One-relator groups with torsion are coherent

Larsen Louder and Henry Wilton*

August 11, 2020

Abstract

We show that any one-relator group $G = F/\langle\langle w\rangle\rangle$ with torsion is coherent – i.e., that every finitely generated subgroup of G is finitely presented – answering a 1974 question of Baumslag in this case.

1 Introduction

Definition 1.1. A group G is *coherent* if every finitely generated subgroup of G is finitely presentable.

A well known question of Baumslag asks whether every one-relator group $F/\langle\langle w\rangle\rangle$ is coherent [Bau74, p. 76]. It is a curious feature of one-relator groups that the case with torsion, in which the relator w is a proper power, is often better behaved than the general case; most famously, one-relator groups with torsion are always hyperbolic [New68], and Wise proved that one-relator groups with torsion are residually finite, indeed linear [Wis12]. In this note we answer Baumslag's question affirmatively for one-relator groups with torsion.

Theorem 1.2. *If G is a one-relator group with torsion – that is, $G \cong F/\langle\langle w^n\rangle\rangle$, for $n > 1$ – then G is coherent.*

In 2003, Wise circulated [Wis03], including a purported proof of the following conjecture, stated as Theorem 4.13 of that paper.

Conjecture 1.3. *If X is a compact 2-complex with non-positive immersions then $\pi_1 X$ is coherent.*

The reader is referred to, for instance, [LW17] or [HW16] for the definition of non-positive immersions. On his webpage, Wise acknowledges that there is a gap (found by Mladen Bestvina) in the proof of [Wis03, Theorem 4.13]. In [Wis05], Wise used Conjecture 1.3 (stated as Theorem 1.5 of that paper) in a proof that Theorem 1.2 follows from the Strengthened Hannah Neumann Conjecture. The latter conjecture has more recently been proved independently by Friedman [Fri15] and Mineyev [Min12].

*Supported by EPSRC Standard Grant EP/L026481/1.

In summary, the results of [Wis05] are conditional on Conjecture 1.3, which remains open, and therefore Theorem 1.2 was not hitherto known unconditionally. After our proof was circulated, we learned from Wise that he has also given an unconditional proof of Theorem 1.2 in [Wis18], a revised version of [Wis03].

Our proof (and also that of [Wis18]) uses Wise's w-cycles conjecture (Theorem 3.1), which was proved independently by the authors [LW17] and by Helfer–Wise [HW16].

The outline of the proof is as follows. We realize G as the fundamental group of a compact, aspherical orbicomplex X. Since one-relator groups are virtually torsion-free, there is a finite-sheeted covering map $X_0 \looparrowright X$ so that $G_0 = \pi_1 X_0$ is torsion free. We then use the w-cycles conjecture to show that, whenever $Y \looparrowright X_0$ is an immersion from a compact two-complex without free faces, the number of two-cells in Y is bounded by the number of generators of $\pi_1(Y)$. In the final step, a folding argument expresses an arbitrary finitely generated subgroup H of G_0 as a direct limit of fundamental groups of 2-complexes with boundedly many 2-cells, and we deduce that H is finitely presented.

Acknowledgements

Thanks to Jim Howie and Hamish Short for pointing out an error in Section 4 of an earlier version.

2 One-relator orbicomplexes

Let F be a finitely generated free group, and $G = F/\langle\langle w^n \rangle\rangle$ a one-relator group, where $w \in F$ is not a proper power. In the usual way, F can be realized as the fundamental group of some finite topological graph Γ, and w by a continuous map $w : S^1 \to \Gamma$. (Since we are only interested in w up to conjugacy, we ignore base points.) Let $D_n \subseteq \mathbb{C}$ be the closed unit disk equipped with a cone point of order n at the origin. The orbicomplex

$$X = \Gamma \cup_w D_n$$

provides a natural model for G, in the sense that G is the (orbifold) fundamental group of X. We call X a *one-relator orbicomplex*. (There is a much more general theory of orbicomplexes – see, for instance, [Hae91] or [BH99, Chapter III.$\mathcal{C}$] – but the one-relator orbicomplexes defined here are sufficient for our purposes.) When $n = 1$, X is a *one-relator complex*.

