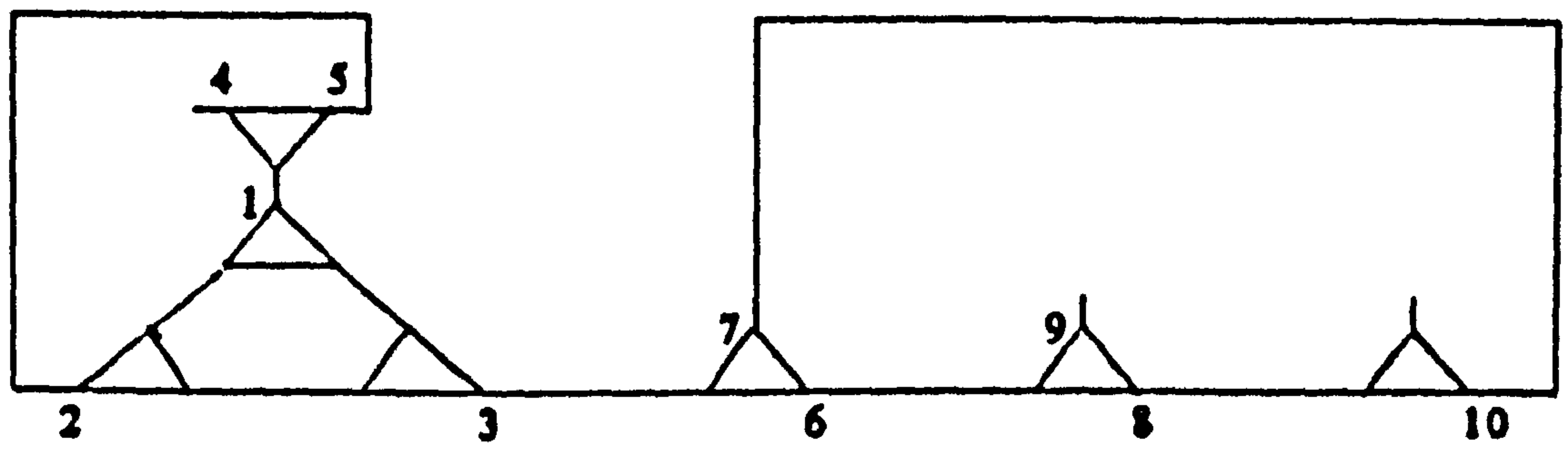
Then, D has eight triangles, with two green 9-cycles, three green points and one green 3-cycle. By Lemma 2.3.1, the 3-cycle is of type (4). Therefore, we can start with a 3-cycle of type (4), and then build this up, noting that there are no red or blue points. We will use the notation used in Case (1).



If $1 \to 2$ or $1 \to 3$, then 2-cycle. ∴ We must have $1 \to \Delta$.



Case (2a) : Assume $2 \to 5$.



If $3 \to 4$, then D would be complete with only four triangles. ∴ $3 \to \Delta$.

If $6 \to 7$, then 7-cycle. If $6 \to 4$, then 12^{+}-cycle. ∴ $6 \to \Delta$.

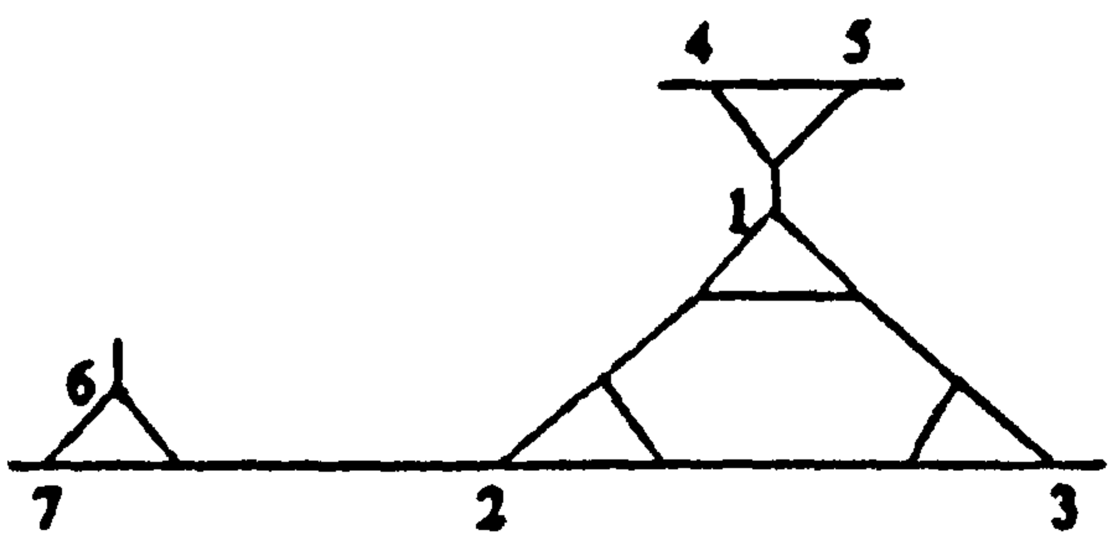
If $8 \to 4$, then 13^{+}-cycle. If $8 \to 7$, then 8-cycle.

If $8 \to 9$, then 10^{+}-cycle. ∴ $8 \to \Delta$.

∴ $7 \to 10$, to complete a 9-cycle, but then we cannot obtain three 1-cycles.

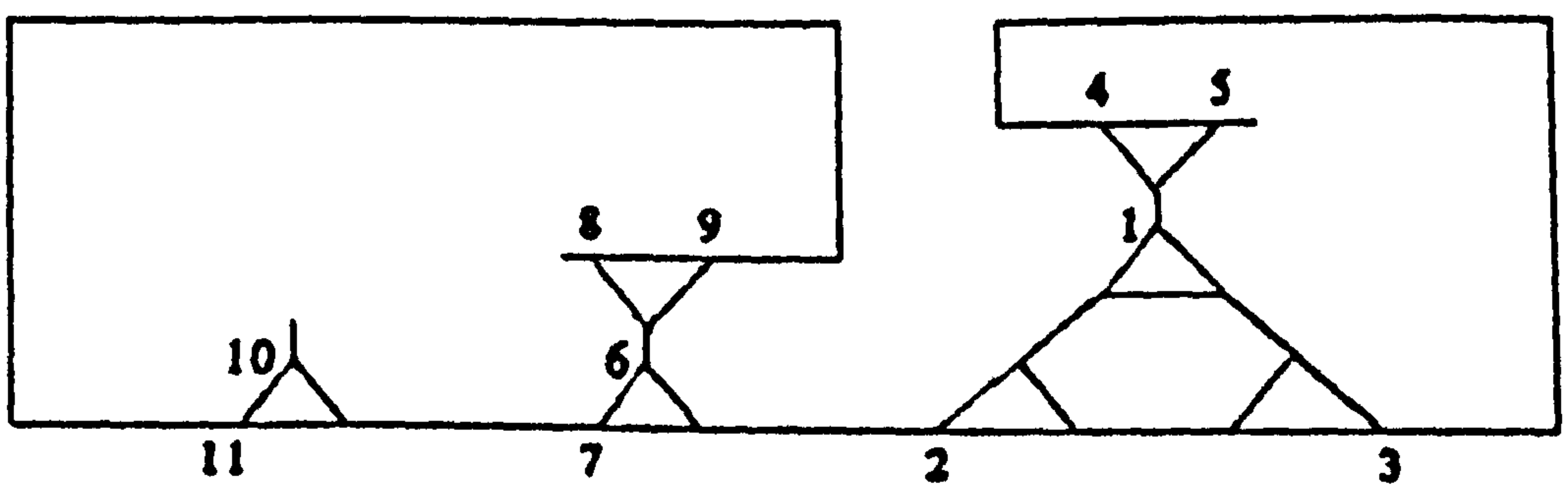
Contradiction. ∴ Case (2a) not possible.

If 2 → 3, then 2-cycle. If 2 → 4, then second 3-cycle. Hence, 2 → Δ.



If 3 → 5 or 3 → 7, then second 3-cycle.

<u>Case (2b) : Assume 3 → 4.</u>



If 6 → 7, then 7-cycle. If 6 → 5, then 12^+-cycle. ∴ 6 → Δ.

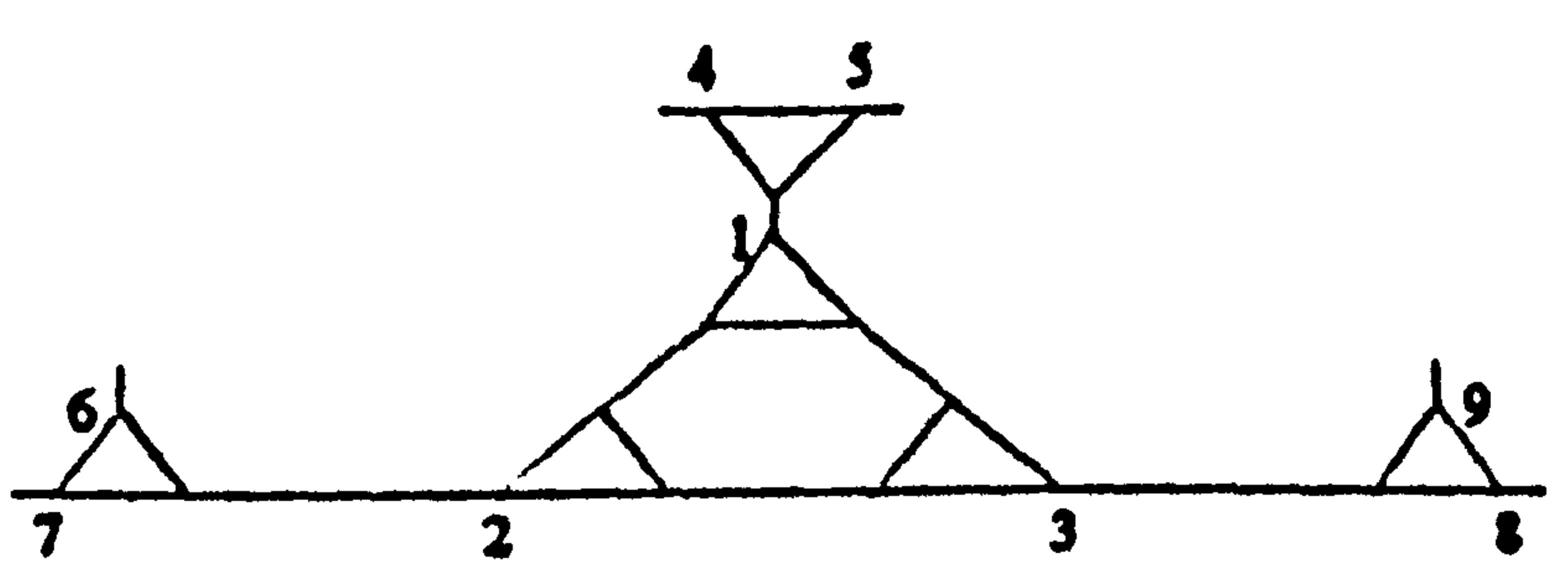
If 7 → 8, then 2-cycle. If 7 → 9, then 8-cycle.

If 7 → 5, then 14^+-cycle. ∴ 7 → Δ.

∴ 9 → 11, to complete a 9-cycle, but then we cannot obtain three 1-cycles.

Contradiction. ∴ Case (2b) not possible.

By the symmetry of the diagram, we can therefore conclude that 3 → 6 is also not possible. ∴ 3 → Δ.

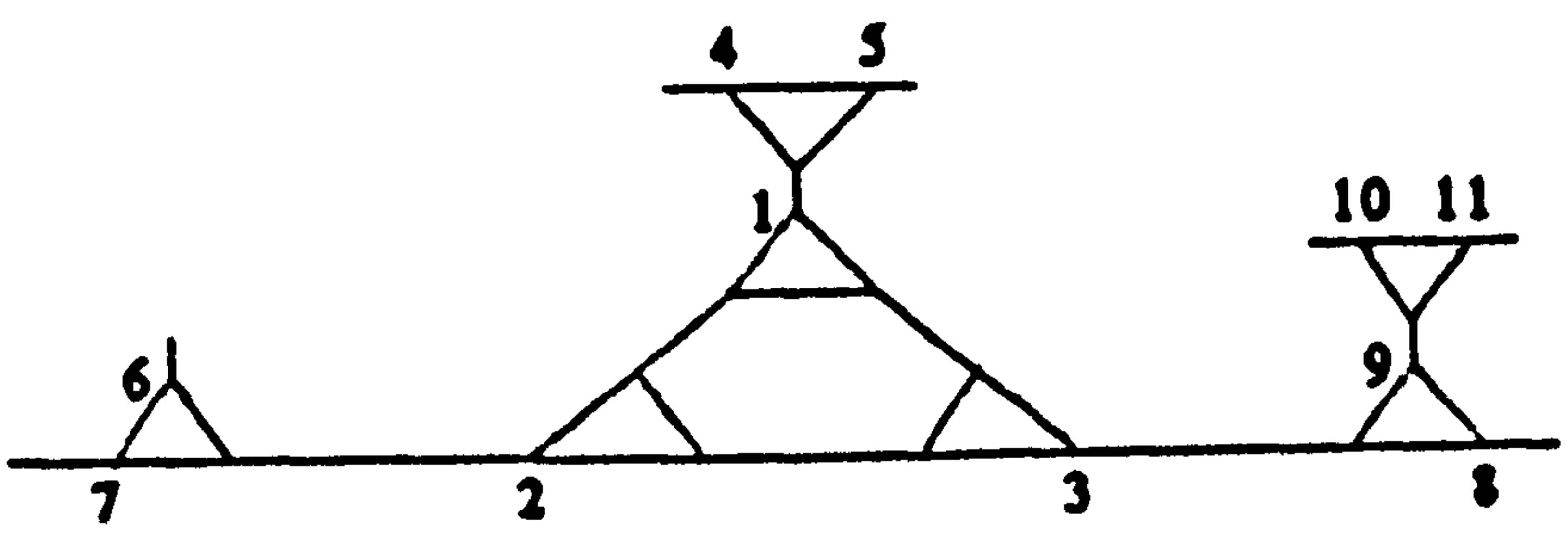


If 9 → 4, then 5 → Δ, 8 → Δ, 6 → 7 ⇒ 7-cycle.

If 9 → 6, then 7 → Δ, 8 → Δ, 4 → 5 ⇒ 8-cycle. If 9 → 5, then 4-cycle.

If 9 → 7, then 6 → Δ, 8 → Δ, 4 → 5 ⇒ 7-cycle.

If $9 \to 8$, then $5 \to \Delta \to 7$ to complete a 9-cycle. But now we cannot obtain three 1-cycles. $\therefore$ $9 \to \Delta$.



If $4 \to \Delta$, then $10 \to 11$, $5 \to 8$ $\Rightarrow$ 7-cycle. If $6 \to \Delta$, then $7 \to 8$ $\Rightarrow$ 4-cycle.

If $5 \to \Delta$, then $6 \to 7$, $4 \to 8$ $\Rightarrow$ 8-cycle. If $10 \to \Delta$, then $8 \to 11$ $\Rightarrow$ 2-cycle.

If $7 \to \Delta$, then $4 \to 5$, $6 \to 8$, $10 \to 11$ $\Rightarrow$ 11-cycle.

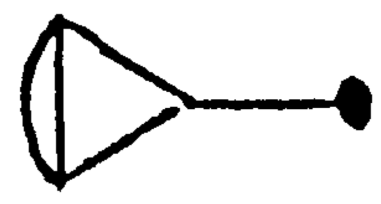
If $8 \to \Delta$, then we obtain four 1-cycles.

If $11 \to \Delta$, then $4 \to 5$, $6 \to 7$, $8 \to 10$ $\Rightarrow$ 13-cycle. Contradiction.

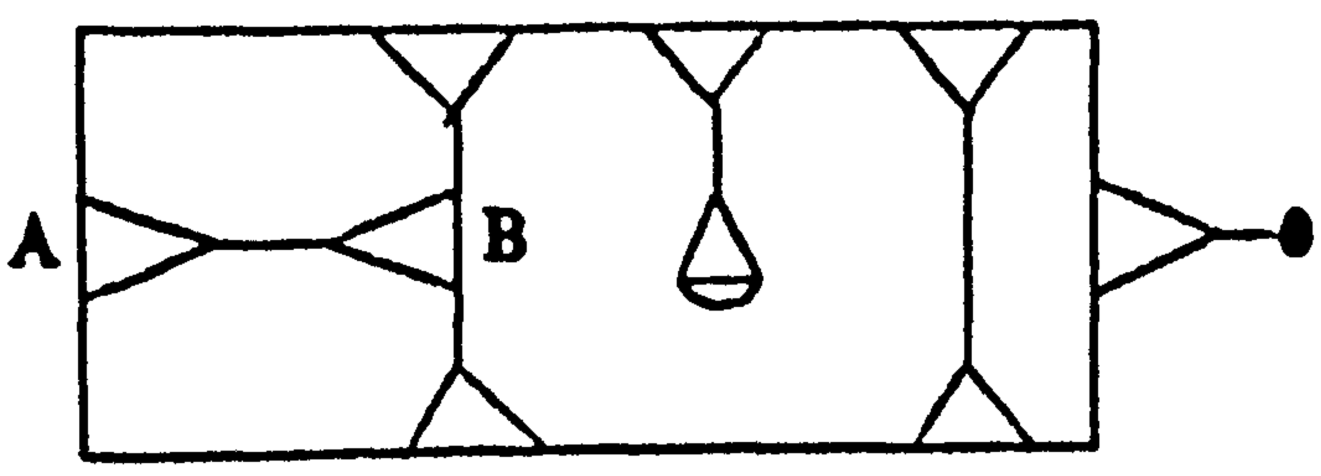
Therefore, there does not exist a coset diagram for (24,0,0,0,3,1).

Case (3) : $(12n+4,0,0,1,1,n+1)$, $n \geq 0$.

$n = 0$ (4,0,0,1,1,1)

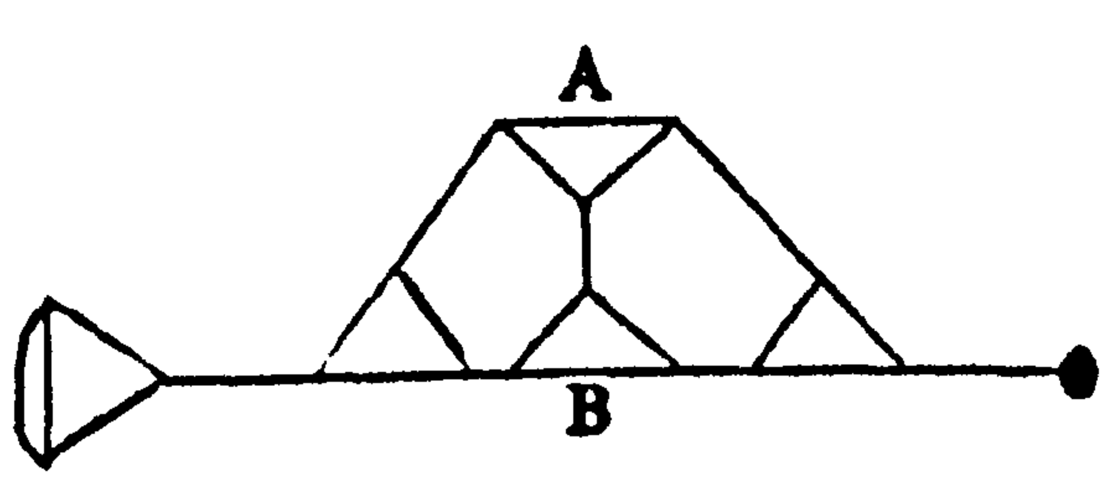


$n = 2$ (28,0,0,1,1,3)



For $n = 4$, we can expand the section between triangles A and B in the diagram for $n = 2$, as we did with Case (1). Similarly for $n = 6, 8, 10, \ldots$

$n = 1$ (16,0,0,1,1,2)



For $n = 3$, we can expand the section between triangles A and B in the diagram for $n = 1$, as we did with Case (1). Similarly for $n = 5, 7, 9, \ldots$

Hence, a coset diagram exists for all $n \geq 0$.

Case (4) : (12n+20,0,0,2,2,n) , n ≥ 0.

From Lemma 2.3.6, we can have X_n (12n+10,0,2,1,1,n) 1(1), ∀ n ≥ 0.

Therefore, X_n 1(1) + X_0 1(1) = (12n+20,0,0,2,2,n), ∀ n ≥ 0.

Hence, a coset diagram exists for all n ≥ 0.

Case (5) : (12n+48,0,0,3,3,n) , n ≥ 0.

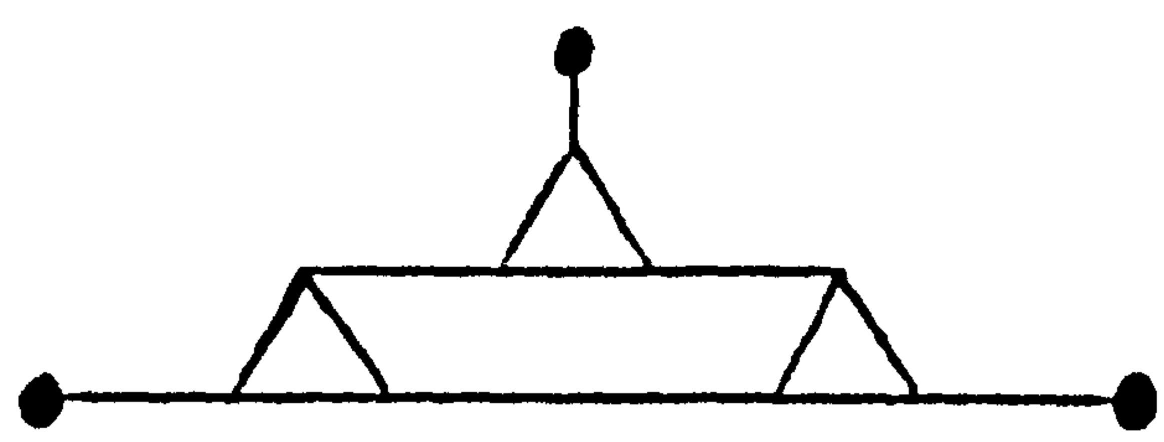
From Lemma 2.3.6, we can have Y_0 (38,0,2,2,2,0) 1(1).

Therefore, X_n 1(1) + Y_0 1(1) = (12n+48,0,0,3,3,n), ∀ n ≥ 0.

