AN INVESTIGATION OF REAL VALUED SEQUENCES

A Thesis

Presented to

the Faculty of the Department of Mathematics

Appalachian State Teachers College

In Partial Fulfillment

of the Requirements for the Degree

Master of Arts

By

James Monroe Boyte

July 1966

DEDICATION

To my wife, Jan

ACKNOWLEDGMENT

I wish to express my sincere appreciation to Dr. Ralph W. Ball for helping make the previous year a most rewarding one at Appalachian State Teachers College. The success of this investigation deserves much credit from the courses taught by him.

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The thesis contains original results with respect to the set of real-valued sequences and with respect to some of its subsets. These results are obtained by defining operations on the sets by means of which algebraic structures are imposed. Emphasis is given to the limit function producing theorems and lemmas of interest. Among these are two theorems of especial interest, viz.,

Theorem 12

$$\text{Reals} \cong \frac{S_c}{\ker(\text{limit})}$$

Theorem 18

There exists a basis for the set of all convergent sequences which contains only one sequence converging to some value other than 0.

TABLE OF CONTENTS

INTRODUCTION

It is my purpose to examine the set of all real valued sequences with respect to certain algebraic structures. Particular consideration will be given to the set of all convergent sequences. Several of the results will be obtained by examining the properties of limit and restricting the domain of limit to the set of sequences converging to zero.

The motivation for doing research in this area arose from the idea that limit might be a ring homomorphism and a linear transformation.

Search at Appalachian failed to disclose information on this topic. Therefore, the following results are regarded as being original:

I. Lemmas 1, 2, 3 and 4

II. Theorems 2, 3, 6, 9, 10, 11, 12, 13, 14, 16, 18, 19, 20 and 21

III. Propositions 1, 2 and 3.

AN INVESTIGATION OF REAL VALUED SEQUENCES

Definition 1

A sequence is a function whose domain is the positive integers.

Hereafter all sequences will have a range that is a subset of the real numbers.

Definition 2

A sequence $\{a_n\}$ converges to its limit a, a real number, if and only if given $\varepsilon > 0$ then there exists a positive integer N such that $n \geq N$ implies $|a_n - a| < \varepsilon$.

Definition 3

A sequence diverges if and only if it does not converge.

Definition 4

A sequence $\{a_n\}$ diverges to $+\infty$ if and only if given a real number B there exists a positive integer N such that $n \geq N$ implies $a_n > B$.

A sequence $\{a_n\}$ diverges to $-\infty$ if and only if given a real number B there exists a positive integer N such that $n \geq N$ implies $a_n < B$.

Theorem 1

The limit of a convergent sequence is unique.[1]

Proof:

Suppose the sequence $\{a_n\}$ converges to a, and $\{a_n\}$ converges to a' such that $a \neq a'$. Put $\varepsilon = |a-a'| > 0$. Since the sequence $\{a_n\}$ converges to a, there exists a positive integer N_1 such that $n \geq N_1$ implies $|a_n-a'| < \frac{\varepsilon}{2}$. Also there exists N_2 such that $n \geq N_2$ implies $|a_n-a| < \frac{\varepsilon}{2}$. Let N equal the maximum $\{N_1, N_2\}$. Then $n \geq N$ implies $|a-a'| = |a-a_n+a_n-a'| \leq |a-a_n| + |a_n-a'| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. Thus we have $a = a'$. This is a contradiction to $a \neq a'$. Hence the limit of a convergent sequence is unique.

Definition 5

Let $\{a_n\}$ and $\{b_n\}$ be sequences and k a real number, then $\{a_n\} \oplus \{b_n\} = \{a_n+b_n\}$, $k\{a_n\} = \{k \cdot a_n\}$ and $\{a_n\} \odot \{b_n\} = \{a_n \cdot b_n\}$.

Lemma 1

The set of all sequences is an abelian group under $\oplus$.

Proof:

Let S be the set of all sequences.

1. Clearly $\oplus$ is a function and $\oplus : S \times S \longrightarrow S$.
2. $(\{a_n\} \oplus \{b_n\}) \oplus \{c_n\} = \{a_n+b_n\} \oplus \{c_n\} = \{(a_n+b_n)+c_n\} = \{a_n+(b_n+c_n)\} = \{a_n\} \oplus \{b_n+c_n\} = \{a_n\} \oplus (\{b_n\} \oplus \{c_n\})$.
3. Let $a_n = 0$ for all n, and denote $\{a_n\} = \{0_n\}$. Then $\{0_n\} \oplus \{b_n\} = \{0_n+b_n\} = \{b_n\}$.

