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Trimmed Lévy processes and their extremal components

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Abstract

We analyze a stochastic process of the form ${}^{(r)}X_t = X_t - \sum_{i=1}^{r} \Delta_t^{(i)}$, where $(X_t)_{t\ge0}$ is a driftless, infinite activity, subordinator on $\mathbb{R}_+$ with its jumps on $[0,t]$ ordered as $\Delta_t^{(1)} \ge \Delta_t^{(2)} \cdots$. The r largest of these are "trimmed" from X_t to give ${}^{(r)}X_t$. When $r \to \infty$, both ${}^{(r)}X_t \downarrow 0$ and $\Delta_t^{(r)} \downarrow 0$ a.s. for each $t > 0$, and it is interesting to study the weak limiting behavior of $\left({}^{(r)}X_t, \Delta_t^{(r)}\right)$ in this case. We term this "large-trimming" behavior, and study the joint convergence of $\left({}^{(r)}X_t, \Delta_t^{(r)}\right)$ as $r \to \infty$ under linear normalization, assuming extreme value-related conditions on the Lévy measure of X_t which guarantee that $\Delta_t^{(r)}$ has a limit distribution with linear normalization. Allowing ${}^{(r)}X_t$ to have random centering and norming in a natural way, we first show that $\left({}^{(r)}X_t, \Delta_t^{(r)}\right)$ has a bivariate normal limiting distribution as $r \to \infty$; then replacing the random normalizations with deterministic normings produces normal, and in some cases, non-normal, limits whose parameters we can specify.

MSC: primary 60G51; 60G52; 60G55; 60G70

Keywords: Trimmed Lévy process; Trimmed subordinator; Subordinator large jumps; Extreme value-related conditions; Large-trimming limits

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1. Introduction

Suppose $(X_t)_{t\geq 0}$ is a driftless subordinator with jump process $(\Delta_t := X_t - X_{t-})_{t>0}$, having infinite Lévy measure Π with tail function $\overline{\Pi}(x) := \Pi\{(x,\infty)\}$, $x > 0$. Thus, X_t has Laplace transform $\mathrm{E}e^{-\lambda X_t} = e^{-t\psi(\lambda)}$, $t \geq 0$, where

$$\psi(\lambda) = \int_{(0,\infty)} (1 - e^{-\lambda x})\Pi(\mathrm{d}x),\ \ \lambda > 0.$$

Let $\Delta_t^{(r)}$ be the rth largest jump of X_t on $[0,t]$, $t > 0$, $r \in \mathbb{N} := \{1, 2, \ldots\}$. The *trimmed subordinator* is defined to be ${}^{(r)}X_t = X_t - \sum_{i=1}^{r} \Delta_t^{(i)}$. See [3] for formal definitions of these quantities. See [16] and [14] for some recent results on the relationship of a Lévy process to its large jumps. In [4] and [15] we analyzed distributional properties of $\Delta_t^{(r)}$ as a function of r and here we continue those studies by considering the joint weak limiting behavior of $\big({}^{(r)}X_t, \Delta_t^{(r)}\big)$ for fixed t as $r \to \infty$. As $r \to \infty$, ${}^{(r)}X_t \downarrow 0$ and $\Delta_t^{(r)} \downarrow 0$ a.s. for each $t > 0$, and, conditionally on $\Delta_t^{(r)}$, we may consider ${}^{(r)}X_t$ as a Lévy process with Lévy measure restricted to $(0, \Delta_t^{(r)})$ (e.g., [18]), hence, with finite variance, conditionally. So as $r \to \infty$ and big jumps are successively removed from ${}^{(r)}X_t$, it is reasonable to expect that it may have a Gaussian weak limit after centering and norming.

Clearly an important component in such a study will be the behavior of the $\Delta_t^{(r)}$, and in what follows we begin in Section 2 by analyzing their convergence as $r \to \infty$, for fixed $t > 0$. The approach we take is to assume conditions on $\overline{\Pi}$ guaranteeing that $\Delta_t^{(r)}$ has a limit distribution under linear normalization, and then prove that a normal limit distribution for ${}^{(r)}X_t$, conditional on the value of $\Delta_t^{(r)}$, also exists as $r \to \infty$. To obtain the normal limit for the conditional distribution we initially apply a natural random centering and random norming. Having established this, we then investigate replacing the random centering and norming with deterministic versions.

We conclude this section with some more preliminary setting up. We always assume

$$\overline{\Pi}(0+) = \lim_{x\downarrow 0} \overline{\Pi}(x) = \infty \tag{1.1}$$

("infinite activity"), for $\overline{\Pi}$, so $\Delta_t^{(r)} > 0$ a.s. for $r \in \mathbb{N}$, $t > 0$. The inverse function $\overline{\Pi}^{\leftarrow}$ to $\overline{\Pi}$ is defined by

$$\overline{\Pi}^{\leftarrow}(x) = \inf\{y > 0 : \overline{\Pi}(y) \leq x\},\ \ x > 0. \tag{1.2}$$

Let $(\Gamma_i)_{i\in\mathbb{N}}$ be distributed as the successive cumulative sums of i.i.d standard exponential random variables. For Borel sets $A \subseteq (0,\infty)$, define point measures $\delta_x(\cdot)$, $x \in (0,\infty)$, by $\delta_x(A) = 1$ or 0 depending on whether $x \in A$ or $x \notin A$. Then the random variable X_t can be constructed from the Poisson random measure $\sum_{i=1}^{\infty} \delta_{\overline{\Pi}^{\leftarrow}(\Gamma_i/t)}$, with mean measure $t\Pi$, which has the points written in decreasing order ([3,6,17,18] and [21, p.139, Ex. 3.38]). So we have, for each $t > 0$, the distributional equivalences

$$X_t \stackrel{\mathrm{D}}{=} \sum_{i=1}^{\infty} \overline{\Pi}^{\leftarrow}(\Gamma_i/t) \quad \text{and} \quad \big(\Delta_t^{(i)}\big)_{i\geq 1} \stackrel{\mathrm{D}}{=} \big(\overline{\Pi}^{\leftarrow}(\Gamma_i/t)\big)_{i\geq 1}. \tag{1.3}$$

We make the simplifying assumption throughout that Π is atomless (equivalently, $\overline{\Pi}$ is continuous on $(0,\infty)$). This means that the inverse function $\overline{\Pi}^{\leftarrow}$ is strictly decreasing on $(0,\infty)$ and the ordered jumps $\Delta X_t^{(1)} \geq \Delta X_t^{(2)} \geq \cdots$ are uniquely defined. We expect that this assumption can be removed using, for example, the formulae in [14], but this would add little of interest to the exposition, so we omit the details.

With this continuity assumption, Prop. 2.3 of [18][2] gives that, conditional on $\Delta_t^{(r)}$, $({}^{(r)}X_t)_{t>0}$ is a subordinator whose Lévy measure is the measure Π restricted to $(0, \Delta_t^{(r)})$. So the conditional characteristic function of ${}^{(r)}X_t$ is

$$\mathrm{E}\big(e^{\mathrm{i}\theta\,{}^{(r)}X_t}\,\big|\,\Delta_t^{(r)}\big) = \exp\Big(t\int_0^{\Delta_t^{(r)}}\big(e^{\mathrm{i}\theta x}-1\big)\Pi(\mathrm{d}x)\Big),\ \ \theta\in\mathbb{R}. \tag{1.4}$$

2. Convergence of $\Delta_t^{(r)}$ as $r \to \infty$

In this section Theorem 2.1 gives necessary and sufficient conditions for the convergence in distribution of $\Delta_t^{(r)}$, linearly normed, as $r \to \infty$, for each $t > 0$. Part of the forward direction of this result was proved in [4] (that (2.1) implies (2.2)). We add in the remaining parts of the proof in Section 3. The symbol "⇒" is used to denote weak convergence of distributions.

Theorem 2.1. *Assume (1.1) and suppose there exist norming functions $a_r > 0$ and centering functions $b_r \in \mathbb{R}$ such that, as $r \to \infty$,*

$$\frac{\Delta_1^{(r)} - b_r}{a_r} \Rightarrow \Delta^{(\infty)}, \tag{2.1}$$

where $\Delta^{(\infty)}$ is a finite nondegenerate random variable. Then, for all $x \in \mathbb{R}$ such that $a_r x + b_r > 0$,

$$\lim_{r\to\infty}\frac{r - \overline{\Pi}\big(a_r x + b_r\big)}{\sqrt{r}} = h(x), \tag{2.2}$$

where $h(x) \in \mathbb{R}$ is a non-decreasing function having the form

$$h(x) = h_\gamma(x) = \begin{cases} 2x, & \text{if } \gamma = 0,\ x\in\mathbb{R},\\ -\dfrac{2}{\gamma}\log(1-\gamma x), & \text{if } \gamma\in\mathbb{R}\setminus\{0\},\ 1-\gamma x>0, \end{cases} \tag{2.3}$$

with a parameter γ which must in our situation satisfy $\gamma \le 0$. The limit rv $\Delta^{(\infty)}$ in (2.1) has distribution

$$\mathrm{P}\big(\Delta^{(\infty)} \le x\big) = \Phi\big(h(x)\big),\ \ x\in\mathbb{R}, \tag{2.4}$$

where $\Phi(x)$ is the standard normal cdf. The functions a_r and b_r may be chosen as follows:

(i) *when $\gamma < 0$, we may take $a_r = |\gamma|\overline{\Pi}^{\leftarrow}(r)$ and $b_r = \overline{\Pi}^{\leftarrow}(r)$;*
(ii) *when $\gamma = 0$, we may take $a_r = 2(\overline{\Pi}^{\leftarrow}(r-\sqrt{r}) - \overline{\Pi}^{\leftarrow}(r))$ and $b_r = \overline{\Pi}^{\leftarrow}(r)$, and then $a_r = o(b_r)$, as $r \to \infty$.*

In either case, $a_r \to 0$ and $b_r \to 0$ as $r \to \infty$. Further, a_r and b_r may be replaced by any finite $\alpha_r > 0$ and $\beta_r \in \mathbb{R}$ such that $\alpha_r \sim c a_r$ for some $c > 0$, and $(\beta_r - b_r)/a_r \to 0$, as $r \to \infty$.

With the choices of a_r and b_r in (i) *and* (ii), *(2.1) implies, generally, for each $t > 0$,*

$$\lim_{r\to\infty}\mathrm{P}\Big(\frac{\Delta_t^{(r)} - b_{r/t}}{a_{r/t}} \le x\Big) = \Phi\big(\sqrt{t}h(x)\big),\ \ x\in\mathbb{R}. \tag{2.5}$$

Conversely, suppose (2.2) holds for some $a_r > 0$ and $b_r \in \mathbb{R}$, with $h(x) \in \mathbb{R}$ a non-decreasing function which takes values $-\infty$ and $+\infty$ at the left and right extremes of its

[2] [18] only gives the case $r = 1$ but this is easily extended to $r \in \mathbb{N}$.

support. Then (2.1) holds for the same a_r and b_r with a finite nondegenerate random variable $\Delta^{(\infty)}$ on the RHS.

Remark 2.1. (a) There is nothing special about the "1" in (2.1); we may replace $\Delta_1^{(r)}$ by $\Delta_{t_0}^{(r)}$ for any given $t_0 > 0$, and the results otherwise remain true as stated. This is clear from the proof in [4].

