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Stability and convergence properties of forced Lur'e systems

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Declaration of any previous Submission of the Work

The material presented here for examination for the award of a higher degree by research has not been incorporated into a submission for another degree.

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Max Edward Gilmore

Declaration of Authorship

I am the author of this thesis, and the work described therein was carried out by myself personally, in collaboration with my supervisors Hartmut Logemann and Chris Guiver

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Max Edward Gilmore

Summary

We investigate stability and convergence properties of forced Lur'e systems, that is, systems comprising a linear system in the forward path, a static nonlinearity in the feedback path and a forcing or input. In both the finite- and infinite-dimensional settings, we develop various sufficient conditions for when such systems are input-to-state stable, incrementally input-to-state stable, and exhibit the converging-input converging-state property. We also study the effect that asymptotically almost periodic inputs have on corresponding state and output trajectories of the aforementioned systems. Finally, we note that we consider very general versions of forced Lur'e systems, and so we are able to apply our results to a variety of applications. For instance, we deduce stability and convergence properties of 'four-block' Lur'e systems, which are forced Lur'e systems where the input and output spaces are split in two and only one part of the output is utilised for feedback and is fed back into one part of the input. We also deduce stability properties of sampled-data integral control systems.

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Contents

Notation

We denote the sets of real and complex numbers by $\mathbb{R}$ and $\mathbb{C}$, respectively. Furthermore, we denote the set of integers by $\mathbb{Z}$ and the set positive integers by $\mathbb{N}$. We define $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$ and $\mathbb{R}_+ := [0, \infty)$. For $\alpha > 0$, we define $\mathbb{D}_\alpha := \{z \in \mathbb{C} : |z| < \alpha\}$ and $\mathbb{E}_\alpha := \{z \in \mathbb{C} : |z| > \alpha\}$, and for $\alpha \in \mathbb{R}$, we define $\mathbb{C}_\alpha := \{s \in \mathbb{C} \ : \ \mathrm{Re}\,(s) > \alpha\}$. For convenience, we label $\mathbb{D} := \mathbb{D}_1$ and $\mathbb{E} := \mathbb{E}_1$. Furthermore, we set $\partial\mathbb{D} := \{z \in \mathbb{C} : |z| = 1\}$ and $\mathrm{clos}(\mathbb{E}) := \{z \in \mathbb{C} : |z| \geq 1\}$.

If $\tau \in \mathbb{Z}_+$, then we define $\underline{\tau} := \{0, 1, \ldots, \tau\}$ and $\overline{\tau} := \{\tau, \tau + 1, \ldots\}$. Furthermore, for $t \in \mathbb{R}$, we define $\lfloor t \rfloor$ to be the greatest integer less than or equal to t, and $\lceil t \rceil$ to be the smallest integer greater than or equal to t. We note that $\lfloor t/2 \rfloor + \lceil t/2 \rceil = t$ for all $t \in \mathbb{Z}_+$.

For two nonempty sets S_1, S_2, we denote the Minkowski sum of S_1 and S_2 by $S_1 + S_2$, that is,

$$S_1 + S_2 = \{s_1 + s_2 : s_1 \in S_1,\ s_2 \in S_2\}.$$

For a normed vector space U, we denote the norm on U by $\|\cdot\|_U$. When U is finite-dimensional, since all norms are equivalent, we will drop the subscript U notation and will just write $\|\cdot\|$. Moreover, for a Hilbert space W, we denote the inner product on W by $\langle \cdot, \cdot \rangle_W$. If $W = \mathbb{C}^n$ or $\mathbb{R}^n$, then we shall simply write $\langle \cdot, \cdot \rangle$ and take the inner product to be the standard complex or real inner product, respectively.

For $\xi \in U$ and $r > 0$, we define

$$\mathbb{B}_U(\xi, r) := \{\zeta \in U : \|\zeta - \xi\|_U < r\} \quad \text{and} \quad \mathrm{clos}\,(\mathbb{B}_U(\xi, r)) := \{\zeta \in U : \|\zeta - \xi\|_U \leq r\},$$

and, when the context is clear, we shall drop the subscript U. Further, in the case that $U = \mathbb{C}^{m \times p}$ or $\mathbb{R}^{m \times p}$, if the context is clear, we shall write $\mathbb{B}_\mathbb{C}(\xi, r) := \mathbb{B}_{\mathbb{C}^{m\times p}}(\xi, r)$ and $\mathbb{B}_\mathbb{R}(\xi, r) := \mathbb{B}_{\mathbb{R}^{m\times p}}(\xi, r)$.

For Banach spaces V, V_1 and V_2, we denote by $\mathcal{L}(V_1, V_2)$ the space of bounded linear operators mapping $V_1 \to V_2$. We set $\mathcal{L}(V) := \mathcal{L}(V, V)$. In addition to this, we denote the *spectrum* of $M \in \mathcal{L}(V)$ by $\sigma(T)$. In the finite-dimensional case, we say that a square matrix M is *Schur* if its spectral radius is less than one, or, equivalently, if $\sigma(M) \subseteq \mathbb{D}$. Additionally, we say that M is *Hurwitz* if every eigenvalue of M has negative real part.

We denote the adjoint of $M \in \mathcal{L}(W)$ by M^*, and, in the case that $W = \mathbb{R}^n$, we shall denote the transpose of M by M^T. If $M \in \mathcal{L}(W)$ is self-adjoint, then we say that M is *positive semi-definite* if $\langle \xi, M\xi \rangle_W \geq 0$ for all $\xi \in W$, and *positive definite* if $\langle \xi, M\xi \rangle_W > 0$ for all $\xi \in W \backslash \{0\}$. In the former, we write that $M \geq 0$, and in the latter we write that $M > 0$. Similarly, we say that M is *negative semi-definite* if $-M \geq 0$, and *negative definite* if $-M > 0$. We write these properties as $M \leq 0$ and $M < 0$, respectively.

For a given function $v : \mathbb{T} \to U$ and $\tau \in \mathbb{T}$, where $\mathbb{T} = \mathbb{R}_+$, $\mathbb{R}$, $\mathbb{Z}_+$ or $\mathbb{Z}$, we define $\Lambda_\tau v : \mathbb{T} \to U$ by $(\Lambda_\tau v)(t) = v(t+\tau)$ for all $t \in \mathbb{T}$. Depending on the sign of τ, we shall call this function a left or right shift of v and may also say that Λ_τ is the left or right shift operator (of magnitude τ). We set $v^+ := \Lambda_1 v$ for all $v : Z \to U$, where $Z = \mathbb{Z}_+$ or $\mathbb{Z}$. Moreover, for $\tau \in \mathbb{Z}_+$ and $u : \mathbb{Z}_+ \to U$, we define the truncation of u of magnitude τ by

$$(\pi_\tau u)(s) := \begin{cases} u(s), & \text{if } s \in \underline{\tau}, \\ 0, & \text{if } s \in \overline{\tau + 1}. \end{cases}$$

If $\alpha > 0$, we define the *Hardy space*

$$H^\infty_\alpha(V) := \{\mathbf{H} : \mathbb{E}_\alpha \to V : \mathbf{H} \text{ is holomorphic and bounded}\},$$

endowed with the norm

$$\|\mathbf{H}\|_{H^\infty_\alpha} := \sup_{z \in \mathbb{E}_\alpha} \|\mathbf{H}(z)\|_V.$$

In the situation that $\alpha = 1$, we set $H^\infty(V) := H^\infty_1(V)$, and if, additionally, $V = \mathbb{C}^{p \times m}$, then we may sometimes write $H^\infty_{p \times m} := H^\infty_1(V)$.

For $Z = \mathbb{Z}_+$ or $\mathbb{Z}$, we denote the set of functions $v : Z \to U$ by U^Z. Moreover, for $p \in [1, \infty)$, we let $\ell^p(\mathbb{Z}_+, U)$ denote the subspace of $U^{\mathbb{Z}_+}$ such that

$$\sum_{k=0}^{\infty} \|v(k)\|_U^p < \infty,$$

endowed with the norm

$$\|v\|_{\ell^p} := \left(\sum_{k=0}^{\infty} \|v(k)\|_U^p \right)^{1/p}.$$

Furthermore, for $\rho > 0$, we define the *weighted* ℓ^2 space

$$\ell^2_\rho(\mathbb{Z}_+, U) := \left\{ v \in \ell^2(\mathbb{Z}_+, U) : \left(\sum_{k=0}^{\infty} \|v(k)\|_U^2 \rho^{2k} \right)^{1/2} < \infty \right\},$$

with norm

$$\|v\|_{\ell^2_\rho} := \left(\sum_{k=0}^{\infty} \|v(k)\|_U^2 \rho^{2k} \right)^{1/2}.$$

We let $\ell^\infty(Z, U)$ be the subspace of U^Z such that $\sup_{k \in Z} \|v(k)\|_U < \infty$, endowed with the norm $\|v\|_{\ell^\infty} := \sup_{k \in Z} \|v(k)\|_U$.

The preimage of a set $W \subseteq U$ under a function $f : U \to U$ is denoted by $f^{-1}(W)$, and in the situation where $W = \{\xi\}$, a singleton set, we shall abuse notation and write $f^{-1}(\xi) = f^{-1}(\{\xi\})$. The cardinality of $f^{-1}(W)$ is denoted by $\# f^{-1}(W)$.

Finally, we abbreviate the phrase "almost everywhere" to "a.e.".

Chapter 1

Introduction

In this thesis, we investigate stability and convergence properties of forced Lur'e systems [81], with a focus on the discrete-time setting. Illustrated in Figure 1.1, Lur'e systems comprise a linear system in the forward path and a static nonlinearity in the feedback path. The study of these types of system constitutes absolute stability theory,

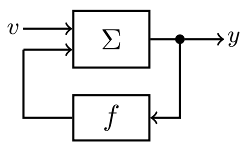

Figure 1.1: *Lur'e system with linear part* Σ*, nonlinearity* f*, output* y *and input* v

which seeks to conclude stability via the interplay of frequency-domain properties of the linear component and sector properties of the nonlinearity. As early as the 1940s, Lyapunov approaches have been used to infer global asymptotic stability of unforced Lur'e systems (see, for example, [53, 66, 71]). Moreover, input/output methods, pioneered by Sandberg and Zames in the 1960s, have been utilised to deduce L^2 and L^∞ stability (see, such as, [31, 121]).

An interesting notion associated with general controlled nonlinear systems is that of input-to-state stability (ISS). Originating in the paper [110], roughly, ISS guarantees a natural boundedness property of the state, in terms of initial conditions and inputs. It is a well-documented area of research, see for example [28, 61, 62], the survey papers [27, 113] and the forthcoming references. In the context of forced Lur'e systems, much work has been completed in determining the extent to which assumptions that guarantee absolute stability, can be extended to also ensure ISS [7, 58, 59, 107, 108, 109]. A well-known quality of ISS systems is the 0-converging-input converging-state property which, roughly, means that when inputs converge to zero, so do corresponding state trajectories. Recently, in the paper [15], a general notion of a converging-input converging-state (CICS) property was defined, and sufficient conditions were presented that guarantee that forced continuous-time Lur'e systems exhibit this property. We note that ISS results from the paper [107] were applied to prove the main results.

A related concept to ISS is incremental ISS, which is concerned with bounding the difference of two state trajectories in terms of the difference of initial conditions and the difference of inputs. For background information regarding incremental stability notions for general nonlinear systems, we refer the reader to [5], which constructs a suite of Lyapunov methods for incremental ISS for finite-dimensional, continuous-time nonlinear control systems. Related ideas, which have been explored in the contexts of contraction methods and convergent systems, can be found in [2, 63, 99] and the references therein.

Frequently in the literature (see, for example, [7, 15, 107, 108]), forced Lur'e systems of the form

$$x^+ = Ax + Bf(y) + Bv, \quad y = Cx, \tag{1.1}$$

or

$$x^+ = Ax + Bf(y) + v, \quad y = Cx, \tag{1.2}$$

or continuous-time analogues, where A, B and C are appropriately sized matrices, v is a forcing function, and f is a nonlinear function, have been considered in an ISS or CICS framework. We call the variables x and y the state and output, respectively. Of key interest to us in this thesis, is the investigation of stability and convergence properties of a more general system, namely

$$x^+ = Ax + Bf(y+w) + B_{\rm e}v, \quad y = Cx + Df(y+w) + D_{\rm e}v, \tag{1.3}$$

where, in addition to the setup for the previous two systems, $B_{\rm e}$, D and $D_{\rm e}$ are appropriately sized matrices and w is a disturbance function. The system (1.3) has potentially nonzero feedthrough (i.e. D need not be 0), forcing arising through external matrices ($B_{\rm e}$ and $D_{\rm e}$) and output disturbances (i.e. w). The inclusion of the function w in (1.3) is to model errors. For example, errors naturally arise in practical applications from the action of feeding back the output. As we shall see in this thesis, systems of the form (1.3) encompass many other (even seemingly more general) systems. One such is the 'four-block' Lur'e system [40, 47], which is a forced Lur'e system where the input and output spaces are split in two, and only one part of the output is utilised for feedback purposes and is fed back into one part of the input. We comment that the investigation of stability and convergence properties of the more general system given by (1.3), is an entirely nontrivial extension of that of (1.1) and (1.2), as we shall prove.

In the forthcoming presentation, we shall be interested not only with finite-dimensional Lur'e systems of the form (1.3), but also with infinite-dimensional ones. By this, we mean that $A, B, B_{\rm e}, C, D$ and $D_{\rm e}$ are bounded linear operators, each mapping from a Banach space to a Banach space, f is a nonlinear vector-valued function, and v and w are also vector-valued.

A further theme of this thesis, is the extent to which almost periodic forcing leads to almost periodic state and output trajectories of forced Lur'e systems. Almost periodic functions are a generalisation of the notion of continuous periodic functions, and were first conceived by Harold Bohr in the 1920s [16]. They are frequently used in the theory of differential and difference equations (see, for example, [34, 52, 105, 128]), and much work has been attributed to the theory. Indeed, several further generalisations of the notion have been developed (see, for example, [14]). In this thesis, we shall concern ourselves with providing sufficient conditions for when state and output trajectories of forced Lur'e systems converge asymptotically to almost periodic functions, when under forcing that converge asymptotically to almost periodic functions.

The layout of the thesis is as follows. In Part I we consider finite-dimensional forced Lur'e systems. In particular, in Chapter 2, we extend the ISS results of [108] to systems of the form (1.3), and then utilise these to obtain discrete-time analogues of the

CICS results of [15], but which holds for systems of the form (1.3). Moreover, we also investigate stability and convergence properties of the output of (1.3), which is a nontrivial exercise in the situation that $D \neq 0$. In [15], the continuous-time version of (1.2) is considered, and in [108], the authors investigate (1.2). Neither paper considers output stability or convergence, owing to the fact that without feedthrough, these are immediately obtained from stability and convergence properties of the state, respectively. In Chapter 3, we present sufficient conditions for when (1.3) exhibits a semi-global version of incremental ISS, and determine hypotheses that guarantee that asymptotically almost periodic inputs generate asymptotically almost periodic state and output trajectories. The final chapter of Part I is Chapter 4, where we once again investigate ISS and CICS of (1.3), but with different hypotheses to those presented in Chapter 2. Indeed, we develop sufficient conditions that allow for potentially superlinear nonlinearities, which differs from Chapter 2, where the nonlinearity is assumed to satisfy a linear bound. We comment that the chapter is inspired by [7], where similar assumptions are used to deduce ISS of the continuous-time version of (1.1).

Moving on to Part II, our attention here concerns infinite-dimensional forced Lur'e systems. In Chapter 5, by imposing stronger assumptions than that given in Part I of this thesis, we deduce exponential incremental ISS and convergence properties of the infinite-dimensional version of (1.3). The final chapter of this thesis is Chapter 6, which utilises the results of Chapter 5 to obtain stability of an infinite-dimensional sampled-data integral control system. We delay giving a thorough background of integral control and sampled-data integral control until Chapter 6, and so we refer the reader there for a review of the relevant literature. What we will say here, however, is that in sampled-data control, a continuous-time system is controlled by a discrete-time controller. One may think of this discrete-time controller as a processor of a digital computer.

Finally, this thesis concludes with several appendices. Appendix A provides a proof for the proposition that the existence of a certain function guarantees ISS of (1.3). Moreover, Appendix B presents a convergence result for a discrete-time control system with an asymptotically stable equilibrium, and Appendix C gives a thorough presentation of almost periodic functions over the time domains $\mathbb{R}, \mathbb{R}_+, \mathbb{Z}$ and $\mathbb{Z}_+$.

We close this introduction by stating that some of the results of this thesis have been either accepted for publication or submitted for publication (see [38]-[42]).

Part I

Finite-dimensional Lur'e systems

Chapter 2

Stability and convergence properties of discrete-time Lur'e systems

In this chapter, we concern ourselves with stability and convergence properties of a class of discrete-time control systems illustrated by Figure 1.1. As mentioned in the introduction, these systems are termed Lur'e systems and their study constitutes absolute stability theory. In the interest of not repeating ourselves, we refer the reader to the introduction for more details. Moreover, we also refer the reader there for background and brief descriptions of ISS and the CICS property. What we shall recall here, is that in the paper [15], a general notion of the CICS property was defined, and sufficient conditions were presented that guarantee that forced continuous-time Lur'e systems, of the form

$$\dot{x} = Ax + Bf(y) + v, \quad x(0) = x^0, \quad y = Cx, \tag{2.1}$$

where A, B and C are appropriately sized matrices, f is a nonlinear function and v is a forcing function, exhibit this property. Interestingly, ISS results from the paper [107] were utilised to prove the main results.

Presently, we concern ourselves with discovering discrete-time analogues of the results of [15], although in a more general setting. To illustrate what we mean by this, we are interested in the discrete-time Lur'e system given by (1.3), where A, B, B_{e}, C, D and D_{e} are appropriately sized matrices, f is a nonlinear function, v is a forcing function and w may be perceived to be an output disturbance. As discussed in the introduction, this is a much more general system than (1.1) and (1.2), and so allows for a larger class of potential applications. Moreover, as we shall see in this chapter, the investigation of the stability and convergence properties of this system is a nontrivial generalistion of that of (1.1) and (1.2). Now, as previously mentioned, the main results of [15] utilise ISS results from [107]. With the intention of acting in a similar manner but for the system (1.3), we shall require discrete-time ISS results for (1.3). The paper [108] provides sufficient criteria for (1.2) being ISS, and [106] gives conditions that guarantee ISS of systems of the form

$$x^+ = Ax + Bf(y) + Bv, \quad y = Cx + Df(y) + Dv.$$

Unfortunately these results cannot be applied to (1.3), and so the present chapter thus involves forming generalised versions of the ISS results of [108], which are then applicable to (1.3). Once we have obtained these results, we shall then apply them to yield discrete-time versions (in the more general setting of (1.3)) of the main results of [15].

In addition to the previous, we also investigate stability and convergence properties related to the output, namely, input-to-state/output stability and the converging-input converging-state/output property. The papers [108] and [15] do not consider these notions, owing to that fact that, for systems without feedthrough, any stability and convergence properties of the state apply immediately to the output. However, when the feedthrough matrix is nonzero, these are nontrivial notions, as we shall see.

The chapter is organised as follows. We begin in Section 2.1 by presenting preliminary definitions and results. Indeed, we give relevant theory concerning linear difference equations and, in particular, discuss linear output feedback via a method commonly referred to as "loopshifting" (see, for example, [44, pp.98-106]). We also use a version of the bounded real lemma in order to obtain results regarding quadratic forms. Moving on to Section 2.2, we prove the previously mentioned input-to-state/output stability results for forced discrete-time Lur'e systems of the form (1.3). In Section 2.3, we concern ourselves with convergence properties of these systems, and this section is the aforementioned discrete-time extension of [15]. Finally, in Section 2.4, we highlight the generality of our forced Lur'e system by applying the previous results to 'four-block' Lur'e systems.

2.1 Preliminaries

In this initial section, we begin by recalling notions of comparison functions and by presenting relevant results. Following this, we collect theory of linear discrete-time systems and then give key results involving quadratic forms that will underpin the main results of the next section. Finally, we discuss the Lur'e system that is central to our study.

2.1.1 Comparison functions

We start with the following definition.

Definition 2.1.1. *We define*

$$\mathcal{K} := \{\alpha : \mathbb{R}_+ \to \mathbb{R}_+ : \alpha(0) = 0,\ \alpha \textit{ strictly increasing and continuous}\},$$

and

$$\mathcal{K}_\infty := \left\{\alpha \in \mathcal{K} : \lim_{s\to\infty} \alpha(s) = \infty\right\}.$$

Furthermore, we define $\mathcal{KL}$ to be the set of functions $\psi : \mathbb{R}_+ \times \mathbb{Z}_+ \to \mathbb{R}_+$ such that: for each fixed $t \in \mathbb{Z}_+$, the function $\psi(\cdot, t) \in \mathcal{K}$; and, for each fixed $s \in \mathbb{R}_+$, the function $\psi(s, \cdot)$ is non-increasing and $\lim_{t\to\infty} \psi(s, t) = 0$.

We comment that comparison functions will appear frequently in the forthcoming stability analysis.

We shall now present a series of properties of comparison functions. We begin with the following two results, which we shall not prove since they are easy to do so.

Lemma 2.1.2. *The addition, multiplication, minimum, maximum or composition of a finite number of functions in $\mathcal{K}$ (respectively, $\mathcal{K}_\infty$) is also in $\mathcal{K}$ (respectively, $\mathcal{K}_\infty$). Furthermore, the inverse of a $\mathcal{K}_\infty$ function exists and is itself a $\mathcal{K}_\infty$ function.*

Lemma 2.1.3. *Let $\alpha \in \mathcal{K}$. Then*

$$\alpha(s_1 + s_2) \leq \alpha(2s_1) + \alpha(2s_2) \quad \forall\, s_1, s_2 \in \mathbb{R}_+.$$

The next two results each present an unboundedness property of certain $\mathcal{K}_\infty$ functions.

Lemma 2.1.4. *Let $\varepsilon > 0$ and $\alpha \in \mathcal{K}_\infty$. The following statements hold.*

(i)
$$\sqrt{(1+\varepsilon)s} - \sqrt{s} \to \infty \text{ as } s \to \infty.$$

(ii)
$$\alpha\left(\sqrt{(1+\varepsilon)s}\right)(1+\varepsilon)s - \alpha\left(\sqrt{s}\right)s \to \infty \text{ as } s \to \infty.$$

Proof. We begin by noting the following two trivial identities. The first is that

$$\sqrt{a+b} \leq \sqrt{a} + \sqrt{b} \quad \forall\, a, b \geq 0, \tag{2.2}$$

which is obtained by recalling that

$$(a+b)^2 \geq a^2 + b^2 \quad \forall\, a, b \geq 0.$$

The second identity is that

$$a - b = \frac{a^2 - b^2}{a+b} \quad \forall\, a, b > 0. \tag{2.3}$$

By combining (2.2) and (2.3), we see that

$$\sqrt{(1+\varepsilon)s} - \sqrt{s} = \frac{\varepsilon s}{\sqrt{(1+\varepsilon)s} + \sqrt{s}} \geq \frac{\varepsilon s}{(2+\sqrt{\varepsilon})\sqrt{s}} = \frac{\varepsilon}{2+\sqrt{\varepsilon}}\sqrt{s} \quad \forall\, s > 0.$$

This tends towards ∞ as $s \to \infty$, hence giving statement (i). As for statement (ii), since $\alpha \in \mathcal{K}_\infty$, α is strictly increasing and so

$$\begin{aligned}\alpha\left(\sqrt{(1+\varepsilon)s}\right)(1+\varepsilon)s - \alpha\left(\sqrt{s}\right)s &\geq \alpha\left(\sqrt{(1+\varepsilon)s}\right)(1+\varepsilon)s - \alpha\left(\sqrt{(1+\varepsilon)s}\right)s \\ &= \alpha\left(\sqrt{(1+\varepsilon)s}\right)\varepsilon s \quad \forall\, s > 0.\end{aligned}$$

This again tends to ∞ as $s \to \infty$, whence completing the proof. □

Lemma 2.1.5. *Let $\alpha, \beta \in \mathcal{K}_\infty$ and define $\gamma \in \mathcal{K}_\infty$ by*

$$\gamma(s) := \min\{\alpha(s), \beta(s)\} \quad \forall\, s \geq 0.$$

If, for some $\varepsilon > 0$,

$$\lim_{s\to\infty} \left(\alpha((1+\varepsilon)s) - \alpha(s)\right) = \infty \quad \textit{and} \quad \lim_{s\to\infty} \left(\beta((1+\varepsilon)s) - \beta(s)\right) = \infty, \tag{2.4}$$

then

$$\lim_{s\to\infty} \left(\gamma((1+\varepsilon)s) - \gamma(s)\right) = \infty. \tag{2.5}$$

Proof. Assume that, for some $\varepsilon > 0$, (2.4) holds. Define $d(s) := \gamma((1+\varepsilon)s) - \gamma(s)$, for all $s \geq 0$. Fix $M \geq 0$ and let $s_\alpha \geq 0$ and $s_\beta \geq 0$ be such that

$$\alpha((1+\varepsilon)s) - \alpha(s) \geq M \quad \text{and} \quad \beta((1+\varepsilon)s) - \beta(s) \geq M,$$

for all s greater than or equal to s_α and s_β, respectively. The existence of such constants is guaranteed by (2.4). We claim that $d(s) \geq M$ for all $s \geq \max\{s_\alpha, s_\beta\}$. To see this, fix $s \geq \max\{s_\alpha, s_\beta\}$ and consider the following four cases. The first two cases comprise (i) $\gamma((1+\varepsilon)s) = \alpha((1+\varepsilon)s)$ and $\gamma(s) = \alpha(s)$, and (ii) $\gamma((1+\varepsilon)s) = \beta((1+\varepsilon)s)$ and $\gamma(s) = \beta(s)$. Both of these cases trivially imply that $d(s) \geq M$. The next case is (iii) $\gamma((1+\varepsilon)s) = \alpha((1+\varepsilon)s)$ and $\gamma(s) = \beta(s)$. This implies that $\alpha((1+\varepsilon)s) \leq \beta((1+\varepsilon)s)$ and $\beta(s) \leq \alpha(s)$. We therefore see that

$$d(s) = \alpha((1+\varepsilon)s) - \beta(s) \geq \alpha((1+\varepsilon)s) - \alpha(s) \geq M.$$

The final case: (iv) $\gamma((1+\varepsilon)s) = \beta((1+\varepsilon)s)$ and $\gamma(s) = \alpha(s)$, is proven similarly and thus is omitted. Since M was arbitrary, we have therefore shown that (2.5) holds, completing the proof. □

The following is a useful relationship between two $\mathcal{K}_\infty$ functions.

Lemma 2.1.6. *Let $\alpha, \beta \in \mathcal{K}_\infty$ and $\mu \geq 0$. Then there exists $\gamma \in \mathcal{K}_\infty$ such that*

$$\gamma(s + \beta(s))(s + \beta(s))^2 \leq \alpha(s) \quad \forall\, s \in [0, \mu].$$

Proof. To begin with, if $\mu = 0$ then the result is trivial and so let us assume that $\mu > 0$. For convenience, define $\psi(s) := s + \beta(s)$ for all $s \geq 0$, and note that $\psi \in \mathcal{K}_\infty$ from Lemma 2.1.2. We further define $\lambda := (\psi(\mu))^2 > 0$ and

$$\gamma(s) := \frac{1}{\lambda}\alpha(\psi^{-1}(s)) \quad \forall\, s \geq 0.$$

We then have, again from Lemma 2.1.2, that $\gamma \in \mathcal{K}_\infty$, and also that

$$\gamma(s) \leq \frac{\alpha(\psi^{-1}(s))}{s^2} = \frac{\alpha(\psi^{-1}(s))}{(\psi(\psi^{-1}(s)))^2} \quad \forall\, s \in (0, \psi(\mu)].$$

This in turn implies that

$$\gamma(\psi(s))(\psi(s))^2 \leq \alpha(s) \quad \forall\, s \in [0, \mu],$$

thus completing the proof. □

We conclude this discussion of comparison functions with the subsequent two lemmas. They can be found in [108, Lemma 14] and [108, Proposition 19] respectively, but are presented here for completeness.

Lemma 2.1.7. *Let $\alpha \in \mathcal{K}_\infty$. The following statements hold.*

(i) *There exists $\gamma \in \mathcal{K}_\infty$ such that*

$$s_1 s_2 \leq s_1 \alpha(s_1) + \gamma(s_2) \quad \forall\, s_1, s_2 \geq 0.$$

(ii) *For every* $\varepsilon > 0$,

$$\alpha(s_1 + s_2) \leq \alpha((1+\varepsilon)s_1) + \alpha((1+\varepsilon^{-1})s_2) \quad \forall\, s_1, s_2 \geq 0.$$

(iii) *Define* $\tilde{\alpha} \in \mathcal{K}_\infty$ *by* $\tilde{\alpha}(s) := \sqrt{s}\alpha(\sqrt{s})$. *For every* $\varepsilon > 0$, *there exists* $\eta \in \mathcal{K}_\infty$ *such that*

$$\tilde{\alpha}(s_1 - s_2) \leq \tilde{\alpha}((1+\varepsilon)s_1) - \eta(s_2) \quad \forall\, s_1 \geq s_2 \geq 0,$$

and $\eta(s)/\sqrt{s} \to \infty$ *as* $s \to \infty$.

Lemma 2.1.8. *Let* $\alpha \in \mathcal{K}_\infty$ *and* $\varepsilon > 0$. *Assume that*

$$\lim_{s\to\infty} (\alpha((1+\varepsilon)s) - \alpha(s)) = \infty$$

and define $\eta : \mathbb{R}_+ \to \mathbb{R}_+$ *by*

$$\eta(s) := \inf_{\sigma\in[0,\infty)} (\alpha((1+\varepsilon)(s+\sigma)) - \alpha(\sigma)) \quad \forall\, s \geq 0.$$

Then $\eta \in \mathcal{K}_\infty$ *and*

$$\alpha(s_1 - s_2) \leq \alpha((1+\varepsilon)s_1) - \eta(s_2) \quad \forall\, s_1 \geq s_2 \geq 0.$$

2.1.2 Linear systems theory and linear output feedback

Our attention now turns towards the theory of linear difference equations. Indeed, we shall consider the following system:

$$\left.\begin{aligned} x^+ &= Ax + Bu + B_\mathrm{e}v, \\ y &= Cx + Du + D_\mathrm{e}v, \end{aligned}\right\} \tag{2.6}$$

where

$$(A, B, B_\mathrm{e}, C, D, D_\mathrm{e}) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{n\times q} \times \mathbb{R}^{p\times n} \times \mathbb{R}^{p\times m} \times \mathbb{R}^{p\times q},$$

$u \in (\mathbb{R}^m)^{\mathbb{Z}_+}$, $v \in (\mathbb{R}^q)^{\mathbb{Z}_+}$ and $n, m, p, q \in \mathbb{N}$. We shall call the variables x and y in (2.6) the state and output, respectively, and we will label u and v as inputs or forcing.

In order to ease notation, we make the following conventions.

Definition 2.1.9. (i) *We set*

$$\mathbb{L} := \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{n\times q} \times \mathbb{R}^{p\times n} \times \mathbb{R}^{p\times m} \times \mathbb{R}^{p\times q}.$$

(ii) *For* $(A, B, B_\mathrm{e}, C, D, D_\mathrm{e}) \in \mathbb{L}$, *we label*

$$\Sigma := (A, B, B_\mathrm{e}, C, D, D_\mathrm{e}).$$

The following remark gives some motivation of the study of (2.6).

Remark 2.1.10. For a given $\Sigma \in \mathbb{L}$, (2.6) encompasses seemingly more general systems. Indeed, the linear difference equation

$$\left.\begin{aligned} x^+ &= Ax + Bu + v_1, \\ y &= Cx + Du + v_2, \end{aligned}\right\}$$

where $v_1 \in (\mathbb{R}^n)^{\mathbb{Z}_+}$ and $v_2 \in (\mathbb{R}^p)^{\mathbb{Z}_+}$, is a special case of (2.6) if we take $B_\mathrm{e} := \begin{pmatrix} I & 0 \end{pmatrix}$, $D_\mathrm{e} := \begin{pmatrix} 0 & I \end{pmatrix}$ and $v := \begin{pmatrix} v_1 & v_2 \end{pmatrix}^T$. $\Diamond$

We now make a series of definitions associated with (2.6), beginning with the following.

Definition 2.1.11. *Let $\Sigma \in \mathbb{L}$. We denote by $\mathbf{G}$ the* transfer function *of Σ (or of* (2.6)*) from u to y, that is,*

$$\mathbf{G}(z) = C(zI - A)^{-1}B + D,$$

for all $z \in \mathbb{C}$ for which the right-hand side makes sense.

As seen in much of the literature (see, for example, [40, 47, 107, 108]), it is often convenient to talk in terms of the *behaviour* of systems of the form (2.6), hence motivating the following definition.

Definition 2.1.12. *Let $\Sigma \in \mathbb{L}$. We define the* behaviour of the linear system (2.6) *as*

$$\mathcal{B}^{\mathrm{lin}}(\Sigma) := \Big\{ (u, v, x, y) \in (\mathbb{R}^m)^{\mathbb{Z}_+} \times (\mathbb{R}^q)^{\mathbb{Z}_+} \times (\mathbb{R}^n)^{\mathbb{Z}_+} \times (\mathbb{R}^p)^{\mathbb{Z}_+} : \\ (u, v, x, y) \text{ satisfies } (2.6) \Big\},$$

and, when the context is clear, we shall suppress this to simply $\mathcal{B}^{\mathrm{lin}}$.

We also give the following terminology.

Definition 2.1.13. *Let $\Sigma \in \mathbb{L}$. We say that Σ is:* controllable *if (A, B) is,* observable *if (A, C) is,* stabilisable *if (A, B) is, and* detectable *if (A, C) is.*

We are now interested in linear output feedback matrices of state-space systems. For motivation of the sequel, let $\Sigma \in \mathbb{L}$, $K \in \mathbb{R}^{m \times p}$ and $(u, v, x, y) \in \mathcal{B}^{\mathrm{lin}}$. If we set $\tilde{u} := u - Ky$, then (2.6) gives that

$$\begin{aligned} x^+ &= Ax + BKy + B\tilde{u} + B_\mathrm{e}v, \\ y &= Cx + DKy + D\tilde{u} + D_\mathrm{e}v. \end{aligned}$$

Moreover, if $I - DK$ is invertible, then the output equation can be written as

$$y = (I - DK)^{-1}Cx + (I - DK)^{-1}D\tilde{u} + (I - DK)^{-1}D_\mathrm{e}v,$$

which in turn implies that

$$\begin{aligned} x^+ = &\left(A + BK(I - DK)^{-1}C\right)x + \left(B + BK(I - DK)^{-1}D\right)\tilde{u} \\ &+ (B_\mathrm{e} + BK(I - DK)^{-1}D_\mathrm{e})v. \end{aligned}$$

We hence see that $(\tilde{u}, v, x, y) \in \mathcal{B}^{\mathrm{lin}}(\tilde{\Sigma})$, where $\tilde{\Sigma} \in \mathbb{L}$ has linear components given by the matrices

$$A + BK(I - DK)^{-1}C, \quad B + BK(I - DK)^{-1}D, \quad B_{\mathrm{e}} + BK(I - DK)^{-1}D_{\mathrm{e}},$$
$$(I - DK)^{-1}Cx, \quad (I - DK)^{-1}D, \quad (I - DK)^{-1}D_{\mathrm{e}}.$$

The previous argument is often termed "*loop shifting*" (see, for example, [44, pp.98-106]), although in classical loopshifting, $B_{\mathrm{e}} = B$ and $D_{\mathrm{e}} = D$.
With the above in mind, we now make the following definition.

Definition 2.1.14. *Let $\mathbb{F} := \mathbb{R}$ or $\mathbb{C}$ and $D \in \mathbb{F}^{m\times p}$. We define the set of* admissible feedback matrices of D *(over $\mathbb{F}$) by*

$$\mathbb{A}_{\mathbb{F}}(D) := \{L \in \mathbb{F}^{m\times p} : I - DL \text{ is invertible}\}.$$

We note that, trivially, $0 \in \mathbb{R}^{m\times p}$ is admissible for all $D \in \mathbb{C}^{p\times m}$ and $\mathbb{A}_{\mathbb{F}}(0) = \mathbb{F}^{m\times p}$.

A characterisation of admissibility is now given.

Lemma 2.1.15. *Let $D, L \in \mathbb{C}^{m\times p}$. Then $I - DL$ is invertible if, and only if, $I - LD$ is invertible, and, in which case,*

$$L(I - DL)^{-1} = (I - LD)^{-1}L. \tag{2.7}$$

Proof. To begin with, assume that $I - DL$ is invertible and note that

$$\begin{aligned}\left(L(I - DL)^{-1}D + I\right)(I - LD) &= L(I - DL)^{-1}D - L(I - DL)^{-1}DLD + I - LD \\ &= L(I - DL)^{-1}\left(I - DL - (I - DL)\right)D + I \\ &= I.\end{aligned}$$

In a similar fashion we may also obtain that $(I - LD)\left(L(I - DL)^{-1}D + I\right) = I$, whence giving that $I - LD$ is invertible. The reverse implication can be proven in a similar manner, and so we leave it to the reader. To conclude the proof, we note the identity

$$(I - LD)L = L - LDL = L(I - DL)$$

gives (2.7) if we apply $(I - LD)^{-1}$ and $(I - DL)^{-1}$ on the left- and right-hand sides, respectively. □

For ease of notation, we make the next definition.

Definition 2.1.16. *Let $\Sigma \in \mathbb{L}$ and $L \in \mathbb{A}_{\mathbb{C}}(D)$. We define the matrices*

$$A^L := A + BL(I - DL)^{-1}C, \quad B^L := B + BL(I - DL)^{-1}D,$$
$$C^L := (I - DL)^{-1}C, \quad D^L := (I - DL)^{-1}D.$$

Remark 2.1.17. Let $\Sigma \in \mathbb{L}$.

(i) If $L \in \mathbb{A}_{\mathbb{C}}(D)$, then from Lemma 2.1.15, we see that

$$B^L = B + B(I - LD)^{-1}LD = B(I - LD)^{-1}. \tag{2.8}$$

In the subsequent work, this alternate definition of B^L will some times be more useful than the original definition, and hence we shall use both interchangeably.

(ii) If $L \in \mathbb{A}_{\mathbb{R}}(D)$, then A^L, B^L, C^L and D^L are all real matrices.

$\Diamond$

The following lemma is a formal declaration of the loopshifting discussion given before Definition 2.1.14.

Lemma 2.1.18. *Let $\Sigma \in \mathbb{L}$ and $L \in \mathbb{A}_{\mathbb{C}}(D)$. Then $(u, v, x, y) \in \mathcal{B}^{\rm lin}$ if, and only if, (u, v, x, y) satisfies*

$$\left.\begin{aligned} x^+ &= A^L x + B^L(u - Ly) + (B_{\rm e} + B^L L D_{\rm e})v\,, \\ y &= C^L x + D^L(u - Ly) + (I - DL)^{-1} D_{\rm e} v\,. \end{aligned}\right\} \tag{2.9}$$

Proof. The proof follows the method outlined in the discussion given immediately prior to Definition 2.1.14, and so we shall be brief. Let $(u, v, x, y) \in \mathcal{B}^{\rm lin}$. Then $(I - DL)y = Cx + D(u - Ly) + D_{\rm e}v$ and so, trivially, $y = C^L x + D^L(u - Ly) + (I - DL)^{-1} D_{\rm e} v$. With this in mind, by recalling Lemma 2.1.15 and (2.8), it is easy to see that

$$\begin{aligned} x^+ &= A^L x + Bu + B_{\rm e}v - BL(I - DL)^{-1}Cx \\ &= A^L x + Bu + B_{\rm e}v - BLy + BLD^L(u - Ly) + B(I - LD)^{-1} L D_{\rm e} v \\ &= A^L x + (B + BLD^L)(u - Ly) + B_{\rm e}v + B^L L D_{\rm e} v \\ &= A^L x + B^L(u - Ly) + (B_{\rm e} + B^L L D_{\rm e})v. \end{aligned}$$

Whence, we have shown that (u, v, x, y) satisfies (2.9). The converse implication can be proven by reversing the previous argument and therefore we omit the proof. $\square$

Again, for ease of notation, we make the subsequent definition.

Definition 2.1.19. *Let $\Sigma \in \mathbb{L}$ and $L \in \mathbb{A}_{\mathbb{C}}(D)$.*

(i) *We denote the linear components of the feedback system* (2.9) *by Σ^L, that is,*

$$\Sigma^L := (A^L, B^L, B_{\rm e} + B^L L D_{\rm e}, C^L, D^L, (I - DL)^{-1} D_{\rm e}).$$

(ii) *We denote by $\mathbf{G}^L$ the transfer function of the system Σ^L (or of* (2.9)*), that is,*

$$\mathbf{G}^L(z) = C^L(zI - A^L)^{-1} B^L + D^L.$$

It is routine to show that $\mathbf{G}^L = \mathbf{G}(I - L\mathbf{G})^{-1}$.

(iii) *For $\mathbb{F} := \mathbb{R}$ or $\mathbb{C}$, we define the set of* stabilising feedback matrices *(over $\mathbb{F}$) by*

$$\mathbb{S}_{\mathbb{F}}(\mathbf{G}) := \{M \in \mathbb{A}_{\mathbb{F}}(D) : \mathbf{G}^M \in H^\infty_{p\times m}\}.$$

The following is, in a sense, an associativity result for admissible feedback matrices.

Lemma 2.1.20. *Let $\Sigma \in \mathbb{L}$ and $L \in \mathbb{A}_{\mathbb{C}}(D)$. For $M \in \mathbb{C}^{m\times p}$, $M \in \mathbb{A}_{\mathbb{C}}(D^L)$ if, and only if, $L + M \in \mathbb{A}_{\mathbb{C}}(D)$, and in which case,*

$$\Sigma^{L+M} = (\Sigma^L)^M \quad \textit{and} \quad (\mathbf{G}^L)^M = \mathbf{G}^{L+M}.$$

An explicit proof of Lemma 2.1.20 is difficult to find in the literature. We therefore present one in order to preserve completeness.

Proof of Lemma 2.1.20. Let $M \in \mathbb{C}^{m\times p}$. By recalling that $I - DL$ is invertible and by considering

$$I - D(L + M) = (I - DL)(I - (I - DL)^{-1}DM) = (I - DL)(I - D^L M), \qquad (2.10)$$

it is clear that $I - D(L + M)$ is invertible if, and only if, $I - D^L M$ is invertible, i.e. $M \in \mathbb{A}_{\mathbb{C}}(D^L)$ if, and only if, $L + M \in \mathbb{A}_{\mathbb{C}}(D)$. We now assume that $M \in \mathbb{A}_{\mathbb{C}}(D^L)$ and seek to show that $\Sigma^{L+M} = (\Sigma^L)^M$. To this end, note that, by using (2.10),

$$\begin{aligned} A^{L+M} &= A + B(L + M)(I - D(L + M))^{-1}C \\ &= A + B(L + M)(I - D^L M)^{-1}C^L. \end{aligned}$$

By combining this with the identity

$$\begin{aligned} BL(I - D^L M)^{-1}C^L &= BL\left(D^L M + I - D^L M\right)(I - D^L M)^{-1}C^L \\ &= BL\left(D^L M(I - D^L M)^{-1} + I\right)C^L \\ &= BLD^L M(I - D^L M)^{-1}C^L + BLC^L, \end{aligned}$$

we obtain that

$$\begin{aligned} A^{L+M} &= A + BLC^L + BLD^L M(I - D^L M)^{-1}C^L + BM(I - D^L M)^{-1}C^L \\ &= A^L + (B + BLD^L)M(I - D^L M)^{-1}C^L \\ &= A^L + B^L M(I - D^L M)^{-1}C^L \\ &= (A^L)^M. \end{aligned}$$

In the interest of brevity, we omit the rest of the proof, since the other identities can be shown in a similar manner. □

Remark 2.1.21. If $\Sigma \in \mathbb{L}$ and $L \in \mathbb{A}_{\mathbb{C}}(D)$, then Lemma 2.1.20 gives that

$$-L \in \mathbb{A}_{\mathbb{C}}(D^L), \quad (\Sigma^L)^{-L} = \Sigma \quad \text{and} \quad (\mathbf{G}^L)^{(-L)} = \mathbf{G}.$$

Therefore, applying an admissible feedback operator L to Σ, and then applying $-L$ to Σ^L, yields the original system, as expected. ◊

The following lemma shows that controllability, observability, stabilisability and detectability are preserved after applying feedback matrices.

Lemma 2.1.22. *Let $\Sigma \in \mathbb{L}$ and $L \in \mathbb{A}_{\mathbb{C}}(D)$. Then Σ is controllable/observable/stabilisable /detectable if, and only if, Σ^L is controllable/observable/stabilisable/detectable.*

Proof. We shall only prove the equivalence for controllable systems, since the proof for observable, stabilisable and detectable systems can be completed in a similar manner. To this end, assume that Σ is controllable. We seek to use the Hautus criterion for controllability (see, such as, [69, Theorem 4.3.3]), and thus, we are required to show that rank $\begin{pmatrix}\lambda I - A^L & B^L\end{pmatrix} = n$ for all $\lambda \in \mathbb{C}$. Indeed, let $\lambda \in \mathbb{C}$ and assume that there exists $\xi \in \mathbb{C}^n$ such that $\xi^* \begin{pmatrix}\lambda I - A^L & B^L\end{pmatrix} = 0$. This implies that

$$\xi^*(\lambda I - A^L) = 0 \text{ and } \xi^* B^L = 0.$$

From (2.8), we then see that $\xi^* B = 0$, which, when combined with the above equality, implies that

$$\lambda \xi^* = \xi^* A^L = \xi^* A.$$

Combining this with the controllability of Σ we obtain that $\xi^* = 0$. Therefore, Σ^L is controllable. Hence, we have shown that applying an output feedback matrix to a controllable system preserves this property. Thus, for the converse implication, if Σ^L is controllable, then the controllability of Σ is obtained immediately by the use of the feedback matrix $-L$ and Lemma 2.1.20 (see also Remark 2.1.21). □

Remark 2.1.23. By combining the previous result with [72, Theorem 2], we see that if $\Sigma \in \mathbb{L}$ is stabilisable and detectable, then

$$\mathbb{S}_{\mathbb{C}}(\mathbf{G}) = \{L \in \mathbb{A}_{\mathbb{C}}(D) : \sigma(A^L) \subset \mathbb{D}\}. \qquad \Diamond$$

We now present a generalisation of [108, Statement (5) of Lemma 6].

Lemma 2.1.24. *Let $\Sigma \in \mathbb{L}$, $r > 0$ and $K \in \mathbb{C}^{m \times p}$. Then*

$$\mathbb{B}_{\mathbb{C}}(K, r) \subseteq \mathbb{S}_{\mathbb{C}}(\mathbf{G}) \iff \|\mathbf{G}^K\|_{H^\infty} \leq 1/r.$$

We shall not provide a proof of this result since an identical argument used in the proof of [108, Statement (5) of Lemma 6] can be used here. Moreover, a discrete-time version of the proof of [46, Proposition 5.6] would also yield the result.

For given $\Sigma \in \mathbb{L}$, $K \in \mathbb{R}^{m \times p}$ and $r > 0$ such that $\mathbb{B}_{\mathbb{C}}(K, r) \subseteq \mathbb{S}_{\mathbb{C}}(\mathbf{G})$, we give the following key assumption which will be used throughout this chapter:

$$\left.\begin{array}{l}\Sigma \text{ is (i) controllable and observable, or (ii) stabilisable and detectable and} \\ r \min\limits_{|z|=1} \|\mathbf{G}^K(z)\| < 1.\end{array}\right\} \qquad \text{(A)}$$

We note that $r \min_{|z|=1} \|\mathbf{G}^K(z)\| < 1$ if, trivially, $r < 1/\|\mathbf{G}^K\|_{H^\infty}$. The motivation for assumption (A) comes from finding solutions to the *bounded real equations*, which we shall explain in the next subsection - see Remark 2.1.28.

2.1.3 The bounded real lemma and quadratic forms

We shall now give some key results that will underpin our development of this chapter. These results can be seen as a generalisation of the preliminary results of [108]. In particular, we use a version of the bounded real lemma to prove a series of results involving quadratic forms.

For a given $\Sigma \in \mathbb{L}$, the bounded real lemma seeks to conclude the existence of matrices P, L and W satisfying

$$\left.\begin{aligned} A^T P A - P + C^T C &= -L^T L, \\ A^T P B + C^T D &= -L^T W, \\ B^T P B + D^T D &= I - W^T W. \end{aligned}\right\} \qquad (2.11)$$

In the literature, the bounded real lemma is presented with differing assumptions. Indeed, for various versions of the result, we refer the reader to references such as [4, 51, 108, 126]. The following is one such version, and is sufficient for our purposes.

Lemma 2.1.25. *Let $\Sigma \in \mathbb{L}$ and assume that $\|\mathbf{G}\|_{H^\infty} \leq 1$. The following statements hold.*

(i) *If Σ is controllable and observable then there exist a positive definite $P = P^T \in \mathbb{R}^{n\times n}$ and matrices L and W such that* (2.11) *holds.*

(ii) *If Σ is stabilisable and detectable and there exists $z_0 \in \mathbb{C}$ such that $|z_0| = 1$ and $\|\mathbf{G}(z_0)\| < 1$, then there exist a positive semi-definite $P = P^T \in \mathbb{R}^{n\times n}$ and matrices L and W such that W is positive definite and* (2.11) *holds.*

We comment that statement (i) is [51, Lemma 3.1]. As for statement (ii), this is [108, Lemma 3] in the situation that $D = 0$, and [106, Lemma 2.2.3] in the potentially nonzero feedthrough case. We highlight that we may apply these results in the current setting since B_{e} and D_{e} play no role in the lemma.

Remark 2.1.26. Interestingly, in [108], an example of $(A, B, C, D) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{p\times n} \times \mathbb{R}^{p\times m}$ is presented that has the property that (A, B) is not controllable but is stabilisable, (A, C) is detectable, $\|\mathbf{G}(z)\| = \|\mathbf{G}\|_{H^\infty} = 1$ for all $z \in \partial\mathbb{D}$, and there does not exist a positive semi-definite $P = P^T \in \mathbb{R}^{n\times n}$ and matrices L and W satisfying (2.11). $\Diamond$

By using the previous lemma, we obtain the following result, which is a generalisation of [108, Corollary 7].

Lemma 2.1.27. *Let $\Sigma \in \mathbb{L}$, $K \in \mathbb{R}^{m\times p}$ and $r > 0$. Assume that $\mathbb{B}_{\mathbb{C}}(K, r) \subseteq \mathbb{S}_{\mathbb{C}}(\mathbf{G})$ and* (A) *holds. Then there exist a positive semi-definite matrix $P = P^T \in \mathbb{R}^{n\times n}$ and constant $\kappa > 0$ such that the function $V : \mathbb{R}^n \to \mathbb{R}_+$, defined by $V(\xi) := \langle P\xi, \xi\rangle$ for all $\xi \in \mathbb{R}^n$, satisfies, for all $(u, v, x, y) \in \mathcal{B}^{\mathrm{lin}}$ and all $t \in \mathbb{Z}_+$,*

$$\begin{aligned} V(x(t+1)) - V(x(t)) &\leq -r^2\|y(t)\|^2 + \|u(t) - Ky(t)\|^2 \\ &\quad + \kappa \left\| \begin{pmatrix} B_{\mathrm{e}}v(t) \\ D_{\mathrm{e}}v(t) \end{pmatrix} \right\| \left(\left\| \begin{pmatrix} B_{\mathrm{e}}v(t) \\ D_{\mathrm{e}}v(t) \end{pmatrix} \right\| + \|u(t) - Ky(t)\| + \|x(t)\| \right). \end{aligned}$$

Proof. We employ similar arguments to those used to prove [108, Corollary 7]. To this end, we seek to apply Lemma 2.1.25 to the system with linear components given by

$$\tilde{\Sigma} := (A^K, rB^K, B_{\mathrm{e}} + B^K K D_{\mathrm{e}}, C^K, rD^K, (I - DK)D_{\mathrm{e}}). \tag{2.12}$$

We note that the transfer function of this system is $r\mathbf{G}^K$ and, from Lemma 2.1.24, $\|r\mathbf{G}^K\|_{H^\infty} \leq 1$. Moreover, since (A) holds, it is readily verified, by use of Lemma 2.1.22, that $\tilde{\Sigma}$ is either (i) controllable and observable, or; (ii) stabilisable and detectable and there exists $z_0 \in \mathbb{C}$ such that $|z_0| = 1$ and $\|r\mathbf{G}(z_0)\| < 1$. Hence, we may indeed apply Lemma 2.1.25 to obtain the existence of a positive semi-definite $Q = Q^T \in \mathbb{R}^{n\times n}$ and matrices L and W and such that

$$\left.\begin{aligned} (A^K)^T Q A^K - Q + (C^K)^T C^K &= -L^T L, \\ (A^K)^T Q B^K + (C^K)^T D^K &= -\frac{1}{r} L^T W, \\ (B^K)^T Q B^K + (D^K)^T D^K &= \frac{1}{r^2}(I - W^T W). \end{aligned}\right\} \tag{2.13}$$

Define $U(\xi) := \langle Q\xi, \xi\rangle$ for all $\xi \in \mathbb{R}^n$. From Lemma 2.1.18, we see that, for all $(u, v, x, y) \in \mathcal{B}^{\rm lin}$ and for all $t \in \mathbb{Z}_+$,

$$U(x(t+1)) = \langle Q(A^K x(t) + B^K(u(t) - Ky(t)) + (B_{\rm e} + B^K K D_{\rm e})v(t)), x(t+1)\rangle.$$

Equivalently, we may write, for all $(u, v, x, y) \in \mathcal{B}^{\rm lin}$ and all $t \in \mathbb{Z}_+$,

$$\begin{aligned} U(x(t+1)) =& \langle QA^K x(t), A^K x(t)\rangle \\ &+ 2\langle QA^K x(t), B^K(u(t) - Ky(t)) + (B_{\rm e} + B^K K D_{\rm e})v(t)\rangle \\ &+ \langle QB^K(u(t) - Ky(t)), B^K(u(t) - Ky(t)) + (B_{\rm e} + B^K K D_{\rm e})v(t)\rangle \\ &+ \langle Q(B_{\rm e} + B^K K D_{\rm e})v(t), B^K(u(t) - Ky(t)) + (B_{\rm e} + B^K K D_{\rm e})v(t)\rangle. \end{aligned}$$

If we now invoke (2.13), we see that, for all $(u, v, x, y) \in \mathcal{B}^{\rm lin}$ and all $t \in \mathbb{Z}_+$,

$$\begin{aligned} U(x(t+1)) \leq& U(x(t)) - \|C^K x(t)\|^2 - \|Lx(t)\|^2 - 2\langle C^K x(t), D^K(u(t) - Ky(t))\rangle \\ &- \frac{2}{r}\langle Lx(t), W(u(t) - Ky(t))\rangle + 2\langle QA^K x(t), (B_{\rm e} + B^K K D_{\rm e})v(t)\rangle \\ &- \|D^K(u(t) - Ky(t))\|^2 + \frac{1}{r^2}\|u(t) - Ky(t)\|^2 - \frac{1}{r^2}\|W(u(t) - Ky(t))\|^2 \\ &+ 2\langle QB^K(u(t) - Ky(t)), (B_{\rm e} + B^K K D_{\rm e})v(t)\rangle \\ &+ \|Q\|\|(B_{\rm e} + B^K K D_{\rm e})v(t)\|^2. \end{aligned}$$

Since, for all $(u, v, x, y) \in \mathcal{B}^{\rm lin}$ and all $t \in \mathbb{Z}_+$,

$$\begin{aligned} -\|C^K x(t)\|^2 &- 2\langle C^K x(t), D^K(u(t) - Ky(t))\rangle - \|D^K(u(t) - Ky(t))\|^2 \\ &= -\|C^K x(t) + D^K(u(t) - Ky(t))\|^2 \\ &= -\|y(t)\|^2 + 2\langle (I - DK)^{-1} D_{\rm e} v(t), C^K x(t) + D^K(u(t) - Ky(t))\rangle \\ &\quad + \|(I - DK)^{-1} D_{\rm e} v(t)\|^2, \end{aligned}$$

we may use the Cauchy-Schwarz inequality to yield, for all $(u, v, x, y) \in \mathcal{B}^{\rm lin}$ and all $t \in \mathbb{Z}_+$,

$$\begin{aligned} &U(x(t+1)) - U(x(t)) \\ &\quad \leq -\|y(t)\|^2 - \left\| Lx(t) + \frac{1}{r} W(u(t) - Ky(t)) \right\|^2 + \frac{1}{r^2}\|u(t) - Ky(t)\|^2 \\ &\qquad + \|(B_{\rm e} + B^K K D_{\rm e})v(t)\| \big(\|Q\|\|(B_{\rm e} + B^K K D_{\rm e})v(t)\| + 2\|QA^K x(t)\| \\ &\qquad\qquad + 2\|QB^K(u(t) - Ky(t))\| \big) \\ &\qquad + \|(I - DK)^{-1} D_{\rm e} v(t)\| \big(\|(I - DK)^{-1} D_{\rm e} v(t)\| + 2\|C^K x(t)\| \\ &\qquad\qquad + 2\|D^K(u(t) - Ky(t))\| \big). \end{aligned}$$

By expanding this further, we obtain that, for all $(u, v, x, y) \in \mathcal{B}^{\rm lin}$ and all $t \in \mathbb{Z}_+$,

$$\begin{aligned} U(x(t+1)) - U(x(t)) \leq& -\|y(t)\|^2 + \frac{1}{r^2}\|u(t) - Ky(t)\|^2 \\ &+ c\left\| \begin{pmatrix} B_{\rm e} v(t) \\ D_{\rm e} v(t) \end{pmatrix} \right\| \left(\left\| \begin{pmatrix} B_{\rm e} v(t) \\ D_{\rm e} v(t) \end{pmatrix} \right\| + \|u(t) - Ky(t)\| + \|x(t)\| \right), \end{aligned}$$

where c is a positive constant such that

$$c \geq 2(1 + \|B^K K\| + \|(I - DK)^{-1}\|) \max\{\|QA^K\|, \|QB^K\|, \|Q\|(1 + \|B^K K\|), \\ \|C^K\|, \|D^K\|, \|(I - DK)^{-1}\|\}.$$

Therefore, by setting $P := r^2 Q$, $\kappa := r^2 c$ and $V(\xi) := \langle P\xi, \xi\rangle$ for all $\xi \in \mathbb{R}^n$, we obtain, for all $(u, v, x, y) \in \mathcal{B}^{\text{lin}}$ and all $t \in \mathbb{Z}_+$,

$$V(x(t+1)) - V(x(t)) \leq -r^2\|y(t)\|^2 + \|u(t) - Ky(t)\|^2 \\ + \kappa \left\| \begin{pmatrix} B_{\text{e}} v(t) \\ D_{\text{e}} v(t) \end{pmatrix} \right\| \left(\left\| \begin{pmatrix} B_{\text{e}} v(t) \\ D_{\text{e}} v(t) \end{pmatrix} \right\| + \|u(t) - Ky(t)\| + \|x(t)\| \right),$$

which completes the proof. □

Remark 2.1.28. We now explain the motivation for (A). In the previous proof, we used (A) to deduce that the assumptions of Lemma 2.1.25 hold for $\tilde{\Sigma}$ given by (2.12). From Remark 2.1.26, we see that if we were to weaken (A) to simply the assertion of stabilisability and detectability, then we would not be able to guarantee the existence of solutions to (2.13). Interestingly, in the continuous-time setting, (A) can indeed be exchanged for simply asserting stabilisability and detectability of the relevant system - see [107]. ♢

The next lemma is a generalisation of [108, Lemma 5].

Lemma 2.1.29. *Let $\Sigma \in \mathbb{L}$. If Σ is detectable, then there exists a positive definite $P = P^T \in \mathbb{R}^{n\times n}$ and $\delta > 0$ such that $V : \mathbb{R}^n \to \mathbb{R}_+$, defined by $V(\xi) := \langle P\xi, \xi\rangle$ for all $\xi \in \mathbb{R}^n$, satisfies, for all $(u, v, x, y) \in \mathcal{B}^{\text{lin}}$ and all $t \in \mathbb{Z}_+$,*

$$V(x(t+1)) - V(x(t)) \leq -\delta\|x(t)\|^2 + \|y(t)\|^2 + \|u(t)\|^2 + \left\| \begin{pmatrix} B_{\text{e}} v(t) \\ D_{\text{e}} v(t) \end{pmatrix} \right\|^2. \tag{2.14}$$

The following proof utilises arguments employed to prove [108, Lemma 5].

Proof of Lemma 2.1.29. To begin with, since Σ is detectable, there exists $H \in \mathbb{R}^{n\times p}$ such that $\sigma(A + HC) \subset \mathbb{D}$. An application of [53, Corollary 3.3.47] then gives the existence of a positive definite $Q = Q^T \in \mathbb{R}^{n\times n}$ which satisfies

$$(A + HC)^T Q(A + HC) - Q = -I. \tag{2.15}$$

We additionally note that, for all $(u, v, x, y) \in \mathcal{B}^{\text{lin}}$,

$$\begin{aligned} x^+ &= Ax + Bu + B_{\text{e}} v \\ &= (A + HC)x - HCx + Bu + B_{\text{e}} v \\ &= (A + HC)x - Hy + (B + HD)u + (B_{\text{e}} + HD_{\text{e}})v. \end{aligned} \tag{2.16}$$

We now define $U : \mathbb{R}^n \to \mathbb{R}_+$ by $U(\xi) := \langle Q\xi, \xi\rangle$ for all $\xi \in \mathbb{R}^n$, and utilise (2.16) so that, for all $(u, v, x, y) \in \mathcal{B}^{\text{lin}}$ and all $t \in \mathbb{Z}_+$,

$$\begin{aligned} U(x(t+1)) = &\langle Q(A + HC)x(t), (A + HC)x(t)\rangle \\ &+ \langle Q(A + HC)x(t), -Hy(t) + (B + HD)u(t) + (B_{\text{e}} + HD_{\text{e}})v(t)\rangle \\ &+ \langle Q\left(-Hy(t) + (B + HD)u(t) + (B_{\text{e}} + HD_{\text{e}})v(t)\right), x(t+1)\rangle. \end{aligned}$$

An application of (2.15) then yields, for all $(u,v,x,y)\in\mathcal{B}^{\mathrm{lin}}$ and all $t\in\mathbb{Z}_+$,

$$\begin{aligned}U(x(t+1)) = U(x(t)) - \|x(t)\|^2 &\\ &+\langle Q(A+HC)x(t), -Hy(t)+(B+HD)u(t)+(B_{\mathrm{e}}+HD_{\mathrm{e}})v(t)\rangle\\ &+\langle Q\left(-Hy(t)+(B+HD)u(t)+(B_{\mathrm{e}}+HD_{\mathrm{e}})v(t)\right), x(t+1)\rangle.\end{aligned}$$

Now, by applying the Cauchy-Schwarz inequality to the above, we obtain the existence of constants $c_1,c_2,c_3,c_4>0$ such that, for all $(u,v,x,y)\in\mathcal{B}^{\mathrm{lin}}$ and all $t\in\mathbb{Z}_+$,

$$\begin{aligned}U(x(t+1)) - U(x(t)) \le -\|x(t)\|^2 &\\ &+c_1\|x(t)\|\left(\|y(t)\|+\|u(t)\|+\left\|\begin{pmatrix}B_{\mathrm{e}}v(t)\\ D_{\mathrm{e}}v(t)\end{pmatrix}\right\|\right)\\ &+c_2\|y(t)\|\left(\|x(t)\|+\|u(t)\|+\left\|\begin{pmatrix}B_{\mathrm{e}}v(t)\\ D_{\mathrm{e}}v(t)\end{pmatrix}\right\|\right)\\ &+c_3\|u(t)\|\left(\|y(t)\|+\|x(t)\|+\left\|\begin{pmatrix}B_{\mathrm{e}}v(t)\\ D_{\mathrm{e}}v(t)\end{pmatrix}\right\|\right)\\ &+c_4\left\|\begin{pmatrix}B_{\mathrm{e}}v(t)\\ D_{\mathrm{e}}v(t)\end{pmatrix}\right\|\left(\|y(t)\|+\|u(t)\|+\|x(t)\|\right).\end{aligned}$$

We let c be a positive constant such that $c^2<1/(3c_1^2+c_2^2+c_3^2+c_4^2)$. From the above inequality, and by also noting the identity: $ab\le 2ab = 2cac^{-1}b\le c^2a^2+b^2/c^2$ for all $a,b\in\mathbb{R}_+$, we obtain that, for all $(u,v,x,y)\in\mathcal{B}^{\mathrm{lin}}$ and all $t\in\mathbb{Z}_+$,

$$\begin{aligned}U(x(t+1)) - U(x(t)) \le &-(1-3c_1^2c^2-c_2^2c^2-c_3^2c^2-c_4^2c^2)\|x(t)\|^2\\ &+\frac{1}{c^2}\left(\|y(t)\|^2+\|u(t)\|^2+\left\|\begin{pmatrix}B_{\mathrm{e}}v(t)\\ D_{\mathrm{e}}v(t)\end{pmatrix}\right\|^2\right)\\ &+\left(\frac{1}{c^2}+2c_2^2c^2\right)\|y(t)\|^2+\frac{1}{c^2}\left(\|u(t)\|^2+\left\|\begin{pmatrix}B_{\mathrm{e}}v(t)\\ D_{\mathrm{e}}v(t)\end{pmatrix}\right\|^2\right)\\ &+\left(\frac{1}{c^2}+2c_3^2c^2\right)\|u(t)\|^2+\frac{1}{c^2}\left(\|y(t)\|^2+\left\|\begin{pmatrix}B_{\mathrm{e}}v(t)\\ D_{\mathrm{e}}v(t)\end{pmatrix}\right\|^2\right)\\ &+\left(\frac{1}{c^2}+2c_4^2c^2\right)\left\|\begin{pmatrix}B_{\mathrm{e}}v(t)\\ D_{\mathrm{e}}v(t)\end{pmatrix}\right\|^2+\frac{1}{c^2}\left(\|y(t)\|^2+\|u(t)\|^2\right).\end{aligned}$$

Setting $d_1 := 4/c^2+2c^2\max\{c_2^2,c_3^2,c_4^2\}>0$ gives, for all $(u,v,x,y)\in\mathcal{B}^{\mathrm{lin}}$ and all $t\in\mathbb{Z}_+$,

$$\begin{aligned}U(x(t+1)) - U(x(t)) \le &-(1-(3c_1^2+c_2^2+c_3^2+c_4^2)c^2)\|x(t)\|^2\\ &+d_1\left(\|y(t)\|^2+\|u(t)\|^2+\left\|\begin{pmatrix}B_{\mathrm{e}}v(t)\\ D_{\mathrm{e}}v(t)\end{pmatrix}\right\|^2\right).\end{aligned}$$

Recalling the choice of $c>0$, we define $d_2:=1-(3c_1^2+c_2^2+c_3^2+c_4^2)c^2>0$ so that, for all $(u,v,x,y)\in\mathcal{B}^{\mathrm{lin}}$ and all $t\in\mathbb{Z}_+$,

$$U(x(t+1)) - U(x(t)) \le -d_2\|x(t)\|^2+d_1\left(\|y(t)\|^2+\|u(t)\|^2+\left\|\begin{pmatrix}B_{\mathrm{e}}v(t)\\ D_{\mathrm{e}}v(t)\end{pmatrix}\right\|^2\right).$$

To conclude the proof, we define $P := d_1^{-1}Q$, $V(\xi) := \langle P\xi, \xi\rangle = d_1^{-1}U(\xi)$ for all $\xi \in \mathbb{R}^n$, and $\delta := d_1^{-1}d_2 > 0$, to yield, for all $(u, v, x, y) \in \mathcal{B}^{\text{lin}}$ and all $t \in \mathbb{Z}_+$, that (2.14) holds. □

2.1.4 The Lur'e system and the initial-value problem

From hereon in, we shall consider the feedback interconnection of (2.6) with $u = f(y+w)$, where $f : \mathbb{R}^p \to \mathbb{R}^m$ is a nonlinearity and $w \in (\mathbb{R}^p)^{\mathbb{Z}_+}$ is an output disturbance (see Figure 2.1). Explicitly, we consider the discrete-time forced Lur'e system of the form

$$\left.\begin{aligned} x^+ &= Ax + Bu + B_{\text{e}}v, \\ y &= Cx + Du + D_{\text{e}}v, \\ u &= f(y + w), \end{aligned}\right\} \tag{2.17}$$

where $\Sigma \in \mathbb{L}$, $v \in (\mathbb{R}^q)^{\mathbb{Z}_+}$, $w \in (\mathbb{R}^p)^{\mathbb{Z}_+}$ and $f : \mathbb{R}^p \to \mathbb{R}^m$. We say that the system is uncontrolled (or undisturbed) if $v = 0$ and $w = 0$.

Remark 2.1.30. (i) We note that in the situation where $I - Df$ is invertible, (2.17) can be expressed succinctly as

$$x^+ = Ax + Bf\left((I - Df)^{-1}Cx + D_{\text{e}}v + w\right) + B_{\text{e}}v.$$

(ii) The relevant literature is mostly concerned with the investigation of forced Lur'e systems that do not involve output disturbances (see, for example, [7, 15, 107, 108]). As motivation for studying (2.17) with the inclusion of an output disturbance w, we note that in practical applications, the action of feeding back the output will naturally produce errors. Therefore, w in (2.17) can encompass these errors. ◊

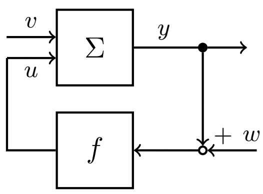

Figure 2.1: *Block diagram of the feedback interconnection of* (2.6) *with* $u = f(y + w)$.

We now make a series of definitions and observations associated with (2.17).

Definition 2.1.31. *Let* $\Sigma \in \mathbb{L}$ *and* $f : \mathbb{R}^p \to \mathbb{R}^m$. *We define the* behaviour *of* (2.17) *as*

$$\mathcal{B}_f(\Sigma) := \Big\{(v, w, x, y) \in (\mathbb{R}^q)^{\mathbb{Z}_+} \times (\mathbb{R}^p)^{\mathbb{Z}_+} \times (\mathbb{R}^n)^{\mathbb{Z}_+} \times (\mathbb{R}^p)^{\mathbb{Z}_+} : (v, w, x, y) \text{ satisfies } (2.17)\Big\}.$$

For ease of notation, we shall write $\mathcal{B} := \mathcal{B}_f(\Sigma)$ *when no ambiguity shall arise.*

Remark 2.1.32. Let $\Sigma \in \mathbb{L}$ and $f : \mathbb{R}^p \to \mathbb{R}^m$.

(i) $(v,w,x,y) \in \mathcal{B}_f(\Sigma)$ if, and only if, $(f(y+w),v,x,y) \in \mathcal{B}^{\mathrm{lin}}(\Sigma)$.

(ii) If $K \in \mathbb{A}_{\mathbb{C}}(D)$, then, by Lemma 2.1.18, $(v,w,x,y) \in \mathcal{B}$ if, and only if, (v,w,x,y) satisfies

$$\begin{aligned} x^+ &= A^K x + B^K(f(y+w) - Ky) + (B_{\mathrm{e}} + B^K K D_{\mathrm{e}})v\,, \\ y &= C^K x + D^K(f(y+w) - Ky) + (I - DK)^{-1} D_{\mathrm{e}} v\,. \end{aligned} \qquad \diamondsuit$$

The following result shows that $\mathcal{B}$ is time-invariant. The proof is trivial, and so we omit it.

Lemma 2.1.33. *Let $\Sigma \in \mathbb{L}$ and $f : \mathbb{R}^p \to \mathbb{R}^m$.*

$$(v,w,x,y) \in \mathcal{B} \implies (\Lambda_\tau v, \Lambda_\tau w, \Lambda_\tau x, \Lambda_\tau y) \in \mathcal{B} \quad \forall\, \tau \in \mathbb{Z}_+.$$

Also associated with (2.17) is an initial-value problem. The rest of the current subsection concerns this.

Definition 2.1.34. *Let $\Sigma \in \mathbb{L}$ and $f : \mathbb{R}^p \to \mathbb{R}^m$. For given $x^0 \in \mathbb{R}^n$, $v \in (\mathbb{R}^q)^{\mathbb{Z}_+}$ and $w \in (\mathbb{R}^p)^{\mathbb{Z}_+}$, we consider the* initial-value problem (IVP)*:*

$$\left.\begin{aligned} x^+ &= Ax + Bu + B_{\mathrm{e}} v, \quad x(0) = x^0, \\ y &= Cx + Du + D_{\mathrm{e}} v, \\ u &= f(y+w). \end{aligned}\right\} \tag{2.18}$$

For given $x^0 \in \mathbb{R}^n$, $v \in (\mathbb{R}^q)^{\mathbb{Z}_+}$ and $w \in (\mathbb{R}^p)^{\mathbb{Z}_+}$, we say that $(x,y) \in (\mathbb{R}^n)^{\mathbb{Z}_+} \times (\mathbb{R}^p)^{\mathbb{Z}_+}$ is a solution to the IVP (2.18)*, if $(v,w,x,y) \in \mathcal{B}$ and $x(0) = x^0$.*

Remark 2.1.35. We note that in the case that $f(0) = 0$, $x^0 = 0$, $v = 0$ and $w = 0$, then, trivially, $(0,0)$ is a solution to the IVP (2.18). $\diamondsuit$

The following proposition presents sufficient conditions for when, given an initial condition and inputs, solutions to the IVP (2.18) exist and when there is at most one solution.

Proposition 2.1.36. *Let $\Sigma \in \mathbb{L}$, $f : \mathbb{R}^p \to \mathbb{R}^m$, $x^0 \in \mathbb{R}^n$, $v \in (\mathbb{R}^q)^{\mathbb{Z}_+}$ and $w \in (\mathbb{R}^p)^{\mathbb{Z}_+}$. Then the following statements hold.*

(i) *If the map $I - Df$ is surjective, then there exists a solution to the IVP* (2.18).

(ii) *If the map $I - Df$ is injective, then there is at most one solution to the IVP* (2.18).