A map of 2-complexes is a *morphism* if it sends n-cells homeomorphically to n-cells, for $n = 0, 1, 2$. A morphism of 2-complexes $Y \to Z$ is an *immersion* if it is a local injection; in this case, we write $Y \looparrowright Z$. If Y is a 2-complex and X is the one-relator orbicomplex defined above, a continuous map $Y \to X$ is a *morphism* if it sends vertices to vertices, edges homeomorphically to edges, and restricts, on each 2-cell, to the standard degree-n map $p_n : D_1 \to D_n$ given by $p_n(z) = z^n$. A morphism $Y \to X$ is an *immersion* if it is locally injective away from the cone points in the 2-cells (again, we write $Y \looparrowright X$), and a *covering* if it is locally a homeomorphism except at the cone points. The next definition plays a crucial role in our argument.

Definition 2.1 (Degree)**.** If $f : Y \looparrowright X$ is an immersion of two-dimensional (orbi)complexes, then the *degree* of f, denoted by $\deg f$, is the minimum number of preimages of a generic point in a 2-cell of X. That is: if X is a 2-complex, then $\deg f$ is the minimum number of preimages of any point in the interior of a 2-cell of X; and if X is a one-relator orbicomplex with 2-cell D_n, then $\deg f$ is the number of preimages of any point in the interior of D_n except 0.

Every one-relator group is virtually torsion free [FKS72], and it follows that the orbi-complex X is finitely covered by a genuine 2-complex. This can be seen using the covering theory for complexes of groups developed in [BH99], but we give a low-tech proof below.

Theorem 2.2 (Unwrapped covers of one-relator complexes)**.** *Let $G = F/\langle\langle w^n \rangle\rangle$ be a one-relator group with w not a proper power, and X the orbicomplex defined above. Then there is a finite-sheeted covering map*

$$X_0 \looparrowright X$$

where X_0 is a compact, connected 2-complex.

Proof. Let X' be $\Gamma \cup_{w^n} D$, the (genuine) two-complex that is the result of gluing a disc D to the graph Γ along the map w^n. Note that there is a natural morphism $X' \to X$ that induces an isomorphism of fundamental groups. Let G_0 be a torsion-free subgroup of finite index in G, and let $X'_0 \to X'$ be the corresponding covering space. Since w has order n in G [MKS04, Corollary 4.11], the 2-cells of X'_0 come in families of cardinality n, such that all the 2-cells in each family have the same attaching map. Let X_0 be the quotient of X'_0 obtained by collapsing each family to a single 2-cell. Picking one 2-cell from each family specifies an inclusion $X_0 \hookrightarrow X'_0$, and the quotient map $X'_0 \to X_0$ is then visibly a retraction, and an isomorphism on fundamental groups by the Seifert–van Kampen theorem. The composition

$$X_0 \hookrightarrow X'_0 \looparrowright X' \to X$$

is the required covering map. ☐

We emphasize that the complex X_0 in the above theorem is a 2-complex, not just an orbicomplex. That is, the covering map $X_0 \to X$ restricts to p_n on each 2-cell. We will call such a cover X_0 *unwrapped*.

3 A bound on the number of 2-cells

A two-complex Y is *reducible* if it has a free face.[1] Writing

$$\partial_Y : \coprod S^1 \to Y^{(1)}$$

for the disjoint union of the attaching maps of the 2-cells, this means that there is an edge e of the 1-skeleton $Y^{(1)}$ such that $\partial_Y^{-1}(e)$ consists of a single edge in $\coprod S^1$. If e is such an edge and Y' is the 2-complex obtained by collapsing the face of Y incident at e, then the natural inclusion map $Y' \hookrightarrow Y$ is a homotopy equivalence, and induces an isomorphism on fundamental groups. Of course, if Y is not reducible it is called *irreducible.*

The main theorem of [LW17] (or [HW16]) can be restated as a result about immersions to one-relator orbicomplexes, as follows.

Theorem 3.1. *Let X be a one-relator orbicomplex, Y a finite 2-complex and $f : Y \looparrowright X$ an immersion. If Y is irreducible then $\chi(Y^{(1)}) + \deg f \leq 0$.*

Proof. This follows from [LW17, Theorem 1.2], with $\Gamma = X^{(1)}$, $\Gamma' = Y^{(1)}$, ρ the restriction of f to Γ', $\Lambda = w$, and $\mathbb{S}$ the disjoint union of the boundaries of the 2-cells of Y. If some edge of $Y^{(1)}$ is not hit by a 2-cell, then we may remove that edge, increasing $\chi(Y^{(1)})$. Otherwise, Y is reducible in the sense of [LW17], so [LW17, Theorem 1.2] applies, taking $\Lambda' = \partial_Y$ and σ the induced map from the boundaries of the 2-cells of Y to the boundary of the 2-cell of X. □

Here, we apply Theorem 3.1 to relate the number of 2-cells of Y to the rank of its fundamental group. (By the *rank* of a group, we mean the minimal cardinality of a generated set.)