(b) Expanding on Cases (i) and (ii) in Theorem 2.1, we can derive alternatively:

(i) when $\gamma < 0$, with $b_r = \overline{\Pi}^{\leftarrow}(r)$, we have for each $t > 0$

$$\frac{\Delta_t^{(r)}}{b_{r/t}} \Rightarrow e^{\gamma N_\Delta/2\sqrt{t}}, \text{ as } r \to \infty; \tag{2.6}$$

(ii) when $\gamma = 0$, with $a_r = 2(\overline{\Pi}^{\leftarrow}(r - \sqrt{r}) - \overline{\Pi}^{\leftarrow}(r))$ and $b_r = \overline{\Pi}^{\leftarrow}(r)$, we have for each $t > 0$

$$\frac{\Delta_t^{(r)} - b_{r/t}}{a_{r/t}} \Rightarrow \frac{N_\Delta}{2\sqrt{t}}, \text{ as } r \to \infty, \tag{2.7}$$

where in (2.6) and (2.7) N_Δ is a standard normal random variable. Further, in case $\gamma = 0$, we have for each $t > 0$

$$a_r = o(b_r) \text{ and } \frac{\Delta_t^{(r)}}{b_{r/t}} \Rightarrow 1, \text{ as } r \to \infty. \tag{2.8}$$

The results in (2.6)–(2.8) are derived in the course of the proof of Theorem 2.1.

(c) Also in the proof of Theorem 2.1 it is seen that (2.2) arises as the necessary and sufficient condition for a distribution to be in the *minimal* domain of attraction of an extreme value distribution. We show in Proposition 4.1 that (2.2) can only hold with $\gamma < 0$ when $\overline{\Pi}(x)$ is slowly varying at 0. When $\gamma = 0$, it is shown in Theorem 5.2 that $\overline{\Pi}$ regularly varying at 0 with index α, $0 \le \alpha \le 1$, implies (2.2). Thus slow variation of $\overline{\Pi}$ at 0 can occur in both cases, $\gamma < 0$ and $\gamma = 0$, and an index of regular variation alone does not suffice to distinguish between the cases.

(d) In [4] it is shown that in (2.5) there is in fact the finite dimensional convergence

$$\left(\frac{\Delta_t^{(r)} - b_{r/t}}{a_{r/t}}\right)_{0<t\le T} \Rightarrow \left(h^{\leftarrow}(t^{-1}B_t)\right)_{0<t\le T}, \text{ as } r \to \infty, \tag{2.9}$$

as random elements in the space of càdlàg functions on $[0, T]$ for each $T > 0$, where $(B_t)_{t\ge 0}$ is a standard Brownian motion. The approach to the limit process, illustrated in Fig. 1, is rather fascinating. Fig. 1 shows a realization of a point process generated from the measure $\overline{\Pi}(x) = x^{-\alpha}$, $x > 0$, with $\alpha = 0.9$, on the interval $[0, 1]$, with the processes of rth largest indicated for various values of r. The corresponding centered and scaled processes are in Fig. 2. The sample path can be compared for large r to that of $\{(2t)^{-1}B_t, 0 \le t \le 1\}$.

3. Proof of Theorem 2.1

Assume (1.1), and that (2.1) holds for functions $a_r > 0$ and $b_r \in \mathbb{R}$, with $\Delta^{(\infty)}$ finite and nondegenerate. Then it was proved in [4, Section 4.2] that (2.2) holds for all $x \in \mathbb{R}$ such that $a_r x + b_r > 0$, with $h(x)$ defined as in (2.3).

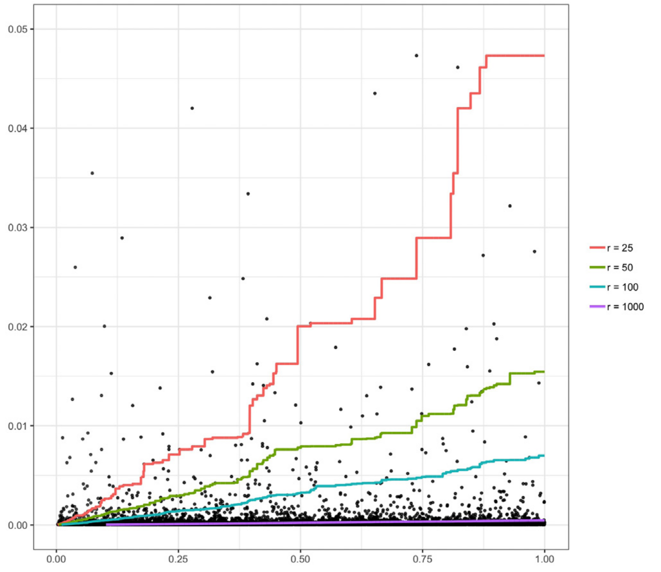

Fig. 1. Realizations of Poisson points (with magnitude less than 0.05) for Lévy measure $\overline{\Pi}(x) = x^{-0.9}$. The corresponding rth record processes $(\Delta_t^{(r)}, 0 \le t \le 1)$ with $r = 25, 50, 100, 1000$ are shown.

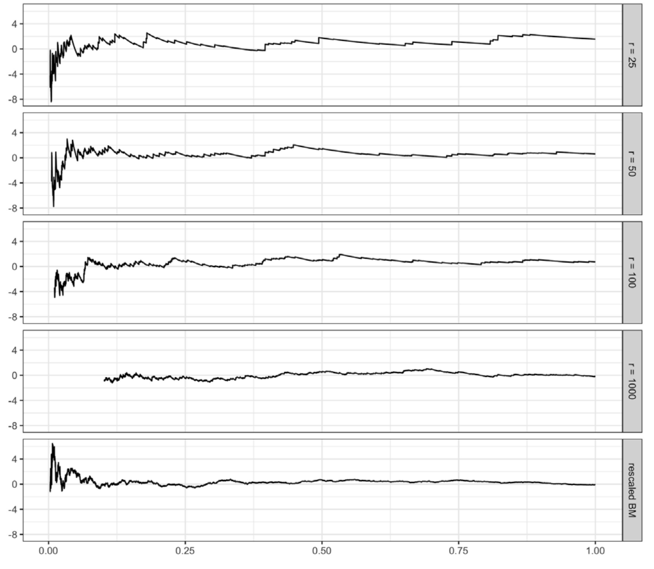

Fig. 2. The corresponding processes to Fig. 1 after centering and scaling by a_r and b_r specified in (9.1). As r increases, the rescaled sample path can be compared to that of rescaled Brownian motion $\{(2t)^{-1} B_t, 0 \le t \le 1\}$ on the last panel.

We can identify the distribution of the limit random variable $\Delta^{(\infty)}$ in terms of the inverse function $h^{\leftarrow}$ of h. From (2.3) this function satisfies, for $y \in \mathbb{R}$,

$$h^{\leftarrow}(y) = h^{\leftarrow}_{\gamma}(y) = \frac{1-e^{-\gamma y/2}}{\gamma} = \begin{cases} y/2, & \text{if } \gamma = 0, \\ \dfrac{1-e^{-\gamma y/2}}{\gamma}, & \text{if } \gamma > 0, \\ \dfrac{e^{|\gamma| y/2}-1}{|\gamma|}, & \text{if } \gamma < 0. \end{cases} \tag{3.1}$$

But now note that, since we assume only positive jumps for the Lévy process, the case $\gamma > 0$ in (2.3) or (3.1) cannot occur. This is because (2.2) means that the function $x \mapsto e^{-\sqrt{\Pi\{(x,\infty)\}}} = e^{-\sqrt{\overline{\Pi}(x)}}$, defined in [4], Sect 4.2, is a distribution function in the minimal domain of attraction, which for the $\gamma > 0$ case would require that function to be regularly varying as $x \to -\infty$. This is impossible of course because Π concentrates on $(0, \infty)$. So we can eliminate the case $\gamma > 0$, and do so from now on.

We note then that $h^{\leftarrow} : \mathbb{R} \mapsto \mathbb{R}_{\gamma}$, where, for $\gamma \le 0$,

$$\mathbb{R}_{\gamma} := \{x \in \mathbb{R} : 1 - \gamma x > 0\} = \begin{cases} \mathbb{R}, & \text{if } \gamma = 0, \\ (-\dfrac{1}{|\gamma|}, \infty), & \text{if } \gamma < 0. \end{cases}$$

Taking inverses in (2.2), we get an equivalent form

$$\lim_{r\to\infty} \frac{\overline{\Pi}^{\leftarrow}(r - y\sqrt{r}) - b_r}{a_r} = h^{\leftarrow}(y), \qquad y \in \mathbb{R}, \tag{3.2}$$

From (3.1) we have $h^{\leftarrow}(0) = 0$, so from (3.2) we deduce for $y \in \mathbb{R}$,

$$\begin{aligned} \lim_{r\to\infty} \frac{\overline{\Pi}^{\leftarrow}(r - y\sqrt{r}) - \overline{\Pi}^{\leftarrow}(r)}{a_r} &= \lim_{r\to\infty} \Big(\frac{\overline{\Pi}^{\leftarrow}(r - y\sqrt{r}) - b_r}{a_r} - \frac{\overline{\Pi}^{\leftarrow}(r) - b_r}{a_r} \Big) \\ &= h^{\leftarrow}(y) - h^{\leftarrow}(0) = h^{\leftarrow}(y). \end{aligned} \tag{3.3}$$

We conclude from (3.3) that for centering constants we may always set $b_r = \overline{\Pi}^{\leftarrow}(r)$. The convergences in (2.2), (3.2) and (3.3) are locally uniform since they are convergences of monotone functions to a continuous limit.

We may understand the form of the limit in (2.5) as follows. By the central limit theorem, we know that the gamma random variable $G_r := (\Gamma_r - r)/\sqrt{r}$, as a standardized cumulative sum of i.i.d standard exponential random variables, tends to a standard normal random variable, as $r \to \infty$. Assume (2.2), so that (3.2) and (3.3) hold. Then, owing to the local uniform convergence in (3.3), we get from (1.3):

$$\begin{aligned} \frac{\Delta_t^{(r)} - b_{r/t}}{a_{r/t}} &= \frac{\overline{\Pi}^{\leftarrow}(\Gamma_r/t) - b_{r/t}}{a_{r/t}} = \frac{\overline{\Pi}^{\leftarrow}\big(r/t + (G_r/\sqrt{t})\sqrt{r/t}\big) - b_{r/t}}{a_{r/t}} \\ &\Rightarrow h^{\leftarrow}\Big(-N_{\Delta}/\sqrt{t}\Big) \stackrel{\mathrm{D}}{=} h^{\leftarrow}\Big(N_{\Delta}/\sqrt{t}\Big). \end{aligned} \tag{3.4}$$

This proves (2.5), and (2.5) of course implies (2.4). Thus in (2.1) we have $\Delta^{(\infty)} \stackrel{\mathrm{D}}{=} h^{\leftarrow}(N_{\Delta})$.

For the next step we want to verify the claimed forms for a_r and b_r. For this we need to introduce some more machinery from regular variation theory.

3.1. Role of the de Haan classes $\mathbf{\Gamma}$ *and* $\mathbf{\Pi}$

Introduce the function $H : [0,\infty) \mapsto [1,\infty)$ defined by

$$H(x) = e^{2\sqrt{x}}, \quad x \geq 0, \tag{3.5}$$

and define the non-increasing function V by

$$V(x) = \overline{\Pi}^{\leftarrow} \circ H^{\leftarrow}(x), \quad x > 1. \tag{3.6}$$

Changing variables gives the representation $\overline{\Pi}^{\leftarrow}(x) = V(H(x))$.