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[1]John M. H. Olmsted, Intermediate Analysis (New York: Appleton-Century-Crofts, Inc., 1956), p. 34.

4. $-\{a_n\} \oplus \{a_n\} = \{-a_n\} \oplus \{a_n\} = \{-a_n+a_n\} = \{0_n\}$.

5. $\{a_n\} \oplus \{b_n\} = \{a_n+b_n\} = \{b_n+a_n\} = \{b_n\} \oplus \{a_n\}$.

<u>Lemma 2</u>

The set of all sequences is a commutative monoid under $\odot$.

Proof:

Let S be the set of all sequences.

1. Clearly $\odot$ is a function and $\odot : S \times S \rightarrow S$.
2. $(\{a_n\} \odot \{b_n\}) \odot \{c_n\} = \{a_n \cdot b_n\} \odot \{c_n\} = \{(a_n \cdot b_n) \cdot c_n\} =$ $\{a_n \cdot (b_n \cdot c_n)\} = \{a_n\} \odot \{b_n \cdot c_n\} = \{a_n\} \odot (\{b_n\} \odot \{c_n\})$.
3. Clearly $\{a_n\}$ where $a_n = 1$ for all n is the identity.
4. $\{a_n\} \odot \{b_n\} = \{a_n \cdot b_n\} = \{b_n \cdot a_n\} = \{b_n\} \cdot \{a_n\}$.

Comment:

A commutative monoid is the strongest conclusion since the set of all sequences has divisors of zero. That is, given $\{a_n\} \neq \{0_n\}$ we cannot say $\{a_n\}^{-1}$ exists.

<u>Lemma 3</u>

The left and right distributive laws of $\oplus$ over $\odot$ hold for all sequences.

Proof:

1. $(\{a_n\}+\{b_n\}) \odot \{c_n\} = \{a_n+b_n\} \odot \{c_n\} = \{(a_n+b_n) \cdot c_n\} =$ $\{a_n \cdot c_n+b_n \cdot c_n\} = \{a_n \cdot c_n\} \oplus \{b_n \cdot c_n\} = (\{a_n\} \odot \{c_n\}) \oplus$ $(\{b_n\} \odot \{c_n\})$.
2. Right distributive property is similar to 1.

<u>Theorem 2</u>

The set of all sequences is a commutative ring with identity.

Proof:

Lemma 1, lemma 2, and lemma 3.

Theorem 3

The set of all sequences is a vector space over the reals.

Proof:

Let S be the set of all sequences. By lemma 1, S is an abelian group under $\oplus$. Clearly $\odot : R \times S \longrightarrow S$. For all $\{a_n\}, \{b_n\} \in S$ and $a, b \in R$ we have:

1. $(a+b)\{b_n\} = \{(a+b)\cdot b_n\} = \{a\cdot b_n + b\cdot b_n\} = \{a\cdot b_n\} \oplus \{b\cdot b_n\} = a \odot \{b_n\} + b \odot \{b_n\}$.
2. $a \odot (\{a_n\} \oplus \{b_n\}) = a \odot (\{a_n+b_n\}) = \{a(a_n+b_n)\} = \{a\cdot a_n + ab_n\} = \{a\cdot a_n\} \oplus \{a\cdot b_n\} = a \odot \{a_n\} \oplus a \odot \{b_n\}$.
3. $(a\cdot b) \odot \{a_n\} = \{(a\cdot b)\cdot a_n\} = \{a(b\cdot a_n)\} = a \odot \{b\cdot a_n\} = a \odot (b \odot \{a_n\})$.
4. $1 \odot \{a_n\} = \{1\cdot a_n\} = \{a_n\}$.

Theorem 4

The sum of two convergent sequences is convergent and the limit of the sum is the sum of the limits.

Proof:

Let $\mathcal{E}$ be given greater than zero. Suppose $\{a_n\}$ converges to a and $\{b_n\}$ converges to b. Then there exists N_1 such that $n \geq N_1$ implies $|a_n - a| < \frac{\mathcal{E}}{2}$ and there exists N_2 such that $n \geq N_2$ implies $|b_n - b| < \frac{\mathcal{E}}{2}$. Let N equal maximum $\{N_1, N_2\}$. Then $|(a_n+b_n)-(a+b)| = |(a_n-a)+(b_n-b)| \leq |a_n-a| + |b_n-b| < \frac{\mathcal{E}}{2} + \frac{\mathcal{E}}{2} = \mathcal{E}$.[2]

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[2]*Ibid.*, p. 35.