Proof. For ease of notation in the sequel, we define $g := I - Df$. We begin with statement (i) and so we assume that g is surjective. This then implies that there exists $z^0 \in \mathbb{R}^p$ such that $g(z^0) = Cx^0 + D_{\mathrm{e}} v(0) + w(0)$. By defining $y^0 := z^0 - w(0)$, we see that

$$y^0 = Cx^0 + Df(y^0 + w(0)) + D_{\mathrm{e}} v(0).$$

We now define $x^1 \in \mathbb{R}^n$ by

$$x^1 := Ax^0 + Bf(y^0 + w(0)) + B_{\mathrm{e}} v(0).$$

By repeating this process, we hence obtain a pair $(x, y) \in (\mathbb{R}^n)^{\mathbb{Z}_+} \times (\mathbb{R}^p)^{\mathbb{Z}_+}$, defined by $x(k) := x^k$ and $y(k) := y^k$ for all $k \in \mathbb{Z}_+$, which is a solution to the IVP (2.18). Our attention now turns to statement (ii), and so we assume instead that g is injective and that $(x_1, y_1), (x_2, y_2) \in (\mathbb{R}^n)^{\mathbb{Z}_+} \times (\mathbb{R}^p)^{\mathbb{Z}_+}$ are both solutions of (2.18). In order to prove equality of the two solutions, we shall utilise an inductive method. To this end, for the initial case we trivially have that $x_1(0) = x^0 = x_2(0)$ and

$$g(y_1(0) + w(0)) = Cx^0 + D_{\mathrm{e}}v(0) + w(0) = g(y_2(0) + w(0)),$$

where injectivity gives $y_1(0) = y_2(0)$. We now assume that $x_1(k) = x_2(k)$ and $y_1(k) = y_2(k)$ for $k \in \mathbb{Z}_+$. Consider

$$\begin{aligned} x_1(k+1) &= Ax_1(k) + Bf(y_1(k) + w(k)) + B_{\mathrm{e}}v(k) \\ &= Ax_2(k) + Bf(y_2(k) + w(k)) + B_{\mathrm{e}}v(k) \\ &= x_2(k+1). \end{aligned}$$

This subsequently gives that

$$\begin{aligned} g(y_1(k+1) + w(k+1)) &= Cx_1(k+1) + D_{\mathrm{e}}v(k+1) + w(k+1) \\ &= Cx_2(k+1) + D_{\mathrm{e}}v(k+1) + w(k+1) \\ &= g(y_2(k+1) + w(k+1)), \end{aligned}$$

which, when combined with the injectivity of g, yields that $y_1(k+1) = y_2(k+1)$, thus completing the proof by induction. □

An application of Lemma 2.1.18 gives the following result. We do not provide a proof, since a modification of the proof of Proposition 2.1.36 will suffice.

Proposition 2.1.37. *Let $\Sigma \in \mathbb{L}$, $f : \mathbb{R}^p \to \mathbb{R}^m$, $x^0 \in \mathbb{R}^n$, $v \in (\mathbb{R}^q)^{\mathbb{Z}_+}$, $w \in (\mathbb{R}^p)^{\mathbb{Z}_+}$ and $K \in \mathbb{A}_{\mathbb{C}}(D)$. Then the following statements hold.*

(i) *If the map $I - D^K(f - K)$ is surjective, then there exists a solution to the IVP* (2.18).

(ii) *If the map $I - D^K(f - K)$ is injective, then there is at most one solution to the IVP* (2.18).

We note that if $D = 0$, then trivially $I - Df$ is bijective, and so Proposition 2.1.36 implies the existence of unique solutions to the IVP (2.18) for some given initial conditions and inputs. The following example demonstrates in the case $D \neq 0$, that if injectivity or surjectivity of $I - Df$ is dropped, then the conclusions of Proposition 2.1.36 need not hold.

Example 2.1.38. Consider (2.18) in the case wherein $n = m = p = q = 1$ and $A = B = C = D = B_{\mathrm{e}} = D_{\mathrm{e}} = 1$. Thus, (2.18) becomes

$$\left.\begin{aligned} x^+ &= x + f(y + w) + v\,, \quad x(0) = x^0 \in \mathbb{R}\,, \\ y &= x + f(y + w) + v\,. \end{aligned}\right\} \tag{2.19}$$

(i) Let $f(\xi) = \xi - e^{\xi}$ for all $\xi \in \mathbb{R}$, and let $x^0 = 0$ and $v(0) = w(0) = 0$. Suppose that (x, y) is a solution to (2.19). Then, in particular, $y(0) = f(y(0))$ and so, $e^{y(0)} = 0$, which is impossible. Therefore, for $x^0 = 0$ and $v(0) = w(0) = 0$, (2.19) has no solutions.

(ii) Let $f(\xi) = 2\xi - \xi^2 - \xi^3$ for all $\xi \in \mathbb{R}$, and let $x^0 = 1$ and $v = w = 0$. Trivially, $(x_1, y_1) \in \mathbb{R}^{\mathbb{Z}_+} \times \mathbb{R}^{\mathbb{Z}_+}$, defined by $x_1(t) = 1 = y_1(t)$ for all $t \in \mathbb{Z}_+$, is a solution to (2.19). Moreover, by setting $y_2(0) = -1$ and computing (x_2, y_2) iteratively, we obtain another solution of (2.19) $\diamondsuit$

2.2 Global asymptotic and input-to-state/output stability

The primary focus of this section is to extend the stability result [108, Theorem 13] to Lur'e systems of the form (2.17). As previously mentioned, [108, Theorem 13] concerns discrete-time systems that are of the form (1.2). Additionally, it is of interest to us in this section to further extend the aforementioned result so that it includes output stability. All of the aforementioned extensions of [108, Theorem 13] are nontrivial, as we shall see.

We split this section into three parts. In the first we present a global asymptotic stability criterion from the literature; in the second we give the main result of this section: the previously described extension of [108, Theorem 13]; and the final part comprises a corollary that guarantees input-to-state stability with bias (see, for example, [59]) of (2.17).

2.2.1 Global asymptotic stability

We begin with the following definition.

Definition 2.2.1. *Let* $\Sigma \in \mathbb{L}$ *and* $f : \mathbb{R}^p \to \mathbb{R}^m$. *We say that* (2.17) *is* globally stable in the large (GS)*, if there exists* $c > 0$ *such that*

$$\|x(t)\| + \|y(t)\| \le c\|x(0)\| \quad \forall\, t \in \mathbb{Z}_+, \forall\, (0, 0, x, y) \in \mathcal{B}.$$

We say that (2.17) *is* globally asymptotically stable in the large (GAS)*, if it is GS and*

$$\lim_{t\to\infty} x(t) = 0 \text{ and } \lim_{t\to\infty} y(t) = 0 \quad \forall\, (0, 0, x, y) \in \mathcal{B}.$$

Remark 2.2.2. We briefly explain why the definition we give of GS contains the words "in the large". This is because, in the literature (see, such as, [53, 77]), 'standard' global stability is defined in a weaker manner. Indeed, a discrete-time global stability notion is often defined as: for all $\varepsilon > 0$, there exists $\delta > 0$ such that, for all $(0, 0, x, y) \in \mathcal{B}$ satisfying $\|x(0)\| \le \delta$, we have $\|x(t)\| \le \varepsilon$ for all $t \in \mathbb{Z}_+$. It is easy to verify that global stability in the large implies this notion of stability. Hence, the stability definition we use is a stronger notion than what is commonly considered in the literature. $\diamondsuit$

The following provides sufficient conditions for when (2.17) is GS and GAS.

Theorem 2.2.3. *Let* $\Sigma \in \mathbb{L}$, $K \in \mathbb{R}^{m\times p}$, $r > 0$ *and* $f : \mathbb{R}^p \to \mathbb{R}^m$. *Assume that* $\mathbb{B}_{\mathbb{C}}(K, r) \subseteq \mathbb{S}_{\mathbb{C}}(\mathbf{G})$, $\|D^K\| < 1/r$ *and* (A) *holds. The following statements hold.*

(i) *If*

$$\|f(\xi) - K\xi\| \le r\|\xi\| \quad \forall\, \xi \in \mathbb{R}^p, \tag{2.20}$$

then (2.17) *is GS.*

(ii) *If f is continuous and*

$$\|f(\xi) - K\xi\| < r\|\xi\| \quad \forall\, \xi \in \mathbb{R}^p \backslash \{0\}, \tag{2.21}$$

then (2.17) *is GAS.*

We shall not provide a proof of Theorem 2.2.3, since it can be deduced from a combination of [106, Proposition 4.2.1] and [106, Proposition 4.2.9]. We will, however, provide some commentary of the result in the following remark.

Remark 2.2.4. (i) We note that in the situation without feedthrough, i.e. $D = 0$, then the condition $\|D^K\| < 1/r$ in the statement of Theorem 2.2.3 is trivially satisfied.

(ii) Statement (ii) of Theorem 2.2.3 may be interpreted as saying that the complexified Aizerman conjecture is true (see, for example [53]). The underlying inspiration of the result is [53, Theorem 5.6.22], where, in a continuous-time setting, the Aizerman conjecture is proven to be true over the complex numbers, for systems with zero feedthrough. ◊

2.2.2 Input-to-state/output stability

Our attention now turns to proving the previously described extension of the input-to-state stability criterion [108, Theorem 13]. To do so, we begin by defining the following.

Definition 2.2.5. *Let $\Sigma \in \mathbb{L}$ and $f : \mathbb{R}^p \to \mathbb{R}^m$.*

(i) *We say that* (2.17) *is* input-to-state stable (ISS)*, if there exist $\psi \in \mathcal{KL}$ and $\phi \in \mathcal{K}$ such that*

$$\|x(t)\| \leq \psi(\|x(0)\|, t) + \phi\left(\max_{s \in \underline{t-1}} \left\| \begin{pmatrix} v(s) \\ w(s) \end{pmatrix} \right\| \right) \quad \forall\, t \in \mathbb{N},\, \forall\, (v, w, x, y) \in \mathcal{B}. \tag{2.22}$$

(ii) *We say that* (2.17) *is* input-to-state/output stable (ISOS)*, if there exist $\psi \in \mathcal{KL}$ and $\phi \in \mathcal{K}$ such that* (2.22) *holds and*

$$\|y(t)\| \leq \psi(\|x(0)\|, t) + \phi\left(\max_{s \in \underline{t}} \left\| \begin{pmatrix} v(s) \\ w(s) \end{pmatrix} \right\| \right) \quad \forall\, t \in \mathbb{Z}_+,\, \forall\, (v, w, x, y) \in \mathcal{B}. \tag{2.23}$$

In the case that $D = 0$, then ISS trivially implies ISOS. However, when $D \neq 0$, ISS does not necessarily imply ISOS. The following example gives a system where $D \neq 0$ and (2.17) is ISS but not ISOS. This hence provides motivation for studying ISOS.

Example 2.2.6. Consider (2.17) where $n = m = p = q = 2$, A is any Schur matrix,

$$B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = D, \qquad B_{\mathrm{e}} = I = D_{\mathrm{e}},$$

and $f : \mathbb{R}^2 \to \mathbb{R}^2$ is the identity map. Since $BC = 0 = BD$ and A is Schur, it is easily checked that (2.17) is ISS. We shall now show that (2.17) is not ISOS. To this end, first let $x^0 \in \mathbb{R}^2$ and let $x := (x^1\; x^2)^T \in (\mathbb{R}^2)^{\mathbb{Z}_+}$ be the unique solution of

$$x^+ = Ax\,, \quad x(0) = x^0\,.$$

By setting $w := (-x^1\ 0)^T \in (\mathbb{R}^2)^{\mathbb{Z}_+}$ and $y := (y^1\ 0)^T \in (\mathbb{R}^2)^{\mathbb{Z}_+}$ for arbitrary $y^1 \in \mathbb{R}^{\mathbb{Z}_+}$, we see that

$$Cx + D(y+w) = \begin{pmatrix} x^1 \\ 0 \end{pmatrix} + \begin{pmatrix} y^1 - x^1 \\ 0 \end{pmatrix} = \begin{pmatrix} y^1 \\ 0 \end{pmatrix} = y,$$

and, since $BC = 0 = BD$ and $Bw = 0$,

$$x^+ = Ax = Ax + B(Cx + D(y+w) + w) = Ax + B(y+w).$$

Hence, $(0, w, x, y) \in \mathcal{B}$. This holds for any $y^1 \in \mathbb{R}^{\mathbb{Z}_+}$ and so (2.17) is not ISOS. $\Diamond$

Before moving on, we present another example to further motivate our investigation of ISS of (2.17). Indeed, in the following, we provide an example of (2.17) which is not ISS, however if we only consider the situation wherein $w = 0$, then (2.17) is ISS in the sense that there exist $\psi \in \mathcal{KL}$ and $\phi \in \mathcal{K}$ such that

$$\|x(t)\| \leq \psi(\|x(0)\|, t) + \phi\left(\max_{s \in \underline{t-1}} \|v(s)\|\right) \quad \forall\, t \in \mathbb{N},\, \forall\, (v, 0, x, y) \in \mathcal{B}. \tag{2.24}$$

This version of ISS is what is discussed in much of the literature.

Example 2.2.7. Let $\Sigma \in \mathbb{L}$ where $n = m = p = q = 2$, A, D and D_{e} are each the zero matrix, B_{e} is the identity matrix, and

$$B = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} = C.$$

Furthermore, define $f : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$f(\xi) = 2\xi_1\xi_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \forall\, \xi = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \in \mathbb{R}^2.$$

Let $v(t) = \begin{pmatrix} 1 & 0 \end{pmatrix}^T = w(t)$ for all $t \in \mathbb{Z}_+$ and consider $(v, w, x, y) \in \mathcal{B}$ with initial condition $x(0) = \begin{pmatrix} 1 & 1 \end{pmatrix}^T$. By writing $x(t) = \begin{pmatrix} x_1(t) & x_2(t) \end{pmatrix}^T$, where $x_1, x_2 \in \mathbb{R}^{\mathbb{Z}_+}$, then we see that

$$x^+ = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} f\left(\begin{pmatrix} 1 \\ x_1 + x_2 \end{pmatrix}\right) + \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad x(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

From this, it is easily checked that $\|x(t)\| \to \infty$ as $t \to \infty$. However, v and w are bounded, which therefore shows that (2.17) is not ISS.
If we now consider (2.17) in the situation that $w = 0$, then, for all $(v, 0, x, y) \in \mathcal{B}$,

$$x^+ = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} f\left(\begin{pmatrix} 0 \\ x_1 + x_2 \end{pmatrix}\right) + v = v,$$

where again we write $x(t) = \begin{pmatrix} x_1(t) & x_2(t) \end{pmatrix}^T$, $x_1, x_2 \in \mathbb{R}^{\mathbb{Z}_+}$. From this, we hence deduce that there trivially exist $\psi \in \mathcal{KL}$ and $\phi \in \mathcal{K}$ such that (2.24) holds. We comment that, in the situation where $w = 0$, since $D = D_{\mathrm{e}} = 0$, (2.17) is of the form (1.2). $\Diamond$

We also present the following definition.

Definition 2.2.8. *Let* $\Sigma \in \mathbb{L}$ *and* $f : \mathbb{R}^p \to \mathbb{R}^m$. *A continuous function* $V : \mathbb{R}^n \to \mathbb{R}_+$ *is said to be an* ISS-Lyapunov function *for* (2.17) *if there exist* $\mathcal{K}_\infty$ *functions* $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ *such that*

$$\alpha_1(\|\xi\|) \le V(\xi) \le \alpha_2(\|\xi\|) \quad \forall\, \xi \in \mathbb{R}^n, \tag{2.25}$$

and, for all $(v, w, x, y) \in \mathcal{B}$ *and all* $t \in \mathbb{Z}_+$,

$$V(x(t+1)) - V(x(t)) \le -\alpha_3(\|x(t)\|) + \alpha_4\left(\left\|\begin{pmatrix} v(t) \\ w(t) \end{pmatrix}\right\|\right). \tag{2.26}$$

It is well-known (see, for example, [61]) that the existence of an ISS-Lyapunov function is necessary and sufficient for determining when a system of the form

$$x^+ = g(x, v), \tag{2.27}$$

where $g : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}^n$ is continuous, is ISS. Unfortunately, we cannot use the work of [61] to deduce that ISS-Lyapunov functions infer ISS of (2.17), since we cannot necessarily write (2.17) in the form of (2.27). However, if we assume that $I - Df$ is invertible and that f and $(I - Df)^{-1}$ are continuous, then (2.17) can indeed be expressed as

$$x^+ = Ax + Bf\left((I - Df)^{-1}(Cx + D_{\mathrm{e}}v + w)\right) + B_{\mathrm{e}}v,$$

and is hence in the form of (2.27). Interestingly, an example of (2.17) being ISS but also such that $I - Df$ is not invertible, is given in Example 2.2.6. Therefore, one may wonder whether or not the existence of an ISS-Lyapunov function guarantees ISS of (2.17) without the extra invertibility and continuity assumptions. The next proposition gives precisely this.

Proposition 2.2.9. *Let* $\Sigma \in \mathbb{L}$ *and* $f : \mathbb{R}^p \to \mathbb{R}^m$. *If there exists an ISS-Lyapunov function* $V : \mathbb{R}^n \to \mathbb{R}_+$, *then* (2.17) *is ISS. Furthermore, the functions* ψ *and* ϕ *in* (2.22) *only depend upon* α_1, α_2, α_3 *and* α_4 *in* (2.25) *and* (2.26).

Proposition 2.2.9 is not difficult to prove, and a similar argument to that used in the proof of [106, Proposition 5.2.4] may be used. For completeness, we provide a proof in Appendix A.

We now present the main result of this section, which shows that, under the same assumptions of [108, Theorem 13], (2.17) is ISOS.

Theorem 2.2.10. *Let* $\Sigma \in \mathbb{L}$, $K \in \mathbb{R}^{m\times p}$, $r > 0$ *and* $\alpha \in \mathcal{K}_\infty$. *Assume that* $\mathbb{B}_{\mathbb{C}}(K, r) \subseteq \mathbb{S}_{\mathbb{C}}(\mathbf{G})$ *and that* (A) *holds. Then there exist* $\psi \in \mathcal{KL}$ *and* $\phi \in \mathcal{K}$ *such that, for all* $f : \mathbb{R}^p \to \mathbb{R}^m$ *satisfying*

$$\|f(\xi) - K\xi\| \le r\|\xi\| - \alpha(\|\xi\|) \quad \forall\, \xi \in \mathbb{R}^p, \tag{2.28}$$

(2.22) *and* (2.23) *hold. In particular, the Lur'e system* (2.17) *is ISOS.*

Before proving Theorem 2.2.10, we provide some commentary in the form of a remark.

Remark 2.2.11. (i) The assumption that $\alpha \in \mathcal{K}_\infty$ in Theorem 2.2.10 cannot be weakened to $\alpha \in \mathcal{K}$. Indeed, we refer the reader to [108] for counterexamples.

(ii) It is worth mentioning that there exist systems which satisfy the assumptions of Theorem 2.2.3, and so are GAS, but are not ISOS. Once again, examples can be found in [108]. ◊

Proof of Theorem 2.2.10. We split the proof into two parts. The first part is a generalisation of the method shown in the proof of [108, Theorem 13] and seeks to show ISS, and the second part addresses ISOS. For the first part, we construct an ISS-Lyapunov function in order to invoke Proposition 2.2.9. To obtain such a Lyapunov function, we will construct two functions $U, V : \mathbb{R}^n \to \mathbb{R}_+$ whose sum is an ISS-Lyapunov function. To this end, we begin with the derivation of U, and invoke Lemma 2.1.29 to obtain the existence of a positive definite matrix $Q = Q^T \in \mathbb{R}^{n\times n}$ and $\delta > 0$ such that the function $U_1 : \mathbb{R}^n \to \mathbb{R}_+$, defined by $U_1(\xi) := \langle Q\xi, \xi\rangle$ for all $\xi \in \mathbb{R}^n$, has the property that, for all $(u, v, x, y) \in \mathcal{B}^{\text{lin}}$ and all $t \in \mathbb{Z}_+$, (2.14) holds but with V replaced by U_1. We fix $f : \mathbb{R}^p \to \mathbb{R}^m$ satisfying (2.28). Since $(v, w, x, y) \in \mathcal{B}_f$ if, and only if, $(f(y + w), v, x, y) \in \mathcal{B}^{\text{lin}}$, we hence obtain that, for all $(v, w, x, y) \in \mathcal{B}$ and all $t \in \mathbb{Z}_+$,

$$U_1(x(t+1))-U_1(x(t)) \leq -\delta\|x(t)\|^2+\|y(t)\|^2+\|f(y(t)+w(t))\|^2+\left\|\begin{pmatrix} B_{\text{e}}v(t) \\ D_{\text{e}}v(t)\end{pmatrix}\right\|^2. \quad (2.29)$$

We now utilise assumption (2.28) so that, for all $(v, w, x, y) \in \mathcal{B}$ and all $t \in \mathbb{Z}_+$,

$$\begin{aligned}\|f(y(t) + w(t))\|^2 &\leq (\|f(y(t)) + w(t)) - K(y(t) + w(t))\| + \|K(y(t) + w(t))\|)^2 \\ &\leq (r + \|K\|)^2\|y(t) + w(t)\|^2.\end{aligned}$$

By combining this with (2.29) and utilising the triangle inequality, we obtain, for all $(v, w, x, y) \in \mathcal{B}$ and all $t \in \mathbb{Z}_+$,

$$\begin{aligned}U_1(x(t + 1)) - U_1(x(t)) \leq &- \delta\|x(t)\|^2 + (\|y(t) + w(t)\| + \|w(t)\|)^2 \\ &+ (r + \|K\|)^2\|y(t) + w(t)\|^2 + (\|B_{\text{e}}\|^2 + \|D_{\text{e}}\|^2)\|v(t)\|^2.\end{aligned}$$

By further expanding the above inequality and using a simple comparison argument, we see that, for all $(v, w, x, y) \in \mathcal{B}$ and all $t \in \mathbb{Z}_+$,

$$\begin{aligned}U_1(x(t + 1)) - U_1(x(t)) \leq &- \delta\|x(t)\|^2 + 4\|y(t) + w(t)\|^2 + 4\|w(t)\|^2 \\ &+ (r + \|K\|)^2\|y(t) + w(t)\|^2 + (\|B_{\text{e}}\|^2 + \|D_{\text{e}}\|^2)\|v(t)\|^2.\end{aligned}$$

Define

$$c_1 := \max\{4 + (r + \|K\|)^2, \|B_{\text{e}}\|^2 + \|D_{\text{e}}\|^2\} > 1,$$

so that, for all $(v, w, x, y) \in \mathcal{B}$ and all $t \in \mathbb{Z}_+$, the above inequality implies

$$U_1(x(t + 1)) - U_1(x(t)) \leq -\delta\|x(t)\|^2 + c_1\left(\|y(t) + w(t)\|^2 + \|w(t)\|^2 + \|v(t)\|^2\right).$$

We now define $\delta_1 := \delta/c_1$ and $U_2 : \mathbb{R}^n \to \mathbb{R}_+$ by $U_2(\xi) := U_1(\xi)/c_1$ for all $\xi \in \mathbb{R}^n$. An application of the previous inequality thus gives, for all $(v, w, x, y) \in \mathcal{B}$ and all $t \in \mathbb{Z}_+$,

$$U_2(x(t + 1)) - U_2(x(t)) \leq -\delta_1\|x(t)\|^2 + \|y(t) + w(t)\|^2 + \left\|\begin{pmatrix} v(t) \\ w(t)\end{pmatrix}\right\|^2. \quad (2.30)$$

Let $c_2 \geq \delta_1$ be such that $U_2(\xi) \leq c_2\|\xi\|^2$ for all $\xi \in \mathbb{R}^n$, and define $b > 1$ such that

$$a := b\left(1 - \frac{\delta_1}{2c_2}\right) < 1.$$

Furthermore, define $\alpha_1 \in \mathcal{K}_\infty$ by

$$\alpha_1(s) := \frac{r}{2}\sqrt{s}\alpha(\sqrt{s}) \quad \forall\, s \geq 0, \tag{2.31}$$

and utilise an application of statement (ii) of Lemma 2.1.7 to obtain the existence of $k > 1$ such that

$$\alpha_1(as_1 + s_2) \leq \alpha_1(s_1) + \alpha_1(ks_2) \quad \forall\, s_1, s_2 \geq 0. \tag{2.32}$$

Statement (iii) of the aforementioned result also yields the existence of $\eta \in \mathcal{K}_\infty$ such that

$$\alpha_1(s_1 - s_2) \leq \alpha_1(bs_1) - \eta(s_2) \quad \forall\, s_1 \geq s_2 \geq 0, \tag{2.33}$$

and

$$\frac{\eta(s)}{\sqrt{s}} \to \infty \text{ as } s \to \infty. \tag{2.34}$$

We subsequently define $U : \mathbb{R}^n \to \mathbb{R}_+$ by $U(\xi) := \alpha_1(cU_2(\xi))$ for all $\xi \in \mathbb{R}^n$, where $c := 1/2bk$. By combining (2.30) with (2.33), we thus have, for all $(v, w, x, y) \in \mathcal{B}$ and all $t \in \mathbb{Z}_+$,

$$\begin{aligned} &U(x(t+1)) = \alpha_1(cU_2(x(t+1))) \\ &\leq \alpha_1\left(c\left(U_2(x(t)) - \delta_1\|x(t)\|^2 + \|y(t)+w(t)\|^2 + \left\|\begin{pmatrix} v(t) \\ w(t)\end{pmatrix}\right\|^2\right)\right) \\ &\leq \alpha_1\left(bc\left(U_2(x(t)) - \frac{\delta_1}{2}\|x(t)\|^2 + \|y(t)+w(t)\|^2 + \left\|\begin{pmatrix} v(t) \\ w(t)\end{pmatrix}\right\|^2\right)\right) - \eta\left(c\frac{\delta_1}{2}\|x(t)\|^2\right). \end{aligned} \tag{2.35}$$

We pause here and note that (2.34) implies the existence of $\mu \in \mathcal{K}_\infty$ such that

$$\eta\left(c\frac{\delta_1}{2}s^2\right) \geq s\mu(s) \quad \forall\, s \geq 0. \tag{2.36}$$

Moreover, recalling that $U_2(\xi) \leq c_2\|\xi\|^2$ for all $\xi \in \mathbb{R}^n$, we see that

$$-\frac{\delta_1}{2}\|\xi\|^2 \leq -\frac{\delta_1}{2c_2}U_2(\xi) \quad \forall\, \xi \in \mathbb{R}^n,$$

which in turn implies,

$$b\left(U_2(\xi) - \frac{\delta_1}{2}\|\xi\|^2\right) \leq bU_2(\xi)\left(1 - \frac{\delta_1}{2c_2}\right) = aU_2(\xi) \quad \forall\, \xi \in \mathbb{R}^n.$$

Therefore, by combining this with (2.35) and (2.36), we have, for all $(v, w, x, y) \in \mathcal{B}$ and all $t \in \mathbb{Z}_+$,

$$U(x(t+1)) \leq \alpha_1\left(acU_2(x(t)) + bc\|y(t)+w(t)\|^2 + bc\left\|\begin{pmatrix} v(t) \\ w(t)\end{pmatrix}\right\|^2\right) - \|x(t)\|\mu(\|x(t)\|).$$

Applying (2.32), along with Lemma 2.1.3 and the definition of c, gives, for all $(v, w, x, y) \in \mathcal{B}$ and all $t \in \mathbb{Z}_+$,

$$\begin{aligned} U(x(t+1)) &\leq \alpha_1 \left(cU_2(x(t))\right) + \alpha_1 \left(bck \left(\|y(t)+w(t)\|^2 + \left\| \begin{pmatrix} v(t) \\ w(t) \end{pmatrix} \right\|^2 \right) \right) \\ &\quad - \|x(t)\|\mu(\|x(t)\|) \\ &\leq U(x(t)) + \alpha_1(\|y(t)+w(t)\|^2) + \alpha_1 \left(\left\| \begin{pmatrix} v(t) \\ w(t) \end{pmatrix} \right\|^2 \right) - \|x(t)\|\mu(\|x(t)\|). \end{aligned} \tag{2.37}$$

With the preamble given at the start of the present proof in mind, we now seek to find V such that $U + V$ is an ISS-Lyapunov function. To this end, invoking Lemma 2.1.27 gives the existence of $\kappa > 0$ and a positive semi-definite matrix $P = P^T \in \mathbb{R}^{n\times n}$ such that the function $V : \mathbb{R}^n \to \mathbb{R}_+$, defined by $V(\xi) := \langle P\xi, \xi\rangle$ for all $\xi \in \mathbb{R}^n$, satisfies, for all $(u, v, x, y) \in \mathcal{B}^{\text{lin}}$ and all $t \in \mathbb{Z}_+$,

$$\begin{aligned} V(x(t+1)) - V(x(t)) &\leq -r^2\|y(t)\|^2 + \|u(t) - Ky(t)\|^2 \\ &\quad + \kappa \left\| \begin{pmatrix} B_{\text{e}}v(t) \\ D_{\text{e}}v(t) \end{pmatrix} \right\| \left(\left\| \begin{pmatrix} B_{\text{e}}v(t) \\ D_{\text{e}}v(t) \end{pmatrix} \right\| + \|u(t) - Ky(t)\| + \|x(t)\| \right). \end{aligned}$$

Since $(v, w, x, y) \in \mathcal{B}_f$ if, and only if, $(f(y + w), v, x, y) \in \mathcal{B}^{\text{lin}}$, we hence obtain that, for all $(v, w, x, y) \in \mathcal{B}$ and all $t \in \mathbb{Z}_+$,

$$\begin{aligned} &V(x(t+1)) - V(x(t)) \leq -r^2\|y(t)\|^2 + \|f(y(t)+w(t)) - Ky(t)\|^2 \\ &\quad + \kappa \left\| \begin{pmatrix} B_{\text{e}}v(t) \\ D_{\text{e}}v(t) \end{pmatrix} \right\| \left(\left\| \begin{pmatrix} B_{\text{e}}v(t) \\ D_{\text{e}}v(t) \end{pmatrix} \right\| + \|f(y(t)+w(t)) - Ky(t)\| + \|x(t)\| \right). \end{aligned} \tag{2.38}$$

We pause here and derive some useful inequalities. As for the first, we use a trivial comparison argument to obtain that

$$\begin{aligned} \|f(\xi+\zeta) - K\xi\|^2 &\leq (\|f(\xi+\zeta) - K(\xi+\zeta)\| + \|K\zeta\|)^2 \\ &\leq 4\|f(\xi+\zeta) - K(\xi+\zeta)\|^2 + 4\|K\zeta\|^2 \quad \forall \xi, \zeta \in \mathbb{R}^p. \end{aligned}$$

By combining this with (2.28), we see that

$$\begin{aligned} \|f(\xi+\zeta) - K\xi\|^2 \leq &4r^2\|\xi+\zeta\|^2 - 8r\|\xi+\zeta\|\alpha(\|\xi+\zeta\|) \\ &+ 4\alpha(\|\xi+\zeta\|)^2 + 4\|K\zeta\|^2 \quad \forall \xi, \zeta \in \mathbb{R}^p. \end{aligned} \tag{2.39}$$

Moreover, since $\|\xi\| \geq |\|\xi+\zeta\| - \|\zeta\||$ for all $\xi, \zeta \in \mathbb{R}^p$,

$$\begin{aligned} \|\xi\|^2 &\geq (\|\zeta\| - \|\xi+\zeta\|)(\|\xi+\zeta\|) - \|\zeta\|) \\ &= 2\|\zeta\|\|\xi+\zeta\| - \|\zeta\|^2 - \|\xi+\zeta\|^2 \quad \forall \xi, \zeta \in \mathbb{R}^p. \end{aligned}$$

Hence,

$$-r^2\|\xi\|^2 \leq -2r^2\|\zeta\|\|\xi+\zeta\| + r^2\|\zeta\|^2 + r^2\|\xi+\zeta\|^2 \quad \forall \xi, \zeta \in \mathbb{R}^p.$$

Now, fix $\xi, \zeta \in \mathbb{R}^p$ and note that if $\|\zeta\| \geq 3\|\xi+\zeta\|$, then

$$-r^2\|\xi\|^2 \leq -5r^2\|\xi+\zeta\|^2 + r^2\|\zeta\|^2,$$

and if $\|\zeta\| \leq 3\|\xi + \zeta\|$, then

$$-r^2\|\xi\|^2 \leq \frac{1}{3}r^2\|\zeta\|^2 + r^2\|\xi + \zeta\|^2.$$

Whence,

$$-r^2\|\xi\|^2 \leq \frac{4}{3}r^2\|\zeta\|^2 - 4r^2\|\xi + \zeta\|^2 \quad \forall\, \xi, \zeta \in \mathbb{R}^p.$$

We now combine this with (2.39) to obtain that

$$\begin{aligned}\|f(\xi+\zeta) - K\xi\|^2 - r^2\|\xi\|^2 \leq & - 8r\|\xi+\zeta\|\alpha(\|\xi+\zeta\|) + 4\alpha(\|\xi+\zeta\|)^2 \\ & + 4\|K\zeta\|^2 + \frac{4}{3}r^2\|\zeta\|^2 \quad \forall\, \xi, \zeta \in \mathbb{R}^p.\end{aligned}$$

Since (2.28) implies that $\alpha(\|\eta\|) \leq r\|\eta\|$ for all $\eta \in \mathbb{R}^p$, the above inequality yields that

$$\begin{aligned}\|f(\xi+\zeta) - K\xi\|^2 - r^2\|\xi\|^2 \leq -4r\|\xi+\zeta\|\alpha(\|\xi+\zeta\|) + \left(4\|K\|^2 + \frac{4}{3}r^2\right)\|\zeta\|^2 \\ \forall\, \xi, \zeta \in \mathbb{R}^p. \qquad (2.40)\end{aligned}$$

Before continuing with (2.38), we note that, from (2.28), for all $(v, w, x, y) \in \mathcal{B}$ and all $t \in \mathbb{Z}_+$,

$$\begin{aligned}&\kappa \left\| \begin{pmatrix} B_{\mathrm{e}}v(t) \\ D_{\mathrm{e}}v(t)\end{pmatrix} \right\| \|f(y(t)+w(t)) - Ky(t)\| \\ &\leq \kappa\sqrt{\|B_{\mathrm{e}}\|^2 + \|D_{\mathrm{e}}\|^2} \left\| \begin{pmatrix} v(t) \\ w(t)\end{pmatrix} \right\| \left(r\|y(t)+w(t)\| + \|K\| \left\| \begin{pmatrix} v(t) \\ w(t)\end{pmatrix} \right\| \right),\end{aligned}$$

and we also invoke statement (i) of Lemma 2.1.7 to yield the existence of $\gamma_1 \in \mathcal{K}_\infty$ such that

$$r\kappa\sqrt{\|B_{\mathrm{e}}\|^2 + \|D_{\mathrm{e}}\|^2}s_1 s_2 \leq \frac{7r}{2}s_1\alpha(s_1) + \gamma_1(s_2) \quad \forall\, s_1, s_2 \geq 0.$$

With the above two inequalities in mind, (2.38) and (2.40) therefore give that, for all $(v, w, x, y) \in \mathcal{B}$ and all $t \in \mathbb{Z}_+$,

$$\begin{aligned}V(x(t+1)) - V(x(t)) \leq & - 4r\|y(t)+w(t)\|\alpha(\|y(t)+w(t)\|) + \left(4\|K\|^2 + \frac{4r^2}{3}\right)\|w(t)\|^2 \\ & + \frac{7r}{2}\|y(t)+w(t)\|\alpha(\|y(t)+w(t)\|) + \gamma_1\left(\left\| \begin{pmatrix} v(t) \\ w(t)\end{pmatrix} \right\|\right) \\ & + \kappa\sqrt{\|B_{\mathrm{e}}\|^2 + \|D_{\mathrm{e}}\|^2}\|K\| \left\| \begin{pmatrix} v(t) \\ w(t)\end{pmatrix} \right\|^2 \\ & + \kappa \left\| \begin{pmatrix} B_{\mathrm{e}}v(t) \\ D_{\mathrm{e}}v(t)\end{pmatrix} \right\| \left(\left\| \begin{pmatrix} B_{\mathrm{e}}v(t) \\ D_{\mathrm{e}}v(t)\end{pmatrix} \right\| + \|x(t)\| \right).\end{aligned}$$

Setting $\gamma_2(s) := \gamma_1(s) + (4\|K\|^2 + 4r^2/3 + \kappa\sqrt{\|B_{\mathrm{e}}\|^2 + \|D_{\mathrm{e}}\|^2}\|K\| + \kappa(\|B_{\mathrm{e}}\|^2 + \|D_{\mathrm{e}}\|^2))s^2$ for all $s \geq 0$, thus gives, for all $(v, w, x, y) \in \mathcal{B}$ and all $t \in \mathbb{Z}_+$,

$$\begin{aligned}V(x(t+1)) - V(x(t)) \leq & - \frac{r}{2}\|y(t)+w(t)\|\alpha(\|y(t)+w(t)\|) + \kappa\|x(t)\| \left\| \begin{pmatrix} B_{\mathrm{e}}v(t) \\ D_{\mathrm{e}}v(t)\end{pmatrix} \right\| \\ & + \gamma_2\left(\left\| \begin{pmatrix} v(t) \\ w(t)\end{pmatrix} \right\|\right).\end{aligned}$$

By recalling the definition of $\alpha_1 \in \mathcal{K}_\infty$ from (2.31), we see that, for all $(v, w, x, y) \in \mathcal{B}$ and all $t \in \mathbb{Z}_+$,

$$V(x(t+1)) - V(x(t)) \leq -\alpha_1(\|y(t) + w(t)\|^2) + \kappa\|x(t)\| \left\| \begin{pmatrix} B_\mathrm{e} v(t) \\ D_\mathrm{e} v(t) \end{pmatrix} \right\| + \gamma_2 \left(\left\| \begin{pmatrix} v(t) \\ w(t) \end{pmatrix} \right\| \right). \tag{2.41}$$

With this in mind, we utilise another application of statement (i) of Lemma 2.1.7 to obtain $\gamma_3 \in \mathcal{K}_\infty$ such that

$$\kappa\sqrt{\|B_\mathrm{e}\|^2 + \|D_\mathrm{e}\|^2} s_1 s_2 \leq \frac{1}{2} s_1 \mu(s_1) + \gamma_3(s_2) \quad \forall\, s_1, s_2 \geq 0.$$

This then subsequently gives, for all $(v, w, x, y) \in \mathcal{B}$ and all $t \in \mathbb{Z}_+$,

$$V(x(t+1)) - V(x(t)) \leq -\alpha_1(\|y(t) + w(t)\|^2) + \frac{1}{2}\|x(t)\|\mu(\|x(t)\|) + \gamma_4 \left(\left\| \begin{pmatrix} v(t) \\ w(t) \end{pmatrix} \right\| \right), \tag{2.42}$$

where $\gamma_4 := \gamma_2 + \gamma_3 \in \mathcal{K}_\infty$.

Finally, we define $W := U + V$ and, for all $s \geq 0$, we further define $\alpha_2(s) := s\mu(s)/2$ and $\alpha_3(s) := \alpha_1(s^2) + \gamma_4(s)$. Invoking (2.37) and (2.42) obtains, for all $(v, w, x, y) \in \mathcal{B}$ and all $t \in \mathbb{Z}_+$,

$$W(x(t+1)) - W(x(t)) \leq -\alpha_2(\|x(t)\|) + \alpha_3 \left(\left\| \begin{pmatrix} v(t) \\ w(t) \end{pmatrix} \right\| \right).$$

Since U_2 is positive definite, let $c_3, c_4 > 0$ be such that $V(\xi) \leq c_3\|\xi\|^2$ and $c_4\|\xi\|^2 \leq U_2(\xi)$ for all $\xi \in \mathbb{R}^n$. Recalling that $c_2\|\xi\|^2 \geq U_2(\xi)$ for all $\xi \in \mathbb{R}^n$, we define $\alpha_4(s) := \alpha_1(c_4 c s^2)$ and $\alpha_5(s) := \alpha_1(c_2 c s^2) + c_3 s^2$ for all $s \geq 0$ so that

$$\alpha_4(\|\xi\|) \leq W(\xi) \leq \alpha_5(\|\xi\|) \quad \forall\, \xi \in \mathbb{R}^n.$$

Therefore, Proposition 2.2.9 yields that the Lur'e system (2.17) is ISS. Moreover, Proposition 2.2.9 also gives that ψ and ϕ in (2.22) only depend on α_2, α_3, α_4 and α_5. An inspection of the previous workings show that these are independent of f.

Our attention now turns to the second part of the proof, that is, showing that the system is ISOS. To this end, by the first part, let $\psi_1 \in \mathcal{KL}$ and $\phi_1 \in \mathcal{K}$ be such that, for all $f : \mathbb{R}^p \to \mathbb{R}^m$ satisfying (2.28),

$$\|x(t)\| \leq \psi_1(\|x(0)\|, t) + \phi_1 \left(\max_{s \in \underline{t-1}} \left\| \begin{pmatrix} v(s) \\ w(s) \end{pmatrix} \right\| \right) \quad \forall\, t \in \mathbb{N},\, \forall\, (v, w, x, y) \in \mathcal{B}. \tag{2.43}$$

Consider that, for all $f : \mathbb{R}^p \to \mathbb{R}^m$, all $(v, w, x, y) \in \mathcal{B}$ and all $t \in \mathbb{Z}_+$, Lemma 2.1.18 implies that

$$\|y(t)\| \leq \|C^K\|\|x(t)\| + \|D^K\|\|f(y(t) + w(t)) - Ky(t)\| + \|(I - DK)^{-1}\|\|D_\mathrm{e}\|\|v(t)\|.$$

If $f : \mathbb{R}^p \to \mathbb{R}^m$ satisfies (2.28), then, for all $(v, w, x, y) \in \mathcal{B}$ and all $t \in \mathbb{Z}_+$,

$$\|y(t)\| \leq \|C^K\|\|x(t)\| + \|D^K\| \left(r\,\mathrm{id} - \alpha\right)\left(\|y(t) + w(t)\|\right) + \left(\|(I - DK)^{-1}\|\|D_\mathrm{e}\| + \|D^K\|\|K\|\right) \left\| \begin{pmatrix} v(t) \\ w(t) \end{pmatrix} \right\|, \tag{2.44}$$

where $\mathrm{id} : \mathbb{R}_+ \to \mathbb{R}_+$ is the identity map. Define $g := (1 - r\|D^K\|)\,\mathrm{id} + \|D^K\|\alpha$ and note that, using Lemma 2.1.24, $r\|D^K\| \le r\|\mathbf{G}^K\| \le 1$. From this we deduce that $g \in \mathcal{K}_\infty$, and so from Lemma 2.1.2, we see that g^{-1} exists and is in $\mathcal{K}_\infty$. If we now rearrange (2.44), we see that, for all $f : \mathbb{R}^p \to \mathbb{R}^m$ satisfying (2.28), all $(v, w, x, y) \in \mathcal{B}$ and all $t \in \mathbb{Z}_+$,

$$\|y(t) + w(t)\| \le g^{-1}\left(\|C^K\|\|x(t)\| + (\|(I - DK)^{-1}\|\|D_{\mathrm{e}}\| + \|D^K\|\|K\|)\left\|\begin{pmatrix} v(t) \\ w(t) \end{pmatrix}\right\| + \|w(t)\|\right).$$

Combining this with (2.43) and the reverse triangle inequality gives that, for all $f : \mathbb{R}^p \to \mathbb{R}^m$ satisfying (2.28), all $(v, w, x, y) \in \mathcal{B}$ and all $t \in \mathbb{Z}_+$,

$$\|y(t)\| \le g^{-1}\left(\|C^K\|\psi_1(\|x(0)\|, t) + \|C^K\|\phi_1\left(\max_{s \in \underline{t}}\left\|\begin{pmatrix} v(s) \\ w(s) \end{pmatrix}\right\|\right) + (1 + \|(I - DK)^{-1}\|\|D_{\mathrm{e}}\| + \|D^K\|\|K\|)\left\|\begin{pmatrix} v(t) \\ w(t) \end{pmatrix}\right\|\right) + \|w(t)\|.$$

We now invoke Lemma 2.1.3 to yield, for all $f : \mathbb{R}^p \to \mathbb{R}^m$ satisfying (2.28), all $(v, w, x, y) \in \mathcal{B}$ and all $t \in \mathbb{Z}_+$,

$$\|y(t)\| \le g^{-1}\left(2\|C^K\|\psi_1(\|x(0)\|, t)\right) + \left\|\begin{pmatrix} v(s) \\ w(s) \end{pmatrix}\right\| + g^{-1}\left(\tilde{c}(\phi_1 + \mathrm{id})\left(\max_{s \in \underline{t}}\left\|\begin{pmatrix} v(s) \\ w(s) \end{pmatrix}\right\|\right)\right),$$

where $\tilde{c} := 2(\|C^K\| + 1 + \|(I - DK)^{-1}\|\|D_{\mathrm{e}}\| + \|D^K\|\|K\|) > 0$. We define $\psi_2 \in \mathcal{KL}$ by $\psi_2(s, t) := g^{-1}\left((2\|C^K\| + 1)\psi_1(s, t)\right)$ for all $(s, t) \in \mathbb{R}_+ \times \mathbb{Z}_+$, and $\phi_2 \in \mathcal{K}$ by

$$\phi_2(s) := s + g^{-1}\left(\tilde{c}(\phi_1 + \mathrm{id})(s)\right),$$

for all $s \ge 0$. Therefore, for all $f : \mathbb{R}^p \to \mathbb{R}^m$ satisfying (2.28), all $(v, w, x, y) \in \mathcal{B}$ and all $t \in \mathbb{Z}_+$,

$$\|y(t)\| \le \psi_2(\|x(0)\|, t) + \phi_2\left(\max_{s \in \underline{t}}\left\|\begin{pmatrix} v(s) \\ w(s) \end{pmatrix}\right\|\right). \tag{2.45}$$

Finally, by defining $\psi \in \mathcal{KL}$ and $\phi \in \mathcal{K}$ by

$$\psi(s, t) := \max\{\psi_1(s, t), \psi_2(s, t)\} \quad \text{and} \quad \phi(s) := \max\{\phi_1(s), \phi_2(s)\} \quad \forall\, s \ge 0, \forall\, t \in \mathbb{Z}_+,$$

we immediately see from (2.43) and (2.45) that the proof is complete. □

We conclude this section with the following example which presents a system that satisfies the assumptions of Theorem 2.2.10.

Example 2.2.12. Consider (2.17) where $n = 2$, $m = p = 1$,

$$A = \begin{pmatrix} \frac{1}{2} & 1 \\ 0 & \frac{1}{2} \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \end{pmatrix}, \quad D = 1,$$

$f : \mathbb{R} \to \mathbb{R}$ is defined by

$$f(\xi) = \frac{2}{3}\mathrm{sign}(\xi)\left(\sqrt{|\xi|+1}-1\right) \quad \forall\, \xi \in \mathbb{R},$$

and q, B_{e} and D_{e} are arbitrary. It is easy to deduce that

$$\mathbf{G}(1) = 3 \quad \text{and} \quad |\mathbf{G}(z)| \leq 3 \quad \forall\, z \in \mathbb{C},\, |z| \geq 1.$$

Whence, we obtain that $\|\mathbf{G}\|_{H^\infty} = 3$. Furthermore, by defining $r = 1/3$, $K = 0$ and $\alpha \in \mathcal{K}_\infty$ by

$$\alpha(s) := \frac{s}{3} - \frac{2}{3}\left(\sqrt{s+1}-1\right) \quad \forall\, s \geq 0,$$

it is straight forward to verify that the assumptions of Theorem 2.2.10 hold. Therefore, in this case, (2.17) is ISOS. ◊

2.2.3 Input-to-state stability with bias

Here, we present a corollary to Theorem 2.2.10 that shows that if the nonlinearity of (2.17) satisfies the sector condition (2.28) but only for all $\xi \in \mathbb{R}^p$ outside of a bounded set, then the same assumptions on the linear system guarantee that (2.17) exhibits a variant of ISS, namely ISS with bias. We now define this notion with respect to (2.17) and, for background reading of ISS with bias, we refer the reader to works such as [59, 108].

Definition 2.2.13. *Let $\Sigma \in \mathbb{L}$ and $f : \mathbb{R}^p \to \mathbb{R}^m$.*

(i) *We say that* (2.17) *is* ISS with bias*, if there exist $\psi \in \mathcal{KL}$, $\phi \in \mathcal{K}$ and $\theta > 0$ such that*

$$\|x(t)\| \leq \psi(\|x(0)\|, t) + \phi\left(\theta + \max_{s \in \underline{t-1}}\left\|\begin{pmatrix} v(s) \\ w(s)\end{pmatrix}\right\|\right) \quad \forall\, t \in \mathbb{N},\, \forall\,(v,w,x,y) \in \mathcal{B}. \tag{2.46}$$

(ii) *We say that* (2.17) *is* ISOS with bias*, if there exist $\psi \in \mathcal{KL}$, $\phi \in \mathcal{K}$ and $\theta > 0$ such that* (2.46) *holds and*

$$\|y(t)\| \leq \psi(\|x(0)\|, t) + \phi\left(\theta + \max_{s \in \underline{t}}\left\|\begin{pmatrix} v(s) \\ w(s)\end{pmatrix}\right\|\right) \quad \forall\, t \in \mathbb{Z}_+,\, \forall\,(v,w,x,y) \in \mathcal{B}. \tag{2.47}$$

(iii) *For a set $S \subseteq \mathbb{R}^p$ and $\xi \in \mathbb{R}^p$, we define*

$$\mathrm{dist}(\xi, S) := \inf\{\|\xi - \zeta\| : \zeta \in S\}.$$

The following is the aforementioned corollary to Theorem 2.2.10. It is a generalisation of [108, Corollary 17] to systems of the form (2.17).

Corollary 2.2.14. *Let $\Sigma \in \mathbb{L}$, $K \in \mathbb{R}^{m\times p}$, $r > 0$ and $\alpha \in \mathcal{K}_\infty$. Assume that $\mathbb{B}_{\mathbb{C}}(K, r) \subseteq \mathbb{S}_{\mathbb{C}}(\mathbf{G})$ and that* (A) *holds. Then there exist $\psi \in \mathcal{KL}$ and $\phi \in \mathcal{K}$ such that, for every nonlinearity $f : \mathbb{R}^p \to \mathbb{R}^m$ that is bounded on bounded sets and satisfies*

$$\|f(\xi) - K\xi\| \leq r\|\xi\| - \alpha(\|\xi\|) \quad \forall\, \xi \in \mathbb{R}^p \text{ with sufficiently large } \|\xi\|, \tag{2.48}$$

it follows that (2.46) *and* (2.47) *hold, where*

$$\theta = \sup_{\xi \in \mathbb{R}^p} \mathrm{dist}\left(f(\xi), \mathbb{B}_{\mathbb{C}}(K\xi, r\|\xi\| - \alpha(\|\xi\|))\right) \in \mathbb{R}_+.$$

Proof. We shall follow the steps laid out in [108, Corollary 17], and so shall only be brief. Indeed, as per [108, Corollary 17], we note that we may assume, without loss of generality, that $\alpha(s) < rs$ for all $s > 0$. With this in mind, we set $\rho(s) := rs - \alpha(s)$ for every $s \geq 0$ so that $\rho(s) > 0$ for all $s > 0$. It is easily seen that $\rho(0) = 0$ and ρ is continuous. Let $f : \mathbb{R}^p \to \mathbb{R}^m$ be bounded on bounded sets and satisfy (2.48). We subsequently define $\hat{f} : \mathbb{R}^p \to \mathbb{R}^m$ by

$$\hat{f}(\xi) := \left.\begin{cases} f(\xi) - K\xi, & \text{if } \|f(\xi) - K\xi\| \leq \rho(\|\xi\|), \\ \frac{f(\xi)-K\xi}{\|f(\xi)-K\xi\|}\rho(\|\xi\|), & \text{if } \|f(\xi) - K\xi\| > \rho(\|\xi\|), \end{cases}\right\} \quad \forall\, \xi \in \mathbb{R}^p,$$

and $\tilde{f} : \mathbb{R}^p \to \mathbb{R}^m$ by $\tilde{f}(\xi) := \hat{f}(\xi) + K\xi$ for all $\xi \in \mathbb{R}^p$. We highlight the fact that $\tilde{f}(\xi) = f(\xi)$ for all $\xi \in \mathbb{R}^p$ such that $\|f(\xi) - K\xi\| \leq \rho(\|\xi\|)$ and also that

$$\|\tilde{f}(\xi) - K\xi\| = \|\hat{f}(\xi)\| \leq \rho(\|\xi\|) \quad \forall\, \xi \in \mathbb{R}^p. \tag{2.49}$$

It is then not difficult to deduce that (see the proof of [108, Corollary 17])

$$\sup_{\xi\in\mathbb{R}^p} \|f(\xi) - \tilde{f}(\xi)\| = \sup_{\xi\in\mathbb{R}^p} \operatorname{dist}\left(f(\xi), \mathbb{B}_{\mathbb{C}}(K\xi, \rho(\|\xi\|))\right) = \theta < \infty. \tag{2.50}$$

To conclude the proof, we note that, for every $(v, w, x, y) \in \mathcal{B}_f(\Sigma)$, it follows that $(\tilde{v}, w, x, y) \in \mathcal{B}_{\tilde{f}}(A, B, \tilde{B}_{\mathrm{e}}, C, D, \tilde{D}_{\mathrm{e}})$, where $\tilde{B}_{\mathrm{e}} := \begin{pmatrix} B & B_{\mathrm{e}} \end{pmatrix}$, $\tilde{D}_{\mathrm{e}} := \begin{pmatrix} D & D_{\mathrm{e}} \end{pmatrix}$ and

$$\tilde{v}(t) := \begin{pmatrix} f(y(t) + w(t)) - \tilde{f}(y(t) + w(t)) \\ v(t) \end{pmatrix} \quad \forall\, t \in \mathbb{Z}_+.$$

In view of (2.49) and (2.50), we invoke Theorem 2.2.10 to obtain $\psi \in \mathcal{KL}$ and $\phi \in \mathcal{K}$ (independent of f) such that (2.46) and (2.47) hold. □

2.3 Convergence properties

In [15], sufficient conditions are given for when continuous-time Lur'e systems, of the form (2.1), exhibit a property known as the *converging-input converging-state property.* Roughly, systems exhibit this notion if, for every vector v^∞, inputs that asymptotically converge to v^∞, generate state trajectories that asymptotically converge to some vector x^∞. Our aim for this section is to present discrete-time analogues of the results of [15], but which also allow for systems with feedthrough and with output disturbances. Moreover, we shall also investigate the extent to which a converging input leads to a converging output.

We split the current section into three, where, in the first part, we collect relevant preliminary results and definitions. In the second part we then investigate convergence properties of (2.17), and then, finally, we concern ourselves with steady-state gain maps in the final portion of the section.

2.3.1 Preliminaries

For an element $x^{\mathrm{e}} \in \mathbb{R}^n$, when the context is clear, we will sometimes abuse notation and interchangeably write x^{e} as an element of $\mathbb{R}^n$ and also the constant function mapping $t \mapsto x^{\mathrm{e}}$, for all $t \in \mathbb{Z}_+$.

Definition 2.3.1. *Let $\Sigma \in \mathbb{L}$ and $f : \mathbb{R}^p \to \mathbb{R}^m$.*

(i) *We say that a quadruple $(v^{\rm e}, w^{\rm e}, x^{\rm e}, y^{\rm e}) \in \mathbb{R}^q \times \mathbb{R}^p \times \mathbb{R}^n \times \mathbb{R}^p$ is an* equilibrium quadruple *of* (2.17)*, if $(v^{\rm e}, w^{\rm e}, x^{\rm e}, y^{\rm e}) \in \mathcal{B}$.*

(ii) *We say that an equilibrium quadruple $(v^{\rm e}, w^{\rm e}, x^{\rm e}, y^{\rm e})$ is* globally asymptotically stable in the large (GAS)*, if there exists $c > 0$ such that, for all $(v^{\rm e}, w^{\rm e}, x, y) \in \mathcal{B}$,*

$$\|x(t) - x^{\rm e}\| + \|y(t) - y^{\rm e}\| \leq c\|x(0) - x^{\rm e}\| \quad \forall\, t \in \mathbb{Z}_+,$$

and

$$\lim_{t\to\infty} x(t) = x^{\rm e} \quad \text{and} \quad \lim_{t\to\infty} y(t) = y^{\rm e}.$$

(iii) *We say that an equilibrium quadruple $(v^{\rm e}, w^{\rm e}, x^{\rm e}, y^{\rm e})$ is* input-to-state stable (ISS) *if there exist $\psi \in \mathcal{KL}$ and $\phi \in \mathcal{K}$ such that, for every $(v, w, x, y) \in \mathcal{B}$,*

$$\|x(t) - x^{\rm e}\| \leq \psi(\|x(0) - x^{\rm e}\|, t) + \phi\left(\max_{s\in\underline{t-1}} \left\| \begin{pmatrix} v(s) - v^{\rm e} \\ w(s) - w^{\rm e} \end{pmatrix} \right\|\right) \quad \forall\, t \in \mathbb{N}. \quad (2.51)$$

(iv) *We say that an equilibrium quadruple $(v^{\rm e}, w^{\rm e}, x^{\rm e}, y^{\rm e})$ is* input-to-state/output stable (ISOS) *if there exist $\psi \in \mathcal{KL}$ and $\phi \in \mathcal{K}$ such that, for every $(v, w, x, y) \in \mathcal{B}$,* (2.51) *holds and*

$$\|y(t) - y^{\rm e}\| \leq \psi(\|x(0) - x^{\rm e}\|, t) + \phi\left(\max_{s\in\underline{t}} \left\| \begin{pmatrix} v(s) - v^{\rm e} \\ w(s) - w^{\rm e} \end{pmatrix} \right\|\right) \quad \forall\, t \in \mathbb{Z}_+.$$

Remark 2.3.2. Let $\Sigma \in \mathbb{L}$ and $f : \mathbb{R}^p \to \mathbb{R}^m$.

(i) If $f(0) = 0$, then it is clear that $(0, 0, 0, 0)$ is an equilibrium quadruple of (2.17).

(ii) If $(0, 0, 0, 0)$ is an equilibrium quadruple of (2.17), then $(0, 0, 0, 0)$ being GAS, ISS or ISOS, is equivalent to (2.17) being GAS, ISS and ISOS, respectively. Indeed, we refer the reader to Definitions 2.2.1 and 2.2.5. ◊

With the previous definitions in mind, we are able to reinterpret Theorems 2.2.3 and 2.2.10 in terms of equilibirum quadruples.

Corollary 2.3.3. *Let $\Sigma \in \mathbb{L}$, $K \in \mathbb{R}^{m\times p}$, $r > 0$ and $f : \mathbb{R}^p \to \mathbb{R}^m$. Assume that $\mathbb{B}_{\mathbb{C}}(K, r) \subseteq \mathbb{S}_{\mathbb{C}}(\mathbf{G})$ and* (A) *holds. The following statements hold.*

(i) *If $\|D^K\| < 1/r$, f is continuous and*

$$\|f(\xi) - K\xi\| < r\|\xi\| \quad \forall\, \xi \in \mathbb{R}^p\backslash\{0\},$$

then $(0, 0, 0, 0)$ is a GAS equilibrium quadruple of (2.17)*.*

(ii) *If there exists $\alpha \in \mathcal{K}_\infty$ such that*

$$\|f(\xi) - K\xi\| \leq \gamma\|\xi\| - \alpha(\|\xi\|) \quad \forall\, \xi \in \mathbb{R}^p,$$

then $(0, 0, 0, 0)$ is an ISOS equilibrium quadruple of (2.17)*.*

Proof. In either case, the assumptions on f guarantee that $f(0) = 0$. As discussed in Remark 2.3.2, this implies that $(0, 0, 0, 0)$ is an equilibrium quadruple of (2.17) and is GAS or ISOS if, and only if, (2.17) is GAS or ISOS, respectively. Theorems 2.2.3 and 2.2.10 hence complete the proof. □

The following function (first explicitly defined in the continuous-time setting in [15]) will be useful in obtaining the existence of equilibrium quadruples.

Definition 2.3.4. *Let $\Sigma \in \mathbb{L}$ and $f : \mathbb{R}^p \to \mathbb{R}^m$. For $K \in \mathbb{S}_{\mathbb{R}}(\mathbf{G})$, we define $F_K : \mathbb{R}^p \to \mathbb{R}^p$ by*

$$F_K(\xi) := \xi - \mathbf{G}^K(1)(f(\xi) - K\xi) \quad \forall\, \xi \in \mathbb{R}^p. \tag{2.52}$$

The next lemma shows a useful invariance property of F_K.

Lemma 2.3.5. *Let $\Sigma \in \mathbb{L}$, $f : \mathbb{R}^p \to \mathbb{R}^m$, $W \subseteq \mathbb{R}^p$ be a non-empty subset and assume that $K \in \mathbb{S}_{\mathbb{R}}(\mathbf{G})$. Then*

$$\begin{aligned} F_K\big(\operatorname{im}(C^K) &+ \operatorname{im}(D^K) + \operatorname{im}\big((I - DK)^{-1}D_{\mathrm{e}}\big) + W\big) \\ &\subseteq \operatorname{im}(C^K) + \operatorname{im}(D^K) + \operatorname{im}\big((I - DK)^{-1}D_{\mathrm{e}}\big) + W, \end{aligned} \tag{2.53}$$

and

$$\begin{aligned} F_K^{-1}\big(\operatorname{im}(C^K) &+ \operatorname{im}(D^K) + \operatorname{im}\big((I - DK)^{-1}D_{\mathrm{e}}\big) + W\big) \\ &\subseteq \operatorname{im}(C^K) + \operatorname{im}(D^K) + \operatorname{im}\big((I - DK)^{-1}D_{\mathrm{e}}\big) + W. \end{aligned} \tag{2.54}$$

Proof. Since $\mathbf{G}^K(1)\xi \in \operatorname{im}(C^K) + \operatorname{im}(D^K)$ for every $\xi \in \mathbb{R}^m$, it is clear that (2.53) holds. Moreover, if $\xi \in F_K^{-1}\big(\operatorname{im}(C^K) + \operatorname{im}(D^K) + \operatorname{im}\big((I - DK)^{-1}D_{\mathrm{e}}\big) + W\big)$ then there exist $\zeta_1 \in \mathbb{R}^n$, $\zeta_2 \in \mathbb{R}^m$, $\zeta_3 \in \mathbb{R}^q$ and $\zeta_4 \in W$ such that

$$\xi - \mathbf{G}^K(1)(f(\xi) - K\xi) = C^K\zeta_1 + D^K\zeta_2 + (I - DK)^{-1}D_{\mathrm{e}}\zeta_3 + \zeta_4.$$

By combining this with the fact that $\mathbf{G}^K(1)(f(\xi) - K\xi) \in \operatorname{im}(C^K) + \operatorname{im}(D^K)$, the proof is complete. □

The following proposition shows that we are able to obtain equilibrium quadruples from certain preimages of the map F_K.

Proposition 2.3.6. *Let $\Sigma \in \mathbb{L}$ be stabilisable and detectable, $f : \mathbb{R}^p \to \mathbb{R}^m$, $v^\infty \in \mathbb{R}^q$ and $w^\infty \in \mathbb{R}^p$. Assume that $K \in \mathbb{S}_{\mathbb{R}}(\mathbf{G})$ and*

$$\mathcal{T}_K := F_K^{-1}\begin{pmatrix} C^K(I - A^K)^{-1}\big(B_{\mathrm{e}} + B^K K D_{\mathrm{e}}\big)\, v^\infty \\ \qquad + (I - DK)^{-1}D_{\mathrm{e}}v^\infty + \big(I + \mathbf{G}^K(1)K\big)\, w^\infty \end{pmatrix} \tag{2.55}$$

is nonempty. Let $z^\infty \in \mathcal{T}_K$ and define $y^\infty := z^\infty - w^\infty$ and

$$x^\infty := (I - A^K)^{-1}\big(B^K(f(z^\infty) - K(z^\infty - w^\infty)) + (B_{\mathrm{e}} + B^K K D_{\mathrm{e}})v^\infty\big)\,. \tag{2.56}$$

Then

$$y^\infty = C^K x^\infty + D^K(f(z^\infty) - K(z^\infty - w^\infty)) + (I - DK)^{-1}D_{\mathrm{e}}v^\infty\,, \tag{2.57}$$

and $(v^\infty, w^\infty, x^\infty, y^\infty)$ is an equilibrium quadruple of (2.17).

Remark 2.3.7. (i) For $\Sigma \in \mathbb{L}$ and $f : \mathbb{R}^p \to \mathbb{R}^m$, Proposition 2.3.6 is motivated by the wish to find $x^\infty \in \mathbb{R}^n$ and $y^\infty \in \mathbb{R}^p$ such that

$$x^\infty = Ax^\infty + Bf(y^\infty + w^\infty) + B_{\mathrm{e}}v^\infty,$$
$$y^\infty = Cx^\infty + Df(y^\infty + w^\infty) + D_{\mathrm{e}}v^\infty,$$

for given $v^\infty \in \mathbb{R}^q$ and $w^\infty \in \mathbb{R}^p$. Since $I - A$ need not be invertible, we include the output feedback matrix K. Indeed, from [72, Theorem 2] (see Remark 2.1.23), the assumptions of Proposition 2.3.6 guarantee that A^K is Schur. In the simple situation wherein $B_{\mathrm{e}} = B$, $D_{\mathrm{e}} = D$, $K = 0$ and $w^\infty = 0$, the condition $\mathcal{T}_K \neq \emptyset$ is equivalent to $F_K^{-1}(\mathbf{G}(1)v^\infty) \neq \emptyset$.

(ii) A continuous-time analogue to Proposition 2.3.6 is given in [15, Proposition 3.1]. Although, as already mentioned, in [15] a much less general system is considered, that is, one without feedthrough and without output disturbances. $\diamondsuit$

Proof of Proposition 2.3.6. For ease of notation in the sequel, we define $f^K(\xi) := f(\xi) - K(\xi)$ for all $\xi \in \mathbb{R}^p$. We begin by recalling from the previous remark that A^K is Schur, and so $(I - A^K)^{-1}$ is well-defined (see Remark 2.3.7). By considering

$$\begin{aligned} C^K x^\infty &= C^K (I - A^K)^{-1} \big(B^K(f(z^\infty) - K(z^\infty - w^\infty)) + (B_{\mathrm{e}} + B^K K D_{\mathrm{e}})v^\infty\big) \\ &= \mathbf{G}^K(1) f^K(z^\infty) - D^K f^K(z^\infty) + C^K(I - A^K)^{-1}\big((B_{\mathrm{e}} + B^K K D_{\mathrm{e}})v^\infty + B^K K w^\infty\big). \end{aligned}$$

By recalling the definition of F_K from (2.52), this becomes

$$\begin{aligned} C^K x^\infty =& z^\infty - F_K(z^\infty) - D^K f^K(z^\infty) \\ &+ C^K(I - A^K)^{-1}\big((B_{\mathrm{e}} + B^K K D_{\mathrm{e}})v^\infty + B^K K w^\infty\big). \end{aligned}$$

Since $z^\infty \in \mathcal{T}_K$, it is then easily checked that

$$\begin{aligned} C^K x^\infty =& z^\infty - D^K f^K(z^\infty) - (I - DK)^{-1} D_{\mathrm{e}} v^\infty - w^\infty - D^K K w^\infty \\ =& y^\infty - D^K(f(y^\infty + w^\infty) - K y^\infty) - (I - DK)^{-1} D_{\mathrm{e}} v^\infty. \end{aligned}$$

Therefore,

$$y^\infty = C^K x^\infty + D^K(f(y^\infty + w^\infty) - K y^\infty) + (I - DK)^{-1} D_{\mathrm{e}} v^\infty, \tag{2.58}$$

which is (2.57). To prove that $(v^\infty, w^\infty, x^\infty, y^\infty)$ is an equilibrium quadruple, we note that

$$x^\infty = A^K x^\infty + B^K(f(y^\infty + w^\infty) - K y^\infty) + (B_{\mathrm{e}} + B^K K D_{\mathrm{e}})v^\infty,$$

which, together with (2.58), completes the proof via an application of Lemma 2.1.18. $\square$

With Proposition 2.3.6 in mind, we see that a useful property in determining equilibrium quadruples is surjectivity of F_K. The next result is a discrete-time analogue of [15, Proposition 4.1], and gives conditions sufficient for guaranteeing that F_K is surjective.

Proposition 2.3.8. *Let $\Sigma \in \mathbb{L}$, $K \in \mathbb{R}^{m\times p}$, $r > 0$, $W \subseteq \mathbb{R}^p$ be a non-empty subset and $f : \mathbb{R}^p \to \mathbb{R}^m$. Assume that $\mathbb{B}_{\mathbb{C}}(K, r) \subseteq \mathbb{S}_{\mathbb{C}}(\mathbf{G})$, and that f satisfies*

$$\|f(\xi + \zeta) - f(\zeta) - K\xi\| < r\|\xi\| \quad \forall\, \zeta \in Y,\ \forall\, \xi \in \mathbb{R}^p\backslash\{0\}, \tag{2.59}$$

where $Y \subseteq \mathrm{im}(C^K) + \mathrm{im}(D^K) + \mathrm{im}\big((I - DK)^{-1} D_{\mathrm{e}}\big) + W$ is a non-empty subset. The following statements hold.

(i) $\#F_K^{-1}(\xi) = 1$ *for all* $\xi \in \mathrm{im}(C^K) + \mathrm{im}(D^K) + \mathrm{im}\,\big((I - DK)^{-1}D_\mathrm{e}\big) + W$ *such that* $F_K^{-1}(\xi) \cap Y \neq \emptyset$.

(ii) *If* f *is continuous and* $\|\mathbf{G}^K(1)\| < 1/r$, *then* F_K *is surjective.*

(iii) *If* f *is continuous and there exists* $\eta \in \mathbb{R}^p$ *such that*

$$r\|\xi\| - \|f(\xi + \eta) - f(\eta) - K\xi\| \to \infty \text{ as } \|\xi\| \to \infty,$$

then F_K *is surjective.*

(iv) *If* $Y = \mathrm{im}(C^K) + \mathrm{im}(D^K) + \mathrm{im}\,\big((I - DK)^{-1}D_\mathrm{e}\big) + W$ *and either of the hypotheses of statements (ii) or (iii) hold, then* $\tilde{F}_K : Y \to Y$, *defined by* $\tilde{F}_K(\xi) := F_K(\xi)$ *for all* $\xi \in Y$, *is a bijection.*

In order to prove Proposition 2.3.8, we shall use [15, Lemma 4.2]. For convenience, we present it here.

Lemma 2.3.9. *Let* $g : \mathbb{R}^p \to \mathbb{R}^m$ *and* $r > 0$.

(i) *If there exists* $\eta \in \mathbb{R}^p$ *such that*

$$r\|\xi\| - \|g(\xi + \eta) - g(\eta)\| \to \infty \text{ as } \|\xi\| \to \infty,$$

then, for all $\zeta \in \mathbb{R}^p$,

$$r\|\xi\| - \|g(\xi + \zeta) - g(\zeta)\| \to \infty \text{ as } \|\xi\| \to \infty.$$

(ii) *If* g *is continuous,* $\|g(\xi)\| < r\|\xi\|$ *for all* $\xi \in \mathbb{R}^p \backslash \{0\}$, *and* $r\|\xi\| - \|g(\xi)\| \to \infty$ *as* $\|\xi\| \to \infty$, *then there exists* $\alpha \in \mathcal{K}_\infty$ *such that*

$$\|g(\xi)\| \leq r\|\xi\| - \alpha(\|\xi\|) \quad \forall\, \xi \in \mathbb{R}^p.$$

Proof of Proposition 2.3.8. The proof is a discrete-time analogue of the proof of [15, Proposition 4.1]. We begin with the observation that if $\mathbf{G}^K(1) = 0$, then F_K is trivially a bijective function. For the rest of this proof, we thus assume that $\mathbf{G}^K(1) \neq 0$ and hence $\|\mathbf{G}^K\|_{H^\infty} \neq 0$.

With regards to proving statement (i), let $\xi \in \mathrm{im}(C^K) + \mathrm{im}(D^K) + \mathrm{im}\,\big((I - DK)^{-1}D_\mathrm{e}\big) + W$ be such that $F_K^{-1}(\xi) \cap Y \neq \emptyset$ and let $\xi_1 \in F_K^{-1}(\xi) \cap Y$ and $\xi_2 \in F_K^{-1}(\xi)$. We seek to show that $\xi_1 = \xi_2$. To this end, first note that

$$\begin{aligned}\|\xi_2 - \xi_1\| &= \|F_K(\xi_2) + \mathbf{G}^K(1)(f(\xi_2) - K\xi_2) - F_K(\xi_1) - \mathbf{G}^K(1)(f(\xi_1) - K\xi_1)\| \\ &\leq \|\mathbf{G}^K\|_{H^\infty}\|f(\xi_2) - f(\xi_1) - K(\xi_2 - \xi_1)\|.\end{aligned}$$

If $\xi_1 \neq \xi_2$, then, by Lemma 2.1.24 and (2.59), we obtain that

$$\|\xi_2 - \xi_1\| < r\|\mathbf{G}^K\|_{H^\infty}\|\xi_2 - \xi_1\| \leq \|\xi_2 - \xi_1\|,$$

which is a contradiction. Whence, $\xi_1 = \xi_2$ and $\#F_K^{-1}(\xi) = 1$, proving statement (i).