The function H is the canonical example of a non-decreasing function in the de Haan class $\mathbf{\Gamma}$ with auxiliary function $f(y) = \sqrt{y}$ [1,7–9,12,21] satisfying

$$\lim_{y\to\infty} \frac{H(y + xf(y))}{H(y)} = e^x, \quad x \in \mathbb{R}. \tag{3.7}$$

This can be verified directly or by reference to [9, p. 248, line −1]. The inverse function $H^{\leftarrow} : [1,\infty) \mapsto [0,\infty)$ to H is $H^{\leftarrow}(y) = \frac{1}{4}\log^2 y,\ y > 1$, and inverting (3.7) shows that $H^{\leftarrow}$ satisfies

$$\lim_{s\to\infty} \frac{H^{\leftarrow}(sy) - H^{\leftarrow}(s)}{f(H^{\leftarrow}(s))} = \log y, \quad y > 0, \tag{3.8}$$

so $H^{\leftarrow}$ is an increasing function in de Haan's function class $\mathbf{\Pi}$ ([1,8,21] or [11, p. 375]). It has slowly varying auxiliary function $g(s) = f \circ H^{\leftarrow}(s) = \sqrt{H^{\leftarrow}(s)} = \frac{1}{2}\log s$ which is the denominator in (3.8). The convergence in (3.8) is uniform in compact intervals of y bounded away from 0. The properties of V we need are in the next proposition. For convenience we occasionally write $a_r = a(r)$ and $b_r = b(r)$ in what follows.

Proposition 3.1. *Assume (2.2) holds for some* $a_r > 0$ *and* $b_r \in \mathbb{R}$. *Then*

(i) *when* $\gamma < 0$, $V(x)$ *is regularly varying at* ∞ *with index* $-|\gamma|/2$ *and*

$$V(x) \sim a \circ H^{\leftarrow}(x)/|\gamma|, \ \text{as } x \to \infty; \tag{3.9}$$

(ii) *when* $\gamma = 0$, *we have* $-V \in \mathbf{\Pi}$ *at* ∞ *with slowly varying auxiliary function* $\frac{1}{2}a \circ H^{\leftarrow}$.

Remark 3.1. When $-V \in \mathbf{\Pi}$ we say, equivalently, $V \in \mathbf{\Pi}_-$, see [13].

Proof of Proposition 3.1. Assume (2.2), so we have (3.2) and (3.3) also. Use (3.6) to write, for $x > 0$, $s > 0$,

$$\frac{V(sx) - V(s)}{a \circ H^{\leftarrow}(s)} = \frac{\overline{\Pi}^{\leftarrow} \circ H^{\leftarrow}(sx) - \overline{\Pi}^{\leftarrow} \circ H^{\leftarrow}(s)}{a \circ H^{\leftarrow}(s)}$$

$$= \frac{\overline{\Pi}^{\leftarrow}\left(H^{\leftarrow}(s) + \left\{\frac{H^{\leftarrow}(sx) - H^{\leftarrow}(s)}{\sqrt{H^{\leftarrow}(s)}}\right\}\sqrt{H^{\leftarrow}(s)}\right) - \overline{\Pi}^{\leftarrow} \circ H^{\leftarrow}(s)}{a \circ H^{\leftarrow}(s)}.$$

Substitute $H^{\leftarrow}(s) = y$ and let $y \to \infty$, applying the uniform convergence in (3.8) followed by (3.3), we get the limit of the RHS as

$$\lim_{y\to\infty} \frac{\overline{\Pi}^{\leftarrow}(y + \sqrt{y}\log x) - \overline{\Pi}^{\leftarrow}(y)}{a(y)} = h^{\leftarrow}(-\log x).$$

Thus, for $x > 0$, using the form of $h^{\leftarrow}$ in (3.1),

$$\lim_{s\to\infty} \frac{V(sx) - V(s)}{a \circ H^{\leftarrow}(s)} = \begin{cases} -\frac{1}{2}\log x, & \text{if } \gamma = 0, \\ \dfrac{x^{-|\gamma|/2} - 1}{|\gamma|}, & \text{if } \gamma < 0. \end{cases} \tag{3.10}$$

Now keep $\gamma < 0$. From [11, Theorem B.2.1, p.372], we get $V(x) \sim a \circ H^{\leftarrow}(x)/|\gamma| \in RV_{-|\gamma|/2}$ at ∞. This proves (3.9) and the regular variation of $V(x)$ at ∞ in Part (i) of Proposition 3.1.

For Part (ii), multiply the limit relation in (3.10) by -1 to see that the non-decreasing function $-V$ is in $\mathbf{\Pi}$ ([11, p. 375]), and in fact is extended regularly varying at ∞ ([11, p. 295]). □

Continuing with the proof of Theorem 2.1, we now apply the results of Proposition 3.1 to get the required properties of a_r and b_r. As derived from (3.3), we set $b_r = \overline{\Pi}^{\leftarrow}(r)$.

(i) Take $\gamma < 0$. From (3.9), $V(x) \sim a \circ H^{\leftarrow}(x)/|\gamma|$, so from (3.6)

$$b_r = \overline{\Pi}^{\leftarrow}(r) = V(H(r)) \sim a_r/|\gamma|. \tag{3.11}$$

Thus (3.4) can be written

$$\frac{\Delta_t^{(r)} - b_{r/t}}{|\gamma| b_{r/t}} \Rightarrow h^{\leftarrow}(N_\Delta/\sqrt{t}),$$

which gives the forms of the a_r and b_r in Part (i) of Theorem 2.1. Using (3.1), the last relation can be rewritten as

$$\frac{\Delta_t^{(r)}}{b_{r/t}} \Rightarrow 1 + |\gamma| h^{\leftarrow}(N_\Delta/\sqrt{t}) \stackrel{\mathrm{D}}{=} e^{\gamma N_\Delta/2\sqrt{t}},$$

which gives (2.6).

(ii) Take $\gamma = 0$. From (3.1) with $\gamma = 0$ and (3.2) with $y = 1$ we get

$$\lim_{r\to\infty} \frac{2(\overline{\Pi}^{\leftarrow}(r - \sqrt{r}) - \overline{\Pi}^{\leftarrow}(r))}{a_r} = 1,$$

and the choice of a_r for Part (ii) of Theorem 2.1 and for (2.7) follows from the convergence to types theorem. Since, by Proposition 3.1, $V \in \mathbf{\Pi}_-$ with auxiliary function $\frac{1}{2} a \circ H^{\leftarrow}$, and the ratio of a non-negative $\mathbf{\Pi}$-function to its auxiliary function tends to ∞ ([11], Cor. B.2.13), we have in this case

$$\lim_{r\to\infty} \frac{b_r}{a_r} = \lim_{r\to\infty} \frac{\overline{\Pi}^{\leftarrow}(r)}{a_r} = \lim_{r\to\infty} \frac{V(H(r))}{a \circ H^{\leftarrow} \circ H(r)} = \infty.$$

Finally, for the forward direction, multiplying (2.7) by $a_{r/t}/b_{r/t}$, which tends to 0 as $r \to \infty$, yields a limit of 0 which is tantamount to saying $\Delta_t^{(r)}/b_{r/t} \Rightarrow 1$, and completing the proof of (2.8).

For the converse part of the proof of Theorem 2.1, suppose (2.2) holds with $h(x) \in \mathbb{R}$ a non-decreasing function $h(x)$ which takes values $-\infty$ and $+\infty$ at its left and right extremes. Then from (1.3), for $x \in \mathbb{R}$,

$$\begin{aligned} P\left(\frac{\Delta_1^{(r)} - b_r}{a_r} \le x\right) &= P(\Delta_1^{(r)} \le x a_r + b_r) \\ &= P\left(\frac{\Gamma_r - r}{\sqrt{r}} \ge \frac{\overline{\Pi}(x a_r + b_r) - r}{\sqrt{r}}\right) \end{aligned}$$

$$\begin{aligned} &\to P(N(0,1) \geq -h(x)) \\ &= P(N(0,1) \leq h(x)) = \Phi(h(x)). \end{aligned}$$

In view of the assumptions on $h(x)$, $\Phi(h(x))$ is the cdf of a proper distribution and so (2.1) holds. □

4. Convergence of $({}^{(r)}X_t, \Delta_t^{(r)})$, random standardization

To begin this section we set out the steps we intend to follow to understand the joint limit behavior of $({}^{(r)}X_t, \Delta_t^{(r)})$ as $r \to \infty$ under (2.2) or, equivalently, (3.2).

1. As discussed, we expect a normal limit as $r \to \infty$ for ${}^{(r)}X_t$ with suitable linear normalizations. We show that this happens for ${}^{(r)}X_t|\Delta_t^{(r)}$ under natural *random* centering and norming (Theorem 4.1).
2. Following that, we extend asymptotic normality of ${}^{(r)}X_t|\Delta_t^{(r)}$ to a joint asymptotic weak limit for $({}^{(r)}X_t, \Delta_t^{(r)})$ in which the limit has independent components. At this stage, ${}^{(r)}X_t$ still has random centering and norming, though $\Delta_t^{(r)}$ has non-random standardization (Corollary 4.1).
3. In Section 5, we note there is a cost to replacing the random centering and norming for ${}^{(r)}X_t$ by deterministic: dependencies and non-normality are introduced into the limit (Theorems 5.1 and 5.2).
4. Proofs of the theorems and further discussion are deferred to Sections 6, 7 and 8. A number of subsidiary propositions are also needed; these are proved in Section 4.1. Section 9 gives an application to the stable subordinator in the case $\gamma = 0$. Section 10 concludes with some final thoughts.

Throughout, we fix $t > 0$ and write $\mathrm{P}^{\Delta_t^{(r)}}(\cdot) = \mathrm{P}(\cdot|\Delta_t^{(r)})$ for the conditional distribution, given $\Delta_t^{(r)}$. For detailed calculations of this and other conditional distributions in general cases see [14].

We will also need truncated first and second moment functions, defined for $x > 0$ by

$$\mu(x) = \int_0^x y\, \Pi(\mathrm{d}y) \quad \text{and} \quad \sigma^2(x) = \int_0^x y^2\, \Pi(\mathrm{d}y). \tag{4.1}$$

Theorem 4.1. *Suppose (X_t) is a driftless subordinator with atomless Lévy tail measure $\overline{\Pi}$ that satisfies (2.1) for deterministic functions $a_r > 0$ and $b_r \in \mathbb{R}$. Then, for each $t > 0$,*

$$\lim_{r\to\infty} \mathrm{P}^{\Delta_t^{(r)}}\left(\frac{{}^{(r)}X_t - \mu(\Delta_t^{(r)})}{\sqrt{t}\,\sigma(\Delta_t^{(r)})} \leq x \right) = \Phi(x), \ \ x \in \mathbb{R}. \tag{4.2}$$

Remark 4.1. By the dominated convergence theorem the convergence in (4.2) holds unconditionally as well, so we also have, when (2.1) holds,

$$\frac{{}^{(r)}X_t - \mu(\Delta_t^{(r)})}{\sigma(\Delta_t^{(r)})} \Rightarrow N(0,t) \stackrel{\mathrm{D}}{=} \sqrt{t}N(0,1), \ \text{as } r \to \infty, \ \text{ for each } t > 0. \tag{4.3}$$

Retaining the random centering and norming, Theorem 4.1 immediately leads to a joint limit distribution for $({}^{(r)}X_t, \Delta_t^{(r)})$. In the following corollary, and throughout, N_X and N_Δ are independent standard normal random variables, being the limits of the standardized ${}^{(r)}X_t$ and $\Delta_t^{(r)}$, with the subscripts on N_X and N_Δ serving to distinguish the components corresponding to each.

Corollary 4.1. *Assume (2.1) holds for some deterministic functions $a_r > 0$ and $b_r \in \mathbb{R}$. Then we have, in $\mathbb{R}^2$, for each $t > 0$ and $\gamma \le 0$,*

$$\left(\frac{{}^{(r)}X_t - \mu(\Delta_t^{(r)})}{\sigma(\Delta_t^{(r)})}, \ \frac{\Delta_t^{(r)} - b_{r/t}}{a_{r/t}} \right) \Rightarrow \left(\sqrt{t} N_X, \ h^{\leftarrow}(N_\Delta/\sqrt{t}) \right), \text{ as } r \to \infty. \tag{4.4}$$

where $h^{\leftarrow} = h_\gamma^{\leftarrow}$ is defined in (3.1).