Theorem 5

If $\{a_n\}$ converges to a, then for $k \in R$, $k \odot \{a_n\}$ converges to $k \cdot a$.[3]

Proof:

Let $\varepsilon > 0$. There is a positive integer N such that $n \geq N$ implies $|a_n - a| < \frac{\varepsilon}{|k|+1}$. Thus $n \geq N$ implies $|ka_n - ka| = |k(a_n - a)| = |k||a_n - a| < |k|\frac{\varepsilon}{|k|+1} = \frac{|k|}{|k|+1} \cdot \varepsilon < \varepsilon$.

Theorem 6

Let S be the set of all sequences and S_c be the set of all convergent sequences then S_c is a subspace of S.

Proof:

For all $\{a_n\}, \{b_n\} \in S$ and $a \in R$ we have, by theorem 4, that $\{a_n\} \oplus \{b_n\} \in S_c$ and by theorem 5 that $a \odot \{a_n\} \in S_c$. It follows that S_c is a subspace of S.[4]

Theorem 7

If sequence $\{a_n\}$ is convergent, then $\{a_n\}$ is bounded.[5]

Proof:

Suppose $\{a_n\}$ converges to a. There exists N such that $n \geq N$ implies $|a_n - a| < 1$. Thus $|a_n| - |a| \leq ||a_n| - |a|| \leq |a_n - a| < 1$. So $n \geq N$ implies $|a_n| < |a| + 1$. Let B equal the maximum $\{|a_1|, |a_2|, \cdots, |a_{N-1}|, |a|+1\}$. Then $\{a_n\} \leq B$ for all n.

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[3] Ibid., p. 37.

[4] A. I. Mal'cev, Foundations of Linear Algebra (San Francisco: W. H. Freeman and Company, 1963), p. 47.

[5] Olmsted, op. cit., p. 35.

Theorem 8

The product of two convergent sequences is convergent and the limit of the product is the product of the limit.[6]

Proof:

Suppose $\{a_n\}$ converges to a and $\{b_n\}$ converges to b. By theorem 7, there exists $B > 0$ such that $|b_n| \leq B$ for all n. Let $\mathcal{E} > 0$. Now pick $N > 0$ such that $n \geq N$ implies $|b_n - b| < \frac{\mathcal{E}}{2(|a|+1)}$ and $n \geq N$ implies $|a_n - a| < \frac{\mathcal{E}}{2B}$. Then $n \geq N$ implies $|a_nb_n - ab| = |a_nb_n - ab_n + ab_n - ab| = |(a_n - a)b_n + a(b_n - b)| \leq |(a_n - a)b_n| + |a(b_n - b)| = |b_n||a_n - a| + |a||b_n - b| \leq |a_n - a|B + |a||b_n - b| < \frac{\mathcal{E}B}{2B} + |a|\frac{\mathcal{E}}{2(|a|+1)} \leq \frac{\mathcal{E}}{2} + \frac{|a|}{|a|+1} \cdot \frac{\mathcal{E}}{2} < \frac{\mathcal{E}}{2} + \frac{\mathcal{E}}{2} = \mathcal{E}$.

Theorem 9

Let S be the set of all sequences and S_c the set of all convergent sequences. Then S_c is a commutative subring of S with identity.

Proof:

Suppose $\{a_n\}$ and $\{b_n\}$ belong to S_c then, by theorem 4 and theorem 5, $\{a_n\} - \{b_n\} = \{a_n\} \oplus \{-b_n\} \in S_c$. By theorem 8, $\{a_n\} \odot \{b_n\} = \{a_n \cdot b_n\} \in S_c$. It follows that S_c is a subring of S.[7] Commutativity is inherited, and $\{a_n\} = \{1_n\}$ is the identity.

Theorem 10

Let S_c be the set of all convergent sequences. Then

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[6] Ibid.

[7] Wilfred E. Barnes, Introduction to Abstract Algebra (Boston: D. C. Heath and Company, 1963), p. 81.

limit is a ring homomorphism mapping the set of all convergent sequences onto the real numbers.