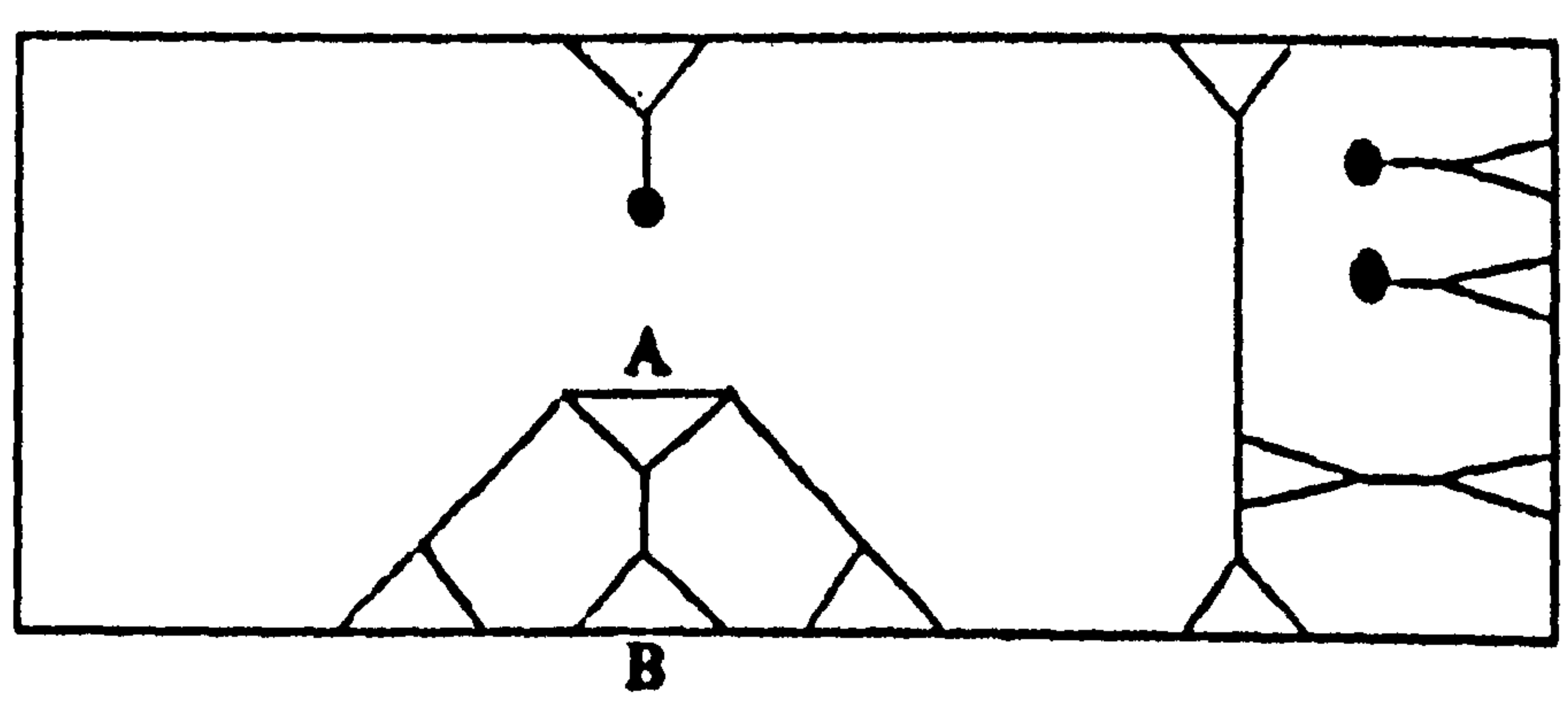
Hence, a coset diagram exists for all n ≥ 0.

Case (6) : (12n,0,0,3,0,n) , n ≥ 1.

n = 1 (12,0,0,3,0,1)

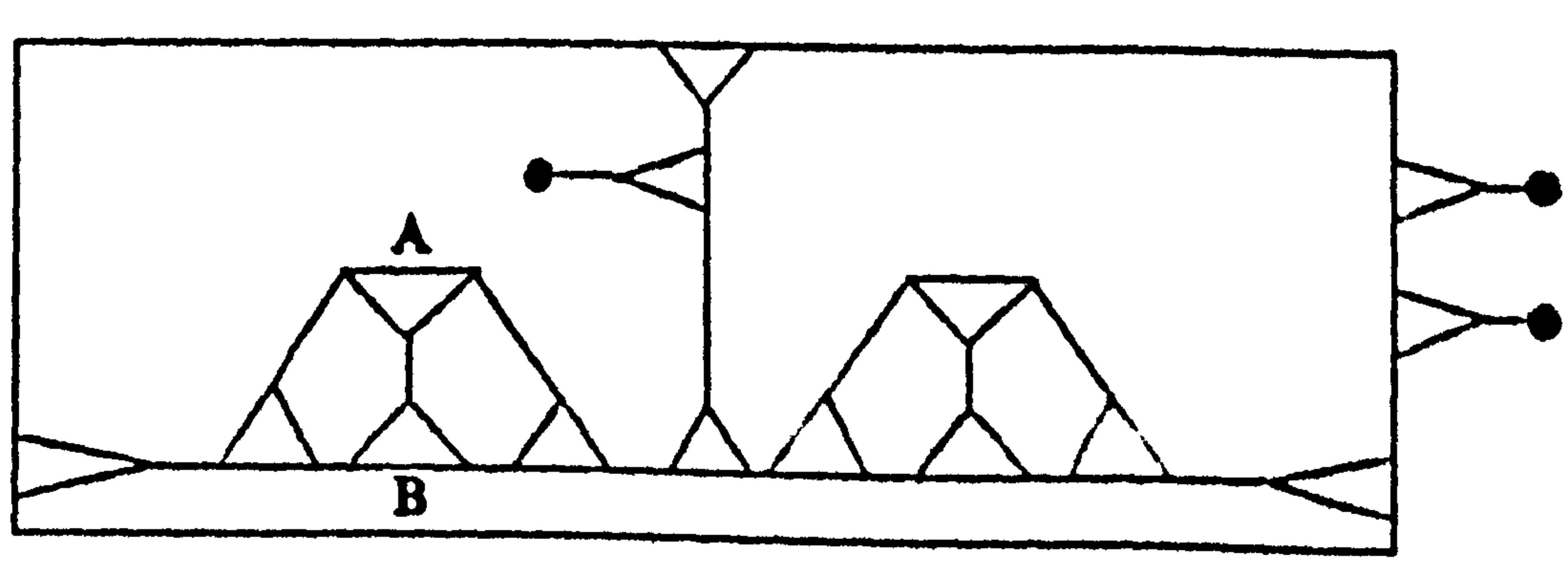


n = 3 (36,0,0,3,0,3)



For n = 5, we can expand the section between triangles A and B in the diagram for n = 3, as we did with Case (1). Similarly for n = 7, 9, 11, ...

n = 4 (48,0,0,3,0,4)



For $n = 6$, we can expand the section between triangles A and B in the diagram for $n = 4$, as we did with Case (1). Similarly for $n = 8, 10, 12, \ldots$

Hence, a coset diagram exists for all $n \geq 3$ and for $n = 1$.

However, no coset diagram exists for $n = 2$, as we now show.

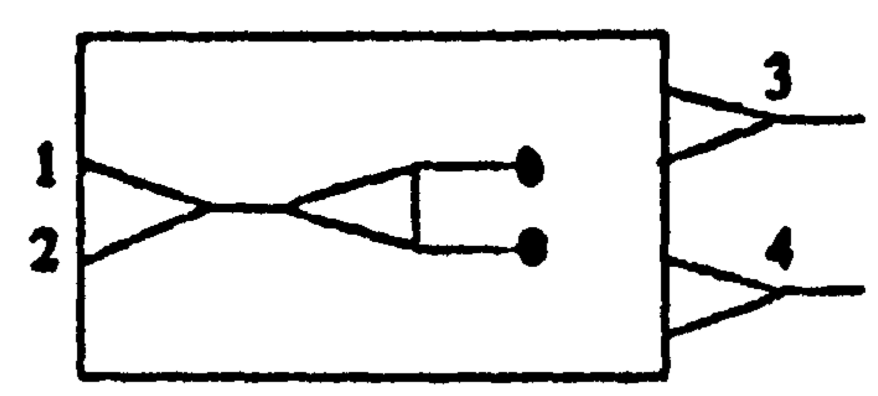
Assume, with a view to obtaining a contradiction, that there exists a coset diagram D for (24,0,0,3,0,2).

Then, D has seven triangles, with two green 9-cycles, three blue points and two green 3-cycles. By Lemma 2.3.1, both 3-cycles must be of type (4). It is not possible to have all three blue points in the same 9-cycle, as together they would contribute at least nine green lines.

If they contributed exactly nine green lines, then the diagram would be completed with only three triangles. Therefore, we must have two blue points in one 9-cycle, and one blue point in the other. Let us start with the two blue points which are in the same 9-cycle. Either the two blue points emanate from the same triangle (contributing seven green lines) or from two separate triangles (contributing six green lines). We can then build this up, noting that there are no red or green points. Notation as in Case (1).

Case (6a) : Assume two blue points emanate from the same triangle.

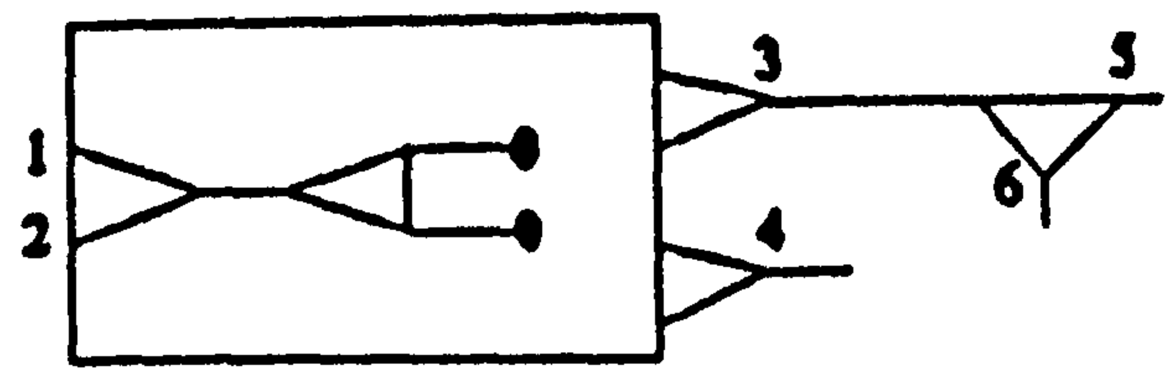
To complete a 9-cycle, we must have $1 \rightarrow \Delta \rightarrow \Delta \rightarrow 2$.



If $3 \rightarrow 4$, then 2-cycle.

If $3 \rightarrow \bullet$, then $4 \rightarrow \Delta \rightarrow \Delta$, but to complete diagram, we must create a 1-cycle.

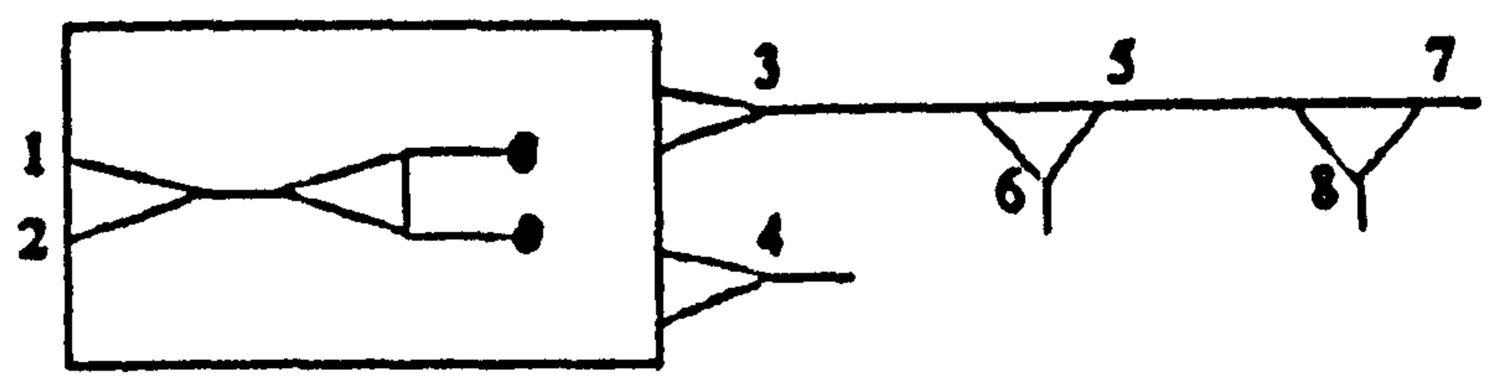
$\therefore\ 3 \rightarrow \Delta$.



If $5 \to 6$, then 1-cycle. If $5 \to 4$, then 4-cycle.

If $5 \to \bullet$, then $4 \to \Delta$, so that 4^+-cycle, 7^+-cycle $\Rightarrow$ 11^+-cycle.

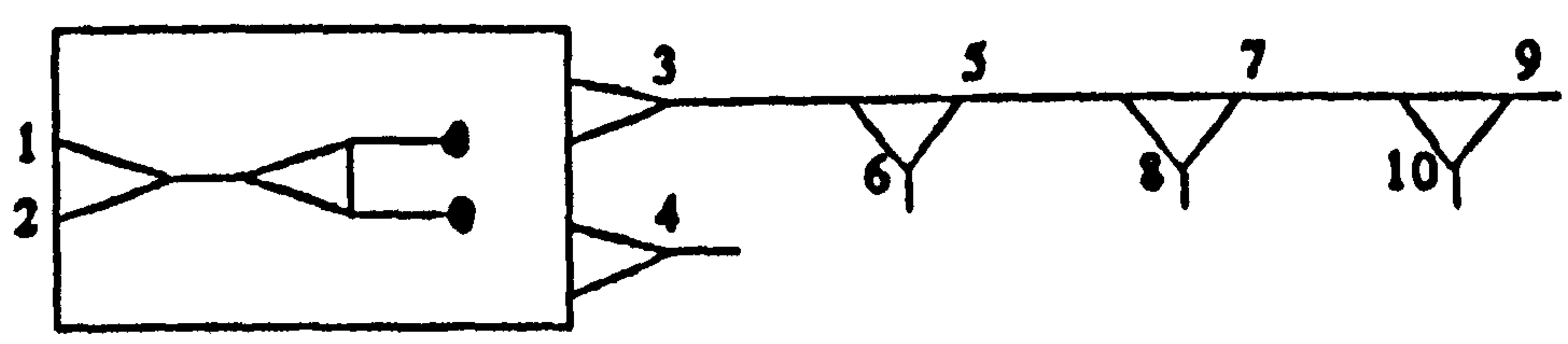
$\therefore$ $5 \to \Delta$.



If $7 \to 8$, then 1-cycle. If $7 \to 4$, then 5-cycle.

If $7 \to 6$, then $8 \to \Delta$, $4 \to \bullet$, and 1-cycle created.

If $7 \to \bullet$, then $6 \to 4$ $\Rightarrow$ $8 \to \Delta$ $\Rightarrow$ 11-cycle. $\therefore$ $7 \to \Delta$.



If $4 \to \bullet$, then 10^+-cycle.

If $6 \to \bullet$, then 6^+-cycle through 4, 6^+-cycle through 8 $\Rightarrow$ 12^+-cycle.

If $8 \to \bullet$, then 6^+-cycle through 4, 5^+-cycle through 10 $\Rightarrow$ 11^+-cycle.

If $9 \to \bullet$, then $4 \to 6$ $\Rightarrow$ $8 \to 10$ $\Rightarrow$ 2-cycle.

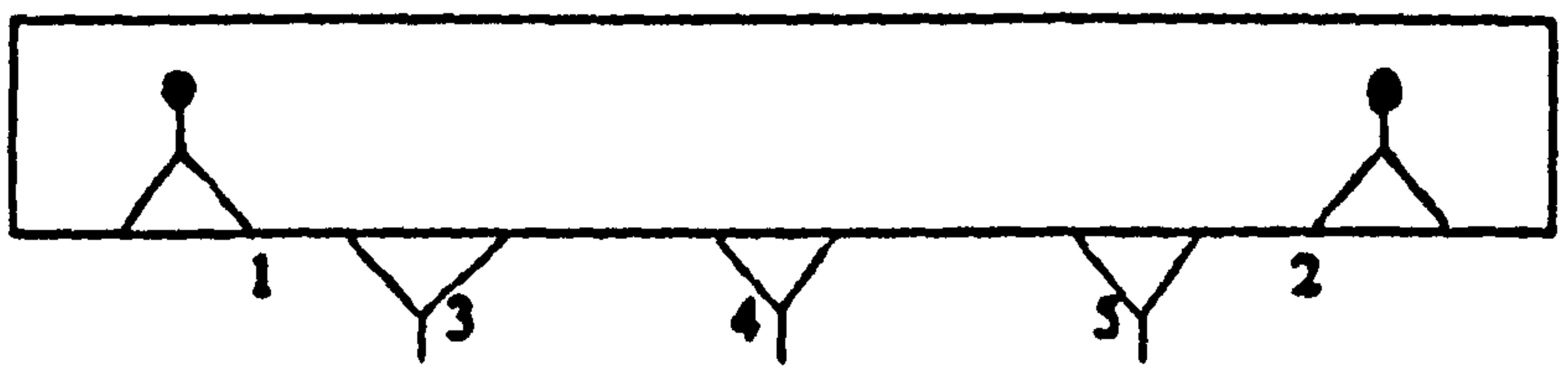
If $10 \to \bullet$, then 6^+-cycle through 4, 4^+-cycle through 9 $\Rightarrow$ 10^+-cycle.

Contradiction. $\therefore$ Case (6a) not possible.

Case (6b) : Assume each blue point emanates from a different triangle.

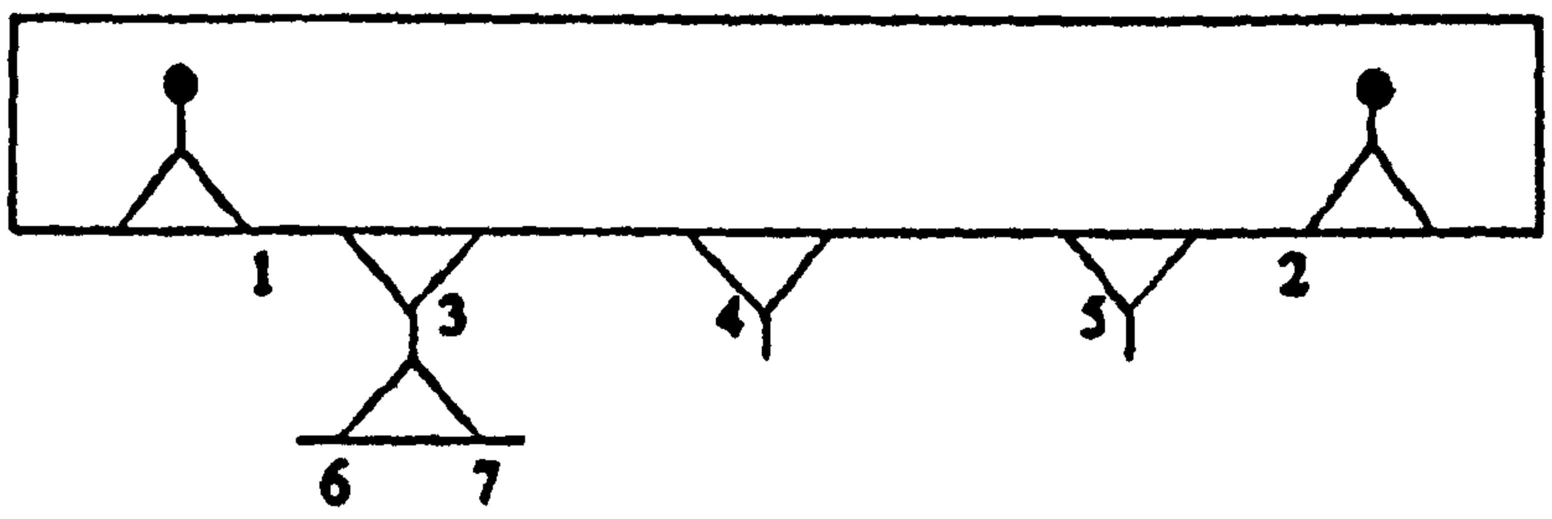
One of the 9-cycles contains one blue point, and the other 9-cycle contains the other two blue points. A triangle with a blue point contributes three green lines to a cycle, so the 9-cycle with the two blue points will have three triangles emanating from it. There are two distinct cases for the positioning of these three triangles.

<u>Case (6b1) : Assume all three triangles lie between 1 and 2.</u>



If 3 → 4, then 2-cycle. If 3 → 5, then 4-cycle.

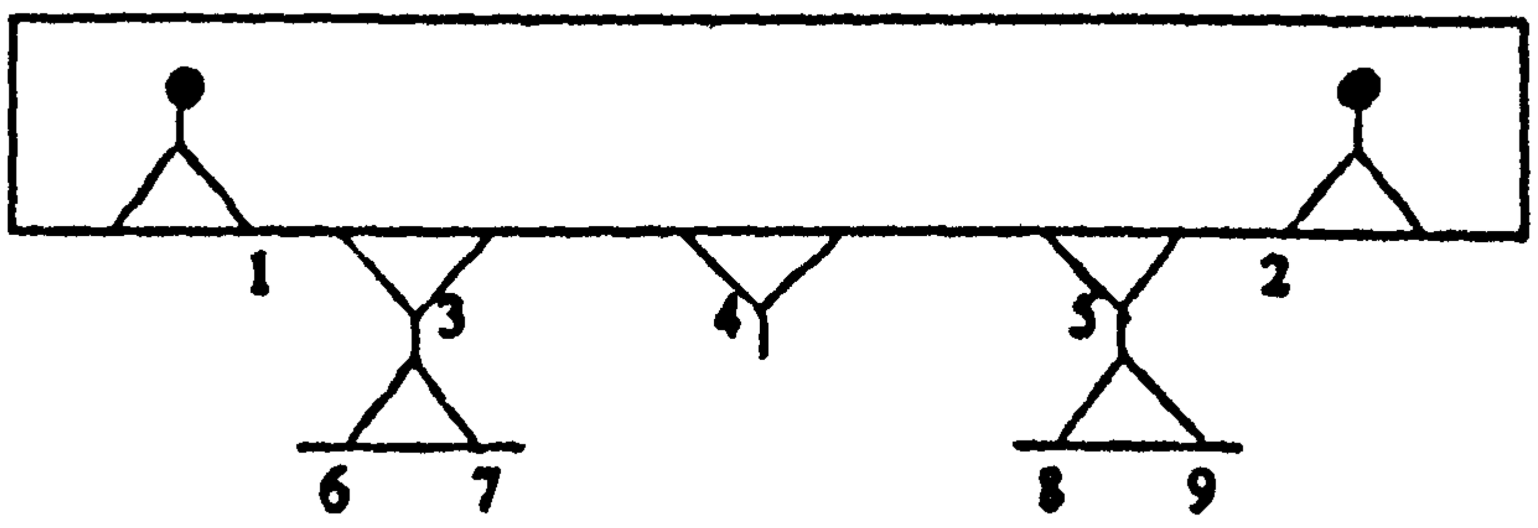
If 3 → ●, then 4 → Δ and 5 → Δ, which creates a 4-cycle. ∴ 3 → Δ.



If 5 → 4, then 2-cycle. If 5 → 6, then 5-cycle.

If 5 → 7, then 4 → Δ ⇒ 6^+-cycle, 7^+-cycle ⇒ 13^+-cycle.

If 5 → ●, then 6 → Δ → 4, to complete the second 9-cycle, but this creates a 2-cycle. ∴ 5 → Δ.



If 6 → ●, then the 8^+-cycle through 7 cannot be converted to a 9-cycle.

If 7 → ●, then 6^+-cycle through 6, 5^+-cycle through 4 ⇒ 11^+-cycle.