4.1. Further implications of the variation of $\overline{\Pi}^{\leftarrow}$

In this subsection we derive some additional, purely analytical, properties of the functions $\mu(x)$ and $\sigma(x)$ defined in (4.1), separating cases when $\gamma < 0$ or $\gamma = 0$.

Case (i): $\gamma < 0$. Suppose throughout that (2.2) holds with $h(x) = h_\gamma(x)$ for $\gamma < 0$ as in (2.3), so by (3.11) we can take $a_r = |\gamma| b_r$ and $b_r = \overline{\Pi}^{\leftarrow}(r)$, and have $\Delta_t^{(r)}/b_{r/t} \Rightarrow e^{\gamma N_\Delta / 2\sqrt{t}}$ as stated in (2.6).

Proposition 4.1. *Assume (2.2) holds for some deterministic functions $a_r > 0$ and $b_r \in \mathbb{R}$ and take $\gamma < 0$ in (2.3).*

(a) *For $p \ge 1$,*

$$\int_0^x y^p \Pi(\mathrm{d}y) \sim \frac{2}{p|\gamma|} x^p \sqrt{\overline{\Pi}(x)}, \ \text{as } x \downarrow 0. \tag{4.5}$$

In particular, when $p = 2$,

$$\sigma^2(x) \sim \frac{1}{|\gamma|} x^2 \sqrt{\overline{\Pi}(x)}, \ \text{as } x \downarrow 0. \tag{4.6}$$

(b) *$\overline{\Pi}(x)$ is slowly varying at 0, $\sigma^2(x)$ is regularly varying at 0 with index 2, and the distribution function $G(x) := e^{-\sqrt{\overline{\Pi}(x)}}$, $x > 0$, is regularly varying at 0 with index $1/|\gamma|$.*

Proof of Proposition 4.1. (a) Assume (2.2) holds with $h = h_\gamma$ and keep $\gamma < 0$ throughout. To see (4.5), use that $\overline{\Pi}^{\leftarrow} = V \circ H$ from (3.6), where V is regularly varying at ∞ with index $-|\gamma|/2$ by (3.9), and $H(x) = e^{2\sqrt{x}}$ is a $\boldsymbol{\Gamma}$ function with auxiliary function $f(t) = \sqrt{t}$ (see (3.7)). Such a composition is again in the class $\boldsymbol{\Gamma}$ [1,8,9,11,20,21], so for $z \in \mathbb{R}$

$$\frac{\overline{\Pi}^{\leftarrow}(r + \sqrt{r}z)}{\overline{\Pi}^{\leftarrow}(r)} = \frac{V\left(\frac{H(r+\sqrt{r}z)}{H(r)} H(r) \right)}{V(H(r))} \to e^{-z|\gamma|/2},$$

or, equivalently, after a change of variable to $w = z|\gamma|/2$,

$$\frac{\overline{\Pi}^{\leftarrow}(r + \frac{2\sqrt{r}}{|\gamma|} w)}{\overline{\Pi}^{\leftarrow}(r)} \to e^{-w}, \quad w \in \mathbb{R}. \tag{4.7}$$

The limit relation (4.7) identifies the auxiliary function of the non-increasing $\boldsymbol{\Gamma}$-varying function $\overline{\Pi}^{\leftarrow}(x)$ as $f_1(r) = \frac{2}{|\gamma|}\sqrt{r}$. Likewise for any $p \ge 1$, $(\overline{\Pi}^{\leftarrow})^p \in \boldsymbol{\Gamma}$ with auxiliary function $f_p(r) = \frac{2}{p|\gamma|}\sqrt{r}$. Auxiliary functions of $\boldsymbol{\Gamma}$-functions are unique up to asymptotic equivalence and also may be constructed in a canonical way (see for example, [11, p.19, eqn. 1.2.5], [1, p.177, Corollary 3.10.5(b)]). Therefore, we may identify the auxiliary function of the $\mathbf{\Gamma}$-function $(\overline{\Pi}^{\leftarrow})^p$ in two asymptotically equivalent ways:

$$f_p(r) \sim \frac{2}{p|\gamma|}\sqrt{r} \quad \text{or} \quad f_p(r) \sim \frac{\int_r^\infty \left(\overline{\Pi}^{\leftarrow}(u)\right)^p \mathrm{d}u}{\left(\overline{\Pi}^{\leftarrow}(r)\right)^p}, \; r \to \infty. \tag{4.8}$$

Using the transformation theorem for integrals (e.g. [2, p. 301]) and (4.8), we can write

$$\int_r^\infty \left(\overline{\Pi}^{\leftarrow}(y)\right)^p \mathrm{d}y = \int_0^{\overline{\Pi}^{\leftarrow}(r)} y^p \Pi(\mathrm{d}y) \sim (\overline{\Pi}^{\leftarrow}(r))^p f_p(r) \sim \frac{2(\overline{\Pi}^{\leftarrow}(r))^p\sqrt{r}}{p|\gamma|}, \; r \to \infty,$$

and a change of variables to $x = \overline{\Pi}^{\leftarrow}(r) \downarrow 0$ gives (4.5) and (4.6).

(b) Invert the limit relation (4.7) to get

$$\lim_{x\downarrow 0} \frac{\overline{\Pi}(xy) - \overline{\Pi}(x)}{(2/|\gamma|)\sqrt{\overline{\Pi}(x)}} = -\log y, \; y > 0.$$

Dividing by $\overline{\Pi}(x)$ instead of $\sqrt{\overline{\Pi}(x)}$, we get zero on the right side in the limit, which shows that $\overline{\Pi}(x)$ is slowly varying at 0, hence $\sigma^2(x)$ is regularly varying at 0 with index 2 by (4.6). Finally, factoring as

$$\overline{\Pi}^{1/2}(xy) - \overline{\Pi}^{1/2}(x) = \frac{\overline{\Pi}(xy) - \overline{\Pi}(x)}{\overline{\Pi}^{1/2}(xy) + \overline{\Pi}^{1/2}(x)} \sim -\frac{1}{|\gamma|}\log y,$$

and using the slow variation of $\overline{\Pi}(x)$, hence of $\overline{\Pi}^{1/2}(x)$, at 0, gives the regular variation of $e^{-\sqrt{\overline{\Pi}(x)}}$ at 0 with index $1/|\gamma|$. □

Case (ii): $\gamma = 0$. Suppose (2.2) holds with $h(x) = h_\gamma(x) = 2x$ in (2.3). From Theorem 2.1 we know in this case we may take $b_r = \overline{\Pi}^{\leftarrow}(r)$ and $a_r = 2(\overline{\Pi}^{\leftarrow}(r - \sqrt{r}) - b_r)$, and then $(\Delta_t^{(r)} - b_{r/t})/a_{r/t} \Rightarrow N_\Delta/2\sqrt{t}$, where N_Δ is a standard normal random variable. Also $a_r/b_r \to 0$ and $\Delta_t^{(r)}/b_{r/t} \Rightarrow 1$.

The following proposition parallels Proposition 4.1 for the $\gamma = 0$ case. Recall the functions H from (3.5) and $V = \overline{\Pi}^{\leftarrow} \circ H^{\leftarrow}$ from (3.6), satisfying $V^{\leftarrow} = H \circ \overline{\Pi}$ and $V \in \mathbf{\Pi}_-$ with slowly varying auxiliary function $\frac{1}{2}a \circ H^{\leftarrow}(s)$.

Proposition 4.2. *Assume (2.2) holds for some deterministic functions $a_r > 0$ and $b_r \in \mathbb{R}$ and take $\gamma = 0$ in (2.3).*

(a) *For $p \geq 1$, there exist $\mathbf{\Pi}$-varying functions $\pi_p(\cdot)$ such that*

$$\int_0^{b_r} u^p \Pi(du) = \pi_p(H(r)) = \pi_p(e^{2\sqrt{r}}), \tag{4.9}$$

where the slowly varying auxiliary function of π_p is $g_p(x) = \frac{1}{2}V^p(x)\log x$.

(b) *As $r \to \infty$,*

$$\frac{\sigma^2(\Delta_t^{(r)})}{\sigma^2(b_{r/t})} \Rightarrow 1. \tag{4.10}$$

Proof of Proposition 4.2. (a) Take $p \ge 1$ and $y > 0$, and recall $H^{\leftarrow}(y) = \frac{1}{4}\log^2 y$. Consider

$$\int_0^y u^p \Pi(\mathrm{d}u) = \int_{\overline{\Pi}(y)}^{\infty} \big(\overline{\Pi}^{\leftarrow}(s)\big)^p \mathrm{d}s = \int_{\overline{\Pi}(y)}^{\infty} (V \circ H(s))^p ds = \int_{H\circ\overline{\Pi}(y)}^{\infty} V^p(v) dH^{\leftarrow}(v)$$
$$= \int_{V^{\leftarrow}(y)}^{\infty} V^p(v) \frac{1}{2} \log v \frac{\mathrm{d}v}{v} = \pi_p\big(V^{\leftarrow}(y)\big), \tag{4.11}$$

where we define

$$\pi_p(y) = \frac{1}{2}\int_y^{\infty} V^p(v) \log v \frac{\mathrm{d}v}{v}. \tag{4.12}$$

Now, V is $\mathbf{\Pi}_{-}$-varying at ∞ and hence slowly varying at ∞, so V^p is slowly varying at ∞, as is $\log v$. Thus the function $\pi_p(\cdot)$ is the integral of a -1-varying function. The indefinite integral of a -1-varying function is $\mathbf{\Pi}$-varying ([10,11], [21, p. 30]). Thus $\pi_p \in \mathbf{\Pi}$ at ∞ and the auxiliary function is $g_p(y) = \frac{1}{2} V^p(y) \log y$.

(b) A $\mathbf{\Pi}$-varying function is always of larger order than its auxiliary function ([11, p. 378]), so

$$\lim_{y\to\infty} \frac{\pi_p(y)}{g_p(y)} = \infty. \tag{4.13}$$

Now we apply these results with $p = 2$. Because of the representation in (4.11), we invert the $\mathbf{\Pi}_{-}$-variation of $V(\cdot)$ in (3.10) with $\gamma = 0$ by writing (3.10) as

$$\frac{-V(sx) - (-V(s))}{\frac{1}{2}a(H^{\leftarrow}(s))} = -\frac{V(sx) - V(s)}{\frac{1}{2}a(\overline{\Pi}(V(s)))} \to \log x, \text{ as } s \to \infty,$$

which inverts as (cf. (3.7) and (3.8))

$$\lim_{y\downarrow 0} \frac{V^{\leftarrow}\big(y - \frac{1}{2}xa(\overline{\Pi}(y))\big)}{V^{\leftarrow}(y)} = e^x.$$

Substituting $y = b_{r/t}$, so $\overline{\Pi}(y) = r/t$, gives, for $x > 0$,

$$\lim_{r\to\infty} \frac{V^{\leftarrow}\big(b_{r/t} + xa_{r/t}\big)}{V^{\leftarrow}(b_{r/t})} = e^{-2x}. \tag{4.14}$$

To show (4.10), use (4.11) with $p = 2$ to write

$$\sigma^2(\Delta_t^{(r)}) - \sigma^2(b_{r/t}) = \pi_2\big(V^{\leftarrow}(\Delta_t^{(r)})\big) - \pi_2\big(V^{\leftarrow}(b_{r/t})\big).$$