Proof:

By theorem 1, limit is a function. The fact that the reals are complete implies limit: $S_c \rightarrow R$. Clearly limit maps S_c onto the reals. Let a_n, b_n belong to S_c, then by theorem 4, $(\{a_n\} \oplus \{b_n\})$limit $= (\{a_n\})$limit $+ (\{b_n\})$limit, and by theorem 8, $(\{a_n\} \odot \{b_n\})$limit $= (\{a_n\})$limit $\cdot (\{b_n\})$limit.

Theorem 11

Limit is a linear transformation mapping the set of all convergent sequences onto the real numbers.

Proof:

Let S_c be the set of all convergent sequences. Suppose $\{a_n\}, \{b_n\}$ belong to S_c and a,b belong to R, then $(a\{a_n\} \oplus b\{b_n\})$limit $= (\{a \cdot a_n\} \oplus \{b \cdot b_n\})$limit $= (\{a \cdot a_n\})$limit $+ (\{b \cdot b_n\})$limit $= (a \odot \{a_n\})$limit $+ (b \odot \{b_n\})$limit $= a \cdot ((\{a_n\})$limit$) + b \cdot ((\{b_n\})$limit$)$. Limit is a function by theorem 1. Thus limit belongs to $\mathcal{L}(S_c, R)$. Clearly limit maps onto the real numbers.

Theorem 12

$$\text{Reals} \cong \frac{S_c}{\ker(\text{limit})}$$

Proof

By theorem 10, limit is a ring homomorphism mapping S_c onto the reals. By fundamental theorem of ring homomorphism, we have $\frac{S_c}{\ker(\text{limit})} \cong (S_c)\text{limit} = R$.[8]

[8]Ibid., p. 90.

Theorem 13

The set of all sequences is a linear algebra over R.

Proof:

Let S be the set of all sequences. By theorem 3, S is a vector space over the reals, and by lemma 2, $\odot : S \times S \rightarrow S$ and is associative. Let $\{a_n\},\{b_n\},\{c_n\}$ belong to S and a,b belong to R. Then $\{a_n\} \odot (a\{c_n\} \oplus b\{b_n\}) = \{a_n\} \odot (\{ac_n\} \oplus \{bb_n\}) = \{a_n\} \odot (\{ac_n+bb_n\}) = \{a_n(ac_n) + a_n(bb_n)\} = \{a(a_nc_n) + b(a_nb_n)\} = \{a(a_nc_n)\} \oplus \{b(a_nb_n)\} = a \odot \{a_nc_n\} \oplus b \odot \{a_nb_n\} = a \odot (\{a_n\} \cdot \{c_n\}) \oplus b \odot (\{a_n\} \odot \{b_n\})$. Similarly, $(a\{a_n\} \oplus b\{b_n\}) \odot \{c_n\} = a(\{a_n\} \odot \{c_n\}) \oplus b(\{b_n\} \odot \{c_n\})$.

Theorem 14

The set of all convergent sequences is a linear algebra over the reals.

Proof:

Let S_c be the set of all convergent sequences. By theorem 6, S_c is a subspace of the set of all sequences. By theorem 9, $\odot : S_c \times S_c \rightarrow S_c$ and is associative. The bilinear property is inherited.

Theorem 15

Let R be a commutative ring with identity $1 \neq 0$. An ideal I in R is proper maximal if and only if $\frac{R}{I}$ is a field.[9]

Theorem 16

The kernel of limit is proper maximal in the set of all convergent sequences.

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[9]Ibid., p. 126.

Proof:

Let S_c be the set of all convergent sequences. The kernel of limit of S_c is an ideal.[10] By theorem 12, $\frac{S_c}{\ker(\text{limit})}$ is a field since the reals is a field. We have S_c a commutative ring with identity. Thus by theorem 15, the theorem follows.

Theorem 17

Let V and W be vector spaces with f a linear transformation mapping V into W. Let $C = \{v_\alpha\}_{\alpha \varepsilon A}$ be a basis for the null space of f. If $D = \{V_\alpha\}_{\alpha \varepsilon B}$ is a set of vectors such that $C \cup D$ is a basis for V with $C \cap D = \emptyset$, then $\{(v_\alpha)f\}_{\alpha \varepsilon B}$ is a basis for the range of f.