We now move on to showing that statement (ii) holds and hence we additionally assume that f is continuous and that $\|\mathbf{G}^K(1)\| < 1/r$. We deduce from this that F_K is continuous. Thus, in order to show that F_K is surjective, we refer to [93, Theorem 9.36] and shall prove that F_K is coercive, i.e.

$$\frac{1}{\|\xi\|}\langle F_K(\xi),\xi\rangle \to \infty \text{ as } \|\xi\| \to \infty.$$

To this end, an application of the Cauchy-Schwarz and triangle inequalities yields, for all $\xi \in \mathbb{R}^p$,

$$\begin{aligned}\frac{1}{\|\xi\|}\langle F_K(\xi),\xi\rangle &= \frac{1}{\|\xi\|}\langle \xi - \mathbf{G}^K(1)(f(\xi)-K\xi),\xi\rangle \\ &= \|\xi\| - \frac{\langle \mathbf{G}^K(1)(f(\xi)-K\xi),\xi\rangle}{\|\xi\|} \\ &\geq \|\xi\| - \|\mathbf{G}^K(1)\|\|f(\xi)-K\xi\|. \end{aligned} \tag{2.60}$$

We now fix $\zeta \in Y$ and utilise (2.59) so that, for all $\xi \in \mathbb{R}^p$,

$$\begin{aligned}\|f(\xi)-f(\zeta)-K\xi\| &\leq \|f(\xi-\zeta+\zeta)-f(\zeta)-K(\xi-\zeta)\| + \|K\zeta\| \\ &\leq r\|\xi-\zeta\| + \|K\zeta\| \\ &\leq r\|\xi\| + (\|K\|+r)\|\zeta\|.\end{aligned}$$

By subsequently combining this with (2.60), we see that, for all $\xi \in \mathbb{R}^p$,

$$\begin{aligned}\frac{1}{\|\xi\|}\langle F_K(\xi),\xi\rangle &\geq \|\xi\| - \|\mathbf{G}^K(1)\|\left(\|f(\xi)-f(\zeta)-K\xi\| + \|f(\zeta)\|\right) \\ &\geq \|\xi\| - \|\mathbf{G}^K(1)\|\left(r\|\xi\| + (\|K\|+r)\|\zeta\| + \|f(\zeta)\|\right) \\ &= (1-r\|\mathbf{G}^K(1)\|)\|\xi\| - \|\mathbf{G}^K(1)\|\left((\|K\|+r)\|\zeta\| + \|f(\zeta)\|\right). \end{aligned} \tag{2.61}$$

Since $1 - r\|\mathbf{G}^K(1)\| > 0$, we obtain the desired coercivity of F_K hence completing the proof of statement (ii).
Turning our attention now to statement (iii), we see that under the additional assumptions of continuity of f and that there exists $\eta \in \mathbb{R}^p$ such that

$$r\|\xi\| - \|f(\xi+\eta)-f(\eta)-K\xi\| \to \infty \text{ as } \|\xi\| \to \infty,$$

statement (i) of Lemma 2.3.9 guarantees that, for every $\zeta \in \mathbb{R}^p$,

$$r\|\xi\| - \|f(\xi+\zeta)-f(\zeta)-K\xi\| \to \infty \text{ as } \|\xi\| \to \infty.$$

We fix $\zeta \in Y$ and note that the hypotheses of statement (ii) of Lemma 2.3.9 hold (with respect to the function $\xi \mapsto f(\xi+\zeta)-f(\zeta)-K\xi$). This then implies that there exists $\alpha \in \mathcal{K}_\infty$ (dependent on ζ) such that

$$\|f(\xi+\zeta)-f(\zeta)-K\xi\| \leq r\|\xi\| - \alpha(\|\xi\|) \quad \forall\, \xi \in \mathbb{R}^p.$$

We now act similarly to the proof of statement (ii) and seek to show that F_K is coercive. Indeed, first note that, for all $\xi \in \mathbb{R}^p$, the above inequality yields that

$$\begin{aligned}\|f(\xi)-f(\zeta)-K\xi\| &\leq \|f(\xi-\zeta+\zeta)-f(\zeta)-K(\xi-\zeta)\| + \|K\zeta\| \\ &\leq r\|\xi-\zeta\| - \alpha(\|\xi-\zeta\|) + \|K\zeta\| \\ &\leq r\|\xi\| - \alpha(\|\xi-\zeta\|) + (\|K\|+r)\|\zeta\|.\end{aligned}$$

By combining this with (2.60), we obtain that, for all $\xi \in \mathbb{R}^p$,

$$\begin{aligned}\frac{1}{\|\xi\|}\langle F_K(\xi), \xi\rangle &\geq \|\xi\| - \|\mathbf{G}^K(1)\| \left(\|f(\xi) - f(\zeta) - K\xi\| + \|f(\zeta)\|\right) \\ &\geq (1 - r\|\mathbf{G}^K(1)\|)\|\xi\| + \|\mathbf{G}^K(1)\|\alpha(\|\xi - \zeta\|) \\ &\qquad - \|\mathbf{G}^K(1)\| \left((\|K\| + r)\|\zeta\| + \|f(\zeta)\|\right).\end{aligned}$$

Since $1 - r\|\mathbf{G}^K(1)\| \geq 1 - r\|\mathbf{G}^K\|_{H^\infty} \geq 0$ by Lemma 2.1.24, and $\alpha \in \mathcal{K}_\infty$, we have that F_K is coercive and hence is surjective.
For the final part of the proof, we aim to show that statement (iv) holds. To do so, we let $Y = \mathrm{im}(C^K) + \mathrm{im}(D^K) + \mathrm{im}\left((I - DK)^{-1}D_{\mathrm{e}}\right) + W$ and assume that either of the hypotheses of statements (ii) or (iii) hold. The map F_K is surjective and so for all $\zeta \in Y$, there exists $\xi \in \mathbb{R}^p$ such that $F_K(\xi) = \zeta$. From Lemma 2.3.5, we see that $\xi \in Y$ and so $\tilde{F}_K(\xi) = \zeta$. Moreover, $\#F_K^{-1}(\zeta) = 1$ from statement (i). Therefore, $\tilde{F}_K$ is bijective. □

We shall now see that if (2.59) in fact holds globally (that is, $Y = \mathbb{R}^p$), then a certain map is bijective with a globally Lipschitz inverse. These properties of this map will be useful in subsequent results (see Theorem 2.3.16).

Proposition 2.3.10. *Let $\Sigma \in \mathbb{L}$, $K \in \mathbb{R}^{m\times p}$, $r > 0$ and $f : \mathbb{R}^p \to \mathbb{R}^m$. Assume that $\mathbb{B}_{\mathbb{C}}(K, r) \subseteq \mathbb{S}_{\mathbb{C}}(\mathbf{G})$, $\|D^K\| < 1/r$ and that f satisfies*

$$\|f(\xi + \zeta) - f(\zeta) - K\xi\| < r\|\xi\| \quad \forall\, \xi, \zeta \in \mathbb{R}^p,\ \xi \neq 0. \tag{2.62}$$

Then $I - D^K(f - K)$ is bijective and $(I - D^K(f - K))^{-1}$ is globally Lipschitz with Lipschitz constant $L := 1/(1 - r\|D^K\|) > 0$.

Proof. For ease of notation in the sequel, we define $g := I - D^K(f - K)$. We begin by showing that g is surjective, and will do this by first showing that the map $I - g$ is a contraction. To this end, from (2.62), for all $\xi_1, \xi_2 \in \mathbb{R}^p$,

$$\begin{aligned}\|(I - g)(\xi_1) - (I - g)(\xi_2)\| &= \|D^K(f - K)(\xi_1) - D^K(f - K)(\xi_2)\| \\ &\leq r\|D^K\|\|\xi_1 - \xi_2\|.\end{aligned}$$

Thus, since $r\|D^K\| < 1$, $I - g$ is a contraction. We now fix $\zeta \in \mathbb{R}^p$ and define $h_\zeta(\xi) := \xi - g(\xi) + \zeta$ for all $\xi \in \mathbb{R}^p$. Since $I - g$ is a contraction, it is clear that h_ζ is also. Therefore, an application of the contraction mapping theorem gives the existence of $\eta \in \mathbb{R}^p$ such that $h_\zeta(\eta) = \eta$. That is, $g(\eta) = \zeta$, whence showing surjectivity.
As for injectivity of g, we utilise a contradiction argument and suppose that $g(\xi_1) = g(\xi_2)$ where $\xi_1, \xi_2 \in \mathbb{R}^p$ are such that $\xi_1 \neq \xi_2$. From this we see that

$$\xi_1 - \xi_2 = D^K(f - K)(\xi_1) - D^K(f - K)(\xi_2),$$

which, when combined with (2.62) and the fact that $r\|D^K\| < 1$, gives

$$\|\xi_1 - \xi_2\| < r\|D^K\|\|\xi_1 - \xi_2\| < \|\xi_1 - \xi_2\|.$$

This is a contradiction, hence yielding that g is injective.
Finally, in order to show that g^{-1} is globally Lipschitz, note that, by (2.62),

$$\|\xi_1 - \xi_2\| \leq \|g(\xi_1) - g(\xi_2)\| + r\|D^K\|\|\xi_1 - \xi_2\| \quad \forall\, \xi_1, \xi_2 \in \mathbb{R}^p.$$

Additionally, since $r\|D^K\| < 1$, we have $L := 1/(1 - r\|D^K\|) > 0$, and so the above inequality gives

$$\|\xi_1 - \xi_2\| \leq L\|g(\xi_1) - g(\xi_2)\| \quad \forall\, \xi_1, \xi_2 \in \mathbb{R}^p.$$

From this, it is clear that g^{-1} is Lipschitz continuous with Lipschitz constant L. □

Remark 2.3.11. By combining Proposition 2.1.37 with Proposition 2.3.10, we see that, under the assumptions of Proposition 2.3.10, for a given $x^0 \in \mathbb{R}^n$, $v \in (\mathbb{R}^q)^{\mathbb{Z}_+}$ and $w \in (\mathbb{R}^p)^{\mathbb{Z}_+}$, the IVP (2.18) has a unique solution. ◊

2.3.2 The converging-input converging-state/output property

As already discussed, the continuous-time result [15, Theorem 4.3] provides sufficient conditions for when systems of the form (2.1) exhibit a notion called the converging-input converging-state property. Our intention here is to prove a discrete-time version of this result, which holds for general systems of the form (2.17). We hence begin by giving the following definition.

Definition 2.3.12. *Let $\Sigma \in \mathbb{L}$ and $f : \mathbb{R}^p \to \mathbb{R}^m$. We say that* (2.17) *has the* converging-input converging-state (CICS) property *if for every $v^\infty \in \mathbb{R}^q$ and $w^\infty \in \mathbb{R}^p$, there exists $x^\infty \in \mathbb{R}^n$ such that $\lim_{t\to\infty} x(t) = x^\infty$ for every $(v, w, x, y) \in \mathcal{B}$ with $\lim_{t\to\infty} v(t) = v^\infty$ and $\lim_{t\to\infty} w(t) = w^\infty$.*

Not only shall we present conditions that guarantee that (2.17) exhibits the CICS property, we will also show that, under these same assumptions, the output converges asymptotically to some vector in $\mathbb{R}^p$, thus motivating the following definition.

Definition 2.3.13. *Let $\Sigma \in \mathbb{L}$ and $f : \mathbb{R}^p \to \mathbb{R}^m$. We say that* (2.17) *has the* converging-input converging-state/output (CICSO) property *if for every $v^\infty \in \mathbb{R}^q$ and $w^\infty \in \mathbb{R}^p$, there exists $x^\infty \in \mathbb{R}^n$ and $y^\infty \in \mathbb{R}^p$ such that $\lim_{t\to\infty} x(t) = x^\infty$ and $\lim_{t\to\infty} y(t) = y^\infty$ for every $(v, w, x, y) \in \mathcal{B}$ with $\lim_{t\to\infty} v(t) = v^\infty$ and $\lim_{t\to\infty} w(t) = w^\infty$.*

Of course, in the situation where the feedthrough is zero, the CICS property implies the CICSO property. The following example shows that there exist systems with nonzero feedthrough that exhibit the CICS property but not the CICSO property.

Example 2.3.14. Consider (2.17) where $n = m = p = q = 2$, $A = D_{\mathrm{e}} = 0 \in \mathbb{R}^{2\times 2}$, $B_{\mathrm{e}} = I \in \mathbb{R}^{2\times 2}$,

$$B = \begin{pmatrix} 1/2 & 0 \\ 0 & 0 \end{pmatrix}, \quad C := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad D := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

and $f : \mathbb{R}^2 \to \mathbb{R}^2$ is the identity map. We note that since $BC = B$ and $BD = 0$, the state equation in (2.17) becomes

$$x^+ = 0 + B(Cx + Dy + Dw + 0 + w) + v = Bx + Bw + v.$$

Since B is a Schur matrix, we hence see, from standard linear control theory, that (2.17) has the CICS property.

We now let $v^\infty \in \mathbb{R}^2$ be fixed and set $w^\infty := 0 \in \mathbb{R}^2$. We additionally define $v, w \in (\mathbb{R}^2)^{\mathbb{Z}_+}$ by $v(t) := v^\infty$ and $w(t) := 0$ for all $t \in \mathbb{Z}_+$, and denote by $x = (x_1 x_2)^T \in (\mathbb{R}^2)^{\mathbb{Z}_+}$ the solution of the initial-value problem

$$x^+ = Bx + v^\infty, \quad x(0) = 0.$$

Finally, we define $y = (y_1\, y_2)^T \in (\mathbb{R}^2)^{\mathbb{Z}_+}$ by

$$y(t) := \begin{pmatrix} x_1(t) \\ (-1)^t \end{pmatrix} \quad \forall\, t \in \mathbb{Z}_+.$$

We then see that

$$Cx + Df(y + w) + D_{\mathrm{e}}v = \begin{pmatrix} x_1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ y_2 \end{pmatrix} = y,$$

and

$$Ax + Bf(y + w) + B_{\mathrm{e}}v = Bx + v^\infty = x^+.$$

Therefore, $(v, w, x, y) \in \mathcal{B}$. To conclude the example, we note that, trivially, $v(t)$ and $w(t)$ converge to v^∞ and w^∞ as $t \to \infty$, respectively, but $y(t)$ does not converge. $\Diamond$

Before giving the next result, we recall what it means for a set of functions to be equi-convergent to a vector.

Definition 2.3.15. *We say that a subset $\mathcal{V} \subseteq (\mathbb{R}^q)^{\mathbb{Z}_+}$ is* equi-convergent *to $v^\infty \in \mathbb{R}^q$ if, for all $\varepsilon > 0$, there exists $\tau \in \mathbb{Z}_+$ such that*

$$\|(\Lambda_\tau v)(t) - v^\infty\| \le \varepsilon \quad \forall\, t \in \mathbb{Z}_+, \forall\, v \in \mathcal{V}.$$

We now present the main result of this section, the aforementioned generalisation of [15, Theorem 4.3] to discrete-time Lur'e systems of the form (2.17).

Theorem 2.3.16. *Let $\Sigma \in \mathbb{L}$, $f : \mathbb{R}^p \to \mathbb{R}^m$, $K \in \mathbb{R}^{m \times p}$, $r > 0$, $W \subseteq \mathbb{R}^p$ be a non-empty subset, $v^\infty \in \mathbb{R}^q$ and $w^\infty \in W$. Assume that $\mathbb{B}_{\mathbb{C}}(K, r) \subseteq \mathbb{S}_{\mathbb{C}}(\mathbf{G})$, (A) holds and that f is continuous and satisfies*

$$\|f(\xi + \zeta) - f(\zeta) - K\xi\| < r\|\xi\| \quad \forall\, \zeta \in Y, \forall\, \xi \in \mathbb{R}^p \backslash \{0\}, \tag{2.63}$$

where $Y \subseteq \mathrm{im}(C^K) + \mathrm{im}(D^K) + \mathrm{im}\left((I - DK)^{-1}D_{\mathrm{e}}\right) + W$ is a non-empty subset such that $\mathcal{T}_K \cap Y \neq \emptyset$, and where $\mathcal{T}_K$ is given by (2.55). Then $\#\mathcal{T}_K = 1$ and $(v^\infty, w^\infty, x^\infty, y^\infty)$ is an equilibrium quadruple, where $y^\infty := z^\infty - w^\infty$ with $\{z^\infty\} = \mathcal{T}_K$, and x^∞ is given by (2.56). Furthermore, the following statements hold.

(i) *If $\|D^K\| < 1/r$, then $(v^\infty, w^\infty, x^\infty, y^\infty)$ is a GAS equilibrium quadruple. Additionally, if $I - D^K(f - K)$ is bijective and $(I - D^K(f - K))^{-1}$ is continuous, then for all $(v, w, x, y) \in \mathcal{B}$ with $\lim_{t\to\infty} v(t) = v^\infty$ and $\lim_{t\to\infty} w(t) = w^\infty$, we have that either $x(t) \to x^\infty$ or $\|x(t)\| \to \infty$ as $t \to \infty$.*

(ii) *If there exists $\eta \in \mathbb{R}^p$ such that*

$$r\|\xi\| - \|f(\xi + \eta) - f(\eta) - K\xi\| \to \infty \text{ as } \|\xi\| \to \infty, \tag{2.64}$$

then $(v^\infty, w^\infty, x^\infty, y^\infty)$ *is an ISOS equilibrium quadruple and there exist* ψ_1, $\psi_2 \in \mathcal{KL}$ *and* $\phi \in \mathcal{K}$ *such that, for all* $(v, w, x, y) \in \mathcal{B}$ *and all* $t \in \mathbb{Z}_+$,

$$\left\| \begin{pmatrix} x(t) - x^\infty \\ y(t) - y^\infty \end{pmatrix} \right\| \leq \psi_1(\|x(0) - x^\infty\|, t) + \psi_2 \left(\max_{s \in \lfloor t/2 \rfloor} \left\| \begin{pmatrix} v(s) - v^\infty \\ w(s) - w^\infty \end{pmatrix} \right\|, t \right) + \phi \left(\max_{s \in \lceil t/2 \rceil} \left\| \begin{pmatrix} (\Lambda_{\lfloor t/2 \rfloor} v)(s) - v^\infty \\ (\Lambda_{\lfloor t/2 \rfloor} w)(s) - w^\infty \end{pmatrix} \right\| \right). \tag{2.65}$$

In particular, for all $(v, w, x, y) \in \mathcal{B}$ *with* $\lim_{t\to\infty} v(t) = v^\infty$ *and* $\lim_{t\to\infty} w(t) = w^\infty$*, we have* $\lim_{t\to\infty} x(t) = x^\infty$ *and* $\lim_{t\to\infty} y(t) = y^\infty$.

Remark 2.3.17. (i) Under the hypotheses of Theorem 2.3.16, we note that x^∞ and y^∞ given by (2.56) and (2.57) do not depend on the choice of K. Indeed, this is shown in the proof of Lemma 2.3.19 situated immediately after the following proof of Theorem 2.3.16. We comment that one can also deduce that x^∞ and y^∞ do not depend upon K from the conclusions of Theorem 2.3.16. For more details of this, we refer the reader to [15, Remark 4.4.(c)].

(ii) In Theorem 2.3.16 we assume that f is continuous. We highlight that this condition is trivially a consequence of (2.63) if

$$Y = \mathrm{im}(C^K) + \mathrm{im}(D^K) + \mathrm{im}\left((I - DK)^{-1} D_\mathrm{e}\right) + W = \mathbb{R}^p.$$

Furthermore, in such a situation, (2.62) holds and so, if $\|D^K\| < 1/r$, it follows from Proposition 2.3.10 that $I - D^K(f - K)$ is bijective and $(I - D^K(f - K))^{-1}$ is globally Lipschitz. Hence, the additional assumptions of statement (i) are redundant.

(iii) The convergence in statement (ii) of Theorem 2.3.16 is uniform in the following sense: given a set of inputs $\mathcal{V} \subseteq (\mathbb{R}^q)^{\mathbb{Z}_+} \times (\mathbb{R}^p)^{\mathbb{Z}_+}$ which is equi-convergent to (v^∞, w^∞) and $\kappa > 0$, the set

$$\left\{ x \in (\mathbb{R}^n)^{\mathbb{Z}_+} : \exists\, (v, w, y) \in \mathcal{V} \times (\mathbb{R}^p)^{\mathbb{Z}_+} \text{ s.t. } (v, w, x, y) \in \mathcal{B} \text{ and } \|x(0)\| + \sup_{t \in \mathbb{Z}_+} \left\| \begin{pmatrix} v(t) \\ w(t) \end{pmatrix} \right\| \leq \kappa \right\}$$

is equi-convergent to x^∞. ◊

Before proving Theorem 2.3.16, we first present the following corollary, which succinctly gives sufficient conditions that guarantee that (2.17) exhibits the CICSO property.

Corollary 2.3.18. *Let* $\Sigma \in \mathbb{L}$, $f : \mathbb{R}^p \to \mathbb{R}^m$, $K \in \mathbb{R}^{m \times p}$, $r > 0$ *and* $W \subseteq \mathbb{R}^p$ *be a non-empty subset. Assume that* $\mathbb{B}_\mathbb{C}(K, r) \subseteq \mathbb{S}_\mathbb{C}(\mathbf{G})$, (A) *holds,* f *is continuous,* (2.63) *holds with* $Y = \mathrm{im}(C^K) + \mathrm{im}(D^K) + \mathrm{im}\left((I - DK)^{-1} D_\mathrm{e}\right) + W$ *and there exists* $\eta \in \mathbb{R}^p$ *such that* (2.64) *holds. Then for each* $v^\infty \in \mathbb{R}^q$ *and* $w^\infty \in W$*, there exists* $x^\infty \in \mathbb{R}^n$ *and* $y^\infty \in \mathbb{R}^p$ *such that* $\lim_{t\to\infty} x(t) = x^\infty$ *and* $\lim_{t\to\infty} y(t) = y^\infty$ *for all* $(v, w, x, y) \in \mathcal{B}$ *where* $\lim_{t\to\infty} v(t) = v^\infty$ *and* $\lim_{t\to\infty} w(t) = w^\infty$. *In particular, if* $W = \mathbb{R}^p$*, then* (2.17) *has the CICSO property.*

Proof. First, fix $v^\infty \in \mathbb{R}^q$ and $w^\infty \in W$ and note that statement (iii) in Proposition 2.3.8 gives that F_K is surjective. Moreover, Lemma 2.3.5 implies that $F_K^{-1}(Y) \subseteq Y$. Combining this with the surjectivity of F_K, we see that $\mathcal{T}_K \cap Y \neq \emptyset$, where $\mathcal{T}_K$ is given by (2.55). Statement (ii) of Theorem 2.3.16 hence completes the proof. □

Proof of Theorem 2.3.16. We begin with an application of statement (i) of Proposition 2.3.8 to obtain that $\#\mathcal{T}_K = 1$, and an application of Proposition 2.3.6 to yield that $(v^\infty, w^\infty, x^\infty, y^\infty)$ is an equilibrium quadruple. We also define $\tilde{f} : \mathbb{R}^p \to \mathbb{R}^m$ by

$$\tilde{f}(\xi) := f(\xi + y^\infty + w^\infty) - f(y^\infty + w^\infty) \quad \forall\, \xi \in \mathbb{R}^p,$$

and highlight that, since $y^\infty + w^\infty = z^\infty \in \mathcal{T}_K \subseteq Y$ and (2.63) holds,

$$\|\tilde{f}(\xi) - K\xi\| < r\|\xi\| \quad \forall\, \xi \in \mathbb{R}^p\backslash\{0\}. \tag{2.66}$$

Our attention now turns to proving statement (i). To this end, we assume that $\|D^K\| < 1/r$ and let $(v^\infty, w^\infty, x, y) \in \mathcal{B}$. We define $\tilde{y} := y - y^\infty$ and $\tilde{x} := x - x^\infty$, and combine Lemma 2.1.18 with Proposition 2.3.6 to yield

$$\begin{aligned}\tilde{x}^+ &= A^K x + B^K\left(f(y + w^\infty) - Ky\right) + (B_\mathrm{e} + B^K K D_\mathrm{e})v^\infty \\ &\quad - A^K x^\infty - B^K\left(f(y^\infty + w^\infty) - Ky^\infty\right) - (B_\mathrm{e} + B^K K D_\mathrm{e})v^\infty \\ &= A^K\tilde{x} + B^K\left(f(\tilde{y} + y^\infty + w^\infty) - f(y^\infty + w^\infty) - K\tilde{y}\right) \\ &= A^K\tilde{x} + B^K\left(\tilde{f}(\tilde{y}) - K\tilde{y}\right).\end{aligned}$$

It can similary be shown that

$$\tilde{y} = C^K\tilde{x} + D^K\left(\tilde{f}(\tilde{y}) - K\tilde{y}\right),$$

which, when combined with Lemma 2.1.18, shows that $(0, 0, \tilde{x}, \tilde{y}) \in \mathcal{B}_{\tilde{f}}$. Therefore, in order to show that $(v^\infty, w^\infty, x^\infty, y^\infty)$ is GAS, it suffices to prove that $(0, 0, 0, 0)$ is a GAS equilibrium quadruple of the system given by (2.17) but with nonlinearity $\tilde{f}$. Indeed this is obtained immediately from Corollary 2.3.3 by recalling (2.66). Hence $(v^\infty, w^\infty, x^\infty, y^\infty)$ is a GAS equilibrium quadruple. To conclude the proof of statement (i), we now additionally assume that $I - D^K(f - K)$ is bijective and $(I - D^K(f - K))^{-1}$ is continuous. Let $(v, w, x, y) \in \mathcal{B}$ with $\lim_{t\to\infty} v(t) = v^\infty$ and $\lim_{t\to\infty} w(t) = w^\infty$. In order to prove the statement, we shall utilise an application of Theorem B.0.2 in Appendix B. To this end, since $(I - D^K(f - K))^{-1}$ exists, it follows from Lemma 2.1.18 that we can write

$$y + w = (I - D^K(f - K))^{-1}\left(C^K x + (I - DK)^{-1}D_\mathrm{e}v + (I + D^K K)w\right).$$

By combining this with the fact that $(I - D^K(f - K))^{-1}$ is continuous, it is easily checked that (2.17) can be written in the form $x^+ = g\left(x, (v, w)^T\right)$, for some continuous function $g : \mathbb{R}^n \times \mathbb{R}^{q+p} \to \mathbb{R}^n$. Explicitly, for all $\xi \in \mathbb{R}^n$ and $\zeta \in \mathbb{R}^{q+p}$, $g(\xi, \zeta)$ is defined by

$$\begin{aligned}g(\xi, \zeta) :&= A^K\xi + \begin{pmatrix} B_\mathrm{e} + B^K K D_\mathrm{e} & B^K K\end{pmatrix}\zeta \\ &\quad + B^K(f - K)\Big((I - D^K(f - K))^{-1}\left(C^K\xi + \begin{pmatrix}(I - DK)^{-1}D_\mathrm{e} & I + D^K K\end{pmatrix}\zeta\right)\Big).\end{aligned}$$

If $\|x(t)\|$ does not tend to infinity as $t \to \infty$, then there exists $M \geq 0$ such that, for every $k \in \mathbb{Z}_+$, there exists $t_k \in \overline{k}$ such that $\|x(t_k)\| \leq M$. Let $\mathcal{C}$ be the compact set defined by

$$\mathcal{C} = \{\xi \in \mathbb{R}^n : \|\xi\| \leq M\}.$$

Using Theorem B.0.2 then gives that $x(t) \to x^\infty$ as $t \to \infty$, as required.
We now prove statement (ii) and so we now assume that there exists $\eta \in \mathbb{R}^p$ such that (2.64) holds. Let $(v, w, x, y) \in \mathcal{B}$ and define $\tilde{y} := y - y^\infty$, $\tilde{x} := x - x^\infty$, $\tilde{v} := v - v^\infty$ and $\tilde{w} := w - w^\infty$. A combination of Lemma 2.1.18 and Proposition 2.3.6 gives

$$\begin{aligned}
\tilde{x}^+ &= A^K x + B^K \left(f(y+w) - Ky\right) + (B_\mathrm{e} + B^K K D_\mathrm{e})v \\
&\quad - A^K x^\infty - B^K \left(f(y^\infty + w^\infty) - Ky^\infty\right) - (B_\mathrm{e} + B^K K D_\mathrm{e})v^\infty \\
&= A^K \tilde{x} + B^K \left(f(\tilde{y} + \tilde{w} + y^\infty + w^\infty) - f(y^\infty + w^\infty) - K\tilde{y}\right) + (B_\mathrm{e} + B^K K D_\mathrm{e})\tilde{v} \\
&= A^K \tilde{x} + B^K \left(\tilde{f}(\tilde{y} + \tilde{w}) - K\tilde{y}\right) + (B_\mathrm{e} + B^K K D_\mathrm{e})\tilde{v}.
\end{aligned}$$

It can similary be shown that

$$\tilde{y} = C^K \tilde{x} + D^K \left(\tilde{f}(\tilde{y} + \tilde{w}) - K\tilde{y}\right) + (I - DK)^{-1} D_\mathrm{e} \tilde{v},$$

and so another application of Lemma 2.1.18 implies that $(\tilde{v}, \tilde{w}, \tilde{x}, \tilde{y}) \in \mathcal{B}_{\tilde{f}}$. Therefore, in order to show that $(v^\infty, w^\infty, x^\infty, y^\infty)$ is an ISOS equilibrium quadruple of the Lur'e system (2.17), it suffices to prove that $(0,0,0,0)$ is an ISOS equilibrium quadruple of the system given by (2.17) but with nonlinearity $\tilde{f}$. To this end, note that, from (2.64) and an application of statement (i) of Lemma 2.3.9,

$$r\|\xi\| - \|\tilde{f}(\xi) - K\xi\| \to \infty \text{ as } \|\xi\| \to \infty.$$

By combining this with (2.66), it follows from statement (ii) of Lemma 2.3.9, that there exists $\alpha \in \mathcal{K}_\infty$ such that

$$\|\tilde{f}(\xi) - K\xi\| \leq r\|\xi\| - \alpha(\|\xi\|) \quad \forall\, \xi \in \mathbb{R}^p.$$

Whence, by Corollary 2.3.3, $(0,0,0,0)$ is an ISOS equilibrium quadruple of the system given by (2.17) but with nonlinearity $\tilde{f}$, which implies that $(v^\infty, w^\infty, x^\infty, y^\infty)$ is an ISOS equilibrium quadruple of (2.17). Therefore, there exist $\psi \in \mathcal{KL}$ and $\phi \in \mathcal{K}$ such that, for all $(v, w, x, y) \in \mathcal{B}$ and all $t \in \mathbb{Z}_+$,

$$\left\| \begin{pmatrix} x(t) - x^\infty \\ y(t) - y^\infty \end{pmatrix} \right\| \leq \psi(\|x(0) - x^\infty\|, t) + \phi\left(\max_{s \in \underline{t}} \left\| \begin{pmatrix} v(s) - v^\infty \\ w(s) - w^\infty \end{pmatrix} \right\| \right). \tag{2.67}$$

A combination of Lemma 2.1.33 alongside the identity $\lceil t/2 \rceil + \lfloor t/2 \rfloor = t$ for all $t \in \mathbb{Z}_+$, thus yields, for all $(v, w, x, y) \in \mathcal{B}$ and all $t \in \mathbb{Z}_+$,

$$\begin{aligned}
\left\| \begin{pmatrix} x(t) - x^\infty \\ y(t) - y^\infty \end{pmatrix} \right\| &= \left\| \begin{pmatrix} (\Lambda_{\lfloor t/2 \rfloor} x)(\lceil t/2 \rceil) - x^\infty \\ (\Lambda_{\lfloor t/2 \rfloor} y)(\lceil t/2 \rceil) - y^\infty \end{pmatrix} \right\| \\
&\leq \psi\left(\|x(\lfloor t/2 \rfloor) - x^\infty\|, \lceil t/2 \rceil\right) + \phi\left(\max_{s \in \underline{\lceil t/2 \rceil}} \left\| \begin{pmatrix} (\Lambda_{\lfloor t/2 \rfloor} v)(s) - v^\infty \\ (\Lambda_{\lfloor t/2 \rfloor} w)(s) - w^\infty \end{pmatrix} \right\| \right).
\end{aligned}$$

Applying another ISOS estimate to the term $\|x(\lfloor t/2 \rfloor) - x^\infty\|$ in the above inequality then obtains, for all $(v, w, x, y) \in \mathcal{B}$ and all $t \in \mathbb{Z}_+$,

$$\left\| \begin{pmatrix} x(t) - x^\infty \\ y(t) - y^\infty \end{pmatrix} \right\| \leq \psi\left(\psi(\|x(0) - x^\infty\|, \lfloor t/2 \rfloor) + \phi\left(\max_{s \in \underline{\lfloor t/2 \rfloor}} \left\| \begin{pmatrix} v(s) - v^\infty \\ w(s) - w^\infty \end{pmatrix} \right\| \right), \lceil t/2 \rceil \right) + \phi\left(\max_{s \in \underline{\lceil t/2 \rceil}} \left\| \begin{pmatrix} (\Lambda_{\lfloor t/2 \rfloor} v)(s) - v^\infty \\ (\Lambda_{\lfloor t/2 \rfloor} w)(s) - w^\infty \end{pmatrix} \right\| \right).$$

Finally, we set $\psi_1(s, t) := \psi\left(2\psi(s, \lfloor t/2 \rfloor), \lceil t/2 \rceil\right)$ and $\psi_2(s, t) := \psi\left(2\phi(s), \lceil t/2 \rceil\right)$ for all $s \in \mathbb{R}_+$ and $t \in \mathbb{Z}_+$, to yield that (2.65) holds, for all $(v, w, x, y) \in \mathcal{B}$ and all $t \in \mathbb{Z}_+$. For all $(v, w, x, y) \in \mathcal{B}$ with $\lim_{t \to \infty} v(t) = v^\infty$ and $\lim_{t \to \infty} w(t) = w^\infty$, it is clear that the right hand side of the previous inequality tends to zero as $t \to \infty$, thus yielding that $\lim_{t \to \infty} x(t) = x^\infty$ and $\lim_{t \to \infty} y(t) = y^\infty$ and completing the proof. □

The following result was mentioned in Remark 2.3.17, and shows that x^∞ and y^∞, given by (2.56) and (2.57), do not depend on the choice of K.

Lemma 2.3.19. *Let $\Sigma \in \mathbb{L}$ be stabilisable and detectable, $f : \mathbb{R}^p \to \mathbb{R}^m$, $K_1, K_2 \in \mathbb{R}^{m \times p}$, $r_1, r_2 > 0$, $W \subseteq \mathbb{R}^p$ be a non-empty subset, $v^\infty \in \mathbb{R}^q$ and $w^\infty \in W$. Assume that, for each $i = 1, 2$, $\mathbb{B}_{\mathbb{C}}(K_i, r_i) \subseteq \mathbb{S}_{\mathbb{C}}(\mathbf{G})$ and that f satisfies*

$$\|f(\xi + \zeta) - f(\zeta) - K_i \xi\| < r_i \|\xi\| \quad \forall \zeta \in Y_i, \forall \xi \in \mathbb{R}^p \backslash \{0\}, \tag{2.68}$$

where $Y_i \subseteq \operatorname{im}(C^{K_i}) + \operatorname{im}(D^{K_i}) + \operatorname{im}\left((I - DK_i)^{-1} D_{\mathrm{e}}\right) + W$ is a non-empty subset such that $\mathcal{T}_{K_i} \cap Y_i \neq \emptyset$. Then $x_1^\infty = x_2^\infty$ and $y_1^\infty = y_2^\infty$, where $y_i^\infty + w^\infty \in \mathcal{T}_{K_i}$ and

$$x_i^\infty := (I - A^{K_i})^{-1} \left(B^{K_i}(f(y_i^\infty + w^\infty) - K_i(y^\infty)) + (B_{\mathrm{e}} + B^{K_i} K_i D_{\mathrm{e}}) v^\infty \right),$$

for each $i = 1, 2$.

Proof. An application of statement (i) of Proposition 2.3.8 gives that $\#\mathcal{T}_{K_i} = 1$ for each $i = 1, 2$, and, from Proposition 2.3.6, we obtain that $(v^\infty, w^\infty, x_i^\infty, y_i^\infty)$ is an equilibrium quadruple for each $i = 1, 2$. Therefore, by setting $\tilde{x} := x_1^\infty - x_2^\infty$ and $\tilde{y} := y_1^\infty - y_2^\infty$, we see that

$$\begin{aligned} \tilde{x} &= A\tilde{x} + B(f(\tilde{y} + y_2^\infty + w^\infty) - f(y_2^\infty + w^\infty)) \\ \tilde{y} &= C\tilde{x} + D(f(\tilde{y} + y_2^\infty + w^\infty) - f(y_2^\infty + w^\infty)). \end{aligned}$$

From this, Lemma 2.1.18 implies that

$$\left. \begin{aligned} \tilde{x} &= A^{K_2}\tilde{x} + B^{K_2}(f(\tilde{y} + y_2^\infty + w^\infty) - f(y_2^\infty + w^\infty) - K_2\tilde{y}) \\ \tilde{y} &= C^{K_2}\tilde{x} + D^{K_2}(f(\tilde{y} + y_2^\infty + w^\infty) - f(y_2^\infty + w^\infty) - K_2\tilde{y}). \end{aligned} \right\} \tag{2.69}$$

Consequently,

$$\begin{aligned} \tilde{y} &= (C^{K_2}(I - A^{K_2})^{-1} B^{K_2} + D^{K_2})(f(\tilde{y} + y_2^\infty + w^\infty) - f(y_2^\infty + w^\infty) - K_2\tilde{y}) \\ &= \mathbf{G}^{K_2}(1)(f(\tilde{y} + y_2^\infty + w^\infty) - f(y_2^\infty + w^\infty) - K_2\tilde{y}). \end{aligned}$$

Since $\|\mathbf{G}^{K_2}(1)\| \leq 1/r$ by Lemma 2.1.24 and since (2.68) holds and $y_2^\infty + w^\infty \in Y_2$, we yield that $\tilde{y} = 0$, that is, $y_1^\infty = y_2^\infty$. Finally, by combining this with (2.69), we obtain that $\tilde{x} = A^{K_2}\tilde{x}$. Since $I - A^{K_2}$ is invertible, this implies that $\tilde{x} = 0$, or that $x_1^\infty = x_2^\infty$, completing the proof. □

In the situation where the signal w in (2.17) is considered to be an output disturbance, obtained say from sampling or actuation errors for example, then it is quite possible that w is not convergent. This idea motivates the following corollary.

Corollary 2.3.20. *Let* $\Sigma \in \mathbb{L}$, $f : \mathbb{R}^p \to \mathbb{R}^m$, $K \in \mathbb{R}^{m\times p}$ *and* $r > 0$. *Assume that* $\mathbb{B}_{\mathbb{C}}(K, r) \subseteq \mathbb{S}_{\mathbb{C}}(\mathbf{G})$, (A) *holds and that* f *satisfies* (2.63) *with* $Y = \mathbb{R}^p$ *and is such that there exists* $\eta \in \mathbb{R}^p$ *such that* (2.64) *holds. Then for each* $v^\infty \in \mathbb{R}^q$ *and* $w^\infty \in \mathbb{R}^p$, *there exists* $x^\infty \in \mathbb{R}^n$, $y^\infty \in \mathbb{R}^p$ *and* $\phi \in \mathcal{K}$ *such that, for all* $(v, w, x, y) \in \mathcal{B}$ *with* $\lim_{t\to\infty} v(t) = v^\infty$ *and* w *bounded,*

$$\limsup_{t\to\infty} \left\| \begin{pmatrix} x(t) - x^\infty \\ y(t) - y^\infty \end{pmatrix} \right\| \leq \phi\left(\limsup_{t\to\infty} \|w(t) - w^\infty\|\right). \tag{2.70}$$

Proof. Let $v^\infty \in \mathbb{R}^q$ and $w^\infty \in \mathbb{R}^p$. Since (2.63) holds with $Y = \mathbb{R}^p$, Proposition 2.3.8 implies that F_K is bijective. Therefore, the hypotheses of Statement (ii) of Theorem 2.3.16 are satisfied with $W = \mathbb{R}^p$. We thus immediately obtain the existence of $x^\infty \in \mathbb{R}^n$, $y^\infty \in \mathbb{R}^p$, ψ_1, $\psi_2 \in \mathcal{KL}$ and $\phi \in \mathcal{K}$ such that (2.65) holds for all $(v, w, x, y) \in \mathcal{B}$ and all $t \in \mathbb{Z}_+$. From this, it is then straightforward to see that (2.70) holds for all $(v, w, x, y) \in \mathcal{B}$ with $\lim_{t\to\infty} v(t) = v^\infty$. □

We now give two examples of systems which satisfy the assumptions of Corollary 2.3.18, and hence exhibit the CICSO property.

Example 2.3.21. Consider (2.17) in the case given in Example 2.2.12, that is, let $n = 2$, $m = p = 1$,

$$A = \begin{pmatrix} \frac{1}{2} & 1 \\ 0 & \frac{1}{2} \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \end{pmatrix}, \quad D = 1,$$

$f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(\xi) = \frac{2}{3}\operatorname{sign}(\xi)\left(\sqrt{|\xi| + 1} - 1\right) \quad \forall\, \xi \in \mathbb{R},$$

and let q, B_{e} and D_{e} be arbitrary. It was shown in Example 2.2.12 that $\|\mathbf{G}\|_{H^\infty} = 3$ and that (A) holds if we take $K = 0$ and $r = 1/3$. Furthermore, since $f'(\xi) \in (0, 1/3)$ for all $\xi \in \mathbb{R}\backslash\{0\}$, an application of [15, Lemma 4.9] implies that (2.63) holds with $Y = \mathbb{R}^p$. Combining this with the fact that f is continuous and that

$$\frac{1}{3}|\xi| - |f(\xi)| \to \infty \text{ as } |\xi| \to \infty,$$

we see that the hypotheses of Corollary 2.3.18 are satisfied with $W = \mathbb{R}^p$. Therefore, (2.17) has the CICSO property, in this case. ◊

Example 2.3.22. Consider (2.17) where $n = 3$, $m = p = q = 1$,

$$A = \begin{pmatrix} \frac{1}{3} & 0 & 1 \\ 0 & \frac{1}{2} & 2 \\ 0 & 0 & \frac{1}{9} \end{pmatrix}, \quad B = \begin{pmatrix} \frac{1}{3} \\ 0 \\ 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}, \quad D = \frac{1}{2},$$

$f : \mathbb{R} \to \mathbb{R}$ is defined by

$$f(\xi) = \operatorname{sign}(\xi)\ln(|\xi| + 1) \quad \forall\, \xi \in \mathbb{R},$$

and q, B_e and D_e are arbitrary. Now, straight forward calculations lead to the finding that

$$\mathbf{G}(1) = 1 \quad \text{and} \quad |\mathbf{G}(z)| \leq 1 \quad \forall\, z \in \mathbb{C},\, |z| \geq 1.$$

We hence obtain that $\|\mathbf{G}\|_{H^\infty} = 1$. Moreover, it is also easy to see that (A) holds with $K = 0$ and $r = 1$. Furthermore, the nonlinearity f is the same as the nonlinearity given in [15, Example 4.12]. There, it is shown that f satisfies (2.63) with $Y = \mathbb{R}^p$, and that

$$|\xi| - |f(\xi)| \to \infty \text{ as } |\xi| \to \infty.$$

By combining all of this together, we see that the assumptions of Corollary 2.3.18 hold with $W = \mathbb{R}^p$. This in turn whence gives that, in this case, (2.17) has the CICSO property. We highlight that, since $f'(0) = 1$, there does not exist $\delta > 0$ such that

$$|f(\xi + \zeta) - f(\zeta)| \leq (1 - \delta)|\xi| \quad \forall\, \xi, \zeta \in \mathbb{R}. \qquad \Diamond$$

The final corollary of Theorem 2.3.16 that we shall present here is a result that guarantees that (2.17) has the CICSO property, but with assumptions that are reminiscent of the circle criterion (see, for example, [66]). Before giving this, we first require the following definition.

Definition 2.3.23. *We say a rational $\mathbb{C}^{m\times m}$-valued function $\mathbf{H}$ is* positive real *if $\mathbf{H}(z) + \mathbf{H}(z)^*$ is positive semi-definite for every $z \in \mathbb{E}$ which is not a pole of $\mathbf{H}$.*

We now give the aforementioned corollary of Theorem 2.3.16, which can be interpreted as a discrete-time generalisation of [15, Corollary 4.15].

Corollary 2.3.24. *Let $\Sigma \in \mathbb{L}$, $f : \mathbb{R}^p \to \mathbb{R}^m$ and $K_1, K_2 \in \mathbb{R}^{m\times p}$ with $K_1 \in \mathbb{A}_{\mathbb{R}}(D)$. Assume that $\mathbf{H} := (I - K_2\mathbf{G})(I - K_1\mathbf{G})^{-1}$ is positive real and that Σ is either* (i) *controllable and observable or,* (ii) *stabilisable and detectable and there exists $z \in \mathbb{C}$ such that $|z| = 1$ and $\mathbf{H}(z) + \mathbf{H}(z)^*$ is positive definite. If, additionally,*

$$\langle f(\xi + \zeta) - f(\zeta) - K_1\xi, f(\xi + \zeta) - f(\zeta) - K_2\xi\rangle < 0 \quad \forall\, \xi, \zeta \in \mathbb{R}^p,\, \xi \neq 0, \tag{2.71}$$

and there exists $\eta \in \mathbb{R}^p$ and $\alpha \in \mathcal{K}_\infty$ such that

$$\langle f(\xi + \eta) - f(\eta) - K_1\xi, f(\xi + \eta) - f(\eta) - K_2\xi\rangle \leq -\alpha(\|\xi\|)\|\xi\| \quad \forall\, \xi \in \mathbb{R}^p,$$

then (2.17) *has the CICSO property.*

Proof. We proceed in an analogous manner to that seen in the proof of [15, Corollary 4.15]. To begin with, we define

$$L := (K_1 - K_2)/2, \quad M := (K_1 + K_2)/2.$$

We then see that, for all $\xi, \zeta \in \mathbb{R}^p$,

$$\begin{aligned}
&\langle f(\xi + \zeta) - f(\zeta) - K_1\xi, f(\xi + \zeta) - f(\zeta) - K_2\xi\rangle \\
&\qquad = \langle f(\xi + \zeta) - f(\zeta) - (M + L)\xi, f(\xi + \zeta) - f(\zeta) - (M - L)\xi\rangle \\
&\qquad = \|f(\xi + \zeta) - f(\zeta) - M\xi\|^2 - \|L\xi\|^2.
\end{aligned}$$

By combining this with (2.71), it is easily deduced that $\ker L = 0$, and hence, that $L^T L$ is invertible. By setting $\tilde{L} = (L^T L)^{-1}L^T$, we see that $\tilde{L}$ is a left inverse of L. With this and Lemma 2.1.18 in mind, we note that $(v, w, x, y) \in \mathcal{B}$ if, and only if, (v, w, x, y) satisfies

$$\begin{aligned} x^+ &= A^{K_1}x + B^{K_1}g(L(y+w)) + B^{K_1}K_1 w + (B_\mathrm{e} + B^{K_1}K_1 D_\mathrm{e})v, \\ y &= C^{K_1}x + D^{K_1}g(L(y+w)) + D^{K_1}K_1 w + (I - DK_1)^{-1}D_\mathrm{e}v, \end{aligned}$$

where $g : \mathbb{R}^m \to \mathbb{R}^m$ is defined by $g(\xi) := f(\tilde{L}\xi) - K_1\tilde{L}\xi$ for all $\xi \in \mathbb{R}^m$. Moreover, by the left-invertibility of L, this implies that $(v, w, x, y) \in \mathcal{B}$ if, and only if, (v, w, x, y) satisfies

$$\left.\begin{aligned} x^+ &= A^{K_1}x + B^{K_1}g(L(y+w)) + B^{K_1}K_1 w + (B_\mathrm{e} + B^{K_1}K_1 D_\mathrm{e})v, \\ Ly &= LC^{K_1}x + LD^{K_1}g(L(y+w)) + LD^{K_1}K_1 w + L(I - DK_1)^{-1}D_\mathrm{e}v. \end{aligned}\right\} \tag{2.72}$$

Therefore, to complete the proof, it is sufficient to show that (2.72) exhibits the CICSO property. Indeed, we shall do this by showing that the relevant assumptions of Corollary 2.3.18 are satisfied. To this end, we define $K := -L\tilde{L}$ and denote by $\mathbf{F}$ the transfer function of (2.72), that is,

$$\mathbf{F}(z) := LC^{K_1}\left(zI - A^{K_1}\right)^{-1}B^{K_1} + LD^{K_1} = L\mathbf{G}^{K_1}(z).$$

By utilising the methods outlined in the proof of [108, Corollary 11], we may obtain that $\mathbb{B}_\mathbb{C}(K, 1) \subseteq \mathbb{S}_\mathbb{C}(\mathbf{F})$, and that the analogous assumption of (A) (with $r = 1$) holds for the system (2.72). In addition to this, by again following similar arguments to that presented in the proofs of [108, Corollary 11 and Corollary 16], it is easy to deduce the existence of $\beta \in \mathcal{K}_\infty$ such that

$$\|g(\xi + L\eta) - g(L\eta) - K\xi\| \le \|\xi\| - \beta(\|\xi\|) \quad \forall\, \xi \in \mathbb{R}^m,$$

and that

$$\|g(\xi + \zeta) - g(\zeta) - K\xi\| < \|\xi\| \quad \forall\, \xi, \zeta \in \mathbb{R}^m,\ \xi \neq 0.$$

We may now invoke Corollary 2.3.18 (with $r = 1$ and $W = \mathbb{R}^p$) to deduce that (2.72), and hence (2.17), exhibits the CICSO property. □

2.3.3 The steady-state gain maps

We shall now concern ourselves with explicitly writing formulae for so-called steady-state gain maps. For motivation of this, let us consider (2.17) in the situation that A is Schur, $f = 0$, $B_\mathrm{e} = B$ and $D_\mathrm{e} = D$. It is easy to see that if $v \in (\mathbb{R}^m)^{\mathbb{Z}_+}$ converges asymptotically to $v^\infty \in \mathbb{R}^m$, then for every initial condition $x^0 \in \mathbb{R}^n$, the solution of the corresponding IVP (2.18), (x, y), is such that x converges to $(I - A)^{-1}Bv^\infty$ and y to $\mathbf{G}(1)v^\infty$. We hence obtain linear maps $v^\infty \mapsto (I - A)^{-1}Bv^\infty$ and $v^\infty \mapsto \mathbf{G}(1)v^\infty$. It is our intention here to generalise these maps for the Lur'e system (2.17). To this end, we make the following definition.

Definition 2.3.25. *Let $\Sigma \in \mathbb{L}$ and $f : \mathbb{R}^p \to \mathbb{R}^m$. If (2.17) has the CICSO property, then the maps*

$$\Gamma_\mathrm{is} : \mathbb{R}^q \times \mathbb{R}^p \to \mathbb{R}^n, \quad (v^\infty, w^\infty) \mapsto x^\infty,$$

and

$$\Gamma_\mathrm{io} : \mathbb{R}^q \times \mathbb{R}^p \to \mathbb{R}^p, \quad (v^\infty, w^\infty) \mapsto y^\infty,$$

are well-defined. We say that Γ_is is the input-to-state steady-state (ISSS) gain *and Γ_io is the* input-to-output steady-state (IOSS) gain*.*

Before presenting a result that gives explicit formulae for the ISSS and IOSS gain maps, we first recall that when under the assumptions of Corollary 2.3.18 with $W = \mathbb{R}^p$, Proposition 2.3.8 gives that F_K is bijective. With this in mind, under the assumptions of Corollary 2.3.18 with $W = \mathbb{R}^p$, we define $G_K : \mathbb{R}^q \times \mathbb{R}^p \to \mathbb{R}^p$ by

$$\left.\begin{aligned} G_K(\xi_1,\xi_2) :=& F_K^{-1}\begin{pmatrix} C^K(I-A^K)^{-1}\left(B_\mathrm{e}+B^KKD_\mathrm{e}\right)\xi_1 \\ \qquad + (I-DK)^{-1}D_\mathrm{e}\xi_1 + \left(I+\mathbf{G}^K(1)K\right)\xi_2 \end{pmatrix} \\ &\forall\,(\xi_1,\xi_2) \in \mathbb{R}^q \times \mathbb{R}^p. \end{aligned}\right\} \tag{2.73}$$

The next proposition gives explicit formulae for the ISSS and IOSS gain maps.

Proposition 2.3.26. *Under the assumptions Corollary 2.3.18 with $W = \mathbb{R}^p$, for all $(\xi_1,\xi_2) \in \mathbb{R}^q \times \mathbb{R}^p$,*

$$\begin{aligned} \Gamma_{\mathrm{is}}(\xi_1,\xi_2) =& (I-A^K)^{-1}\left(B^K(f-K)G_K(\xi_1,\xi_2) + B^KK\xi_2 \right. \\ & \left. + (B_\mathrm{e}+B^KKD_\mathrm{e})\xi_1\right), \end{aligned} \tag{2.74}$$

and

$$\Gamma_{\mathrm{io}}(\xi_1,\xi_2) = G_K(\xi_1,\xi_2) - \xi_2, \tag{2.75}$$

where G_K is given by (2.73)*.*

We shall not provide a proof of Proposition 2.3.26, since the result can be obtained from a combination of Proposition 2.3.6, Proposition 2.3.8 and Corollary 2.3.18.

Remark 2.3.27. In the simple case wherein $D = 0 = D_\mathrm{e}$ and $B_\mathrm{e} = I$, then Proposition 2.3.26 tells us that, under the assumptions and notation of Corollary 2.3.18 with $W = \mathbb{R}^p$, for all $\xi \in \mathbb{R}^q$,

$$\Gamma_{\mathrm{is}}(\xi,0) = (I-A^K)^{-1}\left(B(f-K)\left(F_K^{-1}(C(I-A^K)^{-1}\xi)\right) + \xi\right),$$

and

$$\Gamma_{\mathrm{io}}(\xi,0) = F_K^{-1}(C(I-A^K)^{-1}\xi),$$

which forms the discrete-time analogue of [15, Equations (4.30) and (4.31)]. If, in addition, A is Schur, $f = 0$ and $K = 0$, then (2.17) simply becomes

$$x^+ = Ax + v, \quad y = Cx,$$

which has transfer function given by $z \mapsto C(zI-A)^{-1}$. In this case, F_K becomes the identity map and $\Gamma_{\mathrm{io}}(\xi,0) = \mathbf{G}(1)\xi$ for all $\xi \in \mathbb{R}^q$, as expected. $\Diamond$

We conclude this section with the following proposition, which is a generalisation of [15, Proposition 4.14] to the discrete-time setting. The result gives sufficient conditions for when the ISSS and IOSS gains are continuous and globally Lipschitz.

Proposition 2.3.28. *Under the assumptions Corollary 2.3.18 with $W = \mathbb{R}^p$, the following statements hold.*

(i) Γ_{is} *and* Γ_{io} *are continuous.*

(ii) *If* $\|\mathbf{G}^K\|_{H^\infty} = 0$*, then* Γ_{is} *and* Γ_{io} *are globally Lipschitz.*

(iii) *If* $\|\mathbf{G}^K\|_{H^\infty} > 0$ *and* $f - K$ *is globally Lipschitz with Lipschitz constant* $\lambda < 1/\|\mathbf{G}^K\|_{H^\infty}$*, then* Γ_{is} *and* Γ_{io} *are globally Lipschitz.*

Proof. In view of Proposition 2.3.26, in order to prove each statement, it is sufficient to show that F_K^{-1} is continuous or globally Lipschitz, respectively. Since, *mutatis mutandis*, this is what is shown in the proof of [15, Proposition 4.14], for brevity we leave the proof to the reader. □

2.4 Application to the four-block problem

In this section, we consider so-called 'four-block' forced discrete-time Lur'e systems which are informally described by Figure 2.2. In Figure 2.2, $\Sigma = (A, B, B_{\text{e}}, C, D, D_{\text{e}}) \in \mathbb{L}$ and the signal y is given by

$$y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

The motivation for studying such systems is that there may be outputs that are of interest but that are not utilised for feedback purposes (i.e. y_1 in Figure 2.2). In the literature, the four-block arrangement has been considered in the infinite-dimensional continuous-time setting in [47] and in the infinite-dimensional discrete-time setting in [40] (see also Chapter 5).

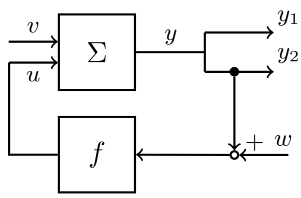

Figure 2.2: *Block diagram of a four-block forced Lur'e system.*

In this section, we shall give analogous results to those seen in the earlier sections, but applicable to four-block forced discrete-time Lur'e systems. We shall do this by rewriting the four-block system in the form of (2.17) and then invoking the previous results. First, however, we need to precisely give the four-block Lur'e system that we will be considering and present some preliminary results.

2.4.1 Preliminaries

Throughout this section, we let $p_1, p_2 \in \mathbb{Z}_+$ be such that

$$p_1 + p_2 = p,$$

and will repeatedly consider the matrices defined in the following definition.

Definition 2.4.1. *For* $i = 1, 2$*, we define* $P_i \in \mathbb{R}^{p_i \times p}$ *by*

$$P_1 := \left(I_{p_1}, 0_{p_1 \times p_2}\right) \quad \text{and} \quad P_2 := \left(0_{p_2 \times p_1}, I_{p_2}\right),$$

where, for $r, s \in \mathbb{Z}_+$*,* I_r *denotes the identity matrix in* $\mathbb{R}^{r \times r}$ *and* $0_{r \times s} \in \mathbb{R}^{r \times s}$ *is the matrix with all entries zero.*

Remark 2.4.2. We note that, for $w_i \in \mathbb{R}^{p_i}$ where $i = 1, 2$,

$$P_i \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = w_i.$$

Thus, the action of P_i may be thought of as "projecting" $\mathbb{R}^p$ to the subspace $\mathbb{R}^{p_i}$. $\Diamond$

The next lemma shows that the previously defined matrices have norm of magnitude 1.

Lemma 2.4.3. *For each $i = 1, 2$, we have that $\|P_i\| = 1$.*

Proof. Fix $i \in \{1, 2\}$ and note that

$$\|P_i\| = \sup_{\|x\|=1} \|P_i x\| \leq \sup_{\|x\|=1} \|x\| = 1.$$

To obtain equality, let $e_i \in \mathbb{R}^p$ be the vector with all zeros except for 1 in either: (i) the first entry if $i = 1$; or, (ii) the last entry if $i = 2$. Then, $\|e_i\| = 1$ and

$$\|P_i\| \geq \|P_i e_i\| = 1,$$

completing the proof. $\square$

We are now ready to precisely give the so-called four-block discrete-time Lur'e system described informally by Figure 2.2. Indeed, we shall consider

$$\left.\begin{aligned} x^+ &= Ax + Bu + B_\mathrm{e} v \\ y &= Cx + Du + D_\mathrm{e} v \\ u &= f(P_i y + w), \end{aligned}\right\} \tag{2.76}$$

where $\Sigma \in \mathbb{L}$, $v \in (\mathbb{R}^q)^{\mathbb{Z}_+}$, $w \in (\mathbb{R}^{p_i})^{\mathbb{Z}_+}$, $f : \mathbb{R}^{p_i} \to \mathbb{R}^m$ and $i = 1, 2$. For the rest of this section, we consider $i \in \{1, 2\}$ to be a fixed quantity.

For ease of notation in the sequel, we make the following definitions.

Definition 2.4.4. *Let $\Sigma \in \mathbb{L}$ and $f : \mathbb{R}^{p_i} \to \mathbb{R}^m$.*

(i) *We label the relevant components of the subsystem of* (2.76) *generated by applying P_i to the output equation, by Σ_i, that is,*

$$\Sigma_i := (A, B, B_\mathrm{e}, P_i C, P_i D, P_i D_\mathrm{e}).$$

(ii) *We define the* behaviour *of* (2.76) *by*

$$\mathcal{B}_f^i(\Sigma) := \Big\{ (v, w, x, y) \in (\mathbb{R}^q)^{\mathbb{Z}_+} \times (\mathbb{R}^{p_i})^{\mathbb{Z}_+} \times (\mathbb{R}^n)^{\mathbb{Z}_+} \times (\mathbb{R}^p)^{\mathbb{Z}_+} : (v, w, x, y) \text{ satisfies } (2.76) \Big\}.$$

For ease of notation, we shall write $\mathcal{B}^i := \mathcal{B}_f^i(\Sigma)$ when no ambiguity shall arise.

Remark 2.4.5. We note that (2.76) can be rewritten in terms of a Lur'e system of the form (2.17). Indeed, by applying P_i to the output equation, (2.76) becomes

$$\begin{aligned} x^+ &= Ax + Bf(P_i y + w) + B_\mathrm{e} v \\ P_i y &= P_i Cx + P_i Df(P_i y + w) + P_i D_\mathrm{e} v. \end{aligned}$$

We recognise this to be a system of the form of (2.17) with linear component given by Σ_i. As mentioned in the preamble of this section, the previous rewritten form of (2.76) will be the key to obtaining analogous results to those seen in the earlier sections. $\diamondsuit$

For given $\Sigma \in \mathbb{L}$, $K_i \in \mathbb{R}^{m\times p_i}$ and $r > 0$ such that $\mathbb{B}_\mathbb{C}(K_i, r) \subseteq \mathbb{S}_\mathbb{C}(P_i\mathbf{G})$, we shall repeatedly make use of the following assumption:

$$\left.\begin{aligned} &\Sigma_i \text{ is (i) controllable and observable, or (ii) stabilisable and detectable and} \\ &r \min_{|z|=1} \|(P_i\mathbf{G})^{K_i}(z)\| < 1. \end{aligned}\right\} \quad (\mathrm{A}_i)$$

Assumption (A_i) is simply assumption (A) but in the context of the subsystem generated by Σ_i. The following two propositions help to relate (A) with (A_i).

Proposition 2.4.6. *Let $\Sigma \in \mathbb{L}$, $K_i \in \mathbb{C}^{m\times p_i}$ and set $K := K_i P_i$. Then $K_i \in \mathbb{A}_\mathbb{C}(P_i D)$ if, and only if, $K \in \mathbb{A}_\mathbb{C}(D)$, and in which case*

$$\begin{aligned} \Sigma_i^{K_i} &= \left(A^{K_i}, B^{K_i}, B_\mathrm{e} + B^{K_i}K_iP_iD_\mathrm{e}, (P_iC)^{K_i}, (P_iD)^{K_i}, (I - P_iDK_i)^{-1}P_iD_\mathrm{e}\right) \\ &= \left(A^K, B^K, B_\mathrm{e} + B^K K D_\mathrm{e}, P_iC^K, P_iD^K, P_i(I - DK)^{-1}D_\mathrm{e}\right), \end{aligned}$$

and

$$(P_i\mathbf{G})^{K_i} = P_i\mathbf{G}^K.$$

Proof. An application of Lemma 2.1.15 immediately gives that $I - P_iDK_i$ is invertible if, and only if, $I - DK_iP_i = I - DK$ is invertible, that is, $K_i \in \mathbb{A}_\mathbb{C}(P_iD)$ if, and only if, $K \in \mathbb{A}_\mathbb{C}(D)$. Under the assumption that $K_i \in \mathbb{A}_\mathbb{C}(P_iD)$, another application of Lemma 2.1.15 yields that

$$(P_iD)^{K_i} = (I - P_iDK_i)^{-1}P_iD = P_i(I - DK_iP_i)^{-1}D = P_iD^K.$$

The rest of the claimed identities can be proven in a similar manner, and thus are left to the reader to verify. $\square$

Proposition 2.4.7. *Let $\Sigma \in \mathbb{L}$, $K_i \in \mathbb{C}^{m\times p_i}$ and set $K := K_iP_i$. The following statements hold.*

(i) *Σ is stabilisable/detectable if Σ_i is stabilisable/detectable.*

(ii) *Σ is controllable/observable if Σ_i is controllable/observable.*

(iii) *If Σ_i is stabilisable and detectable, then $K_i \in \mathbb{S}_\mathbb{C}(P_i\mathbf{G})$ if, and only if, $K \in \mathbb{S}_\mathbb{C}(\mathbf{G})$.*

Proof. The first two statements hold trivially, and so we shall only prove statement (iii). To this end, assume that Σ_i is stabilisable and detectable and note that, from statement (i), we have that Σ is stabilisable and detectable. Hence, a combination of [72, Theorem 2], Lemma 2.1.22 and Proposition 2.4.6 yields

$$\begin{aligned} K_i \in \mathbb{S}_{\mathbb{C}}(P_i\mathbf{G}) &\iff (P_i\mathbf{G})^{K_i} \in H^\infty_{p_i\times m} \\ &\iff A^{K_i} \text{ is Schur} \\ &\iff A^{K} \text{ is Schur} \\ &\iff \mathbf{G}^K \in H^\infty_{p\times m} \\ &\iff K \in \mathbb{S}_{\mathbb{C}}(\mathbf{G}), \end{aligned}$$

which completes the proof. □

Associated with (2.76) is the following initial-value-problem (IVP), which will be our focus for the rest of this subsection.

Definition 2.4.8. *Let $\Sigma \in \mathbb{L}$ and $f : \mathbb{R}^{p_i} \to \mathbb{R}^m$. For given $x^0 \in \mathbb{R}^n$, $v \in (\mathbb{R}^q)^{\mathbb{Z}_+}$ and $w \in (\mathbb{R}^{p_i})^{\mathbb{Z}_+}$, we consider the IVP:*

$$\left.\begin{aligned} x^+ &= Ax + Bu + B_{\mathrm{e}}v, \quad x(0) = x^0, \\ y &= Cx + Du + D_{\mathrm{e}}v, \\ u &= f(P_iy + w). \end{aligned}\right\} \tag{2.77}$$

For given $x^0 \in \mathbb{R}^n$, $v \in (\mathbb{R}^q)^{\mathbb{Z}_+}$ and $w \in (\mathbb{R}^{p_i})^{\mathbb{Z}_+}$, we say that $(x, y) \in (\mathbb{R}^n)^{\mathbb{Z}_+} \times (\mathbb{R}^p)^{\mathbb{Z}_+}$ is a solution to (2.77) *if $(v, w, x, y) \in \mathcal{B}^i$ and $x(0) = x^0$.*

We wish to present a result that guarantees the existence and uniqueness of solutions of (2.77). With Proposition 2.1.36 in mind, we give the following result.

Proposition 2.4.9. *Let $\Sigma \in \mathbb{L}$ and $f : \mathbb{R}^{p_i} \to \mathbb{R}^m$. The following statements hold.*

(i) $P_i(I - DfP_i) = (I - P_iDf)P_i$.

(ii) *$I - DfP_i$ is injective if, and only if, $I - P_iDf$ is injective.*

(iii) *$I - DfP_i$ is surjective if, and only if, $I - P_iDf$ is surjective.*

Proof. Statement (i) is trivial and so we shall only prove statements (ii) and (iii). For ease of notation in the sequel, we define $g := I - DfP_i$ and $g_i := I - P_iDf$. Without loss of generality, we assume that $i = 1$. To prove statement (ii), we assume that g is injective and that $g_1(\xi_1) = g_1(\zeta_1)$, where $\xi_1, \zeta_1 \in \mathbb{R}^{p_1}$. By setting $\xi_2 := P_2Df(\xi_1) \in \mathbb{R}^{p_2}$, we see that

$$g\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix} = \begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix} - Df(\xi_1) = \begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix} - \begin{pmatrix}P_1Df(\xi_1)\\ P_2Df(\xi_1)\end{pmatrix} = \begin{pmatrix}g_1(\xi_1)\\ 0\end{pmatrix}.$$

Proceeding similarly, it can be checked that

$$g\begin{pmatrix}\zeta_1\\ \zeta_2\end{pmatrix} = \begin{pmatrix}g_1(\zeta_1)\\ 0\end{pmatrix},$$

where $\zeta_2 := P_2Df(\zeta_1) \in \mathbb{R}^{p_2}$. The injectivity of g then yields that $\xi_1 = \zeta_1$ and, hence, that g_1 is injective. For the converse implication, assume that g_1 is injective and that

$$g\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix} = g\begin{pmatrix}\zeta_1\\ \zeta_2\end{pmatrix},$$

where $\xi_1, \zeta_1 \in \mathbb{R}^{p_1}$ and $\xi_2, \zeta_2 \in \mathbb{R}^{p_2}$. From this, we deduce that $g_1(\xi_1) = g_1(\zeta_1)$, which subequently gives that $\xi_1 = \zeta_1$. Additionally, we also obtain that $\xi_2 - P_2Df(\xi_1) = \zeta_2 - P_2Df(\zeta_1) = \zeta_2 - P_2Df(\xi_1)$, which implies that $\xi_2 = \zeta_2$, whence obtaining injectivity of g and proving statement (ii). Turning our attention now to statement (iii), assume that g_1 is surjective and let $\xi_1 \in \mathbb{R}^{p_1}$ and $\xi_2 \in \mathbb{R}^{p_2}$. By the surjectivity of g_1, there exists $\zeta_1 \in \mathbb{R}^{p_1}$ such that $g_1(\zeta_1) = \xi_1$. Subsequently defining $\zeta_2 := \xi_2 + P_2Df(\zeta_1) \in \mathbb{R}^{p_2}$, leads to the conclusion that

$$g\begin{pmatrix}\zeta_1\\ \zeta_2\end{pmatrix} = \begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix},$$

and that g is surjective. The converse statement is trivially true, and hence we omit the proof. □

The following is immediately obtained from a combination of Proposition 2.1.36 with Remark 2.4.5 and Proposition 2.4.9.

Corollary 2.4.10. *Let $\Sigma \in \mathbb{L}$, $f : \mathbb{R}^{p_i} \to \mathbb{R}^m$, $x^0 \in \mathbb{R}^n$, $v \in (\mathbb{R}^q)^{\mathbb{Z}_+}$ and $w \in (\mathbb{R}^{p_i})^{\mathbb{Z}_+}$. The following statements hold.*

(i) *If the map $I - DfP_i$ is surjective, then there exists a solution to the IVP* (2.77).

(ii) *If the map $I - DfP_i$ is injective, then there is at most one solution to the IVP* (2.77).

2.4.2 Stability and convergence properties

We now present a number of corollaries of the previous sections, but for (2.76). Before doing so however, we make some natural analogous definitions of those seen earlier, applied to (2.76).

Definition 2.4.11. *Let $\Sigma \in \mathbb{L}$ and $f : \mathbb{R}^{p_i} \to \mathbb{R}^m$.*

(i) *We say that* (2.76) *is* globally asymptotically stable in the large (GAS)*, if there exists $c > 0$ such that*

$$\|x(t)\| + \|y(t)\| \leq c\|x(0)\| \quad \forall\, t \in \mathbb{Z}_+,\, \forall\, (0, 0, x, y) \in \mathcal{B}^i$$

and

$$\lim_{t\to\infty} x(t) = 0 \text{ and } \lim_{t\to\infty} y(t) = 0 \quad \forall\, (0, 0, x, y) \in \mathcal{B}^i.$$

(ii) *We say that* (2.76) *is* input-to-state/output stable (ISOS)*, if there exist $\psi \in \mathcal{KL}$ and $\phi \in \mathcal{K}$ such that*

$$\|x(t)\| \leq \psi(\|x(0)\|, t) + \phi\left(\max_{s\in\underline{t-1}} \left\|\begin{pmatrix}v(s)\\ w(s)\end{pmatrix}\right\|\right) \quad \forall\, t \in \mathbb{N},\, \forall\, (v, w, x, y) \in \mathcal{B}^i. \tag{2.78}$$

and

$$\|y(t)\| \leq \psi(\|x(0)\|, t) + \phi\left(\max_{s\in\underline{t}} \left\|\begin{pmatrix}v(s)\\ w(s)\end{pmatrix}\right\|\right) \quad \forall\, t \in \mathbb{Z}_+,\, \forall\, (v, w, x, y) \in \mathcal{B}^i. \tag{2.79}$$

(iii) *We say that* (2.76) *has the* converging-input converging-state/output (CICSO) property *if for every* $v^\infty \in \mathbb{R}^q$ *and* $w^\infty \in \mathbb{R}^{p_i}$*, there exists* $x^\infty \in \mathbb{R}^n$ *and* $y^\infty \in \mathbb{R}^p$ *such that* $\lim_{t\to\infty} x(t) = x^\infty$ *and* $\lim_{t\to\infty} y(t) = y^\infty$ *for every* $(v, w, x, y) \in \mathcal{B}^i$ *with* $\lim_{t\to\infty} v(t) = v^\infty$ *and* $\lim_{t\to\infty} w(t) = w^\infty$.

The first corollary that we shall give highlights sufficient conditions for when (2.76) is GAS.

Corollary 2.4.12. *Let* $\Sigma \in \mathbb{L}$, $K_i \in \mathbb{R}^{m\times p_i}$, $r > 0$ *and* $f : \mathbb{R}^{p_i} \to \mathbb{R}^m$. *Assume that* $\mathbb{B}_{\mathbb{C}}(K_i, r) \subseteq \mathbb{S}_{\mathbb{C}}(P_i\mathbf{G})$, $\|(P_iD)^{K_i}\| < 1/r$ *and that* (A_i) *holds. If* f *is continuous and*

$$\|f(\xi) - K_i\xi\| < r\|\xi\| \quad \forall\, \xi \in \mathbb{R}^{p_i}\backslash\{0\},$$

then (2.76) *is GAS.*

Proof. We begin by applying P_i to the output equation in (2.76) to obtain that

$$\begin{aligned} x^+ &= Ax + Bf(P_iy + w) + B_\mathrm{e}v \\ P_iy &= P_iCx + P_iDf(P_iy + w) + P_iD_\mathrm{e}v. \end{aligned}$$

The assumptions of Theorem 2.2.3 hold in the context of this system, and so we obtain the existence of $c > 0$ such that, for all $(0, 0, x, y) \in \mathcal{B}_f(\Sigma_i)$,

$$\|x(t)\| + \|y(t)\| \le c\|x(0)\| \quad \forall\, t \in \mathbb{Z}_+,$$

and $\lim_{t\to\infty} x(t) = 0 = \lim_{t\to\infty} y(t)$. Since $(0, 0, x, y) \in \mathcal{B}^i$ implies that $(0, 0, x, P_iy) \in \mathcal{B}_f(\Sigma_i)$, this gives that, for all $(0, 0, x, y) \in \mathcal{B}^i$,

$$\|x(t)\| + \|P_iy(t)\| \le c\|x(0)\| \quad \forall\, t \in \mathbb{Z}_+, \tag{2.80}$$

and $\lim_{t\to\infty} x(t) = 0 = \lim_{t\to\infty} P_iy(t)$. To complete the proof, it is sufficient to show that there exists a constant $b > 0$ such that, for all $(0, 0, x, y) \in \mathcal{B}^i$,

$$\|y(t)\| \le b\|x(0)\| \quad \forall\, t \in \mathbb{Z}_+, \quad \text{and} \quad \lim_{t\to\infty} y(t) = 0. \tag{2.81}$$

To show this, let $(0, 0, x, y) \in \mathcal{B}^i$. Now, by Proposition 2.4.6, $K := K_iP_i \in \mathbb{A}_{\mathbb{C}}(D)$, and hence, by Lemma 2.1.18,

$$y = C^Kx + D^K(f(P_iy) - K_iP_iy).$$

We then have

$$\|y(t)\| \le \|C^K\|\|x(t)\| + r\|D^K\|\|P_iy(t)\| \quad \forall\, t \in \mathbb{Z}_+,$$

which, along with (2.80), implies the existence of $b > 0$ such that (2.81) holds, and completes the proof. □

The next two results present hypotheses that guarantee that (2.76) is ISOS and exhibits the CICSO property, respectively. We shall not provide a proof of these two results, since the method to do so follows in a similar manner to the previous proof. Indeed, in order to prove the subsequent corollaries, one should rewrite (2.76) in terms of a system of the form (2.17) (see Remark 2.4.5), invoke the relevant result from the previous sections (either Theorem 2.2.10 or Corollary 2.3.18), and then verify that the result holds for the 'entire' output y (i.e. not just for P_iy).