From (2.7) write $(\Delta_t^{(r)} - b_{r/t})/a_{r/t} = \xi_t^{(r)}$, so that $\xi_t^{(r)} \Rightarrow N_\Delta/2\sqrt{t}$ as $r \to \infty$, and remember that $b_{r/t} = \overline{\Pi}^{\leftarrow}(r/t)$. The previous difference then becomes

$$\pi_2\big(V^{\leftarrow}(b_{r/t} + a_{r/t}\xi_t^{(r)})\big) - \pi_2\big(V^{\leftarrow}(b_{r/t})\big) = \pi_2\Big(\frac{V^{\leftarrow}(b_{r/t} + a_{r/t}\xi_t^{(r)})}{V^{\leftarrow}(b_{r/t})} V^{\leftarrow}(b_{r/t})\Big)$$
$$- \pi_2\big(V^{\leftarrow}(b_{r/t})\big).$$

Applying the definition of $\mathbf{\Pi}$-variation and (4.14) we get

$$\frac{\sigma^2(\Delta_t^{(r)}) - \sigma^2(b_{r/t})}{g_2(V^{\leftarrow}(b_{r/t}))} \Rightarrow \frac{N_\Delta}{\sqrt{t}}. \tag{4.15}$$

Since

$$\frac{\sigma^2(b_{r/t})}{g_2(V^{\leftarrow}(b_{r/t}))} = \frac{\pi_2(V^{\leftarrow}(b_{r/t}))}{g_2(V^{\leftarrow}(b_{r/t}))} \to \infty,$$

by (4.13), we have proved (4.10), since if we divide (4.15) by something of larger order (namely, $\sigma^2(b_{r/t})$), we get a limit of 0. □

5. Convergence of $({}^{(r)}X_t, \Delta_t^{(r)})$, deterministic standardization

Next we need to understand the effect of replacing the random centering and norming by deterministic counterparts. The treatment is broken up according to the cases of the constant γ in (2.3). Recall the definitions of $\mu(x)$ and $\sigma(x)$ in (4.1).

Theorem 5.1. *Suppose (2.2) holds for some deterministic functions $a_r > 0$ and $b_r \in \mathbb{R}$ and define γ as in (2.3). As before, $t > 0$ is fixed, and N_X and N_Δ are independent $N(0,1)$.*

(i) *When $\gamma < 0$, we have, as $r \to \infty$, with $b_r = \overline{\Pi}^{\leftarrow}(r)$,*

$$\left(\frac{{}^{(r)}X_t - t\mu(\Delta_t^{(r)})}{\sigma(b_{r/t})}, \frac{\Delta_t^{(r)}}{b_{r/t}}\right) \Rightarrow \left(\sqrt{t}N_X e^{\gamma N_\Delta/2\sqrt{t}}, e^{\gamma N_\Delta/2\sqrt{t}}\right), \tag{5.1}$$

and also a version with deterministic centering and norming:

$$\left(\frac{{}^{(r)}X_t - t\mu(b_{r/t})}{b_{r/t}\sqrt{r}}, \frac{\Delta_t^{(r)}}{b_{r/t}}\right) \Rightarrow \left(\frac{2t}{|\gamma|}(e^{\gamma N_\Delta/2\sqrt{t}} - 1), e^{\gamma N_\Delta/2\sqrt{t}}\right), \text{ as } r \to \infty. \tag{5.2}$$

(ii) *When $\gamma = 0$, we have, as $r \to \infty$, with $a_r = 2(\overline{\Pi}^{\leftarrow}(r-\sqrt{r}) - \overline{\Pi}^{\leftarrow}(r))$ and $b_r = \overline{\Pi}^{\leftarrow}(r)$,*

$$\left(\frac{{}^{(r)}X_t - t\mu(\Delta_t^{(r)})}{\sigma(b_{r/t})}, \frac{\Delta_t^{(r)} - b_{r/t}}{a_{r/t}}\right) \Rightarrow \left(\sqrt{t}N_X, \frac{N_\Delta}{2\sqrt{t}}\right), \text{ as } r \to \infty. \tag{5.3}$$

Remark 5.1.

(a) Note that when $\gamma < 0$, we no longer have independence of the components in the limits in (5.1) and (5.2) when we replace the random norming by the deterministic one. The norming constants are conveniently written in terms of $b_{r/t}$ in (5.1) and (5.2), but recall that $b_r \sim a_r/|\gamma|$ (by (3.11)) in this $\gamma < 0$ case, and $\sigma^2(x)$ is regularly varying with index 2 as $x \downarrow 0$ by (4.6), so they are easily rewritten in terms of $a_{r/t}$.

(b) When $\gamma = 0$, we can always make the norming deterministic, as in (5.3); however, this is not in general the case for the centering; replacing $\mu(\Delta_t^{(r)})$ with $\mu(b_{r/t})$ in (5.3) is only possible under some subsidiary conditions. A detailed discussion is given in Section 8. For the special case when $\overline{\Pi} \in RV_{-\alpha}$ at 0 for $0 \le \alpha \le 1$, however, we can specify the joint limiting distribution of ${}^{(r)}X$ and $\Delta_t^{(r)}$ precisely, as in the following theorem.

Theorem 5.2. *Suppose $\overline{\Pi}$ is regularly varying at 0 with index $-\alpha$ for an $\alpha \in [0,1]$. Then (2.2) and (2.3) hold with the case $\gamma = 0$ in (2.3), for functions $b_r = \overline{\Pi}^{\leftarrow}(r)$ and $a_r = 2(\overline{\Pi}^{\leftarrow}(r-\sqrt{r}) - \overline{\Pi}^{\leftarrow}(r)) \sim (2/\alpha)r^{-1/2}b_r$.*

Further, fix $t > 0$ and let $c_\alpha := \alpha/(2-\alpha)$.

(i) *Suppose $0 < \alpha \le 1$, so that $0 < c_\alpha \le 1$. Then*

$$\left(\frac{{}^{(r)}X_t - t\mu(b_{r/t})}{\sigma(b_{r/t})}, \frac{\Delta_t^{(r)} - b_{r/t}}{a_{r/t}}\right) \Rightarrow \left(\sqrt{t}N_X + \frac{\sqrt{t}N_\Delta}{\sqrt{c_\alpha}}, \frac{N_\Delta}{2\sqrt{t}}\right). \tag{5.4}$$

(ii) *Suppose $\alpha = 0$, so $c_\alpha = 0$ and $\overline{\Pi}$ is slowly varying at 0. Then*

$$\left(\frac{{}^{(r)}X_t - t\mu(b_{r/t})}{\sqrt{r}\,b_{r/t}}, \frac{\Delta_t^{(r)} - b_{r/t}}{a_{r/t}}\right) \Rightarrow \left(N_X, \frac{N_\Delta}{2\sqrt{t}}\right). \tag{5.5}$$

Remark 5.2. Restricting X to be a subordinator, as we do, requires $\int_0^1 x\Pi(\mathrm{d}x) < \infty$, or, equivalently, $\int_0^1 \overline{\Pi}(x)\mathrm{d}x < \infty$, which forces $0 \le \alpha \le 1$ in Theorem 5.2. The case $\alpha = 1$ is possible, take for example $\overline{\Pi}(x) = \mathbf{1}_{\{0<x<e^{-1}\}}/x(\log x)^2$.

Proofs of Theorems 5.1 and 5.2 are deferred to Sections 7 and 8.

6. Proofs of Theorem 4.1 and Corollary 4.1

In this section we first prove the conditioned limit theorem, Theorem 4.1, using random centering and norming; this is followed by the proof of Corollary 4.1. Throughout, assume X is a driftless subordinator on $(0,\infty)$ with atomless Lévy measure Π on $(0,\infty)$.

Proof of Theorem 4.1. Suppose (2.1), hence that the conclusions in Theorem 2.1, hold. From (1.4), the conditional characteristic function of the centered and normed ${}^{(r)}X_t$ is

$$\mathrm{E}\Big(\exp\Big(\mathrm{i}\theta\frac{{}^{(r)}X_t - t\mu(\Delta_t^{(r)})}{\sigma(\Delta_t^{(r)})}\Big)\,\Big|\,\Delta_t^{(r)}\Big)$$
$$= \exp\Big(t\int_0^{\Delta_t^{(r)}}\Big(e^{\mathrm{i}\theta u/\sigma(\Delta_t^{(r)})} - 1 - \mathrm{i}\theta u/\sigma(\Delta_t^{(r)})\Big)\Pi(\mathrm{d}u)\Big),\ \theta\in\mathbb{R}. \tag{6.1}$$

Thus for (4.2) it is enough to show, for each $t > 0$, $\theta \in \mathbb{R}$,

$$\Big|t\int_0^{\Delta_t^{(r)}}\Big(e^{\mathrm{i}\theta u/\sigma(\Delta_t^{(r)})} - 1 - \mathrm{i}\theta u/\sigma(\Delta_t^{(r)})\Big)\Pi(\mathrm{d}u) + \frac{t}{2}\theta^2\Big| \to 0,\ \text{as } r\to\infty. \tag{6.2}$$

Noting that, by (4.1),

$$\frac{\theta^2}{2\sigma^2(\Delta_t^{(r)})}\int_0^{\Delta_t^{(r)}} y^2\Pi(\mathrm{d}y) = \frac{1}{2}\theta^2,$$

the left hand side of (6.2) equals

$$t\Big|\int_0^{\Delta_t^{(r)}}\Big(e^{\mathrm{i}\theta u/\sigma(\Delta_t^{(r)})} - 1 - \mathrm{i}\theta u/\sigma(\Delta_t^{(r)})\Big)\Pi(\mathrm{d}u) - \int_0^{\Delta_t^{(r)}}\Big(-\frac{1}{2}\theta^2\Big)\frac{u^2}{\sigma^2(\Delta_t^{(r)})}\Pi(\mathrm{d}u)\Big|.$$

Using the inequality $|e^{\mathrm{i}\theta} - 1 - \mathrm{i}\theta - \frac{(\mathrm{i}\theta)^2}{2}| \le |\theta|^3/3!$, $\theta\in\mathbb{R}$, this is bounded above by

$$\frac{t|\theta|^3}{3!}\int_0^{\Delta_t^{(r)}}\frac{u^3}{\sigma^3(\Delta_t^{(r)})}\Pi(\mathrm{d}u) \le \frac{t|\theta|^3}{3!}\frac{\Delta_t^{(r)}}{\sigma(\Delta_t^{(r)})}. \tag{6.3}$$

So it suffices to show that $\Delta_t^{(r)}/\sigma(\Delta_t^{(r)}) \Rightarrow 0$ when (2.1) holds. When $\gamma < 0$, this is immediate from (4.6), since $\overline{\Pi}^{\leftarrow}(0+) = \infty$. When $\gamma = 0$, (2.8) and Proposition 4.2 yield $\Delta_t^{(r)}/b_{r/t} \Rightarrow 1$ and $\sigma^2(\Delta_t^{(r)})/\sigma^2(b_{r/t}) \Rightarrow 1$. Therefore

$$\frac{(\Delta_t^{(r)})^2}{\sigma^2(\Delta_t^{(r)})} = \frac{(\Delta_t^{(r)})^2}{b_{r/t}^2}\times\frac{b_{r/t}^2}{\sigma^2(b_{r/t})}\times\frac{\sigma^2(b_{r/t})}{\sigma^2(\Delta_t^{(r)})} = (1+o_p(1))\frac{b_{r/t}^2}{\sigma^2(b_{r/t})}. \tag{6.4}$$