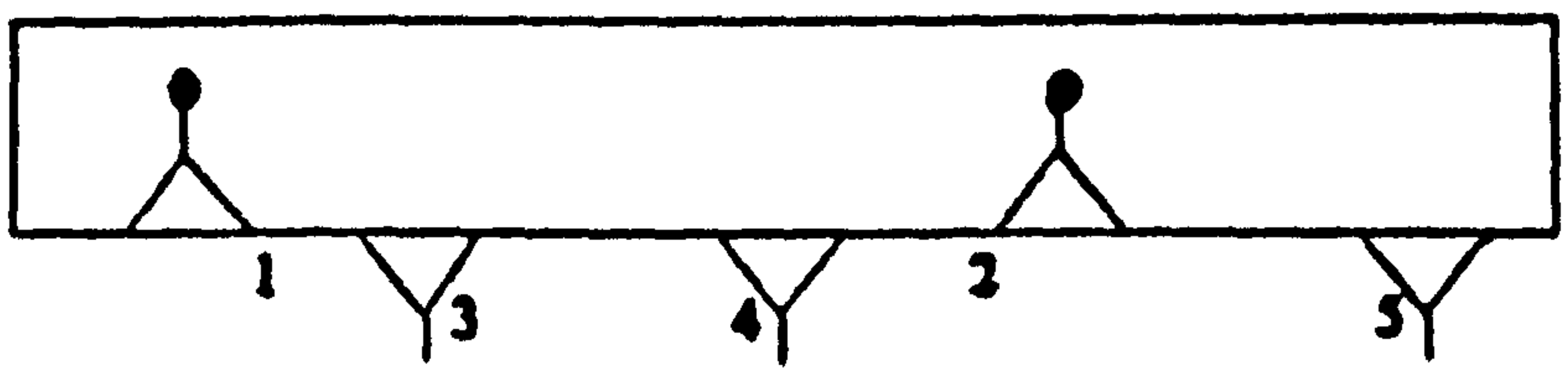
If 4 → ●, then 6^+-cycle through 6, 7^+-cycle through 8 ⇒ 13^+-cycle.

If 8 → ●, then 6^+-cycle through 6, 5^+-cycle through 9 ⇒ 11^+-cycle.

If 9 → ●, then the 8^+-cycle through 6 cannot be converted to a 9-cycle.

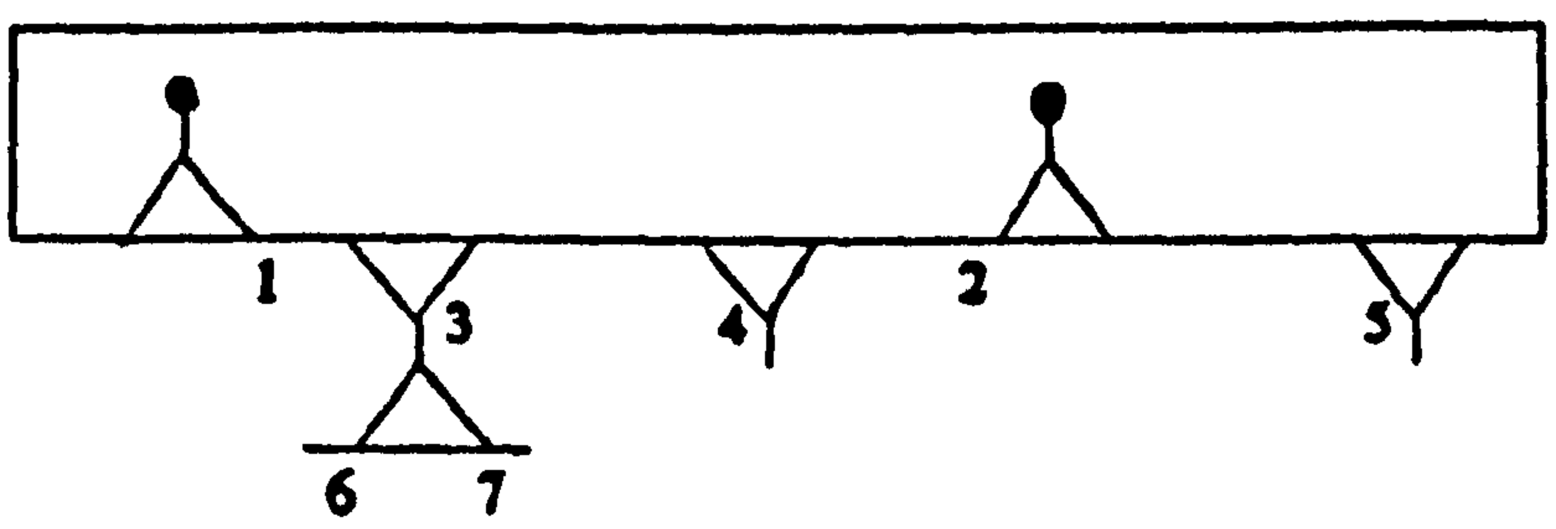
Contradiction. ∴ Case (6b1) not possible.

Case (6b2) : Assume two triangles lie between 1 and 2.



If 3→4, then 2-cycle. If 3→5, then 4→Δ→Δ→● ⇒ 2-cycle.

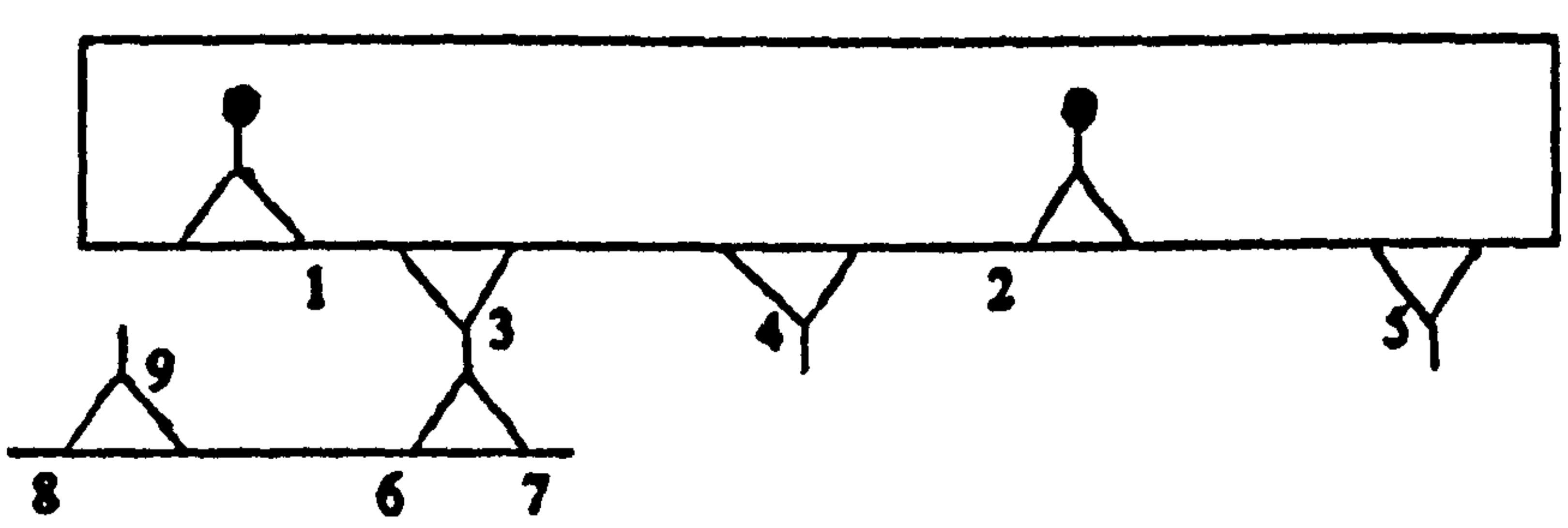
If 3→●, then 4→Δ ⇒ 4^+-cycle, 7^+-cycle ⇒ 11^+-cycle. ∴ 3→Δ.



If 6→7, then 1-cycle. If 6→5, then 4-cycle.

If 6→4, then 4^+-cycle, 7^+-cycle ⇒ 11^+-cycle.

If 6→●, then 7→Δ ⇒ 4→5 ⇒ 1-cycle. ∴ 6→Δ.



If 4→●, then 5^+-cycle through 9, 7^+-cycle through 5 ⇒ 12^+-cycle.

If 5→●, then 4→9 ⇒ 7→8 ⇒ 2-cycle. If 8→●, then 4→5 ⇒ 8-cycle.

If 7→●, then 5^+-cycle through 9, 6^+-cycle through 4 ⇒ 11^+-cycle.

If 9→●, then 8→4 ⇒ 5→7 ⇒ 15-cycle.

Contradiction. ∴ Case (6b2) not possible.

Therefore, Case (6b) not possible.

Therefore, there does not exist a coset diagram for (24,0,0,3,0,2).

Hence, no such S exists. □

THEOREM 2.3.9 Every specification (u,p,e,f,g_1,g_3) satisfying the genus formula (2.1.2), corresponds to a subgroup of (finite) index u in Δ_9, with the following exceptions :

(a) $(12n+9,0,1,0,0,n+3)$, $\forall n \geq 0$.

(b) (24,0,0,0,0,5)

(c) (24,0,0,0,3,1)

(d) (24,0,0,3,0,2).

Proof : From Lemmas 2.3.2 to 2.3.8, we know there exists a coset diagram for every specification (u,p,e,f,g_1,g_3) satisfying (2.1.2), with exceptions (a), (b), (c) and (d).

From Lemma 2.1 in [18], there is a correspondence between subgroups of index u in Δ_9 and u point coset diagrams for Δ_9. The theorem follows immediately. □

CHAPTER 3

(2,3,13) TRIANGLE GROUP

§3.1 GENUS FORMULA

The genus formula can be derived from Theorem 2 in [15]. We get

$2p - 2 + e(1 - \frac{1}{2}) + f(1 - \frac{1}{3}) + g(1 - \frac{1}{13}) = u(1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{13})$,

which simplifies to

$$7u = 156(p - 1) + 39e + 52f + 72g \qquad (3.1.1)$$

This is the genus formula for subgroups of index u in Δ_{13}.

§3.2 SPECIFICATION

For Δ_{13}, we define a *specification* to be a list of non-negative integers (u,p,e,f,g), with $u \geq 1$, which satisfies the genus formula (3.1.1).

If a coset diagram with specification (u,p,e,f,g) exists, then there will be u vertices, including e red points, f blue points and g green points.

THEOREM 3.2.1 The genus formula (3.1.1) has a solution for each $u \geq 104$, but not for $u = 103$.

Proof : $u \equiv e \pmod 4$, from (3.1.1).

Without loss of generality, let $p = 0$ and $e \leq 3$.

Then there are four cases :

(i) $u = 4v$, $e = 0$ (iii) $u = 4v + 2$, $e = 2$

(ii) $u = 4v + 1$, $e = 1$ (iv) $u = 4v + 3$, $e = 3$.

case (i) : $u = 4v$, $e = p = 0$. Substitute values in (3.1.1).

$$28v + 156 = 52f + 72g$$

$$\therefore \quad 7v + 39 = 13f + 18g$$

By Result 1.2.1, this is solvable if $7v + 39 \geq 204$, i.e. $u \geq 95$.

case (ii) : $u = 4v + 1$, $e = 1$, $p = 0$. Substitute values in (3.1.1).

$$28v + 124 = 52f + 72g$$

$$\therefore \quad 7v + 31 = 13f + 18g$$

By Result 1.2.1, this is solvable if $7v + 31 \geq 204$, i.e. $u \geq 100$.

case (iii) : $u = 4v + 2$, $e = 2$, $p = 0$. Substitute values in (3.1.1).

$$28v + 92 = 52f + 72g$$

$$\therefore \quad 7v + 23 = 13f + 18g$$

By Result 1.2.1, this is solvable if $7v + 23 \geq 204$, i.e. $u \geq 106$.

case (iv) : $u = 4v + 3$, $e = 3$, $p = 0$. Substitute values in (3.1.1).

$$28v + 60 = 52f + 72g$$

$$\therefore \quad 7v + 15 = 13f + 18g$$

By Result 1.2.1, this is solvable if $7v + 15 \geq 204$, i.e. $u \geq 111$.

From cases (i), (ii), (iii) and (iv), we deduce that (3.1.1) has a solution for each $u \geq 111$. Using (3.1.1), a computer program was developed to determine all the solutions for (3.1.1) for $u \leq 110$. This program and its output are shown in APPENDIX C. From the output, we see that solutions of (3.1.1) exist for $u = 104, 105, \ldots, 110$, but not for 103. These values can also be checked by hand using (3.1.1).

The specifications listed in the program output for $u = 104, 105, \ldots, 110$ do satisfy (3.1.1).

For $u = 103$, we substitute this value into (3.1.1) and re-arrange to get

$$877 - 72g = 13(12p + 3e + 4f) \qquad (3.2.1)$$

Now we can conclude that $g \leq 12$ since $g, p, e, f \geq 0$. Next, we put each possible value of g $(0, 1, \ldots, 12)$ into (3.2.1). The RHS is divisible by 13, but the LHS is only divisible by 13 when $g = 12$, in which case we have

$$1 = 12p + 3e + 4f \qquad (3.2.2)$$

Now, p, e and f are non-negative integers, so clearly (3.2.2) has no solution. This implies that (3.1.1) has no solution for $u = 103$. □

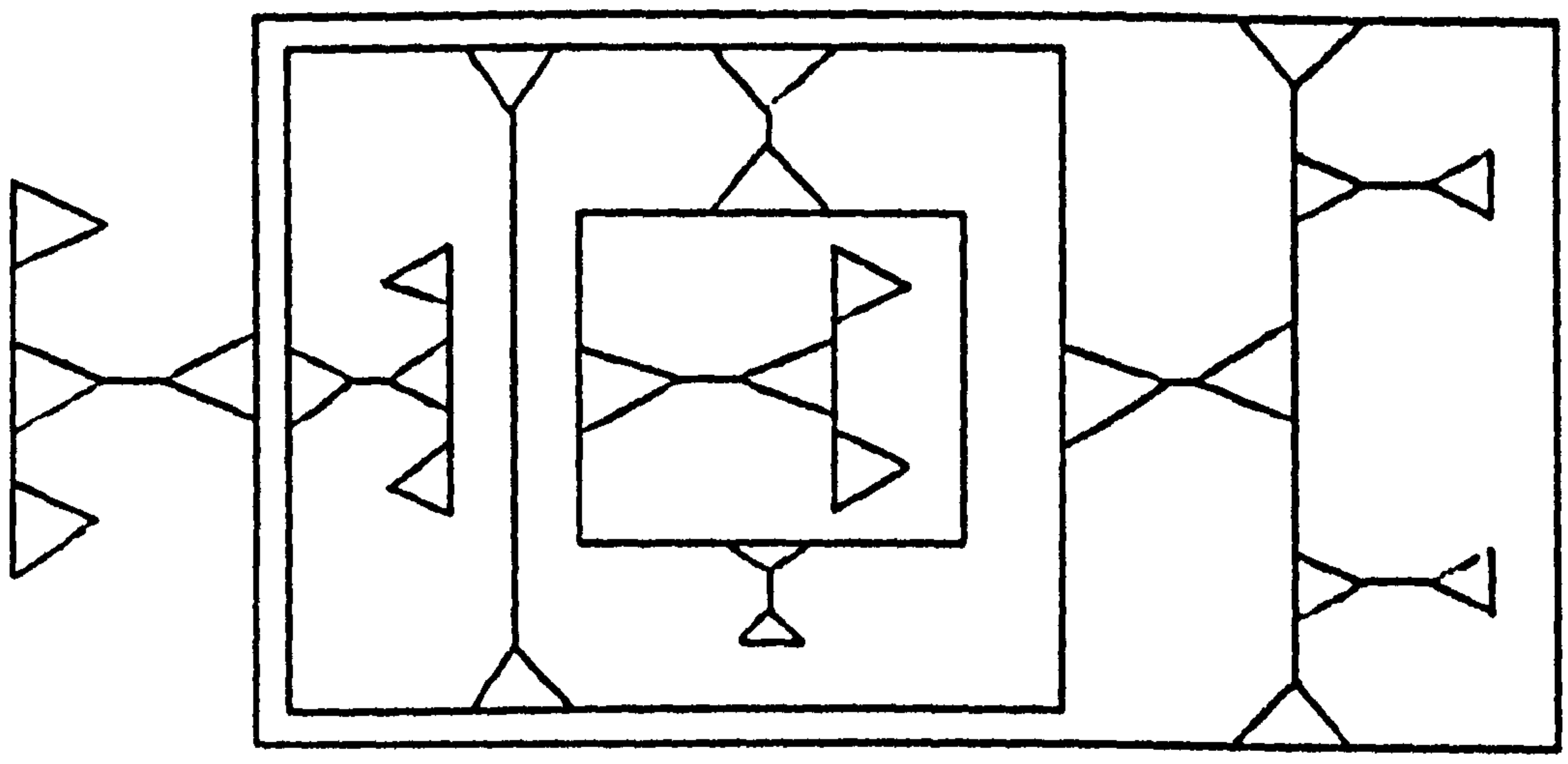
§3.3 SUBGROUPS OF FINITE INDEX IN Δ_{13}

Before we can prove Theorem 3.3.6, we will need the following five lemmas. Note that some specifications will be used in more than one lemma.

LEMMA 3.3.1 If $S(u,p,e,f,g)$ satisfies (3.1.1) and $e \geq 4$, then there exists a coset diagram with specification S which is $n(1)$ where $n \geq 2$.

Proof : Assume S is a counter-example with $p + e + f + g$ minimal. We want to show that no such S exists.

E_1 (78,0,18,0,0) 9(1)

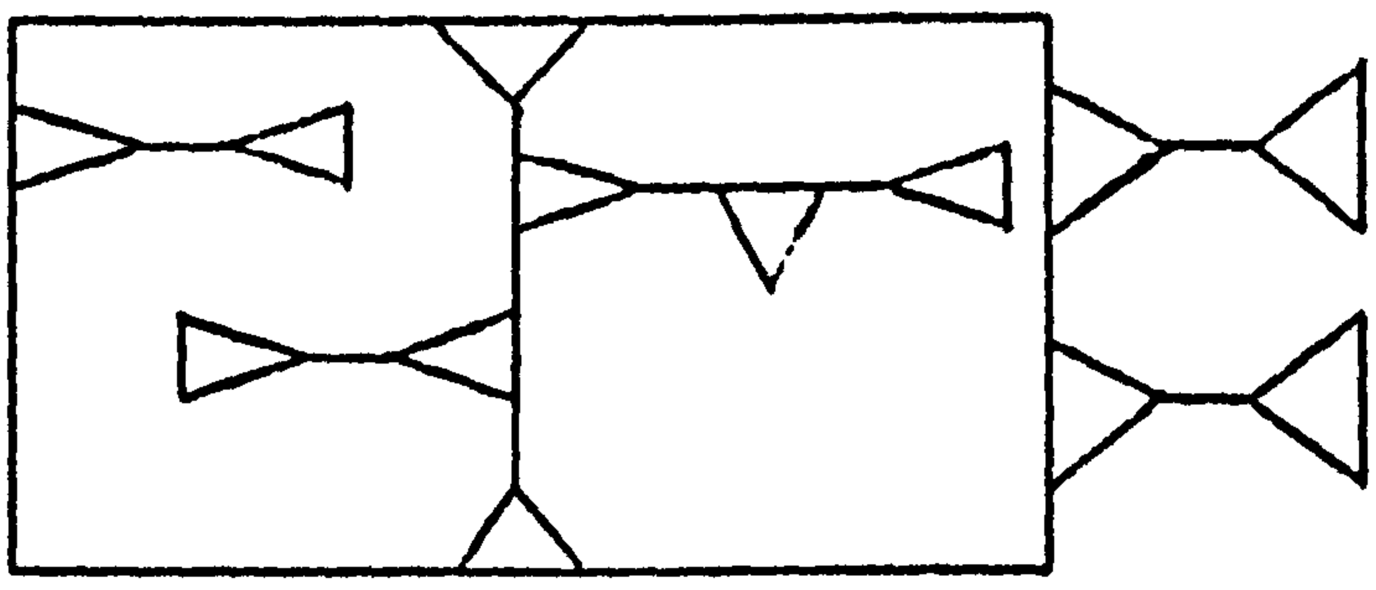


$E_1 + E_1$ = (156,0,32,0,0) 16(1). Now, 1-compose seven times to get

P_1 (156,7,4,0,0) 2(1).

If $p \geq 7$ and D(u−156,p−7,e,f,g) satisfies (3.1.1), then $D + P_1$ has specification S which is n(1), $n \geq 2$. Therefore, S has $p < 7$.

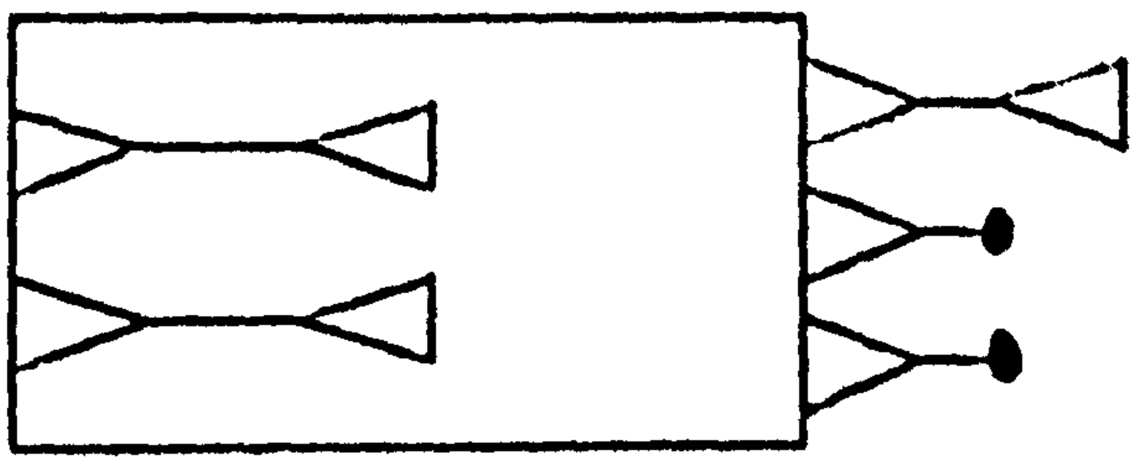
E_2 (39,0,11,0,0) 5(1)



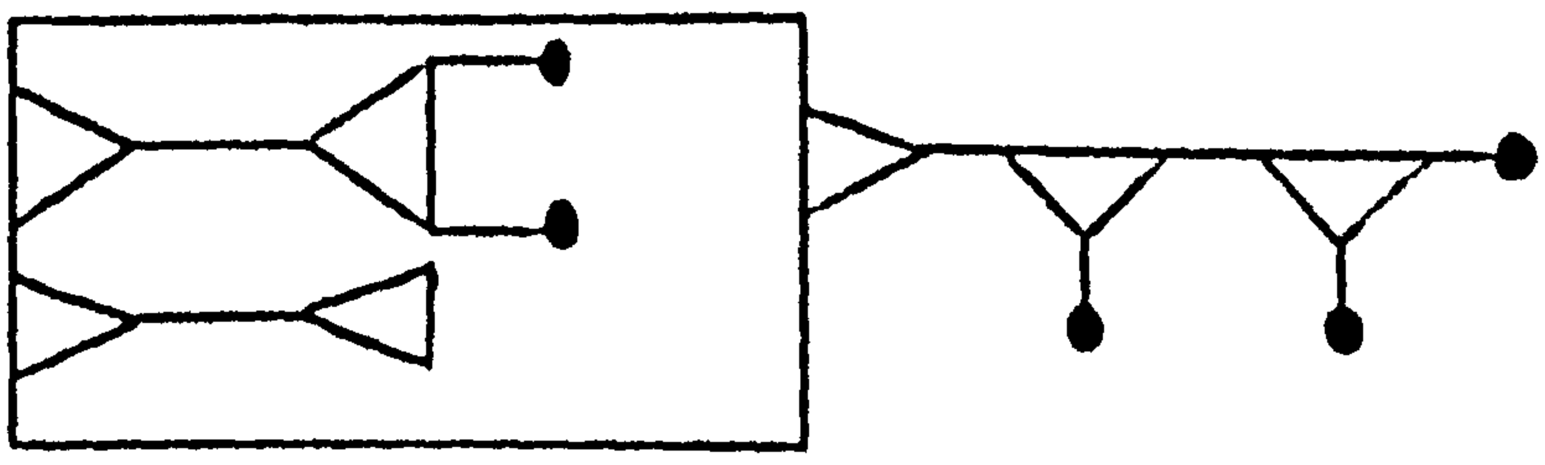
If $e \geq 11$ and D(u−39,p,e−7,f,g) satisfies (3.1.1), then $D + E_2$ has specification S which is n(1), $n \geq 2$. Therefore. S has $e < 11$.