Corollary 2.4.13. *Let $\Sigma \in \mathbb{L}$, $K_i \in \mathbb{R}^{m\times p_i}$, $r > 0$ and $\alpha \in \mathcal{K}_\infty$. Assume that $\mathbb{B}_{\mathbb{C}}(K_i, r) \subseteq \mathbb{S}_{\mathbb{C}}(P_i\mathbf{G})$ and (A_i) holds. Then there exists $\psi \in \mathcal{KL}$ and $\phi \in \mathcal{K}$ such that, for all $f : \mathbb{R}^{p_i} \to \mathbb{R}^m$ satisfying*

$$\|f(\xi) - K_i\xi\| \le r\|\xi\| - \alpha(\|\xi\|) \quad \forall\, \xi \in \mathbb{R}^{p_i},$$

(2.78) *and* (2.79) *hold. In particular, the Lur'e system* (2.76) *is ISOS.*

Corollary 2.4.14. *Let $\Sigma \in \mathbb{L}$, $f : \mathbb{R}^{p_i} \to \mathbb{R}^m$, $K_i \in \mathbb{R}^{m\times p_i}$, $r > 0$ and $W \subseteq \mathbb{R}^{p_i}$ be a nonempty subset. Assume that $\mathbb{B}_{\mathbb{C}}(K_i, r) \subseteq \mathbb{S}_{\mathbb{C}}(P_i\mathbf{G})$, (A_i) holds and that f is continuous, satisfies*

$$\|f(\xi+\zeta) - f(\zeta) - K_i\xi\| < r\|\xi\| \quad \forall\, \zeta \in Y,\, \xi \in \mathbb{R}^{p_i}\backslash\{0\},$$

with $Y = \mathrm{im}((P_iC)^{K_i}) + \mathrm{im}((P_iD)^{K_i}) + \mathrm{im}\left((I - P_iDK_i)^{-1}P_iD_{\mathrm{e}}\right) + W$, and there exists $\eta \in \mathbb{R}^{p_i}$ such that

$$r\|\xi\| - \|f(\xi+\zeta) - f(\zeta) - K_i\xi\| \to \infty \text{ as } \|\xi\| \to \infty.$$

Then for each $v^\infty \in \mathbb{R}^q$ and $w^\infty \in W$, there exists $x^\infty \in \mathbb{R}^n$ and $y^\infty \in \mathbb{R}^p$ such that $\lim_{t\to\infty} x(t) = x^\infty$ and $\lim_{t\to\infty} y(t) = y^\infty$ for all $(v, w, x, y) \in \mathcal{B}^i$ where $\lim_{t\to\infty} v(t) = v^\infty$ and $\lim_{t\to\infty} w(t) = w^\infty$. In particular, if $W = \mathbb{R}^{p_i}$, then the Lur'e system (2.76) *has the CICSO property.*

Remark 2.4.15. We comment that every other result of the previous sections can also be generalised to the four-block setting. So as to avoid repetition, we omit these results and leave them to the reader to formulate. ◊

2.5 Notes and references

As we have already discussed, the contents of this chapter can be split into two main parts: the first comprising generalisations of the main results of [108] that are extended to include output stability and to systems of the form (2.17); and the second being a discrete-time analogue of [15], but again which also extends the results to systems of the form (2.17) and which also considers output convergence. We note that these generalisations can additionally be completed in the continuous-time setting. Explicitly, one could, via analogous methods to those conducted in the previous sections, generalise the main results of [107] and [15] to continuous-time versions of (2.17). The rationale for considering the discrete-time setting in this chapter is to further differentiate it from the known works.

We now discuss the following relevant references. To begin with, as already mentioned, [106, Theorem 5.3.1] and [108, Theorem 13] utilise the same assumptions as Theorem 2.2.10 to deduce ISS estimates for discrete-time Lur'e systems which are far more restrictive than (2.17). Moreover, the issue of ISOS is not considered in either [106] or [108]. We again comment that the generalisation we have provided is nontrivial, which we evidenced with several examples given throughout the chapter.

Continuing with the theme of ISOS, a relevant paper from the literature is [115] which concerns input to output stability of continuous-time control systems. The authors provide relationships between various output stability notions. Further to this, in [67], it is shown that an exponential ISS property holds for discrete-time Lur'e systems in the context of observer-based output feedback. Neither paper directly overlaps with the current work. ISOS of forced infinite-dimensional discrete- and continuous-time Lur'e systems is also considered in the papers [40] (see also Chapter 5) and [47], respectively. These papers use stronger assumptions than those seen in the main results of this chapter to deduce stronger stability notions, particular cases of which yield ISOS and CICS. In [40] and Chapter 5, exponential weighting and small-gain arguments are used, whereas Lyapunov arguments were used in the current chapter. It would be interesting to see if infinite-dimensional versions of the results of this chapter hold.

We also mention the paper [7] since, in the continuous-time setting, ISS for forced Lur'e systems (without feedthrough and without external and output disturbances) is investigated. This work differs from ours since the assumptions utilised to deduce ISS concern positive real conditions on the linear system and unboundedness properties on the nonlinearity.

Regarding the results of Section 2.3 which concern the CICS and CICSO properties, there is little else apart from (of course) [15] that is relevant to the current work. The references [5, 100, 112] deal with CICS-like properties of general nonlinear systems, with a discrete-time analogue to [112] being used in the proof of Theorem 2.3.16. We also refer the reader to [20, 104]. The closest of these to the curent setting is [104, Statement 1) of Theorem], which concerns the steady state error of Lur'e systems in the single-input single-output setting, when under the conditions of the circle criterion.

Chapter 3

Semi-global incremental stability properties and convergence under almost periodic forcing of Lur'e systems

In this chapter, we define a semi-global notion of incremental ISS for discrete-time forced Lur'e systems of the form (2.17), and provide sufficient conditions for when these systems exhibit this property. We also investigate convergence properties of (2.17), accumulating in a result that guarantees that asymptotically almost periodic inputs generate asymptotically almost periodic state and output trajectories. In the interest of not repeating ourselves, we refer the reader to the introduction for background reading and brief descriptions of incremental ISS and almost periodicity. Moreover, a thorough presentation of almost periodicity is provided in Appendix C. Of key relevance to the present chapter, however, is that there are many generalisations of the original Bohr definition of almost periodicity (see, for example, [14]). One such is the notion of Stepanov almost periodicity. In addition to the previously mentioned outcomes of this chapter, we shall also investigate the effect on state trajectories of continuous-time forced Lur'e systems that have Stepanov almost periodic inputs. We consider continuous-time systems for this, since Stepanov almost periodicity is only relevant in that setting.

This chapter is outlined as follows. We begin with Section 3.1, which is based upon the work achieved in [39]. We split the section in four, beginning with a discussion of some preliminary results. We then introduce semi-global incremental ISS, and move on to developing criteria that guarantees that Lur'e systems of the form (2.17) exhibit this property. The final two portions of the section comprise our investigation into convergence properties of trajectories of these systems when under almost periodic forcing, and an application of the previous results to the four-block setting, respectively. The results of Section 3.2 are based off those in [41]. We split the section into three main parts, with the first explicitly describing the continuous-time forced Lur'e systems of interest to us in the section, and providing preliminary results. We then move onto presenting conditions that are sufficient to conclude various incremental stability properties. Similar to the previous section, we then apply these results to obtain convergence properties of state trajectories when inputs are Stepanov almost periodic.

3.1 Incremental and convergence properties of discrete-time Lur'e systems

As previously mentioned, this section largely comprises the work of [39].

3.1.1 Preliminaries

Throughout this section, we shall concern ourselves with the system given by (2.17) situated in Chapter 2, that is, a discrete-time forced Lur'e system with potentially non-zero feedthrough and which allows for output disturbances. Explicitly, we recall the system to be

$$\left.\begin{aligned} x^+ &= Ax + Bu + B_{\mathrm{e}}v, \\ y &= Cx + Du + D_{\mathrm{e}}v, \\ u &= f(y + w), \end{aligned}\right\} \tag{2.17}$$

where (see Definition 2.1.9)

$$\Sigma := (A, B, B_{\mathrm{e}}, C, D, D_{\mathrm{e}}) \in \mathbb{L} := \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{n\times q} \times \mathbb{R}^{p\times n} \times \mathbb{R}^{p\times m} \times \mathbb{R}^{p\times q},$$

$v \in (\mathbb{R}^q)^{\mathbb{Z}_+}$, $w \in (\mathbb{R}^p)^{\mathbb{Z}_+}$, $f : \mathbb{R}^p \to \mathbb{R}^m$ and $n, m, p, q \in \mathbb{N}$. We assume all notation associated with this system which was defined and discussed in Section 2.1. In particular, for (2.17), we denote the transfer function by $\mathbf{G}$, the behaviour by $\mathcal{B}$, and the set of stabilising feedback matrices over $\mathbb{C}$ and $\mathbb{R}$ by $\mathbb{S}_{\mathbb{C}}(\mathbf{G})$ and $\mathbb{S}_{\mathbb{R}}(\mathbf{G})$, respectively (see Definition 2.1.19). Finally, for $\Sigma \in \mathbb{L}$, $K \in \mathbb{R}^{m\times p}$ and $r > 0$ such that $\mathbb{B}_{\mathbb{C}}(K, r) \subseteq \mathbb{S}_{\mathbb{C}}(\mathbf{G})$, we recall assumption (A) which we will regularly make use of:

$$\left.\begin{array}{l} \Sigma \text{ is (i) controllable and observable, or (ii) stabilisable and detectable and} \\ r \min_{|z|=1} \|\mathbf{G}^K(z)\| < 1. \end{array}\right\} \tag{A}$$

Once again, we highlight that a thorough presentation of preliminary results concerning (2.17) was conducted in Section 2.1. This included the investigation of linear output feedback via 'loop-shifting', time-invariance principles and the associated initial-value problem. Moreover, the generality of (2.17), and in particular how it encompasses many other systems, was also highlighted. In the interest of brevity, we do not repeat ourselves, and instead refer the reader to Section 2.1 for more detail.

We shall now present two results that underpin the main results of this section. We begin with the following.

Lemma 3.1.1. *Let $h : \mathbb{R}^p \to \mathbb{R}^m$ be continuous and $\Gamma \subseteq \mathbb{R}^p$ be compact. Then the function $\xi \mapsto \sup_{\zeta\in\Gamma} \|h(\xi + \zeta) - h(\zeta)\|$ is continuous.*

An explicit reference of the above is difficult to locate in the literature, and hence we provide a proof here.

Proof of Lemma 3.1.1. Let $\xi \in \mathbb{R}^p$ and let $(\xi_k)_{k\in\mathbb{N}} \subseteq \mathbb{R}^p$ be a sequence such that $\lim_{k\to\infty} \xi_k = \xi$. We begin with the observation that, for all $k \in \mathbb{N}$, there exists $\zeta_k \in \Gamma$ such that

$$\sup_{\zeta\in\Gamma} \|h(\xi_k + \zeta) - h(\zeta)\| = \|h(\xi_k + \zeta_k) - h(\zeta_k)\|, \tag{3.1}$$

and there exists $\zeta_\xi \in \Gamma$ such that

$$\sup_{\zeta\in\Gamma} \|h(\xi + \zeta) - h(\zeta)\| = \|h(\xi + \zeta_\xi) - h(\zeta_\xi)\|. \tag{3.2}$$

Here, we have utilised the continuity of h and the compactness of Γ. Again by using the compactness of Γ, we may assume, without loss of generality, that $(\zeta_k)_{k\in\mathbb{N}}$ converges to $\tilde{\zeta} \in \Gamma$. We now fix $\varepsilon > 0$ and note that, as a consequence of h being continuous, there exists $\delta_1 > 0$ such that, for all $\eta \in \mathbb{R}^p$,

$$\|\eta - \xi\| < \delta_1 \implies \Big| \|h(\xi + \zeta_\xi) - h(\zeta_\xi)\| - \|h(\eta + \zeta_\xi) - h(\zeta_\xi)\| \Big| \leq \varepsilon. \tag{3.3}$$

Furthermore, again since h is continuous, there exists $\delta_2 > 0$ such that, for all $\eta_1, \eta_2 \in \mathbb{R}^p$,

$$\|\eta_1 - \xi - \tilde{\zeta}\|, \|\eta_2 - \tilde{\zeta}\| < \delta_2 \implies \Big| \|h(\eta_1) - h(\eta_2)\| - \|h(\xi + \tilde{\zeta}) - h(\tilde{\zeta})\| \Big| \leq \varepsilon. \tag{3.4}$$

We subsequently set $\delta := \min\{\delta_1, \delta_2\}/2 > 0$ and consider k large enough so that $\|\xi_k - \xi\| + \|\zeta_k - \tilde{\zeta}\| < \delta$. We then see that, from (3.2) and (3.3),

$$\begin{aligned} \sup_{\zeta\in\Gamma} \|h(\xi + \zeta) - h(\zeta)\| &= \|h(\xi + \zeta_\xi) - h(\zeta_\xi)\| \\ &\leq \|h(\xi_k + \zeta_\xi) - h(\zeta_\xi)\| + \varepsilon \\ &\leq \sup_{\zeta\in\Gamma} \|h(\xi_k + \zeta) - h(\zeta)\| + \varepsilon. \end{aligned} \tag{3.5}$$

Similarly, we utilise (3.1) and (3.4) to obtain that

$$\begin{aligned} \sup_{\zeta\in\Gamma} \|h(\xi_k + \zeta) - h(\zeta)\| &= \|h(\xi_k + \zeta_k) - h(\zeta_k)\| \\ &\leq \|h(\xi + \tilde{\zeta}) - h(\tilde{\zeta})\| + \varepsilon \\ &\leq \sup_{\zeta\in\Gamma} \|h(\xi + \zeta) - h(\zeta)\| + \varepsilon. \end{aligned} \tag{3.6}$$

Therefore, a combination of (3.5) and (3.6) yields, for all k sufficiently large enough,

$$\left| \sup_{\zeta\in\Gamma} \|h(\xi + \zeta) - h(\zeta)\| - \sup_{\zeta\in\Gamma} \|h(\xi_k + \zeta) - h(\zeta)\| \right| \leq \varepsilon.$$

Since ε was arbitrary, this shows that $\xi \mapsto \sup_{\zeta\in\Gamma} \|h(\xi + \zeta) - h(\zeta)\|$ is continuous, and hence completes the proof. □

The second preliminary result that we provide here is a version of Theorem 2.2.10 that holds for systems of the form (2.17) but which have a time-varying nonlinearity. That is, an input-to-state/output stability criterion for the system:

$$\left.\begin{aligned} x(t+1) &= Ax(t) + Bg(t, y(t) + w(t)) + B_\mathrm{e} v(t) \\ y(t) &= Cx(t) + Dg(t, y(t) + w(t)) + D_\mathrm{e} v(t) \end{aligned}\right\} \quad \forall\, t \in \mathbb{Z}_+\,, \tag{3.7}$$

where $\Sigma \in \mathbb{L}$, $v \in (\mathbb{R}^q)^{\mathbb{Z}_+}$, $w \in (\mathbb{R}^p)^{\mathbb{Z}_+}$ and $g : \mathbb{Z}_+ \times \mathbb{R}^p \to \mathbb{R}^m$.

Corollary 3.1.2. *Let $\Sigma \in \mathbb{L}$, $K \in \mathbb{R}^{m\times p}$, $r > 0$ and $\alpha \in \mathcal{K}_\infty$. Assume that $\mathbb{B}_{\mathbb{C}}(K,r) \subseteq \mathbb{S}_{\mathbb{C}}(\mathbf{G})$ and* (A) *holds. Then there exist $\psi \in \mathcal{KL}$ and $\phi \in \mathcal{K}$ such that, for all $g : \mathbb{Z}_+ \times \mathbb{R}^p \to \mathbb{R}^m$ satisfying*

$$\sup_{t\in\mathbb{Z}_+} \|g(t,\xi) - K\xi\| \le r\|\xi\| - \alpha(\|\xi\|) \quad \forall\, \xi \in \mathbb{R}^p, \tag{3.8}$$

and for every $(v,w,x,y) \in (\mathbb{R}^q)^{\mathbb{Z}_+} \times (\mathbb{R}^p)^{\mathbb{Z}_+} \times (\mathbb{R}^n)^{\mathbb{Z}_+} \times (\mathbb{R}^p)^{\mathbb{Z}_+}$ satisfying (3.7),

$$\|x(t)\| \le \psi(\|x(0)\|,t) + \phi\left(\sup_{s\in\underline{t-1}} \left\|\begin{pmatrix} v(s)\\ w(s)\end{pmatrix}\right\|\right) \quad \forall\, t \in \mathbb{N}, \tag{3.9}$$

and

$$\|y(t)\| \le \psi(\|x(0)\|,t) + \phi\left(\sup_{s\in\underline{t}} \left\|\begin{pmatrix} v(s)\\ w(s)\end{pmatrix}\right\|\right) \quad \forall\, t \in \mathbb{Z}_+. \tag{3.10}$$

We shall not provide a proof of Corollary 3.1.2, since the result can be proven identically, *mutatis mutandis*, to the proof of Theorem 2.2.10. To give some more detail here: since (3.8) holds uniformly in time, by following the steps laid out in the proof of Theorem 2.2.10, we may obtain the existence of an ISS-Lyapunov function (see Definition 2.2.8). Furthermore, by also inspecting the proof of Proposition 2.2.9, situated in Appendix A, we see that this ISS-Lyapunov function is sufficient for obtaining (3.9). As for (3.10), again by using the uniformity of (3.8) in time and following the final steps of the proof of Theorem 2.2.10, the claim is proven. Finally, the uniformity of ϕ and ϕ with respect to g, v, w, x and y, follows from the fact that ψ and ϕ depend only upon $\alpha_1, \alpha_2, \alpha_3$ and α_4 in (2.25) and (2.26), which in turn depend only upon Σ, K, r and α.

3.1.2 Incremental stability properties

Our attention now turns towards incremental stability properties of (2.17). As previously mentioned, here we shall develop a notion called *semi-global incremental input-to-state stability* and present a theorem that gives sufficient conditions for when (2.17) exhibits this type of stability. We also provide a corollary of this theorem which has assumptions reminiscent of the well-known circle criterion (see, for example, [66]). To this end, we begin with the following definition.

Definition 3.1.3. *Let $\Sigma \in \mathbb{L}$ and $f : \mathbb{R}^p \to \mathbb{R}^m$.*

(i) *We say that* (2.17) *is* semi-globally incrementally input-to-state stable *if for any $R > 0$, there exist $\psi \in \mathcal{KL}$ and $\phi \in \mathcal{K}$ such that, for all $(v_i,w_i,x_i,y_i) \in \mathcal{B}$ with $\|x_i(0)\| + \left\|(v_i\, w_i)^T\right\|_{\ell^\infty} \le R$, $i = 1,2$,*

$$\|x_1(t)-x_2(t)\| \le \psi\left(\|x_1(0) - x_2(0)\|,t\right)+\phi\left(\sup_{s\in\underline{t-1}} \left\|\begin{pmatrix} v_1(s) - v_2(s)\\ w_1(s) - w_2(s)\end{pmatrix}\right\|\right) \quad \forall t \in \mathbb{N}. \tag{3.11}$$

(ii) *We say that* (2.17) *is* semi-globally incrementally input-to-state/output stable *if for any $R > 0$, there exist $\psi \in \mathcal{KL}$ and $\phi \in \mathcal{K}$ such that, for all $(v_i,w_i,x_i,y_i) \in \mathcal{B}$ with $\|x_i(0)\| + \left\|(v_i\, w_i)^T\right\|_{\ell^\infty} \le R$, $i = 1,2$, it follows that* (3.11) *holds and*

$$\|y_1(t)-y_2(t)\| \le \psi\left(\|x_1(0) - x_2(0)\|,t\right)+\phi\left(\sup_{s\in\underline{t}} \left\|\begin{pmatrix} v_1(s) - v_2(s)\\ w_1(s) - w_2(s)\end{pmatrix}\right\|\right) \quad \forall t \in \mathbb{Z}_+. \tag{3.12}$$

Remark 3.1.4. Although the previously defined stability notion is semi-global, it is suitable for almost all practical applications. This is because all relevant initial conditions and inputs are likely to have their norm bounded by some $R > 0$. We refer the reader to papers such as [5, 40, 47] and to Chapter 5, for various notions of global incremental stability. ◊

The next theorem is the main stability result of this section.

Theorem 3.1.5. *Let $\Sigma \in \mathbb{L}$, $f : \mathbb{R}^p \to \mathbb{R}^m$, $K \in \mathbb{R}^{m\times p}$, $r > 0$ and $Z \subseteq \mathbb{R}^p$ be a nonempty and closed subset. Assume that $\mathbb{B}_{\mathbb{C}}(K, r) \subseteq \mathbb{S}_{\mathbb{C}}(\mathbf{G})$,* (A) *holds, f is continuous,*

$$\|f(\xi + \zeta) - f(\zeta) - K\xi\| < r\|\xi\| \quad \forall\, \zeta \in Z, \forall\, \xi \in \mathbb{R}^p\backslash\{0\}, \tag{3.13}$$

and there exists $\eta \in \mathbb{R}^p$ such that

$$r\|\xi\| - \|f(\xi + \eta) - f(\eta) - K\xi\| \to \infty \text{ as } \|\xi\| \to \infty. \tag{3.14}$$

Then for any $R > 0$, there exist $\psi \in \mathcal{KL}$ and $\phi \in \mathcal{K}$ such that (3.11) *and* (3.12) *both hold for all $(v_1, w_1, x_1, y_1) \in \mathcal{B}$ satisfying $\|x_1(0)\| + \big\|(v_1\, w_1)^T\big\|_{\ell^\infty} \leq R$ and $y_1(t) + w_1(t) \in Z$ for all $t \in \mathbb{Z}_+$, and all $(v_2, w_2, x_2, y_2) \in \mathcal{B}$.*

As a direct consequence of Theorem 3.1.5, we obtain the following.

Corollary 3.1.6. *Let $\Sigma \in \mathbb{L}$, $f : \mathbb{R}^p \to \mathbb{R}^m$, $K \in \mathbb{R}^{m\times p}$ and $r > 0$. Assume that $\mathbb{B}_{\mathbb{C}}(K, r) \subseteq \mathbb{S}_{\mathbb{C}}(\mathbf{G})$,* (A) *holds, f satisfies* (3.13) *with $Z = \mathbb{R}^p$, and there exists $\eta \in \mathbb{R}^p$ such that* (3.14) *holds. Then* (2.17) *is semi-globally incrementally ISOS.*

Before proving Theorem 3.1.5, we first provide some commentary of its hypotheses.

Remark 3.1.7. (i) In the situation that $Z = \mathbb{R}^p$, if (3.13) holds, then f is continuous, therefore making the assumption redundant.

(ii) Under the assumptions of Theorem 3.1.5 with $Z = \mathbb{R}^p$, we obtain from Corollary 2.3.18 that (2.17) has the CICSO property. Moreover, in such a situation, it follows from Lemma 2.3.9, that for every $\zeta \in \mathbb{R}^p$, there exists $\alpha_\zeta \in \mathcal{K}_\infty$ such that

$$\|f(\xi + \zeta) - f(\zeta) - K\xi\| \leq r\|\xi\| - \alpha_\zeta(\|\xi\|) \quad \forall\, \xi \in \mathbb{R}^p.$$

In particular, if $f(0) = 0$, then

$$\|f(\xi) - K\xi\| \leq r\|\xi\| - \alpha_0(\|\xi\|) \quad \forall\, \xi \in \mathbb{R}^p,$$

which we recognise as the main assumption imposed on the nonlinearity in Theorem 2.2.10 (see also [108, Theorem 13]). Hence, in this case, the assumptions of Theorem 3.1.5 guarantee ISOS of (2.17).

(iii) If the assumptions of Theorem 3.1.5 are strengthened so that there exists $\delta > 0$ such that

$$\|f(\xi + \zeta) - f(\zeta) - K\xi\| \leq (r - \delta)\|\xi\| \quad \forall\, \xi, \zeta \in \mathbb{R}^p, \tag{3.15}$$

then [40, Theorem 3.2] (see also Theorem 5.2.5 situated later in Chapter 5) yields that (2.17) is *exponentially incrementally input-to-state/output stable* (see Definition 5.2.3). We highlight that there exist Lur'e systems that satisfy the assumptions of Theorem 3.1.5, but for which there does not exist $\delta > 0$ such that (3.15) holds - see Examples 3.1.8 and 3.1.9 situated after the proof of Theorem 3.1.5. ◊

Proof of Theorem 3.1.5. We seek to apply Corollary 3.1.2. To begin with, however, since Z is assumed to be nonempty, we let $z^* \in Z$. By invoking (3.14) and statement (i) of Lemma 2.3.9, we obtain that

$$r\|\xi\| - \|f(\xi+\zeta) - f(\zeta) - K\xi\| \to \infty \text{ as } \|\xi\| \to \infty\,, \quad \forall\, \zeta \in \mathbb{R}^p.$$

Consequently,

$$r\|\xi\| - \|f(\xi+z^*) - f(z^*) - K\xi\| \to \infty \text{ as } \|\xi\| \to \infty\,. \tag{3.16}$$

By setting $\tilde{f}(\xi) := f(\xi+z^*) - f(z^*)$ for all $\xi \in \mathbb{R}^p$, it is clear from the continuity of f, (3.13) and (3.16), that $\tilde{f} - K : \mathbb{R}^p \to \mathbb{R}^m$ is continuous, $\|\tilde{f}(\xi) - K\xi\| < r\|\xi\|$ for all $\xi \in \mathbb{R}^p\backslash\{0\}$, and $r\|\xi\| - \|\tilde{f}(\xi) - K\xi\| \to \infty$ as $\|\xi\| \to \infty$. Hence, $\tilde{f} - K$ satisfies the hypotheses of statement (ii) of Lemma 2.3.9 and so, there exists $\alpha_* \in \mathcal{K}_\infty$ such that

$$\|f(\xi+z^*) - f(z^*) - K\xi\| = \|\tilde{f}(\xi) - K\xi\| \le r\|\xi\| - \alpha_*(\|\xi\|) \quad \forall\, \xi \in \mathbb{R}^p\,. \tag{3.17}$$

We shall use (3.17) to establish the existence of $\alpha_1 \in \mathcal{K}_\infty$ and $s_1 > 0$ such that

$$\|f(\xi) - K\xi\| \le r\|\xi\| - \alpha_1(\|\xi\|) \quad \forall\, \xi \in \mathbb{R}^p \text{ s.t. } \|\xi\| \ge s_1\,. \tag{3.18}$$

For which purpose, we note that, for all $\xi \in \mathbb{R}^p$,

$$\begin{aligned}\|f(\xi) - K\xi\| &\le \|f(\xi - z^* + z^*) - f(z^*) - K(\xi - z^*)\| + \|f(z^*) - Kz^*\| \\ &\le r\|\xi - z^*\| - \alpha_*(\|\xi - z^*\|) + \|f(z^*) - Kz^*\|\,,\end{aligned}$$

where we have used (3.17). Defining $\tilde{\alpha} \in \mathcal{K}_\infty$ by

$$\tilde{\alpha}(s) := \begin{cases} \alpha_*(s - \|z^*\|), & \text{if } s \ge \|z^*\| + 1, \\ \alpha_*(1)s/(\|z^*\| + 1), & \text{if } 0 \le s < \|z^*\| + 1, \end{cases}$$

we obtain that, for all $\xi \in \mathbb{R}^p$ such that $\|\xi\| \ge \|z^*\| + 1$,

$$\|f(\xi) - K\xi\| \le r\|\xi\| - \tilde{\alpha}(\|\xi\|) + r\|z^*\| + \|f(z^*) - Kz^*\|.$$

Therefore, if we let $s_1 \ge \|z^*\| + 1$ be such that $\tilde{\alpha}(s) > r\|z^*\| + \|f(z^*) - Kz^*\|$ for all $s \ge s_1$, and define $\alpha_1 \in \mathcal{K}_\infty$ by

$$\alpha_1(s) := \begin{cases} \tilde{\alpha}(s) - r\|z^*\| - \|f(z^*) - Kz^*\|, & \text{if } s \ge s_1, \\ (\tilde{\alpha}(s_1) - r\|z^*\| - \|f(z^*) - Kz^*\|)s/s_1, & \text{if } 0 \le s < s_1, \end{cases}$$

we obtain that (3.18) holds.
Next, fix $R > 0$ and combine (3.18) with Corollary 2.2.14 to obtain the existence of $\tilde{\psi} \in \mathcal{KL}$, $\tilde{\phi} \in \mathcal{K}$ and $\theta > 0$ such that (2.46) and (2.47) hold (but with ψ and ϕ replaced by $\tilde{\psi}$ and $\tilde{\phi}$, respectively). By setting $\rho := \tilde{\psi}(R, 0) + \tilde{\phi}(\theta + R) + R > 0$, we see, from (2.46) and (2.47), that, for all $(v, w, x, y) \in \mathcal{B}$ with $\|x(0)\| + \|(v\, w)^T\|_{\ell^\infty} \le R$,

$$\|y(t) + w(t)\| \le \|y(t)\| + R \le \rho \quad \forall\, t \in \mathbb{Z}_+\,. \tag{3.19}$$

Without loss of generality, we assume that $\rho \ge \|z^*\|$. Subsequently, the set

$$W := \{\xi \in \mathbb{R}^p : \|\xi\| \le \rho\} \cap Z \subseteq \mathbb{R}^p\,,$$

is nonempty since $z^* \in Z$. Moreover, W is compact since Z is closed. We claim that

$$r\|\xi\| - \sup_{\zeta \in W} \|f(\xi + \zeta) - f(\zeta) - K\xi\| \to \infty \text{ as } \|\xi\| \to \infty. \tag{3.20}$$

To avoid interruption of the argument, we relegate the validation of (3.20) to the end of the proof.
Continuing, by invoking (3.13), the continuity of f and compactness of W, we conclude that

$$\sup_{\zeta \in W} \|f(\xi + \zeta) - f(\zeta) - K\xi\| < r\|\xi\| \quad \forall\, \xi \in \mathbb{R}^p \backslash \{0\}\,. \tag{3.21}$$

Moreover, by Lemma 3.1.1, the function $\xi \mapsto \sup_{\zeta \in W} \|f(\xi+\zeta) - f(\zeta) - K\xi\|$ is continuous which, in conjunction with (3.20) and (3.21) and an application of statement (ii) of Lemma 2.3.9, shows that there exists $\alpha \in \mathcal{K}_\infty$ such that

$$\sup_{\zeta \in W} \|f(\xi + \zeta) - f(\zeta) - K\xi\| \le r\|\xi\| - \alpha(\|\xi\|) \quad \forall\, \xi \in \mathbb{R}^p\,. \tag{3.22}$$

Let $(v_1, w_1, x_1, y_1) \in \mathcal{B}$ be such that $\|x_1(0)\| + \|(v_1\, w_1)^T\|_{\ell^\infty} \le R$ and $y_1(t) + w_1(t) \in Z$ for all $t \in \mathbb{Z}_+$. Define $g : \mathbb{Z}_+ \times \mathbb{R}^p \to \mathbb{R}^m$ by

$$g(t, \xi) := f\left(\xi + y_1(t) + w_1(t)\right) - f\left(y_1(t) + w_1(t)\right) \quad \forall\, (t, \xi) \in \mathbb{Z}_+ \times \mathbb{R}^p.$$

From (3.19) and (3.22),

$$\sup_{t \in \mathbb{Z}_+} \|g(t, \xi) - K\xi\| \le \sup_{\zeta \in W} \|f(\xi + \zeta) - f(\zeta) - K\xi\| \le r\|\xi\| - \alpha(\|\xi\|) \quad \forall\, \xi \in \mathbb{R}^p,$$

that is, (3.8) holds. Moreover, for $(v_2, w_2, x_2, y_2) \in \mathcal{B}$, if we define $v := v_2 - v_1$, $w := w_2 - w_1$, $x := x_2 - x_1$ and $y := y_2 - y_1$, it is easily checked that (v, w, x, y) satisfies (3.7). An application of Corollary 3.1.2 then yields the existence of $\psi \in \mathcal{KL}$ and $\phi \in \mathcal{K}$ such that (3.11) and (3.12) both hold for all $(v_2, w_2, x_2, y_2) \in \mathcal{B}$. Since ψ and ϕ depend only on Σ, K, r and α (and not on (v_1, w_1, x_1, y_1)), we see that in fact (3.11) and (3.12) both hold for all $(v_1, w_1, x_1, y_1) \in \mathcal{B}$ satisfying $\|x_1(0)\| + \left\|(v_1\, w_1)^T\right\|_{\ell^\infty} \le R$ and $y_1(t) + w_1(t) \in Z$ for all $t \in \mathbb{Z}_+$, and all $(v_2, w_2, x_2, y_2) \in \mathcal{B}$.
All that is left to prove is (3.20). To that end, since f is continuous, and W is compact, it follows that for each $\xi \in \mathbb{R}^p$, there exists $\zeta_\xi \in W$ such that

$$\sup_{\zeta \in W} \|f(\xi + \zeta) - f(\zeta) - K\xi\| = \|f(\xi + \zeta_\xi) - f(\zeta_\xi) - K\xi\|\,. \tag{3.23}$$

If we write, for all $\xi \in \mathbb{R}^p$,

$$\begin{aligned}\|f(\xi + \zeta_\xi) - f(\zeta_\xi) - K\xi\| \le \|f(\xi + \zeta_\xi - z^* + z^*) - f(z^*) - K(\xi + \zeta_\xi - z^*)\| \\ + \|f(z^*)\| + \|K\zeta_\xi\| + \|Kz^*\| + \|f(\zeta_\xi)\|,\end{aligned}$$

and use (3.17), we see that

$$\begin{aligned}\|f(\xi + \zeta_\xi) - f(\zeta_\xi) - K\xi\| \le r\|\xi\| + r\|\zeta_\xi - z^*\| - \alpha_*(\|\xi + \zeta_\xi - z^*\|) \\ + \|f(z^*)\| + \|K\zeta_\xi\| + \|Kz^*\| + \|f(\zeta_\xi)\|.\end{aligned} \tag{3.24}$$

Since $\alpha_* \in \mathcal{K}_\infty$, by the reverse triangle inequality we have that

$$\alpha_*(\|\xi + \zeta_\xi - z^*\|) \ge \alpha_*(\|\xi\| - \|\zeta_\xi - z^*\|) \quad \forall\, \xi \in \mathbb{R}^p,\ \|\xi\| \ge \|\zeta_\xi - z^*\|\,. \tag{3.25}$$

Hence, by recalling that $\|\zeta_\xi - z^*\| \le \rho + \|z^*\|$ for all $\xi \in \mathbb{R}^p$ and by combining (3.23)–(3.25), we obtain, for all $\xi \in \mathbb{R}^p$ with $\|\xi\| \ge \rho + \|z^*\|$,

$$r\|\xi\| - \sup_{\zeta \in W} \|f(\xi+\zeta) - f(\zeta) - K\xi\| \ge -r\rho - r\|z^*\| + \alpha_*(\|\xi\| - \rho - \|z^*\|) - \|f(z^*)\| - \|K\zeta_\xi\| - \|Kz^*\| - \|f(\zeta_\xi)\|.$$

Since $\|\zeta_\xi\| \le \rho$ for all $\xi \in \mathbb{R}^p$, and f is continuous, it follows that $\|f(\zeta_\xi)\|$ is bounded and so the right-hand side of the above converges to ∞ as $\|\xi\| \to \infty$. This is precisely (3.20). □

To illustrate Theorem 3.1.5, we present two examples. They each provide a system that satisfies the hypotheses of Theorem 3.1.5, but are such that there does not exist $\delta > 0$ such that (3.15) holds.

Example 3.1.8. Consider the simple case of the Lur'e system (2.17) with

$$A := \frac{1}{2}\begin{pmatrix} 2 & 1 & 0 \\ -1 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad B := \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad C := \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}, \quad D = 0,$$

B_e and D_e arbitrary, and $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(\xi) := \frac{1}{2}\operatorname{sign}(\xi)\ln(1 + |\xi|) \quad \forall\, \xi \in \mathbb{R}.$$

It is easy to check that A is Schur,

$$\mathbf{G}(z) = \frac{1/4}{(z - 1/2)^3},$$

and

$$|\mathbf{G}(z)| \le 2 = |\mathbf{G}(1)| \quad \forall\, z \in \operatorname{clos}(\mathbb{E}).$$

We hence deduce that

$$\|\mathbf{G}\|_{H^\infty} = |\mathbf{G}(1)| = 2.$$

Moreover, it can easily be verified that (A) holds with $r = 1/2$ and $K = 0$. Since $f'(0) = 1/2$, it can also be checked that there does not exist $\delta > 0$ such that

$$|f(\xi+\zeta) - f(\zeta)| \le \left(\frac{1}{2} - \delta\right)|\xi| \quad \forall\, \xi, \zeta \in \mathbb{R},$$

that is, f does not satisfy the assumptions of [40, Theorem 3.2]. However, since f is continuously differentiable with

$$f'(0) = \frac{1}{2} \quad \text{and} \quad f'(\xi) \in \left(0, \frac{1}{2}\right) \quad \forall\, \xi \in \mathbb{R}\backslash\{0\},$$

it follows from [15, Lemma 4.9] that (3.13) holds with $Z = \mathbb{R}^p$, $K = 0$ and $r = 1/2$. Finally,

$$\frac{1}{2}|\xi| - |f(\xi)| \to \infty \text{ as } |\xi| \to \infty$$

and so (3.14) is satisfied with $K = 0$, $r = 1/2$ and $\eta = 0$. Therefore, Theorem 3.1.5 gives that (2.17) is semi-globally incrementally input-to-state stable. ♢

We comment that the nonlinearity given in the previous example is a modification of that found in [15, Example 4.12].

Example 3.1.9. Consider again a simple version of (2.17). Here we take

$$A := \begin{pmatrix} 1/10 & 1 \\ 0 & 1/2 \end{pmatrix}, \quad B := \begin{pmatrix} 1 \\ 1/2 \end{pmatrix}, \quad C := \begin{pmatrix} 0 & 1 \end{pmatrix}, \quad D = 0,$$

B_{e} and D_{e} arbitrary, and $f : \mathbb{R} \to \mathbb{R}$ as

$$f(\xi) := \begin{cases} \xi/(e-1), & \text{if } |\xi| \leq e-1, \\ \operatorname{sign}(\xi)\left(|\xi| - e + 3 - \sqrt{|\xi| - e + 2}\right), & \text{if } |\xi| > e-1. \end{cases}$$

Trivially, A is Schur. Moreover,

$$\mathbf{G}(z) = \frac{1/2}{z - 1/2},$$

and

$$|\mathbf{G}(z)| \leq 1 = |\mathbf{G}(1)| \quad \forall\, z \in \operatorname{clos}(\mathbb{E}).$$

Therefore,

$$\|\mathbf{G}\|_{H^\infty} = |\mathbf{G}(1)| = 1,$$

and (A) holds with $r = 1$ and $K = 0$. We note that since

$$\frac{1}{|\xi|}|f(\xi) - f(0)| \to 1 \text{ as } |\xi| \to \infty,$$

there does not exist $\delta > 0$ such that

$$|f(\xi + \zeta) - f(\zeta)| \leq (1 - \delta)\,|\xi| \quad \forall\, \xi, \zeta \in \mathbb{R}.$$

Whence, f does not satisfy the assumptions of [40, Theorem 3.2]. However, as in Example 3.1.8, since f is continuously differentiable with

$$f'(\xi) \in (0,1) \quad \forall\, \xi \in \mathbb{R},$$

it follows from [15, Lemma 4.9] that (3.13) holds with $Z = \mathbb{R}^p$, $K = 0$ and $r = 1$. Finally,

$$|\xi| - |f(\xi)| \to \infty \text{ as } |\xi| \to \infty$$

and so (3.14) is satisfied with $K = 0$, $r = 1$ and $\eta = 0$. We thus conclude that the hypotheses of Theorem 3.1.5 are satisfied (with $Z = \mathbb{R}^p$, $K = 0$ and $r = 1$) and so (2.17) is semi-globally incrementally input-to-state stable. $\diamondsuit$

We conclude our discussion of incremental stability properties of (2.17) with a corollary of Theorem 3.1.5. This corollary presents sufficient conditions, reminiscent of the well-known circle-criterion (see, for example, [66]), for when (2.17) is semi-globally incrementally ISOS. Before giving this result, we recall from Definition 2.3.23 that a $\mathbb{C}^{m\times m}$-valued rational function $\mathbf{H}$ is *positive real* if $\mathbf{H}(z)+\mathbf{H}(z)^*$ is positive semi-definite for every $z \in \mathbb{E}$ which is not a pole of $\mathbf{H}$.

Corollary 3.1.10. *Let* $\Sigma \in \mathbb{L}$, $f : \mathbb{R}^p \to \mathbb{R}^m$ *and* $K_1, K_2 \in \mathbb{R}^{m \times p}$ *with* $K_1 \in \mathbb{A}_{\mathbb{R}}(D)$. *Assume that* $\mathbf{H} := (I - K_2\mathbf{G})(I - K_1\mathbf{G})^{-1}$ *is positive real and that* Σ *is either* (i) *controllable and observable or,* (ii) *stabilisable and detectable and there exists* $z \in \mathbb{C}$ *such that* $|z| = 1$ *and* $\mathbf{H}(z) + \mathbf{H}(z)^*$ *is positive definite. If, additionally,*

$$\langle f(\xi+\zeta) - f(\zeta) - K_1\xi, f(\xi+\zeta) - f(\zeta) - K_2\xi \rangle < 0 \quad \forall \xi, \zeta \in \mathbb{R}^p,\ \xi \neq 0, \tag{3.26}$$

and there exists $\eta \in \mathbb{R}^p$ *and* $\alpha \in \mathcal{K}_\infty$ *such that*

$$\langle f(\xi+\eta) - f(\eta) - K_1\xi, f(\xi+\eta) - f(\eta) - K_2\xi \rangle \leq -\alpha(\|\xi\|)\|\xi\| \quad \forall \xi \in \mathbb{R}^p,$$

then (2.17) *is semi-globally incrementally ISOS.*

Remark 3.1.11. The assumptions of Corollary 3.1.10 are the same as those of Corollary 2.3.24, where it was deduced that (2.17) exhibits the CICSO property. ◊

We shall not provide a proof of Corollary 3.1.10, since the result can be proven by using the proof of Corollary 2.3.24, but invoking Theorem 3.1.5 instead of Corollary 2.3.18.

3.1.3 Discrete-time Lur'e systems subject to almost periodic forcing

We now provide a corollary of Theorem 3.1.5 that asserts that the hypotheses of Theorem 3.1.5 are sufficient for obtaining that asymptotically almost periodic inputs generate asymptotically almost periodic state and output trajectories. A thorough presentation of almost periodic functions is given in Appendix C. Indeed, we define the notion of almost and asymptotically almost periodic functions over various time-domains, give relevant theory, and discuss relationships between almost periodicity and other notions. Therefore, in the interest of not repeating ourselves, we will not define almost and asymptotically almost periodic functions here, or give much relevant theory, and instead refer the reader to the appendix. For additional background reading, we also refer the reader to literature such as [1, 14, 17, 21, 22, 52].

However, for convenience, we recall: that $AP(Z, \mathbb{R}^n)$ denotes the space of almost periodic functions mapping $Z \to \mathbb{R}^n$, where $Z = \mathbb{Z}$ or $\mathbb{Z}_+$; for $\varepsilon > 0$, the set of ε-periods of an almost periodic function v^{ap} is denoted by $P(v^{\mathrm{ap}}, \varepsilon)$; the space of asymptotically almost periodic functions $\mathbb{Z}_+ \to \mathbb{R}^n$ is denoted by $AAP(\mathbb{Z}_+, \mathbb{R}^n)$; and $c_0(\mathbb{Z}_+, \mathbb{R}^n)$ denotes the space of all functions $\mathbb{Z}_+ \to \mathbb{R}^n$ that converge to zero as their argument tends to infinity. In addition to this, we also recall that every $v^{\mathrm{ap}} \in AP(\mathbb{Z}_+, \mathbb{R}^n)$ has a unique extension $v^{\mathrm{ap}}_{\mathrm{e}} \in AP(\mathbb{Z}, \mathbb{R}^n)$ such that

$$v^{\mathrm{ap}}_{\mathrm{e}}(t) = v^{\mathrm{ap}}(t) \quad \forall\, t \in \mathbb{Z}_+ \quad \text{and} \quad \sup_{t\in\mathbb{Z}} \|v^{\mathrm{ap}}_{\mathrm{e}}(t)\| = \sup_{t\in\mathbb{Z}_+} \|v^{\mathrm{ap}}(t)\|.$$

Indeed, as discussed in Appendix C, if we follow an idea seen in [12, Remark on p.318] and define $v^{\mathrm{ap}}_{\mathrm{e}}$ by

$$v^{\mathrm{ap}}_{\mathrm{e}}(t) = \lim_{k\to\infty} v^{\mathrm{ap}}(t + \tau_k) \quad \forall\, t \in \mathbb{Z}, \tag{3.27}$$

where, for each $k \in \mathbb{N}$, $\tau_k \in P(v^{\mathrm{ap}}, 1/k)$ and $\tau_k \nearrow \infty$ as $k \to \infty$, then it can be shown (see Lemma C.1.6 and Corollary C.1.33) that this function is a well-defined almost periodic extension of v^{ap} and satisfies $\sup_{t\in\mathbb{Z}} \|v^{\mathrm{ap}}_{\mathrm{e}}(t)\| = \sup_{t\in\mathbb{Z}_+} \|v^{\mathrm{ap}}(t)\|$. With this in mind, we obtain the following theorem, which is a finite-dimensional version of Theorem C.1.35 in Appendix C.

Theorem 3.1.12. *The map $AP(\mathbb{Z}_+, \mathbb{R}^n) \to AP(\mathbb{Z}, \mathbb{R}^n)$ which maps v^{ap} to $v_{\mathrm{e}}^{\mathrm{ap}}$, where $v_{\mathrm{e}}^{\mathrm{ap}}$ is given by* (3.27)*, is an isometric isomorphism.*

We are now ready to present the previously discussed corollary of Theorem 3.1.5, which concerns the convergence of trajectories with (asymptotically) almost periodic inputs.

Corollary 3.1.13. *Let $\Sigma \in \mathbb{L}$, $f : \mathbb{R}^p \to \mathbb{R}^m$, $K \in \mathbb{R}^{m\times p}$, $r > 0$, $v^{\mathrm{ap}} \in AP(\mathbb{Z}_+, \mathbb{R}^q)$ and $w^{\mathrm{ap}} \in AP(\mathbb{Z}_+, \mathbb{R}^p)$. Assume that $\mathbb{B}_{\mathbb{C}}(K, r) \subseteq \mathbb{S}_{\mathbb{C}}(\mathbf{G})$,* (A) *holds, there exists $(\tilde{x}, \tilde{y}) \in (\mathbb{R}^n)^{\mathbb{Z}_+} \times (\mathbb{R}^p)^{\mathbb{Z}_+}$ such that $(v^{\mathrm{ap}}, w^{\mathrm{ap}}, \tilde{x}, \tilde{y}) \in \mathcal{B}$, f satisfies* (3.13) *with $Z = \mathbb{R}^p$, and there exists $\eta \in \mathbb{R}^p$ such that* (3.14) *holds. Then the following statements hold.*

(i) *There exists a unique pair $(x^{\mathrm{ap}}, y^{\mathrm{ap}}) \in AP(\mathbb{Z}_+, \mathbb{R}^n) \times AP(\mathbb{Z}_+, \mathbb{R}^p)$ such that $(v^{\mathrm{ap}}, w^{\mathrm{ap}}, x^{\mathrm{ap}}, y^{\mathrm{ap}}) \in \mathcal{B}$ and, for all $(v, w, x, y) \in \mathcal{B}$ such that $v - v^{\mathrm{ap}} \in c_0(\mathbb{Z}_+, \mathbb{R}^q)$ and $w - w^{\mathrm{ap}} \in c_0(\mathbb{Z}_+, \mathbb{R}^p)$,*

$$\lim_{t\to\infty} \|x(t) - x^{\mathrm{ap}}(t)\| = 0 = \lim_{t\to\infty} \|y(t) - y^{\mathrm{ap}}(t)\|. \tag{3.28}$$

Furthermore, for all $\varepsilon > 0$, there exists $\delta > 0$ such that $P(v^{\mathrm{ap}}, \delta) \cap P(w^{\mathrm{ap}}, \delta) \subseteq P(x^{\mathrm{ap}}, \varepsilon) \cap P(y^{\mathrm{ap}}, \varepsilon)$. In particular, if v^{ap} and w^{ap} are τ-periodic, then x^{ap} and y^{ap} are τ-periodic.

(ii) *The pair of almost periodic extensions $(x_{\mathrm{e}}^{\mathrm{ap}}, y_{\mathrm{e}}^{\mathrm{ap}})$ of $(x^{\mathrm{ap}}, y^{\mathrm{ap}})$ to $\mathbb{Z}$, is the unique bounded pair of functions that satisfy*

$$\left.\begin{aligned} z_1(t+1) &= Az_1(t) + Bf(z_2(t) + w_{\mathrm{e}}^{\mathrm{ap}}(t)) + B_{\mathrm{e}}v_{\mathrm{e}}^{\mathrm{ap}}(t), \\ z_2(t) &= Cz_1(t) + Df(z_2(t) + w_{\mathrm{e}}^{\mathrm{ap}}(t)) + D_{\mathrm{e}}v_{\mathrm{e}}^{\mathrm{ap}}(t), \end{aligned}\right\} \quad \forall\, t \in \mathbb{Z}. \tag{3.29}$$

Remark 3.1.14. (i) Under the assumptions of Corollary 3.1.13, asymptotically almost periodic inputs generate asymptotically almost periodic state and output trajectories.

(ii) Under the assumptions of Corollary 3.1.13, if, additionally, $\|D^K\| < 1/r$, then Proposition 2.3.10 gives that $I - D^K(f - K)$ is bijective. By combining this with Proposition 2.1.37, we see that, for every $x^0 \in \mathbb{R}^n$, $v \in (\mathbb{R}^q)^{\mathbb{Z}_+}$, $w \in (\mathbb{R}^p)^{\mathbb{Z}_+}$, there is a unique solution to the IVP given by (2.18). Hence, in this case, the assumption in Corollary 3.1.13 asserting the existence of $(\tilde{x}, \tilde{y}) \in (\mathbb{R}^n)^{\mathbb{Z}_+} \times (\mathbb{R}^p)^{\mathbb{Z}_+}$ such that $(v^{\mathrm{ap}}, w^{\mathrm{ap}}, \tilde{x}, \tilde{y}) \in \mathcal{B}$, is redundant. We further note that, trivially, $\|D^K\| < 1/r$ if $D = 0$.

(iii) Instead of imposing in Corollary 3.1.13 the assumptions of Theorem 3.1.5, we could impose the hypotheses of Corollary 3.1.10, and statements (i) and (ii) remain valid. ◊

Proof of Corollary 3.1.13. We begin by proving statement (i). To this end, we use Propositions 2.3.6 and 2.3.8 to yield that, for every $v^* \in \mathbb{R}^q$ and $w^* \in \mathbb{R}^p$, there exist $x^* \in \mathbb{R}^n$ and $y^* \in \mathbb{R}^p$ such that (v^*, w^*, x^*, y^*) is an equilibrium quadruple (see Definition 2.3.1). We now fix such a quadruple (v^*, w^*, x^*, y^*). Let $(v^{\mathrm{ap}}, w^{\mathrm{ap}}, \tilde{x}, \tilde{y}) \in \mathcal{B}$ and note that, since v^{ap} and w^{ap} are almost periodic, v^{ap} and w^{ap} are bounded (see Lemma C.1.31). Let $R_1 > 0$ be such that

$$\|\tilde{x}(0)\| + \left\| \begin{pmatrix} v^{\mathrm{ap}} \\ w^{\mathrm{ap}} \end{pmatrix} \right\|_{\ell^\infty}, \|x^*\| + \left\| \begin{pmatrix} v^* \\ w^* \end{pmatrix} \right\| \le R_1. \tag{3.30}$$

Hence, by setting $Z := \mathbb{R}^p$, Theorem 3.1.5 yields the existence of $\psi_1 \in \mathcal{KL}$ and $\phi_1 \in \mathcal{K}$ (dependent on R_1) such that

$$\left\| \begin{pmatrix} \tilde{x}(t) - x^* \\ \tilde{y}(t) - y^* \end{pmatrix} \right\| \leq \psi_1 \left(\|\tilde{x}(0) - x^*\|, t \right) + \phi_1 \left(\max_{s \in \underline{t}} \left\| \begin{pmatrix} v^{\mathrm{ap}}(s) - v^* \\ w^{\mathrm{ap}}(s) - w^* \end{pmatrix} \right\| \right) \quad \forall\, t \in \mathbb{Z}_+ .$$

Combining this with (3.30), we see that

$$\left\| \begin{pmatrix} \tilde{x}(t) - x^* \\ \tilde{y}(t) - y^* \end{pmatrix} \right\| \leq \psi_1 \left(2R_1, 0 \right) + \phi_1 \left(2R_1 \right) \quad \forall\, t \in \mathbb{Z}_+ ,$$

whence $\tilde{x}$ and $\tilde{y}$ are bounded. Thus, as a consequence, there exists $R > 0$ such that

$$\|\tilde{x}\|_{\ell^\infty} + \left\| \begin{pmatrix} v^{\mathrm{ap}} \\ w^{\mathrm{ap}} \end{pmatrix} \right\|_{\ell^\infty} \leq R . \tag{3.31}$$

Since v^{ap} and w^{ap} are almost periodic, by Lemma C.1.37, we may obtain a sequence $(\tau_k)_{k \in \mathbb{N}} \subseteq \mathbb{Z}_+$ such that

$$\tau_k \in P\left(v^{\mathrm{ap}}, \frac{1}{2k} \right) \cap P\left(w^{\mathrm{ap}}, \frac{1}{2k} \right) \quad \forall\, k \in \mathbb{N} \quad \text{and} \quad \tau_k \to \infty \text{ as } k \to \infty. \tag{3.32}$$

Inspired by an argument from the proof of [5, Proposition 4.4], we claim that $(\Lambda_{\tau_k} \tilde{x})_{k \in \mathbb{N}}$ and $(\Lambda_{\tau_k} \tilde{y})_{k \in \mathbb{N}}$ are Cauchy sequences in $\ell^\infty(\mathbb{Z}_+, \mathbb{R}^n)$ and $\ell^\infty(\mathbb{Z}_+, \mathbb{R}^p)$, respectively. To show this, we first invoke Theorem 3.1.5 to obtain $\psi \in \mathcal{KL}$ and $\phi \in \mathcal{K}$ (dependent on R) such that, for all $(v_i, w_i, x_i, y_i) \in \mathcal{B}$ with $\|x_i(0)\| + \left\|(v_i\; w_i)^T\right\|_{\ell^\infty} \leq R$, $i = 1, 2$,

$$\left\| \begin{pmatrix} x_1(t) - x_2(t) \\ y_1(t) - y_2(t) \end{pmatrix} \right\| \leq \psi \left(\|x_1(0) - x_2(0)\|, t \right) + \phi \left(\sup_{s \in \underline{t}} \left\| \begin{pmatrix} v_1(s) - v_2(s) \\ w_1(s) - w_2(s) \end{pmatrix} \right\| \right) \quad \forall\, t \in \mathbb{Z}_+. \tag{3.33}$$

Subsequently, we let $\varepsilon > 0$ and $k, l \in \mathbb{N}$ be sufficiently large so that

$$\psi(2R, \tau_k), \psi(2R, \tau_l) \leq \frac{\varepsilon}{2} \quad \text{and} \quad \phi\left(\frac{1}{k} + \frac{1}{l} \right) \leq \frac{\varepsilon}{2},$$

and, without loss of generality, assume that $\tau_l \geq \tau_k$. Then, for all $t \in \mathbb{Z}_+$,

$$\begin{aligned} &\sup_{s \in \underline{\tau_k}} \left\| \begin{pmatrix} v^{\mathrm{ap}}(s+t) - v^{\mathrm{ap}}(s+t+\tau_l-\tau_k) \\ w^{\mathrm{ap}}(s+t) - w^{\mathrm{ap}}(s+t+\tau_l-\tau_k) \end{pmatrix} \right\| \\ &\qquad \leq \sup_{s \in \underline{\tau_k}} \left\| \begin{pmatrix} v^{\mathrm{ap}}(s+t) - v^{\mathrm{ap}}(s+t+\tau_l) \\ w^{\mathrm{ap}}(s+t) - w^{\mathrm{ap}}(s+t+\tau_l) \end{pmatrix} \right\| \\ &\qquad\quad + \sup_{s \in \underline{\tau_k}} \left\| \begin{pmatrix} v^{\mathrm{ap}}(s+t+\tau_l) - v^{\mathrm{ap}}(s+t+\tau_l-\tau_k) \\ w^{\mathrm{ap}}(s+t+\tau_l) - w^{\mathrm{ap}}(s+t+\tau_l-\tau_k) \end{pmatrix} \right\| \\ &\qquad \leq \frac{1}{l} + \frac{1}{k}. \end{aligned}$$

Hence, for all $t \in \mathbb{Z}_+$,

$$\phi \left(\sup_{s \in \underline{\tau_k}} \left\| \begin{pmatrix} v^{\mathrm{ap}}(s+t) - v^{\mathrm{ap}}(s+t+\tau_l-\tau_k) \\ w^{\mathrm{ap}}(s+t) - w^{\mathrm{ap}}(s+t+\tau_l-\tau_k) \end{pmatrix} \right\| \right) \leq \frac{\varepsilon}{2},$$

which, when combined with Lemma 2.1.33, (3.31) and (3.33), implies, for all $t \in \mathbb{Z}_+$,

$$\begin{aligned}\left\| \begin{pmatrix} (\Lambda_{\tau_k}\tilde{x})(t) - (\Lambda_{\tau_l}\tilde{x})(t) \\ (\Lambda_{\tau_k}\tilde{y})(t) - (\Lambda_{\tau_l}\tilde{y})(t) \end{pmatrix} \right\| &= \left\| \begin{pmatrix} (\Lambda_t\tilde{x})(\tau_k) - (\Lambda_{t+\tau_l-\tau_k}\tilde{x})(\tau_k) \\ (\Lambda_t\tilde{y})(\tau_k) - (\Lambda_{t+\tau_l-\tau_k}\tilde{y})(\tau_k) \end{pmatrix} \right\| \\ &\le \psi(\|\tilde{x}(t) - \tilde{x}(t+\tau_l-\tau_k)\|, \tau_k) + \frac{\varepsilon}{2} \\ &\le \psi(2R, \tau_k) + \frac{\varepsilon}{2} \\ &\le \varepsilon. \end{aligned}$$

Whence, we have shown that $(\Lambda_{\tau_k}\tilde{x})_{k\in\mathbb{N}}$ and $(\Lambda_{\tau_k}\tilde{y})_{k\in\mathbb{N}}$ are Cauchy sequences in $\ell^\infty(\mathbb{Z}_+, \mathbb{R}^n)$ and $\ell^\infty(\mathbb{Z}_+, \mathbb{R}^p)$, respectively. They therefore converge to functions $x^{\mathrm{ap}} \in \ell^\infty(\mathbb{Z}_+, \mathbb{R}^n)$ and $y^{\mathrm{ap}} \in \ell^\infty(\mathbb{Z}_+, \mathbb{R}^p)$, respectively. To show that $(v^{\mathrm{ap}}, w^{\mathrm{ap}}, x^{\mathrm{ap}}, y^{\mathrm{ap}}) \in \mathcal{B}$, we note that, for all $k \in \mathbb{Z}_+$ and $t \in \mathbb{Z}_+$, by Lemma 2.1.33,

$$\begin{aligned}(\Lambda_{\tau_k}\tilde{x})(t+1) &= A(\Lambda_{\tau_k}\tilde{x})(t) + Bf((\Lambda_{\tau_k}\tilde{y})(t) + (\Lambda_{\tau_k}w^{\mathrm{ap}})(t)) + B_{\mathrm{e}}(\Lambda_{\tau_k}v^{\mathrm{ap}})(t), \\ (\Lambda_{\tau_k}\tilde{y})(t) &= C(\Lambda_{\tau_k}\tilde{x})(t) + Df((\Lambda_{\tau_k}\tilde{y})(t) + (\Lambda_{\tau_k}w^{\mathrm{ap}})(t)) + D_{\mathrm{e}}(\Lambda_{\tau_k}v^{\mathrm{ap}})(t).\end{aligned}$$

Since f is continuous, if we use (3.32) and take the limit as $k \to \infty$, we see that $(v^{\mathrm{ap}}, w^{\mathrm{ap}}, x^{\mathrm{ap}}, y^{\mathrm{ap}}) \in \mathcal{B}$.

To show that $x^{\mathrm{ap}} \in AP(\mathbb{Z}_+, \mathbb{R}^n)$ and $y^{\mathrm{ap}} \in AP(\mathbb{Z}_+, \mathbb{R}^p)$, we fix $\varepsilon > 0$ and note that, since $\phi(0) = 0$ and ϕ is continuous, there exists $\delta_1 > 0$ such that

$$\phi(s) \le \varepsilon \quad \forall\, s \in [0, \delta_1]. \tag{3.34}$$

Let $\delta := \delta_1/2$ and let $\tau \in P(v^{\mathrm{ap}}, \delta) \cap P(w^{\mathrm{ap}}, \delta)$, which exists by Lemma C.1.37. Then, by combining Lemma 2.1.33 with (3.31), (3.33) and (3.34), we see that, for all $t \in \mathbb{Z}_+$ and all $k \in \mathbb{N}$,

$$\begin{aligned}\left\| \begin{pmatrix} (\Lambda_{\tau_k}\tilde{x})(t) - (\Lambda_{\tau_k}\tilde{x})(t+\tau) \\ (\Lambda_{\tau_k}\tilde{y})(t) - (\Lambda_{\tau_k}\tilde{y})(t+\tau) \end{pmatrix} \right\| &= \left\| \begin{pmatrix} (\Lambda_t\tilde{x})(\tau_k) - (\Lambda_{t+\tau}\tilde{x})(\tau_k) \\ (\Lambda_t\tilde{y})(\tau_k) - (\Lambda_{t+\tau}\tilde{y})(\tau_k) \end{pmatrix} \right\| \\ &\le \psi\left(\|\tilde{x}(t) - \tilde{x}(t+\tau)\|, \tau_k\right) + \varepsilon. \end{aligned} \tag{3.35}$$

Since $(\tau_k)_{k\in\mathbb{N}}$ converges to ∞ as $k \to \infty$ and $\tilde{x}$ is bounded, (3.35) hence yields, for all $t \in \mathbb{Z}_+$,

$$\left\| \begin{pmatrix} x^{\mathrm{ap}}(t) - x^{\mathrm{ap}}(t+\tau) \\ y^{\mathrm{ap}}(t) - y^{\mathrm{ap}}(t+\tau) \end{pmatrix} \right\| \le \varepsilon,$$

showing that $\tau \in P(x^{\mathrm{ap}}, \varepsilon) \cap P(y^{\mathrm{ap}}, \varepsilon)$. It follows that $P(v^{\mathrm{ap}}, \delta) \cap P(w^{\mathrm{ap}}, \delta) \subseteq P(x^{\mathrm{ap}}, \varepsilon) \cap P(y^{\mathrm{ap}}, \varepsilon)$. Since $P(v^{\mathrm{ap}}, \delta) \cap P(w^{\mathrm{ap}}, \delta)$ is relatively dense in $\mathbb{Z}_+$ (see Defintion C.1.29 and Lemma C.1.37), we see that $P(x^{\mathrm{ap}}, \varepsilon) \cap P(y^{\mathrm{ap}}, \varepsilon)$ is relatively dense in $\mathbb{Z}_+$, showing that $x^{\mathrm{ap}} \in AP(\mathbb{Z}_+, \mathbb{R}^n)$ and $y^{\mathrm{ap}} \in AP(\mathbb{Z}_+, \mathbb{R}^p)$.

In order to establish (3.28), let $(v, w, x, y) \in \mathcal{B}$ be such that $v - v^{\mathrm{ap}} \in c_0(\mathbb{Z}_+, \mathbb{R}^q)$ and $w - w^{\mathrm{ap}} \in c_0(\mathbb{Z}_+, \mathbb{R}^p)$. Obviously, v and w are bounded, and so, as in the beginning of this proof, an application of Theorem 3.1.5 yields that x and y are bounded. Subsequently, let $\tilde{R} > 0$ be such that

$$\|x\|_{\ell^\infty} + \left\| \begin{pmatrix} v \\ w \end{pmatrix} \right\|_{\ell^\infty}, \|x^{\mathrm{ap}}\|_{\ell^\infty} + \left\| \begin{pmatrix} v^{\mathrm{ap}} \\ w^{\mathrm{ap}} \end{pmatrix} \right\|_{\ell^\infty} \le \tilde{R}\,.$$

Another application of Theorem 3.1.5 guarantees the existence of $\tilde{\psi} \in \mathcal{KL}$ and $\tilde{\phi} \in \mathcal{K}$ (dependent upon $\tilde{R}$) such that (3.33) holds for all $(v_i, w_i, x_i, y_i) \in \mathcal{B}$ with $\|x_i(0)\| + \left\|(v_i\, w_i)^T\right\|_{\ell^\infty} \leq \tilde{R}$, $i = 1, 2$, but with ψ and ϕ replaced by $\tilde{\psi}$ and $\tilde{\phi}$, respectively. In particular, by setting $\tilde{v} := v - v^{\mathrm{ap}}$ and $\tilde{w} := w - w^{\mathrm{ap}}$ and using Lemma 2.1.33, we see that, for all $t \in \mathbb{Z}_+$,

$$\left\|\begin{pmatrix} x(t) - x^{\mathrm{ap}}(t) \\ y(t) - y^{\mathrm{ap}}(t) \end{pmatrix}\right\| = \left\|\begin{pmatrix} (\Lambda_{\lfloor t/2 \rfloor} x)(\lceil t/2 \rceil) - (\Lambda_{\lfloor t/2 \rfloor} x^{\mathrm{ap}})(\lceil t/2 \rceil) \\ (\Lambda_{\lfloor t/2 \rfloor} y)(\lceil t/2 \rceil) - (\Lambda_{\lfloor t/2 \rfloor} y^{\mathrm{ap}})(\lceil t/2 \rceil) \end{pmatrix}\right\|$$
$$\leq \tilde{\psi}\left(\|x(\lfloor t/2 \rfloor) - x^{\mathrm{ap}}(\lfloor t/2 \rfloor)\|, \lceil t/2 \rceil\right) + \tilde{\phi}\left(\sup_{s \in \underline{\lceil t/2 \rceil}} \left\|\begin{pmatrix} (\Lambda_{\lfloor t/2 \rfloor} \tilde{v})(s) \\ (\Lambda_{\lfloor t/2 \rfloor} \tilde{w})(s) \end{pmatrix}\right\|\right).$$

By applying (3.33) (again with ψ and ϕ replaced by $\tilde{\psi}$ and $\tilde{\phi}$, respectively) to the term $\|x(\lfloor t/2 \rfloor) - x^{\mathrm{ap}}(\lfloor t/2 \rfloor)\|$ in the above inequality, we deduce that, for all $t \in \mathbb{N}$,

$$\left\|\begin{pmatrix} x(t) - x^{\mathrm{ap}}(t) \\ y(t) - y^{\mathrm{ap}}(t) \end{pmatrix}\right\| \leq \tilde{\psi}\left(\tilde{\psi}(\|x(0) - x^{\mathrm{ap}}(0)\|, \lfloor t/2 \rfloor) + \tilde{\phi}\left(\sup_{s \in \underline{\lfloor t/2 \rfloor}} \left\|\begin{pmatrix} \tilde{v}(s) \\ \tilde{w}(s) \end{pmatrix}\right\|\right), \lceil t/2 \rceil\right)$$
$$+ \tilde{\phi}\left(\sup_{s \in \underline{\lceil t/2 \rceil}} \left\|\begin{pmatrix} (\Lambda_{\lfloor t/2 \rfloor} \tilde{v})(s) \\ (\Lambda_{\lfloor t/2 \rfloor} \tilde{w})(s) \end{pmatrix}\right\|\right).$$

Finally, the right hand side of the above inequality converges to 0 as $t \to \infty$, showing that (3.28) holds. It is easily seen, by a combination of (3.28) with Lemma C.1.39, that $(x^{\mathrm{ap}}, y^{\mathrm{ap}})$ is the unique almost periodic pair such that $(v^{\mathrm{ap}}, w^{\mathrm{ap}}, x^{\mathrm{ap}}, y^{\mathrm{ap}}) \in \mathcal{B}$, completing the proof of statement (i).

We proceed to prove statement (ii). To this end, first note that the almost periodic extensions $x_{\mathrm{e}}^{\mathrm{ap}}$ and $y_{\mathrm{e}}^{\mathrm{ap}}$ of x^{ap} and y^{ap} to $\mathbb{Z}$, respectively, are bounded. We shall now show that $(x_{\mathrm{e}}^{\mathrm{ap}}, y_{\mathrm{e}}^{\mathrm{ap}})$ satisfies (3.29). To this end, note that since $\Lambda_1 x_{\mathrm{e}}^{\mathrm{ap}}$ is the almost periodic extension of $\Lambda_1 x^{\mathrm{ap}}$, we hence see that $\Lambda_1 x_{\mathrm{e}}^{\mathrm{ap}}$ is also the almost periodic extension of $Ax^{\mathrm{ap}} + Bf(y^{\mathrm{ap}} + w^{\mathrm{ap}}) + B_{\mathrm{e}} v^{\mathrm{ap}}$. Moreover, since $y_{\mathrm{e}}^{\mathrm{ap}} + w_{\mathrm{e}}^{\mathrm{ap}}$ is almost periodic (see Corollary C.1.34) and (3.13) holds with $Z = \mathbb{R}^p$, it follows that $f(y_{\mathrm{e}}^{\mathrm{ap}} + w_{\mathrm{e}}^{\mathrm{ap}})$ is almost periodic. Consequently, $Ax_{\mathrm{e}}^{\mathrm{ap}} + Bf(y_{\mathrm{e}}^{\mathrm{ap}} + w_{\mathrm{e}}^{\mathrm{ap}}) + B_{\mathrm{e}} v_{\mathrm{e}}^{\mathrm{ap}}$ is also an almost periodic extension of $Ax^{\mathrm{ap}} + Bf(y^{\mathrm{ap}} + w^{\mathrm{ap}}) + B_{\mathrm{e}} v^{\mathrm{ap}}$. Hence, by the uniqueness of almost periodic extensions (see Theorem 3.1.12),

$$\Lambda_1 x_{\mathrm{e}}^{\mathrm{ap}} = Ax_{\mathrm{e}}^{\mathrm{ap}} + Bf(y_{\mathrm{e}}^{\mathrm{ap}} + w_{\mathrm{e}}^{\mathrm{ap}}) + B_{\mathrm{e}} v_{\mathrm{e}}^{\mathrm{ap}}.$$

In a similar manner, one can easily show that

$$y_{\mathrm{e}}^{\mathrm{ap}} = Cx_{\mathrm{e}}^{\mathrm{ap}} + Df(y_{\mathrm{e}}^{\mathrm{ap}} + w_{\mathrm{e}}^{\mathrm{ap}}) + D_{\mathrm{e}} v_{\mathrm{e}}^{\mathrm{ap}}.$$

Therefore, $(x_{\mathrm{e}}^{\mathrm{ap}}, y_{\mathrm{e}}^{\mathrm{ap}})$ satisfies (3.29).

To show that $(x_{\mathrm{e}}^{\mathrm{ap}}, y_{\mathrm{e}}^{\mathrm{ap}})$ is the unique bounded pair that satisfies (3.29) on $\mathbb{Z}$, let $(\hat{x}, \hat{y}) \in (\mathbb{R}^n)^{\mathbb{Z}} \times (\mathbb{R}^p)^{\mathbb{Z}}$ be another bounded pair satisfying (3.29). Let $\hat{R} > 0$ be such that

$$\|x_{\mathrm{e}}^{\mathrm{ap}}\|_{\ell^\infty} + \|\hat{x}\|_{\ell^\infty} + \left\|\begin{pmatrix} v_{\mathrm{e}}^{\mathrm{ap}} \\ w_{\mathrm{e}}^{\mathrm{ap}} \end{pmatrix}\right\|_{\ell^\infty} \leq \hat{R},$$

and apply Theorem 3.1.5 to obtain the existence of $\psi_{\mathrm{e}} \in \mathcal{KL}$ and $\phi_{\mathrm{e}} \in \mathcal{K}$ (dependent upon $\hat{R}$) such that (3.33) holds for all $(v_i, w_i, x_i, y_i) \in \mathcal{B}$ with $\|x_i(0)\| + \|(v_i w_i)^T\|_{\ell^\infty} \leq \hat{R}$,

$i = 1, 2$, but with ψ and ϕ replaced by ψ_{e} and ϕ_{e}, respectively. Let $\varepsilon > 0$ and $t \in \mathbb{Z}$ and choose $\tau \in \mathbb{Z}$ such that $\tau \leq t$ and

$$\psi_{\mathrm{e}}(2\hat{R}, t - \tau) \leq \varepsilon.$$

Since the restrictions of $(\Lambda_\tau v_{\mathrm{e}}^{\mathrm{ap}}, \Lambda_\tau w_{\mathrm{e}}^{\mathrm{ap}}, \Lambda_\tau x_{\mathrm{e}}^{\mathrm{ap}}, \Lambda_\tau y_{\mathrm{e}}^{\mathrm{ap}})$ and $(\Lambda_\tau v_{\mathrm{e}}^{\mathrm{ap}}, \Lambda_\tau w_{\mathrm{e}}^{\mathrm{ap}}, \Lambda_\tau \hat{x}, \Lambda_\tau \hat{y})$ to $\mathbb{Z}_+$ are in $\mathcal{B}$ and satisfy $\|(\Lambda_\tau x_{\mathrm{e}}^{\mathrm{ap}})(0)\| + \|(\Lambda_\tau \hat{x})(0)\| + \|(\Lambda_\tau v_{\mathrm{e}}^{\mathrm{ap}} \; \Lambda_\tau w_{\mathrm{e}}^{\mathrm{ap}})^T\|_{\ell^\infty} \leq \hat{R}$, it follows from (3.33) (with ψ and ϕ replaced by ψ_{e} and ϕ_{e}, respectively) that

$$\begin{aligned}\left\| \begin{pmatrix} x_{\mathrm{e}}^{\mathrm{ap}}(t) - \hat{x}(t) \\ y_{\mathrm{e}}^{\mathrm{ap}}(t) - \hat{y}(t) \end{pmatrix} \right\| &= \left\| \begin{pmatrix} (\Lambda_\tau x_{\mathrm{e}}^{\mathrm{ap}})(t-\tau) + (\Lambda_\tau \hat{x})(t-\tau) \\ (\Lambda_\tau y_{\mathrm{e}}^{\mathrm{ap}})(t-\tau) + (\Lambda_\tau \hat{y})(t-\tau) \end{pmatrix} \right\| \\ &\leq \psi_{\mathrm{e}}(\|x_{\mathrm{e}}^{\mathrm{ap}}(\tau) - \hat{x}(\tau)\|, t - \tau) \\ &\leq \psi_{\mathrm{e}}(2\hat{R}, t-\tau) \\ &\leq \varepsilon.\end{aligned}$$

Since ε was arbitrary, we see that $x_{\mathrm{e}}^{\mathrm{ap}}(t) = \hat{x}(t)$ and $y_{\mathrm{e}}^{\mathrm{ap}}(t) = \hat{y}(t)$ and, since t was also arbitrary, it follows that $x_{\mathrm{e}}^{\mathrm{ap}} = \hat{x}$ and $y_{\mathrm{e}}^{\mathrm{ap}} = \hat{y}$, completing the proof. □

Remark 3.1.15. Assume the hypotheses of Theorem 3.1.5 hold with $Z = \mathbb{R}^p$ and, for simplicity, that $D = 0$. Consider the (non-autonomous) system

$$z(t+1) = g(t, z(t)), \tag{3.36}$$

where

$$g(t, \xi) := A\xi + Bf(C\xi + D_{\mathrm{e}} v^{\mathrm{ap}}(t) + w^{\mathrm{ap}}(t)) + B_{\mathrm{e}} v^{\mathrm{ap}}(t) \quad \forall\, (t, \xi) \in \mathbb{Z} \times \mathbb{R}^n,$$

$v^{\mathrm{ap}} \in AP(\mathbb{Z}, \mathbb{R}^q)$ and $w^{\mathrm{ap}} \in AP(\mathbb{Z}, \mathbb{R}^p)$. For $(t_0, x^0) \in \mathbb{Z} \times \mathbb{R}^n$, we denote the solution to (3.36) with initial state x^0 at time t_0 by $x(\cdot; t_0, x^0)$, which is defined on $\overline{t_0}$. Corollary 3.1.13 yields the existence of a unique bounded solution $x_{\mathrm{b}} \in (\mathbb{R}^n)^{\mathbb{Z}}$ of (3.36). If we now apply Theorem 3.1.5 and use methods similar to those employed in the proof of statement (ii) of Corollary 3.1.13, we obtain that for all $R > 0$, there exists $\psi \in \mathcal{KL}$ such that, for all $(t_0, x^0) \in \mathbb{Z} \times \mathbb{R}^n$ with $\|x^0\| \leq R$,

$$\|x(t; t_0, x^0) - x_{\mathrm{b}}(t)\| \leq \psi(\|x^0 - x_{\mathrm{b}}(t_0)\|, t - t_0) \quad \forall\, t \in \mathbb{Z},\, t \geq t_0.$$

This shows that (3.36) satisfies a semi-global version of the definition of a uniformly convergent system given in [99]. ◊

Example 3.1.16. The systems given in Example 3.1.8 and Example 3.1.9 were both shown to satisfy the assumptions of Theorem 3.1.5. Therefore, Corollary 3.1.13 yields that asymptotically almost periodic inputs generate asymptotically almost periodic state trajectories, in both systems. ◊

We conclude this section with the following two corollaries. Before doing so, we recall from Lemma C.1.39 that every $v \in AAP(\mathbb{Z}_+, \mathbb{R}^n)$ has a unique decomposition $v = v^{\mathrm{ap}} + v^0$, where $v^{\mathrm{ap}} \in AP(\mathbb{Z}_+, \mathbb{R}^n)$ and $v^0 \in c_0(\mathbb{Z}_+, \mathbb{R}^n)$. In the sequel, we shall call v^{ap} the "almost periodic part of v".