The RHS of (6.4) converges to 0 as $r \to \infty$ for the following reason: we can use (4.9) (and the definition of the function g_p following (4.9)) to write

$$\lim_{r\to\infty} \frac{\sigma^2(b_{r/t})}{b_{r/t}^2} = \lim_{r\to\infty} \frac{\pi_2(H(r/t))}{(\overline{\Pi}^{\leftarrow}(r/t))^2} = \lim_{x\to\infty} \frac{\pi_2(x)}{V^2(x)} = \lim_{x\to\infty} \frac{\pi_2(x)}{V^2(x)\frac{1}{2}\log x}\Big(\frac{1}{2}\log x\Big)$$
$$= \lim_{x\to\infty} \frac{\pi_2(x)}{g_2(x)}\Big(\frac{1}{2}\log x\Big) = \infty \text{ (by (4.13))}.$$

Thus, via (6.4), the right hand side of (6.3) tends to 0, completing the proof of Theorem 4.1. □

Proof of Corollary 4.1. Recall that N_X and N_Δ are independent $N(0,1)$ and define

$$Z_X^{(r)}(t) = \frac{{}^{(r)}X_t - t\mu(\Delta_t^{(r)})}{\sigma(\Delta_t^{(r)})} \quad \text{and} \quad Z_\Delta^{(r)}(t) = \frac{\Delta_t^{(r)} - b_{r/t}}{a_{r/t}},$$

and suppose f, g are non-negative continuous functions bounded by 1. Then from (2.5) and (4.2) we have, as $r \to \infty$,

$$\mathrm{E} f\big(Z_X^{(r)}(t)\big) \to \mathrm{E} f(\sqrt{t}N_X) \quad \text{and} \quad \mathrm{E} g\big(Z_\Delta^{(r)}(t)\big) \to \mathrm{E} g(h^{\leftarrow}(N_\Delta/\sqrt{t})).$$

Writing

$$\Big|\mathrm{E}\Big(g(Z_\Delta^{(r)}(t))\,\mathrm{E}^{\Delta^{(r)}}\big(f(Z_X^{(r)}(t))\big)\Big) - \mathrm{E} g(Z_\Delta^{(r)}(t))\,\mathrm{E} f(\sqrt{t}N_X)\Big|$$
$$= \Big|\mathrm{E}\Big(g(Z_\Delta^{(r)}(t))\mathrm{E}^{\Delta^{(r)}}\big(f(Z_X^{(r)}(t)) - \mathrm{E} f(\sqrt{t}N_X)\big)\Big)\Big|$$
$$\le \mathrm{E}\Big|\mathrm{E}^{\Delta_t^{(r)}}\big(f(Z_X^{(r)}(t))\big) - \mathrm{E} f(\sqrt{t}N_X)\Big| \to 0, \text{ as } r \to \infty,$$

we get by dominated convergence

$$\mathrm{E} f(Z_X^{(r)}(t))\, g(Z_\Delta^{(r)}(t)) = \mathrm{E}(g(Z_\Delta^{(r)}(t))\,\mathrm{E}^{\Delta^{(r)}}\big(f(Z_X^{(r)}(t))\big)$$
$$\to \mathrm{E}\big(f(\sqrt{t}N_X)\big)\,\mathrm{E}\big(g\big(h^{\leftarrow}(N_\Delta/\sqrt{t})\big)\big).$$

Using this we can complete the proof of (4.4). □

7. Proof of Theorem 5.1

Throughout, assume (2.2) and (2.3), hence the conclusions contained in Theorem 2.1.

(i) When $\gamma < 0$ we set $b_r = \overline{\Pi}^{\leftarrow}(r)$ and $a_r = |\gamma| b_r$, and then by (2.6)

$$\frac{\Delta_t^{(r)}}{b_{r/t}} \Rightarrow Y_t := e^{\gamma N_\Delta/2\sqrt{t}}, \text{ as } r \to \infty. \tag{7.1}$$

Now $\sigma^2(x)$ is regularly varying at 0 with index 2 (see Proposition 4.1), so $\sigma(\Delta_t^{(r)})/\sigma(b_{r/t}) \Rightarrow Y_t$ as $r \to \infty$. This together with the joint convergence in Corollary 4.1 implies via continuous mapping that

$$\Big(\frac{{}^{(r)}X_t - t\mu(\Delta_t^{(r)})}{\sigma(b_{r/t})}, \frac{\Delta_t^{(r)}}{b_{r/t}}\Big) = \Big(\frac{{}^{(r)}X_t - t\mu(\Delta_t^{(r)})}{\sigma(\Delta_t^{(r)})} \times \frac{\sigma(\Delta_t^{(r)})}{\sigma(b_{r/t})}, \frac{\Delta_t^{(r)}}{b_{r/t}}\Big)$$
$$\Rightarrow \big(\sqrt{t}N_X Y_t,\ Y_t\big) = \big(\sqrt{t}N_X e^{\gamma N_\Delta/2\sqrt{t}},\ e^{\gamma N_\Delta/2\sqrt{t}}\big),$$

where (N_X, N_Δ) are i.i.d standard normal random variables, and $(\Delta_t^{(r)} - b_{r/t})/a_{r/t}$ in (4.4) can be replaced by $\Delta_t^{(r)}/b_{r/t}$ using that $a_r = b_r|\gamma|$ as we assumed. So we have replaced the random with a deterministic norming.

Now consider the effect of changing the random centering to a deterministic one in the first component. Again using $a_r = |\gamma| b_r$, convert (2.2) to vague convergence ([19] Section 3.4) on $(0, \infty)$, as

$$\frac{\Pi(b_{r/t}\mathrm{d}u)}{\sqrt{r/t}} \xrightarrow{v} \frac{2}{|\gamma|}\frac{\mathrm{d}u}{u}, \quad u > 0,$$

which gives, for each $x > 0$,

$$\int_1^x u \frac{\Pi(b_{r/t}\mathrm{d}u)}{\sqrt{r/t}} \to \int_1^x \frac{2}{|\gamma|}\mathrm{d}u = \frac{2}{|\gamma|}(x-1),$$

locally uniformly on $(0, \infty)$ since the LHS is a family of monotone functions converging to a continuous limit. Since $\Delta_t^{(r)}/b_{r/t} \Rightarrow Y_t$ as $r \to \infty$, we have by the continuous mapping theorem,

$$\frac{\mu(\Delta_t^{(r)}) - \mu(b_{r/t})}{b_{r/t}\sqrt{r/t}} = \int_1^{\Delta_t^{(r)}/b_{r/t}} u \frac{\Pi(b_{r/t}\mathrm{d}u)}{\sqrt{r/t}} \Rightarrow \frac{2}{|\gamma|}(Y_t - 1).$$

From (4.6),

$$\sigma(b_{r/t}) = \sqrt{\sigma^2(b_{r/t})} \sim b_{r/t}\frac{(r/t)^{1/4}}{\sqrt{|\gamma|}}, \text{ as } r \to \infty. \tag{7.2}$$

Thus $b_{r/t}\sqrt{r}/\sigma(b_{r/t}) \to \infty$ and we have

$$\begin{aligned}\frac{{}^{(r)}X_t - t\mu(b_{r/t})}{b_{r/t}\sqrt{r}} &= \Big(\frac{{}^{(r)}X_t - t\mu(\Delta_t^{(r)})}{\sigma(b_{r/t})}\Big) \times \frac{\sigma(b_{r/t})}{b_{r/t}\sqrt{r}} + t\Big(\frac{\mu(\Delta_t^{(r)}) - \mu(b_{r/t})}{b_{r/t}\sqrt{r/t}}\Big)\\ &= o_p(1) + t\Big(\frac{\mu(\Delta_t^{(r)}) - \mu(b_{r/t})}{b_{r/t}\sqrt{r/t}}\Big) \Rightarrow \frac{2t}{|\gamma|}(Y_t - 1), \text{ as } r \to \infty.\end{aligned}$$

Since we have joint convergence with $\Delta_t^{(r)}$ we get (5.2).

(ii) When $\gamma = 0$, by Proposition 4.2 we have $\sigma(\Delta_t^{(r)})/\sigma(b_{r/t}) \Rightarrow 1$ with $b_r = \overline{\Pi}^{\leftarrow}(r)$. Then from Corollary 4.1,

$$\Big(\frac{{}^{(r)}X_t - t\mu(\Delta_t^{(r)})}{\sigma(b_{r/t})}, \frac{\Delta_t^{(r)} - b_{r/t}}{a_{r/t}}\Big) \Rightarrow \Big(\sqrt{t}N_X, \frac{N_\Delta}{2\sqrt{t}}\Big), \tag{7.3}$$

By Theorem 2.1 we may choose $a_r = 2(\overline{\Pi}^{\leftarrow}(r - \sqrt{r}) - \overline{\Pi}^{\leftarrow}(r))$. □

8. Proof of Theorem 5.2

In the present section we investigate what happens when we attempt to replace the random centering for ${}^{(r)}X_t$ by deterministic centering in the case $\gamma = 0$. Assume (2.2) holds with $h = h_\gamma$ for $\gamma = 0$. As displayed in (7.3), we may replace the random norming for ${}^{(r)}X_t$ by the deterministic norming $\sigma(b_{r/t})$, and it is natural to try to replace $\mu(\Delta_t^{(r)})$ with $\mu(b_{r/t})$ by a method similar to the one used in the proof of (5.2). We begin with some preliminary remarks to illustrate the issues arising.

With the notation in (4.11), and setting $p = 1$, we can write

$$\mu(\Delta_t^{(r)}) - \mu(b_{r/t}) = \pi_1\big(V^{\leftarrow}(\Delta_t^{(r)})\big) - \pi_1\big(V^{\leftarrow}(b_{r/t})\big).$$

Recall from (3.4) that we may write $(\Delta_t^{(r)} - b_{r/t})/a_{r/t} = \xi_t^{(r)} \Rightarrow N_\Delta/2\sqrt{t}$. The previous difference thus becomes

$$\pi_1\big(V^{\leftarrow}(a_{r/t}\xi_t^{(r)} + b_{r/t})\big) - \pi_1\big(V^{\leftarrow}(b_{r/t})\big) = \pi_1\Big(\frac{V^{\leftarrow}(a_{r/t}\xi_t^{(r)} + b_{r/t})}{V^{\leftarrow}(b_{r/t})} V^{\leftarrow}(b_{r/t})\Big) - \pi_1\big(V^{\leftarrow}(b_{r/t})\big). \tag{8.1}$$

By Proposition 4.2, $\pi_1(\cdot)$ is $\boldsymbol{\Pi}$-varying with auxiliary function $g_1(x) = \frac{1}{2}V(x)\log x$. Apply this to the RHS of (8.1) in conjunction with (4.14) to get

$$\frac{\mu(\Delta_t^{(r)}) - \mu(b_{r/t})}{g_1(V^{\leftarrow}(b_{r/t}))} \Rightarrow \frac{N_\Delta}{\sqrt{t}}. \tag{8.2}$$

To replace $\mu(\Delta_t^{(r)})$ with $\mu(b_{r/t})$ in (7.3) thus requires that the difference in (8.2) be compared with $\sigma(b_{r/t})$. The cleanest result would be if the difference were $o(\sigma(b_{r/t}))$ as $r \to \infty$, but this is not always the case and the final form of the joint limit with deterministic centering and norming in general depends on the behavior of the limit of

$$\lim_{r\to\infty} \frac{\sigma^2(b_{r/t})}{g_1^2(H(r/t))} = \lim_{r\to\infty} \frac{\pi_2(H(r/t))}{g_1^2(H(r/t))}, \tag{8.3}$$

assuming there is indeed a limit. (Note that the $\boldsymbol{\Pi}$-function $\pi_2(\cdot)$ has auxiliary function g_2 and not g_1^2 so we cannot rely on (4.13) here.) Recall that $H(x) = e^{2\sqrt{x}}$ and $V(x) = \overline{\Pi}^{\leftarrow}(H^{\leftarrow}(x))$ (see (3.5) and (3.6)). In the numerator of (8.3), by (4.12)

$$\begin{aligned}
\pi_2(H(r/t)) &= \frac{1}{2}\int_{H(r/t)}^{\infty} V^2(y)\log y \frac{\mathrm{d}y}{y} \\
&= \frac{1}{2}\int_{r/t}^{\infty} (\overline{\Pi}^{\leftarrow}(v))^2 \log H(v) \frac{\mathrm{d}H(v)}{H(v)} \\
&= \int_{r/t}^{\infty} (\overline{\Pi}^{\leftarrow}(v))^2 \mathrm{d}v.
\end{aligned}$$