Corollary 3.2. *Let $f : Y \looparrowright X$ be an immersion from a finite, irreducible 2-complex Y to a one-relator orbicomplex X. Then*

$$\chi(Y) + (n-1)|\{\text{2-cells in } Y\}| \leq 0 \ .$$

In particular,

$$|\{\text{2-cells in } Y\}| \leq \frac{\mathrm{rk}(\pi_1(Y)) - 1}{n-1} < \mathrm{rk}(\pi_1(Y))$$

if $n > 1$.

Proof. By Theorem 3.1, $\chi(Y^{(1)}) + \deg(f) \leq 0$. Since f restricts to p_n on each 2-cell of Y, it follows that

$$\deg f = n|\{\text{2-cells in } Y\}|$$

[1] Note that this definition is slightly stronger than the definition given in [LW17], where a 2-complex was called 'reducible' if it has a free face or a free edge. The definition given here is more convenient in this context, since the complexes Y_i produced by Lemma 4.4 are irreducible in this sense, but not in the sense of [LW17].

since X is one-relator.

The Euler characteristic of Y is the Euler characteristic of $Y^{(1)}$ plus the number of two-cells in Y, so Theorem 3.1 implies the first assertion. The second assertion now follows from the first, using the trivial fact that

$$1 - \mathrm{rk}(\pi_1 Y) \leq \chi(Y)$$

for any connected 2-complex Y. ∎

In the case with torsion, Corollary 3.2 gives a bound on the the number of 2-cells of an immersion in terms of the rank of the fundamental group. In order to make a connection to arbitrary finitely generated subgroups of G, we use folding, in the spirit of Stallings.

4 Folding

Folding was introduced by Stallings to study free groups and their subgroups. The next lemma extends [Sta83, Algorithm 5.4] to the context of 2-complexes and their morphisms.

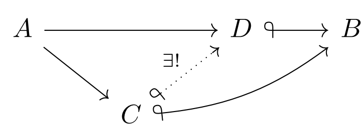
Lemma 4.1. *Any morphism of finite 2-complexes $A \to B$ factors as*

$$A \to C \looparrowright B$$

where $A \to C$ is surjective and π_1-surjective. Furthermore, if $A \to B$ factors as

$$A \to D \looparrowright B$$

then there is a unique immersion $C \looparrowright D$ so that the following diagram commutes.



Proof. Folding shows that the map of 1-skeleta factors as

$$A^{(1)} \to C^{(1)} \looparrowright B^{(1)}$$

where $A^{(1)} \to C^{(1)}$ is surjective and π_1-surjective. We now construct C by pushing the attaching maps of the 2-cells of A forward to $C^{(1)}$ and identifying any 2-cells with the same image in B and equal attaching maps. The resulting map $A \to C$ is surjective and π_1-surjective. We next check that the natural map $C \to B$ is an immersion.

Since $C \to B$ is a morphism, it can only fail to be locally injective at a point $p \in C$ if two higher-dimensional cells incident at p have the same image in B. The map of 1-skeleta is an immersion, so this can only occur if two 2-cells c_1, c_2 in C, incident at p, have the same image in B. Because the attaching maps of c_1 and c_2 agree at p and $C^{(1)} \to B^{(1)}$ is an immersion, it follows that the attaching maps of c_1 and c_2 agree everywhere. Therefore, c_1 and c_2 are equal in C by construction.

It remains to prove the universal property. This fact is standard for graphs, which defines the required immersion of 1-skeleta $C^{(1)} \looparrowright D^{(1)}$ uniquely. Let c be a 2-cell in C and let a_1, a_2 be preimages of c in A. By construction, a_1 and a_2 have the same boundary in $C^{(1)}$ and the same image in B. Therefore, their images d_1, d_2 respectively in D have the same boundary in $D^{(1)}$ and the same image in B. But $D \looparrowright B$ is an immersion, and it follows that $d_1 = d_2$. Therefore, we may extend the map $C^{(1)} \to D^{(1)}$ across c in a unique way, as required. □

A *free edge* of a 2-complex is an edge of the 1-skeleton that is not in the image of the attaching map of any 2-cell. The next result appeals to the Scott lemma (which plays a crucial role in the proof of coherence for 3-manifold groups) to represent a finitely generated, freely indecomposable subgroup by an immersion from a compact 2-complex without free edges or faces.