Corollary 3.1.17. *Let $\Sigma \in \mathbb{L}$, $f : \mathbb{R}^p \to \mathbb{R}^m$, $K \in \mathbb{R}^{m\times p}$ and $r > 0$. Assume that $\mathbb{B}_{\mathbb{C}}(K, r) \subseteq \mathbb{S}_{\mathbb{C}}(\mathbf{G})$, (A) holds, $\|D^K\| < 1/r$, f satisfies (3.13) with $Z = \mathbb{R}^p$, and there exists $\eta \in \mathbb{R}^p$ such that (3.14) holds. Then, for all $R > 0$, there exists $\phi \in \mathcal{K}$ such that, for all $(v_i, w_i, x_i, y_i) \in \mathcal{B}$ with $v_i \in AAP(\mathbb{Z}_+, \mathbb{R}^q)$, $w_i \in AAP(\mathbb{Z}_+, \mathbb{R}^p)$ and $\|x_i(0)\| + \|(v_i\, w_i)^T\|_{\ell^\infty} \le R$, $i = 1, 2$*

$$\left\| \begin{pmatrix} x_1^{\mathrm{ap}} - x_2^{\mathrm{ap}} \\ y_1^{\mathrm{ap}} - y_2^{\mathrm{ap}} \end{pmatrix} \right\|_{\ell^\infty} \le \phi\left(\left\| \begin{pmatrix} v_1^{\mathrm{ap}} - v_2^{\mathrm{ap}} \\ w_1^{\mathrm{ap}} - w_2^{\mathrm{ap}} \end{pmatrix} \right\|_{\ell^\infty} \right),$$

where v_i^{ap}, w_i^{ap}, x_i^{ap} and y_i^{ap} are the almost periodic parts of v_i, w_i, x_i and y_i, respectively, $i = 1, 2$.

Remark 3.1.18. The hypotheses of Corollary 3.1.17 imply that those of Corollary 3.1.13 hold, since, as mentioned in Remark 3.1.14, the assumption $\|D^K\| < 1/r$ along with the fact that (3.13) holds with $Z = \mathbb{R}^p$, guarantees that for every $x^0 \in \mathbb{R}^n$, $v \in (\mathbb{R}^q)^{\mathbb{Z}_+}$, $w \in (\mathbb{R}^p)^{\mathbb{Z}_+}$, there is a unique solution to the IVP given by (2.18). ♢

Proof of Corollary 3.1.17. Fix $R > 0$ and let $v^* \in \mathbb{R}^q$ and $w^* \in \mathbb{R}^p$. Applications of Propositions 2.3.6 and 2.3.8 give the existence of $x^* \in \mathbb{R}^n$ and $y^* \in \mathbb{R}^p$ such that (v^*, w^*, x^*, y^*) is an equilibrium quadruple. Let $R_1 := \max\{R, \|(x^*\, y^*)^T\| + \|(v^*\, w^*)^T\|\} > 0$. We now apply Theorem 3.1.5 to obtain the existence of $\psi_1 \in \mathcal{KL}$ and $\phi_1 \in \mathcal{K}$ (dependent upon R_1) such that (3.33) holds for all $(v_i, w_i, x_i, y_i) \in \mathcal{B}$ with $\|x_i(0)\| + \left\|(v_i\, w_i)^T\right\|_{\ell^\infty} \le R_1$, $i = 1, 2$, with ψ and ϕ replaced by ψ_1 and ϕ_1, respectively. For $(v, w, x, y) \in \mathcal{B}$ with $\|x(0)\| + \|(v\, w)^T\|_{\ell^\infty} \le R$, if we use the above with $(v_1, w_1, x_1, y_1) = (v, w, x, y)$ and $(v_2, w_2, x_2, y_2) = (v^*, w^*, x^*, y^*)$, we then see that

$$\left\| \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \right\| \le \psi_1(R + R_1, 0) + \phi_1(R + R_1) + R_1 =: R_2 \quad \forall\, t \in \mathbb{Z}_+.$$

Hence, by setting $R_3 := R_2 + R > 0$, we have that $\|x\|_{\ell^\infty} + \|(v\, w)^T\|_{\ell^\infty} \le R_3$ for all $(v, w, x, y) \in \mathcal{B}$ with $\|x(0)\| + \|(v\, w)^T\|_{\ell^\infty} \le R$.

With this in mind, we once again apply Theorem 3.1.5 to obtain the existence of $\psi \in \mathcal{KL}$ and $\phi \in \mathcal{K}$ (dependent on R_3) such that, for all $(v_i, w_i, x_i, y_i) \in \mathcal{B}$ with $\|x_i(0)\| + \|(v_i\, w_i)^T\|_{\ell^\infty} \le R_3$, $i = 1, 2$, it follows that (3.33) holds.

Let $(v_i, w_i, x_i, y_i) \in \mathcal{B}$ with $v_i \in AAP(\mathbb{Z}_+, \mathbb{R}^q)$, $w_i \in AAP(\mathbb{Z}_+, \mathbb{R}^p)$ and $\|x_i(0)\| + \|(v_i\, w_i)^T\|_{\ell^\infty} \le R$, $i = 1, 2$. Let $v_i^{\mathrm{ap}} \in AP(\mathbb{Z}_+, \mathbb{R}^q)$, $w_i^{\mathrm{ap}} \in AP(\mathbb{Z}_+, \mathbb{R}^p)$, $\tilde{v}_i \in c_0(\mathbb{Z}_+, \mathbb{R}^q)$ and $\tilde{w}_i \in c_0(\mathbb{Z}_+, \mathbb{R}^p)$ be such that $v_i = v_i^{\mathrm{ap}} + \tilde{v}_i$ and $w_i = w_i^{\mathrm{ap}} + \tilde{w}_i$, $i = 1, 2$. From Corollary 3.1.13, we see that x_i and y_i are also asymptotically almost periodic for each $i = 1, 2$. We similarly let $x_i^{\mathrm{ap}} \in AP(\mathbb{Z}_+, \mathbb{R}^n)$, $y_i^{\mathrm{ap}} \in AP(\mathbb{Z}_+, \mathbb{R}^p)$, $\tilde{x}_i \in c_0(\mathbb{Z}_+, \mathbb{R}^n)$ and $\tilde{y}_i \in c_0(\mathbb{Z}_+, \mathbb{R}^p)$ be such that $x_i = x_i^{\mathrm{ap}} + \tilde{x}_i$ and $y_i = y_i^{\mathrm{ap}} + \tilde{y}_i$, $i = 1, 2$. Now, by noting that $\|x_i\|_{\ell^\infty} + \|(v_i\, w_i)^T\| \le R_3$, $i = 1, 2$, we obtain from (3.33) and Lemma 2.1.33 that, for all $t \in \mathbb{Z}_+$,

$$\begin{aligned} \left\| \begin{pmatrix} x_1(t) - x_2(t) \\ y_1(t) - y_2(t) \end{pmatrix} \right\| &= \left\| \begin{pmatrix} (\Lambda_{\lfloor t/2 \rfloor} x_1)(\lceil t/2 \rceil) - (\Lambda_{\lfloor t/2 \rfloor} x_2)(\lceil t/2 \rceil) \\ (\Lambda_{\lfloor t/2 \rfloor} y_1)(\lceil t/2 \rceil) - (\Lambda_{\lfloor t/2 \rfloor} y_2)(\lceil t/2 \rceil) \end{pmatrix} \right\| \\ &\le \psi\left(\|x_1(\lfloor t/2 \rfloor) - x_2(\lfloor t/2 \rfloor)\|, \lceil t/2 \rceil \right) \\ &\quad + \phi\left(\sup_{s \in \underline{\lceil t/2 \rceil}} \left\| \begin{pmatrix} (\Lambda_{\lfloor t/2 \rfloor} v_1)(s) - (\Lambda_{\lfloor t/2 \rfloor} v_2)(s) \\ (\Lambda_{\lfloor t/2 \rfloor} w_1)(s) - (\Lambda_{\lfloor t/2 \rfloor} w_2)(s) \end{pmatrix} \right\| \right). \end{aligned}$$

By recalling that x_1 and x_2 are bounded by R_3, this gives, for all $t \in \mathbb{Z}_+$,

$$\begin{aligned}\left\| \begin{pmatrix} x_1(t) - x_2(t) \\ y_1(t) - y_2(t) \end{pmatrix} \right\| \leq & \psi\left(2R_3, \lceil t/2 \rceil\right) + \phi\left(2 \left\| \begin{pmatrix} v_1^{\mathrm{ap}} - v_2^{\mathrm{ap}} \\ w_1^{\mathrm{ap}} - w_2^{\mathrm{ap}} \end{pmatrix} \right\|_{\ell^\infty}\right) \\ & + \phi\left(2 \sup_{s \in \underline{\lceil t/2 \rceil}} \left\| \begin{pmatrix} (\Lambda_{\lfloor t/2 \rfloor} \tilde{v}_1)(s) - (\Lambda_{\lfloor t/2 \rfloor} \tilde{v}_2)(s) \\ (\Lambda_{\lfloor t/2 \rfloor} \tilde{w}_1)(s) - (\Lambda_{\lfloor t/2 \rfloor} \tilde{w}_2)(s) \end{pmatrix} \right\|\right), \end{aligned} \tag{3.37}$$

where we have used Lemma 2.1.3. Fix $\varepsilon > 0$ and let $T \in \mathbb{Z}_+$ be such that, for all $t \in \overline{T}$,

$$\begin{aligned}\varepsilon \geq & \left\| \begin{pmatrix} \tilde{x}_1(t) - \tilde{x}_2(t) \\ \tilde{y}_1(t) - \tilde{y}_2(t) \end{pmatrix} \right\| + \phi\left(2 \sup_{s \in \underline{\lceil t/2 \rceil}} \left\| \begin{pmatrix} (\Lambda_{\lfloor t/2 \rfloor} \tilde{v}_1)(s) - (\Lambda_{\lfloor t/2 \rfloor} \tilde{v}_2)(s) \\ (\Lambda_{\lfloor t/2 \rfloor} \tilde{w}_1)(s) - (\Lambda_{\lfloor t/2 \rfloor} \tilde{w}_2)(s) \end{pmatrix} \right\|\right) \\ & + \psi\left(2R_3, \lceil t/2 \rceil\right). \end{aligned}$$

Then, from (3.37), an application of Lemma C.1.32 gives

$$\left\| \begin{pmatrix} x_1^{\mathrm{ap}} - x_2^{\mathrm{ap}} \\ y_1^{\mathrm{ap}} - y_2^{\mathrm{ap}} \end{pmatrix} \right\|_{\ell^\infty} \leq \varepsilon + \phi\left(2 \left\| \begin{pmatrix} v_1^{\mathrm{ap}} - v_2^{\mathrm{ap}} \\ w_1^{\mathrm{ap}} - w_2^{\mathrm{ap}} \end{pmatrix} \right\|_{\ell^\infty}\right).$$

Since ε was arbitrary, the proof is thus complete. □

For the second, and final corollary, let us recall the CICSO property, which was given in Definition 2.3.13: for $\Sigma \in \mathbb{L}$ and $f : \mathbb{R}^p \to \mathbb{R}^m$, we say that (2.17) has the CICSO property if for every $v^\infty \in \mathbb{R}^q$ and $w^\infty \in \mathbb{R}^p$, there exists $x^\infty \in \mathbb{R}^n$ and $y^\infty \in \mathbb{R}^p$ such that $\lim_{t\to\infty} x(t) = x^\infty$ and $\lim_{t\to\infty} y(t) = y^\infty$ for every $(v, w, x, y) \in \mathcal{B}$ with $\lim_{t\to\infty} v(t) = v^\infty$ and $\lim_{t\to\infty} w(t) = w^\infty$.

Corollary 3.1.19. *Let $\Sigma \in \mathbb{L}$, $f : \mathbb{R}^p \to \mathbb{R}^m$, $K \in \mathbb{R}^{m\times p}$ and $r > 0$. Assume that $\mathbb{B}_{\mathbb{C}}(K, r) \subseteq \mathbb{S}_{\mathbb{C}}(\mathbf{G})$, (A) holds, f satisfies (3.13) with $Z = \mathbb{R}^p$, and there exists $\eta \in \mathbb{R}^p$ such that (3.14) holds. Then (2.17) exhibits the CICSO property.*

Remark 3.1.20. We note that Corollary 3.1.19 is not novel, since it is a particular case of Corollary 2.3.18. However, we include a proof of the result below, since it is interesting to see the result proven via an application of Corollary 3.1.13. We have now seen the result proven in two different ways. ◊

Proof of Corollary 3.1.19. Let $v^\infty \in \mathbb{R}^q$ and $w^\infty \in \mathbb{R}^p$. An application of Propositions 2.3.6 and 2.3.8 yields the existence of $x^\infty \in \mathbb{R}^n$ and $y^\infty \in \mathbb{R}^p$ such that $(v^\infty, w^\infty, x^\infty, y^\infty)$ is an equilibrium quadruple (see Definition 2.3.1). We recall that here we are abusing notation and interpreting vectors as constant functions. With this in mind, trivially, $(v^\infty, w^\infty, x^\infty, y^\infty) \in AP(\mathbb{Z}_+, \mathbb{R}^q) \times AP(\mathbb{Z}_+, \mathbb{R}^p) \times AP(\mathbb{Z}_+, \mathbb{R}^n) \times AP(\mathbb{Z}_+, \mathbb{R}^p)$. Therefore, the hypotheses of Corollary 3.1.13 are satisfied (with $v^{\mathrm{ap}} := v^\infty$ and $w^{\mathrm{ap}} := w^\infty$), and so we obtain that (x^∞, y^∞) is the unique almost periodic pair such that $(v^\infty, w^\infty, x^\infty, y^\infty) \in \mathcal{B}$ and, for all $(v, w, x, y) \in \mathcal{B}$ such that $v - v^\infty \in c_0(\mathbb{Z}_+, \mathbb{R}^q)$ and $w - w^\infty \in c_0(\mathbb{Z}_+, \mathbb{R}^p)$,

$$\lim_{t\to\infty} \|x^\infty - x^\infty(t)\| = 0 = \lim_{t\to\infty} \|y^\infty - y^\infty(t)\|.$$

Since v^∞ and w^∞ were arbitrary, the proof is complete. □

3.1.4 Application to the four-block problem

Here, we present analgous results of the previous sections, related to four-block forced discrete-time Lur'e systems, which are described informally by Figure 2.2. Since the four-block system is defined in Section 2.4 and a great deal of attention is attributed to its basic properties, we shall only briefly recall this system here and, if the reader seeks more information, refer them to Section 2.4. To that end, in this subsection, we let $p_1, p_2 \in \mathbb{Z}_+$ be such that

$$p_1 + p_2 = p,$$

and, for $i = 1, 2$, define $P_i \in \mathbb{R}^{p_i \times p}$ by

$$P_1 := \left(I_{p_1}, 0_{p_1 \times p_2}\right) \quad \text{and} \quad P_2 := \left(0_{p_2 \times p_1}, I_{p_2}\right),$$

where, for $r, s \in \mathbb{Z}_+$, I_r denotes the identity matrix in $\mathbb{R}^{r \times r}$ and $0_{r \times s} \in \mathbb{R}^{r \times s}$ is the matrix with all entries zero. In the sequel, we consider $i \in \{1, 2\}$ to be a fixed quantity and shall consider (2.76), which is explicitly given by

$$\left.\begin{aligned} x^+ &= Ax + Bu + B_{\mathrm{e}}v, \\ y &= Cx + Du + D_{\mathrm{e}}v, \\ u &= f(P_i y + w), \end{aligned}\right\} \tag{2.76}$$

where $\Sigma \in \mathbb{L}$, $v \in (\mathbb{R}^q)^{\mathbb{Z}_+}$, $w \in (\mathbb{R}^{p_i})^{\mathbb{Z}_+}$ and $f : \mathbb{R}^{p_i} \to \mathbb{R}^m$. As in Definition 2.4.4, we set $\Sigma_i := (A, B, B_{\mathrm{e}}, P_i C, P_i D, P_i D_{\mathrm{e}})$ and denote the behaviour of (2.76) by $\mathcal{B}^i_f(\Sigma)$, which we will reduce to $\mathcal{B}^i$ when the context is clear.

As discussed in Remark 2.4.5, the key to obtaining the following results is that (2.76) can be rewritten in the form of (2.17). To see this, we simply apply P_i to the output equation, so that (2.76) becomes

$$\left.\begin{aligned} x^+ &= Ax + Bf(P_i y + w) + B_{\mathrm{e}}v \\ P_i y &= P_i Cx + P_i D f(P_i y + w) + P_i D_{\mathrm{e}} v. \end{aligned}\right\} \tag{3.38}$$

For given $\Sigma \in \mathbb{L}$, $K_i \in \mathbb{R}^{m \times p_i}$ and $r > 0$ such that $\mathbb{B}_{\mathbb{C}}(K_i, r) \subseteq \mathbb{S}_{\mathbb{C}}(P_i \mathbf{G})$, let us recall assumption (A_i) (which is simply the assumption (A), but in the context of the subsystem generated by Σ_i):

$$\left.\begin{array}{l} \Sigma_i \text{ is (i) controllable and observable, or (ii) stabilisable and detectable and} \\ r \min\limits_{|z|=1} \|(P_i \mathbf{G})^{K_i}(z)\| < 1. \end{array}\right\} \tag{A_i}$$

We now define semi-global incremental ISS and ISOS of (2.76).

Definition 3.1.21. *Let* $\Sigma \in \mathbb{L}$ *and* $f : \mathbb{R}^{p_i} \to \mathbb{R}^m$.

(i) *We say that* (2.76) *is* semi-globally incrementally input-to-state stable *if for any* $R > 0$, *there exist* $\psi \in \mathcal{KL}$ *and* $\phi \in \mathcal{K}$ *such that* (3.11) *holds for all* $(v_j, w_j, x_j, y_j) \in \mathcal{B}^i$ *with* $\|x_j(0)\| + \left\|(v_j\ w_j)^T\right\|_{\ell^\infty} \leq R$, $j = 1, 2$.

(ii) *We say that* (2.76) *is* semi-globally incrementally input-to-state/output stable *if for any* $R > 0$, *there exist* $\psi \in \mathcal{KL}$ *and* $\phi \in \mathcal{K}$ *such that* (3.11) *and* (3.12) *hold for all* $(v_j, w_j, x_j, y_j) \in \mathcal{B}^i$ *with* $\|x_j(0)\| + \left\|(v_j\ w_j)^T\right\|_{\ell^\infty} \leq R$, $j = 1, 2$.

The following theorem is an analogue of Theorem 3.1.5.

Theorem 3.1.22. *Let $\Sigma \in \mathbb{L}$, $f : \mathbb{R}^{p_i} \to \mathbb{R}^m$, $K_i \in \mathbb{R}^{m\times p_i}$, $r > 0$ and $Z \subseteq \mathbb{R}^{p_i}$ be a nonempty and closed subset. Assume that $\mathbb{B}_{\mathbb{C}}(K_i, r) \subseteq \mathbb{S}_{\mathbb{C}}(P_i\mathbf{G})$, (A_i) holds, f is continuous,*

$$\|f(\xi+\zeta) - f(\zeta) - K_i\xi\| < r\|\xi\| \quad \forall \zeta \in Z, \forall \xi \in \mathbb{R}^{p_i}\backslash\{0\}, \tag{3.39}$$

and there exists $\eta \in \mathbb{R}^{p_i}$ such that

$$r\|\xi\| - \|f(\xi+\eta) - f(\eta) - K_i\xi\| \to \infty \text{ as } \|\xi\| \to \infty. \tag{3.40}$$

Then for any $R > 0$, there exist $\psi \in \mathcal{KL}$ and $\phi \in \mathcal{K}$ such that (3.11) *and* (3.12) *both hold for all $(v_1, w_1, x_1, y_1) \in \mathcal{B}^i$ satisfying $\|x_1(0)\| + \|(v_1\, w_1)^T\|_{\ell^\infty} \le R$ and $y_1(t) + w_1(t) \in Z$ for all $t \in \mathbb{Z}_+$, and all $(v_2, w_2, x_2, y_2) \in \mathcal{B}^i$.*

Proof. Let $R > 0$. As previously mentioned, we apply P_i to the output equation of (2.76) so that it becomes (3.38). We now recognise this to be of the form (2.17) with linear component Σ_i. We also note that the corresponding assumptions of Theorem 3.1.5 are satisfied with respect to this system. Hence, as a consequence, there exist $\psi \in \mathcal{KL}$ and $\phi \in \mathcal{K}$ such that (3.11) and (3.12) both hold for all $(v_1, w_1, x_1, y_1) \in \mathcal{B}_f(\Sigma_i)$ satisfying $\|x_1(0)\| + \|(v_1\, w_1)^T\|_{\ell^\infty} \le R$ and $y_1(t) + w_1(t) \in Z$ for all $t \in \mathbb{Z}_+$, and all $(v_2, w_2, x_2, y_2) \in \mathcal{B}_f(\Sigma_i)$. Since $(v, w, x, y) \in \mathcal{B}^i$ implies that $(v, w, x, P_i y) \in \mathcal{B}_f(\Sigma_i)$, then for all $(v_1, w_1, x_1, y_1) \in \mathcal{B}^i$ satisfying $\|x_1(0)\| + \|(v_1\, w_1)^T\|_{\ell^\infty} \le R$ and $y_1(t) + w_1(t) \in Z$ for all $t \in \mathbb{Z}_+$, and all $(v_2, w_2, x_2, y_2) \in \mathcal{B}^i$, it follows that (3.11) holds and

$$\|P_i y_1(t) - P_i y_2(t)\| \le \psi\left(\|x_1(0) - x_2(0)\|, t\right) + \phi\left(\sup_{s\in\underline{t}} \left\| \begin{pmatrix} v_1(s) - v_2(s) \\ w_1(s) - w_2(s) \end{pmatrix} \right\| \right) \ \forall\, t \in \mathbb{Z}_+. \tag{3.41}$$

Fix $(v_1, w_1, x_1, y_1) \in \mathcal{B}^i$ satisfying $\|x_1(0)\| + \|(v_1\, w_1)^T\|_{\ell^\infty} \le R$ and $y_1(t) + w_1(t) \in Z$ for all $t \in \mathbb{Z}_+$, and fix $(v_2, w_2, x_2, y_2) \in \mathcal{B}^i$. To complete the proof, it suffices to shows that (3.12) holds. To that end, we note that, by Proposition 2.4.6, $K := K_i P_i \in \mathbb{A}_{\mathbb{R}}(D)$, and hence, by Lemma 2.1.18,

$$\begin{aligned} y_1 - y_2 =& C^K(x_1 - x_2) + D^K(f(P_i y_1 + w_1) - f(P_i y_2 + w_2) - K(y_1 - y_2)) \\ &+ (I - DK)^{-1} D_{\mathrm{e}}(v_1 - v_2). \end{aligned}$$

An application of (3.39) then yields that, for all $t \in \mathbb{Z}_+$,

$$\|y_1(t) - y_2(t)\| \le \|C^K\|\|x_1(t) - x_2(t)\| + r\|D^K\|\|P_i y_1(t) - P_i y_2(t)\| + c\left\| \begin{pmatrix} v_1(t) - v_2(t) \\ w_1(t) - w_2(t) \end{pmatrix} \right\|,$$

for some $c > 0$. By combining this with (3.11) and (3.41), and by relabelling ψ and ϕ, the proof is complete. ☐

The next result is an analogue of Corollary 3.1.13.

Corollary 3.1.23. *Let $\Sigma \in \mathbb{L}$, $f : \mathbb{R}^{p_i} \to \mathbb{R}^m$, $K_i \in \mathbb{R}^{m\times p_i}$, $r > 0$, $v^{\mathrm{ap}} \in AP(\mathbb{Z}_+, \mathbb{R}^q)$ and $w^{\mathrm{ap}} \in AP(\mathbb{Z}_+, \mathbb{R}^{p_i})$. Assume that $\mathbb{B}_{\mathbb{C}}(K_i, r) \subseteq \mathbb{S}_{\mathbb{C}}(P_i\mathbf{G})$, (A_i) holds, there exists $(\tilde{x}, \tilde{y}) \in (\mathbb{R}^n)^{\mathbb{Z}_+} \times (\mathbb{R}^{p_i})^{\mathbb{Z}_+}$ such that $(v^{\mathrm{ap}}, w^{\mathrm{ap}}, \tilde{x}, \tilde{y}) \in \mathcal{B}^i$, f satisfies* (3.39) *with $Z = \mathbb{R}^{p_i}$, and there exists $\eta \in \mathbb{R}^{p_i}$ such that* (3.40) *holds. Then the following statements hold.*

(i) *There exists a unique pair* $(x^{\mathrm{ap}}, y^{\mathrm{ap}}) \in AP(\mathbb{Z}_+, \mathbb{R}^n) \times AP(\mathbb{Z}_+, \mathbb{R}^p)$ *such that* $(v^{\mathrm{ap}}, w^{\mathrm{ap}}, x^{\mathrm{ap}}, y^{\mathrm{ap}}) \in \mathcal{B}^i$ *and, for all* $(v, w, x, y) \in \mathcal{B}^i$ *such that* $v - v^{\mathrm{ap}} \in c_0(\mathbb{Z}_+, \mathbb{R}^q)$ *and* $w - w^{\mathrm{ap}} \in c_0(\mathbb{Z}_+, \mathbb{R}^{p_i})$*,*

$$\lim_{t\to\infty} \|x(t) - x^{\mathrm{ap}}(t)\| = 0 = \lim_{t\to\infty} \|y(t) - y^{\mathrm{ap}}(t)\|. \tag{3.42}$$

Furthermore, for all $\varepsilon > 0$*, there exists* $\delta > 0$ *such that* $P(v^{\mathrm{ap}}, \delta) \cap P(w^{\mathrm{ap}}, \delta) \subseteq P(x^{\mathrm{ap}}, \varepsilon) \cap P(y^{\mathrm{ap}}, \varepsilon)$*. In particular, if* v^{ap} *and* w^{ap} *are* τ*-periodic, then* x^{ap} *and* y^{ap} *are* τ*-periodic.*

(ii) *The pair of almost periodic extensions* $(x_{\mathrm{e}}^{\mathrm{ap}}, y_{\mathrm{e}}^{\mathrm{ap}})$ *of* $(x^{\mathrm{ap}}, y^{\mathrm{ap}})$ *to* $\mathbb{Z}$*, is the unique bounded pair of functions that satisfy*

$$\left.\begin{aligned} z_1(t+1) &= Az_1(t) + Bf(P_i z_2(t) + w_{\mathrm{e}}^{\mathrm{ap}}(t)) + B_{\mathrm{e}} v_{\mathrm{e}}^{\mathrm{ap}}(t), \\ z_2(t) &= Cz_1(t) + Df(P_i z_2(t) + w_{\mathrm{e}}^{\mathrm{ap}}(t)) + D_{\mathrm{e}} v_{\mathrm{e}}^{\mathrm{ap}}(t), \end{aligned}\right\} \quad \forall\, t \in \mathbb{Z}. \tag{3.43}$$

Proof. We begin with statement (i) and act in a similar manner as that done in the proof of Theorem 3.1.22. Indeed, we note that the assumptions of Corollary 3.1.13 hold in the context of the system given by (3.38). Therefore, there exists a unique pair $(x^{\mathrm{ap}}, \tilde{y}^{\mathrm{ap}}) \in AP(\mathbb{Z}_+, \mathbb{R}^n) \times AP(\mathbb{Z}_+, \mathbb{R}^{p_i})$ such that $(v^{\mathrm{ap}}, w^{\mathrm{ap}}, x^{\mathrm{ap}}, \tilde{y}^{\mathrm{ap}}) \in \mathcal{B}_f(\Sigma_i)$ and, for all $(v, w, x, y) \in \mathcal{B}_f(\Sigma_i)$ such that $v - v^{\mathrm{ap}} \in c_0(\mathbb{Z}_+, \mathbb{R}^q)$ and $w - w^{\mathrm{ap}} \in c_0(\mathbb{Z}_+, \mathbb{R}^{p_i})$,

$$\lim_{t\to\infty} \|x(t) - x^{\mathrm{ap}}(t)\| = 0 = \lim_{t\to\infty} \|y(t) - \tilde{y}^{\mathrm{ap}}(t)\|. \tag{3.44}$$

Furthermore, for all $\varepsilon > 0$, there exists $\delta > 0$ such that $P(v^{\mathrm{ap}}, \delta) \cap P(w^{\mathrm{ap}}, \delta) \subseteq P(x^{\mathrm{ap}}, \varepsilon) \cap P(\tilde{y}^{\mathrm{ap}}, \varepsilon)$. In particular, if v^{ap} and w^{ap} are τ-periodic, then x^{ap} and $\tilde{y}^{\mathrm{ap}}$ are τ-periodic.
We define $y^{\mathrm{ap}} \in (\mathbb{R}^p)^{\mathbb{Z}}$ by

$$y^{\mathrm{ap}} := Cx^{\mathrm{ap}} + Df(\tilde{y}^{\mathrm{ap}} + w^{\mathrm{ap}}) + D_{\mathrm{e}} v^{\mathrm{ap}},$$

and note that, as a consequence, $P_i y^{\mathrm{ap}} = \tilde{y}^{\mathrm{ap}}$ and so $(v^{\mathrm{ap}}, w^{\mathrm{ap}}, x^{\mathrm{ap}}, y^{\mathrm{ap}}) \in \mathcal{B}^i$. Moreover, again from the definition of y^{ap}, since x^{ap}, $\tilde{y}^{\mathrm{ap}}$, v^{ap} and w^{ap} are almost periodic and f satisfies (3.39) with $Z = \mathbb{R}^{p_i}$, it is easy to deduce that $y^{\mathrm{ap}} \in AP(\mathbb{Z}_+, \mathbb{R}^p)$. Let $(v, w, x, y) \in \mathcal{B}^i$ be such that $v - v^{\mathrm{ap}} \in c_0(\mathbb{Z}_+, \mathbb{R}^q)$ and $w - w^{\mathrm{ap}} \in c_0(\mathbb{Z}_+, \mathbb{R}^{p_i})$. Since $(v, w, x, P_i y) \in \mathcal{B}_f(\Sigma_i)$, (3.44) gives that

$$\lim_{t\to\infty} \|x(t) - x^{\mathrm{ap}}(t)\| = 0 = \lim_{t\to\infty} \|P_i y(t) - \tilde{y}^{\mathrm{ap}}(t)\|. \tag{3.45}$$

In order to show that (3.42) holds, it is therefore sufficient to show that $\lim_{t\to\infty} \|y(t) - y^{\mathrm{ap}}(t)\| = 0$. To see this, note that

$$y - y^{\mathrm{ap}} = C(x - x^{\mathrm{ap}}) + D\left(f(P_i y + w) - f(\tilde{y}^{\mathrm{ap}} + w^{\mathrm{ap}})\right) + D_{\mathrm{e}}(v - v^{\mathrm{ap}}),$$

and so, since f satisfies (3.39) with $Z = \mathbb{R}^{p_i}$ and (3.45) holds, it is clear that $\lim_{t\to\infty} \|y(t) - y^{\mathrm{ap}}(t)\| = 0$. As for uniqueness of the pair $(x^{\mathrm{ap}}, y^{\mathrm{ap}})$, this easily follows from a combination of (3.42) with Lemma C.1.39.
Finally, let $\varepsilon > 0$ and set

$$\tilde{\varepsilon} := \frac{\varepsilon}{\|C\| + 2(r + \|K\|)\|D\| + \|D_{\mathrm{e}}\| + 1} \in (0, \varepsilon).$$

Then, from Corollary 3.1.13, we obtain the existence of $\delta > 0$ such that $P(v^{\mathrm{ap}}, \delta) \cap P(w^{\mathrm{ap}}, \delta) \subseteq P(x^{\mathrm{ap}}, \tilde{\varepsilon}) \cap P(\tilde{y}^{\mathrm{ap}}, \tilde{\varepsilon})$. Without loss of generality, we may assume that $\delta \leq \tilde{\varepsilon}$. Let $\tau \in P(v^{\mathrm{ap}}, \delta) \cap P(w^{\mathrm{ap}}, \delta)$ and note that

$$\begin{aligned}\|y^{\mathrm{ap}}(t+\tau) - y^{\mathrm{ap}}(t)\| \leq & \|C\| \|x^{\mathrm{ap}}(t+\tau) - x^{\mathrm{ap}}(t)\| \\ & + \|D\| \|f(\tilde{y}^{\mathrm{ap}}(t+\tau) + w^{\mathrm{ap}}(t+\tau)) - f(\tilde{y}^{\mathrm{ap}}(t) + w^{\mathrm{ap}}(t))\| \\ & + \|D_{\mathrm{e}}\| \|v^{\mathrm{ap}}(t+\tau) - v^{\mathrm{ap}}(t)\|.\end{aligned}$$

Since f satisfies (3.39) with $Z = \mathbb{R}^{p_i}$, this implies that

$$\begin{aligned}\|y^{\mathrm{ap}}(t+\tau) - y^{\mathrm{ap}}(t)\| & \leq \|C\|\tilde{\varepsilon} + \|D_{\mathrm{e}}\|\delta \\ & \quad + (r + \|K_i\|)\|D\| \|\tilde{y}^{\mathrm{ap}}(t+\tau) - \tilde{y}^{\mathrm{ap}}(t) + w^{\mathrm{ap}}(t+\tau) - w^{\mathrm{ap}}(t)\| \\ & \leq \|C\|\tilde{\varepsilon} + (r + \|K_i\|)\|D\|(\tilde{\varepsilon} + \delta) + \|D_{\mathrm{e}}\|\tilde{\varepsilon} \\ & \leq \varepsilon.\end{aligned}$$

We have therefore shown that $\tau \in P(x^{\mathrm{ap}}, \tilde{\varepsilon}) \cap P(y^{\mathrm{ap}}, \varepsilon) \subseteq P(x^{\mathrm{ap}}, \varepsilon) \cap P(y^{\mathrm{ap}}, \varepsilon)$, which completes the proof of statement (i).
Now that statement (i) is given to be true, statement (ii) can be proven in a similar manner to that shown in the proof of Corollary 3.1.13. We thus omit this, and leave it to the reader. □

Remark 3.1.24. Analogous results of Corollaries 3.1.10 and 3.1.17 can also be obtained for the four-block system given by (2.76). They can be proven via similar methods to those seen in the previous proofs, and so, in the interest of brevity, we do not explicitly give these, and will instead leave them to the reader to formulate and verify. ◊

3.2 Convergence properties for continuous-time Lur'e systems with Stepanov almost periodic forcing

As briefly discussed at the start of this chapter, the focus of the current section is to present an analogous result to Corollary 3.1.13, but for Stepanov almost periodic functions (see, for example, [14] or Appendix C) instead of (Bohr) almost periodic functions. Since Stepanov almost periodicity is only relevant in the continuous-time setting, we consider continuous-time forced Lur'e systems here. We comment that this section is based off the work of [41].

3.2.1 Preliminaries

In this initial subsection, we introduce notation, explicitly give the Lur'e system of our current focus, and present relevant theory. We make note that since a lot of this notation and theory is a continuous-time version of a particular case of that introduced in Chapter 2, we will be very brief. Moreover, we further note that the following notation and setup is unique to this section, and should not be confused with that developed in the rest of this thesis.

We begin by denoting by $\mathcal{KL}^{\mathrm{c}}$ the set of functions $\psi : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ such that: for each fixed $t \in \mathbb{R}_+$, the function $\psi(\cdot, t) \in \mathcal{K}$; and for each fixed $s \in \mathbb{R}_+$, the function

$\psi(s,\cdot)$ is non-increasing and $\lim_{t\to\infty}\psi(s,t)=0$. The superscript "c" illustrates that it is the "continuous-time version" of $\mathcal{KL}$ (see Definition 2.1.1), since here the second element of each function takes values in $\mathbb{R}_+$ instead of $\mathbb{Z}_+$. Continuing, we define $\mathbb{L}^{\mathrm{c}} := \mathbb{R}^{n\times n}\times\mathbb{R}^{n\times m}\times\mathbb{R}^{p\times n}$ and consider the controlled and observed linear system

$$\dot{x} = Ax + Bu + v, \quad x(0) = x^0 \in \mathbb{R}^n, \quad y = Cx, \tag{3.46}$$

where $(A,B,C)\in\mathbb{L}^{\mathrm{c}}$, $u\in L^\infty_{\mathrm{loc}}(\mathbb{R}_+,\mathbb{R}^m)$ and $v\in L^\infty_{\mathrm{loc}}(\mathbb{R}_+,\mathbb{R}^n)$. As in Definition 2.1.11, we denote the transfer function of $(A,B,C)\in\mathbb{L}^{\mathrm{c}}$ by $\mathbf{G}$, that is, $\mathbf{G}(s)=C(sI-A)^{-1}B$. Moreover, for $(A,B,C)\in\mathbb{L}^{\mathrm{c}}$ and $K\in\mathbb{C}^{m\times p}$, we denote by $\mathbf{G}^K$ the transfer function of the system given by (A^K,B,C), as in Definition 2.1.19. We recall that, in this case, $A^K = A+BKC$ (see Definition 2.1.16). For $\mathbb{F}=\mathbb{C}$ or $\mathbb{R}$, we denote the (continuous-time) set of stabilising output feedback matrices (over $\mathbb{F}$) for $(A,B,C)\in\mathbb{L}^{\mathrm{c}}$ by $\mathbb{S}^{\mathrm{c}}_{\mathbb{F}}(\mathbf{G})$, that is,

$$\mathbb{S}^{\mathrm{c}}_{\mathbb{F}}(\mathbf{G}) := \left\{K\in\mathbb{F}^{m\times p} : \mathbf{G}^K \text{ is holomorphic and bounded on } \mathbb{C}_0\right\}.$$

Furthermore, for $K\in\mathbb{C}^{m\times p}$ and $r>0$, we obtain from [107, Lemma 2.1] that $\mathbb{B}_{\mathbb{C}}(K,r)\subseteq\mathbb{S}^{\mathrm{c}}_{\mathbb{C}}(\mathbf{G})$ if, and only if, $\sup_{s\in\mathbb{C}_0}\|\mathbf{G}^K(s)\|\le 1/r$.

Application of the feedback law $u=f(y)$ to the linear system (3.46), where $f:\mathbb{R}^p\to\mathbb{R}^m$ is a locally Lipschitz nonlinearity, leads to the closed-loop system

$$\dot{x} = Ax + Bf(Cx) + v. \tag{3.47}$$

From sources such as [19, 33, 114] and [106, Appendix D], for every $x^0\in\mathbb{R}^n$ and $v\in L^\infty_{\mathrm{loc}}(\mathbb{R}_+,\mathbb{R}^n)$, the local Lipschitz property of f guarantees a unique maximal solution of (3.47) defined on a maximal interval of existence $[0,\omega)\subseteq[0,\infty)$. It is well-known (see, for example, [114, Proposition C.3.6, p.481]) that if $\omega<\infty$, then the solution x satisfies

$$\lim_{t\to\omega}\|x(t)\| = \infty. \tag{3.48}$$

In subsequent results, we shall impose assumptions on (3.47) that, as a consquence, will ensure that solutions are defined globally (see Remark 3.2.6). With this in mind, and for given $(A,B,C)\in\mathbb{L}^{\mathrm{c}}$ and locally Lipschitz $f:\mathbb{R}^p\to\mathbb{R}^m$, we define the behaviour of (3.47) by

$$\mathcal{B}^{\mathrm{c}}_f(A,B,C) := \Big\{(v,x)\in L^\infty_{\mathrm{loc}}(\mathbb{R}_+,\mathbb{R}^n)\times W^{1,1}_{\mathrm{loc}}(\mathbb{R}_+,\mathbb{R}^n) \ : \ (v,x) \text{ satisfies (3.47) a.e. on } \mathbb{R}_+\Big\},$$

where, again, the superscript c indicates the continuous-time setting. When the context is clear, we shall supress $\mathcal{B}^{\mathrm{c}}_f(A,B,C)$ to simply $\mathcal{B}^{\mathrm{c}}$. We note that $\mathcal{B}^{\mathrm{c}}$ is left-shift invariant, that is,

$$(v,x)\in\mathcal{B}^{\mathrm{c}} \implies (\Lambda_\tau v,\Lambda_\tau x)\in\mathcal{B}^{\mathrm{c}} \quad \forall\,\tau\in\mathbb{R}_+. \tag{3.49}$$

Finally, we also define the "bi-lateral" behaviour of the Lur'e system (3.47) by

$$\tilde{\mathcal{B}}^{\mathrm{c}}_f(A,B,C) := \Big\{(v,x)\in L^\infty_{\mathrm{loc}}(\mathbb{R},\mathbb{R}^n)\times W^{1,1}_{\mathrm{loc}}(\mathbb{R},\mathbb{R}^n) \ : \ (v,x) \text{ satisfies (3.47) a.e. on } \mathbb{R}\Big\},$$

and again, when the context is clear, we shall supress this to $\tilde{\mathcal{B}}^{\mathrm{c}}$. Similar to $\mathcal{B}^{\mathrm{c}}$, it is true that $\tilde{\mathcal{B}}^{\mathrm{c}}$ is shift invariant, that is,

$$(v, x) \in \tilde{\mathcal{B}}^{\mathrm{c}} \implies (\Lambda_\tau v, \Lambda_\tau x) \in \tilde{\mathcal{B}}^{\mathrm{c}} \quad \forall \tau \in \mathbb{R}. \tag{3.50}$$

For the rest of this subsection, we consider the following continuous-time Lur'e system with time-varying nonlinearity:

$$\dot{x}(t) = Ax(t) + Bg(t, Cx(t)) + v(t), \quad x(0) = x^0 \in \mathbb{R}^n, \tag{3.51}$$

where $(A, B, C) \in \mathbb{L}^{\mathrm{c}}$, $v \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^n)$, and g belongs to the space $\mathcal{C}$ comprising all functions $h : \mathbb{R}_+ \times \mathbb{R}^p \to \mathbb{R}^m$ such that: (i) $h(\cdot\,, \xi)$ is measurable for every $\xi \in \mathbb{R}^p$; (ii) $h(t, \cdot)$ is continuous for almost every $t \in \mathbb{R}_+$; (iii) for every $\xi \in \mathbb{R}^p$ there exists a locally integrable function $\beta : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\|h(t, \xi)\| \leq \beta(t) \quad \text{a.e. } t \in \mathbb{R}_+;$$

and (iv) for every $\eta \in \mathbb{R}^p$, there exists $\rho > 0$ and a locally integrable function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\|h(t, \xi) - h(t, \zeta)\| \leq \alpha(t)\|\xi - \zeta\| \quad \forall\, t \in \mathbb{R}_+, \forall\, \xi, \zeta \in \mathbb{R}^p \text{ s.t. } \|\xi - \eta\|, \|\zeta - \eta\| \leq \rho.$$

We make this assumption on g so that solutions of (3.51) are guaranteed to exist (and are unique) for every initial condition x^0 and input v (see, for example, [33, 114]).

Lemma 3.2.1. *Let $(A, B, C) \in \mathbb{L}^{\mathrm{c}}$, $K \in \mathbb{R}^{m \times p}$, $r > 0$ and $\alpha \in \mathcal{K}_\infty$. Assume that (A, B, C) is stabilsable and detectable and $\mathbb{B}_{\mathbb{C}}(K, r) \subseteq \mathbb{S}_{\mathbb{C}}(\mathbf{G})$. Then there exist $\psi, \tilde{\psi} \in \mathcal{KL}^{\mathrm{c}}$ and $\phi, \tilde{\phi} \in \mathcal{K}$ such that, for all functions $g \in \mathcal{C}$ satisfying*

$$\operatorname*{ess\,sup}_{t \in \mathbb{R}_+} \|g(t, \xi) - K\xi\| \leq r\|\xi\| - \alpha(\|\xi\|) \quad \forall\, \xi \in \mathbb{R}^p, \tag{3.52}$$

and for every $(v, x) \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^n) \times W^{1,1}_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^n)$ satisfying (3.51) almost everywhere on $\mathbb{R}_+$, it follows that

$$\|x(t)\| \leq \tilde{\psi}(\|x(0)\|, t) + \tilde{\phi}\left(\operatorname*{ess\,sup}_{0 \leq s \leq t} \|v(s)\|\right) \quad \forall\, t \in \mathbb{R}_+, \tag{3.53}$$

and

$$\|x(t)\| \leq \psi(\|x(0)\|, t) + \phi\left(\int_0^t \|v(s)\| ds\right) \quad \forall\, t \in \mathbb{R}_+, \tag{3.54}$$

Lemma 3.2.1 extends [107, Theorem 3.2] to the time-varying case.

Proof of Lemma 3.2.1. For ease of notation in the sequel, for given $g \in \mathcal{C}$, $t \geq 0$ and $\xi, \zeta \in \mathbb{R}^n$, we define

$$F_g(t, \xi, \zeta) := A\xi + Bg(t, C\xi) + \zeta.$$

By identically following the steps of the proof of [107, Theorem 3.2], and by using (3.52), we are able to obtain the existence of a continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}_+$ and $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathcal{K}_\infty$ (which depend only upon A, B, C, K, r and α) such that

$$\alpha_1(\|\xi\|) \leq V(\xi) \leq \alpha_2(\|\xi\|) \quad \forall\, \xi \in \mathbb{R}^n, \tag{3.55}$$

and, for all $g \in \mathcal{C}$ satisfying (3.52), almost every $t \geq 0$ and all $\xi, \zeta \in \mathbb{R}^n$,

$$\langle (\nabla V)(\xi), F_g(t, \xi, \zeta) \rangle \leq -\alpha_3(\|\xi\|) + \alpha_4(\|\zeta\|). \tag{3.56}$$

The existence of such functions guarantees the existence of functions $\tilde{\psi} \in \mathcal{KL}^{\mathrm{c}}$ and $\tilde{\phi} \in \mathcal{K}$ such that (3.53) holds for all $(v, x) \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^n) \times W^{1,1}_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^n)$ satisfying (3.51) almost everywhere on $\mathbb{R}_+$. Indeed, this can be shown by mirroring the proof of [77, Theorem 5.41]. Moreover, again from the proof of [77, Theorem 5.41], it is clear that $\tilde{\psi}$ and $\tilde{\phi}$ depend only upon $V, \alpha_1, \alpha_2, \alpha_3$ and α_4, which in turn only depend upon A, B, C, K, r and α. We now proceed to show that (3.54) holds for all $g \in \mathcal{C}$ satisfying (3.52) and all $(v, x) \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^n) \times W^{1,1}_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^n)$ satisfying (3.51) almost everywhere on $\mathbb{R}_+$. To this end, we note that as a consequence of (3.56), we have that, for all $g \in \mathcal{C}$ satisfying (3.52), almost every $t \geq 0$ and all $\xi \in \mathbb{R}^n$,

$$\langle (\nabla V)(\xi), F_g(t, \xi, 0) \rangle \leq -\alpha_3(\|\xi\|). \tag{3.57}$$

We now follow an argument given in the proof of [6, Lemma IV.10]. Thus, we obtain from (3.57) that, for all $g \in \mathcal{C}$ satisfying (3.52), almost every $t \geq 0$ and all $\xi, \zeta \in \mathbb{R}^n$,

$$\begin{aligned}\langle (\nabla V)(\xi), F_g(t, \xi, \zeta) \rangle &= \langle (\nabla V)(\xi), F_g(t, \xi, 0) \rangle + \langle (\nabla V)(\xi), F_g(t, 0, \zeta) \rangle \\ &\leq -\alpha_3(\|\xi\|) + \|(\nabla V)(\xi)\| \|\zeta\|. \end{aligned} \tag{3.58}$$

Since ∇V is continuous, we see that $\kappa \in \mathcal{K}$, where

$$\kappa(s) := s + \max_{\|\xi\| \leq s} \|(\nabla V)(\xi)\| \quad \forall\, s \geq 0.$$

Therefore, (3.58) yields that, for all $g \in \mathcal{C}$ satisfying (3.52), almost every $t \geq 0$ and all $\xi, \zeta \in \mathbb{R}^n$,

$$\langle (\nabla V)(\xi), F_g(t, \xi, \zeta) \rangle \leq -\alpha_3(\|\xi\|) + \kappa(\|\xi\|)\|\zeta\|.$$

By acting identically to the proof of [6, Proposition II.5], we obtain a positive definite (not necessarily proper, i.e. not necessarily radially unbounded), continuously differentiable function $W : \mathbb{R}^n \to \mathbb{R}_+$ and a positive definite function $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ (both dependent only upon A, B, C, K, r and α) such that, for all $g \in \mathcal{C}$ satisfying (3.52), almost every $t \geq 0$ and all $\xi, \zeta \in \mathbb{R}^n$,

$$\langle (\nabla W)(\xi), F_g(t, \xi, \zeta) \rangle \leq -\rho(\|\xi\|) + \|\zeta\|. \tag{3.59}$$

We now claim the existence of a positive definite, proper function $U : \mathbb{R}^n \to \mathbb{R}_+$ and a positive constant c such that, for all $g \in \mathcal{C}$ satisfying (3.52), almost every $t \geq 0$ and all $\xi, \zeta \in \mathbb{R}^n$,

$$\langle (\nabla U)(\xi), F_g(t, \xi, \zeta) \rangle \leq c\|\zeta\|. \tag{3.60}$$

However, we shall not explicitly provide a proof of this here, since identical methods (ensuring to use the assumption (3.52) that is uniform in time) to that shown in steps 1 and 2 of the proof of [45, Theorem 3.1] will lead to the desired outcome. Finally, by combining (3.59) and (3.60), we see that, for all $g \in \mathcal{C}$ satisfying (3.52), almost every $t \geq 0$ and all $\xi, \zeta \in \mathbb{R}^n$,

$$\langle (\nabla (U + W))(\xi), F_g(t, \xi, \zeta) \rangle \leq -\rho(\|\xi\|) + (c + 1)\|\zeta\|. \tag{3.61}$$

From [6, Proof of 2 $\implies$ 1 in Theorem 1, p. 1088], we see that this is sufficient for the existence of $\psi \in \mathcal{KL}^{\mathrm{c}}$ and $\phi \in \mathcal{K}$ such that (3.54) holds for all $(v, x) \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^n) \times W^{1,1}_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^n)$ satisfying (3.51) almost everywhere on $\mathbb{R}_+$. We also note, by inspection of the previous cited proof, that the functions $\psi \in \mathcal{KL}^{\mathrm{c}}$ and $\phi \in \mathcal{K}$ only depend upon U, W, ρ and c. Since U, W, ρ and c each only depend upon A, B, C, K, r and α, the proof is complete. □

3.2.2 Incremental stability properties

In Section 3.1, we developed a semi-global incremental ISS result that allowed us to obtain convergence properties for forced discrete-time Lur'e systems with (Bohr) almost periodic inputs - see Theorem 3.1.5 and Corollary 3.1.13, respectively. As already mentioned, the focus of the present section is to provide an analogue of Corollary 3.1.13 but which concerns Stepanov almost periodic functions. To do so, we act similarly to the previous section and first develop a semi-global incremental stability result. To that end, we begin with the following definition.

Definition 3.2.2. *Let $(A, B, C) \in \mathbb{L}^{\mathrm{c}}$ and $f : \mathbb{R}^p \to \mathbb{R}^m$ be locally Lipschitz.*

(i) *We say that* (3.47) *is* semi-globally incrementally integral input-to-state stable (iISS) *(with linear iISS gain) if for any $R > 0$, there exist $\psi \in \mathcal{KL}^{\mathrm{c}}$ and $\phi \in \mathcal{K}$ such that, for all $(v_i, x_i) \in \mathcal{B}^{\mathrm{c}}$ with $\|x_i(0)\| + \|v_i\|_{L^\infty} \le R$, $i = 1, 2$,*

$$\|x_1(t) - x_2(t)\| \le \psi\left(\|x_1(0) - x_2(0)\|, t\right) + \phi\left(\int_0^t \|v_1(s) - v_2(s)\| ds\right) \quad \forall\, t \ge 0.$$

(ii) *We say that* (3.47) *is* semi-globally incrementally input-to-state stable (ISS) *if for any $R > 0$, there exist $\psi \in \mathcal{KL}^{\mathrm{c}}$ and $\phi \in \mathcal{K}$ such that, for all $(v_i, x_i) \in \mathcal{B}^{\mathrm{c}}$ with $\|x_i(0)\| + \|v_i\|_{L^\infty} \le R$, $i = 1, 2$,*

$$\|x_1(t) - x_2(t)\| \le \psi\left(\|x_1(0) - x_2(0)\|, t\right) + \phi\left(\operatorname*{ess\,sup}_{0 \le s \le t} \|v_1(s) - v_2(s)\| ds\right) \quad \forall t \ge 0.$$

Remark 3.2.3. Integral ISS was first introduced in [111], and is a weaker notion than ISS, with the latter implying the former. For more details we refer the reader to literature such as [6, 45, 60, 113]. ◊

The following is the aforementioned incremental stability result concerning (3.47).

Proposition 3.2.4. *Let $(A, B, C) \in \mathbb{L}^{\mathrm{c}}$, $f : \mathbb{R}^p \to \mathbb{R}^m$ be locally Lipschitz, $K \in \mathbb{R}^{m \times p}$ and $r > 0$. Assume that (A, B, C) is stabilisable and detectable, $\mathbb{B}_{\mathbb{C}}(K, r) \subseteq \mathbb{S}^{\mathrm{c}}_{\mathbb{C}}(\mathbf{G})$,* (3.13) *holds with $Z = \operatorname{im} C$, and there exists $\eta \in \mathbb{R}^p$ such that* (3.14) *holds. Then, for every $R > 0$, there exist $\psi, \tilde{\psi} \in \mathcal{KL}^{\mathrm{c}}$ and $\phi, \tilde{\phi} \in \mathcal{K}$ such that, for all $(v_1, x_1) \in \mathcal{B}^{\mathrm{c}}$ satisfying $\|x_1(0)\| + \|v_1\|_{L^\infty} \le R$, all $(v_2, x_2) \in \mathcal{B}^{\mathrm{c}}$, and all $t \ge t_0 \ge 0$,*

$$\|x_1(t) - x_2(t)\| \le \tilde{\psi}(\|x_1(t_0) - x_2(t_0)\|, t - t_0) + \tilde{\phi}\left(\operatorname*{ess\,sup}_{t_0 \le s \le t} \|v_1(s) - v_2(s)\|\right), \tag{3.62}$$

and

$$\|x_1(t) - x_2(t)\| \le \psi(\|x_1(t_0) - x_2(t_0)\|, t - t_0) + \phi\left(\int_{t_0}^t \|v_1(s) - v_2(s)\| ds\right). \tag{3.63}$$

Before proving Proposition 3.2.4, we first state the following immediate corollary.

Corollary 3.2.5. *Let $(A, B, C) \in \mathbb{L}^c$, $f : \mathbb{R}^p \to \mathbb{R}^m$ be locally Lipschitz, $K \in \mathbb{R}^{m\times p}$ and $r > 0$. Assume that (A, B, C) is stabilisable and detectable, $\mathbb{B}_{\mathbb{C}}(K, r) \subseteq \mathbb{S}^c_{\mathbb{C}}(\mathbf{G})$, (3.13) holds with $Z = \operatorname{im} C$ and that there exists $\eta \in \mathbb{R}^p$ such that (3.14) holds. Then (3.47) is semi-globally incrementally iISS and ISS.*

Proof of Proposition 3.2.4. We begin by defining $\tilde{f} : \mathbb{R}^p \to \mathbb{R}^m$ by $\tilde{f}(\xi) := f(\xi) - f(0)$ for all $\xi \in \mathbb{R}^p$. In a similar fashion to that done in the proof of Theorem 3.1.5, one may see that, as a consquence of (3.13) and (3.14), there exists $\alpha_0 \in \mathcal{K}_\infty$ such that

$$\|\tilde{f}(\xi) - K\xi\| \le r\|\xi\| - \alpha_0(\|\xi\|) \quad \forall\, \xi \in \mathbb{R}^p. \tag{3.64}$$

Subsequently, [107, Theorem 3.2] guarantees that the Lur'e system with linear component (A, B, C) and nonlinearity $\tilde{f}$ is ISS (in the continuous-time sense - see [107, Equation (3.2)]). Noting that if $(v, x) \in \mathcal{B}^c_f$, then $(v + Bf(0), x) \in \mathcal{B}^c_{\tilde{f}}$, it follows that for given $R > 0$, there exists $\rho > 0$, such that $\|Cx\|_{L^\infty} \le \rho$ for all $(v, x) \in \mathcal{B}^c$ satisfying $\|v\|_{L^\infty} + \|x(0)\| \le R$. We now fix $R > 0$ and let ρ be such a previously described positive constant. Define

$$W := \{\xi \in \mathbb{R}^p : \|\xi\| \le \rho\} \cap \operatorname{im} C \subseteq \mathbb{R}^p.$$

Since W is compact, we may deduce that (3.20) and (3.21) hold by proceeding identically to that done in the proof of Theorem 3.1.5 (with $z^* = 0$). Moreover, by Lemma 3.1.1, the function $\xi \mapsto \sup_{\zeta\in W} \|f(\xi + \zeta) - f(\zeta) - K\xi\|$ is continuous which, in conjunction with (3.20) and (3.21) and an application of statement (ii) of Lemma 2.3.9, implies the existence of $\alpha \in \mathcal{K}_\infty$ such that (3.22) holds. Now, with this $\mathcal{K}_\infty$ function α, we apply Lemma 3.2.1 in order to deduce the existence of $\psi, \tilde{\psi} \in \mathcal{KL}^c$ and $\phi, \tilde{\phi} \in \mathcal{K}$ such that, for all functions $g \in \mathcal{C}$ satisfying (3.52) and for every $(v, x) \in L^\infty_{\text{loc}}(\mathbb{R}_+, \mathbb{R}^n) \times W^{1,1}_{\text{loc}}(\mathbb{R}_+, \mathbb{R}^n)$ satisfying (3.51) almost everywhere on $\mathbb{R}_+$, it follows that (3.53) and (3.54) both hold.

We now let $(v_1, x_1), (v_2, x_2) \in \mathcal{B}^c$ be such that $\|v_1\|_{L^\infty} + \|x_1(0)\| \le R$, and let $t_0 \ge 0$. Note that from the first part of this proof, $\|Cx_1\|_{L^\infty} \le \rho$. Define $g : \mathbb{R}_+ \times \mathbb{R}^p \to \mathbb{R}^m$ by

$$g(t, \xi) := f\left(\xi + C(\Lambda_{t_0}x_1)(t)\right) - f\left(C(\Lambda_{t_0}x_1)(t)\right) \quad \forall\, (t, \xi) \in \mathbb{R}_+ \times \mathbb{R}^p.$$

By recalling (3.22) and the fact that $\|C(\Lambda_{t_0}x_1)\|_{L^\infty} \le \rho$, we see that (3.52) holds. Furthermore, it is easy to check that $g \in \mathcal{C}$ and, by using (3.49), that $(\Lambda_{t_0}v_2 - \Lambda_{t_0}v_1, \Lambda_{t_0}x_2 - \Lambda_{t_0}x_1)$ satisfies (3.51) almost everywhere on $\mathbb{R}_+$. Hence, from Lemma 3.2.1, we obtain that, for all $t \in \mathbb{R}_+$,

$$\|x_1(t + t_0) - x_2(t + t_0)\| \le \tilde{\psi}(\|x_1(t_0) - x_2(t_0)\|, t) + \tilde{\phi}\left(\operatorname*{ess\,sup}_{t_0\le s\le t+t_0} \|v_1(s) - v_2(s)\|\right),$$

and

$$\|x_1(t + t_0) - x_2(t + t_0)\| \le \psi(\|x_1(t_0) - x_2(t_0)\|, t) + \phi\left(\int_{t_0}^{t+t_0} \|v_1(s) - v_2(s)\| ds\right).$$

These two equations imply that (3.62) and (3.63) both hold for all $t \ge t_0$, thus completing the proof. $\square$

Remark 3.2.6. We showed in the previous proof that, when under the assumptions of Proposition 3.2.4, for given $x^0 \in \mathbb{R}^n$ and $v \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^n)$, the unique solution of (3.47) is defined everywhere. Indeed, as a consquence of (3.64) we saw that the Lur'e system with linear component (A, B, C) and nonlinearity $\tilde{f}$, where $\tilde{f}(\xi) := f(\xi) - f(0)$ for all $\xi \in \mathbb{R}^p$, is ISS, and so solutions to that system are defined globally. Hence, if $(v, x) \in \mathcal{B}^{\mathrm{c}}_f$, then $(v + Bf(0), x) \in \mathcal{B}^{\mathrm{c}}_{\tilde{f}}$, and so x is defined globally. $\diamondsuit$

3.2.3 The response to Stepanov almost periodic inputs

Our attention now turns towards the main contribution of this section, namely an analogue of Corollary 3.1.13, but which concerns Stepanov almost periodic functions. As in the previous section, we shall be extremely brief in our setup, since a very thorough presentation of (Bohr) almost periodic and Stepanov almost periodic functions is given in Appendix C. However, for convenience to the reader, we recall the following for a Banach space X and time domain $R = \mathbb{R}$ or $\mathbb{R}_+$. First, $AP(R, X)$ denotes the space of almost periodic functions mapping $R \to X$, where $R = \mathbb{R}$ or $\mathbb{R}_+$ and, for $\varepsilon > 0$, the set of ε-periods of an almost periodic function v^{ap} is denoted by $P(v^{\mathrm{ap}}, \varepsilon)$ (see Definition C.1.1). The set of almost periodic functions is a closed subspace of the space of bounded uniformly continuous functions endowed with the sup-norm. Moreover, the space of asymptotically almost periodic functions $\mathbb{R}_+ \to X$ is denoted by $AAP(\mathbb{R}_+, X)$ and $C_0(\mathbb{R}_+, X)$ denotes the space of all continuous functions $\mathbb{R}_+ \to X$ that converge to zero as their argument converges to infinity. As in the discrete-time setting, we also recall that every $v^{\mathrm{ap}} \in AP(\mathbb{R}_+, X)$ has a unique extension $v^{\mathrm{ap}}_{\mathrm{e}} \in AP(\mathbb{R}, X)$ such that

$$v^{\mathrm{ap}}_{\mathrm{e}}(t) = v^{\mathrm{ap}}(t) \quad \forall\, t \in \mathbb{R}_+ \quad \text{and} \quad \sup_{t\in\mathbb{R}} \|v^{\mathrm{ap}}_{\mathrm{e}}(t)\|_X = \sup_{t\in\mathbb{R}_+} \|v^{\mathrm{ap}}(t)\|_X.$$

For the details of this, we refer the reader to Appendix C, and in particular, Lemma C.1.6. Briefly, we mention that if we follow an idea seen in [12, Remark on p.318] and define $v^{\mathrm{ap}}_{\mathrm{e}}$ by

$$v^{\mathrm{ap}}_{\mathrm{e}}(t) = \lim_{k\to\infty} v^{\mathrm{ap}}(t + \tau_k) \quad \forall\, t \in \mathbb{R}, \tag{3.65}$$

where, for each $k \in \mathbb{N}$, $\tau_k \in P(v^{\mathrm{ap}}, 1/k)$ and $\tau_k \nearrow \infty$ as $k \to \infty$, then it is not difficult to prove that this function is a well-defined almost periodic extension of v^{ap} and satisfies $\sup_{t\in\mathbb{R}} \|v^{\mathrm{ap}}_{\mathrm{e}}(t)\|_X = \sup_{t\in\mathbb{R}_+} \|v^{\mathrm{ap}}(t)\|_X$. With this in mind, we obtain the following theorem, which is Theorem C.1.11 in Appendix C.

Theorem 3.2.7. *The map $AP(\mathbb{R}_+, X) \to AP(\mathbb{R}, X)$ which maps v^{ap} to $v^{\mathrm{ap}}_{\mathrm{e}}$, where $v^{\mathrm{ap}}_{\mathrm{e}}$ is given by* (3.65)*, is an isometric isomorphism.*

We also mention here that for a function $v^{\mathrm{ap}} \in AP(\mathbb{R}, X)$ and $\lambda \in \mathbb{R}$, it follows that

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} e^{-i\lambda t} v^{\mathrm{ap}}(t)dt,$$

exists (see Corollary C.1.23). Moreover, it is well-known that the set of $\lambda \in \mathbb{R}$ such that the above limit is nonzero, denoted by $\Lambda(v^{\mathrm{ap}})$, is countable (see for example, [1, III, p.22]). The elements of $\Lambda(v^{\mathrm{ap}})$ are called the characteristic exponents of v^{ap} [1]. Furthermore, the module $\mathrm{mod}(v^{\mathrm{ap}})$ is the smallest additive group of real numbers that contains $\Lambda(v^{\mathrm{ap}})$. That is, $\mathrm{mod}(v^{\mathrm{ap}})$ is the set of all real numbers which are a finite linear combination of the elements of $\Lambda(v^{\mathrm{ap}})$ with integer coefficients (see Definition C.1.26).

Let us also briefly recall notation and concepts concerning Stepanov almost periodicity. Let $v \in L^1_{\mathrm{loc}}(R, \mathbb{R}^n)$, where $R = \mathbb{R}$ or $\mathbb{R}_+$, and $\varepsilon > 0$. As in Definition C.2.1, we say that $\tau \in R$ is an ε-period of v (in the sense of Stepanov) if

$$\sup_{a \in R} \int_a^{a+1} \|v(s+\tau) - v(s)\| ds \leq \varepsilon.$$

The set of ε-periods of v (in the sense of Stepanov) is denoted by $P_1(v, \varepsilon)$. We say that v is almost periodic in the sense of Stepanov if, for every $\varepsilon > 0$, the set $P_1(v, \varepsilon)$ is relatively dense in R. The set of all functions in $L^1_{\mathrm{loc}}(R, \mathbb{R}^n)$ that are almost periodic in the sense of Stepanov is denoted by $S^1(R, \mathbb{R}^n)$. We further recall from Lemma C.2.8, that $S^1(R, \mathbb{R}^n)$ is a closed subspace of the Banach space of uniformly locally integrable functions, which is defined by

$$UL^1_{\mathrm{loc}}(R, \mathbb{R}^n) := \left\{ v \in L^1_{\mathrm{loc}}(R, \mathbb{R}^n) : \sup_{a \in R} \int_a^{a+1} \|v(s)\| ds < \infty \right\},$$

and is endowed with the Stepanov norm

$$\|v\|_{\mathrm{S}} := \sup_{a \in R} \int_a^{a+1} \|v(s)\| ds.$$

For more details on $UL^1_{\mathrm{loc}}(R, \mathbb{R}^n)$ we refer the reader to Appendix C, and in particular to Remark C.2.2 and Lemma C.2.3. The final thing we shall mention here is that, for a function $v \in L^1_{\mathrm{loc}}(R, \mathbb{R}^n)$ the Bochner transform of v is denoted by $\tilde{v} : R \to L^1([0,1], \mathbb{R}^n)$, that is

$$\big(\tilde{v}(t)\big)(s) := v(t+s) \quad \forall\, t \in R,\ \forall\, s \in [0,1].$$

We note that $\tilde{v}$ is continuous and, furthermore, $v \in S^1(R, \mathbb{R}^n)$ if, and only if, $\tilde{v} \in AP(R, L^1([0,1], \mathbb{R}^n))$ (see Definition C.2.1, Lemma C.2.5 and [1, pp.77-78]). As with (Bohr) almost periodic functions, we may obtain a relationship between $S^1(\mathbb{R}_+, \mathbb{R}^n)$ and $S^1(\mathbb{R}, \mathbb{R}^n)$. Indeed, one can, for each $v \in S^1(\mathbb{R}_+, \mathbb{R}^n)$, construct an extension which is Stepanov almost periodic on R. We denote this extension by v^{e}. Moreover, we obtain the following, which is a particular case of Theorem C.2.11,

Theorem 3.2.8. *The map $S^1(\mathbb{R}_+, \mathbb{R}^n) \to S^1(\mathbb{R}, \mathbb{R}^n)$ which maps v to v^{e}, is an isometric isomorphism.*

For $v \in S^1(R, \mathbb{R}^n)$, one may also define $\Lambda(v)$ and $\mathrm{mod}(v)$. For details of this, and for relevant results, we once again refer the reader to Appendix C (see Theorem C.2.16).

We are now in the position to state and prove the main result of this section: the aforementioned analogue of Corollary 3.1.13.

Theorem 3.2.9. *Let $(A, B, C) \in \mathbb{L}^{\mathrm{c}}$, $f : \mathbb{R}^p \to \mathbb{R}^m$ be locally Lipschitz, $K \in \mathbb{R}^{m \times p}$, $r > 0$ and $w \in S^1(\mathbb{R}_+, \mathbb{R}^n) \cap L^\infty(\mathbb{R}_+, \mathbb{R}^n)$. Assume that (A, B, C) is stabilisable and detectable, $\mathbb{B}_{\mathbb{C}}(K, r) \subseteq \mathbb{S}^{\mathrm{c}}_{\mathbb{C}}(\mathbf{G})$,* (3.13) *holds with $Z = \mathbb{R}^p$ and there exists $\eta \in \mathbb{R}^p$ such that* (3.14) *holds. Then the following statements hold.*

(i) *There exists a unique* $x^{\rm ap} \in AP(\mathbb{R}_+, \mathbb{R}^n)$ *such that* $(w, x^{\rm ap}) \in \mathcal{B}^{\rm c}$ *and, for all* $(v, x) \in \mathcal{B}^{\rm c}$ *satisfying that either:* (a) $v - w \in L^1(\mathbb{R}_+, \mathbb{R}^n)$*; or* (b) $v \in L^\infty(\mathbb{R}_+, \mathbb{R}^n)$ *and* $\operatorname{ess\,sup}_{s \geq t} \|v(s) - w(s)\| \to 0$ *as* $t \to \infty$*; we have that*

$$\lim_{t\to\infty} \|x(t) - x^{\rm ap}(t)\| = 0. \tag{3.66}$$

Furthermore, for all $\varepsilon > 0$*, there exists* $\delta > 0$ *such that* $P_1(w, \delta) \subset P(x^{\rm ap}, \varepsilon)$*. In particular, if* w *is periodic with period* τ*, then* $x^{\rm ap}$ *is* τ*-periodic.*

(ii) *The function* $x^{\rm ap}_{\rm e}$ *is the unique bounded function such that* $(w^{\rm e}, x^{\rm ap}_{\rm e}) \in \tilde{\mathcal{B}}^{\rm c}$*. Moreover,* $\operatorname{mod}(x^{\rm ap}_{\rm e}) \subset \operatorname{mod}(w^{\rm e})$.