$E_3 = E_2(1)$ = (39,1,7,0,0) 3(1).

F_1 (26,0,6,2,0) 3(1)



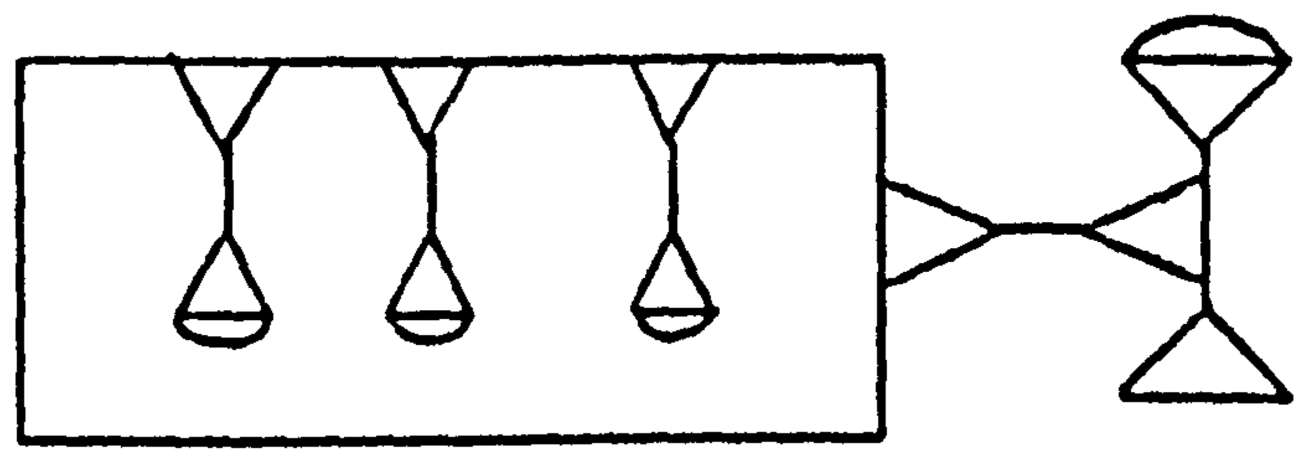
F_2 (26,0,2,5,0) 1(1)



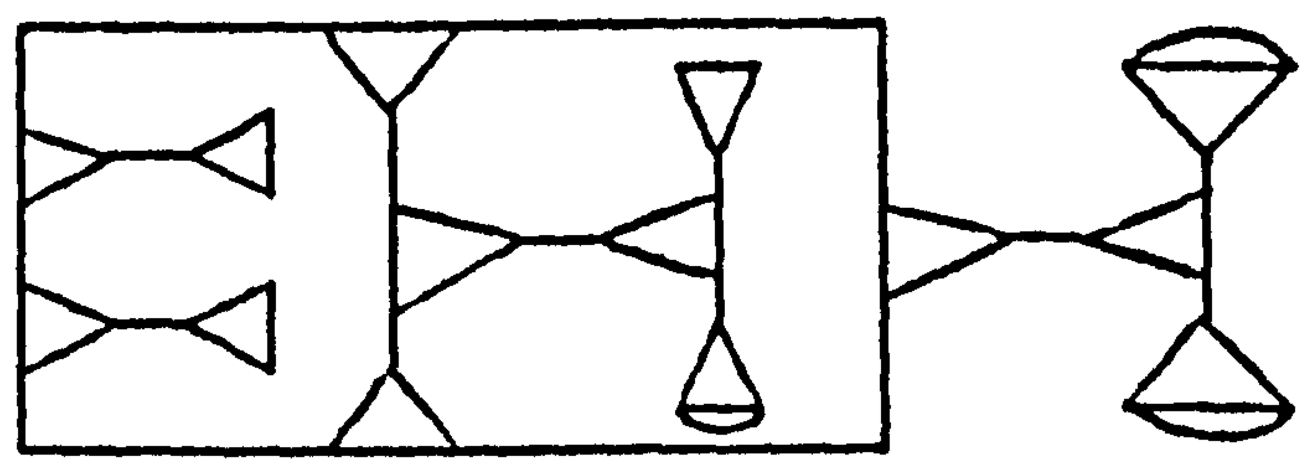
F_3 (52,0,4,7,0) 2(1) = F_1 3(1) + F_2 1(1).

If $f \geq 7$ and $D(u-52,p,e,f-7,g)$ satisfies (3.1.1), then $D + F_3$ has specification S which is n(1), $n \geq 2$. Therefore, S has $f < 7$.

G_1 (30,0,2,0,4) 1(1)



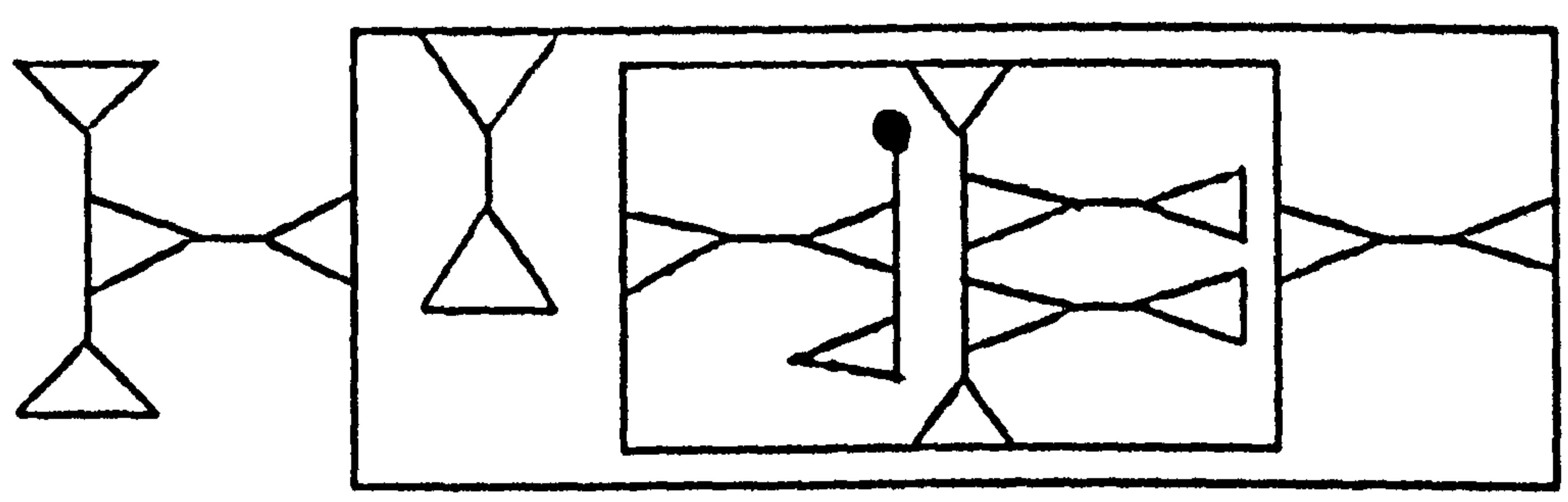
G_2 (42,0,6,0,3) 3(1)



G_3(72,0,4,0,7) 2(1) = G_1 1(1) + G_2 3(1).

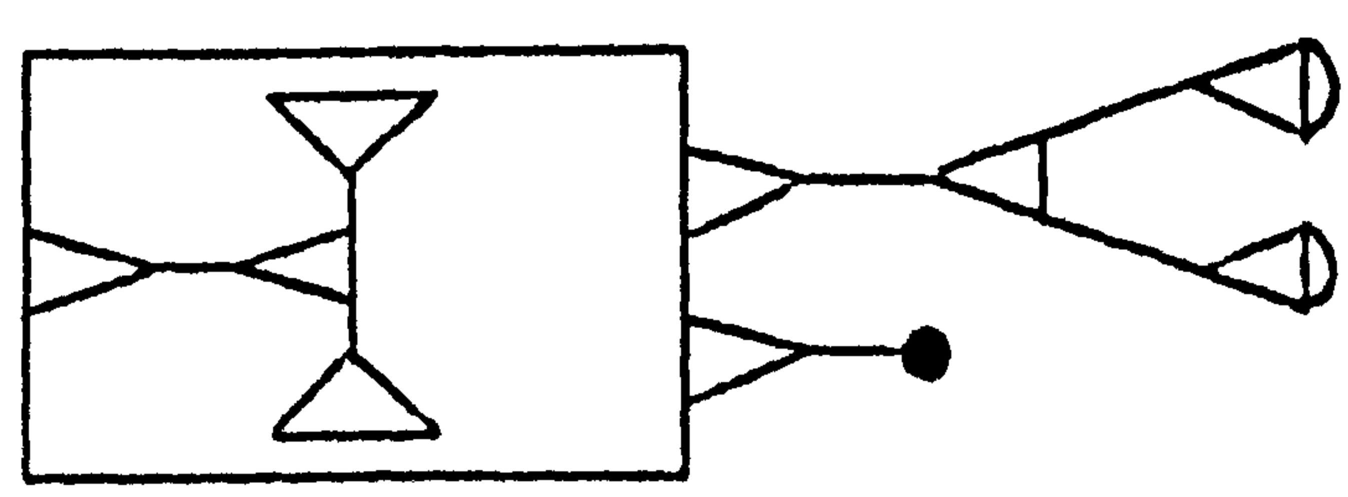
If $g \geq 7$ and $D(u-72,p,e,f,g-7)$ satisfies (3.1.1), then $D + G_3$ has specification S which is n(1), $n \geq 2$. Therefore, S has $g < 7$.

F_4 (52,0,12,1,0)

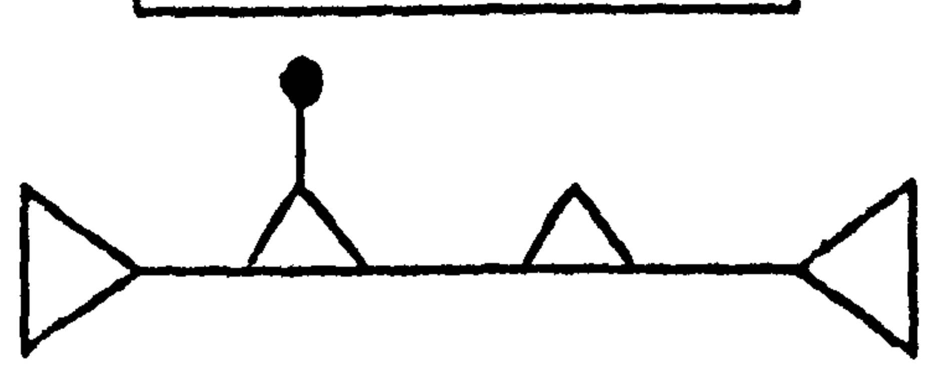


Now, 1-compose F_4 twice to get P_2(52,2,4,1,0) 2(1).

G_4 (28,0,4,1,2) 2(1)



E_4 (13,0,5,1,0) 2(1)



For $f \geq 1$,

if $p \geq 2$ and $D(u-52,p-2,e,f-1,g)$ satisfies (3.1.1),

then $D + P_2$ has specification S which is n(1), $n \geq 2$,

and, if $e \geq 5$ and $D(u-13,p,e-1,f-1,g)$ satisfies (3.1.1),

then $D + E_4$ has specification S which is n(1), $n \geq 2$, and, if $g \geq 2$ and $D(u-28,p,e,f-1,g-2)$ satisfies (3.1.1),

then $D + G_4$ has specification S which is n(1), $n \geq 2$.

Therefore, for $f \geq 1$, S has $p < 2$, $e = 4$ and $g < 2$.

We now know that a minimal S would have one of the following two forms

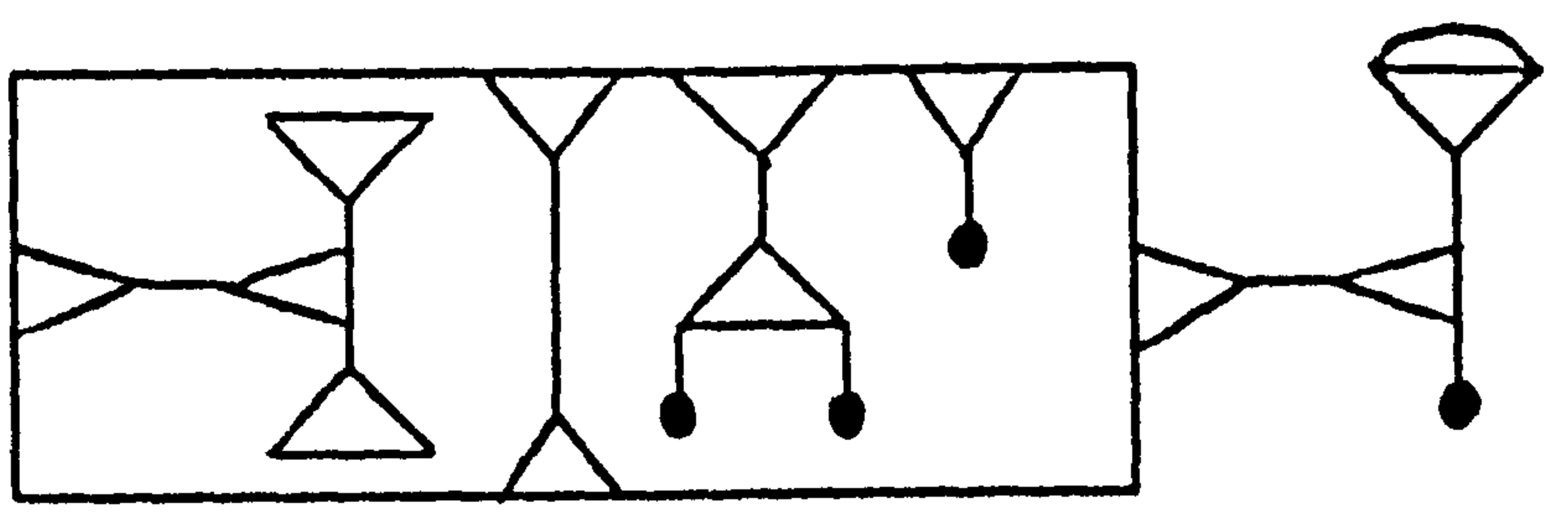
(u,p,4,f,g)	:	$p < 2$, $1 \leq f \leq 6$, $g < 2$	(3.3.1)
(u,p,e,0,g)	:	$p < 7$, $e < 11$, $g < 7$	(3.3.2)

Case (3.3.1) : Put $e = 4$, $p = g = 0$ in (3.1.1) to get

$7u = 52f \quad \therefore \quad f \equiv 0 \pmod 7$, which has no solution for $1 \leq f \leq 6$.

With $e = 4$, $p = 0$, $g = 1$, we get $7u = 52f + 72 \quad \therefore \quad f \equiv 4 \pmod 7 \quad \therefore \quad f = 4$.

F_5 (40,0,4,4,1) 2(1)

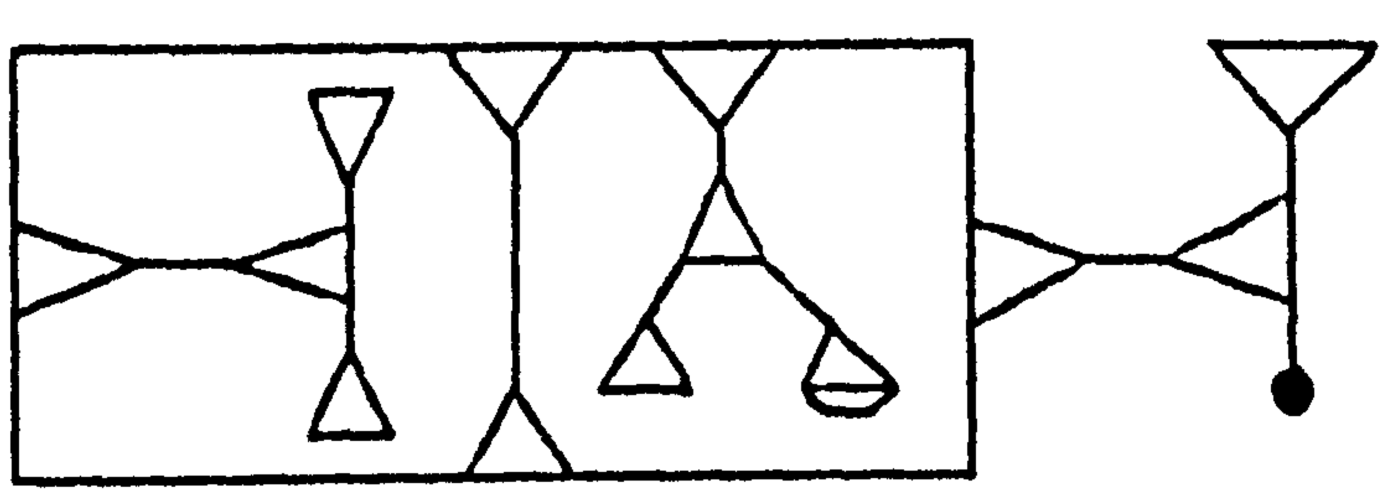


With $e = 4$, $p = 1$, $g = 0$, we get $7u = 52f + 156 \quad \therefore \quad f \equiv 4 \pmod 7 \quad \therefore \quad f = 4$.

$F_1 + F_1 = (52,0,8,4,0)$ 4(1). Now 1-compose once to get $F_6(52,1,4,4,0)$ 2(1).

With $e = 4$, $p = g = 1$, we get $7u = 52f + 228 \quad \therefore \quad f \equiv 1 \pmod 7 \quad \therefore \quad f = 1$.

(40,0,8,1,1) 4(1)



Now 1-compose once to get $F_7(40,1,4,1,1)$ 2(1). This completes Case (3.3.1).

For $p \geq 1$,

if $e \geq 7$ and $D(u-39,p-1,e-3,f,g)$ satisfies (3.1.1),

then $D + E_3$ has specification S which is n(1), $n \geq 2$.

Therefore, for $p \geq 1$, S has $e < 7$.

Hence, Case (3.3.2) can be split into the following two cases

(u,0,e,0,g)	:	$4 \le e \le 10,\ g < 7$	(3.3.3)
(u,p,e,0,g)	:	$1 \le p \le 6,\ 4 \le e \le 6,\ g < 7$	(3.3.4)

<u>Case (3.3.3)</u> : Put $p = f = 0$ in (3.1.1) to get $7u = 72g + 39(e - 4)$.

Replacing e by 4 (resp. 5, 6, ..., 10) in this equation and noting $g < 7$,

$e = 4\ :\ 7u = 72g \quad \therefore \quad g \equiv 0 \pmod 7 \quad \therefore \quad g = 0 \quad \therefore \quad u < 1.$

$e = 5\ :\ 7u = 72g + 39 \quad \therefore \quad g \equiv 5 \pmod 7 \quad \therefore \quad g = 5 \quad \therefore \quad u = 57.$

$e = 6\ :\ 7u = 72g + 78 \quad \therefore \quad g \equiv 3 \pmod 7 \quad \therefore \quad g = 3 \quad \therefore \quad u = 42.$

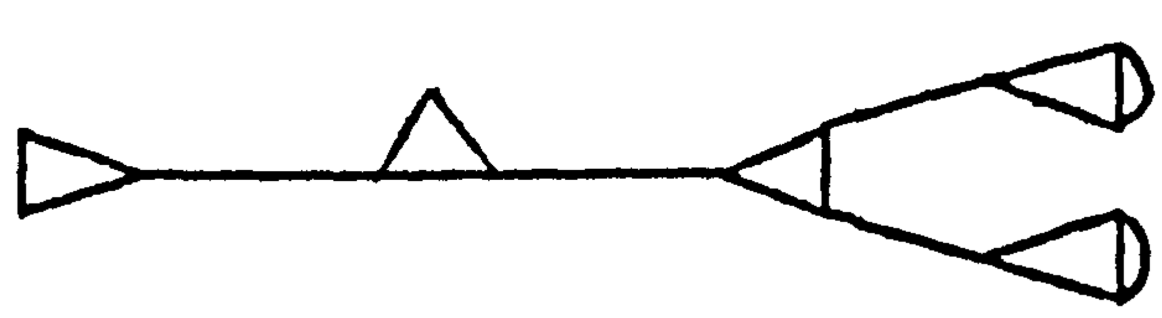
$e = 7\ :\ 7u = 72g + 117 \quad \therefore \quad g \equiv 1 \pmod 7 \quad \therefore \quad g = 1 \quad \therefore \quad u = 27.$

$e = 8\ :\ 7u = 72g + 156 \quad \therefore \quad g \equiv 6 \pmod 7 \quad \therefore \quad g = 6 \quad \therefore \quad u = 84.$

$e = 9\ :\ 7u = 72g + 195 \quad \therefore \quad g \equiv 4 \pmod 7 \quad \therefore \quad g = 4 \quad \therefore \quad u = 69.$

$e = 10\ :\ 7u = 72g + 234 \quad \therefore \quad g \equiv 2 \pmod 7 \quad \therefore \quad g = 2 \quad \therefore \quad u = 54.$

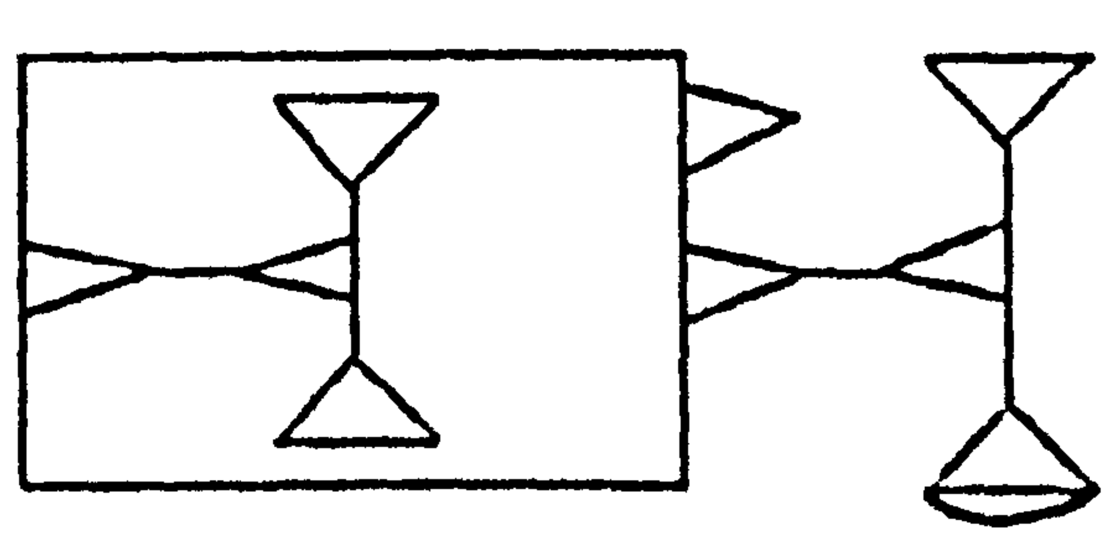
G_5 (15,0,3,0,2) 1(1)



$e = 5$: (57,0,5,0,5) 2(1) = $G_5 + G_2$.