Lemma 4.2. *Let X be a 2-complex, $G = \pi_1(X)$, and H a non-trivial, finitely generated, freely indecomposable subgroup of G. There is an immersion from a compact irreducible 2-complex without free edges $f : Y \looparrowright X$ such that $f_*\pi_1 Y$ is conjugate to H.*

Proof. By the Scott lemma [Sco73, Lemma 2.2], there is a surjection from a finitely presented group $H' \twoheadrightarrow H$ that does not factor through a free product. Since H' is finitely presented, there is a morphism of combinatorial 2-complexes $Z' \to X$ that represents the composition $H' \to H \to G$. By Lemma 4.1, this morphism factors as

$$Z' \to Z \looparrowright X$$

where $Z' \to Z$ is π_1-surjective. By the conclusion of the Scott lemma, $\pi_1 Z$ is freely indecomposable. Let $Y' \subseteq Z$ be a deformation retract of Z obtained by iteratively collapsing free faces. Since $\pi_1 Y' = \pi_1 Z$ is freely indecomposable, any free edges of Y' are separating, and one complementary component of each free edge is simply connected. Let Y be the unique non-simply-connected component obtained from Y' by deleting free edges, and let $f : Y \looparrowright X$ be the natural immersion. Then Y is as required: $f_*\pi_1 Y$ is conjugate to $f_*\pi_1 Y' = H$, and Y has neither free faces nor free edges. □

The next two lemmas show that a finitely generated subgroup can be represented by a direct limit of immersions of irreducible 2-complexes. We start with the freely indecomposable case.

Lemma 4.3. *Let X be a 2-complex, $G = \pi_1(X)$, and $H \leq G$ a finitely generated, freely indecomposable subgroup. Then there is a sequence of π_1-surjective immersions of compact, connected two-complexes*

$$Y_0 \looparrowright Y_1 \looparrowright \cdots \looparrowright Y_i \looparrowright \cdots X$$

with the following properties.

(i) Each Y_i is irreducible.

(ii) $H = \varinjlim \pi_1 Y_i$

(iii) The number of free edges of Y_i is uniformly bounded.

Proof. If H is trivial then so is the result. Otherwise, let $f : Y \looparrowright X$ be the immersion guaranteed by Lemma 4.2, so that $f_* \pi_1 Y$ is conjugate to H. Let $f_0 : Y_0 \looparrowright X$ be a wedge of Y with an interval I so that, fixing a basepoint at the end of the interval, $f_{0*}\pi_1 Y_0 = H$. Let (r_i) be an enumeration of representatives of the conjugacy classes of $\ker f_{0*}$, where each r_i is an immersed combinatorial loop in the 1-skeleton of $Y \subseteq Y_0$.

We now construct the immersions $Y_i \looparrowright Y_{i+1}$ inductively, assuming that $\{r_0, \ldots, r_i\}$ represent elements of the kernel of $\pi_1 Y_i \to G$. Let $E_{i+1} \to X$ be a reduced van Kampen diagram for r_{i+1}, and let $Z = Y_i \cup_{r_{i+1}} E_{i+1}$. Since r_{i+1} did not cross any free edges of Y_i, Z does not have any free faces. We now apply Lemma 4.1 to the natural map $Z \to X$, which yields

$$Z \to Y_{i+1} \looparrowright X$$

for the desired 2-complex Y_{i+1}. Next, we prove properties (i), (ii) and (iii).

By construction, Y_0 has no free faces. Therefore, we may prove (i) by induction: assuming that Y_i has no free faces, we claim that Y_{i+1} has no free faces. Suppose by way of contradiction that an edge e is a free face of a 2-cell c in Y_{i+1}. Since Y_i immerses into Y_{i+1}, and Y_i has no free faces, it follows that e is not in the image of Y_i. Therefore, e is the image of an edge e' in the interior of E_{i+1}. The two neighbouring 2-cells of E_{i+1} both map to c folding across e', contradicting the hypothesis that E_{i+1} is reduced.

Property (ii) is immediate by construction. For (iii), simply note that any free edges of Y_i are the image of an edge in the interval I. □

Finally, we deal with possibly freely decomposable subgroups, by appealing to the Grushko decomposition.

Lemma 4.4. *Let X be a 2-complex, $G = \pi_1(X)$, and $H \le G$ a finitely generated subgroup. Then there is a sequence of π_1-surjective immersions of compact, connected two-complexes*

$$Y_0 \looparrowright Y_1 \looparrowright \cdots \looparrowright Y_i \looparrowright \cdots X$$

with the following properties.