Also $g_1(x) = \frac{1}{2}V(x)\log x$, so the denominator in (8.3) is $g_1^2(H(r/t)) = (\overline{\Pi}^{\leftarrow}(r/t))^2(r/t)$. Thus the limit in (8.2) is

$$\lim_{r\to\infty} \frac{\pi_2(H(r/t))}{g_1^2(H(r/t))} = \lim_{z\to\infty} \frac{\int_z^\infty \big(\overline{\Pi}^{\leftarrow}(v)\big)^2 \mathrm{d}v}{z\big(\overline{\Pi}^{\leftarrow}(z)\big)^2}. \tag{8.4}$$

An easy example to show that $(\mu(\Delta_t^{(r)}) - \mu(b_{r/t}))/\sigma(b_{r/t})$ does not always vanish is the stable subordinator analyzed in detail in the next section. In this example we have

$$\overline{\Pi}(x) = x^{-\alpha}\mathbf{1}_{\{x>0\}} \text{ and } \overline{\Pi}^{\leftarrow}(v) = v^{-1/\alpha}\mathbf{1}_{\{v>0\}}, \text{ for an } \alpha \in (0,1). \tag{8.5}$$

The ratio on the right of (8.4) is in fact constant now:

$$\frac{\int_z^\infty v^{-2/\alpha}\mathrm{d}v}{z(z^{-2/\alpha})} = \frac{\alpha}{2-\alpha}.$$

More generally, if $\overline{\Pi}(x)$ is regularly varying at 0 with index $-\alpha$, then $(\overline{\Pi}^{\leftarrow}(z))^2$ is regularly varying at 0 with index $-2/\alpha$, so by Karamata's theorem for integrals (e.g., [1, p. 27])

$$\lim_{z\to\infty} \frac{\int_z^\infty \big(\overline{\Pi}^{\leftarrow}(v)\big)^2 dv}{z\big(\overline{\Pi}^{\leftarrow}(z)\big)^2} = \lim_{x\to 0} \frac{\int_0^x u^2\Pi(\mathrm{d}u)}{x^2\overline{\Pi}(x)} =: c_\alpha. \tag{8.6}$$

Since $0 < \alpha < 1$, we have $0 < c_\alpha < 1$. Conversely, if for general $\overline{\Pi}$ the limit on the LHS of (8.6) exists with value $c \in (0, \infty)$, the converse part of Karamata's theorem ([1, p. 30]) tells us that $\left(\overline{\Pi}^{\leftarrow}(z)\right)^2$ is regularly varying at ∞ with index $-(c^{-1}+1)$, which implies $\overline{\Pi}^{\leftarrow}(z)$ is regularly varying with index $-(c^{-1}+1)/2$ at ∞. Set $1/\alpha = (c^{-1}+1)/2$. Then for $\overline{\Pi}$ to correspond to a subordinator, we need $\alpha \le 1$, which makes $c \le 1$.

Following this path leads us to formulate Theorem 5.2 as we state it in Section 5, giving the joint limiting distribution of ${}^{(r)}X_t$ and $\Delta_t^{(r)}$ in this particular case. Based on the technology previously developed we can now prove that theorem.

Proof of Theorem 5.2. Assume that $\overline{\Pi}$ is regularly varying at 0 with index $-\alpha$, $0 \le \alpha \le 1$, or, equivalently, $\overline{\Pi}^{\leftarrow}(z)$ is regularly varying at ∞ with index $-1/\alpha$ (rapid variation if $\alpha = 0$). With the choice of functions $a_r = 2(\overline{\Pi}^{\leftarrow}(r-\sqrt{r}) - \overline{\Pi}^{\leftarrow}(r))$ and $b_r = \overline{\Pi}^{\leftarrow}(r)$, as suggested by Part (ii) of Theorem 2.1, that (2.2) and (2.3) hold with $\gamma = 0$ in (2.3), and that $a_r \sim (2/\alpha) r^{-1/2} b_r$ as $r \to \infty$, are easily checked.

First keep $0 < \alpha \le 1$, so $0 < c_\alpha \le 1$. Write

$$\frac{{}^{(r)}X_t - t\mu(b_{r/t})}{\sigma(b_{r/t})} = \frac{{}^{(r)}X_t - t\mu(\Delta_t^{(r)})}{\sigma(b_{r/t})} + \frac{t\big(\mu(\Delta_t^{(r)}) - \mu(b_{r/t})\big)}{\sigma(b_{r/t})}. \tag{8.7}$$

The second term on the RHS equals

$$\frac{t(\mu(\Delta_t^{(r)}) - \mu(b_{r/t}))}{g_1(H(r/t))} \times \frac{g_1(H(r/t))}{\sigma(b_{r/t})} \Rightarrow \frac{tN_\Delta}{\sqrt{t}} \times \frac{1}{\sqrt{c_\alpha}}, \text{ as } r \to \infty,$$

where the convergence for the first term on the left follows from (8.2), and, for the second, we have $g_1(H(r/t))/\sigma(b_{r/t}) \to 1/\sqrt{c_\alpha}$ by (8.3), (8.4) and (8.6). Together with (5.3), this proves (5.4).

When (8.6) holds with $\alpha = c_\alpha = 0$, then $\left(\overline{\Pi}^{\leftarrow}(z)\right)^2$ is rapidly varying at infinity ([8, p. 26]) so the same is true for $\overline{\Pi}^{\leftarrow}(z)$. Then by (8.3) and (8.4) we find

$$\lim_{r\to\infty} \frac{\sigma^2(b_{r/t})}{g_1^2(H(r/t))} = 0.$$

Divide on the left side of (8.7) by $g_1(H(r/t))$ instead of $\sigma(b_{r/t})$. Then by (8.2) we see that (5.4) becomes

$$\left(\frac{{}^{(r)}X_t - t\mu(b_{r/t})}{g_1(H(r/t))}, \ \frac{\Delta_t^{(r)} - b_{r/t}}{a_{r/t}}\right) \Rightarrow \left(\sqrt{t}N_X, \ \frac{N_\Delta}{2\sqrt{t}}\right).$$

Unpacking the notation shows that $g_1 \circ H(r/t) = b_{r/t}\sqrt{r/t}$, so we deduce (5.5). □

9. Example: Stable subordinator, case $\gamma = 0$

In this section we derive some asymptotic properties of the means and variances of a driftless subordinator (S_t) having ordered jumps $\Delta S_t^{(1)} \ge \Delta S_t^{(2)} \ge \cdots$ and r-trimmed version $({}^{(r)}S_t)$. With these we can replace the standardization in Part (ii) of Theorem 5.2 with one based on means and variances, for this process. This is done in Theorem 9.1. It will be seen in the proof of Proposition 9.1 that these moments are finite once r is large enough.[3]

[3] In fact, ${}^{(r)}S_t$ and $\Delta S_t^{(r)}$ have finite moments of any order, if r is chosen large enough.

Consider the stable subordinator having no drift and Lévy tail satisfying (8.5). The numerator of the left side of (3.2) is then

$$(r-y\sqrt{r})^{-1/\alpha}-r^{-1/\alpha}=r^{-1/\alpha}\Big(\big(1-\frac{y}{\sqrt{r}}\big)^{-1/\alpha}-1\Big)\sim\frac{r^{-1/\alpha-1/2}y}{\alpha},\ \text{as } r\to\infty,$$

for each $y\in\mathbb{R}$, so (3.2) holds if we take

$$h^{\leftarrow}(y)=y/2,\quad a_r=2r^{-1/\alpha-1/2}/\alpha,\quad\text{and}\quad b_r=r^{-1/\alpha}, \tag{9.1}$$

in it. Thus we are in the $\gamma=0$ case.

Proposition 9.1. *Suppose* (S_t) *is a driftless subordinator with ordered jumps* $\Delta S_t^{(1)}\geq\Delta S_t^{(2)}\geq\cdots$ *and tail measure satisfying* (8.5). *Fix* $0<\alpha<1$ *and* $t>0$ *throughout. Then*

(i) *we have*

$$\lim_{r\to\infty}\sqrt{r}\big(r^{1/\alpha}\mathrm{E}(\Delta S_t^{(r)})-t^{1/\alpha}\big)=0\ \text{and}\ \lim_{r\to\infty}r^{2/\alpha+1}\mathrm{Var}(\Delta S_t^{(r)})=2t^{2/\alpha}/\alpha^2. \tag{9.2}$$

Also, (ii),

$$\lim_{r\to\infty}\sqrt{r}\,\big(r^{1/\alpha-1}\mathrm{E}\big({}^{(r)}S_t\big)-t^{1/\alpha}\alpha/(1-\alpha)\big)=0 \tag{9.3}$$

and

$$\lim_{r\to\infty}r^{2/\alpha-1}\mathrm{Var}({}^{(r)}S_t)=\frac{t^{2/\alpha}(4-\alpha)}{2-\alpha}. \tag{9.4}$$

Proof of Proposition 9.1. (i) Keep $0<\alpha<1$ and $r>1/\alpha$. From (1.3) we find

$$\begin{aligned}\mathrm{E}\big(\Delta S_t^{(r)}\big)&=\mathrm{E}\big(\overline{\Pi}^{\leftarrow}(\Gamma_r/t)\big)=\int_0^\infty\overline{\Pi}^{\leftarrow}(y/t))\mathrm{P}(\Gamma_r\in\mathrm{d}y)\\&=\int_0^\infty\Big(\frac{y}{t}\Big)^{-1/\alpha}\frac{y^{r-1}e^{-y}}{\Gamma(r)}\mathrm{d}y\\&=t^{1/\alpha}\frac{\Gamma(r-1/\alpha)}{\Gamma(r)}.\end{aligned}$$

(Note this shows $\mathrm{E}(\Delta S_t^{(r)})=t^{1/\alpha}\mathrm{E}(\Delta S_1^{(r)})$ are both finite for $r>1/\alpha$.) Similar calculations give

$$\mathrm{Var}(\Delta S_t^{(r)})=t^{2/\alpha}\mathrm{Var}(\Delta S_1^{(r)})=t^{2/\alpha}\left(\frac{\Gamma(r-2/\alpha)}{\Gamma(r)}-\frac{\Gamma^2(r-1/\alpha)}{\Gamma^2(r)}\right).$$

Using a version of Stirling's formula with remainder in the form

$$\Gamma(r)=\sqrt{2\pi}(r-1)^{r-1/2}e^{-(r-1)}e^{\varepsilon(r)},\quad\text{where}\quad\frac{1}{12r}<\varepsilon(r)<\frac{1}{12(r-2)} \tag{9.5}$$

(e.g., [5], p.66), gives (9.2) after some lengthy but routine calculations.