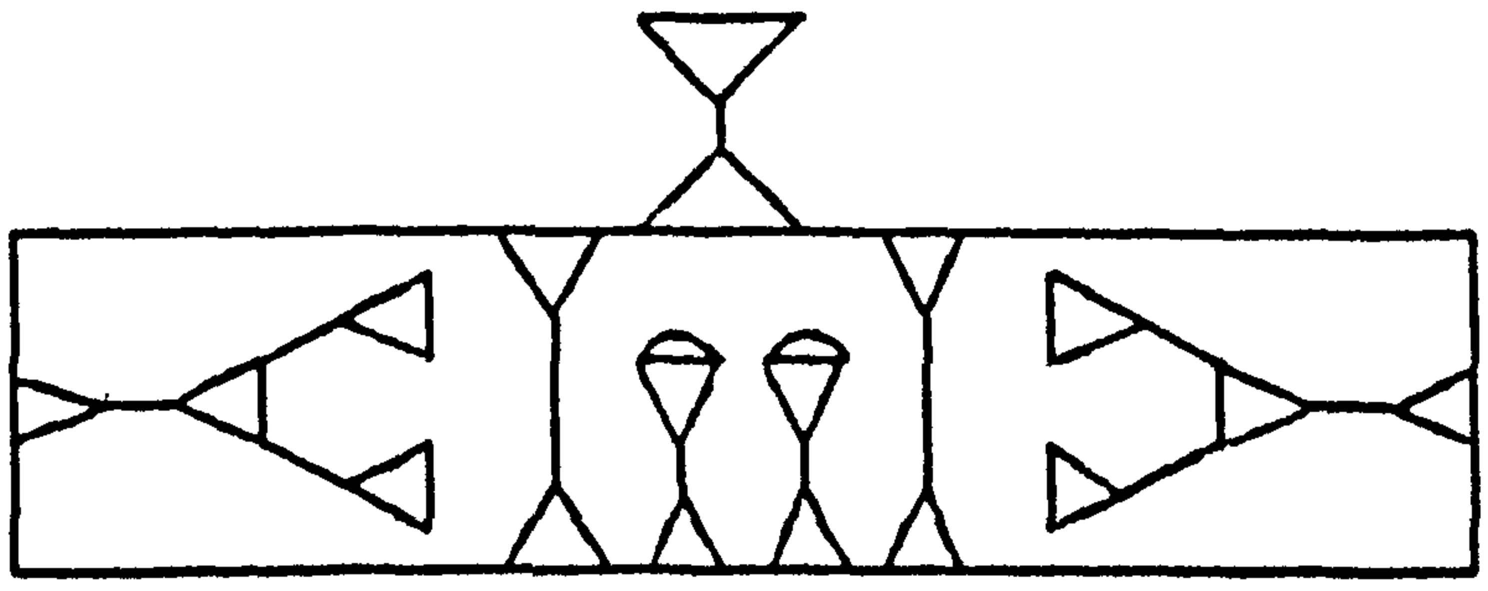
$e = 6$: (42,0,6,0,3) 3(1) = G_2.

$e = 7$: E_5 (27,0,7,0,1) 3(1)



$e = 8$: E_6 (84,0,8,0,6) 4(1) = $G_2 + G_2$.

$e = 9$: E_7 (69,0,9,0,4) 4(1) = $E_5 + G_2$.

$e = 10 : E_8(54,0,10,0,2)\,5(1)$ 

This completes Case (3.3.3).

Case (3.3.4) : This case can be split into three cases :-

Case (1) : e = 4 $(u,p,4,0,g) : 1 \leq p \leq 6,\ 0 \leq g \leq 6.$

Put $e = 4$, $f = 0$ in (3.1.1) to get $7u = 72g + 156p$.

$p = 1 : 7u = 72g + 156 \quad \therefore \quad g \equiv 6 \pmod 7 \quad \therefore \quad g = 6 \quad \therefore \quad u = 84.$

$p = 2 : 7u = 72g + 312 \quad \therefore \quad g \equiv 5 \pmod 7 \quad \therefore \quad g = 5 \quad \therefore \quad u = 96.$

$p = 3 : 7u = 72g + 468 \quad \therefore \quad g \equiv 4 \pmod 7 \quad \therefore \quad g = 4 \quad \therefore \quad u = 108.$

$p = 4 : 7u = 72g + 624 \quad \therefore \quad g \equiv 3 \pmod 7 \quad \therefore \quad g = 3 \quad \therefore \quad u = 120.$

$p = 5 : 7u = 72g + 780 \quad \therefore \quad g \equiv 2 \pmod 7 \quad \therefore \quad g = 2 \quad \therefore \quad u = 132.$

$p = 6 : 7u = 72g + 936 \quad \therefore \quad g \equiv 1 \pmod 7 \quad \therefore \quad g = 1 \quad \therefore \quad u = 144.$

$p = 1 : P_3(84,1,4,0,6)\,2(1) = E_6(1).$

1-compose E_8 twice to get $P_4(54,2,2,0,2)\,1(1)$.

$p = 2 : (96,2,4,0,5)\,2(1) = P_4 + G_2.$

$E_8 + E_8 = (108,0,16,0,4)\,8(1)$. 1-compose three times to get

$p = 3 : P_5(108,3,4,0,4)\,2(1).$

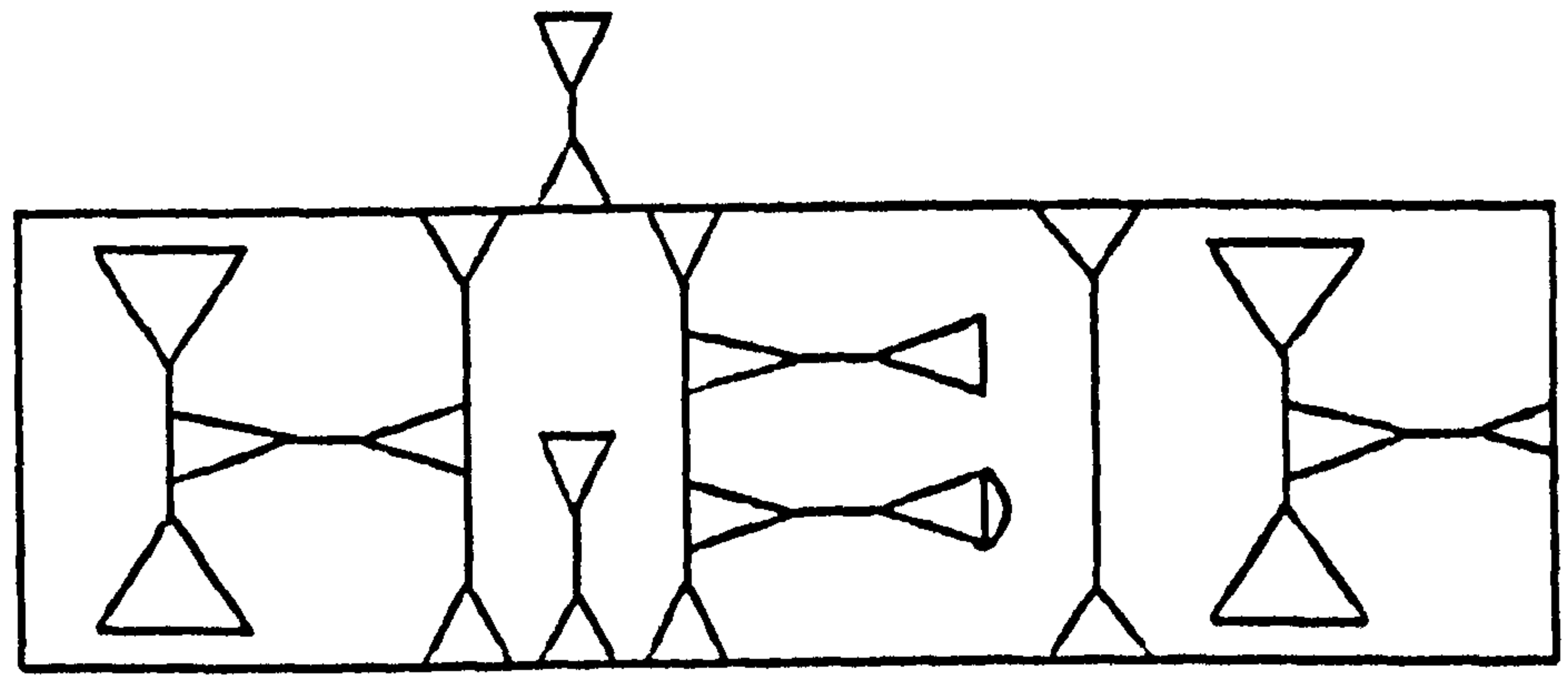
$E_1 + G_2 = (120,0,20,0,3)\,10(1)$. 1-compose four times to get

$p = 4 : P_6(120,4,4,0,3)\,2(1).$

$E_1 + E_8 = (132,0,24,0,2)\,12(1)$. 1-compose five times to get

$p = 5 : P_7(132,5,4,0,2)\,2(1).$

Q_1 (66,0,14,0,1) 7(1)



$Q_1 + E_1 = (144,0,28,0,1)\ 14(1)$. 1-compose six times to get

$p = 6$: $P_8(144,6,4,0,1)\ 2(1)$.

<u>Case (2) : $e = 5$</u> $(u,p,5,0,g)$: $1 \leq p \leq 6,\ 0 \leq g \leq 6$.

Put $e = 5$, $f = 0$ in (3.1.1) to get $7u = 72g + 156p + 39$.

$p = 1$: $7u = 72g + 195$	$\therefore$ $g \equiv 4 \pmod 7$	$\therefore$ $g = 4$	$\therefore$ $u = 69$.
$p = 2$: $7u = 72g + 351$	$\therefore$ $g \equiv 3 \pmod 7$	$\therefore$ $g = 3$	$\therefore$ $u = 81$.
$p = 3$: $7u = 72g + 507$	$\therefore$ $g \equiv 2 \pmod 7$	$\therefore$ $g = 2$	$\therefore$ $u = 93$.
$p = 4$: $7u = 72g + 663$	$\therefore$ $g \equiv 1 \pmod 7$	$\therefore$ $g = 1$	$\therefore$ $u = 105$.
$p = 5$: $7u = 72g + 819$	$\therefore$ $g \equiv 0 \pmod 7$	$\therefore$ $g = 0$	$\therefore$ $u = 117$.
$p = 6$: $7u = 72g + 975$	$\therefore$ $g \equiv 6 \pmod 7$	$\therefore$ $g = 6$	$\therefore$ $u = 201$.

$p = 1$: $(69,1,5,0,4)\ 2(1) = E_7(1)$.

$E_3 + G_2 = (81,1,9,0,3)\ 4(1)$. 1-compose once to get

$p = 2$: $Q_2(81,2,5,0,3)\ 2(1)$.

$E_1 + G_5 = (93,0,17,0,2)\ 8(1)$. 1-compose three times to get

$p = 3$: $Q_3(93,3,5,0,2)\ 2(1)$.

$E_1 + E_5 = (105,0,21,0,1)\ 10(1)$. 1-compose four times to get

$p = 4$: $Q_4(105,4,5,0,1)\ 2(1)$.

$E_1 + E_2 = (117,0,25,0,0)\ 12(1)$. 1-compose five times to get

$p = 5$: $Q_5(117,5,5,0,0)\ 2(1)$.

$E_7 + P_7 = (201,5,9,0,6)\ 4(1)$. 1-compose once to get

$p = 6$: $Q_6(201,6,5,0,6)\ 2(1)$.

Case (3) : e = 6 (u,p,6,0,g) : $1 \le p \le 6$, $0 \le g \le 6$.

Put e = 6, f = 0 in (3.1.1) to get 7u = 72g + 156p + 78.

$p = 1$: $7u = 72g + 234$ $\therefore$ $g \equiv 2 \pmod 7$ $\therefore$ $g = 2$ $\therefore$ $u = 54$.

$p = 2$: $7u = 72g + 390$ $\therefore$ $g \equiv 1 \pmod 7$ $\therefore$ $g = 1$ $\therefore$ $u = 66$.

$p = 3$: $7u = 72g + 546$ $\therefore$ $g \equiv 0 \pmod 7$ $\therefore$ $g = 0$ $\therefore$ $u = 78$.

$p = 4$: $7u = 72g + 702$ $\therefore$ $g \equiv 6 \pmod 7$ $\therefore$ $g = 6$ $\therefore$ $u = 162$.

$p = 5$: $7u = 72g + 858$ $\therefore$ $g \equiv 5 \pmod 7$ $\therefore$ $g = 5$ $\therefore$ $u = 174$.

$p = 6$: $7u = 72g + 1014$ $\therefore$ $g \equiv 4 \pmod 7$ $\therefore$ $g = 4$ $\therefore$ $u = 186$.

p = 1 : (54,1,6,0,2) 3(1) = $E_8(1)$.

1-compose Q_1(66,0,14,0,1) 7(1) twice to get

p = 2 : (66,2,6,0,1) 3(1).

1-compose E_1(78,0,18,0,0) 9(1) three times to get

p = 3 : (78,3,6,0,0) 3(1).

$E_8 + P_5$ = (162,3,10,0,6) 5(1). 1-compose once to get

p = 4 : (162,4,6,0,6) 3(1).

$E_8 + P_6$ = (174,4,10,0,5) 5(1). 1-compose once to get

p = 5 : (174,5,6,0,5) 3(1).

$Q_1 + P_6$ = (186,4,14,0,4) 7(1). 1-compose twice to get

p = 6 : (186,6,6,0,4) 3(1).

Hence, no such S exists. □

<u>LEMMA 3.3.2</u> If S (u,p,e,f,g) satisfies (3.1.1) and e = 3, then there exists a coset diagram with specification S which is 1(1).

Proof : Assume S is a counter-example with $p + f + g$ minimal. We want to show that no such S exists.

$P_9(39,2,3,0,0)\ 1(1) = E_3\,(1)$.

If $p \geq 2$ and $D\,(u-39,p-2,4,f,g)$ satisfies (3.1.1), then D can be 2(1) by Lemma 3.3.1 so that $D + P_9$ has specification S which is 1(1).

Therefore, S has $p < 2$.

$F_8(39,0,3,6,0)\ 1(1) = E_4 + F_2$.

If $f \geq 6$ and $D\,(u-39,p,4,f-6,g)$ satisfies (3.1.1), then D can be 2(1) by Lemma 3.3.1 so that $D + F_8$ has specification S which is 1(1).

Therefore, S has $f < 6$.

A diagram has already been exhibited for $G_5\,(15,0,3,0,2)\ 1(1)$.

If $g \geq 2$ and $D\,(u-15,p,4,f,g-2)$ satisfies (3.1.1), then D can be 2(1) by Lemma 3.3.1 so that $D + G_5$ has specification S which is 1(1).

Therefore, S has $g < 2$.

We now know that a minimal S would have one of the following four forms

(u,0,3,f,0)	:	$f \leq 5$	(3.3.5)
(u,0,3,f,1)	:	$f \leq 5$	(3.3.6)
(u,1,3,f,0)	:	$f \leq 5$	(3.3.7)
(u,1,3,f,1)	:	$f \leq 5$	(3.3.8)

Case (3.3.5) (u,0,3,f,0) : $0 \leq f \leq 5$.

Put $e = 3$, $p = g = 0$ in (3.1.1) to get $7u = 52f - 39$. $\therefore$ $f \equiv 6 \pmod 7$.

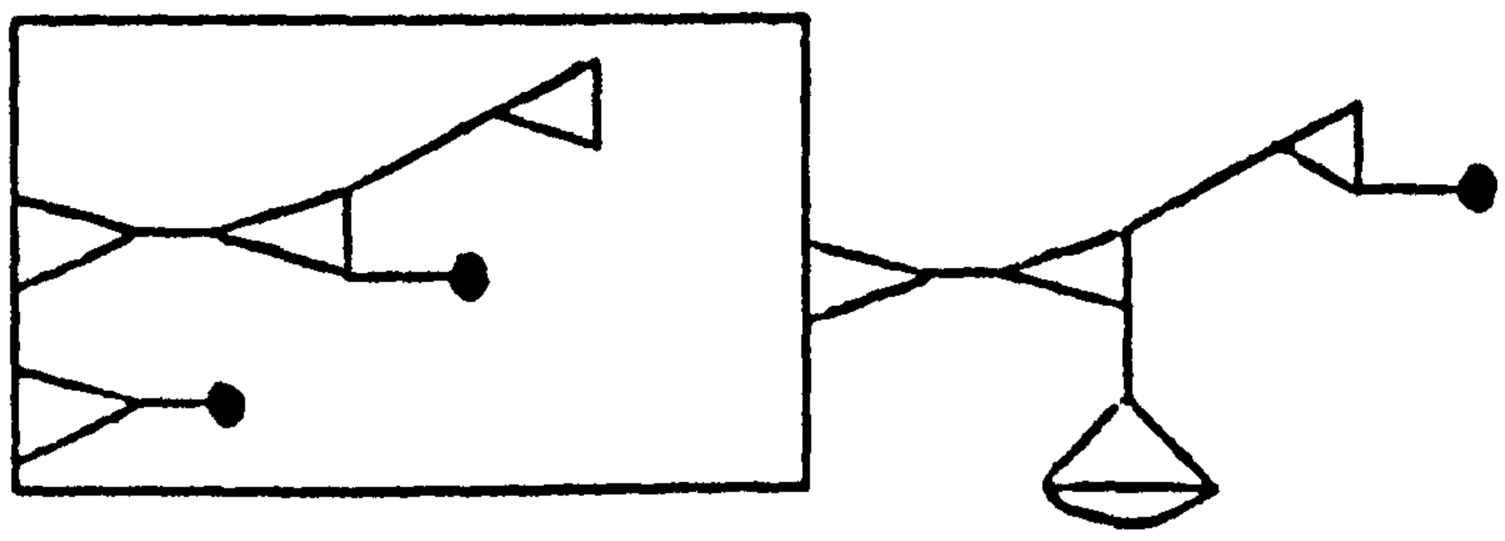
This has no solution for $0 \leq f \leq 5$.

Case (3.3.6) (u,0,3,f,1) : $0 \leq f \leq 5$.

Put $e = 3$, $p = 0$, $g = 1$ in (3.1.1) to get $7u = 52f + 33$. $\therefore$ $f \equiv 3 \pmod 7$.

$\therefore$ $f = 3$ $\therefore$ $7u = 189$ $\therefore$ $u = 27$.

(27,0,3,3,1) 1(1)



Case (3.3.7) (u,1,3,f,0) : $0 \le f \le 5$.

Put $e = 3$, $p = 1$, $g = 0$ in (3.1.1) to get $7u = 52f + 117$. $\therefore$ $f \equiv 3 \pmod 7$.

$\therefore$ $f = 3$ $\therefore$ $7u = 273$ $\therefore$ $u = 39$.

$E_4 + F_1 = (39,0,7,3,0)\ 3(1)$. Now 1-compose once to get (39,1,3,3,0) 1(1).

Case (3.3.8) (u,1,3,f,1) : $0 \le f \le 5$.

Put $e = 3$, $p = g = 1$ in (3.1.1) to get $7u = 52f + 189$. $\therefore$ $f \equiv 0 \pmod 7$.

$\therefore$ $f = 0$ $\therefore$ $7u = 189$ $\therefore$ $u = 27$.

$(27,1,3,0,1)\ 1(1) = E_5(1)$.

Hence, no such S exists. □

LEMMA 3.3.3 If S (u,p,e,f,g) satisfies (3.1.1) and $e = 2$, then there exists a coset diagram with specification S which is 1(1).

Proof : Assume S is a counter-example with $p + f + g$ minimal. We want to show that no such S exists.

A diagram has already been exhibited for E_1 (78,0,18,0,0) 9(1).

1-compose four times to get Q_7 (78,4,2,0,0) 1(1).

If $p \ge 4$ and D (u-78,p-4,4,f,g) satisfies (3.1.1), then D can be 2(1) by Lemma 3.3.1 so that $D + Q_7$ has specification S which is 1(1).

Therefore, S has $p < 4$.

A diagram has already been exhibited for F_2 (26,0,2,5,0) 1(1).

If $f \geq 5$ and D (u−26,p,4,f−5,g) satisfies (3.1.1), then D can be 2(1) by Lemma 3.3.1 so that D + F_2 has specification S which is 1(1).

Therefore, S has $f < 5$.

A diagram has already been exhibited for G_1 (30,0,2,0,4) 1(1).

If $g \geq 4$ and D (u−30,p,4,f,g−4) satisfies (3.1.1), then D can be 2(1) by Lemma 3.3.1 so that D + G_1 has specification S which is 1(1).

Therefore, S has $g < 4$.

Q_8 (26,1,2,2,0) 1(1) = F_1(1). G_6 (42,1,2,0,3) 1(1) = G_2(1).

For $p \geq 1$,

if $f \geq 2$ and D (u−26,p−1,4,f−2,g) satisfies (3.1.1), then D can be 2(1) by Lemma 3.3.1, so that D + Q_8 has specification S which is 1(1),

and, if $g \geq 3$ and D (u−42,p−1,4,f,g−3) satisfies (3.1.1), then D can be 2(1) by Lemma 3.3.1, so that D + G_6 has specification S which is 1(1).

Therefore, for $p \geq 1$, S has $f < 2$ and $g < 3$.

We now know that a minimal S would have one of the following four forms

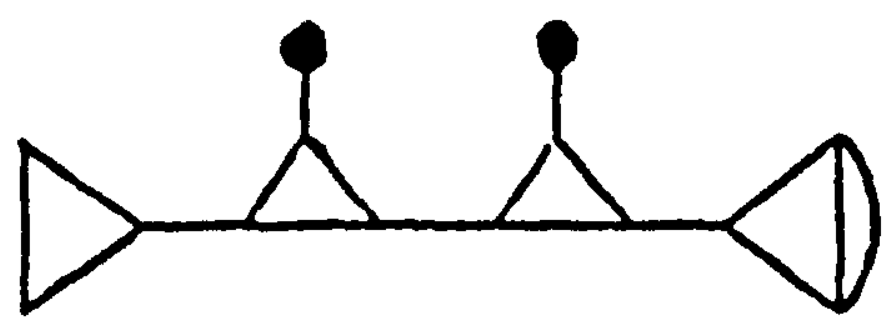
(u,0,2,f,g)	:	$f \leq 4$, $g \leq 3$	(3.3.9)
(u,1,2,f,g)	:	$f \leq 1$, $g \leq 2$	(3.3.10)
(u,2,2,f,g)	:	$f \leq 1$, $g \leq 2$	(3.3.11)
(u,3,2,f,g)	:	$f \leq 1$, $g \leq 2$	(3.3.12)

<u>Case (3.3.9)</u> (u,0,2,f,g) : $0 \leq f \leq 4$, $0 \leq g \leq 3$.

Put $e = 2$, $p = 0$ in (3.1.1) to get $7u = 52f + 72g - 78$.

$g = 0$: $7u = 52f - 78$	∴	$f \equiv 5 \pmod 7$	∴	No solution.		
$g = 1$: $7u = 52f - 6$	∴	$f \equiv 2 \pmod 7$	∴	$f = 2$	∴	$u = 14$.
$g = 2$: $7u = 52f + 66$	∴	$f \equiv 6 \pmod 7$	∴	No solution.		
$g = 3$: $7u = 52f + 138$	∴	$f \equiv 3 \pmod 7$	∴	$f = 3$	∴	$u = 42$.