(i) Each Y_i is irreducible.

(ii) $H = \varinjlim \pi_1 Y_i$

(iii) The number of edges of Y_i that are not incident at a 2-cell is uniformly bounded.

Proof. Let $H = H_0 * \cdots * H_p * F_q$ be the Grushko decomposition of H. For each $1 \leq r \leq p$, let

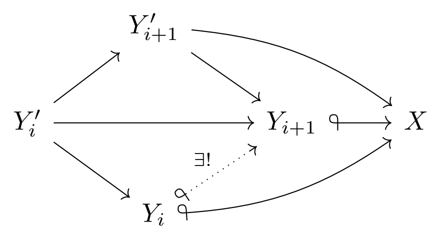
$$Y_0^r \looparrowright Y_1^r \looparrowright \cdots \looparrowright Y_i^r \looparrowright \cdots X$$

be the sequence provided by applying Lemma 4.3 to H_r. Let $Z \looparrowright X$ be a graph immersed in the 1-skeleton with $\pi_1 Z = F_q$. For each i, let

$$Y_i' = Y_i^1 \vee \cdots \vee Y_i^p \vee Z$$

and let $Y_i \looparrowright X$ be the immersion obtained by applying Lemma 4.1 to Y_i'.

The required immersion $Y_i \looparrowright Y_{i+1}$ exists by the universal property of Lemma 4.1: see the following commutative diagram.



Properties (i), (ii) and (iii) are clear from the construction. ☐

We are now ready to prove our main result.

Proof of Theorem 1.2. Realize G as the fundamental group of a one-relator orbicomplex X. Let H be a finitely generated subgroup of G. Let $G_0 \leq G$ be a torsion-free subgroup of finite index, corresponding to the unwrapped cover $X_0 \looparrowright X$ provided by Theorem 2.2. Since a finite extension of a finitely presented group is finitely presented, we may replace H by $H \cap G_0$, and so assume that $H \leq G_0$. Consider the sequence of immersions

$$Y_0 \looparrowright Y_1 \looparrowright \cdots \looparrowright Y_i \looparrowright \cdots X_0$$

provided by Lemma 4.4, taking X_0 for X. By Corollary 3.2, the number of 2-cells of each Y_i is bounded. Each 2-cell of Y_i is a copy of the unique 2-cell of X, hence has boundary of bounded length. Combining this with item (iii) of Lemma 4.4, we see that the number of 1-cells (and hence also 0-cells) of Y_i is also bounded. Since X_0 is finite, there are only finitely many combinatorial types of immersions $Y_i \looparrowright X_0$. Because $Y_i \looparrowright X_0$ factors through $Y_{i+1} \looparrowright X_0$, there is an infinite subsequence

$$Y_{i_1} \looparrowright Y_{i_2} \looparrowright \cdots \looparrowright Y_{i_j} \looparrowright \cdots X_0$$

so that each map $Y_{i_j} \looparrowright Y_{i_{j+1}}$ is a homeomorphism; therefore, $H = \varinjlim \pi_1 Y_{i_j} = \pi_1(Y_{i_1})$ is finitely presented, as required. ☐

5 Groups with good stackings

The results of [LW17] apply equally well to a class of groups which is rather larger than the class of one-relator groups.

Definition 5.1 (Stacking)**.** Let X be a 2-dimensional orbicomplex and let

$$\Lambda : \coprod S^1 \to X^{(1)}$$

be the coproduct of the attaching maps of the 2-cells. A *stacking* of X is a lift of Λ to an embedding

$$\widehat{\Lambda} : \mathbb{S} \equiv \coprod S^1 \hookrightarrow X^{(1)} \times \mathbb{R} \ ;$$

write $\widehat{\Lambda}(x) = (\Lambda(x), h(x))$. A stacking is called *good* if, for each component S of the domain of Λ, there is a point $a \in S$ so that $h(a) \geq h(x)$ for all $x \in \mathbb{S}$ with $\Lambda(a) = \Lambda(x)$, and there is also a point $b \in S$ so that so that $h(b) \leq h(x)$ for all $x \in \mathbb{S}$ with $\Lambda(b) = \Lambda(x)$.

The results of [LW17] apply to the fundamental groups of orbicomplexes with good stackings.