Proof. We begin with applications of [15, Propositions 3.1 and 4.1] to yield that for every $v^* \in \mathbb{R}^n$, there exists $x^* \in \mathbb{R}^n$ such that $(v^*, x^*) \in \mathcal{B}^{\rm c}$, where we abuse notation and consider a vector to be a constant function. We fix such a pair (v^*, x^*). Let $(w, x) \in \mathcal{B}^{\rm c}$ and set $R_1 := \|x(0)\| + \|w\|_{L^\infty} + \|x^*\| + \|v^*\|$. By applying Proposition 3.2.4 and noting that $t \mapsto x^*$ is a bounded function, we are able to deduce that x is bounded. Let $R > 0$ be such that $\|x\|_{L^\infty} + \|w\|_{L^\infty} \leq R$. We now apply Proposition 3.2.4 a further time to obtain the existence of $\psi, \tilde{\psi} \in \mathcal{KL}^{\rm c}$ and $\phi, \tilde{\phi} \in \mathcal{K}$ (dependent on R) such that (3.62) and (3.63) both hold for all $(v_1, x_1) \in \mathcal{B}^{\rm c}$ satisfying $\|x_1(0)\| + \|v_1\|_{L^\infty} \leq R$, all $(v_2, x_2) \in \mathcal{B}^{\rm c}$, and all $t \geq t_0 \geq 0$.
Since the rest of the proof consists of many different parts, we label each step.
Step 1: Construction of $x^{\rm ap}$. Since $w \in S^1(\mathbb{R}_+, \mathbb{R}^n)$, we choose a non-decreasing sequence $(\tau_k)_{k\in\mathbb{N}} \subseteq \mathbb{R}_+$ such that

$$\tau_k \in P_1(w, 1/k^2) \quad \text{and} \quad \tau_k > k \qquad \forall\, k \in \mathbb{N}.$$

We are going to show that $(\Lambda_{\tau_k} x)_{k\in\mathbb{N}}$ is a Cauchy sequence in the Banach space of bounded continuous functions mapping $\mathbb{R}_+ \to \mathbb{R}^n$. To this end, we first note that

$$\begin{aligned} \int_a^{a+k} \|w(t+\tau_k) - w(t)\| dt &= \sum_{j=1}^{k} \int_{a+j-1}^{a+j} \|w(t+\tau_k) - w(t)\| dt \\ &\leq \frac{1}{k} \quad \forall\, a \geq 0,\ \forall\, k \in \mathbb{N}. \end{aligned} \tag{3.67}$$

Moreover, since $(\Lambda_\tau w, \Lambda_\tau x) \in \mathcal{B}^{\rm c}$ for all $\tau \geq 0$ by (3.49), and that $\|\Lambda_\tau x\|_{L^\infty} + \|\Lambda_\tau w\|_{L^\infty} \leq R$ for every $\tau \geq 0$, it follows from (3.63) that

$$\begin{aligned} \|(\Lambda_\sigma x)(s) - (\Lambda_{\sigma+\tau} x)(s)\| \leq\ & \psi(\|x(\sigma + s_0) - x(\sigma + \tau + s_0)\|, s - s_0) \\ & + \phi\left(\int_{s_0}^{s} \|(\Lambda_\sigma w)(\eta) - (\Lambda_{\sigma+\tau} w)(\eta)\| d\eta\right) \\ & \forall\, s \geq s_0 \geq 0,\ \forall\, \sigma, \tau \geq 0. \end{aligned} \tag{3.68}$$

Trivially, for $k, \ell \in \mathbb{N}$ with $k \geq \ell$,

$$(\Lambda_{\tau_\ell} x)(t) - (\Lambda_{\tau_k} x)(t) = (\Lambda_t x)(\tau_\ell) - (\Lambda_{t+\tau_k-\tau_\ell} x)(\tau_\ell) \quad \forall\, t \geq 0,$$

and so, setting

$$I(t; k, \ell) := \int_{\tau_\ell - \ell}^{\tau_\ell} \|(\Lambda_t w)(\eta) - (\Lambda_{t+\tau_k-\tau_\ell} w)(\eta)\| d\eta \quad \forall\, t \geq 0,$$

and invoking (3.68) (with $s = \tau_\ell$, $s_0 = \tau_\ell - \ell$, $\sigma = t$ and $\tau = \tau_k - \tau_\ell$), we arrive at

$$\begin{aligned}\|(\Lambda_{\tau_\ell}x)(t) - (\Lambda_{\tau_k}x)(t)\| \leq &\psi(\|x(t+\tau_\ell-\ell) - x(t+\tau_k-\ell)\|, \ell) + \phi(I(t;k,\ell)) \\ &\forall\, t \geq 0,\ \forall\, k,\ell \in \mathbb{N} \text{ s.t. } k \geq \ell. \end{aligned} \tag{3.69}$$

Now, for all $t \geq 0$ and all $k, \ell \in \mathbb{N}$ such that $k \geq \ell$,

$$I(t;k,\ell) \leq \int_{\tau_\ell-\ell}^{\tau_\ell} \|(\Lambda_t w)(\eta) - (\Lambda_{t+\tau_k}w)(\eta)\| d\eta + \int_{\tau_\ell-\ell}^{\tau_\ell} \|(\Lambda_{t+\tau_k}w)(\eta) - (\Lambda_{t+\tau_k-\tau_\ell}w)(\eta)\| d\eta,$$

which, after changing variables, implies that

$$I(t;k,\ell) \leq \int_{t+\tau_\ell-\ell}^{t+\tau_\ell-\ell+k} \|(w(\eta) - (\Lambda_{\tau_k}w)(\eta)\| d\eta + \int_{t+\tau_k-\ell}^{t+\tau_k} \|(\Lambda_{\tau_\ell}w)(\eta) - w(\eta)\| d\eta.$$

Consequently, by (3.67),

$$I(t;k,\ell) \leq \frac{1}{k} + \frac{1}{\ell} \quad \forall\, t \geq 0,\ \forall\, k,\ell \in \mathbb{N} \text{ s.t. } k \geq \ell.$$

By combining this with (3.69), we see that

$$\|(\Lambda_{\tau_\ell}x)(t) - (\Lambda_{\tau_k}x)(t)\| \leq \psi(2R, \ell) + \phi\left(\frac{1}{k} + \frac{1}{\ell}\right) \quad \forall\, t \geq 0,\ \forall\, k,\ell \in \mathbb{N} \text{ s.t. } k \geq \ell.$$

This shows that $(\Lambda_{\tau_k}x)_{k\in\mathbb{N}}$ is a Cauchy sequence in the space of bounded continuous functions mapping $\mathbb{R}_+ \to \mathbb{R}^n$, the limit of which we denote by x^{ap}. We note that $\|x^{\mathrm{ap}}\|_{L^\infty} + \|w\|_{L^\infty} \leq R$.

Step 2: Almost periodicity of x^{ap}. To show that $x^{\mathrm{ap}} \in AP(\mathbb{R}_+, \mathbb{R}^n)$, let $\varepsilon > 0$ and choose $T \in \mathbb{N}$ and $a > 0$ such that

$$\psi(2R, T) \leq \frac{\varepsilon}{2} \quad \text{and} \quad \phi(a) \leq \frac{\varepsilon}{2}. \tag{3.70}$$

Furthermore, let $\tau \in P_1(w, a/T)$ and note that

$$(\Lambda_{\tau_\ell}x)(t+\tau) - (\Lambda_{\tau_\ell}x)(t) = (\Lambda_{t+\tau}x)(\tau_\ell) - (\Lambda_t x)(\tau_\ell) \quad \forall\, \ell \in \mathbb{N},\ \forall\, t \geq 0.$$

Thus, by invoking (3.68) (with $s = \tau_\ell$, $s_0 = \tau_\ell - T$, and $\sigma = t$),

$$\begin{aligned}\|(\Lambda_{\tau_\ell}x)(t+\tau) - (\Lambda_{\tau_\ell}x)(t)\| \leq &\psi(\|x(t+\tau_\ell-T) - x(t+\tau+\tau_\ell-T)\|, T) \\ &+ \phi\left(\int_{\tau_\ell-T}^{\tau_\ell} \|(\Lambda_t w)(\eta) - (\Lambda_{t+\tau}w)(\eta)\| d\eta\right) \\ &\forall\, t \geq 0,\ \forall\, \ell \in \mathbb{N} \text{ s.t. } \tau_\ell \geq T.\end{aligned}$$

Now,

$$\begin{aligned}\int_{\tau_\ell-T}^{\tau_\ell} \|(\Lambda_t w)(\eta) - (\Lambda_{t+\tau}w)(\eta)\| d\eta &= \int_{t+\tau_\ell-T}^{t+\tau_\ell} \|w(\eta) - (\Lambda_\tau w)(\eta)\| d\eta \\ &= \sum_{j=1}^{T} \int_{t+\tau_\ell-T+j-1}^{t+\tau_\ell-T+j} \|w(\eta) - w(\eta+\tau)\| d\eta \\ &\leq a \quad \forall\, t \geq 0,\ \forall\, \ell \in \mathbb{N} \text{ s.t. } \tau_\ell \geq T,\end{aligned}$$

where the last inequality follows from the choice of τ. Therefore, by combining the previous two inequalities with (3.70), we see that

$$\|(\Lambda_{\tau_\ell}x)(t+\tau) - (\Lambda_{\tau_\ell}x)(t)\| \le \psi(2R,T) + \phi(a) \le \varepsilon \quad \forall\, t \ge 0,\ \forall\, \ell \in \mathbb{N} \text{ s.t. } \tau_\ell \ge T.$$

By then letting $\ell \to \infty$, we see that

$$\|x^{\mathrm{ap}}(t) - x^{\mathrm{ap}}(t+\tau)\| \le \varepsilon \quad \forall\, t \ge 0,$$

and so $\tau \in P(x^{\mathrm{ap}}, \varepsilon)$. Since $\tau \in P_1(w, a/T)$ was arbitrary, we conclude that

$$P_1(w, a/T) \subset P(x^{\mathrm{ap}}, \varepsilon), \tag{3.71}$$

showing that $P(x^{\mathrm{ap}}, \varepsilon)$ is relatively dense in $\mathbb{R}_+$. Consequently, $x^{\mathrm{ap}} \in AP(\mathbb{R}_+, \mathbb{R}^n)$.
Step 3: Trajectory property of (w, x^{ap}). To show that $(w, x^{\mathrm{ap}}) \in \mathcal{B}^{\mathrm{c}}$, let $T \in \mathbb{N}$ be arbitrary and note that

$$\int_0^T \|(\Lambda_{\tau_\ell}w)(s) - w(s)\| ds \le \frac{T}{\ell^2} \quad \forall\, \ell \in \mathbb{N}.$$

Hence, $\Lambda_{\tau_\ell}w \to w$ in $L^1([0,T], \mathbb{R}^n)$ as $\ell \to \infty$. Furthermore, invoking (3.13) with $Z = \mathbb{R}^p$ and the fact that $\big(\Lambda_{\tau_\ell}x\big)_{\ell\in\mathbb{N}}$ converges uniformly to x^{ap} on $\mathbb{R}_+$, we see that the sequence $\big(f(C\Lambda_{\tau_\ell}x)\big)_{\ell\in\mathbb{N}}$ converges uniformly to $f(Cx^{\mathrm{ap}})$ on $\mathbb{R}_+$, and consequently, $f(C\Lambda_{\tau_\ell}x) \to f(Cx^{\mathrm{ap}})$ in $L^1([0,T], \mathbb{R}^n)$ as $\ell \to \infty$. Therefore, since, for all $\ell \in \mathbb{N}$ and all $t \in [0,T]$,

$$(\Lambda_{\tau_\ell}x)(t) = (\Lambda_{\tau_\ell}x)(0) + A\int_0^t (\Lambda_{\tau_\ell}x)(s)ds + B\int_0^t f(C(\Lambda_{\tau_\ell}x)(s))ds + \int_0^t (\Lambda_{\tau_\ell}w)(s)ds$$

it follows that in the limit, as $\ell \to \infty$,

$$x^{\mathrm{ap}}(t) = x^{\mathrm{ap}}(0) + A\int_0^t x^{\mathrm{ap}}(s)ds + B\int_0^t f(Cx^{\mathrm{ap}}(s))ds + \int_0^t w(s)ds \quad \forall\, t \in [0,T].$$

As this holds for every $T \in \mathbb{N}$, we have that $x^{\mathrm{ap}} \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^n)$ and

$$\dot{x}^{\mathrm{ap}}(t) = Ax^{\mathrm{ap}}(t) + Bf(Cx^{\mathrm{ap}}(t)) + w(t) \quad \text{for a.e. } t \ge 0,$$

showing that $(w, x^{\mathrm{ap}}) \in \mathcal{B}^{\mathrm{c}}$.
Step 4: Uniqueness of x^{ap} *within* $AP(\mathbb{R}_+, \mathbb{R}^n)$. Assume that $\tilde{x} \in AP(\mathbb{R}_+, \mathbb{R}^n)$ is such that $(w, \tilde{x}) \in \mathcal{B}^{\mathrm{c}}$. From an application of (3.63), we see that $\|\tilde{x}(t) - x^{\mathrm{ap}}(t)\| \to 0$ as $t \to \infty$. By combining this with Lemma C.1.15 and the fact that $\tilde{x} - x^{\mathrm{ap}}$ is almost periodic (see Corollary C.1.10), we conclude that $\tilde{x} = x^{\mathrm{ap}}$.
Step 5: Proof of (3.66). Let $(v, \tilde{x}) \in \mathcal{B}^{\mathrm{c}}$ and initially let us assume that $v - w \in L^1(\mathbb{R}_+, \mathbb{R}^n)$. Let $\varepsilon > 0$ and choose $t_0, t_1 \ge 0$ such that

$$\phi\left(\int_{t_0}^\infty \|w(s) - v(s)\| ds\right) \le \frac{\varepsilon}{2} \quad \text{and} \quad \psi(\|x^{\mathrm{ap}}(t_0) - \tilde{x}(t_0)\|, t_1) \le \frac{\varepsilon}{2},$$

where the existence of a suitable t_0 follows from the assumption that $v - w \in L^1(\mathbb{R}_+, \mathbb{R}^n)$. Then, by combining this with (3.63) and the fact that $\|x^{\mathrm{ap}}\|_{L^\infty} + \|w\|_{L^\infty} \le R$, we see that $\|x^{\mathrm{ap}}(t) - \tilde{x}(t)\| \le \varepsilon$ for all $t \ge t_0 + t_1$, showing that $\|x^{\mathrm{ap}}(t) - \tilde{x}(t)\| \to 0$ as $t \to \infty$.

If we now instead assume that $v \in L^\infty(\mathbb{R}_+, \mathbb{R}^n)$ and $\operatorname{ess\,sup}_{s \geq t} \|v(s) - w(s)\| \to 0$ as $t \to \infty$, then a similar argument, except by utilising (3.62) instead of (3.63), leads to the conclusion that $\|x^{\mathrm{ap}}(t) - \tilde{x}(t)\| \to 0$ as $t \to \infty$.
Step 6: Proof of statement (ii). For convenience, we set

$$g(\xi) := A\xi + Bf(C\xi) \quad \forall\, \xi \in \mathbb{R}^n.$$

We know from Step 2 (see (3.71)) that, for each $k \in \mathbb{N}$, there exists $\delta_k \in (0, 1/k)$ such that

$$P_1(w, \delta_k) \subset P(x^{\mathrm{ap}}, 1/k) \quad \forall\, k \in \mathbb{N}. \tag{3.72}$$

Moreover, it follows from the fact that (3.13) holds with $Z = \mathbb{R}^p$, that there exists $\kappa > 0$ such that

$$P(x^{\mathrm{ap}}, 1/k) \subset P(g \circ x^{\mathrm{ap}}, \kappa/k) \quad \forall\, k \in \mathbb{N}.$$

Hence, by combining this with (3.72),

$$P_1(w, \delta_k) \subset P(g \circ x^{\mathrm{ap}}, \kappa/k) \subset P_1(g \circ x^{\mathrm{ap}}, \kappa/k) \quad \forall\, k \in \mathbb{N}. \tag{3.73}$$

Let $a < 0$ be fixed and let $\tau_k \in P_1(w, \delta_k) \cap [-a, \infty)$. We note that, for all $t \in [a, 0]$,

$$x_{\mathrm{e}}^{\mathrm{ap}}(t + \tau_k) - x_{\mathrm{e}}^{\mathrm{ap}}(a + \tau_k) = x^{\mathrm{ap}}(t + \tau_k) - x^{\mathrm{ap}}(a + \tau_k) = \int_{a+\tau_k}^{t+\tau_k} g(x^{\mathrm{ap}}(s)) + w(s)ds.$$

From this, we deduce that

$$\begin{aligned} x_{\mathrm{e}}^{\mathrm{ap}}(t + \tau_k) - x_{\mathrm{e}}^{\mathrm{ap}}(a + \tau_k) &= \int_a^t \big(g(x^{\mathrm{ap}}(s + \tau_k)) + w(s + \tau_k)\big)ds \\ &= \int_a^t \big(g(x_{\mathrm{e}}^{\mathrm{ap}}(s + \tau_k)) + w^{\mathrm{e}}(s + \tau_k)\big)ds \;\; \forall\, t \in [a, 0]. \end{aligned} \tag{3.74}$$

Since (see Lemma C.1.6 and Lemma C.2.9), for all $k \in \mathbb{N}$,

$$P_1(w, \delta_k) \subset P_1(w^{\mathrm{e}}, \delta_k),\; P(x^{\mathrm{ap}}, 1/k) \subset P(x_{\mathrm{e}}^{\mathrm{ap}}, 1/k),\; P_1(g \circ x^{\mathrm{ap}}, \kappa/k) \subset P_1(g \circ x_{\mathrm{e}}^{\mathrm{ap}}, \kappa/k),$$

by (3.72) and (3.73), we obtain that

$$\tau_k \in P_1(w^{\mathrm{e}}, \delta_k) \cap P(x_{\mathrm{e}}^{\mathrm{ap}}, 1/k) \cap P_1(g \circ x_{\mathrm{e}}^{\mathrm{ap}}, \kappa/k) \quad \forall\, k \in \mathbb{N},$$

Therefore, letting $k \to \infty$ in (3.74) yields that

$$x_{\mathrm{e}}^{\mathrm{ap}}(t) - x_{\mathrm{e}}^{\mathrm{ap}}(a) = \int_a^t \big(g(x_{\mathrm{e}}^{\mathrm{ap}}(s)) + w^{\mathrm{e}}(s)\big)ds \quad \forall\, t \in [a, 0].$$

Since $a < 0$ was arbitrary, we conclude that

$$\dot{x}_{\mathrm{e}}^{\mathrm{ap}}(t) = g(x_{\mathrm{e}}^{\mathrm{ap}}(t)) + w^{\mathrm{e}}(t) = Ax_{\mathrm{e}}^{\mathrm{ap}}(t) + Bf(Cx_{\mathrm{e}}^{\mathrm{ap}}(t)) + w^{\mathrm{e}}(t) \quad \text{for a.e. } t \leq 0,$$

establishing that $(w^{\mathrm{e}}, x_{\mathrm{e}}^{\mathrm{ap}}) \in \tilde{\mathcal{B}}^{\mathrm{c}}$.
We now show that $x_{\mathrm{e}}^{\mathrm{ap}}$ is the unique bounded function defined on $\mathbb{R}$ such that $(w^{\mathrm{e}}, x_{\mathrm{e}}^{\mathrm{ap}}) \in \tilde{\mathcal{B}}^{\mathrm{c}}$. To do so, we first show that

$$\|w^{\mathrm{e}}\|_{L^\infty(\mathbb{R}, \mathbb{R}^n)} = \|w\|_{L^\infty(\mathbb{R}_+, \mathbb{R}^n)}. \tag{3.75}$$

To that end, let $a \leq -1$ be arbitrary and let $N \in \mathbb{N}$ be such that $\tau_k \geq -a$ for all $k \in \overline{N}$. We then have, for all $k \in \overline{N}$, $w^{\mathrm{e}}(\tau_k + s) = w(\tau_k + s)$ for almost every $s \in [a, a+1]$. Thus,

$$\int_a^{a+1} \|w^{\mathrm{e}}(s) - w(\tau_k + s)\| ds = \int_a^{a+1} \|w^{\mathrm{e}}(s) - w^{\mathrm{e}}(\tau_k + s)\| ds \leq \frac{1}{k} \quad \forall\, k \in \overline{K}.$$

This converges to zero as $k \to \infty$, and so we may obtain a subsequence (which, for ease of notation, has the same label) such that $w^{\mathrm{e}}(s) - w(\tau_k + s)$ converges pointwise to 0 almost everywhere on $[a, a+1]$ as $k \to \infty$. Therefore, $\|w^{\mathrm{e}}(s)\| \leq \|w\|_{L^\infty}$ for almost every $s \in [a, a+1]$. Since $a \leq -1$ was arbitrary, it follows that (3.75) holds. With this established, we now let $\zeta : \mathbb{R} \to \mathbb{R}^n$ be bounded and assume that $(w^{\mathrm{e}}, \zeta) \in \tilde{\mathcal{B}}^{\mathrm{c}}$. We will show that $\zeta = x_{\mathrm{e}}^{\mathrm{ap}}$. To begin with, by (3.50), for any $\tau \in \mathbb{R}$, the restrictions of the pairs $(\Lambda_\tau w^{\mathrm{e}}, \Lambda_\tau x_{\mathrm{e}}^{\mathrm{ap}})$ and $(\Lambda_\tau w^{\mathrm{e}}, \Lambda_\tau \zeta)$ to $\mathbb{R}_+$ are in $\mathcal{B}^{\mathrm{c}}$. Moreover, since $\|(\Lambda_\tau x_{\mathrm{e}}^{\mathrm{ap}})|_{\mathbb{R}_+}\|_{L^\infty} + \|(\Lambda_\tau w^{\mathrm{e}})|_{\mathbb{R}_+}\|_{L^\infty} \leq R$ for any $\tau \in \mathbb{R}$ (see Theorem 3.2.7 and (3.75)), an application of (3.63) gives that

$$\|(\Lambda_\tau \zeta)(s) - (\Lambda_\tau x_{\mathrm{e}}^{\mathrm{ap}})(s)\| \leq \psi(\|(\Lambda_\tau \zeta)(0) - (\Lambda_\tau x_{\mathrm{e}}^{\mathrm{ap}})(0)\|, s) \quad \forall\, s \geq 0,\ \forall\, \tau \in \mathbb{R}.$$

Now let $t \in \mathbb{R}$ and $\varepsilon > 0$. Choosing $\tau \leq t$ such that $\psi(\|x_{\mathrm{e}}^{\mathrm{ap}}\|_{L^\infty} + \|\zeta\|_{L^\infty}, t - \tau) \leq \varepsilon$ and applying the above inequality (with $s = t - \tau$) leads to

$$\begin{aligned}\|\zeta(t) - x_{\mathrm{e}}^{\mathrm{ap}}(t)\| &= \|(\Lambda_\tau \zeta)(t-\tau) - (\Lambda_\tau x_{\mathrm{e}}^{\mathrm{ap}})(t-\tau)\| \\ &\leq \psi(\|\zeta(\tau) - x_{\mathrm{e}}^{\mathrm{ap}}(\tau)\|, t-\tau) \\ &\leq \psi(\|x_{\mathrm{e}}^{\mathrm{ap}}\|_{L^\infty} + \|\zeta\|_{L^\infty}, t-\tau) \\ &\leq \varepsilon.\end{aligned}$$

As $\varepsilon > 0$ was arbitrary, it follows that $\zeta(t) = x_{\mathrm{e}}^{\mathrm{ap}}(t)$. Finally, since $t \in \mathbb{R}$ was arbitrary, we obtain that $\zeta = x_{\mathrm{e}}^{\mathrm{ap}}$.
Finally, from (3.71), we have shown that for every $\varepsilon > 0$, there exists $\delta > 0$ such that $P_1(w, \delta) \subset P(x^{\mathrm{ap}}, \varepsilon)$. If we then apply Lemma C.1.6 and Lemma C.2.9, we see that for every $\varepsilon > 0$, there exists $\delta > 0$ such that $P_1(w^{\mathrm{e}}, \delta) \subset P(x_{\mathrm{e}}^{\mathrm{ap}}, \varepsilon)$. Since $x_{\mathrm{e}}^{\mathrm{ap}}$ is almost periodic, it is also Stepanov almost periodic, and so we may utilise Theorem C.2.16 to yield that $\mathrm{mod}(x_{\mathrm{e}}^{\mathrm{ap}}) \subset \mathrm{mod}(w^{\mathrm{e}})$, thus completing the proof. □

3.3 Notes and references

With regards to the main stability results of this chapter, i.e. Theorem 3.1.5 and Proposition 3.2.4, the papers with results that are closest to these are [40] (see also Chapter 5) and [47], where criteria are given that guarantee that infinite-dimensional Lur'e systems are exponentially incrementally ISS in the discrete- and continuous-time settings, respectively. Exponential incremental ISS is a stronger stability notion than that investigated in this chapter. However, we make note that the assumptions imposed on the nonlinearity in the aforementioned papers are considerably stronger than what we impose in the stability results of this chapter. Indeed, in [40] and [47], a 'strict contraction' assumption is used, that is, an assumption of the form:

$$\|f(\xi + \zeta) - f(\zeta) - K\xi\|_U \leq (r - \delta)\|\xi\|_Y \quad \forall\, \xi, \zeta \in Y, \tag{3.76}$$

where $f : Y \to U$ is a function, Y and U are Hilbert spaces, K is a stabilising feedback operator, $r > 0$ is such that every element of $\mathbb{B}(K, r)$ stabilises the system, and $\delta > 0$. For an illustration that the hypotheses of [40] and [47] are stronger, we refer the reader to Examples 3.1.8 and 3.1.9. Although they are in the discrete-time setting, it is also easy to construct examples in the continuous-time setting.

We now compare to related results in the literature, the results of this chapter concerning the convergence of trajectories of systems with Bohr and Stepanov almost periodic inputs, i.e. Corollary 3.1.13 and Theorem 3.2.9. The most relevant results in this context are [103, 105, 128], which are all in the continuous-time setting. A state-space and Lyapunov approach is used in [128], whilst the analysis in [103, 105] is based on input-output methods. An inspection of the assumptions imposed in [103, Theorem 2], [105, Theorem 1] and [128, Theorem 1] shows that they are equivalent to the existence of a $\delta > 0$ such that (3.76) holds for some stabilising feedback matrix K and $r > 0$. As already mentioned, this assumption is stronger than what we impose in the aforementioned results of this chapter. We further mention that [103, 105, 128] consider feedback systems that are considerably more restrictive than that considered presently, with, in particular, [105, 128] being in the single-input single-output setting. Moreover, none of the papers address the case of Stepanov almost periodic forcing.

The final paper that we mention here is [38], of which the author of this thesis coauthored. The article may be considered to be an infinite-dimensional version of Section 3.2. However, the assumptions imposed on the nonlinearity comprise a strict contraction of the form (3.76). As already mentioned, this is much more restrictive than the hypotheses used in Section 3.2

Chapter 4

Stability and convergence properties of Lur'e systems with potentially superlinear nonlinearity

In [7, Theorem 1], ISS is investigated for forced continuous-time Lur'e systems of the variety

$$\dot{x} = Ax + Bf(Cx) + Bv, \quad x(0) = x^0 \in \mathbb{R}^n, \quad y = Cx, \tag{4.1}$$

where A, B, C are matrices of appropriate dimensions and v is an input or forcing. There, it is shown that a condition on the linear part of the system, similar to positive realness (see, for example, [3]), together with certain unboundedness assumptions on the nonlinearity, guarantees input-to-state stability (ISS) - see Section 2.2.2. Interestingly, the assumptions on the nonlinearity allow for potentially 'superlinear' functions. This differs from other ISS work on forced Lur'e systems, such as [107, 108] and Chapter 2, which all assume a linear bound on the nonlinearity of the system.

In this chapter, we investigate whether an analogue to [7, Theorem 1] holds for discrete-time versions of (4.1), and, moreover, whether similar assumptions can be used to infer ISS of much more general systems, given explicitly by (2.17). As discussed in Chapter 2, (2.17) allows for non-zero feedthrough, the entry of inputs not through the linear components of the system, and output disturbances. As such, it encompasses many other types of system. Not only shall we discuss ISS, we shall also develop sufficient criteria for when the aforementioned system exhibits the CICS property - see Section 2.3.2. Since convergence properties were not considered in [7], the forthcoming investigation further separates this work from the literature. Finally, in this chapter, we also discuss various versions of the positive real lemma. For background regarding the positive real lemma in the discrete-time setting, we refer the reader to [51, 54, 68, 119, 125, 127].

The layout of the chapter is as follows. In Section 4.1 we compile various preliminary results regarding the inner products and bounds of functions, and also give versions of the positive real lemma. In Section 4.2, we present the main stability results of this chapter. These comprise three main theorems and subsequent corollaries. The corollaries include ISS criterions which are expressed in the form of positive and strict positive real assumptions. Finally, in Section 4.3, we investigate the CICS property and develop sufficient conditions, which allow for superlinear nonlinearities, that guarantee the CICS property for systems of the form (2.17).

4.1 Preliminaries

In this initial section, we collect some useful preliminary results which will be key to our development of the rest of the chapter. We split this section into three, with the first and second parts comprising lemmas that concern bounds of functions and inner products of functions, respectively. The last portion of this section contains theory regarding positive and strict positive real transfer functions, accumulating in forming so-called positive and strict positive real lemmas.

4.1.1 Bounds of functions

We begin with the following result concerning a supremum of a continuous function.

Lemma 4.1.1. *Let $h : \mathbb{R}^m \to \mathbb{R}_+$ and $\sigma_1, \sigma_2 \in \mathbb{R}_+$ with $\sigma_1 \leq \sigma_2$. If h is bounded on bounded sets and continuous on $\{\xi \in \mathbb{R}^m : \sigma_1 \leq \|\xi\| \leq \sigma_2\}$, then $\phi : \mathbb{R}_+ \to \mathbb{R}_+$, defined by*

$$\phi(s) := \sup\{h(\xi) : \xi \in \mathbb{R}^m,\ \|\xi\| \leq s\} \quad \forall\, s \geq 0,$$

is well-defined and continuous on $[\sigma_1, \sigma_2]$.

Proof. To begin with, we note that since h is bounded on bounded sets, $\phi(s)$ is well-defined for each $s \geq 0$ and ϕ is nondecreasing. To show continuity on the prescribed interval, we fix $\varepsilon > 0$ and $s \in [\sigma_1, \sigma_2]$. Now, since h is uniformly continuous on $S := \{\xi \in \mathbb{R}^m : \sigma_1 \leq \|\xi\| \leq \sigma_2\}$, there exists $\tilde{\delta} > 0$ such that

$$|h(\xi) - h(\zeta)| < \varepsilon \quad \forall\, \xi, \zeta \in S \text{ s.t. } \|\xi - \zeta\| \leq \tilde{\delta}. \tag{4.2}$$

We claim that there exists $\delta_1 > 0$ such that

$$\phi(s) - \phi(t) = |\phi(s) - \phi(t)| < \varepsilon \quad \forall\, t \in (s - \delta_1, s] \cap [\sigma_1, \sigma_2]. \tag{4.3}$$

In order to prove this claim, we suppose that it is false. Therefore, $s > \sigma_1$ and, for all $\delta_1 > 0$, there exists $t \in (s - \delta_1, s] \cap [\sigma_1, \sigma_2]$ such that $\phi(s) - \phi(t) \geq \varepsilon$. In particular, by setting $\delta_1 := \min\{\tilde{\delta}, s - \sigma_1\} > 0$, we see that there exists $t \in (s - \delta_1, s] \subseteq (\sigma_1, \sigma_2]$ such that

$$\phi(s) - \phi(t) \geq \varepsilon. \tag{4.4}$$

Now, since $\phi(s) > \phi(t)$, it is clear that

$$\phi(s) = \sup\{h(\xi) : \xi \in \mathbb{R}^m,\ t \leq \|\xi\| \leq s\}. \tag{4.5}$$

Moreover, by combining (4.5) with the continuity of h on $\{\xi \in \mathbb{R}^m : t \leq \|\xi\| \leq s\}$, we see that $\phi(s) = h(\xi)$ for some $\xi \in \mathbb{R}^m$ such that $\|\xi\| \in [t, s]$. By noting that $t > \sigma_1 \geq 0$ and so $\xi \neq 0$, we let $\zeta \in \mathbb{R}^m$ be defined by $\zeta := t\xi/\|\xi\|$. We then see that $\zeta \in S$, $\|\xi - \zeta\| < \tilde{\delta}$ and $\|\zeta\| = t$. By invoking (4.2), we then yield that $h(\xi) < \varepsilon + h(\zeta)$. A combination of this with (4.4) then gives that

$$\varepsilon + \phi(t) \leq \phi(s) = h(\xi) < \varepsilon + h(\zeta) \leq \varepsilon + \phi(t),$$

which is a contradiction. Thus, we have shown that there does exist $\delta_1 > 0$ such that (4.3) holds. By operating in a very similar manner, it is possible to prove that there exists $\delta_2 > 0$ such that

$$\phi(t) - \phi(s) = |\phi(s) - \phi(t)| < \varepsilon \quad \forall\, t \in [s, s+\delta_2) \cap [\sigma_1, \sigma_2]. \tag{4.6}$$

By combining (4.3) and (4.6), we see that ϕ is continuous at s, and since s was arbitrary, the proof is complete. □

The subsequent two results give conditions for when a function can be bounded above by a $\mathcal{K}_\infty$ function (see Section 2.1.1).

Lemma 4.1.2. *Let $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ be non-decreasing, continuous on $[0, \varepsilon]$ for some $\varepsilon > 0$, and be such that $\phi(0) = 0$. Then there exists $\gamma \in \mathcal{K}_\infty$ such that $\phi(s) \le \gamma(s)$ for all $s \ge 0$.*

Proof. We begin by defining the function

$$\gamma_1(s) := \begin{cases} \max\left\{\phi(s), \frac{s}{\varepsilon}\int_s^{s+1}\phi(\sigma)d\sigma\right\}, & \text{if } s \in [0,\varepsilon], \\ \int_s^{s+1}\phi(\sigma)d\sigma, & \text{if } s > \varepsilon. \end{cases}$$

Note that γ_1 is continuous, non-decreasing and $\gamma_1(0) = 0$. Further to this, $\gamma_1(s) \ge \phi(s)$ for all $s \ge 0$, and so, by defining

$$\gamma(s) := s + \gamma_1(s) \quad \forall\, s \ge 0,$$

we obtain that $\gamma \in \mathcal{K}_\infty$ and that $\phi(s) \le \gamma(s)$ for all $s \ge 0$, thus completing the proof. □

Corollary 4.1.3. *Let $f : \mathbb{R}^m \to \mathbb{R}^m$ be such that $f(0) = 0$. Assume that f is bounded on bounded sets and that there exists $\varepsilon > 0$ such that f is continuous on $\{\xi \in \mathbb{R}^m : \|\xi\| \le \varepsilon\}$. Then there exists $\gamma \in \mathcal{K}_\infty$ such that $\|f(\xi)\| \le \gamma(\|\xi\|)$ for all $\xi \in \mathbb{R}^m$.*

Proof. First, define $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$\phi(s) := \sup\{\|f(\xi)\| : \xi \in \mathbb{R}^m,\ \|\xi\| \le s\} \quad \forall\, s \ge 0.$$

Note that $\phi(0) = 0$, ϕ is non-decreasing and, by Lemma 4.1.1, ϕ is continuous on $[0, \varepsilon]$. Hence, by Lemma 4.1.2, there exists $\gamma \in \mathcal{K}_\infty$ such that $\phi(s) \le \gamma(s)$ for all $s \ge 0$. In particular, $\|f(\xi)\| \le \phi(\|\xi\|) \le \gamma(\|\xi\|)$ for all $\xi \in \mathbb{R}^m$. □

The following example highlights a simple situation where losing the assumption of boundedness on bounded sets in Corollary 4.1.3 yields that the conclusion does not hold.

Example 4.1.4. Consider $g : \mathbb{R} \to \mathbb{R}$ defined by

$$g(\xi) = \begin{cases} \xi, & \text{if } \xi < 0, \\ e^{\frac{1}{1-\xi}} - e^1, & \text{if } \xi \in [0,1), \\ \xi, & \text{if } \xi \ge 1. \end{cases}$$

Note that g is continuous in a neighbourhood around 0 but g is unbounded on $[0, 1]$. Moreover, we claim that there does not exist $\gamma \in \mathcal{K}_\infty$ such that $|g(\xi)| \le \gamma(|\xi|)$ for all $\xi \in \mathbb{R}$. Indeed, if there was such a γ, then, trivially, γ would be continuous on the compact interval $[0, 1]$ but would be unbounded, which is a contradiction. ◊

The following lemma gives a useful property of an infimum of continuous functions.

Lemma 4.1.5. *Let $h : \mathbb{R}^m \to \mathbb{R}_+$ and $\sigma_1, \sigma_2 \in \mathbb{R}_+$ with $\sigma_1 \leq \sigma_2$. If h is continuous on $\{\xi \in \mathbb{R}^m : \sigma_1 \leq \|\xi\| \leq \sigma_2\}$, then $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous on $[\sigma_1, \sigma_2]$, where ϕ is defined by*

$$\phi(s) := \inf\{h(\xi) : \xi \in \mathbb{R}^m,\ \|\xi\| \geq s\} \quad \forall\, s \geq 0.$$

Proof. To begin with, we note that since h is bounded from below, $\phi(s)$ is well-defined for each $s \geq 0$. Moreover, ϕ is nondecreasing. In order to show continuity on $[\sigma_1, \sigma_2]$, we fix $\varepsilon > 0$ and $s \in [\sigma_1, \sigma_2]$. As in the proof of Lemma 4.1.1, since h is uniformly continuous on $S := \{\xi \in \mathbb{R}^m : \sigma_1 \leq \|\xi\| \leq \sigma_2\}$, there exists $\tilde{\delta} > 0$ such that

$$|h(\xi) - h(\zeta)| < \varepsilon \quad \forall\, \xi, \zeta \in S \text{ s.t. } \|\xi - \zeta\| \leq \tilde{\delta}. \tag{4.7}$$

Acting similarly to how we did in the proof of Lemma 4.1.1, we claim that there exists $\delta_1 > 0$ such that

$$\phi(s) - \phi(t) = |\phi(s) - \phi(t)| < \varepsilon \quad \forall\, t \in (s - \delta_1, s] \cap [\sigma_1, \sigma_2]. \tag{4.8}$$

We suppose this is not the case, in order to then obtain a contradiction and prove that the claim is true. Indeed, if the claim is false, then $s > \sigma_1$ and, for all $\delta_1 > 0$, there exists $t \in (s - \delta_1, s] \cap [\sigma_1, \sigma_2]$ such that $\phi(s) - \phi(t) \geq \varepsilon$. In particular, by setting $\delta_1 := \min\{\tilde{\delta}, s - \sigma_1\} > 0$, we see that there exists $t \in (s - \delta_1, s] \subseteq (\sigma_1, \sigma_2]$ such that

$$\phi(s) - \phi(t) \geq \varepsilon. \tag{4.9}$$

From this we deduce, in particular, that $\phi(s) > \phi(t)$ and so

$$\phi(t) = \inf\{h(\xi) : \xi \in \mathbb{R}^m,\ t \leq \|\xi\| \leq s\}. \tag{4.10}$$

We now combine (4.10) with the continuity of h on $\{\xi \in \mathbb{R}^m : t \leq \|\xi\| \leq s\}$ to obtain the existence of $\zeta \in \mathbb{R}^m$ such that $\|\zeta\| \in [t, s]$ and $\phi(t) = h(\zeta)$. We also note that $t > \sigma_1 \geq 0$, which implies that $\zeta \neq 0$. Subsequently, we let $\xi \in \mathbb{R}^m$ be defined by $\xi := s\zeta/\|\zeta\|$. As a consequence of this definition, we see that $\xi \in S$, $\|\xi - \zeta\| < \tilde{\delta}$ and $\|\xi\| = s$. An application of (4.7) then yields that $h(\xi) < \varepsilon + h(\zeta)$, which, when combined with (4.9) then gives that

$$\phi(s) \geq \varepsilon + \phi(t) = \varepsilon + h(\zeta) > h(\xi) \geq \phi(s),$$

which is a contradiction. We have therefore shown that there does exist $\delta_1 > 0$ such that (4.8) holds. By a very similar manner, it is not difficult to show that there also exists $\delta_2 > 0$ such that

$$\phi(t) - \phi(s) = |\phi(s) - \phi(t)| < \varepsilon \quad \forall\, t \in [s, s + \delta_2) \cap [\sigma_1, \sigma_2]. \tag{4.11}$$

Finally, a combination of (4.8) and (4.11) gives that ϕ is continuous at s. The proof is thus complete, since s was chosen arbitrarily. □

We now give a slightly different version of Lemma 4.1.5. We shall not provide a proof, since the result can be proven in a similar manner to the previous proof.

Lemma 4.1.6. *Let $h : \mathbb{R}^m \to \mathbb{R}_+$ and $\lambda > 0$. If h is continuous, then $\phi : [0, \lambda] \to \mathbb{R}_+$ is continuous, where ϕ is defined by*

$$\phi(s) := \inf\{h(\xi) : \xi \in \mathbb{R}^m,\, s \le \|\xi\| \le \lambda\} \quad \forall\, s \in [0, \lambda].$$

The next lemma asserts that the quotient of functions satisfying certain properties can be bounded below by a $\mathcal{K}_\infty$ function.

Lemma 4.1.7. *Let $h, g : \mathbb{R}^m \to \mathbb{R}_+$. If h and g are continuous, zero at zero, positive away from zero and*

$$\frac{h(\xi)}{g(\xi)} \to \infty \text{ as } \|\xi\| \to \infty,$$

then there exists $\alpha \in \mathcal{K}_\infty$ such that

$$\frac{h(\xi)}{g(\xi)} \ge \alpha(\|\xi\|) \quad \forall\, \xi \in \mathbb{R}^m \backslash \{0\}.$$

The following proof takes some inspiration from the proof of [15, Statement 2 of Lemma 4.2].

Proof of Lemma 4.1.7. Let $\lambda > 0$. Since g is continuous, let $M > 0$ be such that $g(\xi) \le M$ for all $\xi \in \mathbb{R}^m$ with $\|\xi\| \le \lambda$. We define the following functions:

$$\beta_1(s) := \begin{cases} \inf\left\{\frac{h(\xi)}{M} : \xi \in \mathbb{R}^m,\, s \le \|\xi\| \le \lambda\right\}, & \text{if } s \in [0, \lambda], \\ \inf\left\{\frac{h(\xi)}{M} : \xi \in \mathbb{R}^m,\, \|\xi\| = \lambda\right\} + s - \lambda, & \text{if } s > \lambda, \end{cases}$$

$$\beta_2(s) := \begin{cases} \frac{s}{\lambda} \inf\left\{\frac{h(\xi)}{g(\xi)} : \xi \in \mathbb{R}^m,\, \|\xi\| \ge \lambda\right\}, & \text{if } s \in [0, \lambda], \\ \inf\left\{\frac{h(\xi)}{g(\xi)} : \xi \in \mathbb{R}^m,\, \|\xi\| \ge s\right\}, & \text{if } s > \lambda. \end{cases}$$

Note that $\beta_1(0) = 0 = \beta_2(0)$, both are nondecreasing and both tend to ∞ as their arguments do. Moreover, a routine argument yields that $\beta_1(s)$ and $\beta_2(s)$ are positive for positive s. It is also true that β_1 and β_2 are both continuous. Indeed, β_1 is continuous by Lemma 4.1.6 and β_2 is continuous by Lemma 4.1.5. Hence, by setting

$$\alpha_i(s) := (1 - e^{-s})\beta_i(s) \quad \forall\, s \ge 0, \forall\, i \in \{1, 2\},$$

we see that $\alpha_i \in \mathcal{K}_\infty$ for each $i = 1, 2$. We now define $\alpha \in \mathcal{K}_\infty$ by defining

$$\alpha(s) := \min\{\alpha_1(s), \alpha_2(s)\} \quad \forall\, s \ge 0.$$

To complete the proof, let $\xi \in \mathbb{R}^m \backslash \{0\}$ and consider the two cases of when $\|\xi\|$ is greater or less than λ. First, assume that $0 < \|\xi\| \le \lambda$. We then have that

$$\alpha(\|\xi\|) \le \alpha_1(\|\xi\|) \le \beta_1(\|\xi\|) = \inf\left\{\frac{h(\zeta)}{M} : \zeta \in \mathbb{R}^m,\, \|\xi\| \le \|\zeta\| \le \lambda\right\} \le \frac{h(\xi)}{M} \le \frac{h(\xi)}{g(\xi)}.$$

For the second case, assume that $\|\xi\| > \lambda$ so that

$$\alpha(\|\xi\|) \le \alpha_2(\|\xi\|) \le \beta_2(\|\xi\|) = \inf\left\{\frac{h(\zeta)}{g(\zeta)} : \zeta \in \mathbb{R}^m,\, \|\zeta\| \ge \|\xi\|\right\} \le \frac{h(\xi)}{g(\xi)},$$

hence completing the proof. □

4.1.2 Results regarding inner products

We now present lemmas concerning the inner products of functions. We begin by recalling a simple result regarding the negative definiteness and stability of matrices. The result is well-known, see, for example, [92, Proposition 3].

Lemma 4.1.8. *Let $A, P \in \mathbb{R}^{n\times n}$, where $P = P^T$ is positive definite. If $A^TPA - P$ is negative definite, then A is Schur.*

The subsequent lemma is a trivial result regarding inner products, but we present it to ensure clarity of subsequent proofs.

Lemma 4.1.9. *Let $(A, B, C, D) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{m\times n} \times \mathbb{R}^{m\times m}$. Then*

$$\left\langle \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \xi \\ \zeta \end{pmatrix}, \begin{pmatrix} \xi \\ \zeta \end{pmatrix} \right\rangle = \langle A\xi, \xi\rangle + \langle B\zeta, \xi\rangle + \langle C\xi, \zeta\rangle + \langle D\zeta, \zeta\rangle \quad \forall\, \xi \in \mathbb{R}^n, \forall\, \zeta \in \mathbb{R}^m.$$

Proof. In the following, we denote the ith component of a vector ξ by ξ_i. Fix $\xi \in \mathbb{R}^n$ and $\zeta \in \mathbb{R}^m$, and note that

$$\begin{aligned} \left\langle \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \xi \\ \zeta \end{pmatrix}, \begin{pmatrix} \xi \\ \zeta \end{pmatrix} \right\rangle &= \left\langle \begin{pmatrix} A\xi + B\zeta \\ C\xi + D\zeta \end{pmatrix}, \begin{pmatrix} \xi \\ \zeta \end{pmatrix} \right\rangle \\ &= \begin{pmatrix} A\xi + B\zeta \\ C\xi + D\zeta \end{pmatrix}^T \begin{pmatrix} \xi \\ \zeta \end{pmatrix} \\ &= \begin{pmatrix} \xi^TA^T + \zeta^TB^T & \xi^TC^T + \zeta^TD^T \end{pmatrix} \begin{pmatrix} \xi \\ \zeta \end{pmatrix} \\ &= \xi^TA^T\xi + \zeta^TB^T\xi + \xi^TC^T\zeta + \zeta^TD^T\zeta \\ &= \langle A\xi, \xi\rangle + \langle B\zeta, \xi\rangle + \langle C\xi, \zeta\rangle + \langle D\zeta, \zeta\rangle, \end{aligned}$$

which therefore completes the proof. □

The following lemma gives sufficient conditions (which will be of key relevance to the assumptions of the main theorems of the next section - Section 4.2) for when a function satisfies $f(0) = 0$.

Lemma 4.1.10. *Let $f : \mathbb{R}^m \to \mathbb{R}^m$. If f is continuous on a neighbourhood containing 0 and*

$$0 \le -\langle \xi, f(\xi)\rangle \quad \forall\, \xi \in \mathbb{R}^m, \tag{4.12}$$

then $f(0) = 0$.

Proof. First, for $\xi \in \mathbb{R}$ and $i \in \{1, \dots, m\}$, we define ξ_i^0 to be the vector in $\mathbb{R}^m$ with zeros everywhere except for the ith component which is given by ξ. Additionally, we let $f_1, f_2, \dots, f_m : \mathbb{R}^m \to \mathbb{R}$ be such that

$$f(\zeta) = \begin{pmatrix} f_1(\zeta) \\ \vdots \\ f_m(\zeta) \end{pmatrix} \quad \forall\, \zeta \in \mathbb{R}^m,$$

and note that, for each $i \in \{1, \ldots, m\}$, f_i is continuous on the same neighbourhood around 0 that f is. Moreover, for each $i \in \{1, \ldots, m\}$, we define $g_i : \mathbb{R} \to \mathbb{R}$ by

$$g_i(\xi) = f_i(\xi_i^0) \quad \forall \, \xi \in \mathbb{R},$$

and again note that, for each $i \in \{1, \ldots, m\}$, g_i is continuous on the same neighbourhood around 0 that f is. Fix $i \in \{1, \ldots, m\}$ and let $\xi \in \mathbb{R}$. Then, using (4.12), we see that

$$0 \leq -\langle \xi_i^0, f(\xi_i^0) \rangle = -\xi f_i(\xi_i^0) = -\xi g_i(\xi).$$

By combining this with the continuity of g_i around zero, we see that $g_i(0) = 0$, which in turn implies that $f_i(0) = 0$. Furthermore, since i was arbitrary, we have that $f(0) = 0$, and the proof is complete. □

The final preliminary result concerning inner products that we present here is given subsequently.

Lemma 4.1.11. *Let $f : \mathbb{R}^m \to \mathbb{R}^m$ be continuous on a neighbourhood containing 0 and be bounded on bounded sets. Furthermore, assume that there exist $\alpha \in \mathcal{K}_\infty$, $c > 0$ and $\mu > 0$ such that*

$$\|\xi\| \alpha(\|\xi\|) \leq -\langle \xi, f(\xi) \rangle \quad \forall \, \xi \in \mathbb{R}^m, \tag{4.13}$$

and

$$\|f(\xi)\| \leq -c\langle \xi, f(\xi) \rangle \quad \forall \, \xi \in \mathbb{R}^m \text{ s.t. } \|\xi\| \geq \mu. \tag{4.14}$$

Then there exist $\varepsilon > 0$ and $\eta \in \mathcal{K}_\infty$ such that

$$\varepsilon \left(\|f(\xi)\| + \|\xi\| \right) \leq -\langle \xi, f(\xi) \rangle \quad \forall \, \xi \in \mathbb{R}^m \text{ s.t. } \|\xi\| \geq \mu, \tag{4.15}$$

and

$$\eta(\|\xi\|)\|\xi\|^2 + \eta(\|f(\xi)\|)\|f(\xi)\|^2 \leq -\langle \xi, f(\xi) \rangle \quad \forall \, \xi \in \mathbb{R}^m \text{ s.t. } \|\xi\| \leq \mu. \tag{4.16}$$

Before proving Lemma 4.1.11, we provide some commentary.

Remark 4.1.12. A particular case of Lemma 4.1.11 is nested inside the proof of [7, Theorem 1]. In [7, Theorem 1], the above is proven for the case that $c = 1$ and continuity is assumed everywhere. In the next section, in Example 4.2.5, we present a function f which satisfies (4.14) with $c = 2$ and $\mu = 1$, but which has the property that there does not exist any $\mu > 0$ such that (4.14) holds for $c = 1$. ◊

Proof of Lemma 4.1.11. We shall act similarly to that done in the proof of [7, Theorem 1]. To this end, we begin by proving (4.15), and so we let $\varepsilon > 0$ be such that

$$\varepsilon \leq \frac{1}{2} \min \left\{ \frac{1}{c}, \alpha(\mu) \right\}.$$

From (4.14), we see that

$$\varepsilon \|f(\xi)\| \leq -\frac{1}{2} \langle \xi, f(\xi) \rangle \quad \forall \, \xi \in \mathbb{R}^m \text{ s.t. } \|\xi\| \geq \mu. \tag{4.17}$$

Moreover, (4.13) gives that

$$\varepsilon \|\xi\| \leq \frac{1}{2} \alpha(\mu) \|\xi\| \leq \frac{1}{2} \alpha(\|\xi\|) \|\xi\| \leq -\frac{1}{2} \langle \xi, f(\xi) \rangle \quad \forall \, \xi \in \mathbb{R}^m \text{ s.t. } \|\xi\| \geq \mu.$$

Thus, by combining this with (4.17), we yield (4.15). As for (4.16), we use applications of Lemma 4.1.10 and Corollary 4.1.3 to obtain the existence of $\beta \in \mathcal{K}_\infty$ such that $\|f(\xi)\| \leq \beta(\|\xi\|)$ for all $\xi \in \mathbb{R}^m$. Now, for $\xi \in \mathbb{R}^m$ and for $\eta \in \mathcal{K}_\infty$ to be chosen later, we see that

$$\begin{aligned}\eta(\|\xi\|)\|\xi\|^2 + \eta(\beta(\|\xi\|))\beta(\|\xi\|)^2 &\leq (\eta(\|\xi\|) + \eta(\beta(\|\xi\|)))\,(\|\xi\|^2 + \beta(\|\xi\|)^2)\\ &\leq 2\eta(\|\xi\| + \beta(\|\xi\|))(\|\xi\|^2 + \beta(\|\xi\|)^2). \qquad (4.18)\end{aligned}$$

Invoking Lemma 2.1.6 then gives the existence of η such that, for all $\xi \in \mathbb{R}^m$ with $\|\xi\| \leq \mu$,

$$2\eta(\|\xi\| + \beta(\|\xi\|))(\|\xi\|^2 + \beta(\|\xi\|)^2) \leq \|\xi\|\alpha(\|\xi\|) \leq -\langle \xi, f(\xi)\rangle, \qquad (4.19)$$

where we have used (4.13). By the choice of β, we may combine (4.18) with (4.19) to yield (4.16), thus completing the proof. □

4.1.3 The positive and strict positive real lemmas

Let us now recall the following linear difference equation seen in Chapter 2 (see (2.6)):

$$\left.\begin{aligned} x^+ &= Ax + Bu + B_\mathrm{e}v,\\ y &= Cx + Du + D_\mathrm{e}v,\end{aligned}\right\} \qquad (2.6)$$

where $(A, B, B_\mathrm{e}, C, D, D_\mathrm{e}) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{n\times q} \times \mathbb{R}^{p\times n} \times \mathbb{R}^{p\times m} \times \mathbb{R}^{p\times q}$, $u \in (\mathbb{R}^m)^{\mathbb{Z}_+}$ and $v \in (\mathbb{R}^q)^{\mathbb{Z}_+}$. We recall that we defined

$$\mathbb{L} := \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{n\times q} \times \mathbb{R}^{p\times n} \times \mathbb{R}^{p\times m} \times \mathbb{R}^{p\times q},$$

and let $\Sigma := (A, B, B_\mathrm{e}, C, D, D_\mathrm{e}) \in \mathbb{L}$. Furthermore, we recall that the transfer function of (2.6) is denoted by $\mathbf{G}$ and the behaviour by $\mathcal{B}^{\mathrm{lin}}$. For a detailed discussion of (2.6), we refer the reader to Section 2.1, where items such as the generality of the system and linear output feedback via 'loop-shifting', are presented.

For the rest of this chapter, we make the following standing assumption:

Assumption. $$m = p.$$

This is the same as saying that (2.6) is "square".

Our attention now concerns the positive and strict positive real lemmas. The positive real lemma (in discrete-time) is given by [54, Lemma 3] for controllable and observable systems. For given $\Sigma \in \mathbb{L}$, it asserts sufficient conditions for when there exist a positive (semi-)definite matrix $P \in \mathbb{R}^{n\times n}$ and matrices L and W satisfying

$$\left.\begin{aligned} A^T PA - P &= -L^T L,\\ A^T PB &= C^T - L^T W,\\ B^T PB &= D + D^T - W^T W.\end{aligned}\right\} \qquad (4.20)$$

Here, we formulate a second result that guarantees the existence of P, L and W satisfying (4.20), but does not assume that the system is controllable and observable. To do so, we first collect some relevant definitions and results. To this end, let us first recall what we mean by a positive real function from Definition 2.3.23. Indeed, a $\mathbb{C}^{m\times m}$-valued rational function $\mathbf{H}$ is *positive real* if $\mathbf{H}(z)+\mathbf{H}(z)^*$ is positive semi-definite for every $z\in\mathbb{E}$ which is not a pole of $\mathbf{H}$.

We now give two results that will be key to our development. The first associates positive realness of a transfer function in terms of the boundedness of a different function related by a Cayley transform (see, for example, [85]). The lemma is well-known in the continuous-time setting (see, for example, [4, 49]). However, it is difficult to find an explicit result for the discrete-time case. Fortunately, the ideas used in the continuous-time literature can be easily applied to here also. For completeness, we shall provide a brief proof.

Lemma 4.1.13. *Let $\Sigma\in\mathbb{L}$. If $\mathbf{G}$ is positive real, then $I+\mathbf{G}(z)$ is invertible for every $z\in\mathbb{E}$, $I+D$ is invertible and $\tilde{\mathbf{G}}:=(I-\mathbf{G})(I+\mathbf{G})^{-1}$ is such that $\|\tilde{\mathbf{G}}\|_{H^\infty}\le 1$ and $\tilde{\mathbf{G}}(z)=\tilde{C}(zI-\tilde{A})^{-1}\tilde{B}+\tilde{D}$ for all $z\in\mathbb{E}$, where*

$$\left.\begin{aligned}\tilde{A}&:=A-B(I+D)^{-1}C, &\quad \tilde{B}&:=\sqrt{2}B(I+D)^{-1},\\ \tilde{C}&:=-\sqrt{2}(I+D)^{-1}C, &\quad \tilde{D}&:=(I-D)(I+D)^{-1}.\end{aligned}\right\}\tag{4.21}$$

Proof. Assume that $\mathbf{G}$ is positive real. As a consequence of this, we obtain that $D+D^T\ge 0$, and so an application of [46, Corollary 2.3] gives that $I+D$ is invertible and $I+\mathbf{G}(z)$ is invertible for every $z\in\mathbb{E}$. Not only this, [46, Corollary 2.3] also yields that $\|(I-D)(I+D)^{-1}\|\le 1$ and $\|\tilde{\mathbf{G}}(z)\|\le 1$ for all $z\in\mathbb{E}$. We hence deduce that $\|\tilde{\mathbf{G}}\|_{H^\infty}\le 1$. We shall now show that $\tilde{\mathbf{G}}(z)=\tilde{C}(zI-\tilde{A})^{-1}\tilde{B}+\tilde{D}$ for all $z\in\mathbb{E}$. To this end, let $\mathbf{H}:=I-\mathbf{G}$ and $L:=I/2\in\mathbb{R}^{m\times m}$. Note that $I-(I-D)L=(I+D)/2$ is invertible, that is, $L\in\mathbb{A}_{\mathbb{C}}(I-D)$ (see Definition 2.1.14). Moreover,

$$\begin{aligned}\tilde{\mathbf{G}}&=(I-\mathbf{G})(I+\mathbf{G})^{-1}\\&=\mathbf{H}(2I-\mathbf{H})^{-1}\\&=L\mathbf{H}(I-L\mathbf{H})^{-1}\\&=L\mathbf{H}^L,\end{aligned}$$

where recall $\mathbf{H}^L=\mathbf{H}(I-L\mathbf{H})$ - see Definition 2.1.19. Furthermore, $\|\mathbf{H}^L\|_{H^\infty_{m\times m}}=2\|L\mathbf{H}^L\|_{H^\infty}=2\|\tilde{\mathbf{G}}\|_{H^\infty}$ and so $\mathbf{H}^L\in H^\infty_{m\times m}$. Therefore, with these facts and Definition 2.1.16 in mind, we see that, for all $z\in\mathbb{E}$,

$$\begin{aligned}\tilde{\mathbf{G}}(z)&=L\mathbf{H}^L(z)\\&=L(I-(I-D)L)^{-1}(I-D)\\&\quad-L(I-(I-D)L)^{-1}C(zI-A+BL(I-(I-D)L)^{-1}C)^{-1}B(I-(I-D)L)^{-1}\\&=(I+D)^{-1}(I-D)-2(I+D)^{-1}C(zI-A+B(I+D)^{-1}C)^{-1}B(I+D)^{-1}\\&=\tilde{D}+\tilde{C}(zI-\tilde{A})^{-1}\tilde{B},\end{aligned}$$

which completes the proof. □

The second preliminary result shows that certain key properties are preserved under the Cayley transform seen in the previous lemma.

Lemma 4.1.14. *Let $\Sigma\in\mathbb{L}$. Under the assumption that $\mathbf{G}$ is positive real, the following statements hold, where $\tilde{\mathbf{G}}:=(I-\mathbf{G})(I+\mathbf{G})^{-1}$, and $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, $\tilde{D}$ are as in (4.21).*

(i) (A, B) *is stabilisable if, and only if,* $(\tilde{A}, \tilde{B})$ *is stabilisable.*

(ii) (A, C) *is detectable if, and only if,* $(\tilde{A}, \tilde{C})$ *is detectable.*

(iii) *If there exists* $\eta \in \partial\mathbb{D}$ *which is not a pole of* $\mathbf{G}$, *then* $\mathbf{G}(\eta) + \mathbf{G}(\eta)^* > 0$ *if, and only if,* $I + \mathbf{G}(\eta)$ *is invertible and* $\|\tilde{\mathbf{G}}(\eta)\| < 1$.

Proof. For statement (i), assume that (A, B) is stabilisable and let $F \in \mathbb{R}^{m\times n}$ be such that $\sigma(A + BF) \subseteq \mathbb{D}$. By defining $\tilde{F} := \frac{1}{\sqrt{2}}C + \frac{1}{\sqrt{2}}(I + D)F \in \mathbb{R}^{m\times n}$, we have

$$\tilde{A} + \tilde{B}\tilde{F} = A - B(I + D)^{-1}C + \sqrt{2}B(I + D)^{-1}\left(\frac{1}{\sqrt{2}}C + \frac{1}{\sqrt{2}}(I + D)F\right) = A + BF,$$

which thus shows the stabilisability of $(\tilde{A}, \tilde{B})$. For the reverse implication, if $\sigma(\tilde{A} + \tilde{B}\tilde{F}) \subseteq \mathbb{D}$ for some $\tilde{F} \in \mathbb{R}^{m\times n}$, then setting $F := (I + D)^{-1}(\sqrt{2}\tilde{F} - C)$ yields that (A, B) is stabilisable. Statement (ii) is proven similarly and thus the proof is omitted. As for statement (iii), we note that if $\mathbf{G}(\eta) + \mathbf{G}(\eta)^* > 0$, then it is easy to show that there exists $\varepsilon > 0$ such that $\langle(\mathbf{G}(\eta) + \mathbf{G}(\eta)^*)\xi, \xi\rangle \geq \varepsilon\|\xi\|^2$ for all $\xi \in \mathbb{R}^m$. Hence, the statement is immediately obtained from [46, Corollary 2.3]. □

The subsequent lemma is the positive real lemma which we alluded to earlier. The result is in two parts: the first is an immediate consequence of [54, Lemma 3] (see also [51, Lemma 4.1]); and the second is a new result which contains sufficient conditions for the existence of P, L and W satisfying (4.20), which do not necessarily require controllability and observability of the overall system.

Lemma 4.1.15. *Let* $\Sigma \in \mathbb{L}$ *and assume that* $\mathbf{G}$ *is positive real. The following statements hold.*

(i) *If* Σ *is controllable and observable then there exist a positive definite* $P = P^T \in \mathbb{R}^{n\times n}$ *and matrices* L *and* W *such that* (4.20) *holds.*

(ii) *If* Σ *is stabilisable and detectable and there exists* $\eta \in \partial\mathbb{D}$, *not a pole of* $\mathbf{G}$, *such that* $\mathbf{G}(\eta) + \mathbf{G}(\eta)^* > 0$, *then there exist a positive semi-definite* $P = P^T \in \mathbb{R}^{n\times n}$ *and matrices* L *and* W *such that* W *is invertible and* (4.20) *holds.*

Before proving Lemma 4.1.15, we first give some commentary.

Remark 4.1.16. (i) In statement (ii) of Lemma 4.1.15, P is positive semi-definite and not necessarily positive definite. Fortunately, this is sufficient for our purposes in the later sections. However, statement (ii) does guarantee that W is invertible.

(ii) Example 4.1.17, situated after the following proof, shows that only assuming stabilisability and detectability of Σ is insufficient in proving that there exist a positive semi-definite $P = P^T \in \mathbb{R}^{n\times n}$ and matrices L and W such that (4.20) holds. ◊

Proof of Lemma 4.1.15. As repeatedly mentioned, if Σ is controllable and observable, then the result follows from [54, Lemma 3] (or from [51, Lemma 4.1]) and P is positive definite. Thus, we assume that Σ is stabilisable and detectable and that there exists $\eta \in \partial\mathbb{D}$, not a pole of $\mathbf{G}$, such that $\mathbf{G}(\eta) + \mathbf{G}(\eta)^* > 0$. Let $\tilde{\mathbf{G}} := (I - \mathbf{G})(I + \mathbf{G})^{-1}$, and let $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, $\tilde{D}$ be given as in (4.21). Lemma 4.1.14 gives that $(\tilde{A}, \tilde{B})$ is stabilisable, $(\tilde{A}, \tilde{C})$ is detectable, $I + \mathbf{G}(\eta)$ is invertible and $\|\tilde{\mathbf{G}}(\eta)\| < 1$. Since Lemma 4.1.13 gives that $\|\tilde{\mathbf{G}}\|_{H^\infty} \leq 1$, an application of Lemma 2.1.25 therefore yields the existence of a positive semi-definite $P \in \mathbb{R}^{n\times n}$ and matrices $\tilde{L}$ and $\tilde{W}$ such that $\tilde{W}$ is positive definite and

$$\left.\begin{aligned} \tilde{A}^T P\tilde{A} - P + \tilde{C}^T\tilde{C} &= -\tilde{L}^T\tilde{L}, \\ \tilde{A}^T P\tilde{B} + \tilde{C}^T\tilde{D} &= -\tilde{L}^T\tilde{W}, \\ \tilde{B}^T P\tilde{B} + \tilde{D}^T\tilde{D} &= I - \tilde{W}^T\tilde{W}. \end{aligned}\right\} \tag{4.22}$$

With this in mind, we set $W := \frac{1}{\sqrt{2}}\tilde{W}(I+D)$ and shall show that

$$B^T PB = D + D^T - W^T W. \tag{4.23}$$

To this end, consider

$$\begin{aligned} \tilde{B}^T P\tilde{B} + \tilde{D}^T\tilde{D} &= 2(I+D^T)^{-1}B^T PB(I+D)^{-1} \\ &\quad + (I+D^T)^{-1}(I-D^T)(I-D)(I+D)^{-1} \\ &= (I+D^T)^{-1}(2B^T PB + I - D - D^T + D^T D)(I+D)^{-1}. \end{aligned}$$

By multiplying the above through by $(I+D^T)$ and $(I+D)$ on the left and right respectively, and by recalling (4.22), we obtain that

$$I + D + D^T + D^T D - (I+D^T)\tilde{W}^T\tilde{W}(I+D) = 2B^T PB + I - (D+D^T) + D^T D,$$

which easily simplifies to (4.23). Our attention now turns to proving that

$$A^T PB = C^T - L^T W, \tag{4.24}$$

where $L := \tilde{L} + W(I+D)^{-1}C$. Indeed, to show this, consider

$$\begin{aligned} \tilde{A}^T P\tilde{B} + \tilde{C}^T\tilde{D} &= (A - B(I+D)^{-1}C)^T P(\sqrt{2}B(I+D)^{-1}) \\ &\quad - \sqrt{2}((I+D)^{-1}C)^T(I-D)(I+D)^{-1} \\ &= \sqrt{2}A^T PB(I+D)^{-1} - \sqrt{2}C^T(I+D^T)^{-1}B^T PB(I+D)^{-1} \\ &\quad - \sqrt{2}C^T(I+D^T)^{-1}(I-D)(I+D)^{-1}. \end{aligned}$$

We now use (4.22), and multiply by $1/\sqrt{2}$ and $(I+D)$ from the right, to obtain that

$$A^T PB - C^T(I+D^T)^{-1}B^T PB - C^T(I+D^T)^{-1}(I-D) = -\tilde{L}^T W. \tag{4.25}$$

We pause here and note that, from (4.23),

$$\begin{aligned} (I+D^T)^{-1}B^T PB &= (I+D^T)^{-1}(D+D^T - W^T W) \\ &= I - (I+D^T)^{-1}(I-D) - (I+D^T)^{-1}W^T W. \end{aligned}$$

Hence, after writing

$$\begin{aligned} A^T PB - C^T(I+D^T)^{-1}B^T PB - C^T(I+D^T)^{-1}(I-D) \\ = A^T PB - C^T((I+D^T)^{-1}B^T PB + (I+D^T)^{-1}(I-D)), \end{aligned}$$

we combine the previous identity with (4.25) to obtain

$$A^T PB - C^T = -\tilde{L}^T W - C^T(I+D^T)^{-1}W^T W,$$

that is, (4.24). To complete the proof, all that is left to show is that

$$A^T PA - P = -L^T L. \tag{4.26}$$

To this end, consider

$$\tilde{A}^T P \tilde{A} - P = (A - B(I + D)^{-1} C)^T P (A - B(I + D)^{-1} C) - P,$$

which may be rewritten as

$$\begin{aligned}\tilde{A}^T P \tilde{A} - P =& A^T PA - P - A^T PB(I + D)^{-1} C - C^T (I + D^T)^{-1} B^T PA \\ &+ C^T (I + D^T)^{-1} B^T PB(I + D)^{-1} C.\end{aligned} \tag{4.27}$$

Moreover, set $M := A^T PB(I + D)^{-1} C$ and note that, using (4.25),

$$M = \left(C^T (I + D^T)^{-1} B^T PB + C^T (I + D^T)^{-1} (I - D) - \tilde{L}^T W \right) (I + D)^{-1} C,$$

which may be expressed as

$$\begin{aligned}M =& \left(C^T (I + D^T)^{-1} B^T PB + C^T (I + D^T)^{-1} \right) (I + D)^{-1} C \\ &- \left(C^T (I + D^T)^{-1} D + L^T W - C^T (I + D^T)^{-1} W^T W \right) (I + D)^{-1} C.\end{aligned} \tag{4.28}$$

Furthermore, we have

$$\begin{aligned}-\tilde{C}^T \tilde{C} - \tilde{L}^T \tilde{L} =& -2C^T (I + D^T)^{-1} (I + D)^{-1} C \\ &- \left(L - W(I + D)^{-1} C \right)^T \left(L - W(I + D)^{-1} C \right).\end{aligned} \tag{4.29}$$

Therefore, by combining (4.27) with (4.28), (4.29) and the identity $\tilde{A}^T P \tilde{A} - P + \tilde{C}^T \tilde{C} = -\tilde{L}^T \tilde{L}$, a series of algebraic simplifications leads to

$$\begin{aligned}A^T PA - P =& - L^T L + C^T (I + D^T)^{-1} B^T PB(I + D)^{-1} C \\ &- C^T (I + D^T)^{-1} (D + D^T)(I + D)^{-1} C \\ &+ C^T (I + D^T)^{-1} W^T W (I + D)^{-1} C.\end{aligned}$$

An application of (4.23) then gives (4.26). Therefore, (4.23), (4.24) and (4.26) combine to yield (4.20). Finally, Lemma 2.1.25 gives that $\tilde{W}$ is positive definite, and hence invertible. Thus, since $I + D$ is also invertible from Lemma 4.1.13, we see that W is invertible and the proof is complete. □

We now give the following example, which presents a system where the transfer function is a scalar positive real function, the underlying system is stabilisable and detectable, $\mathbf{G}(z) + \mathbf{G}(z)^* = 0$ for all $z \in \partial\mathbb{D}$ which aren't poles, and there does not exist a positive semi-definite $P = P^T \in \mathbb{R}^{n \times n}$ and matrices L and W such that (4.20) holds. This hence shows that statement (ii) of Lemma 4.1.15 is justified in its hypotheses.

Example 4.1.17. Let $\Sigma \in \mathbb{L}$, where $n = 2$, $m = 1$,

$$A := \begin{pmatrix} -1 & -1 \\ 0 & \frac{1}{2} \end{pmatrix}, \quad B := \frac{2}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad C := -\frac{2}{\sqrt{2}} \begin{pmatrix} 1 & 1 \end{pmatrix}, \quad D := 1,$$

and B_{e} and D_{e} are arbitrary. It is easily checked that (A, B) is stabilisable, (A, C) is detectable, the transfer function is positive real and that $\mathbf{G}(z) + \mathbf{G}(z)^* = 0$ for all $z \in \partial\mathbb{D}\backslash\{-1\}$. Furthermore, we suppose for contradiction that there exist a positive semi-definite $P = P^T \in \mathbb{R}^{2\times 2}$ and matrices L and W such that (4.20) holds. We write

$$P = \begin{pmatrix} p_1 & p_2 \\ p_2 & p_4 \end{pmatrix}, \quad L = \begin{pmatrix} l_{1,1} & l_{1,2} \\ \vdots & \vdots \\ l_{q,1} & l_{q,2} \end{pmatrix} \quad \text{and} \quad W = \begin{pmatrix} w_1 \\ \vdots \\ w_q \end{pmatrix},$$

where $q \in \mathbb{N}$ and $p_i, l_{j,k}, w_j \in \mathbb{R}$ for all $i \in \{1, 2, 4\}$, $j \in \{1, \ldots, q\}$ and $k \in \{1, 2\}$. By using (4.20), we obtain the following three identities. The first is

$$\begin{pmatrix} 0 & p_1 - \frac{3}{2}p_2 \\ p_1 - \frac{3}{2}p_2 & p_1 - p_2 - \frac{3}{4}p_4 \end{pmatrix} = A^T PA - P = -L^T L \tag{4.30}$$

the second is given by

$$\frac{2}{\sqrt{2}} \begin{pmatrix} 1 - p_1 \\ 1 - p_1 + \frac{1}{2}p_2 \end{pmatrix} = A^T PB - C^T = -L^T W \tag{4.31}$$

and finally

$$2(p_1 - 1) = B^T PB - 2D = -W^T W. \tag{4.32}$$

From (4.30), we obtain that $l_{1,1} = l_{2,1} = \cdots = l_{q,1} = 0$. Subsequently, (4.31) yields that $p_1 = 1$ and this in turn gives, from (4.30) again, that $p_2 = 2/3$. However, substituting $p_1 = 1$ into (4.32) yields that $w_1 = \cdots = w_q = 0$, and then by combining this with (4.31) we see that $p_2 = 0$, hence contradicting that p_2 also equals $2/3$. ◊

We now investigate a so-called strict positive real lemma. Before discussing this further, we first define what we mean by strict positive realness.

Definition 4.1.18. *Let $\Sigma \in \mathbb{L}$. We say that $\mathbf{G}$ is* strictly positive real *if there exists $\varepsilon \in (0, 1)$ such that $z \mapsto \mathbf{G}(\varepsilon z)$ is positive real.*

Remark 4.1.19. (i) Strict positive realness can be thought of as positive realness on a 'larger domain'. A trivial example of a strict positive real transfer function is $\mathbf{G}(z) = (z - \varepsilon)/(z + \varepsilon)$, where $\varepsilon \in (0, 1)$.

(ii) Quite often in the literature (for example in [51]), $\mathbf{G}$ is defined to be strictly positive real if the poles of $\mathbf{G}$ are in $\mathbb{D}$ and $\mathbf{G}(e^{i\theta}) + \mathbf{G}(e^{i\theta})^* > 0$ for all $\theta \in [0, 2\pi]$. The definition we use, as given in Definition 4.1.18, is actually weaker. Indeed, as discussed in [119], transfer functions of the form

$$\frac{z - \varepsilon}{z + \varepsilon} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},$$

where $\varepsilon \in (0, 1)$, are strictly positive real in the sense of Definition 4.1.18, but are singular at every z. Additionally, [119, Theorem 2.2] proves that the two aforementioned notions of strict positive realness are equivalent, provided that $\mathbf{G}(z) + \mathbf{G}(z^{-1})^T$ has full rank almost everywhere. ◊

The following is a strict positive real lemma, and is a consequence of [51, Lemma 4.2]. However, we shall still provide a brief proof of the result, since we need to verify that the assumptions used below imply that the hypotheses of [51, Lemma 4.2] hold. This is because, as mentioned in Remark 4.1.19, there are various, not equivalent, definitions of strict positive real functions.