$g = 1$: Q_9 (14,0,2,2,1) 1(1)



$g = 3$: (42,0,2,3,3) 1(1) = $G_4 + Q_9$.

<u>Case (3.3.10)</u> (u,1,2,f,g) : $0 \le f \le 1$, $0 \le g \le 2$.

Put e = 2, p = 1 in (3.1.1) to get $7u = 52f + 72g + 78$.

$f = 0$: $7u = 72g + 78$ $\therefore$ $g \equiv 3 \pmod 7$ $\therefore$ No solution.

$f = 1$: $7u = 72g + 130$ $\therefore$ $g \equiv 5 \pmod 7$ $\therefore$ No solution.

<u>Case (3.3.11)</u> (u,2,2,f,g) : $0 \le f \le 1$, $0 \le g \le 2$.

Put e = 2, p = 2 in (3.1.1) to get $7u = 52f + 72g + 234$.

$f = 0$: $7u = 72g + 234$ $\therefore$ $g \equiv 2 \pmod 7$ $\therefore$ $g = 2$ $\therefore$ $u = 54$.

(54,2,2,0,2) 1(1) = P_4.

$f = 1$: $7u = 72g + 286$ $\therefore$ $g \equiv 4 \pmod 7$ $\therefore$ No solution.

<u>Case (3.3.12)</u> (u,3,2,f,g) : $0 \le f \le 1$, $0 \le g \le 2$.

Put e = 2, p = 3 in (3.1.1) to get $7u = 52f + 72g + 390$.

$f = 0$: $7u = 72g + 390$ $\therefore$ $g \equiv 1 \pmod 7$ $\therefore$ $g = 1$ $\therefore$ $u = 66$.

1-compose Q_1(66,0,14,0,1) 7(1) three times to get (66,3,2,0,1) 1(1).

$f = 1$: $7u = 72g + 442$ $\therefore$ $g \equiv 3 \pmod 7$ $\therefore$ No solution.

Hence, no such S exists. □

<u>LEMMA 3.3.4</u> If S (u,p,e,f,g) satisfies (3.1.1) and e = 1, then there exists a coset diagram with specification S.

Proof : Assume S is a counter-example with $p + f + g$ minimal. We want to show that no such S exists.

If $p \geq 1$ and D(u,p−1,5,f,g) satisfies (3.1.1), then D can be 2(1) by Lemma 3.3.1, so we can 1-compose D once to get a diagram with specification S.

Therefore, S has $p < 1$. i.e. S has $p = 0$.

A diagram has already been exhibited for F_2 (26,0,2,5,0) 1(1).

If $f \geq 5$ and D(u−26,p,3,f−5,g) satisfies (3.1.1), then D can be 1(1) by Lemma 3.3.2, so that $D + F_2$ has a diagram with specification S.

Therefore, S has $f < 5$.

A diagram has already been exhibited for G_5 (15,0,3,0,2) 1(1).

If $g \geq 2$ and D(u−15,p,2,f,g−2) satisfies (3.1.1), then D can be 1(1) by Lemma 3.3.3, so that $D + G_5$ has a diagram with specification S.

Therefore, S has $g < 2$.

We now know that a minimal S would have one of the following two forms

(u,0,1,f,0) : $0 \leq f \leq 4$ (3.3.13)

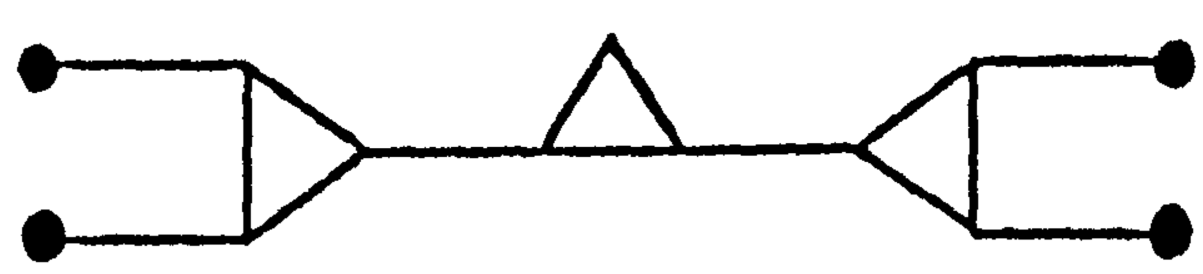
(u,0,1,f,1) : $0 \leq f \leq 4$ (3.3.14)

Case (3.3.13) (u,0,1,f,0) : $0 \leq f \leq 4$.

Put $e = 1$, $p = g = 0$ in (3.1.1) to get $7u = 52f - 117$.

$\therefore$ $f \equiv 4 \pmod 7$ $\therefore$ $f = 4$. $\therefore$ $u = 13$.

(13,0,1,4,0)



Case (3.3.14) (u,0,1,f,1) : $0 \leq f \leq 4$.

Put $p = 0$, $e = g = 1$ in (3.1.1) to get $7u = 52f - 45$.

$\therefore$ $f \equiv 1 \pmod 7$ $\therefore$ $f = 1$. $\therefore$ $u = 1$.

(1,0,1,1,1) The diagram consists of a red point with a blue loop.

Hence, no such S exists. □

<u>LEMMA 3.3.5</u> If $S(u,p,e,f,g)$ satisfies (3.1.1) and $e = 0$, then there exists a coset diagram with specification S.

<u>Proof :</u> Assume S is a counter-example with $p + f + g$ minimal. We want to show that no such S exists.

If $p \geq 1$ and $D(u,p-1,4,f,g)$ satisfies (3.1.1), then D can be 2(1) by Lemma 3.3.1, so we can 1-compose D once to get a diagram with specification S.

Therefore, S has $p < 1$. i.e. S has $p = 0$.

A diagram has already been exhibited for $F_2\,(26,0,2,5,0)$ 1(1).

If $f \geq 5$ and $D(u-26,p,2,f-5,g)$ satisfies (3.1.1), then D can be 1(1) by Lemma 3.3.3, so that $D + F_2$ has a diagram with specification S.

Therefore, S has $f < 5$.

A diagram has already been exhibited for $G_1\,(30,0,2,0,4)$ 1(1).

If $g \geq 4$ and $D(u-30,p,2,f,g-4)$ satisfies (3.1.1), then D can be 1(1) by Lemma 3.3.3, so that $D + G_1$ has a diagram with specification S.

Therefore, S has $g < 4$.

We now know that a minimal S would have the following form

$(u,0,0,f,g) \quad : \quad 0 \leq f \leq 4,\ 0 \leq g \leq 3$

Put $p = e = 0$ in (3.1.1) to get $7u = 52f + 72g - 156$.

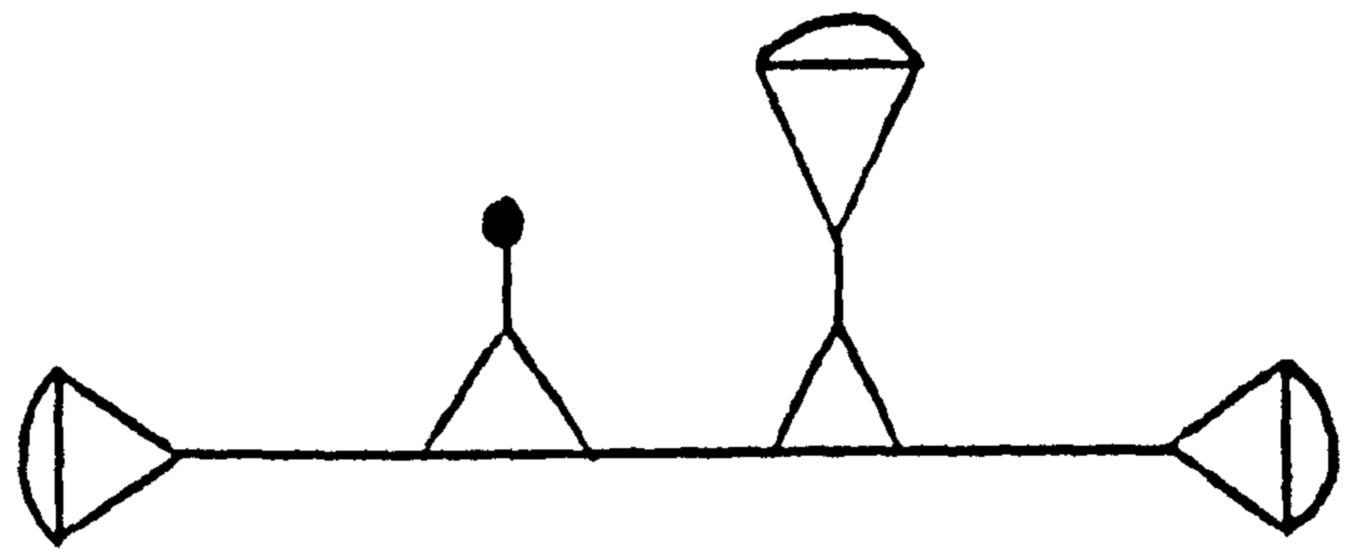
$g = 0$: $7u = 52f - 156$ $\therefore$ $f \equiv 3 \pmod 7$ $\therefore$ $f = 3$ $\therefore$ $u < 1$.

$g = 1$: $7u = 52f - 84$ $\therefore$ $f \equiv 0 \pmod 7$ $\therefore$ $f = 0$ $\therefore$ $u < 1$.

$g = 2$: $7u = 52f - 12$ $\therefore$ $f \equiv 4 \pmod 7$ $\therefore$ $f = 4$ $\therefore$ $u = 28$.

$(28,0,0,4,2) = Q_9 + Q_9$.

$g = 3$: $7u = 52f + 60$ $\therefore$ $f \equiv 1 \pmod 7$ $\therefore$ $f = 1$ $\therefore$ $u = 16$.

(16,0,0,1,3) 

Hence, no such S exists. □

<u>THEOREM 3.3.6</u> Every specification (u,p,e,f,g), satisfying the genus formula (3.1.1), corresponds to a subgroup of (finite) index u in Δ_{13}.

<u>Proof :</u> From Lemmas 3.3.1, 3.3.2, 3.3.3, 3.3.4 and 3.3.5, we know there exists a coset diagram for every specification (u,p,e,f,g) satisfying (3.1.1).
From Lemma 2.1 in [18], there is a correspondence between subgroups of index u in Δ_{13} and u point coset diagrams for Δ_{13}. The theorem follows immediately. □

APPENDIX A

The following computer program is in BASIC. Its function is to give a list of all the specifications of the form (u,p,e,f,g), satisfying the genus formula (1.1.1) for each value of u such that $u \leq 101$. We include $u = 0$ for observation. The output aided the proof of Theorem 1.2.2.

PROGRAM

```
10  LPRINT " SPECIFICATIONS FOR (2,3,11) SUBGROUPS WITH U < 102":LPRINT
20  LPRINT "   N  POINTS GENUS  RED  BLUE  GREEN"
30  N = 0
40  FOR U = 0 TO 101
50    A = (5 * U) + 132 : GMAX = INT(A/60)
60    FOR G = 0 TO GMAX
70      B = (A - (60 * G))/11 : IF B < > INT(B) OR B < 0 THEN 230
80      IF B = 1 OR B = 2 OR B = 5 THEN 230
90      FOR P = 0 TO INT(B/12)
100     FOR E = 0 TO INT(B/3)
110     FOR F = 0 TO INT(B/4)
120       IF (12 * P) + (3 * E) + (4 * F) < > B THEN 200
130       N = N + 1
140       H = N : K = 4   : GOSUB 260 : LPRINT " (";
150       H = U : K = 11  : GOSUB 260 : LPRINT ",";
160       H = P  : K = 16  : GOSUB 260 : LPRINT ",";
170       H = E : K = 22  : GOSUB 260 : LPRINT ",";
180       H = F  : K = 28  : GOSUB 260 : LPRINT ",";
```

```
190        H = G : K = 34  : GOSUB 260 : LPRINT " )"
200      NEXT F
210      NEXT E
220      NEXT P
230    NEXT G
240 NEXT U
250 END
260 REM   subroutine to align output
270 IF H < 10  THEN X = 1 : GOTO 300
280 IF H < 100 THEN X = 2 : GOTO 300
290 X = 3
300 LPRINT TAB(K - X);H;
310 RETURN
```

RESULTS

To save space the output will be arranged in four columns.

1 (0,0,0,3,0)	2 (0,0,4,0,0)	3 (0,1,0,0,0)	4 (1,0,1,1,1)
5 (11,0,3,2,0)	6 (12,0,0,3,1)	7 (12,0,4,0,1)	8 (12,1,0,0,1)
9 (13,0,1,1,2)	10 (22,0,2,4,0)	11 (22,0,6,1,0)	12 (22,1,2,1,0)
13 (23,0,3,2,1)	14 (24,0,0,3,2)	15 (24,0,4,0,2)	16 (24,1,0,0,2)
17 (25,0,1,1,3)	18 (33,0,1,6,0)	19 (33,0,5,3,0)	20 (33,0,9,0,0)
21 (33,1,1,3,0)	22 (33,1,5,0,0)	23 (33,2,1,0,0)	24 (34,0,2,4,1)
25 (34,0,6,1,1)	26 (34,1,2,1,1)	27 (35,0,3,2,2)	28 (36,0,0,3,3)
29 (36,0,4,0,3)	30 (36,1,0,0,3)	31 (37,0,1,1,4)	32 (44,0,0,8,0)
33 (44,0,4,5,0)	34 (44,0,8,2,0)	35 (44,1,0,5,0)	36 (44,1,4,2,0)
37 (44,2,0,2,0)	38 (45,0,1,6,1)	39 (45,0,5,3,1)	40 (45,0,9,0,1)
41 (45,1,1,3,1)	42 (45,1,5,0,1)	43 (45,2,1,0,1)	44 (46,0,2,4,2)

45	(46,0,6,1,2)	46	(46,1,2,1,2)	47	(47,0,3,2,3)	48	(48,0,0,3,4)
49	(48,0,4,0,4)	50	(48,1,0,0,4)	51	(49,0,1,1,5)	52	(55,0,3,7,0)
53	(55,0,7,4,0)	54	(55,0,11,1,0)	55	(55,1,3,4,0)	56	(55,1,7,1,0)
57	(55,2,3,1,0)	58	(56,0,0,8,1)	59	(56,0,4,5,1)	60	(56,0,8,2,1)
61	(56,1,0,5,1)	62	(56,1,4,2,1)	63	(56,2,0,2,1)	64	(57,0,1,6,2)
65	(57,0,5,3,2)	66	(57,0,9,0,2)	67	(57,1,1,3,2)	68	(57,1,5,0,2)
69	(57,2,1,0,2)	70	(58,0,2,4,3)	71	(58,0,6,1,3)	72	(58,1,2,1,3)
73	(59,0,3,2,4)	74	(60,0,0,3,5)	75	(60,0,4,0,5)	76	(60,1,0,0,5)
77	(61,0,1,1,6)	78	(66,0,2,9,0)	79	(66,0,6,6,0)	80	(66,0,10,3,0)
81	(66,0,14,0,0)	82	(66,1,2,6,0)	83	(66,1,6,3,0)	84	(66,1,10,0,0)
85	(66,2,2,3,0)	86	(66,2,6,0,0)	87	(66,3,2,0,0)	88	(67,0,3,7,1)
89	(67,0,7,4,1)	90	(67,0,11,1,1)	91	(67,1,3,4,1)	92	(67,1,7,1,1)
93	(67,2,3,1,1)	94	(68,0,0,8,2)	95	(68,0,4,5,2)	96	(68,0,8,2,2)
97	(68,1,0,5,2)	98	(68,1,4,2,2)	99	(68,2,0,2,2)	100	(69,0,1,6,3)
101	(69,0,5,3,3)	102	(69,0,9,0,3)	103	(69,1,1,3,3)	104	(69,1,5,0,3)
105	(69,2,1,0,3)	106	(70,0,2,4,4)	107	(70,0,6,1,4)	108	(70,1,2,1,4)
109	(71,0,3,2,5)	110	(72,0,0,3,6)	111	(72,0,4,0,6)	112	(72,1,0,0,6)
113	(73,0,1,1,7)	114	(77,0,1,11,0)	115	(77,0,5,8,0)	116	(77,0,9,5,0)
117	(77,0,13,2,0)	118	(77,1,1,8,0)	119	(77,1,5,5,0)	120	(77,1,9,2,0)
121	(77,2,1,5,0)	122	(77,2,5,2,0)	123	(77,3,1,2,0)	124	(78,0,2,9,1)
125	(78,0,6,6,1)	126	(78,0,10,3,1)	127	(78,0,14,0,1)	128	(78,1,2,6,1)
129	(78,1,6,3,1)	130	(78,1,10,0,1)	131	(78,2,2,3,1)	132	(78,2,6,0,1)
133	(78,3,2,0,1)	134	(79,0,3,7,2)	135	(79,0,7,4,2)	136	(79,0,11,1,2)
137	(79,1,3,4,2)	138	(79,1,7,1,2)	139	(79,2,3,1,2)	140	(80,0,0,8,3)
141	(80,0,4,5,3)	142	(80,0,8,2,3)	143	(80,1,0,5,3)	144	(80,1,4,2,3)
145	(80,2,0,2,3)	146	(81,0,1,6,4)	147	(81,0,5,3,4)	148	(81,0,9,0,4)
149	(81,1,1,3,4)	150	(81,1,5,0,4)	151	(81,2,1,0,4)	152	(82,0,2,4,5)

153 (82,0,6,1,5)	154 (82,1,2,1,5)	155 (83,0,3,2,6)	156 (84,0,0,3,7)
157 (84,0,4,0,7)	158 (84,1,0,0,7)	159 (85,0,1,1,8)	160 (88,0,0,13,0)
161 (88,0,4,10,0)	162 (88,0,8,7,0)	163 (88,0,12,4,0)	164 (88,0,16,1,0)
165 (88,1,0,10,0)	166 (88,1,4,7,0)	167 (88,1,8,4,0)	168 (88,1,12,1,0)
169 (88,2,0,7,0)	170 (88,2,4,4,0)	171 (88,2,8,1,0)	172 (88,3,0,4,0)
173 (88,3,4,1,0)	174 (88,4,0,1,0)	175 (89,0,1,11,1)	176 (89,0,5,8,1)
177 (89,0,9,5,1)	178 (89,0,13,2,1)	179 (89,1,1,8,1)	180 (89,1,5,5,1)
181 (89,1,9,2,1)	182 (89,2,1,5,1)	183 (89,2,5,2,1)	184 (89,3,1,2,1)
185 (90,0,2,9,2)	186 (90,0,6,6,2)	187 (90,0,10,3,2)	188 (90,0,14,0,2)
189 (90,1,2,6,2)	190 (90,1,6,3,2)	191 (90,1,10,0,2)	192 (90,2,2,3,2)
193 (90,2,6,0,2)	194 (90,3,2,0,2)	195 (91,0,3,7,3)	196 (91,0,7,4,3)
197 (91,0,11,1,3)	198 (91,1,3,4,3)	199 (91,1,7,1,3)	200 (91,2,3,1,3)
201 (92,0,0,8,4)	202 (92,0,4,5,4)	203 (92,0,8,2,4)	204 (92,1,0,5,4)
205 (92,1,4,2,4)	206 (92,2,0,2,4)	207 (93,0,1,6,5)	208 (93,0,5,3,5)
209 (93,0,9,0,5)	210 (93,1,1,3,5)	211 (93,1,5,0,5)	212 (93,2,1,0,5)
213 (94,0,2,4,6)	214 (94,0,6,1,6)	215 (94,1,2,1,6)	216 (95,0,3,2,7)
217 (96,0,0,3,8)	218 (96,0,4,0,8)	219 (96,1,0,0,8)	220 (97,0,1,1,9)
221 (99,0,3,12,0)	222 (99,0,7,9,0)	223 (99,0,11,6,0)	224 (99,0,15,3,0)
225 (99,0,19,0,0)	226 (99,1,3,9,0)	227 (99,1,7,6,0)	228 (99,1,11,3,0)
229 (99,1,15,0,0)	230 (99,2,3,6,0)	231 (99,2,7,3,0)	232 (99,2,11,0,0)
233 (99,3,3,3,0)	234 (99,3,7,0,0)	235 (99,4,3,0,0)	236 (100,0,0,13,1)
237 (100,0,4,10,1)	238 (100,0,8,7,1)	239 (100,0,12,4,1)	240 (100,0,16,1,1)
241 (100,1,0,10,1)	242 (100,1,4,7,1)	243 (100,1,8,4,1)	244 (100,1,12,1,1)
245 (100,2,0,7,1)	246 (100,2,4,4,1)	247 (100,2,8,1,1)	248 (100,3,0,4,1)
249 (100,3,4,1,1)	250 (100,4,0,1,1)	251 (101,0,1,11,2)	252 (101,0,5,8,2)
253 (101,0,9,5,2)	254 (101,0,13,2,2)	255 (101,1,1,8,2)	256 (101,1,5,5,2)
257 (101,1,9,2,2)	258 (101,2,1,5,2)	259 (101,2,5,2,2)	260 (101,3,1,2,2)

APPENDIX B

The following computer program is in BASIC. Its function is to give a list of all the specifications of the form (u,p,e,f,g_1,g_3), satisfying the genus formula (2.1.2) for each value of u such that $u \leq 38$. We include $u = 0$ for observation. The output aided the proof of Theorem 2.2.1.

PROGRAM

```
10  LPRINT " SPECIFICATIONS FOR (2,3,9) SUBGROUPS WITH U < 39":LPRINT
20  LPRINT "   N  POINTS GENUS  RED  BLUE  GN(1)  GN(3)"
30  N = 0
40  FOR U = 0 TO 38
50    A = U + 36 : G1MAX = INT(A/16) : G3MAX = INT(A/12)
60    FOR G1 = 0 TO G1MAX
70    FOR G3 = 0 TO G3MAX
80      B = (A - (16 * G1)-(12 * G3))/3 : IF B < > INT(B) OR B < 0 THEN 260
90      IF B = 1 OR B = 2 OR B = 5 THEN 260
100     FOR P = 0 TO INT(B/12)
110     FOR E = 0 TO INT(B/3)
120     FOR F = 0 TO INT(B/4)
130       IF (12 * P) + (3 * E) + (4 * F) < > B THEN 230
140       IF (U - F)/3 < > INT((U - F)/3) THEN 230
150       N = N + 1
160       H = N  : K = 4   : GOSUB 300 : LPRINT " (";
170       H = U  : K = 11  : GOSUB 300 : LPRINT ",";
180       H = P  : K = 16  : GOSUB 300 : LPRINT ",";
```

```
190        H = E  : K = 22  : GOSUB 300 : LPRINT ",";
200        H = F  : K = 28  : GOSUB 300 : LPRINT ",";
210        H = G1 : K = 34  : GOSUB 300 : LPRINT ",";
220        H = G3 : K = 40  : GOSUB 300 : LPRINT " )"
230      NEXT F
240      NEXT E
250      NEXT P
260    NEXT G3
270    NEXT G1
280 NEXT U
290 END
300 REM   subroutine to align output
310 IF H < 10  THEN X = 1 : GOTO 340
320 IF H < 100 THEN X = 2 : GOTO 340
330 X = 3
340 LPRINT TAB(K - X);H;
350 RETURN
```

RESULTS

To save space the output will be arranged in four columns.