Definition 5.2. We say that a group has a *good stacking* if it is the fundamental group of a compact, 2-dimensional orbicomplex that admits a good stacking. We say it has a *good branched stacking* if it has a good stacking, and every 2-cell has a cone point of index at least 2.

Every one-relator group admits a good stacking [LW17, Lemma 3.4], which is branched if the group has torsion. Corollary 3.2 applies to groups with a good branched stacking. The proof of Theorem 1.2 applies verbatim to groups with good branched stackings, except that groups with good branched stackings are not known to admit unwrapped covers – that is, the analogue of Theorem 2.2 is unknown.

However, we conjecture that Wise's proof that one-relator groups with torsion are residually finite goes through for groups with branched good stackings.

Conjecture 5.3. *If G has a good branched stacking then G is hyperbolic, and has a virtual quasiconvex hierarchy, in the sense of [Wis12].*

By the results of [Wis12], Conjecture 5.3 would imply that every such group is virtually torsion-free, and hence coherent by the same proof as Theorem 1.2.

References

[Bau74] Gilbert Baumslag, *Some problems on one-relator groups*, Proceedings of the Second International Conference on the Theory of Groups (Australian Nat. Univ., Canberra, 1973) (Berlin), Lecture Notes in Math., vol. 372, Springer, 1974, pp. 75—-81.

[BH99] Martin R. Bridson and André Haefliger, *Metric spaces of non-positive curvature*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 319, Springer-Verlag, Berlin, 1999.

[FKS72] J. Fischer, A. Karrass, and D. Solitar, *On one-relator groups having elements of finite order*, Proc. Amer. Math. Soc. **33** (1972), 297–301. MR 0311780

[Fri15] Joel Friedman, *Sheaves on graphs, their homological invariants, and a proof of the Hanna Neumann conjecture: with an appendix by Warren Dicks*, Mem. Amer. Math. Soc. **233** (2015), no. 1100, xii+106, With an appendix by Warren Dicks. MR 3289057

[Hae91] André Haefliger, *Complexes of groups and orbihedra*, Group theory from a geometrical viewpoint (Trieste, 1990), World Sci. Publ., River Edge, NJ, 1991, pp. 504–540. MR 1170375

[HW16] Joseph Helfer and Daniel T. Wise, *Counting cycles in labeled graphs: the nonpositive immersion property for one-relator groups*, Int. Math. Res. Not. IMRN (2016), no. 9, 2813–2827.

[LW17] Larsen Louder and Henry Wilton, *Stackings and the W-cycles Conjecture*, Canad. Math. Bull. **60** (2017), no. 3, 604–612.

[Min12] Igor Mineyev, *Submultiplicativity and the Hanna Neumann conjecture*, Ann. of Math. (2) **175** (2012), no. 1, 393–414. MR 2874647

[MKS04] Wilhelm Magnus, Abraham Karrass, and Donald Solitar, *Combinatorial group theory*, second ed., Dover Publications, Inc., Mineola, NY, 2004, Presentations of groups in terms of generators and relations. MR 2109550

[New68] B. B. Newman, *Some results on one-relator groups*, Bull. Amer. Math. Soc. **74** (1968), 568–571.

[Sco73] G. P. Scott, *Finitely generated 3-manifold groups are finitely presented*, Journal of the London Mathematical Society. Second Series **6** (1973), 437–440.

[Sta83] John R. Stallings, *Topology of finite graphs*, Invent. Math. **71** (1983), no. 3, 551–565.

[Wis03] Daniel T. Wise, *Coherence, local-indicability and non-positive immersions*, Preprint, `http://www.gidon.com/dani/tl.cgi?athe=pspapers/NonPositiveCoherence.ps`, 2003.

[Wis05] Daniel T. Wise, *The coherence of one-relator groups with torsion and the Hanna Neumann conjecture*, Bull. London Math. Soc. **37** (2005), no. 5, 697–705. MR 2164831

[Wis12] D. T. Wise, *The structure of groups with a quasi-convex hierarchy*, Preprint, `http://goo.gl/3ctNvX`, April 2012.

[Wis18] Daniel T. Wise, *Coherence, local-indicability and non-positive immersions*, Personal communication, 2018.

L. Louder, Department of Mathematics, University College London, Gower Street, London WC1E 6BT, UK
E-mail address: `l.louder@ucl.ac.uk`

H. Wilton, DPMMS, Centre for Mathematical Sciences, Wilberforce Road, Cambridge CB3 0WB, UK
E-mail address: `h.wilton@maths.cam.ac.uk`