Lemma 4.1.20. *Let $\Sigma \in \mathbb{L}$. If Σ is controllable and observable, $\mathbf{G}$ is strictly positive real and $D+D^T > 0$, then there exist matrices $P, R \in \mathbb{R}^{n\times n}$, $L \in \mathbb{R}^{m\times n}$ and $W \in \mathbb{R}^{m\times m}$ such that P, R and W are all positive definite, and*

$$\left.\begin{aligned} A^TPA - P &= -L^TL - R, \\ A^TPB &= C^T - L^TW, \\ B^TPB &= D + D^T - W^TW. \end{aligned}\right\} \tag{4.33}$$

Remark 4.1.21. We note that in the case that $m = 1$, as long as $\mathbf{G}$ is positive real and not the zero function, then [24, Lemma 6.1] yields that $D > 0$. ◊

Proof of Lemma 4.1.20. Since $\mathbf{G}$ is strictly positive real, we have, in particular, that $\mathbf{G}$ is holomorphic on $\{z \in \mathbb{C} : |z| \geq 1\}$. From this, we obtain that $\mathbf{G} \in H^\infty_{m\times m}$. By combining this with the controllability and observability of Σ, as a consequence, we see that A is Schur (see, for example, [72, Theorem 2]). In addition to this, by combining the strict positive realness of $\mathbf{G}$ with the assumption that $D + D^T > 0$, we claim that

$$\mathbf{G}(z) + \mathbf{G}(z)^* > 0 \quad \forall\, z \in \mathbb{E}_\varepsilon, \tag{4.34}$$

where $\varepsilon \in (0, 1)$ is such that the function $z \mapsto \mathbf{G}(\varepsilon z)$ is positive real. We delay the verification of this claim until the proof of Lemma 5.2.17 in Chapter 5. We do this, because there we shall prove it in a more general setting.
Moving on, with the stability of A in mind, (4.34) allows us to apply [51, Lemma 4.2] to yield the existence of positive definite matrices $P, R \in \mathbb{R}^{n\times n}$ such that

$$D + D^T - B^TPB > 0$$

and

$$P = A^TPA + (A^TPB - C^T)(D + D^T - B^TPB)^{-1}(B^TPA - C) + R.$$

We now invoke [9, Theorem 7.35, p.226] to yield a positive definite matrix $W \in \mathbb{R}^{m\times m}$ such that $W^TW = D + D^T - B^TPB > 0$. By setting $L := -(W^T)^{-1}(B^TPA - C)$, we complete the proof. □

4.2 Stability properties

This section contains our main stability results of this chapter. We split the section into three. In the first part, we recall relevant theory and give a key preliminary result; in the second, we give three stability theorems; and in the third, we reformulate these in terms of positive and strict positive real assumptions, by utilising the outcomes of the previous section.

4.2.1 System and definitions

From hereon in this chapter, we consider the feedback interconnection of (2.6) with $u = f(y+w)$, where $f : \mathbb{R}^m \to \mathbb{R}^m$ and $w \in (\mathbb{R}^m)^{\mathbb{Z}_+}$ is an output disturbance. This system (in the not necessarily square case) is given by (2.17) situated in Chapter 2. For completeness, let us explicitly recall it here also:

$$\left.\begin{aligned} x^+ &= Ax + Bu + B_\mathrm{e}v, \\ y &= Cx + Du + D_\mathrm{e}v, \\ u &= f(y+w), \end{aligned}\right\} \tag{2.17}$$

where $\Sigma \in \mathbb{L}$, $v \in (\mathbb{R}^q)^{\mathbb{Z}_+}$, $w \in (\mathbb{R}^m)^{\mathbb{Z}_+}$ and $f : \mathbb{R}^m \to \mathbb{R}^m$. We hereby recall all notation and all stability notions associated with this system, which were defined and discussed in Chapter 2. For example, for $\Sigma \in \mathbb{L}$ and $f : \mathbb{R}^m \to \mathbb{R}^m$, we denote the behaviour of (2.17) by $\mathcal{B}$ and say that (2.17) is *input-to-state stable (ISS)*, if there exist $\psi \in \mathcal{KL}$ and $\phi \in \mathcal{K}$ such that

$$\|x(t)\| \le \psi(\|x(0)\|, t) + \phi\left(\max_{s \in \underline{t-1}} \left\| \begin{pmatrix} v(s) \\ w(s) \end{pmatrix} \right\| \right) \quad \forall\, t \in \mathbb{N}, \forall\, (v, w, x, y) \in \mathcal{B}. \tag{4.35}$$

Since a thorough presentation of results regarding (2.17) was given in Section 2.1, we shall not repeat ourselves, and instead refer the reader there for more details. What we shall mention here, however, is the generality of (2.17) when compared to the literature. Frequently, B_e and D_e are assumed to equal B and D, respectively, and w is assumed to be the zero function. Additionally, more often than not, the feedthrough matrix D is taken to be the zero matrix. Interestingly, Example 2.2.7 presented a version of (2.17) which is not ISS, but if we only consider the situation wherein $w = 0$, then, in that case, there exist $\psi \in \mathcal{KL}$ and $\phi \in \mathcal{K}$ such that

$$\|x(t)\| \le \psi(\|x(0)\|, t) + \phi\left(\max_{s \in \underline{t-1}} \|v(s)\|\right) \quad \forall\, t \in \mathbb{N}, \forall\, (v, 0, x, y) \in \mathcal{B}. \tag{4.36}$$

This justifies our investigation of ISS for the general system (2.17).

We conclude this initial subsection with the following key result.

Lemma 4.2.1. *Let $\Sigma \in \mathbb{L}$ and $P \in \mathbb{R}^{n \times n}$ be such that $P = P^T$. Then $U : \mathbb{R}^n \to \mathbb{R}$, defined by $U(\xi) := \langle \xi, P\xi \rangle$ for all $\xi \in \mathbb{R}^n$, satisfies, for all $t \in \mathbb{Z}_+$ and all $(u, v, x, y) \in \mathcal{B}^{lin}$,*

$$\begin{aligned} U(x(t+1)) - U(x(t)) = &\left\langle \begin{pmatrix} A^TPA - P & A^TPB - C^T \\ B^TPA - C & B^TPB - (D + D^T) \end{pmatrix} \begin{pmatrix} x(t) \\ u(t) \end{pmatrix}, \begin{pmatrix} x(t) \\ u(t) \end{pmatrix} \right\rangle \\ &+ 2\langle Cx(t) + Du(t), u(t) \rangle + 2\langle x(t), A^TPB_\mathrm{e}v(t) \rangle \\ &+ 2\langle Bu(t), PB_\mathrm{e}v(t) \rangle + \langle B_\mathrm{e}v(t), PB_\mathrm{e}v(t) \rangle \end{aligned} \tag{4.37}$$

Proof. Let $(u, v, x, y) \in \mathcal{B}^{\mathrm{lin}}$ and $t \in \mathbb{Z}_+$. We then have

$$\begin{aligned} U(x(t+1)) - U(x(t)) = &\langle Ax(t) + Bu(t) + B_\mathrm{e}v(t), P(Ax(t) + Bu(t) + B_\mathrm{e}v(t)) \rangle \\ &- \langle x(t), Px(t) \rangle, \end{aligned}$$

which in turn implies that

$$\begin{aligned}U(x(t+1)) - U(x(t)) = {}& \langle x(t), (A^T P A - P)x(t)\rangle + 2\langle x(t), A^T P B u(t)\rangle \\ &+ 2\langle x(t), A^T P B_{\mathrm{e}} v(t)\rangle + \langle u(t), B^T P B u(t)\rangle \\ &+ 2\langle B u(t), P B_{\mathrm{e}} v(t)\rangle + \langle B_{\mathrm{e}} v(t), P B_{\mathrm{e}} v(t)\rangle.\end{aligned}$$

Furthermore, an application of Lemma 4.1.9 yields

$$\begin{aligned}&\left\langle \begin{pmatrix} A^T P A - P & A^T P B - C^T \\ B^T P A - C & B^T P B - (D + D^T)\end{pmatrix} \begin{pmatrix} x(t) \\ u(t)\end{pmatrix}, \begin{pmatrix} x(t) \\ u(t)\end{pmatrix}\right\rangle \\ &\quad = \langle (A^T P A - P)x(t), x(t)\rangle + \langle (A^T P B - C^T)u(t), x(t)\rangle \\ &\qquad + \langle (B^T P A - C)x(t), u(t)\rangle + \langle (B^T P B - (D + D^T))u(t), u(t)\rangle.\end{aligned}$$

By combining the previous two identities, it is easy to see that (4.37) holds. Since (u, v, x, y) and t are arbitrary, the proof is thus complete. □

4.2.2 Input-to-state stability

We now present three stability theorems. The first gives sufficient conditions for when (2.17), with B_{e} and D_{e} replaced by B and D respectively, satisfies (4.36) for some $\psi \in \mathcal{KL}$ and $\phi \in \mathcal{K}$. This result may be percieved to be a discrete-time analogue of [7, Theorem 1]. The second and third results give hypotheses that guarantee that (2.17) (without the previous restrictions on B_{e} and D_{e}) is ISS by using slightly stronger assumptions. Interestingly, this differs from the work of Section 2.2.2, where it is shown that the same assumptions used in [108, Theorem 13] also yield ISS for the general system (2.17). We shall discuss the aforementioned stronger hypotheses after giving the first result, which we do so now.

Theorem 4.2.2. *Let $\Sigma \in \mathbb{L}$ be such that $B_{\mathrm{e}} = B$ and $D_{\mathrm{e}} = D$, and let $f : \mathbb{R}^m \to \mathbb{R}^m$. Assume that Σ is detectable and that there exists a positive semi-definite $P = P^T \in \mathbb{R}^{n \times n}$ such that*

$$\begin{pmatrix} A^T P A - P & A^T P B - C^T \\ B^T P A - C & B^T P B - (D + D^T)\end{pmatrix} \leq 0. \tag{4.38}$$

Furthermore, assume that f is continuous on a neighbourhood containing 0, f is bounded on bounded sets and that there exist $\alpha \in \mathcal{K}_\infty$, $c > 0$ and $\mu > 0$ such that

$$\|\xi\|\alpha(\|\xi\|) \leq -\langle \xi, f(\xi)\rangle \quad \forall\, \xi \in \mathbb{R}^m, \tag{4.39}$$

and

$$\|f(\xi)\| \leq -c\langle \xi, f(\xi)\rangle \quad \forall\, \xi \in \mathbb{R}^m \text{ s.t. } \|\xi\| \geq \mu. \tag{4.40}$$

Then there exist $\psi \in \mathcal{KL}$ and $\phi \in \mathcal{K}$ such that

$$\|x(t)\| \leq \psi(\|x(0)\|, t) + \phi\left(\max_{s \in \underline{t-1}} \|v(s)\|\right) \quad \forall\, t \in \mathbb{N},\ \forall\, (v, 0, x, y) \in \mathcal{B}. \tag{4.41}$$

Before proving Theorem 4.2.2, we provide some commentary in the form of a remark, and two examples.

Remark 4.2.3. (i) Theorem 4.2.2 is a discrete-time generalisation of [7, Theorem 1], but with weaker assumptions. Indeed, the first improvement concerns the linear matrix inequality (LMI) given by (4.38). In [7, Theorem 1], it is assumed that $A^T P + PA \leq 0$, $PB = C^T$ and $D = 0$. The discrete-time analogue to these assumptions, would be that $A^T PA - P \leq 0$, $A^T PB = C^T$ and $B^T PB = (D + D^T)$. We claim that the LMI is a weaker assumption than this and refer the reader to Example 4.2.4, situated after the current remark, which gives a situation where the LMI (4.38) holds for some $P \geq 0$, but that $A^T PB \neq C^T$ for all $P \geq 0$. The second improvement in the hypotheses, is that [7, Theorem 1] assumes the existence of $\mu > 0$ such that (4.40) holds for $c = 1$. Example 4.2.5, once again located after the present remark, gives a function f that satisfies (4.40) with $\mu = 1$ and $c = 2$, but which does not satisfy (4.40) for any $\mu > 0$ in the situation that $c = 1$.

(ii) The assumptions imposed on the nonlinearity f in Theorem 4.2.2 do not restrict the function to be linearly bounded, as in Chapter 2 (see also [107] and [108]). Therefore, we see that f may be 'superlinear', and indeed should be thought of as such, since this is the most interesting case. Moreover, Lemma 4.1.10 guarantees that, under the assumptions of Theorem 4.2.2, $f(0) = 0$. An archetypal example of a function satisfying the hypotheses of Theorem 4.2.2 is $f : \mathbb{R} \to \mathbb{R}$, $f(\xi) := -\xi^3$. For further (multi-dimensional) examples, a collection is given after the proof of Theorem 4.2.2.

(iii) In the single-input single-output case where $m = 1$, so long as (4.39) holds, then (4.40) is redundant, since the inequality trivially holds for $\mu = 1 = c$. In the multi-input multi-output setting where $m > 1$, this is not true. Indeed, a counterexample is given in [7, Remark 2].

(iv) We highlight that in the forthcoming proof of Theorem 4.2.2, we use repeatedly that f can be bounded above by a $\mathcal{K}_\infty$ function. If the assumption of boundedness of f on bounded sets is dropped, then this cannot be guaranteed. Indeed, consider g given in Example 4.1.4 and set $f = -g$. It is clear that $\alpha \in \mathcal{K}_\infty$ defined by $\alpha(s) := s$ for all $s \geq 0$, is such that (4.39) holds, however, there does not exist such a $\mathcal{K}_\infty$ function bounding f from above. ◊

Example 4.2.4. Let $\Sigma \in \mathbb{L}$ where $n = 2$, $m = 1$,

$$A := \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{pmatrix}, \quad B := \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad C := \begin{pmatrix} 9 & 1 \end{pmatrix}, \quad D := \frac{13}{2},$$

and B_e and D_e are arbitrary. By defining

$$P := \begin{pmatrix} 12 & 3 \\ 3 & 1 \end{pmatrix},$$

it is easily checked that P is positive definite and (4.38) holds. However, we claim that there does not exist a semi-definite $P = P^T \in \mathbb{R}^{n \times n}$ such that $A^T PB = C^T$. Indeed, if so, then writing

$$P := \begin{pmatrix} p_1 & p_2 \\ p_2 & p_4 \end{pmatrix},$$

where $p_1, p_2, p_4 \in \mathbb{R}$, we see that

$$0 = A^T PB - C^T = \begin{pmatrix} \frac{1}{2}p_1 - 9 \\ -1 \end{pmatrix},$$

hence yielding a contradiction. ◊

Example 4.2.5. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$f(\xi) := \begin{cases} -\|\xi\| \begin{pmatrix} \frac{1}{2\|\xi\|} & -\sin\left(\cos^{-1}\left(\frac{1}{2\|\xi\|}\right)\right) \\ \sin\left(\cos^{-1}\left(\frac{1}{2\|\xi\|}\right)\right) & \frac{1}{2\|\xi\|} \end{pmatrix} \xi, & \text{if } \|\xi\| \geq 1, \\ -\|\xi\| \begin{pmatrix} \frac{1}{2} & -\sin\left(\cos^{-1}\left(\frac{1}{2}\right)\right) \\ \sin\left(\cos^{-1}\left(\frac{1}{2}\right)\right) & \frac{1}{2} \end{pmatrix} \xi, & \text{if } \|\xi\| < 1. \end{cases}$$

We comment that f can be thought of as a linear extension of a rotation matrix which, for each $\xi \in \mathbb{R}^2$ with $\|\xi\| \geq 1$, rotates ξ at the variable angle of $\cos^{-1}(1/2\|\xi\|)$. In the sequel, for $\xi \in \mathbb{R}^2$, we shall write $\xi = \begin{pmatrix} \xi_1 & \xi_2 \end{pmatrix}^T$ where $\xi_1, \xi_2 \in \mathbb{R}$. Consider that, for all $\xi \in \mathbb{R}^2$ with $\|\xi\| \geq 1$,

$$\begin{aligned} -\langle \xi, f(\xi) \rangle &= \|\xi\| \left(\frac{1}{2\|\xi\|}\xi_1^2 - \sin\left(\cos^{-1}\left(\frac{1}{2\|\xi\|}\right)\right)\xi_1\xi_2 \right. \\ &\qquad \left. + \sin\left(\cos^{-1}\left(\frac{1}{2\|\xi\|}\right)\right)\xi_1\xi_2 + \frac{1}{2\|\xi\|}\xi_2^2 \right) \\ &= \frac{1}{2}\|\xi\|^2. \end{aligned} \tag{4.42}$$

In a similar fashion, it can also be shown that, for all $\xi \in \mathbb{R}^2$ with $\|\xi\| < 1$,

$$-\langle \xi, f(\xi) \rangle = \frac{1}{2}\|\xi\|^3.$$

Hence, by defining $\alpha \in \mathcal{K}_\infty$ by $\alpha(s) := \min\{s, s^2\}/2$ for all $s \geq 0$, we see that (4.39) holds. In addition to this, note that, for all $\xi \in \mathbb{R}^2$ with $\|\xi\| \geq 1$,

$$\begin{aligned} \|f(\xi)\|^2 = \|\xi\|^2 &\left(\frac{1}{4\|\xi\|^2}\xi_1^2 - \frac{1}{\|\xi\|}\sin\left(\cos^{-1}\left(\frac{1}{2\|\xi\|}\right)\right)\xi_1\xi_2 + \sin^2\left(\cos^{-1}\left(\frac{1}{2\|\xi\|}\right)\right)\xi_2^2 \right. \\ &\left. \frac{1}{4\|\xi\|^2}\xi_2^2 + \frac{1}{\|\xi\|}\sin\left(\cos^{-1}\left(\frac{1}{2\|\xi\|}\right)\right)\xi_1\xi_2 + \sin^2\left(\cos^{-1}\left(\frac{1}{2\|\xi\|}\right)\right)\xi_1^2 \right) \\ = \|\xi\|^2 &\left(\frac{1}{4} + \sin^2\left(\cos^{-1}\left(\frac{1}{2\|\xi\|}\right)\right)\|\xi\|^2 \right). \end{aligned}$$

Since $\sin^2(\theta) = 1 - \cos^2(\theta)$ for all $\theta \in \mathbb{R}$, this implies that, for all $\xi \in \mathbb{R}^2$ with $\|\xi\| \geq 1$,

$$\|f(\xi)\|^2 = \|\xi\|^2 \left(\frac{1}{4} + \left(1 - \frac{1}{4\|\xi\|^2}\right)\|\xi\|^2 \right) = \|\xi\|^4.$$

By combining this with (4.42), we hence deduce that

$$\|f(\xi)\| = -2\langle \xi, f(\xi) \rangle \quad \forall\, \xi \in \mathbb{R}^2 \text{ s.t. } \|\xi\| \geq 1.$$

Therefore, (4.40) holds with $c = 2$ and $\mu = 1$. Furthermore, we also deduce that (4.40) does not hold for any $\mu > 0$ in the situation that $c = 1$, hence highlighting the differences of the assumptions of Theorem 4.2.2 and [7, Theorem 1]. ◊

We shall now prove Theorem 4.2.2, but first we make note that the idea for the method of the proof is obtained from the proof of [7, Theorem 1]. Moreover, some of the arguments we use take inspiration from some ideas seen in [108, Theorem 13].

Proof of Theorem 4.2.2. We seek $U, V : \mathbb{R}^n \to \mathbb{R}_+$ and $\gamma_1, \gamma_2 \in \mathcal{K}_\infty$ such that

$$(U+V)(x(t+1)) - (U+V)(x(t)) \leq -\gamma_1(\|x(t)\|) + \gamma_2(\|v(t)\|) \quad \forall\, t \in \mathbb{Z}_+,\ \forall\,(v,0,x,y) \in \mathcal{B}, \tag{4.43}$$

and constants $\lambda_1, \lambda_2 > 0$ such that

$$\lambda_1 \|\xi\|^2 \leq (U+V)(\xi) \leq \lambda_2 \|\xi\|^2 \quad \forall\, \xi \in \mathbb{R}^n. \tag{4.44}$$

By following an identical argument to the proof of Proposition 2.2.9 situated in Appendix A, we see that the previous is sufficient for the existence of $\psi \in \mathcal{KL}$ and $\phi \in \mathcal{K}$ such that (4.41) holds.
We begin by defining $U : \mathbb{R}^n \to \mathbb{R}_+$ by $U(\xi) := \langle \xi, P\xi \rangle$ for all $\xi \in \mathbb{R}^n$. Since $B_{\mathrm{e}} = B$ and $D_{\mathrm{e}} = D$, we see that $(f(y)+v, 0, x, y) \in \mathcal{B}^{\mathrm{lin}}$ for all $(v,0,x,y) \in \mathcal{B}$. Therefore, we invoke Lemma 4.2.1 to yield, for all $(v,0,x,y) \in \mathcal{B}$ and all $t \in \mathbb{Z}_+$,

$$\begin{aligned} &U(x(t+1)) - U(x(t)) \\ &\quad = \left\langle \begin{pmatrix} A^T PA - P & A^T PB - C^T \\ B^T PA - C & B^T PB - (D + D^T) \end{pmatrix} \begin{pmatrix} x(t) \\ f(y(t)) + v(t) \end{pmatrix}, \begin{pmatrix} x(t) \\ f(y(t)) + v(t) \end{pmatrix} \right\rangle \\ &\qquad + 2\langle Cx(t) + Df(y(t)) + Dv(t), f(y(t)) + v(t) \rangle. \end{aligned}$$

A use of (4.38) subsequently gives

$$U(x(t+1)) - U(x(t)) \leq 2\langle y(t), f(y(t)) \rangle + 2\langle y(t), v(t) \rangle \quad \forall\, t \in \mathbb{Z}_+,\ \forall\,(v,0,x,y) \in \mathcal{B}. \tag{4.45}$$

We now claim that

$$2\langle y(t), v(t) \rangle \leq -\langle y(t), f(y(t)) \rangle + 2\alpha^{-1}(2\|v(t)\|)\|v(t)\| \quad \forall\, t \in \mathbb{Z}_+,\ \forall\,(v,0,x,y) \in \mathcal{B}. \tag{4.46}$$

Indeed to see this, fix $t \in \mathbb{Z}_+$ and $(v,0,x,y) \in \mathcal{B}$, and consider the following two cases. If $\|v(t)\| \geq \frac{1}{2}\alpha(\|y(t)\|)$, then

$$2\langle y(t), v(t) \rangle \leq 2\|y(t)\|\|v(t)\| \leq 2\alpha^{-1}(2\|v(t)\|)\|v(t)\|, \tag{4.47}$$

and if $\|v(t)\| \leq \frac{1}{2}\alpha(\|y(t)\|)$, then an application of (4.39) provides

$$2\langle y(t), v(t) \rangle \leq 2\|y(t)\|\|v(t)\| \leq \|y(t)\|\alpha(\|y(t)\|) \leq -\langle y(t), f(y(t)) \rangle. \tag{4.48}$$

Therefore, by combining (4.47) with (4.48), we see that the (4.46) holds. Continuing with (4.45), we thus have that

$$U(x(t+1)) - U(x(t)) \leq \langle y(t), f(y(t)) \rangle + 2\alpha^{-1}(2\|v(t)\|)\|v(t)\| \quad \forall\, t \in \mathbb{Z}_+,\ \forall\,(v,0,x,y) \in \mathcal{B}. \tag{4.49}$$

We now construct a function $V : \mathbb{R}^n \to \mathbb{R}_+$ such that $U + V$ satisfies the properties discussed at the beginning of the proof. To this end, note that since Σ is detectable, we may apply Lemma 2.1.29 to yield the existence of a positive definite matrix $\tilde{P} = \tilde{P}^T \in \mathbb{R}^{n\times n}$ and $\delta_1 > 0$ such that $V_1 : \mathbb{R}^n \to \mathbb{R}_+$, defined by $V_1(\xi) := \langle \xi, \tilde{P}\xi\rangle$ for all $\xi \in \mathbb{R}^n$, has the property that (2.14) holds for all $(u, v, x, y) \in \mathcal{B}^{\mathrm{lin}}$ and all $t \in \mathbb{Z}_+$, with V and δ replaced by V_1 and δ_1, respectively. In particular, for all $(v, 0, x, y) \in \mathcal{B}$ and all $t \in \mathbb{Z}_+$, again since $(f(y) + v, 0, x, y) \in \mathcal{B}^{\mathrm{lin}}$, (2.14) gives that

$$V_1(x(t+1)) - V_1(x(t)) \leq -\delta_1\|x(t)\|^2 + \|f(y(t)) + v(t)\|^2 + \|y(t)\|^2. \tag{4.50}$$

By noting that, for all $(v, 0, x, y) \in \mathcal{B}$ and all $t \in \mathbb{Z}_+$,

$$\begin{aligned}\|f(y(t)) + v(t)\|^2 &\leq \|f(y(t))\|^2 + \|v(t)\|^2 + 2\|f(y(t))\|\|v(t)\| \\ &\leq 4\|f(y(t))\|^2 + 4\|v(t)\|^2,\end{aligned}$$

we set $\delta := \delta_1/4$ and define $V_2 : \mathbb{R}^n \to \mathbb{R}_+$ by $V_2(\xi) = V_1(\xi)/4$ for all $\xi \in \mathbb{R}^n$. We then have, from (4.50), that

$$V_2(x(t+1)) - V_2(x(t)) \leq -\delta\|x(t)\|^2 + \|f(y(t))\|^2 + \|v(t)\|^2 + \|y(t)\|^2 \quad \forall\, t \in \mathbb{Z}_+,\ \forall\,(v, 0, x, y) \in \mathcal{B}. \tag{4.51}$$

We pause here for a moment, and invoke Lemma 4.1.11 to obtain the existence of $\varepsilon > 0$ and $\eta \in \mathcal{K}_\infty$ such that (4.15) and (4.16) hold. We also define $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$\rho(s) := \min\left\{\varepsilon\sqrt{\frac{s}{4}},\, \eta\left(\sqrt{\frac{s}{4}}\right)\frac{s}{4}\right\} \quad \forall\, s \geq 0,$$

and note that Lemma 2.1.2 gives that $\rho \in \mathcal{K}_\infty$. Let $a_1 \geq \delta$ be such that $V_2(\xi) \leq a_1\|\xi\|^2$ for all $\xi \in \mathbb{R}^n$, and, furthermore, choose $a_2 > 1$ such that

$$a_3 := a_2\left(1 - \frac{\delta}{2a_1}\right) < 1.$$

Statement (ii) of Lemma 2.1.7 yields the existence of $a_4 > 1$ such that

$$\rho(a_3 s_1 + s_2) \leq \rho(s_1) + \rho(a_4 s_2) \quad \forall\, s_1, s_2 \geq 0. \tag{4.52}$$

Moreover, Lemma 2.1.4 combined with Lemma 2.1.5 yields that

$$\lim_{s\to\infty}(\rho(a_2 s) - \rho(s)) = \infty.$$

By utilising this along with Lemma 2.1.8, we yield the existence of $\tilde{\gamma}_1 \in \mathcal{K}_\infty$ such that

$$\rho(s_1 - s_2) \leq \rho(a_2 s_1) - \tilde{\gamma}_1(s_2) \quad \forall\, s_1 \geq s_2 \geq 0. \tag{4.53}$$

Continuing with (4.51), let $a := 1/(a_2 a_4)$ and define $V(\xi) := \rho(aV_2(\xi))$ for all $\xi \in \mathbb{R}^n$, so that, for all $(v, 0, x, y) \in \mathcal{B}$ and all $t \in \mathbb{Z}_+$,

$$V(x(t+1)) \leq \rho(a(V_2(x(t)) - \delta\|x(t)\|^2 + \|f(y(t))\|^2 + \|v(t)\|^2 + \|y(t)\|^2)).$$

Combining this with (4.53) then gives that, for all $(v, 0, x, y) \in \mathcal{B}$ and all $t \in \mathbb{Z}_+$,

$$V(x(t+1)) \leq \rho\left(aa_2\left(V_2(x(t)) - \frac{\delta}{2}\|x(t)\|^2 + \|f(y(t))\|^2 + \|v(t)\|^2 + \|y(t)\|^2\right)\right) - \tilde{\gamma}_1\left(a\frac{\delta}{2}\|x(t)\|^2\right). \tag{4.54}$$

Furthermore, since $V_2(\xi) \leq a_1\|\xi\|^2$ for all $\xi \in \mathbb{R}^n$, we see that

$$a_2\left(V_2(\xi) - \frac{\delta}{2}\|\xi\|^2\right) \leq a_2\left(1 - \frac{\delta}{2a_1}\right)V_2(\xi) = a_3V_2(\xi) \quad \forall\, \xi \in \mathbb{R}^n,$$

and thus, by substituting this into (4.54), we obtain that, for all $(v, 0, x, y) \in \mathcal{B}$ and all $t \in \mathbb{Z}_+$,

$$V(x(t+1)) \leq \rho\left(aa_3V_2(x(t)) + aa_2\left(\|f(y(t))\|^2 + \|v(t)\|^2 + \|y(t)\|^2\right)\right) - \tilde{\gamma}_1\left(a\frac{\delta}{2}\|x(t)\|^2\right).$$

We now invoke (4.52) and recall that $1 = aa_2a_4$ so that the above inequality implies, for all $(v, 0, x, y) \in \mathcal{B}$ and all $t \in \mathbb{Z}_+$,

$$V(x(t+1)) \leq V(x(t)) + \rho\left(\|f(y(t))\|^2 + \|v(t)\|^2 + \|y(t)\|^2\right) - \tilde{\gamma}_1\left(a\frac{\delta}{2}\|x(t)\|^2\right). \tag{4.55}$$

For $(v, 0, x, y) \in \mathcal{B}$ and $t \in \mathbb{Z}_+$, if $\|y(t)\| \geq \mu$, then (4.15) gives that

$$\rho(4\|y(t)\|^2) + \rho(4\|f(y(t))\|^2) \leq \varepsilon(\|y(t)\| + \|f(y(t))\|) \leq -\langle y(t), f(y(t))\rangle,$$

and if $\|y(t)\| \leq \mu$, then (4.16) yields

$$\begin{aligned}\rho(4\|y(t)\|^2) + \rho(4\|f(y(t))\|^2) &\leq \eta(\|y(t)\|)\|y(t)\|^2 + \eta(\|f(y(t))\|)\|f(y(t))\|^2 \\ &\leq -\langle y(t), f(y(t))\rangle.\end{aligned}$$

By combining these inequalities with (4.55), and also by recalling Lemma 2.1.3, we see that, for all $(v, 0, x, y) \in \mathcal{B}$ and all $t \in \mathbb{Z}_+$,

$$\begin{aligned}V(x(t+1)) - V(x(t)) &\leq -\tilde{\gamma}_1\left(a\frac{\delta}{2}\|x(t)\|^2\right) + \rho(4\|f(y(t))\|^2) + \rho(4\|y(t)\|^2) \\ &\quad + \rho(2\|v(t)\|^2) \\ &\leq -\tilde{\gamma}_1\left(a\frac{\delta}{2}\|x(t)\|^2\right) + \rho(2\|v(t)\|^2) - \langle y(t), f(y(t))\rangle.\end{aligned}$$

Hence, we define $\gamma_1 \in \mathcal{K}_\infty$ by $\gamma_1(s) := \tilde{\gamma}_1(a\frac{\delta}{2}s^2)$ for all $s \geq 0$, so that

$$V(x(t+1)) - V(x(t)) \leq -\gamma_1(\|x(t)\|) + \rho(2\|v(t)\|^2) - \langle y(t), f(y(t))\rangle \quad \forall\, t \in \mathbb{Z}_+, \forall\,(v, 0, x, y) \in \mathcal{B}. \tag{4.56}$$

To conclude the proof, by considering (4.49) and (4.56), we see that (4.43) holds, where $\gamma_2 \in \mathcal{K}_\infty$ is defined by $\gamma_2(s) := 2\alpha^{-1}(2s)s + \rho(2s^2)$ for all $s \geq 0$. Moreover, since $\tilde{P}$ is positive definite, it is clear that there exist $\lambda_1, \lambda_2 > 0$ such that (4.43) holds. Therefore, as mentioned at the start of this proof, these two properties of $U + V$ are sufficient to imply the existence of $\psi \in \mathcal{KL}$ and $\phi \in \mathcal{K}$ such that (4.41) holds. □

The subsequent example gives two classes of functions that satisfy the assumptions of Theorem 4.2.2 that are relevant to the nonlinearity f.

Example 4.2.6. Let $f : \mathbb{R}^m \to \mathbb{R}^m$ to be defined and let $k \in \mathbb{R}_+$ be arbitrary.

(i) If we define f by

$$f(\xi) := -\|\xi\|^k \xi \quad \forall \xi \in \mathbb{R}^m,$$

then it is clear that f satisfies (4.39) and (4.40) with $\alpha(s) := s^{k+1}$ for all $s \geq 0$, and $c = \mu = 1$, respectively.

(ii) Let $m = 2$. We define f by

$$f(\xi) := -\|\xi\|^k \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \xi \quad \forall \xi \in \mathbb{R}^2.$$

It is easy to conclude that

$$-\langle \xi, f(\xi) \rangle = \|\xi\|^k \left\langle \xi, \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \xi \right\rangle = \|\xi\|^{k+2} \quad \forall \xi \in \mathbb{R}^2,$$

and that

$$\|f(\xi)\| = \sqrt{(\xi_1 - \xi_2)^2 + (\xi_1 + \xi_2)^2} \|\xi\|^k = \sqrt{2} \|\xi\|^{k+1} \quad \forall \xi = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \in \mathbb{R}^2.$$

Therefore, f satisfies (4.39) and (4.40) with $\alpha(s) := s^{k+1}$ for all $s \geq 0$, and $c = 1$ and $\mu = \sqrt{2}$, respectively. ◊

In the following, we present two examples of (2.17) that satisfy the assumptions of Theorem 4.2.2.

Example 4.2.7. Consider (2.17) where $n = 2$, $m = 1$,

$$A := \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{pmatrix}, \quad B := \begin{pmatrix} 1 \\ 0 \end{pmatrix} =: B_{\mathrm{e}}, \quad C := \begin{pmatrix} 9 & 1 \end{pmatrix}, \quad D := \frac{13}{2} =: D_{\mathrm{e}},$$

and

$$f(\xi) = \operatorname{sign}(\xi)(1 - e^{|\xi|}) \quad \forall \xi \in \mathbb{R}.$$

As in Example 4.2.4, it is easily checked that

$$P := \begin{pmatrix} 12 & 3 \\ 3 & 1 \end{pmatrix}$$

is a positive semi-definite matrix such that (4.38) holds. Moreover,

$$-\xi f(\xi) = |\xi|(e^{|\xi|} - 1) \geq |\xi|^2 = |\xi| \alpha(|\xi|),$$

where $\alpha \in \mathcal{K}_\infty$ is defined by $\alpha(s) := s$ for all $s \geq 0$. Therefore, (4.39) is satisfied and so Theorem 4.2.2 yields that there exist $\psi \in \mathcal{KL}$ and $\phi \in \mathcal{K}$ such that (4.41) holds. (Here, we have used that (4.39) implies that there exist $c, \mu > 0$ such that (4.40) holds if $m = 1$ (see Remark 4.2.3).) ◊

Example 4.2.8. Consider (2.17) with $n = m = 2$, $A = I/2$, where I is the identity matrix,

$$B = B_{\mathrm{e}} = C = D = D_{\mathrm{e}} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

and

$$f(\xi) = -\|\xi\|^2 \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \xi \quad \forall\, \xi \in \mathbb{R}^2.$$

We see from Example 4.2.6 that f satisfies (4.39) and (4.40) for some $\alpha \in \mathcal{K}_\infty$ and $c, \mu > 0$. We shall now show that there exists a positive semi-definite matrix $P \in \mathbb{R}^{2\times 2}$ such that (4.38) holds. To this end, we define $P = 2I$. It can easily be shown that $A^T PA - P = -3I/2$ and $A^T PB - C^T = 0 = B^T PB - (D + D^T)$. Therefore, (4.38) is satisfied and so, from Theorem 4.2.2, we obtain that there exist $\psi \in \mathcal{KL}$ and $\phi \in \mathcal{K}$ such that (4.41) holds. ◊

We now focus our attention on the case where B and D need not equal B_{e} and D_{e} respectively, and where w need not be 0. To this end, the next result is the second theorem that we mentioned at the start of this subsection. The result shows that if we impose the assumptions of Theorem 4.2.2, but additionally assume an extra condition on the nonlinearity, then (2.17) is ISS.

Theorem 4.2.9. *Let $\Sigma \in \mathbb{L}$ and $f : \mathbb{R}^m \to \mathbb{R}^m$. Assume that Σ is detectable and that there exists a positive semi-definite $P = P^T \in \mathbb{R}^{n\times n}$ such that* (4.38) *holds. Furthermore, assume that f is continuous, there exists $\alpha \in \mathcal{K}_\infty$ such that* (4.39) *holds and*

$$-\frac{\langle \xi, f(\xi)\rangle}{\|f(\xi)\|} \to \infty \text{ as } \|\xi\| \to \infty. \tag{4.57}$$

Then (2.17) *is ISS.*

Before giving the proof of Theorem 4.2.9, we first give some commentary and an example which presents a suite of functions that satisfy (4.57).

Remark 4.2.10. (i) Apart from the extra continuity assumption, the statements of Theorem 4.2.2 and Theorem 4.2.9 are identical with the exception of the assumptions (4.40) and (4.57). We claim that if $f : \mathbb{R}^m \to \mathbb{R}^m$ is continuous and there exists $\alpha \in \mathcal{K}_\infty$ such that (4.39) holds, then (4.57) implies (4.40). Indeed, to see this, note that Lemma 4.1.7 and Lemma 4.1.10 together guarantee the existence of $\gamma \in \mathcal{K}_\infty$ such that

$$\gamma(\|\xi\|) \le -\frac{\langle \xi, f(\xi)\rangle}{\|f(\xi)\|} \quad \forall\, \xi \in \mathbb{R}^m\backslash\{0\}.$$

From this, we see that there exists $\mu > 0$ such that (4.40) holds with $c = 1$. We note, from Example 4.2.5, that (4.39) and (4.40) holding for some $\alpha \in \mathcal{K}_\infty$, $c > 0$ and $\mu > 0$, does not imply (4.57).

(ii) In the case that $m = 1$, (4.39) implies (4.57), and is hence redundant in this case. We therefore conclude that in the single-input single-output setting, the statements of Theorem 4.2.2 and Theorem 4.2.9 are identical. Hence, we now learn that the assumptions of Theorem 4.2.2, in the case that $m = 1$, in fact guarantee ISS of (2.17) (without any extra assumption). ◊

Example 4.2.11. Let $f : \mathbb{R}^m \to \mathbb{R}^m$ be defined by

$$f(\xi) := -g(\xi)M\xi \quad \forall\, \xi \in \mathbb{R}^m,$$

where $g : \mathbb{R}^m \to \mathbb{R}_+$ and $M \in \mathbb{R}^{m\times m}$ are such that $g(\xi) > 0$ for all $\xi \in \mathbb{R}^m\backslash\{0\}$ and

$$M + M^T > 0.$$

Let $\lambda := \min\{\sigma(M + M^T)\} > 0$. We note that, for all $\xi \in \mathbb{R}^m\backslash\{0\}$,

$$-\frac{\langle \xi, f(\xi)\rangle}{\|f(\xi)\|} = g(\xi)\frac{\langle \xi, M\xi\rangle}{\|g(\xi)M\xi\|} = \frac{\langle \xi, (M + M^T)\xi\rangle}{2\|M\xi\|} \geq \frac{\lambda\|\xi\|^2}{2\|M\xi\|} \geq \frac{\lambda}{2\|M\|}\|\xi\|.$$

Therefore, f satisfies (4.57). For a specific example of such a function f, we refer the reader to statement (ii) of Example 4.2.6. ◊

We now present the proof of Theorem 4.2.9. As with the previous proof, we note that the following is partly inspired by the proof of [108, Theorem 13].

Proof of Theorem 4.2.9. We begin the proof by applying Lemmas 4.1.7 and 4.1.10 to guarantee the existence of $\tilde{\alpha} \in \mathcal{K}_\infty$ such that

$$\tilde{\alpha}(\|\xi\|) \leq -\frac{\langle \xi, f(\xi)\rangle}{\|f(\xi)\|} \quad \forall\, \xi \in \mathbb{R}^m\backslash\{0\}. \tag{4.58}$$

Furthermore, we combine Lemma 4.1.10 and Corollary 4.1.3 to obtain $\beta \in \mathcal{K}_\infty$ such that $\|f(\xi)\| \leq \beta(\|\xi\|)$ for all $\xi \in \mathbb{R}^m$. Define $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$\rho(s) := \min\left\{\frac{1}{3}\sqrt{\frac{s}{16}}\alpha\left(\sqrt{\frac{s}{16}}\right), \sqrt{\frac{s}{16}}\tilde{\alpha}\left(\beta^{-1}\left(\sqrt{\frac{s}{4}}\right)\right)\right\},$$

and note that, from Lemma 2.1.2, $\rho \in \mathcal{K}_\infty$. Now, via the detectability of Σ, an application of Lemma 2.1.29 yields the existence of a positive definite matrix $\tilde{P} = \tilde{P}^T \in \mathbb{R}^{n\times n}$ and $\delta > 0$ such that $V_1 : \mathbb{R}^n \to \mathbb{R}_+$, defined by $V_1(\xi) := \langle \xi, \tilde{P}\xi\rangle$ for all $\xi \in \mathbb{R}^n$, has the property that, for all $(v, w, x, y) \in \mathcal{B}$ and all $t \in \mathbb{Z}_+$,

$$\begin{aligned} V_1(x(t + 1)) - V_1(x(t)) \leq& -\delta\|x(t)\|^2 + \|f(y(t) + w(t))\|^2 + \left\|\begin{pmatrix} B_{\mathrm{e}}v(t) \\ D_{\mathrm{e}}v(t)\end{pmatrix}\right\|^2 \\ &+ \|y(t)\|^2. \end{aligned} \tag{4.59}$$

We let $a_1 \geq \delta$ be such that $V_1(\xi) \leq a_1\|\xi\|^2$ for all $\xi \in \mathbb{R}^n$, and, furthermore, choose $a_2 > 1$ such that

$$a_3 := a_2\left(1 - \frac{\delta}{2a_1}\right) < 1.$$

An application of statement (ii) of Lemma 2.1.7 gives that there exists $a_4 > 1$ such that

$$\rho(a_3 s_1 + s_2) \leq \rho(s_1) + \rho(a_4 s_2) \quad \forall\, s_1, s_2 \geq 0, \tag{4.60}$$

and statement (iii) of Lemma 2.1.7 yields that there exists $\tilde{\gamma}_1 \in \mathcal{K}_\infty$ such that

$$\rho(s_1 - s_2) \leq \rho(a_2 s_1) - \tilde{\gamma}_1(s_2) \quad \forall\, s_1 \geq s_2 \geq 0, \tag{4.61}$$

and

$$\frac{\tilde{\gamma}_1(s)}{\sqrt{s}} \to \infty \quad \text{as } s \to \infty. \tag{4.62}$$

Let $a := 1/(a_2 a_4)$ and define $V : \mathbb{R}^m \to \mathbb{R}_+$ by $V(\xi) := \rho(aV_1(\xi))$ for all $\xi \in \mathbb{R}^n$. Recalling (4.59), we see that, for all $(v, w, x, y) \in \mathcal{B}$ and all $t \in \mathbb{Z}_+$,

$$V(x(t+1)) \leq \rho\left(a\left(V_1(x(t)) - \delta\|x(t)\|^2 + \|f(y(t)+w(t))\|^2 + \left\|\begin{pmatrix} B_\mathrm{e}v(t) \\ D_\mathrm{e}v(t)\end{pmatrix}\right\|^2 + \|y(t)\|^2\right)\right).$$

We combine this with (4.61) to yield that, for all $(v, w, x, y) \in \mathcal{B}$ and all $t \in \mathbb{Z}_+$,

$$V(x(t+1)) \leq \rho\left(aa_2\left(V_1(x(t)) - \frac{\delta}{2}\|x(t)\|^2 + \|f(y(t)+w(t))\|^2 + \left\|\begin{pmatrix} B_\mathrm{e}v(t) \\ D_\mathrm{e}v(t)\end{pmatrix}\right\|^2 + \|y(t)\|^2\right)\right) - \tilde{\gamma}_1\left(a\frac{\delta}{2}\|x(t)\|^2\right). \tag{4.63}$$

Also, by invoking (4.62) and by applying Lemma 4.1.7, we obtain the existence of $\tilde{\gamma}_2 \in \mathcal{K}_\infty$ such that

$$\tilde{\gamma}_1\left(a\frac{\delta}{2}s^2\right) \geq s\tilde{\gamma}_2(s) \quad \forall\, s \geq 0.$$

Moreover, since $V_1(\xi) \leq a_1\|\xi\|^2$ for all $\xi \in \mathbb{R}^n$, we see that

$$a_2\left(V_1(\xi) - \frac{\delta}{2}\|\xi\|^2\right) \leq a_2\left(1 - \frac{\delta}{2a_1}\right)V_1(\xi) = a_3 V_1(\xi) \quad \forall\, \xi \in \mathbb{R}^n.$$

Thus, by combining the previous two inequalities with (4.63), we yield that, for all $(v, w, x, y) \in \mathcal{B}$ and all $t \in \mathbb{Z}_+$,

$$V(x(t+1)) \leq \rho\left(aa_3V_1(x(t)) + aa_2\left(\|f(y(t)+w(t))\|^2 + \left\|\begin{pmatrix} B_\mathrm{e}v(t) \\ D_\mathrm{e}v(t)\end{pmatrix}\right\|^2 + \|y(t)\|^2\right)\right) - \|x(t)\|\tilde{\gamma}_2\left(\|x(t)\|\right).$$

If we now use (4.60) and recall that $aa_2a_4 = 1$, we see that, for all $(v, w, x, y) \in \mathcal{B}$ and all $t \in \mathbb{Z}_+$,

$$V(x(t+1)) \leq V(x(t)) + \rho\left(\|f(y(t)+w(t))\|^2 + \left\|\begin{pmatrix} B_\mathrm{e}v(t) \\ D_\mathrm{e}v(t)\end{pmatrix}\right\|^2 + \|y(t)\|^2\right) - \|x(t)\|\tilde{\gamma}_2\left(\|x(t)\|\right). \tag{4.64}$$

We pause here and claim two inequalities. The first is that, for all $(v, w, x, y) \in \mathcal{B}$ and all $t \in \mathbb{Z}_+$,

$$\rho(4\|y(t)\|^2) \leq -\frac{1}{2}\langle y(t)+w(t), f(y(t)+w(t))\rangle + \frac{1}{2}\|w(t)\|\alpha(\|w(t)\|), \tag{4.65}$$

and the second is that, for all $(v, w, x, y) \in \mathcal{B}$ and all $t \in \mathbb{Z}_+$,

$$\rho(4\|f(y(t)+w(t))\|^2) \le -\frac{1}{2}\langle y(t)+w(t), f(y(t)+w(t))\rangle. \tag{4.66}$$

Turning our attention first to proving (4.65), fix $(v, w, x, y) \in \mathcal{B}$ and $t \in \mathbb{Z}_+$ and consider

$$\begin{aligned}\rho(4\|y(t)\|^2) &\le \frac{1}{6}\|y(t)\|\alpha\left(\frac{\|y(t)\|}{2}\right)\\ &\le \frac{1}{6}(\|y(t)+w(t)\|+\|w(t)\|)\alpha\left(\frac{\|y(t)+w(t)\|+\|w(t)\|}{2}\right)\\ &\le \frac{1}{6}(\|y(t)+w(t)\|+\|w(t)\|)(\alpha(\|y(t)+w(t)\|)+\alpha(\|w(t)\|)),\end{aligned} \tag{4.67}$$

where we have used Lemma 2.1.3. By considering cases of when $\|w(t)\|$ is greater and less than $\|y(t)+w(t)\|$, we see that

$$\begin{aligned}&\|y(t)+w(t)\|\alpha(\|w(t)\|)+\|w(t)\|\alpha(\|y(t)+w(t)\|)\\ &\qquad\le 2\|y(t)+w(t)\|\alpha(\|y(t)+w(t)\|)+2\|w(t)\|\alpha(\|w(t)\|),\end{aligned}$$

which, when combined with (4.67) and (4.39), yields that

$$\begin{aligned}\rho(4\|y(t)\|^2) &\le \frac{1}{2}\|y(t)+w(t)\|\alpha(\|y(t)+w(t)\|)+\frac{1}{2}\|w(t)\|\alpha(\|w(t)\|)\\ &\le -\frac{1}{2}\langle y(t)+w(t), f(y(t)+w(t))\rangle+\frac{1}{2}\|w(t)\|\alpha(\|w(t)\|),\end{aligned}$$

which is precisely (4.65). We now move onto showing (4.66). To this end, we again fix $(v, w, x, y) \in \mathcal{B}$ and $t \in \mathbb{Z}_+$ and note that

$$\rho(4\|f(y(t)+w(t))\|^2) \le \frac{1}{2}\tilde{\alpha}(\beta^{-1}(\|f(y(t)+w(t))\|))\|f(y(t)+w(t))\|.$$

By using (4.58) and the fact that $\|f(\xi)\| \le \beta(\|\xi\|)$ for all $\xi \in \mathbb{R}^m$, this becomes

$$\begin{aligned}\rho(4\|f(y(t)+w(t))\|^2) &\le \frac{1}{2}\tilde{\alpha}(\|y(t)+w(t)\|)\|f(y(t)+w(t))\|\\ &\le -\frac{1}{2}\langle y(t)+w(t), f(y(t)+w(t))\rangle,\end{aligned}$$

hence giving (4.66). We now continue with (4.64) and utilise (4.65) and (4.66) along with Lemma 2.1.3, to yield, for all $(v, w, x, y) \in \mathcal{B}$ and all $t \in \mathbb{Z}_+$,

$$\begin{aligned}V(x(t+1))-V(x(t)) &\le -\|x(t)\|\tilde{\gamma}_2(\|x(t)\|)+\rho\left(2\left\|\begin{pmatrix}B_{\mathrm{e}}v(t)\\ D_{\mathrm{e}}v(t)\end{pmatrix}\right\|^2\right)\\ &\quad-\langle y(t)+w(t), f(y(t)+w(t))\rangle+\frac{1}{2}\|w(t)\|\alpha(\|w(t)\|).\end{aligned} \tag{4.68}$$

To complete the proof, we seek to obtain $U : \mathbb{R}^n \to \mathbb{R}_+$ such that $U+V$ is an ISS-Lyapunov function with the aim of invoking Proposition 2.2.9. To this end, define $U : \mathbb{R}^n \to \mathbb{R}_+$ by $U(\xi) := \langle \xi, P\xi\rangle$ for all $\xi \in \mathbb{R}^n$, and note that since $(f(y+w), v, x, y) \in \mathcal{B}^{\mathrm{lin}}$ for all $(v, w, x, y) \in \mathcal{B}$, Lemma 4.2.1 yields that (4.37) holds for all $(v, w, x, y) \in \mathcal{B}$ and all $t \in \mathbb{Z}_+$, where $u = f(y+w)$. Therefore, by combining (4.37) with (4.38), we see that, for all $(v,w,x,y) \in \mathcal{B}$ and all $t \in \mathbb{Z}_+$,

$$U(x(t+1)) - U(x(t)) \le 2\langle y(t) - D_{\rm e}v(t), f(y(t)+w(t))\rangle + \|P\|\|B_{\rm e}v(t)\|^2 + 2\|A^TP\|\|x(t)\|\|B_{\rm e}v(t)\| + 2\|B^TP\|\|f(y(t)+w(t))\|\|B_{\rm e}v(t)\|. \quad (4.69)$$

We now combine (4.69) with statement (i) of Lemma 2.1.7 to obtain the existence of $\tilde{\gamma}_3 \in \mathcal{K}_\infty$ such that, for all $(v,w,x,y) \in \mathcal{B}$ and all $t \in \mathbb{Z}_+$,

$$U(x(t+1)) - U(x(t)) \le \frac{1}{2}\|x(t)\|\tilde{\gamma}_2\left(\|x(t)\|\right) + \tilde{\gamma}_3\left(\|v(t)\|\right) + 2\|w(t)\|\|f(y(t)+w(t))\| + 2\langle y(t)+w(t), f(y(t)+w(t))\rangle + 2(\|B^TP\|\|B_{\rm e}\| + \|D_{\rm e}\|)\|v(t)\|\|f(y(t)+w(t))\|.$$

By defining $c := 2(1+\|B^TP\|\|B_{\rm e}\|+\|D_{\rm e}\|) > 0$, we then see that, for all $(v,w,x,y) \in \mathcal{B}$ and all $t \in \mathbb{Z}_+$,

$$U(x(t+1)) - U(x(t)) \le \frac{1}{2}\|x(t)\|\tilde{\gamma}_2(\|x(t)\|) + \tilde{\gamma}_3\left(\left\|\begin{pmatrix} v(t) \\ w(t)\end{pmatrix}\right\|\right) + 2\langle y(t)+w(t), f(y(t)+w(t))\rangle + c\|f(y(t)+w(t))\|\left\|\begin{pmatrix} v(t) \\ w(t)\end{pmatrix}\right\|. \quad (4.70)$$

We pause here for a moment, and claim that there exists $\tilde{\gamma}_4 \in \mathcal{K}_\infty$ such that, for all $(v,w,x,y) \in \mathcal{B}$ and all $t \in \mathbb{Z}_+$,

$$c\|f(y(t)+w(t))\|\left\|\begin{pmatrix} v(t) \\ w(t)\end{pmatrix}\right\| \le -\langle y(t)+w(t), f(y(t)+w(t))\rangle + \tilde{\gamma}_4\left(\left\|\begin{pmatrix} v(t) \\ w(t)\end{pmatrix}\right\|\right). \quad (4.71)$$

Indeed, to see this, fix $(v,w,x,y) \in \mathcal{B}$ and $t \in \mathbb{Z}_+$, and begin with the case that

$$\left\|\begin{pmatrix} v(t) \\ w(t)\end{pmatrix}\right\| \le \frac{1}{c}\tilde{\alpha}(\|y(t)+w(t)\|),$$

where we recall $\tilde{\alpha}$ from (4.58). We then have that

$$c\|f(y(t)+w(t))\|\left\|\begin{pmatrix} v(t) \\ w(t)\end{pmatrix}\right\| \le \tilde{\alpha}(\|y(t)+w(t)\|)\|f(y(t)+w(t))\| \le -\langle y(t)+w(t), f(y(t)+w(t))\rangle.$$

If instead the converse were to hold, then

$$c\|f(y(t)+w(t))\|\left\|\begin{pmatrix} v(t) \\ w(t)\end{pmatrix}\right\| \le c\beta(\|y(t)+w(t)\|)\left\|\begin{pmatrix} v(t) \\ w(t)\end{pmatrix}\right\| \le c\beta\left(\tilde{\alpha}^{-1}\left(c\left\|\begin{pmatrix} v(t) \\ w(t)\end{pmatrix}\right\|\right)\right)\left\|\begin{pmatrix} v(t) \\ w(t)\end{pmatrix}\right\|,$$

where we recall that $\beta \in \mathcal{K}_\infty$ is such that $\|f(\xi)\| \le \beta(\|\xi\|)$ for all $\xi \in \mathbb{R}^m$. We therefore see that a combination of the two cases proves (4.71). By continuing with (4.70), (4.71) gives that, for all $(v,w,x,y) \in \mathcal{B}$ and all $t \in \mathbb{Z}_+$,

$$U(x(t+1)) - U(x(t)) \le \frac{1}{2}\|x(t)\|\tilde{\gamma}_2\left(\|x(t)\|\right) + \langle y(t)+w(t), f(y(t)+w(t))\rangle + \tilde{\gamma}_5\left(\left\|\begin{pmatrix} v(t) \\ w(t)\end{pmatrix}\right\|\right), \quad (4.72)$$

where $\tilde{\gamma}_5(s) := \tilde{\gamma}_4(s) + \tilde{\gamma}_3(s)$ for all $s \geq 0$. To conclude the proof, we set $W := U + V$ and refer to (4.68) and (4.72) so that, for all $(v, w, x, y) \in \mathcal{B}$ and all $t \in \mathbb{Z}_+$,

$$W(x(t+1)) - W(x(t)) \leq -\gamma_1 (\|x(t)\|) + \gamma_2 \left(\left\| \begin{pmatrix} v(t) \\ w(t) \end{pmatrix} \right\| \right),$$

where $\gamma_1, \gamma_2 \in \mathcal{K}_\infty$ are defined by $\gamma_1(s) := \frac{1}{2} s \tilde{\gamma}_2(s)$ and $\gamma_2(s) := \tilde{\gamma}_5(s) + \rho(2(\|B_\mathrm{e}\|^2 + \|D_\mathrm{e}\|^2)s^2) + s\alpha(s)/2$ for all $s \geq 0$. Hence, since $\tilde{P}$ is positive definite, W is an ISS-Lyapunov function and so, by Propositon 2.2.9, the proof is complete. □

We saw in Theorem 4.2.9 that if we strengthen the nonlinearity assumptions of Theorem 4.2.2, then we obtain that (2.17) is ISS. The next, and final, stability result shows that if we instead strengthen the linear assumptions, namely the LMI (4.38), then (2.17) is ISS.

Theorem 4.2.12. *Let $\Sigma \in \mathbb{L}$ and $f : \mathbb{R}^m \to \mathbb{R}^m$. Assume that there exists a positive definite $P = P^T \in \mathbb{R}^{n \times n}$ such that*

$$\begin{pmatrix} A^T PA - P & A^T PB - C^T \\ B^T PA - C & B^T PB - (D + D^T) \end{pmatrix} < 0, \tag{4.73}$$

and $0 \leq -\langle \xi, f(\xi) \rangle$ for all $\xi \in \mathbb{R}^m$. Then (2.17) is ISS.

Before proving Theorem 4.2.12, we give the following remark and two examples.

Remark 4.2.13. (i) The LMI (4.73) is a stronger assumption than (4.38). Indeed, Example 4.2.14, given after this remark, presents a system that satisfies (4.38) for some $P \geq 0$, but not (4.73) for any $P > 0$.

(ii) We highlight that the strength of (4.73) allows us to impose only a very weak assumption on the nonlinearity. Indeed, the assumption $0 \leq -\langle \xi, f(\xi) \rangle$ for all $\xi \in \mathbb{R}^m$ is weaker than the assumptions on f utilised in the previous two theorems, and we refer the reader to Example 4.2.15, situated after the current remark, for an illustration of this.

(iii) From Lemma 4.1.9, we see that as a consequence of (4.73), $A^T PA - P$ is negative definite. An application of Lemma 4.1.8 then yields that A is Schur. ◊

Example 4.2.14. Consider the linear system from Example 4.2.8. Namely, consider $\Sigma \in \mathbb{L}$ with $n = m = 2$, $A = I/2$, where I is the identity matrix, and

$$B = B_\mathrm{e} = C = D = D_\mathrm{e} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

We saw in Example 4.2.8 that Σ satisfies (4.38) with $P = 2I$. However, we shall now show that (4.73) is not satisfied for any positive definite matrix $P \in \mathbb{R}^{2 \times 2}$. To this end, suppose such a matrix exists and denote it by

$$P = \begin{pmatrix} p_1 & p_2 \\ p_2 & p_4 \end{pmatrix},$$

where $p_1, p_2, p_4 \in \mathbb{R}$. An application of Lemma 4.1.9 then yields that $B^T PB - (D + D^T)$ is a negative definite matrix. However,

$$B^T PB - (D + D^T) = \begin{pmatrix} p_1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} p_1 - 2 & 0 \\ 0 & 0 \end{pmatrix},$$

which is not negative definite. We hence have a contradiction, which thus implies that (4.73) does not hold for any positive definite matrix $P \in \mathbb{R}^{2\times 2}$. $\Diamond$

Example 4.2.15. Consider (2.17) where $n = 2$, $m = 1$,

$$A = \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \end{pmatrix}, \quad D = 2,$$

B_{e} and D_{e} are arbitrary, and $f : \mathbb{R} \to \mathbb{R}$ is defined by

$$f(\xi) = \begin{cases} -\mathrm{sign}(\xi)\sqrt{|\xi|}, & \text{if } |\xi| \le 1 \\ -\mathrm{sign}(\xi), & \text{if } |\xi| > 1. \end{cases}$$

We further define $P = 2I \in \mathbb{R}^{2\times 2}$, where I is the identity matrix. Trivially, P is positive definite and it is not difficult to verify that (4.73) holds and that $0 \le -\langle \xi, f(\xi)\rangle$ for all $\xi \in \mathbb{R}^m$. Therefore, Theorem 4.2.12 guarantees that (2.17) is ISS. However, f certainly does not satisfy (4.39) for any $\alpha \in \mathcal{K}_\infty$, and so Theorem 4.2.2 cannot be applied. $\Diamond$

Proof of Theorem 4.2.12. Define $U : \mathbb{R}^n \to \mathbb{R}_+$ by $U(\xi) := \langle \xi, P\xi\rangle$ for all $\xi \in \mathbb{R}^n$. Note that since $(f(y + w), v, x, y) \in \mathcal{B}^{\mathrm{lin}}$ for all $(v, w, x, y) \in \mathcal{B}$, Lemma 4.2.1 yields that (4.37) holds for all $(v, w, x, y) \in \mathcal{B}$ and all $t \in \mathbb{Z}_+$ and where $u = f(y + w)$. Moreover, by (4.73), we obtain the existence of $\lambda > 0$ such that, for all $\xi \in \mathbb{R}^n$ and $\zeta \in \mathbb{R}^m$,

$$\left\langle \begin{pmatrix} A^T PA - P & A^T PB - C^T \\ B^T PA - C & B^T PB - (D + D^T) \end{pmatrix} \begin{pmatrix} \xi \\ \zeta \end{pmatrix}, \begin{pmatrix} \xi \\ \zeta \end{pmatrix} \right\rangle \le -\lambda \left\| \begin{pmatrix} \xi \\ \zeta \end{pmatrix} \right\|^2.$$

Hence, by combining this with (4.37), we obtain that, for all $(v, w, x, y) \in \mathcal{B}$ and all $t \in \mathbb{Z}_+$,

$$\begin{aligned} U(x(t+1)) - U(x(t)) \le &- \lambda \left\| \begin{pmatrix} x(t) \\ f(y(t) + w(t)) \end{pmatrix} \right\|^2 + 2\langle y(t) - D_{\mathrm{e}}v(t), f(y(t) + w(t))\rangle \\ &+ 2\|A^T P\|\|x(t)\|\|B_{\mathrm{e}}v(t)\| + \|P\|\|B_{\mathrm{e}}v(t)\|^2 \\ &+ 2\|B^T P\|\|f(y(t) + w(t))\|\|B_{\mathrm{e}}v(t)\|. \end{aligned} \tag{4.74}$$

We shall now show that U is an ISS-Lyapunov function (see Definition 2.2.8) and then invoke Proposition 2.2.9 to complete the proof. To this end, we shall first show that there exist $\gamma_1, \gamma_2 \in \mathcal{K}_\infty$ such that, for all $(v, w, x, y) \in \mathcal{B}$ and all $t \in \mathbb{Z}_+$,

$$U(x(t+1)) - U(x(t)) \le -\gamma_1(\|x(t)\|) + \gamma_2\left(\left\| \begin{pmatrix} v(t) \\ w(t) \end{pmatrix} \right\| \right). \tag{4.75}$$

Indeed, to prove this, fix $(v, w, x, y) \in \mathcal{B}$ and $t \in \mathbb{Z}_+$ and note that, by considering separate cases of when $4\|A^T P\|\|B_{\mathrm{e}}v(t)\|$ is greater or less than $\lambda\|x(t)\|$, we obtain that

$$\begin{aligned} 2\|A^T P\|\|x(t)\|\|B_{\mathrm{e}}v(t)\| &\le \frac{\lambda}{2}\|x(t)\|^2 + \frac{8}{\lambda}\|A^T P\|^2\|B_{\mathrm{e}}v(t)\|^2 \\ &\le \frac{\lambda}{2}\|x(t)\|^2 + \frac{8}{\lambda}\|A^T P\|^2\|B_{\mathrm{e}}\|^2 \left\| \begin{pmatrix} v(t) \\ w(t) \end{pmatrix} \right\|^2. \end{aligned} \tag{4.76}$$

Furthermore,

$$\begin{aligned} 2\langle y(t)-D_{\mathrm{e}}v(t), f(y(t)+w(t))\rangle &+ 2\|B^TP\|\|f(y(t)+w(t))\|\|B_{\mathrm{e}}v(t)\| \\ &\leq 2\langle y(t)+w(t), f(y(t)+w(t))\rangle \\ &\quad + 2\left(\|w(t)\| + \|D_{\mathrm{e}}v(t)\| + \|B^TP\|\|B_{\mathrm{e}}v(t)\|\right)\|f(y(t)+w(t))\| \\ &\leq 2\langle y(t)+w(t), f(y(t)+w(t))\rangle \\ &\quad + 2\left(1+\|D_{\mathrm{e}}\| + \|B^TP\|\|B_{\mathrm{e}}\|\right)\left\|\begin{pmatrix} v(t) \\ w(t)\end{pmatrix}\right\| \|f(y(t)+w(t))\|. \end{aligned}$$

Therefore, by considering separate cases of when

$$\left\|\begin{pmatrix} v(t) \\ w(t)\end{pmatrix}\right\| \text{is greater or less than } \frac{\lambda\|f(y(t)+w(t))\|}{4(1+\|D_{\mathrm{e}}\|+\|B^TP\|\|B_{\mathrm{e}}\|)},$$

we see that

$$\begin{aligned} 2\langle y(t)-D_{\mathrm{e}}v(t), f(y(t)+w(t))\rangle &+ 2\|B^TP\|\|f(y(t)+w(t))\|\|B_{\mathrm{e}}v(t)\| \\ &\leq 2\langle y(t)+w(t), f(y(t)+w(t))\rangle + \frac{\lambda}{2}\|f(y(t)+w(t))\|^2 \\ &\quad + \frac{8}{\lambda}(1+\|D_{\mathrm{e}}\|+\|B^TP\|\|B_{\mathrm{e}}\|)^2\left\|\begin{pmatrix} v(t) \\ w(t)\end{pmatrix}\right\|^2. \end{aligned} \tag{4.77}$$

Hence, we define $\gamma_2 \in \mathcal{K}_\infty$ by

$$\gamma_2(s) := \frac{8}{\lambda}\left((1+\|D_{\mathrm{e}}\|+\|B^TP\|\|B_{\mathrm{e}}\|)^2 + \|A^TP\|^2\|B_{\mathrm{e}}\|^2\right)s^2 + \|P\|\|B_{\mathrm{e}}\|^2 s^2 \quad \forall\, s \geq 0,$$

so that (4.76) and (4.77), when combined with (4.74) and the identity

$$\left\|\begin{pmatrix} \xi \\ \zeta\end{pmatrix}\right\|^2 = \|\xi\|^2 + \|\zeta\|^2 \quad \forall\, \xi \in \mathbb{R}^n, \forall\, \zeta \in \mathbb{R}^m,$$

yield

$$\begin{aligned} U(x(t+1)) - U(x(t)) \leq &-\frac{\lambda}{2}\|x(t)\|^2 - \frac{\lambda}{2}\|f(y(t)+w(t))\|^2 \\ &+ 2\langle y(t)+w(t), f(y(t)+w(t))\rangle + \gamma_2\left(\left\|\begin{pmatrix} v(t) \\ w(t)\end{pmatrix}\right\|\right). \end{aligned}$$

Since $\langle \xi, f(\xi)\rangle \leq 0$ for all $\xi \in \mathbb{R}^m$, by defining $\gamma_1 \in \mathcal{K}_\infty$ by $\gamma_1(s) := \lambda s^2/2$ for all $s \geq 0$, it is clear that (4.75) holds. Moreover, since γ_1 and γ_2 only depend on Σ and P (and hence not on (v, w, x, y) or t), we see that (4.75) holds for all $(v, w, x, y) \in \mathcal{B}$ and all $t \in \mathbb{Z}_+$. Finally, since P is positive definite, it is easy to check that there exist $c_1, c_2 > 0$ such that

$$c_1\|\xi\|^2 \leq U(\xi) \leq c_2\|\xi\|^2 \quad \forall\, \xi \in \mathbb{R}^n.$$

Therefore, by combining this with the fact that (4.75) holds for all $(v, w, x, y) \in \mathcal{B}$ and all $t \in \mathbb{Z}_+$, we see that U is indeed an ISS-Lyapunov function, which, by Proposition 2.2.9, completes the proof. □

We conclude this subsection by summarising the assumptions of the previous three theorems. Indeed, we refer the reader to Table 4.1 for an illustration. We recall that (4.73) implies (4.38). Moreover, (4.39) guarantees that $0 \leq -\langle \xi, f(\xi)\rangle$, $\forall\, \xi \in \mathbb{R}^m$, and (4.39) together with (4.57) implies (4.40).

	Linear assumptions		Nonlinear assumptions			
Theorem	(4.38)	(4.73)	(4.39)	(4.40)	(4.57)	$0 \le -\langle \xi, f(\xi) \rangle$, $\forall \xi \in \mathbb{R}^m$
4.2.2	✓		✓	✓		
4.2.9	✓		✓		✓	
4.2.12		✓				✓

Table 4.1: *Summary of the assumptions of the three theorems of Section 4.2. The symbol ✓ means that the relevant assumption is imposed in the corresponding theorem.*

4.2.3 Positive real assumptions

Our attention now turns towards utilising the positive and strict positive real lemmas, that is, Lemmas 4.1.15 and 4.1.20, in order to rewrite the three stability theorems of the previous subsection in terms of positive and strictly positive real assumptions on the transfer function. In particular, we shall express the LMIs given by (4.38) and (4.73) in terms of positive and strict positive real assumptions, respectively. To this end, we begin with the following two corollaries, which are each an immediate consequence of Lemma 4.1.15 combined with Theorem 4.2.2 and Theorem 4.2.9, respectively. Indeed, the existence of a positive semi-definite matrix $P \in \mathbb{R}^{n\times n}$ and matrices L and W satisfying (4.20) trivially implies that (4.38) holds.

Corollary 4.2.16. *Let $\Sigma \in \mathbb{L}$ with $B_{\mathrm{e}} = B$ and $D_{\mathrm{e}} = D$, and let $f : \mathbb{R}^m \to \mathbb{R}^m$. Assume that $\mathbf{G}$ is positive real and that either:* (i) *Σ is controllable and observable, or;* (ii) *Σ is stabilisable and detectable and there exists $\eta \in \partial\mathbb{D}$, not a pole of $\mathbf{G}$, such that $\mathbf{G}(\eta) + \mathbf{G}(\eta)^* > 0$. Furthermore, assume that f is continuous on a neighbourhood containing 0, f is bounded on bounded sets and that there exist $\alpha \in \mathcal{K}_\infty$, $c > 0$ and $\mu > 0$ such that the nonlinearity f satisfies* (4.39) *and* (4.40)*. Then there exist $\psi \in \mathcal{KL}$ and $\phi \in \mathcal{K}$ such that* (4.41) *holds.*

Corollary 4.2.17. *Let $\Sigma \in \mathbb{L}$ and let $f : \mathbb{R}^m \to \mathbb{R}^m$. Assume that $\mathbf{G}$ is positive real and that either:* (i) *Σ is controllable and observable, or;* (ii) *Σ is stabilisable and detectable and there exists $\eta \in \partial\mathbb{D}$, not a pole of $\mathbf{G}$, such that $\mathbf{G}(\eta) + \mathbf{G}(\eta)^* > 0$. Furthermore, assume that f is continuous, there exists $\alpha \in \mathcal{K}_\infty$ such that* (4.39) *holds and f satisfies* (4.57)*. Then* (2.17) *is ISS.*

The next example illustrates that the assumptions of the previous two corollaries are stronger than the assumptions given in Theorem 4.2.2 and Theorem 4.2.9, respectively.

Example 4.2.18. Let $\Sigma \in \mathbb{L}$, where $n = 2$, $m = 1$,

$$A := \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -1 \end{pmatrix}, \quad B := \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad C := \begin{pmatrix} 0 & -2 \end{pmatrix}, \quad D := 1,$$

and B_{e} and D_{e} are arbitrary. By defining

$$P := \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix},$$

is can easily be verified that P is positive definite and

$$\begin{pmatrix} A^T PA - P & A^T PB - C^T \\ B^T PA - C & B^T PB - (D + D^T) \end{pmatrix} = \begin{pmatrix} -\frac{3}{4} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

hence giving that the LMI (4.38) holds. Moreover, Σ is detectable, and so the assumptions of Theorems 4.2.2 and 4.2.9, relevant to Σ, hold. However, is it easily checked that Σ is not controllable. Additionally, since

$$\mathbf{G}(z) = \frac{z-1}{z+1} \quad \forall\, z \in \mathbb{C}\backslash\{-1\},$$

it is simple to check that $\mathbf{G}(z) + \mathbf{G}(z)^* = 0$ on $\partial\mathbb{D}\backslash\{-1\}$. We therefore see that Σ does not satisfy the relevant hypotheses of Corollaries 4.2.16 and 4.2.17. $\Diamond$

The final corollary of this section is obtained from a combination of Theorem 4.2.12 with Lemma 4.1.20.

Corollary 4.2.19. *Let $\Sigma \in \mathbb{L}$ and $f : \mathbb{R}^m \to \mathbb{R}^m$. Assume that Σ is controllable and observable, $\mathbf{G}$ is strictly positive real, $D + D^T > 0$ and $0 \leq -\langle \xi, f(\xi)\rangle$ for all $\xi \in \mathbb{R}^m$. Then* (2.17) *is ISS.*

Proof. In order to invoke Theorem 4.2.12, we only need to verify the existence of a positive definite matrix $P = P^T \in \mathbb{R}^{n\times n}$ such that (4.73) holds. To this end, note that since the hypotheses of Lemma 4.1.20 hold, we obtain the existence of matrices $P, R \in \mathbb{R}^{n\times n}$, $L \in \mathbb{R}^{m\times n}$ and $W \in \mathbb{R}^{m\times m}$ such that P, R and W are all positive definite, and (4.33) holds. Let

$$\begin{pmatrix} \xi \\ \zeta \end{pmatrix} \in \mathbb{R}^{n+m}\backslash\{0\},$$

where $\xi \in \mathbb{R}^n$ and $\zeta \in \mathbb{R}^m$. For ease of notation in the sequel, set

$$M := \begin{pmatrix} A^TPA - P & A^TPB - C^T \\ B^TPA - C & B^TPB - (D + D^T) \end{pmatrix} \quad \text{and} \quad \eta := \begin{pmatrix} \xi \\ \zeta \end{pmatrix}.$$

By using Lemma 4.1.9, we see that

$$\langle M\eta, \eta\rangle = \langle (A^TPA - P)\xi, \xi\rangle + 2\langle (A^TPB - C^T)\zeta, \xi\rangle + \langle (B^TPB - (D + D^T))\zeta, \zeta\rangle.$$

Furthermore, if we now invoke (4.33), we obtain that

$$\begin{aligned} \langle M\eta, \eta\rangle &= -\|L\xi\|^2 - \langle R\xi, \xi\rangle - 2\langle L\xi, W\zeta\rangle - \|W\zeta\|^2 \\ &= -\|L\xi + W\zeta\|^2 - \langle R\xi, \xi\rangle. \end{aligned} \tag{4.78}$$

We consider the following two cases. First, if $\xi \neq 0$, then $\langle R\xi, \xi\rangle > 0$ and so (4.78) gives that $\langle M\eta, \eta\rangle < 0$. In the second case of when $\xi = 0$, we deduce therefore that $\zeta \neq 0$ and so, since W is positive definite, $\|W\zeta\| > 0$. Hence, from (4.78), we yield that $\langle M\eta, \eta\rangle < 0$, which completes the proof since ξ and ζ were arbitrary. $\square$

We conclude this section with the following trivial example, which shows that the hypotheses of Theorem 4.2.12 are weaker than the hypotheses of Corollary 4.2.19.