1 (0,0,0,3,0,0)	2 (0,0,4,0,0,0)	3 (0,1,0,0,0,0)	4 (0,0,0,0,0,3)
5 (1,0,1,1,1,0)	6 (3,0,3,0,0,1)	7 (4,0,0,1,1,1)	8 (6,0,2,0,0,2)
9 (9,0,1,3,0,0)	10 (9,0,5,0,0,0)	11 (9,1,1,0,0,0)	12 (9,0,1,0,0,3)
13 (10,0,2,1,1,0)	14 (12,0,0,3,0,1)	15 (12,0,4,0,0,1)	16 (12,1,0,0,0,1)
17 (12,0,0,0,0,4)	18 (12,0,0,0,3,0)	19 (13,0,1,1,1,1)	20 (15,0,3,0,0,2)
21 (16,0,0,1,1,2)	22 (18,0,2,3,0,0)	23 (18,0,6,0,0,0)	24 (18,1,2,0,0,0)
25 (18,0,2,0,0,3)	26 (19,0,3,1,1,0)	27 (20,0,0,2,2,0)	28 (21,0,1,3,0,1)

29 (21,0,5,0,0,1)	30 (21,1,1,0,0,1)	31 (21,0,1,0,0,4)	32 (21,0,1,0,3,0)
33 (22,0,2,1,1,1)	34 (24,0,0,3,0,2)	35 (24,0,4,0,0,2)	36 (24,1,0,0,0,2)
37 (24,0,0,0,0,5)	38 (24,0,0,0,3,1)	39 (25,0,1,1,1,2)	40 (27,0,3,3,0,0)
41 (27,0,7,0,0,0)	42 (27,1,3,0,0,0)	43 (27,0,3,0,0,3)	44 (28,0,0,4,1,0)
45 (28,0,4,1,1,0)	46 (28,1,0,1,1,0)	47 (28,0,0,1,1,3)	48 (29,0,1,2,2,0)
49 (30,0,2,3,0,1)	50 (30,0,6,0,0,1)	51 (30,1,2,0,0,1)	52 (30,0,2,0,0,4)
53 (30,0,2,0,3,0)	54 (31,0,3,1,1,1)	55 (32,0,0,2,2,1)	56 (33,0,1,3,0,2)
57 (33,0,5,0,0,2)	58 (33,1,1,0,0,2)	59 (33,0,1,0,0,5)	60 (33,0,1,0,3,1)
61 (34,0,2,1,1,2)	62 (36,0,0,6,0,0)	63 (36,0,4,3,0,0)	64 (36,0,8,0,0,0)
65 (36,1,0,3,0,0)	66 (36,1,4,0,0,0)	67 (36,2,0,0,0,0)	68 (36,0,0,3,0,3)
69 (36,0,4,0,0,3)	70 (36,1,0,0,0,3)	71 (36,0,0,0,0,6)	72 (36,0,0,0,3,2)
73 (37,0,1,4,1,0)	74 (37,0,5,1,1,0)	75 (37,1,1,1,1,0)	76 (37,0,1,1,1,3)
77 (38,0,2,2,2,0)			

APPENDIX C

The following computer program is in BASIC. Its function is to give a list of all the specifications of the form (u,p,e,f,g), satisfying the genus formula (3.1.1) for each value of u such that $u \leq 110$. We include $u = 0$ for observation. The output aided the proof of Theorem 3.2.1.

PROGRAM

```
10  LPRINT " SPECIFICATIONS FOR (2,3,13) SUBGROUPS WITH U < 111":LPRINT
20  LPRINT "   N  POINTS GENUS  RED  BLUE  GREEN"
30  N = 0
40  FOR U = 0 TO 110
50    A = (7 * U) + 156 : GMAX = INT(A/72)
60    FOR G = 0 TO GMAX
70      B = (A - (72 * G))/13 : IF B < > INT(B) OR B < 0 THEN 230
80      IF B = 1 OR B = 2 OR B = 5 THEN 230
90      FOR P = 0 TO INT(B/12)
100     FOR E = 0 TO INT(B/3)
110     FOR F = 0 TO INT(B/4)
120       IF (12 * P) + (3 * E) + (4 * F) < > B THEN 200
130       N = N + 1
140       H = N : K = 4   : GOSUB 260 : LPRINT " (";
150       H = U : K = 11  : GOSUB 260 : LPRINT ",";
160       H = P : K = 16  : GOSUB 260 : LPRINT ",";
170       H = E : K = 22  : GOSUB 260 : LPRINT ",";
180       H = F : K = 28  : GOSUB 260 : LPRINT ",";
```

```
190         H = G : K = 34  : GOSUB 260 : LPRINT " )"
200       NEXT F
210       NEXT E
220       NEXT P
230    NEXT G
240 NEXT U
250 END
260 REM    subroutine to align output
270 IF H < 10   THEN X = 1 : GOTO 300
280 IF H < 100 THEN X = 2 : GOTO 300
290 X = 3
300 LPRINT TAB(K - X);H;
310 RETURN
```

RESULTS

1 (0,0,0,3,0)	2 (0,0,4,0,0)	3 (0,1,0,0,0)	4 (1,0,1,1,1)
5 (13,0,1,4,0)	6 (13,0,5,1,0)	7 (13,1,1,1,0)	8 (14,0,2,2,1)
9 (15,0,3,0,2)	10 (16,0,0,1,3)	11 (26,0,2,5,0)	12 (26,0,6,2,0)
13 (26,1,2,2,0)	14 (27,0,3,3,1)	15 (27,0,7,0,1)	16 (27,1,3,0,1)
17 (28,0,0,4,2)	18 (28,0,4,1,2)	19 (28,1,0,1,2)	20 (29,0,1,2,3)
21 (30,0,2,0,4)	22 (39,0,3,6,0)	23 (39,0,7,3,0)	24 (39,0,11,0,0)
25 (39,1,3,3,0)	26 (39,1,7,0,0)	27 (39,2,3,0,0)	28 (40,0,0,7,1)
29 (40,0,4,4,1)	30 (40,0,8,1,1)	31 (40,1,0,4,1)	32 (40,1,4,1,1)
33 (40,2,0,1,1)	34 (41,0,1,5,2)	35 (41,0,5,2,2)	36 (41,1,1,2,2)
37 (42,0,2,3,3)	38 (42,0,6,0,3)	39 (42,1,2,0,3)	40 (43,0,3,1,4)
41 (44,0,0,2,5)	42 (45,0,1,0,6)	43 (52,0,0,10,0)	44 (52,0,4,7,0)
45 (52,0,8,4,0)	46 (52,0,12,1,0)	47 (52,1,0,7,0)	48 (52,1,4,4,0)
49 (52,1,8,1,0)	50 (52,2,0,4,0)	51 (52,2,4,1,0)	52 (52,3,0,1,0)

53	(53,0,1,8,1)	54	(53,0,5,5,1)	55	(53,0,9,2,1)	56	(53,1,1,5,1)
57	(53,1,5,2,1)	58	(53,2,1,2,1)	59	(54,0,2,6,2)	60	(54,0,6,3,2)
61	(54,0,10,0,2)	62	(54,1,2,3,2)	63	(54,1,6,0,2)	64	(52,2,2,0,2)
65	(55,0,3,4,3)	66	(55,0,7,1,3)	67	(55,1,3,1,3)	68	(56,0,0,5,4)
69	(56,0,4,2,4)	70	(56,1,0,2,4)	71	(57,0,1,3,5)	72	(57,0,5,0,5)
73	(57,1,1,0,5)	74	(58,0,2,1,6)	75	(60,0,0,0,8)	76	(65,0,1,11,0)
77	(65,0,5,8,0)	78	(65,0,9,5,0)	79	(65,0,13,2,0)	80	(65,1,1,8,0)
81	(65,1,5,5,0)	82	(65,1,9,2,0)	83	(65,2,1,5,0)	84	(65,2,5,2,0)
85	(65,3,1,2,0)	86	(66,0,2,9,1)	87	(66,0,6,6,1)	88	(66,0,10,3,1)
89	(66,0,14,0,1)	90	(66,1,2,6,1)	91	(66,1,6,3,1)	92	(66,1,10,0,1)
93	(66,2,2,3,1)	94	(66,2,6,0,1)	95	(66,3,2,0,1)	96	(67,0,3,7,2)
97	(67,0,7,4,2)	98	(67,0,11,1,2)	99	(67,1,3,4,2)	100	(67,1,7,1,2)
101	(67,2,3,1,2)	102	(68,0,0,8,3)	103	(68,0,4,5,3)	104	(68,0,8,2,3)
105	(68,1,0,5,3)	106	(68,1,4,2,3)	107	(68,2,0,2,3)	108	(69,0,1,6,4)
109	(69,0,5,3,4)	110	(69,0,9,0,4)	111	(69,1,1,3,4)	112	(69,1,5,0,4)
113	(69,2,1,0,4)	114	(70,0,2,4,5)	115	(70,0,6,1,5)	116	(70,1,2,1,5)
117	(71,0,3,2,6)	118	(72,0,0,3,7)	119	(72,0,4,0,7)	120	(72,1,0,0,7)
121	(73,0,1,1,8)	122	(78,0,2,12,0)	123	(78,0,6,9,0)	124	(78,0,10,6,0)
125	(78,0,14,3,0)	126	(78,0,18,0,0)	127	(78,1,2,9,0)	128	(78,1,6,6,0)
129	(78,1,10,3,0)	130	(78,1,14,0,0)	131	(78,2,2,6,0)	132	(78,2,6,3,0)
133	(78,2,10,0,0)	134	(78,3,2,3,0)	135	(78,3,6,0,0)	136	(78,4,2,0,0)
137	(79,0,3,10,1)	138	(79,0,7,7,1)	139	(79,0,11,4,1)	140	(79,0,15,1,1)
141	(79,1,3,7,1)	142	(79,1,7,4,1)	143	(79,1,11,1,1)	144	(79,2,3,4,1)
145	(79,2,7,1,1)	146	(79,3,3,1,1)	147	(80,0,0,11,2)	148	(80,0,4,8,2)
149	(80,0,8,5,2)	150	(80,0,12,2,2)	151	(80,1,0,8,2)	152	(80,1,4,5,2)
153	(80,1,8,2,2)	154	(80,2,0,5,2)	155	(80,2,4,2,2)	156	(80,3,0,2,2)
157	(81,0,1,9,3)	158	(81,0,5,6,3)	159	(81,0,9,3,3)	160	(81,0,13,0,3)

161 (81,1,1,6,3)	162 (81,1,5,3,3)	163 (81,1,9,0,3)	164 (81,2,1,3,3)
165 (81,2,5,0,3)	166 (81,3,1,0,3)	167 (82,0,2,7,4)	168 (82,0,6,4,4)
169 (82,0,10,1,4)	170 (82,1,2,4,4)	171 (82,1,6,1,4)	172 (82,2,2,1,4)
173 (83,0,3,5,5)	174 (83,0,7,2,5)	175 (83,1,3,2,5)	176 (84,0,0,6,6)
177 (84,0,4,3,6)	178 (84,0,8,0,6)	179 (84,1,0,3,6)	180 (84,1,4,0,6)
181 (84,2,0,0,6)	182 (85,0,1,4,7)	183 (85,0,5,1,7)	184 (85,1,1,1,7)
185 (86,0,2,2,8)	186 (87,0,3,0,9)	187 (88,0,0,1,10)	188 (91,0,3,13,0)
189 (91,0,7,10,0)	190 (91,0,11,7,0)	191 (91,0,15,4,0)	192 (91,0,19,1,0)
193 (91,1,3,10,0)	194 (91,1,7,7,0)	195 (91,1,11,4,0)	196 (91,1,15,1,0)
197 (91,2,3,7,0)	198 (91,2,7,4,0)	199 (91,2,11,1,0)	200 (91,3,3,4,0)
201 (91,3,7,1,0)	202 (91,4,3,1,0)	203 (92,0,0,14,1)	204 (92,0,4,11,1)
205 (92,0,8,8,1)	206 (92,0,12,5,1)	207 (92,0,16,2,1)	208 (92,1,0,11,1)
209 (92,1,4,8,1)	210 (92,1,8,5,1)	211 (92,1,12,2,1)	212 (92,2,0,8,1)
213 (92,2,4,5,1)	214 (92,2,8,2,1)	215 (92,3,0,5,1)	216 (92,3,4,2,1)
217 (92,4,0,2,1)	218 (93,0,1,12,2)	219 (93,0,5,9,2)	220 (93,0,9,6,2)
221 (93,0,13,3,2)	222 (93,0,17,0,2)	223 (93,1,1,9,2)	224 (93,1,5,6,2)
225 (93,1,9,3,2)	226 (93,1,13,0,2)	227 (93,2,1,6,2)	228 (93,2,5,3,2)
229 (93,2,9,0,2)	230 (93,3,1,3,2)	231 (93,3,5,0,2)	232 (93,4,1,0,2)
233 (94,0,2,10,3)	234 (94,0,6,7,3)	235 (94,0,10,4,3)	236 (94,0,14,1,3)
237 (94,1,2,7,3)	238 (94,1,6,4,3)	239 (94,1,10,1,3)	240 (94,2,2,4,3)
241 (94,2,6,1,3)	242 (94,3,2,1,3)	243 (95,0,3,8,4)	244 (95,0,7,5,4)
245 (95,0,11,2,4)	246 (95,1,3,5,4)	247 (95,1,7,2,4)	248 (95,2,3,2,4)
249 (96,0,0,9,5)	250 (96,0,4,6,5)	251 (96,0,8,3,5)	252 (96,0,12,0,5)
253 (96,1,0,6,5)	254 (96,1,4,3,5)	255 (96,1,8,0,5)	256 (96,2,0,3,5)
257 (96,2,4,0,5)	258 (96,3,0,0,5)	259 (97,0,1,7,6)	260 (97,0,5,4,6)
261 (97,0,9,1,6)	262 (97,1,1,4,6)	263 (97,1,5,1,6)	264 (97,2,1,1,6)
265 (98,0,2,5,7)	266 (98,0,6,2,7)	267 (98,1,2,2,7)	268 (99,0,3,3,8)

269 (99,0,7,0,8)	270 (99,1,3,0,8)	271 (100,0,0,4,9)	272 (100,0,4,1,9)
273 (100,1,0,1,9)	274 (101,0,1,2,10)	275 (102,0,2,0,11)	276 (104,0,0,17,0)
277 (104,0,4,14,0)	278 (104,0,8,11,0)	279 (104,0,12,8,0)	280 (104,0,16,5,0)
281 (104,0,20,2,0)	282 (104,1,0,14,0)	283 (104,1,4,11,0)	284 (104,1,8,8,0)
285 (104,1,12,5,0)	286 (104,1,16,2,0)	287 (104,2,0,11,0)	288 (104,2,4,8,0)
289 (104,2,8,5,0)	290 (104,2,12,2,0)	291 (104,3,0,8,0)	292 (104,3,4,5,0)
293 (104,3,8,2,0)	294 (104,4,0,5,0)	295 (104,4,4,2,0)	296 (104,5,0,2,0)
297 (105,0,1,15,1)	298 (105,0,5,12,1)	299 (105,0,9,9,1)	300 (105,0,13,6,1)
301 (105,0,17,3,1)	302 (105,0,21,0,1)	303 (105,1,1,12,1)	304 (105,1,5,9,1)
305 (105,1,9,6,1)	306 (105,1,13,3,1)	307 (105,1,17,0,1)	308 (105,2,1,9,1)
309 (105,2,5,6,1)	310 (105,2,9,3,1)	311 (105,2,13,0,1)	312 (105,3,1,6,1)
313 (105,3,5,3,1)	314 (105,3,9,0,1)	315 (105,4,1,3,1)	316 (105,4,5,0,1)
317 (105,5,1,0,1)	318 (106,0,2,13,2)	319 (106,0,6,10,2)	320 (106,0,10,7,2)
321 (106,0,14,4,2)	322 (106,0,18,1,2)	323 (106,1,2,10,2)	324 (106,1,6,7,2)
325 (106,1,10,4,2)	326 (106,1,14,1,2)	327 (106,2,2,7,2)	328 (106,2,6,4,2)
329 (106,2,10,1,2)	330 (106,3,2,4,2)	331 (106,3,6,1,2)	332 (106,4,2,1,2)
333 (107,0,3,11,3)	334 (107,0,7,8,3)	335 (107,0,11,5,3)	336 (107,0,15,2,3)
337 (107,1,3,8,3)	338 (107,1,7,5,3)	339 (107,1,11,2,3)	340 (107,2,3,5,3)
341 (107,2,7,2,3)	342 (107,3,3,2,3)	343 (108,0,0,12,4)	344 (108,0,4,9,4)
345 (108,0,8,6,4)	346 (108,0,12,3,4)	347 (108,0,16,0,4)	348 (108,1,0,9,4)
349 (108,1,4,6,4)	350 (108,1,8,3,4)	351 (108,1,12,0,4)	352 (108,2,0,6,4)
353 (108,2,4,3,4)	354 (108,2,8,0,4)	355 (108,3,0,3,4)	356 (108,3,4,0,4)
357 (108,4,0,0,4)	358 (109,0,1,10,5)	359 (109,0,5,7,5)	360 (109,0,9,4,5)
361 (109,0,13,1,5)	362 (109,1,1,7,5)	363 (109,1,5,4,5)	364 (109,1,9,1,5)
365 (109,2,1,4,5)	366 (109,2,5,1,5)	367 (109,3,1,1,5)	368 (110,0,2,8,6)
369 (110,0,6,5,6)	370 (110,0,10,2,6)	371 (110,1,2,5,6)	372 (110,1,6,2,6)
373 (110,2,2,2,6)			

APPENDIX D

Additional Background Material

Although the methods used are group-theoretic (except for the argument on pp 69–70), it is important to realise that the groups Γ and Δ_n can be viewed in a variety of ways. In this appendix, we discuss these ways and indicate how they relate.

1. Geometry.

The group $PGL_2(\mathbb{R})$ acts on the extended complex plane $(\mathbb{C} \cup \{\infty\})$ in the following way :

$$\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right](z) = \frac{az+b}{cz+d},$$

with the obvious modifications when $z = \infty$ or $cz + d = 0$. The map is orientation-preserving (resp. reversing) according as $ad - bc$ is positive (resp. negative).

The subgroup of orientation-preserving is $PSL_2(\mathbb{R})$. This acts on the upper half-plane : $H = \{ z \in \mathbb{C} : \mathrm{Im}(z) > 0 \}$.
In fact, it is the isometry group for the Poincaré model of hyperbolic geometry (the "lines" being vertical Euclidean lines and arcs of Euclidean circles orthogonal to the real axis).

Our groups Γ and Δ_n and their subgroups are Fuchsian groups (i.e. act *discretely* on H). It is thus possible to choose a set of representatives of the orbits in a "nice" way, viz we have a closed connected set D_G for G such that

(1) $\forall z \in H$, $\exists w \in D_G$, $g \in G$ with $g(w) = z$.

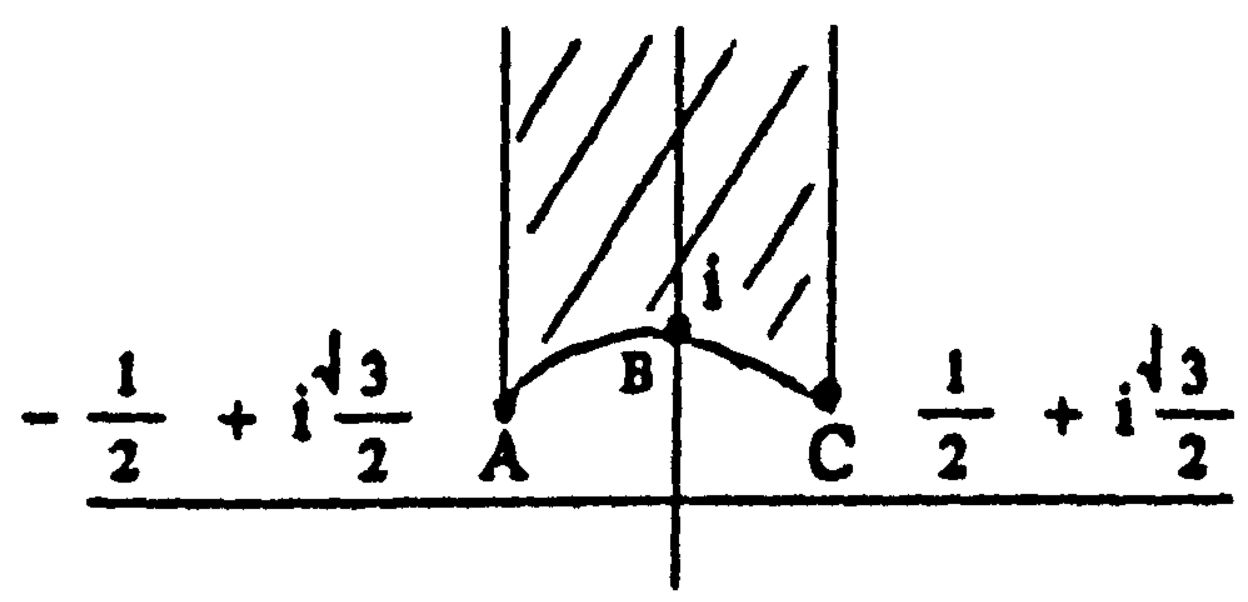
(2) If $z, w \in D_G^o$, $z \neq w$, then there is no $g \in G$ with $g(w) = z$.

(Here X^o denotes the interior of X).

With care, we can choose D_G to be a hyperbolic *polygon*. From (1), (2) above, we note that it is possible for an element of G to map boundary points of D_G to other boundary points. In fact, G is generated by such elements, and a presentation can be obtained in this fashion. (In this there is one generator for each pair of edges identified by a g, and one additional one).

If we identify the sides of D_G as suggested by the g above, then we get a Riemann surface of some genus p.

Example Γ has D_Γ as shown



$z : Z \to Z + 1$ identifies the vertical edges,

$x : Z \to -1/Z$ identifies the arcs AB, BC.

It is easy to see that $x^2 = 1$, $y^3 = 1$ where $p = tu^{-1}$, and we have the presentation

$$< x, y, z : x^2 = y^3 = xyz = 1 >.$$

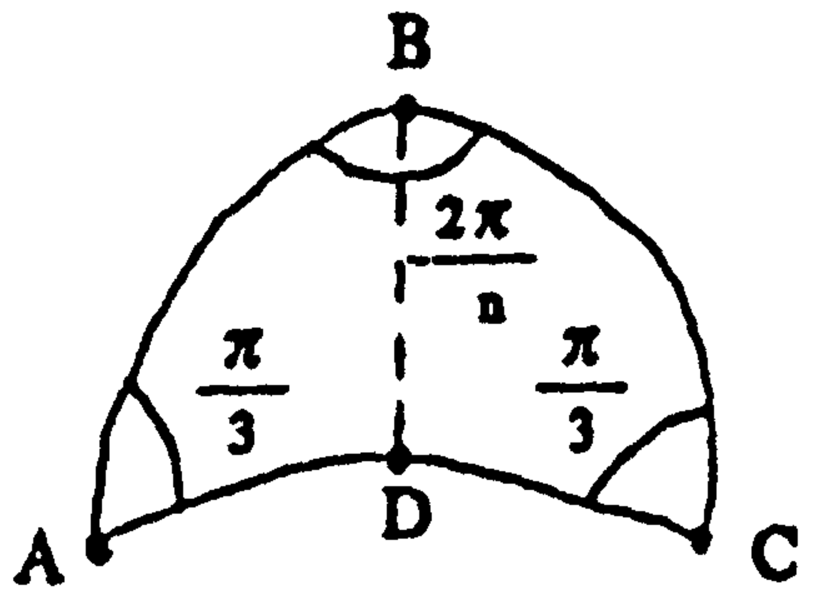
The "fundamental domain" shown in the Introduction is for the group generated by reflections R_2, R_3, R_n. These are orientation-reversing, so generate a group $\tilde{\Delta}_n \subseteq PGL_2(\mathbb{R})$.