Proof:

Let w belong to the range of f. There exists a $v \varepsilon V$ such that $(v)f = w$. Now $v = \sum_{i=1}^{n} a_i v_{\alpha_i}$. Then $w = (v)f = \left(\sum_{i=1}^{n} a_i v_{\alpha_i}\right) f = \sum_{i=1}^{n} a_i (v_{\alpha_i})f = \sum_{\alpha_i \not\varepsilon A} a_i (v_{\alpha_i})f$, since $\alpha_i \varepsilon A$ implies $(v_{\alpha_i})f = 0$. So $\{(v_\alpha)f\}_{\alpha \varepsilon B}$ spans the range of f. Let $\sum_{i=1}^{n} a_i (v_{\alpha_i})f = 0$ with $\alpha_i \varepsilon B$. Let $v = \sum_{i=1}^{n} a_i v_{\alpha_i}$. Then $(v)f = \left(\sum_{i=1}^{n} a_i v_{\alpha_i}\right) f = \sum_{i=1}^{n} a_i (v_{\alpha_i})f = 0$. So v is in the null space of f. Thus there exists $\alpha_j \varepsilon A$ such that $v = \sum_{j=1}^{m} b_j v_{\alpha_i} = \sum_{i=1}^{n} a_i v_{\alpha_i}$ with $\alpha_i \varepsilon B$. Hence $\sum_{j=1}^{m} b_j v_{\alpha_j} + \sum_{i=1}^{n} -a_i v_{\alpha_i} = 0$ implies $a_i = 0$ for all i. So

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[10] Ibid., p. 90.

$\{(v_\alpha)f\}_{\alpha \varepsilon B}$ is a linearly independent set.

Comment:

The linear transformation f, restricted to $\{N_\alpha\}_{\alpha \varepsilon B}$, is one to one.

Theorem 18

There exists a basis for the set of all convergent sequences which contains only one sequence converging to some value other than 0.

Proof:

Consider the basis $\{v_\alpha\}_{\alpha \varepsilon A \cup B}$ in theorem 17. We have shown that $\{(v_\alpha)f\}_{\alpha \varepsilon B}$ is a basis for the range. Now considering f = limit and $\{v_\alpha\}_{\alpha \varepsilon A \cup B}$ a basis for the set of convergent sequences, we have $\{(v_\alpha)\text{limit}\}_{\alpha \varepsilon B}$ is a basis for the reals. The basis for the reals consists of one number $\neq 0$. Thus there exists a single $N_\alpha \varepsilon \{N_\alpha\}_{\alpha \varepsilon B}$ since limit restricted to $\{N_\alpha\}_{\alpha \varepsilon B}$ is one to one. Hence there exists only one sequence in $\{v_\alpha\}_{\alpha \varepsilon A \cup B}$ which converges to something other than zero.

Comment:

A sequence converges if and only if it can be written as a finite linear combination of convergent sequences with at most one sequence in the combination not converging to zero.

Proposition 1

The identity sequence would serve as the sequence in theorem 18 not converging to zero.

Definition 6

The cardinality of the natural numbers is N_0 and the cardinality of the real numbers is $\mathfrak{c}$.

Theorem 19

The dimension of the null space of limit is at least N_0.

Proof:

Let A be the set of all sequences having the number 1 in exactly one entry and zeros elsewhere. Clearly A is a linearly independent subset of the set of all sequences converging to zero and the cardinality of A is N_0.

Lemma 4

Let S be the space of all sequences, then S has cardinality $\mathfrak{c}$.

Proof:

The cardinality of the reals is $\mathfrak{c}$. Let S be the set of all sequences, then $S = R \times R \times R \cdots$ (N_0 times). Thus $|S| = \mathfrak{c} \cdot \mathfrak{c} \cdot \mathfrak{c} \cdots$ (N_0 times) $= \mathfrak{c}^{N_0} = (2^{N_0})^{N_0} = 2^{N_0 \cdot N_0} = 2^{\mathfrak{c}} = \mathfrak{c}$.

Theorem 20

Let S be the space of all sequences, S_c the space of all convergent sequences, and S_N the null space of limit, then the cardinality for the basis of each is at least N_0 and at most $\mathfrak{c}$.

Proof:

Apply theorem 19, and lemma 4.

Theorem 21

Let S_N be the null space of limit and S_c the set of all convergent sequences, then the dimension of S_N is equal to the dimension of S_c.

Proof:

If dimension of $S_N = N_0$ then by applying theorem 18, $N_0 + 1 = N_0$ is the dimension for S_c. If dimension of $S_N = \mathfrak{c}$ then $\mathfrak{c} + 1 = \mathfrak{c}$ is the dimension for S_c.

Proposition 2

The dimension for the null space of limit is N_0.

Proposition 3

The dimension for the space of all sequences is $\mathfrak{c}$.

B I B L I O G R A P H Y

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