(ii) By differentiating (1.4) with respect to θ, then setting $\theta=0$, we get

$$\mathrm{E}\big({}^{(r)}X_t\big|\Delta_t^{(r)}\big)=t\int_0^{\Delta_t^{(r)}}x\Pi(\mathrm{d}x) \tag{9.6}$$

and

$$\mathrm{E}\big(({}^{(r)}X_t)^2\big|\Delta_t^{(r)}\big)=t\int_0^{\Delta_t^{(r)}}x^2\Pi(\mathrm{d}x)+\Big(t\int_0^{\Delta_t^{(r)}}x\Pi(\mathrm{d}x)\Big)^2, \tag{9.7}$$

hence

$$\operatorname{Var}\left({}^{(r)}X_t \,\middle|\, \Delta_t^{(r)}\right) = t \int_0^{\Delta_t^{(r)}} x^2 \Pi(\mathrm{d}x). \tag{9.8}$$

Then using (1.3) and similar calculations as for Part (i) we find, once $r > 1/\alpha - 1$,

$$\mathrm{E}\left({}^{(r)}S_t\right) = \frac{a_\alpha t^{1/\alpha}}{\Gamma(r)} \int_0^\infty v^{1-1/\alpha} e^{-v} v^{r-1} \mathrm{d}v = a_\alpha t^{1/\alpha} \frac{\Gamma(r+1-1/\alpha)}{\Gamma(r)},$$

where $a_\alpha := \alpha/(1-\alpha)$. Hence

$$\sqrt{r}\left(r^{1/\alpha-1}\,\mathrm{E}\left({}^{(r)}S_t\right) - t^{1/\alpha}a_\alpha\right) = a_\alpha t^{1/\alpha} a\sqrt{r}\left(\frac{r^{1/\alpha-1}\Gamma(r+1-1/\alpha)}{\Gamma(r)} - 1\right),$$

and (9.3) follows from this after use of Stirling's formula again and more lengthy but routine calculations.

Similar lengthy calculations give

$$\begin{aligned}
&\operatorname{Var}({}^{(r)}S_t) \\
&= \frac{t^{2/\alpha}}{\Gamma(r)} \int_0^\infty \left(\frac{\alpha}{2-\alpha} v^{1-2/\alpha} + a_\alpha^2 v^{2-2/\alpha}\right) e^{-v} v^{r-1} \mathrm{d}v - a_\alpha^2 t^{2/\alpha} \frac{\Gamma^2(r+1-1/\alpha)}{\Gamma^2(r)} \\
&= \frac{\alpha t^{2/\alpha}\Gamma(r+1-2/\alpha)}{(2-\alpha)\Gamma(r)} + a_\alpha^2 t^{2/\alpha}\left(\frac{\Gamma(r+2-2/\alpha)}{\Gamma(r)} - \frac{\Gamma^2(r+1-1/\alpha)}{\Gamma^2(r)}\right).
\end{aligned} \tag{9.9}$$

Use of Stirling's formula (9.5) and more calculations give (9.4) from this. □

Theorem 9.1. *Suppose (S_t) is a driftless subordinator with tail measure satisfying (8.5). Keep $t > 0$ fixed throughout. Then, as $r \to \infty$,*

$$\left(\frac{{}^{(r)}S_t - \mathrm{E}({}^{(r)}S_t)}{\sqrt{\operatorname{Var}({}^{(r)}S_t)}}, \frac{\Delta S_t^{(r)} - \mathrm{E}(\Delta S_t^{(r)})}{\sqrt{\operatorname{Var}(\Delta S_t^{(r)})}}\right) \xrightarrow{\mathrm{D}} (N_1, N_2), \tag{9.10}$$

where (N_1, N_2) is bivariate $\mathbf{N}(\mathbf{0}, \mathbf{\Sigma})$ with

$$\mathbf{\Sigma} := \begin{bmatrix} \dfrac{2}{4-\alpha} & \sqrt{\dfrac{2-\alpha}{2(4-\alpha)}} \\ \sqrt{\dfrac{2-\alpha}{2(4-\alpha)}} & \dfrac{1}{2} \end{bmatrix}. \tag{9.11}$$

Remark 9.1. Note that the limit distribution in (9.10) does not depend on t, so the natural standardization produces a very easily interpretable result.

Proof of Theorem 9.1. Write

$$\frac{{}^{(r)}S_t - \mathrm{E}({}^{(r)}S_t)}{\sqrt{\operatorname{Var}({}^{(r)}S_t)}} = \left(\frac{{}^{(r)}S_t - t\mu(b_{r/t})}{\sigma(b_{r/t})}\right)\frac{\sigma(b_{r/t})}{\sqrt{\operatorname{Var}({}^{(r)}S_t)}} + \frac{t\int_0^{b_{r/t}} x\Pi(\mathrm{d}x) - \mathrm{E}({}^{(r)}S_t)}{\sqrt{\operatorname{Var}({}^{(r)}S_t)}}. \tag{9.12}$$

From (4.1) and the regular variation of $\overline{\Pi}(x)$ we deduce

$$\sigma(x) \sim \sqrt{\frac{\alpha}{2-\alpha}} x^{1-\alpha/2}, \ x \downarrow 0$$

(see also (8.6)). Thus, using (5.4), the expression in brackets on the RHS of (9.12) tends in distribution to $\sqrt{t}(N_X + N_\Delta/\sqrt{c_\alpha})$, where $c_\alpha = \alpha/(2-\alpha)$. Recalling $b_r = r^{-1/\alpha}$ and the variance estimate in (9.4), the factor multiplying the bracket in (9.12) is

$$\frac{\sigma(b_{r/t})}{\sqrt{\mathrm{Var}({}^{(r)}S_t)}} \sim \frac{(r/t)^{-1/\alpha+1/2}\sqrt{\alpha/(2-\alpha)}}{r^{-1/\alpha+1/2}t^{1/\alpha}\sqrt{4-\alpha}/\sqrt{2-\alpha}} = \frac{\sqrt{\alpha}}{\sqrt{4-\alpha}}t^{-1/2}.$$

In the second summand on the RHS of (9.12), the integral is

$$t\int_0^{(r/t)^{-1/\alpha}} x\Pi(\mathrm{d}x) = \alpha t\int_0^{(r/t)^{-1/\alpha}} x^{-\alpha}\mathrm{d}x = \frac{\alpha t^{1/\alpha}r^{1-1/\alpha}}{1-\alpha},$$

while from (9.3),

$$\mathrm{E}\big({}^{(r)}S_t\big) = t^{1/\alpha}r^{1-1/\alpha}\alpha/(1-\alpha) + o(r^{1/2-1/\alpha}).$$

This together with (9.4) shows that the second summand in (9.12) is o(1) and hence the LHS of (9.12) tends to $\sqrt{\alpha/(4-\alpha)}N_X + \sqrt{(2-\alpha)/(4-\alpha)}N_\Delta$.

Next write

$$\frac{\Delta S_t^{(r)} - \mathrm{E}(\Delta S_t^{(r)})}{\sqrt{\mathrm{Var}(\Delta S_t^{(r)})}} = \left(\frac{\Delta S_t^{(r)} - b_{r/t}}{a_{r/t}}\right)\frac{a_{r/t}}{\sqrt{\mathrm{Var}(\Delta S_t^{(r)})}} + \frac{b_{r/t} - \mathrm{E}(\Delta S_t^{(r)})}{\sqrt{\mathrm{Var}(\Delta S_t^{(r)})}}. \tag{9.13}$$

By (5.4), the expression in brackets on the RHS tends to $N_\Delta/(2\sqrt{t})$. Recalling $a_r = 2r^{-1/\alpha-1/2}/\alpha$ and the variance estimate in (9.2), the factor multiplying the bracket is

$$\frac{a_{r/t}}{\sqrt{\mathrm{Var}(\Delta_t^{(r)})}} \sim \frac{2r^{-1/\alpha-1/2}\,t^{1/\alpha+1/2}/\,\alpha}{r^{-1/\alpha-1/2}\,\sqrt{2}\,t^{1/\alpha}/\,\alpha} = \sqrt{2}.$$

In the second summand on the RHS of (9.13), $b_r = r^{-1/\alpha}$ and $\mathrm{E}(\Delta S_t^{(r)})$ can be substituted by the first moment estimate in (9.3). We deduce that the second summand in (9.13) is o(1) and hence the LHS of (9.13) tends to $N_\Delta/\sqrt{2}$. □

10. Final thoughts

It is tempting to conjecture that functional weak limit theorems may hold for standardized versions of $({}^{(r)}X_t, \Delta_t^{(r)})$ as functions of t when $r \to \infty$. As we noted in Remark 2.1, in Proposition 4.2 of [4] it is shown that there is in fact finite dimensional convergence in (2.9). But a proof of the full functional convergence seems hard.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

References

[1] N.H. Bingham, C.M. Goldie, J.L. Teugels, Regular Variation, in: Encyclopedia of Mathematics and its Applications, vol. 27, Cambridge University Press, Cambridge, 1989.
[2] P. Brémaud, Point processes and queues, Springer-Verlag, New York-Berlin, 1981, Martingale dynamics, Springer Series in Statistics.
[3] B. Buchmann, Y. Fan, R.A. Maller, Functional laws for trimmed Lévy processes, J. Appl. Probab. 54 (3) (2017) 873–889.

[4] B. Buchmann, R. Maller, S. Resnick, Processes of rth largest, Extremes 21 (2018) 485–508, http://dx.doi.org/10.1007/s10687-018-0308-x.
[5] W. Feller, An Introduction to Probability Theory and Its Applications, Vol. 1, third ed., Wiley, New York, 1957.
[6] T. Ferguson, M. Klass, A representation theorem of independent increment processes without Gaussian component., Ann. Math. Stat. 43 (1972) 1634–1643.
[7] J.L. Geluk, L. de Haan, Regular Variation, Extensions and Tauberian Theorems, in: CWI Tract, vol. 40, Stichting Mathematisch Centrum, Centrum voor Wiskunde en Informatica, Amsterdam, 1987.
[8] L. de Haan, On Regular Variation and Its Application to the Weak Convergence of Sample Extremes, Mathematisch Centrum Amsterdam, 1970.
[9] L. de Haan, Equivalence classes of regularly varying functions, Stochastic Process. Appl. 2 (1974) 243–259.
[10] L. de Haan, An Abel-Tauber theorem for Laplace transforms, J. Lond. Math. Soc. (2) 13 (3) (1976) 537–542.
[11] L. de Haan, A. Ferreira, Extreme Value Theory: An Introduction, Springer-Verlag, New York, 2006.
[12] L. de Haan, S. Resnick, Almost sure limit points of record values, J. Appl. Probab. 10 (1973) 528–542.
[13] L. de Haan, S. Resnick, Conjugate π-variation and process inversion, Ann. Probab. 7 (6) (1979) 1028–1035.
[14] Y. Ipsen, P. Kevei, R. Maller, Convergence to stable limits for ratios of trimmed Lévy processes and their jumps, Markov Process. Rel. Fields 24 (2018) 539–562.
[15] Y. Ipsen, R. Maller, S. Resnick, Ratios of ordered points of point processes with regularly varying intensity measures, Stochastic Process. Appl. 129 (2019) 205–222.
[16] P. Kevei, D. Mason, The limit distribution of ratios of jumps and sums of jumps of subordinators, ALEA-Latin Amer. J. Probab. Math. Stat. 11 (2) (2014) 631–642.
[17] R. LePage, M. Woodroofe, J. Zinn, Convergence of a stable distribution via order statistics, Ann. Probab. 9 (1981) 624–632.
[18] S. Resnick, Point processes, regular variation and weak convergence, Adv. Appl. Probab. 18 (1986) 66–138.
[19] S. Resnick, Extreme Values, Regular Variation and Point Processes, Springer-Verlag, New York, 1987.
[20] S. Resnick, Heavy Tail Phenomena: Probabilistic and Statistical Modeling, Springer Series in Operations Research and Financial Engineering, Springer-Verlag, New York, 2007, ISBN: 0-387-24272-4.
[21] S. Resnick, Extreme Values, Regular Variation and Point Processes, Springer, New York, 2008, Reprint of the 1987 original.