We call this the extended (2,3,n)-triangle group. The name "triangle group" comes from this picture.

Our group Δ_n consists of all the orientation-preserving elements of $\tilde{\Delta}_n$ (so comprises the products of an *even* number of R_i). It is clearly generated by $x = R_nR_3$, $y = R_2R_n$, $z = R_2R_3$ $(= yx)$ of orders 2, 3, n.

If G is a discrete group, and H a subgroup of index u in G, then we may take as D_H a (connected) collection of u images of D_G. In particular, if G is Γ or Δ_n, then the images of D_G *triangulate* the surface obtained from D_H. The "genus formula" is just the usual formula which computes the Euler characteristic using a triangulation.

Since Δ_n has index 2 in $\tilde{\Delta}_n$, we can take a fundamental domain $D_{\Delta n}$



where ABD is a domain for $\tilde{\Delta}_n$ (with angle $\frac{\pi}{2}$ at D) and BDC is the reflection of ABD in BD.

Now suppose that $G \le \Delta_n$ has index u. Then we may take D_G as u images of $D_{\Delta n}$. The elements "g" which identify the sides of D_G are of two types :

(1) g identifies an adjacent pair of edges of D_G. Then g fixes the common vertex V (obviously Δ_n-equivalent to A, B, C or D). The order of g is then

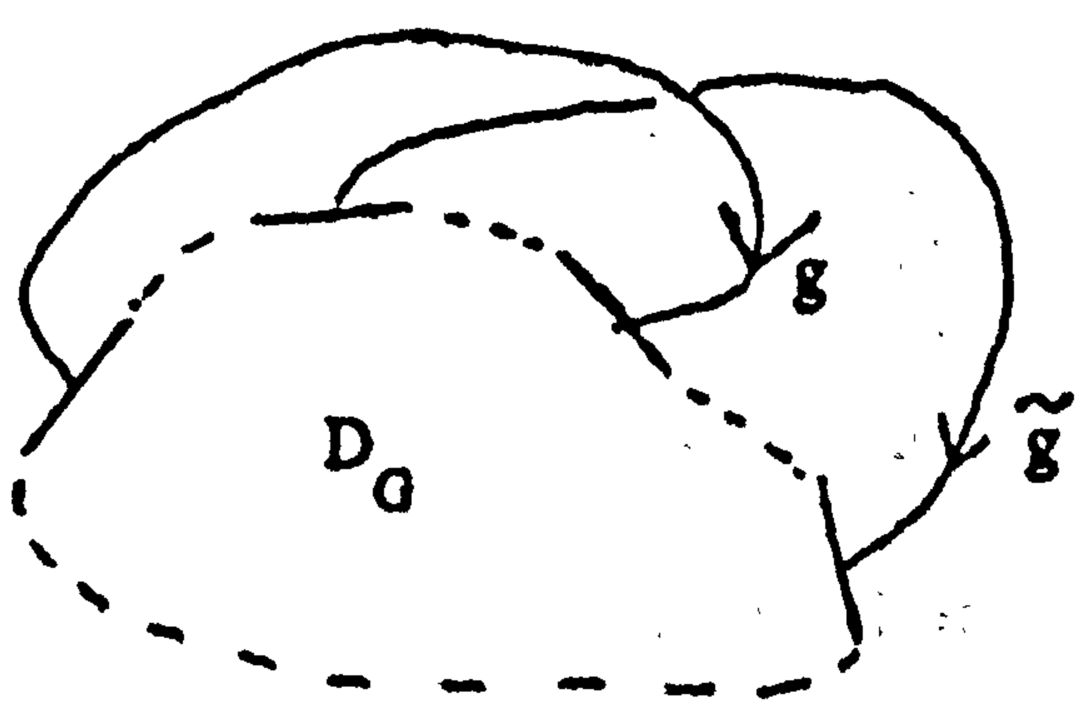
(a) 2 if $V \sim D$

(b) 3 if $V \sim A$ or C

(c) k (where $k \backslash n$) if $V \sim B$

In the last case, D_G has $\frac{n}{k}$ copies of $D_{\Delta n}$ at V so angle is $\frac{2\pi}{k}$.

(2) g identifies a pair with at least one edge intervening in each direction. Then there is a "companion" $\tilde{g} \in G$, where $\tilde{g}$ identifies a pair interlocking those for g, i.e.



(the dotted lines denote a sequence, possibly null, of edges of D_G)

After a bit of rearrangement, we get a presentation

$$< x_1, \ldots, x_e, y_1, \ldots, y_f, z_1, \ldots, z_h, g_1, \tilde{g}_1, \ldots, g_p, \tilde{g}_p :$$
$$1 = x_i^2 = y_j^3 = z_l^{k(l)} = \Pi x_i \Pi y_j \Pi z_l \Pi[g_m, \tilde{g}_m] > \qquad (*)$$

where [a,b] denotes the commutator, and each k(l) is a divisor of n.

NB (1) The constants e, f, p are those in the genus formula.

(2) The "h" is the h for the associated subgroup of level u in Γ.

(3) The $\frac{n}{k(l)}$ are the cusp-widths for the associated subgroup.

2. Group Theory.

Here we stick mainly to Γ, since the group structure is simpler. We have

$$\Gamma = C_2 \times C_3$$

where $\times$ denotes the free product. Standard result then shows that each subgroup G has the form

$$G = \underbrace{C_2 \times \ldots \times C_2}_{e} \times \underbrace{C_3 \times \ldots \times C_3}_{f} \times \underbrace{C_\infty \times \ldots \times C_\infty}_{t}$$

Here, e and f are as usual, and $t = 2g + h - 1$.

With care, h and the cusp-widths can be recovered (there are h conjugacy classes in G which are Γ-conjugate to $<z^m>$ for some m, etc.).

NB This is not a particularly nice way to obtain the genus formula, or the standard presentation (*).

It is not so easy to deal with Δ_n in this fashion; it is easier to recall that Δ_n is a homomorphic image of Γ and handle it this way.

3. Complex Analysis.

Here we again use the action of a discrete group G on $\tilde{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.

We say that f is an *automorphic function* for G if

(1) $\forall z \in \mathbb{C}$, $g \in G$, $f(g(z)) = f(z)$

(2) f is "analytic" (with suitable definitions at ∞).

Clearly, the automorphic functions for G form a field $\mathbb{C}_G$. If $H \leq G$, then each automorphic function for G will be automorphic for H (see (1) above), so $\mathbb{C}_H$ will be an *extension field* of $\mathbb{C}_G$. In fact, if H has index u in G, then $\mathbb{C}_H$ is an extension of degree u of $\mathbb{C}_G$.

Once again, the genus formula can be recovered by looking at how $\mathbb{C}_H$ ramifies over $\mathbb{C}_G$. In general, this is not an attractive way to obtain e, f, etc.!

One special case we use is that when the group G has genus 0. In this case, the field can be generated by a single function ζ, i.e. $\mathbb{C}_G = \mathbb{C}(\zeta)$.
Such a ζ is a Hauptmodul. There are many choices for ζ, but each is a bilinear transform of any other. The situation we meet in p69 is where G has genus 0, and has index u in Γ. Let j denote the usual Hauptmodul for Γ (normalised so $j(\frac{1}{2} \pm i\frac{\sqrt{3}}{2}) = 0$, $j(i) = 1728$, j has a pole at ∞), and let ζ denote a Hauptmodul for Γ. As in [1], since $j \in \mathbb{C}_G = \mathbb{C}(\zeta)$,

$$j = \frac{P(\zeta)}{Q(\zeta)}$$

where P, Q are polynomials, $\partial P, \partial Q \leq u$, and at least one is of degree u.

The ramifications must be over points where $j = 0, 1728$ or ∞ and so must correspond to factorisation of P ($j = 0$), $P-1728Q$ ($j = 1728$) or Q ($j = \infty$). In this case the "genus formula" is quite easy to obtain.

<u>Footnote</u> If $G \leq \Gamma$ is of index u, let c(G) denote the intersection of all conjugates of G. Then,

(1) c(G) is the largest subgroup of G normal in Γ.

(2) c(G) has index at most u! (i.e. finite) - which can be seen by looking at action on cosets.

(3) $\Gamma/c(G)$ is isomorphic to the Galois group of $\mathbb{C}_G$ as an extension of $\mathbb{C}_\Gamma$.

APPENDIX E

In this Appendix we define *n(a)* and describe *composition* of diagrams.

1. n(a)

A diagram with specification (u,p,e,f,g), which has *n* pairs of red points which are separated by *a* green lines in some direction, is defined to be *n(a)*.

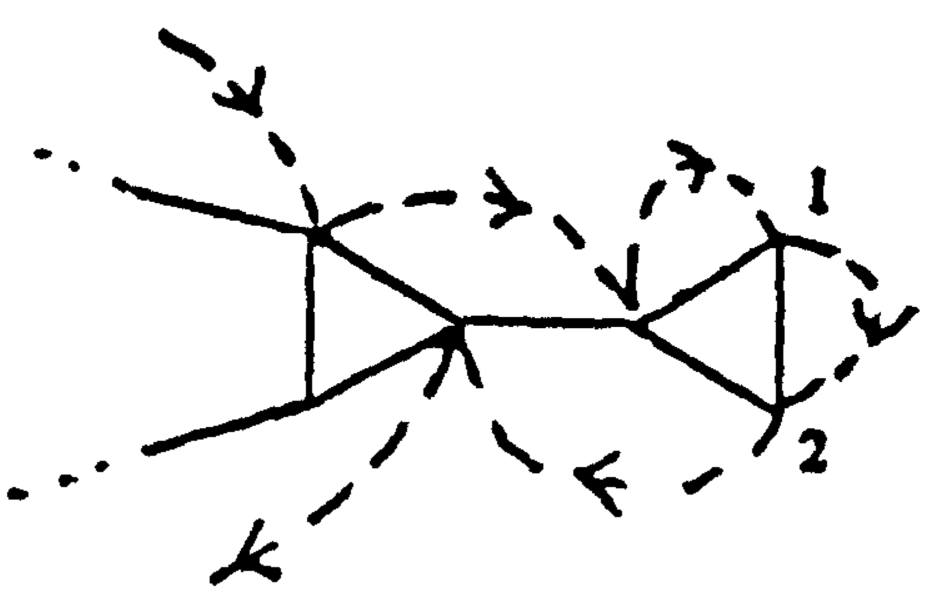
For example, the diagram on p6 for B with specification (66,0,14,0,0) can be described as 5(1) 2(2), since there are five pairs of red points which would each be separated by one green line, and two pairs of red points which would each be separated by two green lines.

2. Composition

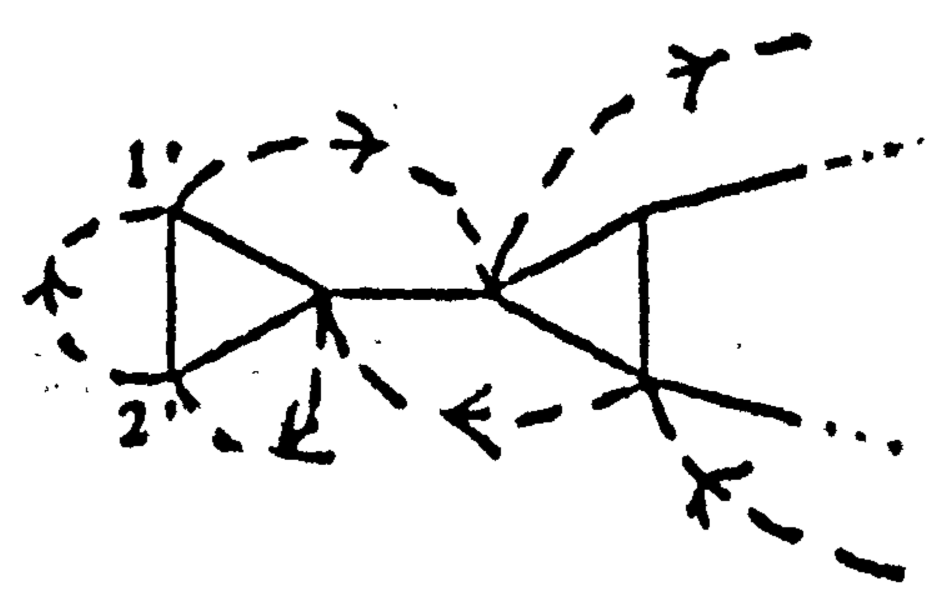
(i) 1-composition (where the two diagrams to be composed each have a triangle with two red points)

Recall : A green line is obtained by following a blue line, then a red line.

A green line will be represented thus : — — >— — —



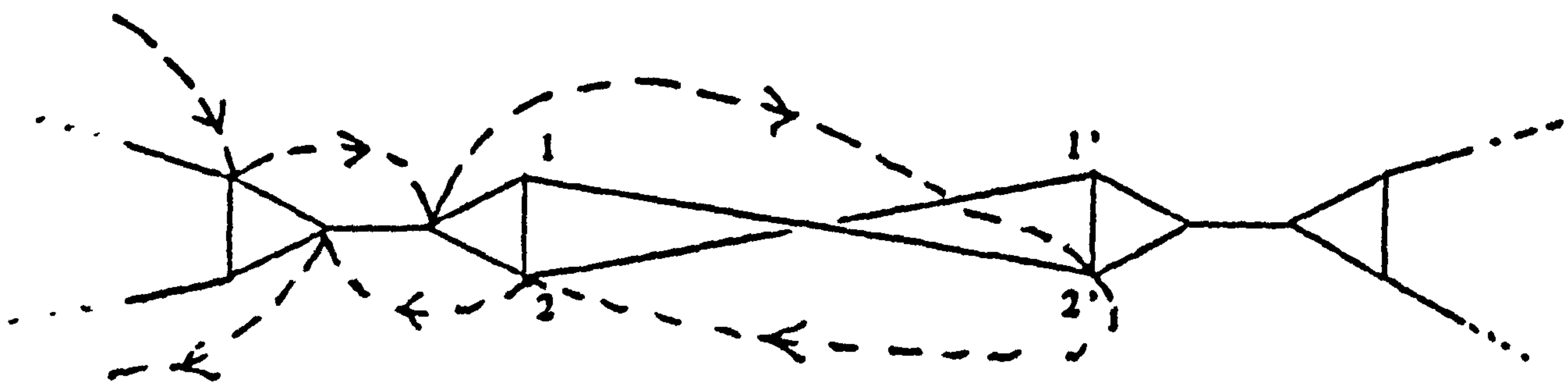
Segment of diagram A



Segment of diagram B

Vertices 1 and 2 in diagram A, and vertices 1' and 2' in diagram B, are red points (loops).

We now demonstrate 1-composition of diagrams A and B. The red points are unravelled and then vertex 1 (resp. 2) is joined with vertex 2' (resp. 1').



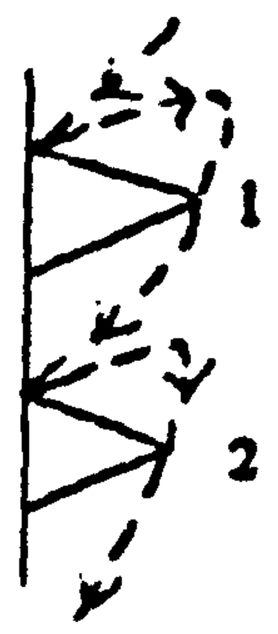
The green cycle from diagram A only is shown here.

Notes : (1) One green cycle in each diagram is affected. Two green lines in each of these cycles pass through a vertex from the other diagram, but the cycle lengths remain the same.

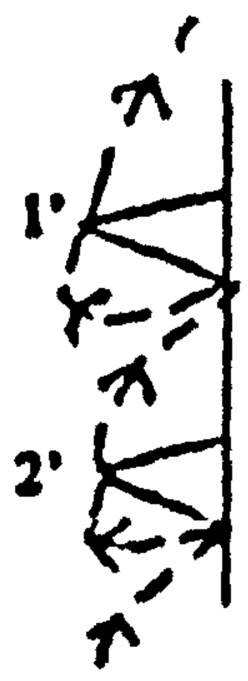
(2) No other green cycles are affected.

(3) Alternatively, we could have composed diagrams A and B by reversing the orientation of all the triangles in either A or B, and then joining vertex 1 (resp. 2) with vertex 1' (resp 2').

(ii) 2-composition (where the two diagrams each have a pair of red points separated by two green lines)



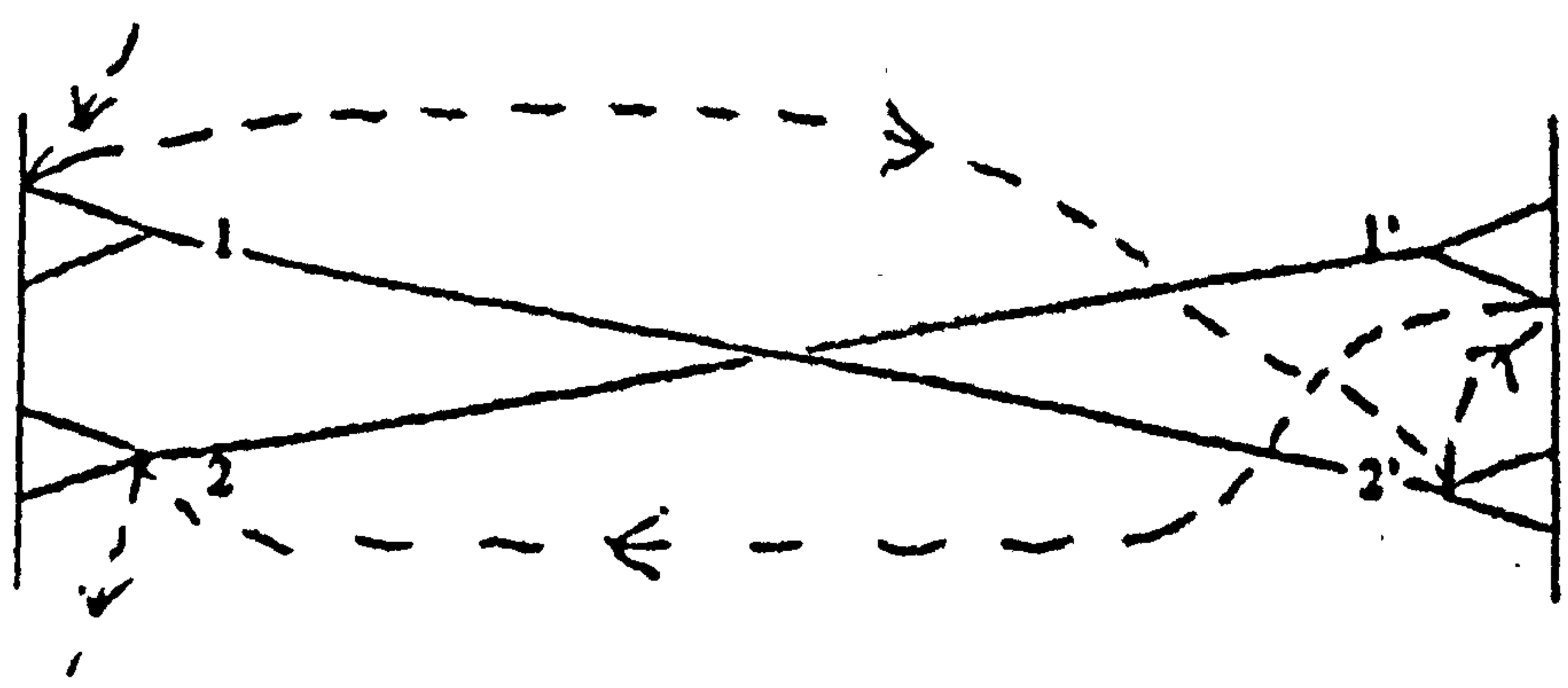
Segment of diagram A



Segment of diagram B

Vertices 1, 1', 2, 2' are red points.

We can 2-compose diagrams A and B by joining 1 with 2', and 2 with 1'.



Only the green cycle from diagram A is shown in the composition.

<u>Notes</u> : (1) One green cycle in each diagram is affected. Three green lines in each of these cycles pass through new vertices, but the cycle lengths remain the same.

(2) No other green cycles are affected.

(3) Alternatively, we could have composed diagrams A and B by reversing the orientation of all the triangles in either A or B, and then joining vertex 1 (resp. 2) with vertex 1' (resp 2').

We have shown cases where two separate diagrams are composed. It is not difficult to see that cycle lengths will also be preserved if we compose within a single diagram. This is even true when all vertices affected lie in a single cycle.

REFERENCES

[1] Atkin A.O.L. & Swinnerton-Dyer H.P.F., Modular Forms on Noncongruence Subgroups

AMS Proc. Sympos. Pure Math. XIX Combinatorics (1971)

[2] Brauer A., On a problem of partitions

American J. of Math. 64 (1942) 299–312

[3] Conder, M.D.E., Generators for Alternating and Symmetric Groups

J. London Math. Soc. (2) 22 (1980) 75–86

[4] Conder, M.D.E., More on Generators for Alternating and Symmetric Groups

Quart. J. Math. 32 (1981) 137-163

[5] Conder, M.D.E., Some Results on Quotients of Triangle Groups

Bull. Austral. Math. Soc. 30 (1984) 73–90

[6] Jones G.A. & Singerman D., Theory of maps on orientable surfaces

Proc. London Math. Soc. (3) 37 (1978) 273-307

[7] Jones G.A. & Singerman D., Complex Functions

Cambridge University Press, 1988

[8] Jones G.A., Triangular maps and non-congruence subgroups of the modular group

Bull. London Math. Soc. 11 (1979) 117-123

[9] MacBeath A.M., Generators of the linear fractional groups

AMS Proc. Sympos. Pure Math. XII Number Theory (1969) 14-32

[10] Magnus W., Noneuclidean Tesselations and their groups

Academic Press, New York (1974)

[11] Millington, M.H., Subgroups of the Classical Modular Group

J. London Math. Soc. (2) 1 (1969) 351-357

[12] Mushtaq Q. & Shaheen F., On the subgroups of polyhedral groups
Math. Japonica 35, No. 5 (1990) 849-853

[13] Neumann P.M., The SQ-universality of some finitely presented groups
J. Austral. Math. Soc. 16 (1973) 1-6

[14] Shimura G., Automorphic Functions and Number Theory
Lecture Notes in Mathematics 54 (1968)

[15] Singerman D., Subgroups of Fuchsian Groups and Finite Permutation Groups
Bull. London Math. Soc. 2 (1970) 319-323

[16] Stothers W.W., Subgroups of the Modular Group
Proc. Camb. Phil. Soc. 75 (1974) 139-153

[17] Stothers W.W., Impossible specifications for the modular group
Manuscripta Math. 13 (1974) 415-428

[18] Stothers W.W., Subgroups of the (2,3,7) Triangle Group
Manuscripta Math. 20 (1977) 323-334

[19] Stothers W.W., The number of subgroups of given index in the modular group
Proc. Royal Soc. Edin. 78A (1977) 105-112

[20] Stothers W.W., Subgroups of infinite index in the modular group
Glasgow Math. J. 19 (1978) 33-43

[21] Stothers W.W., Subgroups of infinite index in the modular group II
Glasgow Math. J. 22 (1981) 101-118

[22] Stothers W.W., Subgroups of infinite index in the modular group III
Glasgow Math. J. 22 (1981) 119-131

[23] Stothers W.W., Polynomial identities and Hauptmoduln
Quart. J. Math. Oxford (2) 32 (1981) 349-370