Example 4.2.20. Let $\Sigma \in \mathbb{L}$ be as in Example 4.2.15. That is, $n = 2$, $m = 1$,

$$A = \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \end{pmatrix}, \quad D = 2,$$

and B_{e} and D_{e} are arbitrary. By defining $P \in \mathbb{R}^{2\times 2}$ to be $2I$, where I is the identity matrix, we see that (4.73) holds. However, we note that Σ is not controllable. Therefore, a system with linear component given by Σ satisfies the linear assumptions of Theorem 4.2.12, but not those of Corollary 4.2.19. $\Diamond$

4.3 Convergence properties

In this section, we shall develop sufficient criteria for when (2.17), with potentially superlinear nonlinearity, exhibits a certain convergence property, namely the converging-input converging-state (CICS) property. This property was discussed in detail in Section 2.3, and was first investigated in [15]. We refer the reader to Section 2.3 for explicit definitions and details.

In both Section 2.3 and [15], the CICS property has been obtained from ISS theorems. Indeed, this was done in Section 2.3 and [15] by, roughly, forming an 'incremental' version of the assumptions relevant to the nonlinearity in Theorem 2.2.10 and [107, Theorem 3.2], respectively, and by making sure that these assumptions also guarantee the existence of an equilibrium corresponding to each constant input. Repeated uses of ISS estimates (of a 'shifted' system) then yielded the CICS property. It is this methodology that we hope to try to reproduce in our current setting, that is, where functions are allowed to be 'superlinear'. With this motivation in mind, we split the current section in two. In the first part we present relevant preliminary results, and in the second, we give sufficient conditions for when (2.17) exhibits the CICS property.

4.3.1 Preliminary results and definitions

The first preliminary result we shall give is the following, which shows that if a function satisfies a relevant limit for one particular value, then it does so for every value.

Lemma 4.3.1. *Let $f : \mathbb{R}^m \to \mathbb{R}^m$ and assume that there exists $\eta \in \mathbb{R}^m$ such that*

$$\min\left\{-\frac{\langle \xi, f(\xi+\eta)-f(\eta)\rangle}{\|\xi\|}, -\frac{\langle \xi, f(\xi+\eta)-f(\eta)\rangle}{\|f(\xi+\eta)-f(\eta)\|}\right\} \to \infty \text{ as } \|\xi\| \to \infty. \tag{4.79}$$

Then for every $\zeta \in \mathbb{R}^m$,

$$\min\left\{-\frac{\langle \xi, f(\xi+\zeta)-f(\zeta)\rangle}{\|\xi\|}, -\frac{\langle \xi, f(\xi+\zeta)-f(\zeta)\rangle}{\|f(\xi+\zeta)-f(\zeta)\|}\right\} \to \infty \text{ as } \|\xi\| \to \infty. \tag{4.80}$$

Proof. Let $\zeta \in \mathbb{R}^m$. We set $R := \|\zeta - \eta\|$ which we assume is positive, else (4.80) trivially holds. A consequence of (4.79) is that we may choose $\mu > R$ sufficiently large so that

$$2R \le -\frac{\langle \xi, f(\xi+\eta)-f(\eta)\rangle}{\|f(\xi+\eta)-f(\eta)\|} \quad \forall\, \xi \in \mathbb{R}^m,\ \|\xi\| \ge \mu\,, \tag{4.81}$$

and so

$$-\|f(\xi+\eta)-f(\eta)\| \ge \frac{1}{2R}\langle \xi, f(\xi+\eta)-f(\eta)\rangle \quad \forall\, \xi \in \mathbb{R}^m,\ \|\xi\| \ge \mu\,. \tag{4.82}$$

Moreover, for each $\xi \in \mathbb{R}^m$ with $\|\xi\| \ge \mu + R > 0$, define $z_\xi = \xi + \zeta - \eta$. We note that $\|z_\xi\| \ge \mu$ and

$$\frac{1}{2} \le 1 - \frac{R}{\|\xi\|} \le \frac{\|z_\xi\|}{\|\xi\|} \le 1 + \frac{R}{\|\xi\|} \le 2\,. \tag{4.83}$$

Now, in order to establish (4.80), we first show that

$$-\frac{\langle \xi, f(\xi+\zeta)-f(\zeta)\rangle}{\|\xi\|} \to \infty \quad \text{as } \|\xi\| \to \infty. \tag{4.84}$$

To this end, we note that, for all $\xi \in \mathbb{R}^m$ with $\|\xi\| \geq \mu + R$,

$$-\langle \xi, f(\xi+\zeta) - f(\zeta)\rangle = -\langle \xi, f(z_\xi+\eta) - f(\eta)\rangle - \langle \xi, f(\eta) - f(\zeta)\rangle. \tag{4.85}$$

By using the Cauchy-Schwarz inequality, we see that, for all $\xi \in \mathbb{R}^m$ with $\|\xi\| \geq \mu + R$,

$$\begin{aligned} -\frac{\langle \xi, f(z_\xi+\eta) - f(\eta)\rangle}{\|\xi\|} &= -\frac{\langle z_\xi, f(z_\xi+\eta) - f(\eta)\rangle}{\|\xi\|} + \frac{\langle \zeta-\eta, f(z_\xi+\eta) - f(\eta)\rangle}{\|\xi\|} \\ &= \frac{\|z_\xi\|}{\|\xi\|}\left(-\frac{\langle z_\xi, f(z_\xi+\eta) - f(\eta)\rangle}{\|z_\xi\|} + \frac{\langle \zeta-\eta, f(z_\xi+\eta) - f(\eta)\rangle}{\|z_\xi\|}\right) \\ &\geq \frac{\|z_\xi\|}{\|\xi\|}\left(-\frac{\langle z_\xi, f(z_\xi+\eta) - f(\eta)\rangle}{\|z_\xi\|} - \|\zeta-\eta\|\frac{\|f(z_\xi+\eta) - f(\eta)\|}{\|z_\xi\|}\right). \end{aligned}$$

By combining this with (4.82), and by also recalling that $\|\zeta - \eta\| = R$, we obtain, for all $\xi \in \mathbb{R}^m$ with $\|\xi\| \geq \mu + R$,

$$\begin{aligned} -&\frac{\langle \xi, f(z_\xi+\eta) - f(\eta)\rangle}{\|\xi\|} \\ &\geq \frac{\|z_\xi\|}{\|\xi\|}\left(-\frac{\langle z_\xi, f(z_\xi+\eta) - f(\eta)\rangle}{\|z_\xi\|} + \frac{\|\zeta-\eta\|}{2R}\frac{\langle z_\xi, f(z_\xi+\eta) - f(\eta)\rangle}{\|z_\xi\|}\right) \\ &= -\frac{1}{2}\frac{\|z_\xi\|}{\|\xi\|}\frac{\langle z_\xi, f(z_\xi+\eta) - f(\eta)\rangle}{\|z_\xi\|}. \end{aligned} \tag{4.86}$$

From (4.83), this implies that, for all $\xi \in \mathbb{R}^m$ with $\|\xi\| \geq \mu + R$,

$$-\frac{\langle \xi, f(z_\xi+\eta) - f(\eta)\rangle}{\|\xi\|} \geq -\frac{1}{4}\frac{\langle z_\xi, f(z_\xi+\eta) - f(\eta)\rangle}{\|z_\xi\|}, \tag{4.87}$$

which converges to infinity as $\|\xi\| \to \infty$. Here, we have used that $-\langle z_\xi, f(z_\xi + \eta) - f(\eta)\rangle > 0$ by (4.81). In light of (4.85), (4.87) and the fact that

$$-\frac{\langle \xi, f(\eta) - f(\zeta)\rangle}{\|\xi\|} \geq -\|f(\eta) - f(\zeta)\| \quad \forall\, \xi \in \mathbb{R}^m\backslash\{0\},$$

we have shown that (4.84) holds. Hence, to complete the proof, it suffices to now show that

$$-\frac{\langle \xi, f(\xi+\zeta) - f(\zeta)\rangle}{\|f(\xi+\zeta) - f(\zeta)\|} \to \infty \quad \text{as } \|\xi\| \to \infty. \tag{4.88}$$

To this end, let $M > 0$. By combining (4.79) and with (4.84), we yield the existence of $\nu > \mu$ such that

$$\min\left\{-\frac{\langle \xi, f(\xi+\zeta) - f(\zeta)\rangle}{\|\xi\|}, -\frac{\langle \xi, f(\xi+\eta) - f(\eta)\rangle}{\|f(\xi+\eta) - f(\eta)\|}\right\} > 4(M + \|f(\eta) - f(\zeta)\|) \quad \forall\, \xi \in \mathbb{R}^m,\ \|\xi\| \geq \nu, \tag{4.89}$$

and so in particular $-\langle \xi, f(\xi+\zeta) - f(\zeta)\rangle > 0$ for all such ξ. Since, from (4.79),

$$\|f(\xi+\eta) - f(\eta)\| \geq -\frac{\langle \xi, f(\xi+\eta) - f(\eta)\rangle}{\|\xi\|} \to \infty \quad \text{as } \|\xi\| \to \infty,$$

without loss of generality, we assume that ν is large enough so that

$$\|f(\xi+\eta)-f(\eta)\| \geq \|f(\eta)-f(\zeta)\| \quad \forall\, \xi \in \mathbb{R}^m,\ \|\xi\| \geq \nu\,. \tag{4.90}$$

Let $\xi \in \mathbb{R}^m$ be such that $\|\xi\| > \nu + R$, so that $\|z_\xi\| \geq \nu$. If $\|f(\xi+\zeta)-f(\zeta)\| \leq \|\xi\|$, then clearly, from (4.89),

$$-\frac{\langle \xi, f(\xi+\zeta)-f(\zeta)\rangle}{\|f(\xi+\zeta)-f(\zeta)\|} \geq -\frac{\langle \xi, f(\xi+\zeta)-f(\zeta)\rangle}{\|\xi\|} > 4(M+\|f(\eta)-f(\zeta)\|) > M\,. \tag{4.91}$$

Alternatively, if $\|f(\xi+\zeta)-f(\zeta)\| > \|\xi\|$, then by recalling (4.85) and by multiplying both sides of (4.86) by $\|\xi\|$ and $1/\|f(\xi+\zeta)-f(\zeta)\|$, we see that

$$-\frac{\langle \xi, f(\xi+\zeta)-f(\zeta)\rangle}{\|f(\xi+\zeta)-f(\zeta)\|} \geq -\frac{1}{2}\frac{\langle z_\xi, f(z_\xi+\eta)-f(\eta)\rangle}{\|f(\xi+\zeta)-f(\zeta)\|} - \|f(\eta)-f(\zeta)\|$$

Therefore, by (4.90),

$$\begin{aligned}-\frac{\langle \xi, f(\xi+\zeta)-f(\zeta)\rangle}{\|f(\xi+\zeta)-f(\zeta)\|} &\geq -\frac{1}{2}\frac{\langle z_\xi, f(z_\xi+\eta)-f(\eta)\rangle}{\|f(z_\xi+\eta)-f(\eta)\|+\|f(\eta)-f(\zeta)\|} - \|f(\eta)-f(\zeta)\| \\ &= \frac{1}{2}\frac{-\frac{\langle z_\xi, f(z_\xi+\eta)-f(\eta)\rangle}{\|f(z_\xi+\eta)-f(\eta)\|}}{1+\frac{\|f(\eta)-f(\zeta)\|}{\|f(z_\xi+\eta)-f(\eta)\|}} - \|f(\eta)-f(\zeta)\| \\ &\geq \frac{1}{4}4(M+\|f(\eta)-f(\zeta)\|) - \|f(\eta)-f(\zeta)\| \\ &= M\,.\end{aligned}$$

Since M was arbitrary, we have shown that (4.88) holds, thus completing the proof. $\square$

The following is an interesting characterisation of functions f which have the property that (4.79) holds for some $\eta \in \mathbb{R}^m$.

Lemma 4.3.2. *A function $f : \mathbb{R}^m \to \mathbb{R}^m$ satisfies that there exists $\eta \in \mathbb{R}^m$ such that* (4.79) *holds, if, and only if, there exists a function $g : \mathbb{R}^m \to \mathbb{R}^m$ and $c \in \mathbb{R}^m$ such that*

$$f(\xi) = g(\xi) + c \quad \forall\, \xi \in \mathbb{R}^m, \tag{4.92}$$

$g(0) = 0$ *and*

$$\min\left\{-\frac{\langle \xi, g(\xi)\rangle}{\|\xi\|}, -\frac{\langle \xi, g(\xi)\rangle}{\|g(\xi)\|}\right\} \to \infty \ \text{as}\ \|\xi\| \to \infty. \tag{4.93}$$

Proof. Let $f : \mathbb{R}^m \to \mathbb{R}^m$ satisfy that there exists $\eta \in \mathbb{R}^m$ such that (4.79) holds. Define $g : \mathbb{R}^m \to \mathbb{R}^m$ by

$$g(\xi) := f(\xi) - f(0) \quad \forall\, \xi \in \mathbb{R}^m.$$

Trivially, $g(0) = 0$ and (4.92) is satisfied with $c := f(0)$. Moreover, an application of Lemma 4.3.1 gives that (4.80) holds for all $\zeta \in \mathbb{R}^m$. In particular, by taking $\zeta = 0$, this yields (4.93).

As for the reverse implication, we now let $g : \mathbb{R}^m \to \mathbb{R}^m$ and $c \in \mathbb{R}^m$ be arbitrary such that $g(0) = 0$ and g satisfies (4.93). We subsequently define $f : \mathbb{R}^m \to \mathbb{R}^m$ via (4.92) and note that

$$\min\left\{-\frac{\langle \xi, f(\xi)-f(0)\rangle}{\|\xi\|}, -\frac{\langle \xi, f(\xi)-f(0)\rangle}{\|f(\xi)-f(0)\|}\right\} = \min\left\{-\frac{\langle \xi, g(\xi)\rangle}{\|\xi\|}, -\frac{\langle \xi, g(\xi)\rangle}{\|g(\xi)\|}\right\},$$

which converges to ∞ as $\|\xi\| \to \infty$ from (4.93). Therefore, (4.79) holds with $\eta = 0$ and the proof is complete. $\square$

Remark 4.3.3. Lemma 4.3.2 shows us that examples of functions $f : \mathbb{R}^m \to \mathbb{R}^m$ satisfying (4.79) for some $\eta \in \mathbb{R}^m$, are easy to find. Indeed, by taking $g : \mathbb{R}^m \to \mathbb{R}^m$ satisfying the relevant conditions given in the statement of Lemma 4.3.2 (see Example 4.2.11), then any constant addition to g yields such a desired function f. $\Diamond$

We now recall the definition of monotone and dissipative functions.

Definition 4.3.4. *Let* $f : \mathbb{R}^m \to \mathbb{R}^m$ *.*

(i) *We say that* f *is* monotone *if*

$$\langle \xi, f(\xi + \zeta) - f(\zeta) \rangle \geq 0 \quad \forall \, \zeta, \xi \in \mathbb{R}^m,$$

and dissipative *if* $-f$ *is monotone.*

(ii) *We say that* f *is* strictly monotone *if*

$$\langle \xi, f(\xi + \zeta) - f(\zeta) \rangle > 0 \quad \forall \, \zeta, \xi \in \mathbb{R}^m, \, \xi \neq 0,$$

and strictly dissipative *if* $-f$ *is strictly monotone.*

For background regarding monotone functions, we refer the reader to works such as [30, 93, 94]. The following lemma is [94, Proposition 12.3, p.534].

Lemma 4.3.5. *Let* $f : \mathbb{R}^m \to \mathbb{R}^m$ *be differentiable. The following statements hold.*

(i) f *is monotone if, and only if, for each* $\xi \in \mathbb{R}^m$*, the Jacobian matrix* $\nabla f(\xi)$ *is positive semi-definite.*

(ii) f *is strictly monotone if, for each* $\xi \in \mathbb{R}^m$*, the Jacobian matrix* $\nabla f(\xi)$ *is positive definite.*

Remark 4.3.6. The definition of (strict) monotonicity and Lemma 4.3.5 together highlight how the notion can be seen as a generalisation of (strictly) increasing functions $\mathbb{R} \to \mathbb{R}$. Similarly, (strictly) dissipative functions can be seen as a generalisation of the notion of (strictly) decreasing functions. $\Diamond$

The next lemma gives sufficient conditions for when the map $\xi \mapsto \xi - \mathbf{G}(1)f(\xi)$ is surjective. This will be useful in determining the existence of equilibrium quadruples of (2.17) (see Definition 2.3.1) in subsequent proofs (see the preamble at the start of this section).

Lemma 4.3.7. *Let* $\Sigma \in \mathbb{L}$ *and* $f : \mathbb{R}^m \to \mathbb{R}^m$*. Assume that* $1 \notin \sigma(A)$*,* $\mathbf{G}(1) + \mathbf{G}(1)^T \geq 0$ *and* f *is dissipative. The following statements hold.*

(i) *The map* $\mathbb{R}^m \to \mathbb{R}^m$, $\xi \mapsto \xi - \mathbf{G}(1)f(\xi)$ *is injective if either* f *is strictly dissipative or* $\mathbf{G}(1) + \mathbf{G}(1)^T > 0$*.*

(ii) *The map* $\mathbb{R}^m \to \mathbb{R}^m$, $\xi \mapsto \xi - \mathbf{G}(1)f(\xi)$ *is surjective if* $\mathbf{G}(1) + \mathbf{G}(1)^T > 0$ *and* f *is continuous.*

(iii) *The map* $\mathbb{R}^m \to \mathbb{R}^m$, $\xi \mapsto \xi - \mathbf{G}(1)f(\xi)$ *is surjective if* f *is continuous, strictly dissipative and there exists* $\eta \in \mathbb{R}^m$ *such that* (4.79) *holds.*

Proof. To prove statement (i), we begin by assuming that f is strictly dissipative. Seeking a contradiction, suppose that $\xi, \eta \in \mathbb{R}^m$ are such that $\xi \neq \eta$ and $\xi - \mathbf{G}(1)f(\xi) = \eta - \mathbf{G}(1)f(\eta)$. This implies that $\xi - \eta = \mathbf{G}(1)(f(\xi) - f(\eta))$, and thus

$$\langle \xi - \eta, f(\xi) - f(\eta)\rangle = \langle \mathbf{G}(1)(f(\xi) - f(\eta)), f(\xi) - f(\eta)\rangle. \tag{4.94}$$

However, the left-hand side is negative by the strict dissipativity of f, and the right-hand side is non-negative by the positive semi-definiteness of $\mathbf{G}(1) + \mathbf{G}(1)^T$. We therefore obtain a contradiction and hence yield that the map in question is injective. Now assume that $\mathbf{G}(1) + \mathbf{G}(1)^T > 0$ and again suppose that $\xi \neq \eta$ and $\xi - \mathbf{G}(1)f(\xi) = \eta - \mathbf{G}(1)f(\eta)$. We note that this supposition necessarily implies that $f(\xi) \neq f(\eta)$, else

$$\xi = \eta + \mathbf{G}(1)f(\xi) - \mathbf{G}(1)f(\eta) = \eta,$$

which would be a contradiction. Therefore, with this in mind, from (4.94) we again obtain a contradiction, since the left-hand side is non-positive by the dissipativity of f, and the right-hand side is positive by the positive definiteness of $\mathbf{G}(1) + \mathbf{G}(1)^T$. We have therefore shown that $\xi \mapsto \xi - \mathbf{G}(1)f(\xi)$ is injective if either f is strictly dissipative or $\mathbf{G}(1) + \mathbf{G}(1)^T > 0$, and completed the proof of statement (i).

We now proceed to prove statement (ii) and so we now assume that $\mathbf{G}(1) + \mathbf{G}(1)^T > 0$ and that f is continuous. For ease of notation, we denote the map $\mathbb{R}^m \to \mathbb{R}^m$, $\xi \mapsto \xi - \mathbf{G}(1)f(\xi)$ by F. To begin with, we note that since $\mathbf{G}(1) + \mathbf{G}(1)^T > 0$, there exists $\lambda > 0$ such that

$$\langle (\mathbf{G}(1) + \mathbf{G}(1)^T)\xi, \xi\rangle \geq \lambda \|\xi\|^2 \quad \forall\, \xi \in \mathbb{R}^m. \tag{4.95}$$

We seek to show that $\|F(\xi)\| \to \infty$ as $\|\xi\| \to \infty$. To this end, let $\xi \in \mathbb{R}^m\backslash\{0\}$ and note that if $f(\xi) \neq 0$, then

$$\begin{aligned}
\|F(\xi)\| &= \frac{\|F(\xi)\|\|f(\xi)\|}{\|f(\xi)\|} \\
&\geq -\frac{\langle F(\xi), f(\xi)\rangle}{\|f(\xi)\|} \\
&= -\frac{\langle \xi, f(\xi)\rangle}{\|f(\xi)\|} + \frac{\langle \mathbf{G}(1)f(\xi), f(\xi)\rangle}{\|f(\xi)\|} \\
&= -\frac{\langle \xi, f(\xi) - f(0)\rangle}{\|f(\xi)\|} - \frac{\langle \xi, f(0)\rangle}{\|f(\xi)\|} + \frac{\langle (\mathbf{G}(1) + \mathbf{G}(1)^T)f(\xi), f(\xi)\rangle}{2\|f(\xi)\|}.
\end{aligned} \tag{4.96}$$

Consider the following two cases. The first case concerns when $2\|\mathbf{G}(1)\|\|f(\xi)\| \geq \|\xi\|$. Since this implies that $f(\xi) \neq 0$, we may combine (4.95) with (4.96) and the dissipativity of f, to obtain that

$$\begin{aligned}
\|F(\xi)\| &\geq -2\|f(0)\|\|\mathbf{G}(1)\| + \frac{\lambda}{2}\|f(\xi)\| \\
&\geq -2\|f(0)\|\|\mathbf{G}(1)\| + \frac{\lambda}{4\|\mathbf{G}(1)\|}\|\xi\|.
\end{aligned} \tag{4.97}$$

Here, we have used that $\|\mathbf{G}(1)\| \neq 0$ by (4.95). Now, for the second case of when

$2\|\mathbf{G}(1)\|\|f(\xi)\| \leq \|\xi\|$, we consider

$$\begin{aligned}\|F(\xi)\| &= \frac{\|F(\xi)\|\|\xi\|}{\|\xi\|} \\ &\geq \frac{\langle F(\xi), \xi\rangle}{\|\xi\|} \\ &= \|\xi\| - \frac{\langle \mathbf{G}(1)f(\xi), \xi\rangle}{\|\xi\|},\end{aligned}$$

which in turn gives that

$$\begin{aligned}\|F(\xi)\| &\geq \|\xi\| - \|\mathbf{G}(1)\|\|f(\xi)\| \\ &\geq \frac{1}{2}\|\xi\|. \end{aligned} \tag{4.98}$$

By combining this with (4.97), we hence deduce that $\|F(\xi)\| \to \infty$ as $\|\xi\| \to \infty$. Therefore, by combining this with the continuity of F and the fact that F is injective by statement (i), we obtain that F is surjective. To give some more detail here, we refer the reader to [30, Theorem 4.3], which says that an injective and continuous function is an open map. From this (as mentioned in the commentary immediately after [30, Theorem 4.3]) we see that $F(\mathbb{R}^m)$ is open and also closed, since for any sequence $(\xi_i)_{i\in\mathbb{Z}_+} \subseteq \mathbb{R}^m$ such that $F(\xi_i) \to \zeta \in \mathbb{R}^m$, unboundedness of F guarantess boundedness of $(\xi_i)_{i\in\mathbb{Z}_+}$ which then guarantees the existence of $\xi \in \mathbb{R}^m$ such that $F(\xi) = \zeta$. By the connectedness of $\mathbb{R}^m$, we see that $F(\mathbb{R}^m) = \mathbb{R}^m$, hence yielding surjectivity.
Finally, our attention turns to statement (iii) and thus we now instead assume that f is continuous, strictly dissipative and there exists $\eta \in \mathbb{R}^m$ such that (4.79) holds. Let $\xi \in \mathbb{R}^m\backslash\{0\}$ and again consider the following two cases. The first case is when $2\|\mathbf{G}(1)\|\|f(\xi)\| \geq \|\xi\|$. In this situation $f(\xi) \neq 0$, and therefore, by invoking (4.96), it is easily checked, by use of the positive semi-definiteness of $\mathbf{G}(1) + \mathbf{G}(1)^T$, that

$$\|F(\xi)\| \geq -\frac{\langle \xi, f(\xi) - f(0)\rangle}{\|f(\xi) - f(0)\| + \|f(0)\|} - 2\|f(0)\|\|\mathbf{G}(1)\|. \tag{4.99}$$

For the second case of when $2\|\mathbf{G}(1)\|\|f(\xi)\| \leq \|\xi\|$, we see again that (4.98) holds. Thus, an application of Lemma 4.3.1 combined with (4.98) and (4.99) implies that $\|F(\xi)\| \to \infty$ as $\|\xi\| \to \infty$. Therefore, again since F is continuous and injective by statement (i), [30, Theorem 4.3] gives that F is surjective and the proof is complete. □

The following provides a useful characterisation of functions $f : \mathbb{R}^m \to \mathbb{R}^m$ that satisfy the assumptions of statement (iii) of Lemma 4.3.7. We shall not provide a proof, since the result is immediately obtained from Lemma 4.3.2.

Corollary 4.3.8. *Let $f : \mathbb{R}^m \to \mathbb{R}^m$. Then f is continuous and strictly dissipative, and there exists $\eta \in \mathbb{R}^m$ such that* (4.79) *holds, if, and only if, there exists a continuous, strictly dissipative function $g : \mathbb{R}^m \to \mathbb{R}^m$ and a constant $c \in \mathbb{R}^m$ such that $g(0) = 0$ and both* (4.92) *and* (4.93) *hold.*

Remark 4.3.9. Since every strictly dissipative function $g : \mathbb{R}^m \to \mathbb{R}^m$ with $g(0) = 0$, also satisfies that $-\langle \xi, g(\xi)\rangle > 0$ for all $\xi \in \mathbb{R}^m\backslash\{0\}$, we see that the function g described in the statement of Corollary 4.3.8 satisfies (4.39) and (4.57) (where we replace f by g) for some $\alpha \in \mathcal{K}_\infty$. Indeed, this can be shown by combining Lemma 4.1.7 with (4.93). Hence, we see that $f : \mathbb{R}^m \to \mathbb{R}^m$ is continuous and strictly dissipative, and there exists $\eta \in \mathbb{R}^m$ such that (4.79) holds, if, and only if, $\xi \mapsto f(\xi) - f(0)$ is strictly dissipative and satisfies the hypotheses imposed on the nonlinearity in Theorem 4.2.9. We recall that Theorem 4.2.9 provides sufficient conditions for when (2.17) is ISS. $\Diamond$

We conclude the current presentation of preliminary results with the following two lemmas.

Lemma 4.3.10. *Let $\Sigma \in \mathbb{L}$. If there exists a positive semi-definite $P = P^T \in \mathbb{R}^{n\times n}$ such that* (4.38) *holds, then for every $z \in \mathbb{C}\backslash\sigma(A)$ with $|z| \geq 1$, $\mathbf{G}(z) + \mathbf{G}(z)^* \geq 0$.*

Lemma 4.3.11. *Let $\Sigma \in \mathbb{L}$. If there exists a positive definite $P = P^T \in \mathbb{R}^{n\times n}$ such that* (4.73) *holds, then for every $z \in \mathbb{C}$ with $|z| \geq 1$, $\mathbf{G}(z) + \mathbf{G}(z)^* > 0$.*

We shall not provide a proof of either lemma, since they are both known in the literature. Indeed, for Lemma 4.3.10, this is one of the implications given in [54, Lemma 3], which utilises the continuous-time version of the result which is given in [3, Theorem 3]. Although the previous two results presuppose a minimal realisation and also that $P > 0$, we note that in the proof of the relevant implication that we are interested in, these properties are not utilised. As for Lemma 4.3.11, we refer the reader to [51, Lemma 4.2] and its continuous-time counterpart [50, Lemma 4.2]. Once again, although minimality is assumed, the proof of the implication we are concerned with does not use this.

4.3.2 The converging-input converging-state property

We are now ready to present conditions which guarantee that (2.17) exhibits the CICS property. Before we do so however, for the benefit of the reader, we briefly recall some definitions associated with (2.17) which were given in Sections 2.3.1 and 2.3.2 (in the non-square case). Indeed, for $\Sigma \in \mathbb{L}$ and $f : \mathbb{R}^m \to \mathbb{R}^m$. We say that (2.17) has the *converging-input converging-state (CICS) property* if for every $v^\infty \in \mathbb{R}^q$ and $w^\infty \in \mathbb{R}^m$, there exists $x^\infty \in \mathbb{R}^n$ such that $\lim_{t\to\infty} x(t) = x^\infty$ for every $(v, w, x, y) \in \mathcal{B}$ with $\lim_{t\to\infty} v(t) = v^\infty$ and $\lim_{t\to\infty} w(t) = w^\infty$. Moreover, a quadruple $(v^{\mathrm{e}}, w^{\mathrm{e}}, x^{\mathrm{e}}, y^{\mathrm{e}}) \in \mathbb{R}^q \times \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^m$ is an *equilibrium quadruple* of (2.17) if $(v^{\mathrm{e}}, w^{\mathrm{e}}, x^{\mathrm{e}}, y^{\mathrm{e}}) \in \mathcal{B}$, where we abuse notation and interchangeably write vectors as constant functions. Furthermore, we say that a subset $\mathcal{V} \subseteq (\mathbb{R}^q)^{\mathbb{Z}_+}$ is *equi-convergent* to $v^\infty \in \mathbb{R}^q$ if, for all $\varepsilon > 0$, there exists $\tau \in \mathbb{Z}_+$ such that

$$\|(\Lambda_\tau v)(t) - v^\infty\| \leq \varepsilon \quad \forall\, t \in \mathbb{Z}_+, \forall\, v \in \mathcal{V}.$$

Finally, for ease of notation in the sequel, we define $F : \mathbb{R}^m \to \mathbb{R}^m$ by $F(\xi) := \xi - \mathbf{G}(1)f(\xi)$ for all $\xi \in \mathbb{R}^m$ and, for a singelton set $\{\xi\}$, where $\xi \in \mathbb{R}^m$, we shall abuse notation and set $F^{-1}(\xi) := F^{-1}(\{\xi\})$ (recall the definition of the preimage given in the notation section).

The following is the first of two main results of this section.

Theorem 4.3.12. *Let $\Sigma \in \mathbb{L}$ and $f : \mathbb{R}^m \to \mathbb{R}^m$. Assume that Σ is detectable, $1 \notin \sigma(A)$ and that there exists a positive semi-definite $P = P^T \in \mathbb{R}^{n\times n}$ such that* (4.38) *holds. Furthermore, assume that f is continuous and strictly dissipative, and that there exists $\eta \in \mathbb{R}^m$ such that* (4.79) *holds. The following statements hold.*

(i) *The map F is bijective.*

(ii) *For all $v^\infty \in \mathbb{R}^q$ and $w^\infty \in \mathbb{R}^m$, by letting $z^\infty \in F^{-1}((C(I-A)^{-1}B_\mathrm{e} + D_\mathrm{e})v^\infty + w^\infty)$, and setting $y^\infty := z^\infty - w^\infty$ and*

$$x^\infty := (I-A)^{-1}\left(Bf(z^\infty) + B_\mathrm{e}v^\infty\right), \tag{4.100}$$

then

$$y^\infty = Cx^\infty + Df(z^\infty) + D_\mathrm{e}v^\infty, \tag{4.101}$$

and $(v^\infty, w^\infty, x^\infty, y^\infty)$ is an equilibrium quadruple of (2.17).

(iii) *Let $v^\infty \in \mathbb{R}^q$ and $w^\infty \in \mathbb{R}^m$. Then there exist $\psi_1, \psi_2 \in \mathcal{KL}$ and $\phi \in \mathcal{K}$ such that, for all $(v, w, x, y) \in \mathcal{B}$ and $t \in \mathbb{Z}_+$,*

$$\begin{aligned}\|x(t) - x^\infty\| \leq & \psi_1(\|x(0) - x^\infty\|, t) + \psi_2\left(\max_{s\in\underline{\lfloor t/2\rfloor}}\left\|\begin{pmatrix} v(s) - v^\infty \\ w(s) - w^\infty\end{pmatrix}\right\|, t\right) \\ & + \phi\left(\max_{s\in\underline{\lceil t/2\rceil}}\left\|\begin{pmatrix}\Lambda_{\lfloor t/2\rfloor}v(s) - v^\infty \\ \Lambda_{\lfloor t/2\rfloor}w(s) - w^\infty\end{pmatrix}\right\|\right),\end{aligned} \tag{4.102}$$

where x^∞ is given by (4.100). In particular, $\lim_{t\to\infty} x(t) = x^\infty$ for all $(v, w, x, y) \in \mathcal{B}$ with $\lim_{t\to\infty} v(t) = v^\infty$ and $\lim_{t\to\infty} w(t) = w^\infty$.

Before proving Theorem 4.3.12, we give the following remark.

Remark 4.3.13. (i) The convergence described in the last line of Theorem 4.3.12 is uniform in the following sense: given a set of inputs $\mathcal{V} \subseteq (\mathbb{R}^q)^{\mathbb{Z}_+} \times (\mathbb{R}^m)^{\mathbb{Z}_+}$ which is equi-convergent to (v^∞, w^∞) and $\kappa > 0$, the set

$$\left\{x \in (\mathbb{R}^n)^{\mathbb{Z}_+} : \exists\,(v, w, y) \in \mathcal{V} \times (\mathbb{R}^m)^{\mathbb{Z}_+} \text{ s.t. } (v, w, x, y) \in \mathcal{B} \right. \\ \left. \text{and } \|x(0)\| + \sup_{t\in\mathbb{Z}_+}\left\|\begin{pmatrix} v(t) \\ w(t)\end{pmatrix}\right\| \leq \kappa\right\}$$

is equi-convergent to x^∞.

(ii) From Corollary 4.3.8 (as discussed in Remark 4.3.9), $f : \mathbb{R}^m \to \mathbb{R}^m$ is continuous and strictly dissipative, and there exists $\eta \in \mathbb{R}^m$ such that (4.79) holds, if, and only if, $\xi \mapsto f(\xi) - f(0)$ is strictly dissipative and satisfies the hypotheses imposed on the nonlinearity in Theorem 4.2.9, which is a result that provides sufficient conditions for when (2.17) is ISS.

(iii) If $f : \mathbb{R}^m \to \mathbb{R}^m$ is continuous and strictly dissipative, we note that in the case that $m = 1$, (4.79) may be reduced to simply saying that

$$-\frac{\xi(f(\xi+\eta) - f(\eta))}{|\xi|} \to \infty \text{ as } |\xi| \to \infty.$$

This is because, since $m = 1$, the strict dissipativity of f guarantees that $\mathrm{sign}(f(\xi+\eta) - f(\eta)) = -\mathrm{sign}(\xi)$ for all $\xi \in \mathbb{R}$. We thus see that

$$-\frac{\xi(f(\xi+\eta) - f(\eta))}{|f(\xi+\eta) - f(\eta)|} = |\xi| \to \infty \text{ as } |\xi| \to \infty$$

as a consequence. ♢

Proof of Theorem 4.3.12. We begin by applying Lemma 4.3.10 to obtain that $\mathbf{G}(1) + \mathbf{G}(1)^T \geq 0$. From this, we utilise statement (i) and statement (iii) of Lemma 4.3.7 so that F is bijective, which gives statement (i). As a consequence, for all $v^\infty \in \mathbb{R}^q$ and all $w^\infty \in \mathbb{R}^m$, $F^{-1}((C(I-A)^{-1}B_\mathrm{e} + D_\mathrm{e})v^\infty + w^\infty)$ is non-empty. Let $v^\infty \in \mathbb{R}^q$ and $w^\infty \in \mathbb{R}^m$ and let $z^\infty \in F^{-1}((C(I-A)^{-1}B_\mathrm{e}+D_\mathrm{e})v^\infty+w^\infty)$. By setting $y^\infty = z^\infty - w^\infty$ and defining x^∞ by (4.100),

$$\begin{aligned} y^\infty &= F(z^\infty) + \mathbf{G}(1)f(z^\infty) - w^\infty \\ &= (C(I-A)^{-1}B_\mathrm{e} + D_\mathrm{e})v^\infty + (C(I-A)^{-1}B + D)f(z^\infty) \\ &= Cx^\infty + Df(z^\infty) + D_\mathrm{e}v^\infty, \end{aligned}$$

which is (4.101). Hence, it is clear that $(v^\infty, w^\infty, x^\infty, y^\infty)$ is an equilibrium quadruple of (2.17), and statement (ii) is proven. Our attention now turns to statement (iii) and so we let $v^\infty \in \mathbb{R}^q$ and $w^\infty \in \mathbb{R}^m$ and define $\tilde{f} : \mathbb{R}^m \to \mathbb{R}^m$ by

$$\tilde{f}(\xi) := f(\xi + y^\infty + w^\infty) - f(y^\infty + w^\infty) \quad \forall\, \xi \in \mathbb{R}^m,$$

where $y^\infty = z^\infty - w^\infty$ and $z^\infty \in F^{-1}((C(I-A)^{-1}B_\mathrm{e} + D_\mathrm{e})v^\infty + w^\infty)$. Since statement (ii) implies that $(v^\infty, w^\infty, x^\infty, y^\infty)$ is an equilibrium quadruple, where x^∞ is given by (4.100), we see that $(v, w, x, y) \in \mathcal{B}$ if, and only if,

$$\begin{aligned} \tilde{x}^+ &= x^+ - x^\infty \\ &= Ax + Bf(y+w) + B_\mathrm{e}v - Ax^\infty - Bf(y^\infty + w^\infty) - B_\mathrm{e}v^\infty \\ &= A\tilde{x} + B\tilde{f}(\tilde{y} + \tilde{w}) + B_\mathrm{e}\tilde{v}, \end{aligned}$$

and

$$\begin{aligned} \tilde{y} &= Cx + Df(y+w) + D_\mathrm{e}v - Cx^\infty - Df(y^\infty + w^\infty) - D_\mathrm{e}v^\infty \\ &= C\tilde{x} + D\tilde{f}(\tilde{y} + w^\infty) + D_\mathrm{e}\tilde{v}, \end{aligned}$$

where $\tilde{y}(t) := y(t) - y^\infty$, $\tilde{x}(t) := x(t) - x^\infty$, $\tilde{v}(t) := v(t) - v^\infty$ and $\tilde{w}(t) := w(t) - w^\infty$ for all $t \in \mathbb{Z}_+$. That is, $(v, w, x, y) \in \mathcal{B}$ if, and only if, $(\tilde{v}, \tilde{w}, \tilde{x}, \tilde{y}) \in \mathcal{B}_{\tilde{f}}$. Since f is strictly dissipative, we see that

$$-\langle \xi, \tilde{f}(\xi)\rangle = -\langle \xi, f(\xi + y^\infty + w^\infty) - f(y^\infty + w^\infty)\rangle > 0 \quad \forall\, \xi \in \mathbb{R}^m \backslash \{0\}. \tag{4.103}$$

Moreover, since (4.79) holds, an application of Lemma 4.3.1 gives that

$$\min\left\{ -\frac{\langle \xi, \tilde{f}(\xi)\rangle}{\|\xi\|}, -\frac{\langle \xi, \tilde{f}(\xi)\rangle}{\|\tilde{f}(\xi)\|} \right\} \to \infty \text{ as } \|\xi\| \to \infty. \tag{4.104}$$

Hence, by combining this with the continuity of f and (4.103), Lemma 4.1.7 yields that there exists $\alpha \in \mathcal{K}_\infty$ such that

$$\alpha(\|\xi\|) \leq -\frac{\langle \xi, \tilde{f}(\xi)\rangle}{\|\xi\|} \quad \forall\, \xi \in \mathbb{R}^m \backslash \{0\}.$$

Therefore, by combining this with (4.104), Theorem 4.2.9 yields the existence of $\psi \in \mathcal{KL}$ and $\phi \in \mathcal{K}$ such that (4.35) holds, but for each element of $\mathcal{B}_{\tilde{f}}$ instead of $\mathcal{B}$. In particular, for all $(v, w, x, y) \in \mathcal{B}$ and all $t \in \mathbb{Z}_+$, by using the left-shift invariance of $\mathcal{B}_{\tilde{f}}$ and that $(\tilde{v}, \tilde{w}, \tilde{x}, \tilde{y}) \in \mathcal{B}_{\tilde{f}}$,

$$\begin{aligned}\|x(t) - x^\infty\| &= \|(\Lambda_{\lfloor t/2 \rfloor} x)(\lceil t/2 \rceil) - x^\infty\| \\ &\le \psi\left(\|x(\lfloor t/2 \rfloor) - x^\infty\|, \lceil t/2 \rceil\right) + \phi\left(\max_{s \in \underline{\lceil t/2 \rceil}} \left\| \begin{pmatrix} \Lambda_{\lfloor t/2 \rfloor} v(s) - v^\infty \\ \Lambda_{\lfloor t/2 \rfloor} w(s) - w^\infty \end{pmatrix} \right\| \right).\end{aligned}$$

If we then take another ISS estimate of the term $\|x(\lfloor t/2 \rfloor) - x^\infty\|$ in the above inequality, we obtain, for all $(v, w, x, y) \in \mathcal{B}$ and all $t \in \mathbb{Z}_+$,

$$\begin{aligned}\|x(t) - x^\infty\| \le &\psi\left(\psi(\|x(0) - x^\infty\|, \lfloor t/2 \rfloor) + \phi\left(\max_{s \in \underline{\lfloor t/2 \rfloor}} \left\| \begin{pmatrix} v(s) - v^\infty \\ w(s) - w^\infty \end{pmatrix} \right\| \right), \lceil t/2 \rceil\right) \\ &+ \phi\left(\max_{s \in \underline{\lceil t/2 \rceil}} \left\| \begin{pmatrix} \Lambda_{\lfloor t/2 \rfloor} v(s) - v^\infty \\ \Lambda_{\lfloor t/2 \rfloor} w(s) - w^\infty \end{pmatrix} \right\| \right).\end{aligned}$$

We now set $\psi_1(s,t) := \psi\left(2\psi(s, \lfloor t/2 \rfloor), \lceil t/2 \rceil\right)$ and $\psi_2(s,t) := \psi\left(2\phi(s), \lceil t/2 \rceil\right)$, to yield, for all $(v, w, x, y) \in \mathcal{B}$ and all $t \in \mathbb{Z}_+$, that (4.102) holds, thus completing the proof. □

The following theorem is the second main result of this section.

Theorem 4.3.14. *Let $\Sigma \in \mathbb{L}$ and $f : \mathbb{R}^m \to \mathbb{R}^m$. Assume that there exists a positive definite $P = P^T \in \mathbb{R}^{n \times n}$ such that (4.73) holds. Furthermore, assume that f is continuous and dissipative. Then statements (i)-(iii) of Theorem 4.3.12 hold.*

Before proving Theorem 4.3.14, we give the following remark.

Remark 4.3.15. (i) Similar to Theorem 4.2.12, the strength of (4.73) holding for some positive definite matrix $P \in \mathbb{R}^{n \times n}$, means that weaker assumptions may be imposed on the nonlinearity than that seen in Theorem 4.3.12.

(ii) In the case where $m = 1$, the assumptions concerning f in Theorem 4.3.14 are simply that f is continuous and decreasing.

(iii) We highlight that we do not need to assume that $1 \notin \sigma(A)$ in Theorem 4.3.14 (as we do in Theorem 4.3.12), since the existence of a positive definite $P = P^T \in \mathbb{R}^{n \times n}$ such that (4.73) holds, implies that A is Schur (see Lemma 4.1.8 and Remark 4.2.13). ◊

Proof of Theorem 4.3.14. We begin with an application of Lemma 4.3.11 to obtain that $\mathbf{G}(1) + \mathbf{G}(1)^T > 0$. We subsequently invoke statement (i) and statement (ii) of Lemma 4.3.7 to obtain that the map $\xi \mapsto \xi - \mathbf{G}(1) f(\xi)$ is bijective. The rest of the proof concerns showing that, for all $v^\infty \in \mathbb{R}^q$ and all $w^\infty \in \mathbb{R}^m$, $(v^\infty, w^\infty, x^\infty, y^\infty)$ is an equilibrium quadruple of (2.17), and to then invoke Theorem 4.2.12 twice to obtain the correct estimates. Since this is very similar to the proof Theorem 4.3.12, we shall omit it and leave it to the reader to verify. □

We now present four corollaries of the previous two theorems. The first two succinctly express assumptions that are sufficient for guaranteeing that (2.17) exhibits the CICS property, and are immediately obtained from Theorem 4.3.12 and Theorem 4.3.14, respectively.

Corollary 4.3.16. *Let $\Sigma \in \mathbb{L}$ and $f : \mathbb{R}^m \to \mathbb{R}^m$. Assume that Σ is detectable, $1 \notin \sigma(A)$ and that there exists a positive semi-definite $P = P^T \in \mathbb{R}^{n\times n}$ such that* (4.38) *holds. Furthermore, assume that f is continuous and strictly dissipative, and that there exists $\eta \in \mathbb{R}^m$ such that* (4.79) *holds. Then* (2.17) *has the CICS property.*

Corollary 4.3.17. *Let $\Sigma \in \mathbb{L}$ and $f : \mathbb{R}^m \to \mathbb{R}^m$. Assume that there exists a positive definite $P = P^T \in \mathbb{R}^{n\times n}$ such that* (4.73) *holds. Furthermore, assume that f is continuous and dissipative. Then* (2.17) *has the CICS property.*

The third and fourth corollaries give assumptions in the form of positive and strict positive real conditions, that assert that (2.17) exhibits the CICS property. The next result is immediately obtained from a combination of Corollary 4.3.16 with Lemma 4.1.15, and the final corollary can be obtained via a combination of Corollary 4.3.17 with Lemma 4.1.20 (see the proof of Corollary 4.2.19 for details on how to prove this).

Corollary 4.3.18. *Let $\Sigma \in \mathbb{L}$ and let $f : \mathbb{R}^m \to \mathbb{R}^m$. Assume that $1 \notin \sigma(A)$, $\mathbf{G}$ is positive real and that either:* (i) *Σ is controllable and observable, or;* (ii) *Σ is stabilisable and detectable and there exists $\eta \in \partial\mathbb{D}$, not a pole of $\mathbf{G}$, such that $\mathbf{G}(\eta) + \mathbf{G}(\eta)^* > 0$. Furthermore, assume that $f : \mathbb{R}^m \to \mathbb{R}^m$ is continuous and strictly dissipative, and that there exists $\eta \in \mathbb{R}^m$ such that* (4.79) *holds. Then* (2.17) *has the CICS property.*

Corollary 4.3.19. *Let $\Sigma \in \mathbb{L}$ and $f : \mathbb{R}^m \to \mathbb{R}^m$. Assume that Σ is controllable and observable, $\mathbf{G}$ is strictly positive real, and $D + D^T > 0$. Furthermore, assume that f is continuous and dissipative. Then* (2.17) *has the CICS property.*

We conclude this section with two examples.

Example 4.3.20. Let $n = m = 2$. Consider (2.17) with linear part given as in Example 4.2.8, that is,

$$A = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad B = B_{\mathrm{e}} = C = D = D_{\mathrm{e}} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

and nonlinearity given by

$$f(\xi) := c - \|\xi\|^2\xi \quad \forall\, \xi \in \mathbb{R}^m,$$

where $c \in \mathbb{R}^m$. Example 4.2.8 showed that there exists a positive semi-definite matrix $P \in \mathbb{R}^{2\times 2}$ such that (4.38) holds. In order to show that f satisfies the relevant assumptions of Corollary 4.3.16, from Corollary 4.3.8, we see that it is sufficient to show that $g : \mathbb{R}^m \to \mathbb{R}^m$, defined by $g(\xi) := -\|\xi\|^2\xi$ for all $\xi \in \mathbb{R}^m$, is continuous, strictly dissipative and satisfies (4.92), (4.93) and $g(0) = 0$, which we do so now. Indeed, g is trivially continuous, zero at zero and satisfies (4.92). Moreover, a combination of Example 4.2.6 and Example 4.2.11 yields that (4.93) holds. Therefore, all that is left to show is that g is strictly dissipative. To prove this, let $\xi, \zeta \in \mathbb{R}^m$ with $\xi \neq 0$. We shall assume that $\zeta \neq -\xi$ and $\zeta \neq 0$, since otherwise,

$$-\langle \xi, g(\xi+\zeta) - g(\zeta)\rangle = \|\xi\|^4 > 0.$$

With this in mind, note that

$$\begin{aligned} -\langle \xi, g(\xi+\zeta) - g(\zeta)\rangle &= -\left\langle \xi, -\|\xi+\zeta\|^2(\xi+\zeta) + \|\zeta\|^2\zeta \right\rangle \\ &= \|\xi+\zeta\|^2\|\xi\|^2 + \left(\|\xi+\zeta\|^2 - \|\zeta\|^2\right)\langle \xi, \zeta\rangle, \end{aligned} \tag{4.105}$$

and consider the following cases. For the first case, assume that $\|\xi+\zeta\|^2 - \|\zeta\|^2 < 0$. This then implies that

$$0 > \|\xi+\zeta\|^2 - \|\zeta\|^2 = \|\xi\|^2 + 2\langle \xi, \zeta\rangle,$$

which in turn implies that $\langle \xi, \zeta\rangle < 0$. Combining this with (4.105) yields that

$$-\langle \xi, g(\xi+\zeta) - g(\zeta)\rangle > 0.$$

For the second case, assume that $\|\xi+\zeta\|^2 - \|\zeta\|^2 \geq 0$. Now, if $\langle \xi, \zeta\rangle \geq 0$, then (4.105) gives that

$$-\langle \xi, g(\xi+\zeta) - g(\zeta)\rangle \geq \|\xi+\zeta\|^2\|\xi\|^2 > 0,$$

where we have utilised that $\xi \neq 0$ and $\xi \neq -\zeta$. Finally, if $\langle \xi, \zeta\rangle < 0$, then since $\|\xi+\zeta\|^2 - \|\zeta\|^2 \geq 0$, we have that

$$\|\xi\|^2 + \langle \xi, \zeta\rangle > \|\xi\|^2 + 2\langle \xi, \zeta\rangle = \|\xi+\zeta\|^2 - \|\zeta\|^2 \geq 0$$

which in turn, by (4.105), yields that

$$-\langle \xi, g(\xi+\zeta) - g(\zeta)\rangle \geq \|\xi+\zeta\|^2 \left(\|\xi\|^2 + \langle \xi, \zeta\rangle\right) - \|\zeta\|^2\langle \xi, \zeta\rangle > 0,$$

where we have used that $\zeta \neq 0$. We have hence shown that $-\langle \xi, g(\xi+\zeta) - g(\zeta)\rangle > 0$ for all $\zeta, \xi \in \mathbb{R}^m$ with $\xi \neq 0$, i.e. that g is strictly dissipative. Therefore, f satisfies the relevant assumptions of Corollary 4.3.16 and so we obtain that (2.17) exhibits the CICS property. ◊

Example 4.3.21. Let $n = 2$ and $m = 1$. Consider (2.17) with linear part described in Example 4.2.15, that is,

$$A = \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \end{pmatrix}, \quad D = 2,$$

and B_{e} and D_{e} are arbitrary. Moreover, we take the nonlinearity f to be given by

$$f(\xi) = c + \begin{cases} -\mathrm{sign}(\xi)\sqrt{|\xi|}, & \text{if } |\xi| \leq 1 \\ -\mathrm{sign}(\xi), & \text{if } |\xi| > 1. \end{cases}$$

where $c \in \mathbb{R}$ is a constant. In Example 4.2.15, it was shown that (4.73) holds for some positive definite matrix $P \in \mathbb{R}^{2\times 2}$. Furthermore, it is easy to see that f is continuous and dissipative. Therefore, Corollary 4.3.17 guarantees that (2.17) has the CICS property. Additionally, we highlight that f does not satisfy the relevant assumptions of Corollary 4.3.16. ◊

4.4 Notes and references

We reiterate that the paper [7] served as motivation for Section 4.2, and indeed the chapter as a whole. To the best of the author's knowledge, there is no discrete-time version of [7] in the literature and so our investigation is well-justified. Furthermore, again to the best of the author's knowledge, the convergence results of Section 4.3, which are applicable to forced Lur'e systems with potentially superlinear nonlinearities, are not found elsewhere.

In Section 4.1.3, we presented a strict positive real lemma that we obtained from [51, Lemma 4.2]. We note that a similar result in the literature is [119, Theorem 3.7]. Indeed, in [119, Theorem 3.7], sufficient conditions are given for the existence of a positive definite matrix P and matrices R and W such that (4.33) holds. We did not use [119, Theorem 3.7] in this chapter since the result does not guarantee that W is positive definite, which is a property we heavily made use of in subsequent applications. We also recall statement (ii) of Lemma 4.1.15, which is a form of the positive real lemma for potentially nonminimal realisations. A noteworthy lemma from the literature is [127, Lemma 6]. The result is what the authors call a "generalized discrete-time positive real lemma", since it holds for nonminimal realisations. The equations derived involve the corresponding reachability matrix. For further reading of positive realness in the discrete-time setting, we refer the reader to the additional sources [51, 54, 68, 125].

Recall Corollary 4.2.19 which presents conditions, in terms of strict positive realness on the transfer function, that guarantee that (2.17) is ISS. A relevant result found in the literature is [51, Theorem 4.1], where the same assumptions are used to conclude asymptotic stability of (2.17). We highlight that this assertion is significantly weaker than ISS, hence showing the strength of Corollary 4.2.19.

We make note that continuous-time analogues of Theorem 4.2.9 and Theorem 4.2.12 are also possible to obtain, thus furthering the work of [7] to systems which have output disturbances, feedthrough, and inputs entering the system externally to the linear system. Moreover, in a similar manner to that already done, one may also develop continuous-time analogues of the convergence results found in Section 4.3. In the interest of brevity, we do not provide these and instead leave them to the reader to formulate.

Part II

Infinite-dimensional Lur'e systems

Chapter 5

Stability and convergence properties of infinite-dimensional discrete-time Lur'e systems

In this chapter, we investigate stability and convergence properties of forced infinite-dimensional, discrete-time Lur'e systems. In particular, we derive conditions that guarantee that such systems, which are infinite-dimensional versions of those described by (2.17), are exponentially incrementally ISS, exhibit the CICS property, and produce asymptotically almost periodic state trajectories when under asymptotically almost periodic forcing. We note that ISS, CICS, incremental stability and almost periodic functions, have already been discussed in this thesis and so, in the interest of not repeating ourselves, we refer the reader to the introduction for a review of the relevant background literature. This chapter is based on the paper [40].

The chapter is structured as follows. We begin in Section 5.1 by giving theory concerning linear, infinite-dimensional difference equations and by introducing the Lur'e system of our current focus. We then utilise these preliminaries to obtain the main stability results of this chapter, which are presented in Section 5.2. Indeed, we provide sufficient conditions for when the aforementioned systems are exponentially incrementally ISS, and then give a series of corollaries with assumptions that are reminiscent of classical absolute stability hypotheses, namely the well-known circle criterion (see, for example, [53]). Following this, in Section 5.3, we investigate certain convergence properties of these infinite-dimensional Lur'e systems. Finally, in Sections 5.4 and 5.5, we present applications of our work by using the previous results to deduce stability and convergence properties of four-block Lur'e systems (see, for example, [40, 47] or Section 2.4) and sampled-data systems (see, for example, [73, 75]), respectively.

Throughout this chapter, we let X and V be complex Banach spaces and Y and U be complex Hilbert spaces. Moreover, we assume that the norm on the cartesian product of two normed spaces W_1 and W_2 is given by

$$\left\| \begin{pmatrix} \xi \\ \zeta \end{pmatrix} \right\|_{W_1 \times W_2} := \sqrt{\|\xi\|_{W_1}^2 + \|\zeta\|_{W_2}^2} \quad \forall\, \xi \in W_1, \forall\, \zeta \in W_2. \tag{5.1}$$

5.1 Preliminaries

We split this initial section into four, and begin with a short presentation of exponential stability of bounded linear operators. Following this, we then give theory regarding infinite-dimensional linear difference equations, and, in particular, investigate linear output feedback. Subsequently, we discuss convolution operators and causality, which culminates in determining a result that is of key importance to the main results of this chapter (see Theorem 5.2.5). Finally, we give the forced infinite-dimensional discrete-time Lur'e system which will be the focus of our stability investigations in later sections.

5.1.1 Exponential stability, stabilisability and detectability

We begin by recalling the following definitions.

Definition 5.1.1. *Let $A \in \mathcal{L}(X)$.*

(i) *We say that A is* exponentially stable *if there exist $M \geq 1$ and $\mu \in (0,1)$ such that*

$$\|A^i\| \leq M\mu^i \quad \forall\, i \in \mathbb{Z}_+. \tag{5.2}$$

(ii) *We define the* exponential growth constant *of A as the infimum of all $\mu > 0$ that satisfy* (5.2)*, for some $M \geq 1$.*

The following lemma comprises two results from the literature regarding the exponential stability of bounded linear operators. Indeed, statement (i) is a direct consequence of [72, Lemma 1] and statement (ii) can be found in, for example, [97, Theorem 18.9, p. 360].

Lemma 5.1.2. *Let $A \in \mathcal{L}(X)$ and denote the spectral radius of A by $\rho(A)$. The following statements hold.*

(i) *A is exponentially stable if, and only if, $\rho(A) < 1$.*

(ii) *$\rho(A) = \lim_{k\to\infty} \|A^k\|^{1/k}$.*

Remark 5.1.3. As a consequence of the previous result, $A \in \mathcal{L}(X)$ is exponentially stable if, and only if, its spectrum is contained in $\mathbb{D}$. Moreover, we also see that the spectral radius of A is precisely the exponential growth constant of A. ◊

We now define the notion of exponential stabilisability and exponential detectability.

Definition 5.1.4. (i) *We say that $(A, B) \in \mathcal{L}(X) \times \mathcal{L}(U, X)$ is* exponentially stabilisable *if there exists $F \in \mathcal{B}(X, U)$ such that $A + BF$ is exponentially stable.*

(ii) *We say that $(A, C) \in \mathcal{L}(X) \times \mathcal{L}(X, Y)$ is* exponentially detectable *if there exists $H \in \mathcal{B}(Y, X)$ such that $A + HC$ is exponentially stable.*

Remark 5.1.5. We comment that, for the most part, we will omit the word "exponential" and shall simply say that a pair of operators is stabilisable or detectable. ◊

We conclude this initial subsection with the following useful, well-known result, the proof of which is straightforward and is thus omitted.

Lemma 5.1.6. *Let $A \in \mathcal{L}(X)$. If A is exponentially stable, then $I - A$ is bijective and its inverse is given by*

$$(I - A)^{-1} = \sum_{j=0}^{\infty} A^j.$$

5.1.2 Linear systems theory and linear output feedback

Our attention now turns to the theory of infinite-dimensional linear controlled and observed difference equations. This work will underpin our development of the rest of this chapter. We shall consider

$$\left.\begin{aligned} x^+ &= Ax + Bu + B_\mathrm{e}v\,, \\ y &= Cx + Du + D_\mathrm{e}v\,, \end{aligned}\right\} \tag{5.3}$$

where $(A, B, B_\mathrm{e}, C, D, D_\mathrm{e}) \in \mathcal{L}(X) \times \mathcal{L}(U, X) \times \mathcal{L}(V, X) \times \mathcal{L}(X, Y) \times \mathcal{L}(U, Y) \times \mathcal{L}(V, Y)$, $u \in U^{\mathbb{Z}_+}$ and $v \in V^{\mathbb{Z}_+}$. The variables x and y in (5.3) are called the state and output, respectively, and u and v are inputs. Occasionally, it will be convenient to identify the linear system (5.3) and the sextuple $(A, B, B_\mathrm{e}, C, D, D_\mathrm{e})$ and to refer to the linear system $(A, B, B_\mathrm{e}, C, D, D_\mathrm{e})$.

Remark 5.1.7. In the case that $X = \mathbb{R}^n$, $U = \mathbb{R}^m$, $Y = \mathbb{R}^p$ and $V = \mathbb{R}^q$, where $n, m, p, q \in \mathbb{N}$, (5.3) is precisely the finite-dimensional controlled and observed linear difference equation given by (2.6). ♢

We now make a series of definitions and give several results regarding (5.3). Many of these definitions and results can be considered to be infinite-dimensional versions of those laid out in Section 2.1.2. We shall therefore be brief here and only go into detail where the infinite-dimensionality is important. The notation we use here should not be confused with its finite-dimensional counterparts seen previously in this thesis.

Definition 5.1.8. (i) *We define*

$$\mathbb{L} := \mathcal{L}(X) \times \mathcal{L}(U, X) \times \mathcal{L}(V, X) \times \mathcal{L}(X, Y) \times \mathcal{L}(U, Y) \times \mathcal{L}(V, Y).$$

(ii) *For* $(A, B, B_\mathrm{e}, C, D, D_\mathrm{e}) \in \mathbb{L}$, *we define*

$$\Sigma := (A, B, B_\mathrm{e}, C, D, D_\mathrm{e}).$$

Remark 5.1.9. For $\Sigma \in \mathbb{L}$, (5.3) encompasses many (seemingly more general) systems. Indeed, we refer the reader to Remark 2.1.10 where we discussed this in the finite-dimensional setting. The idea there applies here also. ♢

We also record the following definitions associated with (5.3).

Definition 5.1.10. *Let* $\Sigma \in \mathbb{L}$.

(i) *We denote by* $\mathbf{G}$ *the* transfer function *of* (5.3) *from* u *to* y, *that is,*

$$\mathbf{G}(z) = C(zI - A)^{-1}B + D,$$

for all $z \in \mathbb{C}$ *for which this makes sense.*

(ii) *We say that* Σ *is* exponentially stable *if* A *is,* stabilisable *if* (A, B) *is, and* detectable *if* (A, C) *is.*

(iii) *We define the* behaviour *of* (5.3) *as*

$$\mathcal{B}^{\text{lin}}(\Sigma) := \Big\{(u, v, x, y) \in U^{\mathbb{Z}_+} \times V^{\mathbb{Z}_+} \times X^{\mathbb{Z}_+} \times Y^{\mathbb{Z}_+} : (u, v, x, y) \text{ satisfies } (5.3)\Big\} ,$$

and, when the context is clear, we will suppress this to just $\mathcal{B}^{\text{lin}}$*.*

The following lemma is [72, Theorem 2], which we present here for completeness.

Lemma 5.1.11. *Let* $\Sigma \in \mathbb{L}$*. Then* Σ *is exponentially stable if, and only if,* Σ *is stabilisable and detectable and* $\mathbf{G} \in H^\infty(\mathcal{L}(U, Y))$*.*

The next result shows that the transfer function of (5.3) is always bounded and holomorphic on the exterior of any open disc in $\mathbb{C}$ centred at 0 with radius greater than the exponential growth constant of A.

Lemma 5.1.12. *Let* $\Sigma \in \mathbb{L}$ *and let* μ *be the exponential growth constant of* A*. Then*

$$\mathbf{G} \in H^\infty_\alpha(\mathcal{L}(U, Y)) \quad \forall\, \alpha > \mu.$$

Whilst Lemma 5.1.12 is a known result, we include a proof here since one in the literature is difficult to locate.

Proof of Lemma 5.1.12. Let $\alpha > \mu$ and note that $\mu/\alpha \in (0, 1)$. Hence, A/α is exponentially stable and so, by using Lemma 5.1.11, we obtain that $\mathbf{G}_\alpha \in H^\infty(\mathcal{L}(U, Y))$, where $\mathbf{G}_\alpha$ denotes the transfer function of the system with linear component given by $(A/\alpha, B, B_{\text{e}}, C, \alpha D, D_{\text{e}})$. We may write $\mathbf{G}_\alpha$ as

$$\mathbf{G}_\alpha(z) = \alpha C(\alpha z I - A)^{-1}B + \alpha D = \alpha \mathbf{G}(\alpha z) \quad \forall\, z \in \mathbb{E},$$

which thus gives that

$$\sup_{z \in \mathbb{E}_\alpha} \mathbf{G}(z) = \sup_{z \in \mathbb{E}_\alpha} \frac{1}{\alpha} \mathbf{G}_\alpha\left(\frac{z}{\alpha}\right) = \sup_{z \in \mathbb{E}} \frac{1}{\alpha} \mathbf{G}_\alpha(z).$$

Therefore, $\mathbf{G} \in H^\infty_\alpha(\mathcal{L}(U, Y))$, hence completing the proof. □

For the rest of this subsection, we will discuss linear output feedback of the linear system (5.3). The process is often termed "*loop shifting*" (see, for example, [44]). For motivation, we refer the reader to Section 2.1.2, where this is discussed in the finite-dimensional setting. The idea of loop-shifting shown there, applies here also. With this motivation in mind, we now give a series of definitions and results. As previously mentioned, for brevity, we shall only explicitly prove the results that are inherently different from the results given in the finite-dimensional setting in Section 2.1.2.

Definition 5.1.13. *Let* $\Sigma \in \mathbb{L}$*. We define the set of* admissible feedback operators *by*

$$\mathbb{A}(D) := \{L \in \mathcal{L}(Y, U) : I - DL \text{ is invertible}\}.$$

The following result gives a characterisation of the definition of admissibility. We shall not prove it here, since an inspection of the proof of Lemma 2.1.15 shows that it also applies in the infinite-dimensional setting.

Lemma 5.1.14. *Let $\Sigma \in \mathbb{L}$. For $L \in \mathcal{L}(Y,U)$, $I - DL$ is invertible if, and only if, $I - LD$ is invertible. Moreover,*

$$L(I - DL)^{-1} = (I - LD)^{-1}L \quad \forall\, L \in \mathbb{A}(D).$$

We now define feedback operators associated with the linear components of (5.3).

Definition 5.1.15. *Let $\Sigma \in \mathbb{L}$ and $L \in \mathbb{A}(D)$. We define the quadruple of feedback operators*

$$A^L := A + BL(I - DL)^{-1}C, \quad B^L := B + BL(I - DL)^{-1}D,$$
$$C^L := (I - DL)^{-1}C, \quad D^L := (I - DL)^{-1}D.$$

Remark 5.1.16. For $\Sigma \in \mathbb{L}$ and $L \in \mathbb{A}(D)$, it can easily be shown, by use of Lemma 5.1.14, that

$$B^L = B(I - LD)^{-1}. \tag{5.4}$$

$\Diamond$

The following lemma gives relations between systems once loop shifting has occured. We omit the proof of the result, since the argument is the same as that given in the proof of Lemma 2.1.18.

Lemma 5.1.17. *Let $\Sigma \in \mathbb{L}$ and $L \in \mathbb{A}(D)$. Then (u, v, x, y) is in $\mathcal{B}^{\mathrm{lin}}$ if, and only if, (u, v, x, y) satisfies*

$$\left.\begin{aligned} x^+ &= A^L x + B^L(u - Ly) + (B_\mathrm{e} + B^L L D_\mathrm{e})v\,, \\ y &= C^L x + D^L(u - Ly) + (I - DL)^{-1}D_\mathrm{e} v\,. \end{aligned}\right\} \tag{5.5}$$

We shall now give the final definitions of this subsection.

Definition 5.1.18. *Let $\Sigma \in \mathbb{L}$ and $L \in \mathbb{A}(D)$.*

(i) *We denote the linear components of the system given by* (5.5) *by Σ^L, that is,*

$$\Sigma^L := (A^L, B^L, B_\mathrm{e} + B^L L D_\mathrm{e}, C^L, D^L, (I - DL)^{-1}D_\mathrm{e}).$$

(ii) *We denote by $\mathbf{G}^L$ the transfer function of the system given by Σ^L, that is,*

$$\mathbf{G}^L(z) = C^L(zI - A^L)^{-1}B^L + D^L.$$

We may express $\mathbf{G}^L$ as $\mathbf{G}^L = \mathbf{G}(I - L\mathbf{G})^{-1}$.

(iii) *We define the set of* stabilising feedback operators *by*

$$\mathbb{S}(\mathbf{G}) := \left\{ M \in \mathbb{A}(D) : \mathbf{G}^M \in H^\infty(\mathcal{L}(U,Y)) \right\}.$$

The next result highlights an associativity condition for admissible feedback operators. Once again, we shall not prove the result since it can be done so identically, *mutatis mutandis*, to the proof of Lemma 2.1.20.

Lemma 5.1.19. *Let $\Sigma \in \mathbb{L}$ and $L \in \mathbb{A}(D)$. For $M \in \mathcal{L}(Y,U)$, it follows that $M \in \mathbb{A}(D^L)$ if, and only if, $L + M \in \mathbb{A}(D)$, and in which case,*

$$\Sigma^{L+M} = (\Sigma^L)^M \quad \textit{and} \quad (\mathbf{G}^L)^M = \mathbf{G}^{L+M}.$$

In the following, we see that applying feedback operators preserves stabilisability and detectability.

Lemma 5.1.20. *Let* $\Sigma \in \mathbb{L}$ *and* $L \in \mathbb{A}(D)$. *The following statements hold.*

(i) Σ *is stabilisable if, and only if,* Σ^L *is stabilisable.*

(ii) Σ *is detectable if, and only if,* Σ^L *is detectable.*

The proof is straightforward and so we leave it to the reader to verify.

We now provide a state-space characterisation of the set of stabilising output feedback operators for stablisable and detectable systems.

Lemma 5.1.21. *Let* $\Sigma \in \mathbb{L}$. *If* Σ *is stabilisable and detectable, then*

$$\mathbb{S}(\mathbf{G}) = \left\{ L \in \mathbb{A}(D) : \sigma(A^L) \subseteq \mathbb{D} \right\}.$$

Proof. Let $L \in \mathbb{A}(D)$. If $\sigma(A^L) \subseteq \mathbb{D}$, then Lemma 5.1.2 implies that A^L is exponentially stable. By combining this with Lemma 5.1.11, we yield that $\mathbf{G}^L \in H^\infty(\mathcal{L}(U,Y))$, or that $L \in \mathbb{S}(\mathbf{G})$. Now assume that $L \in \mathbb{S}(\mathbf{G})$. We begin with an application of Lemma 5.1.20 to obtain that Σ^L is stabilisable and detectable. Hence, from Lemma 5.1.11, we see that A^L is exponentially stable, or, equivalently (see Lemma 5.1.2), that $\sigma(A^L) \subseteq \mathbb{D}$. □

The final result that we present here is the following, and is given in the continuous-time setting in [46, Proposition 5.6].

Lemma 5.1.22. *Let* $\Sigma \in \mathbb{L}$, $L \in \mathcal{L}(Y,U)$ *and* $r > 0$. *Then*

$$\mathbb{B}(L,r) \subseteq \mathbb{S}(\mathbf{G}) \iff \|\mathbf{G}^L\|_{H^\infty} \leq 1/r.$$

A method similar to that used to prove [46, Proposition 5.6] may be applied to prove Lemma 5.1.22. For clarity, we briefly provide such a proof.

Proof of Lemma 5.1.22. To begin with, let us assume that $\|\mathbf{G}^L\|_{H^\infty} \leq 1/r$. Let $K \in \mathbb{B}(L,r)$ and define $M := K - L$, which has norm less than r. Consequently, $\|M\mathbf{G}^L\|_{H^\infty} < 1$ which implies that $I - M\mathbf{G}^L(z)$ is invertible for every $z \in \mathbb{E} \cup \{\infty\}$. In particular, $M \in \mathbb{A}(D^L)$. We claim that $M \in \mathbb{S}(\mathbf{G}^L)$. To see this, note that

$$\|(I - M\mathbf{G}^L(z))^{-1}\| \leq \frac{1}{1 - \|M\mathbf{G}^L(z)\|} \leq \frac{1}{1 - \|M\mathbf{G}^L\|_{H^\infty}} \quad \forall\, z \in \mathbb{E} \cup \{\infty\}.$$

Hence, $(\mathbf{G}^L)^M = \mathbf{G}^L(I - M\mathbf{G}^L)^{-1} \in H^\infty(\mathcal{L}(U,Y))$, that is, $M \in \mathbb{S}(\mathbf{G}^L)$. A subsequent application of Lemma 5.1.19 then yields that $K \in \mathbb{A}(D)$ and $\mathbf{G}^K = (\mathbf{G}^L)^M \in H^\infty(\mathcal{L}(U,Y))$. We have therefore shown that $K \in \mathbb{S}(\mathbf{G})$, which completes the first part of the proof.

As for the converse, let us now assume that $\mathbb{B}(L,r) \subseteq \mathbb{S}(\mathbf{G})$. Seeking a contradiction, suppose that $\|\mathbf{G}^L\|_{H^\infty} > 1/r$. This then implies the existence of $z_0 \in \mathbb{E}$ such that $\|\mathbf{G}^L(z_0)\| > 1/r$. Acting in the same manner as that done in the proof of [46, Proposition 5.6], we let $(u_j)_{j\in\mathbb{N}} \subseteq U$ be such that $\|u_j\|_U = 1$ for all $j \in \mathbb{N}$ and $\|\mathbf{G}^L(z_0)u_j\|_Y \to \|\mathbf{G}^L(z_0)\|$ as $j \to \infty$. We also define

$$v_j := \frac{1}{\|\mathbf{G}^L(z_0)u_j\|_Y}\mathbf{G}^L(z_0)u_j \in Y \quad \forall\, j \in \mathbb{N},$$

and $K_j : Y \to U$ by

$$K_j\xi := \frac{\langle \xi, v_j \rangle_Y}{\|\mathbf{G}^L(z_0)u_j\|_Y} u_j \quad \forall\, \xi \in Y, \forall\, j \in \mathbb{N}.$$

It is easy to verify that $\|K_j\| = 1/\|\mathbf{G}^L(z_0)u_j\|_Y \to 1/\|\mathbf{G}^L(z_0)\| < r$ as $j \to \infty$. Moreover, it is also easy to check that $(I - K_j\mathbf{G}^L(z_0))u_j = 0$ for all $j \in \mathbb{N}$. Therefore, for j large enough, we have that $K_j \in \mathbb{B}(0, r)$ and $K_j \notin \mathbb{S}(\mathbf{G}^L)$. However, for all $j \in \mathbb{N}$ large enough, by setting $M_j := K_j + L$, it is easily seen that $M_j \in \mathbb{B}(L, r) \subseteq \mathbb{S}(\mathbf{G})$. Hence, again from Lemma 5.1.19, $(\mathbf{G}^L)^{K_j} = \mathbf{G}^{M_j} \in H^\infty(\mathcal{L}(U, Y)$ for all $j \in \mathbb{N}$ sufficiently large, which yields the desired contradiction. We thus have that $\|\mathbf{G}^L\|_{H^\infty} \leq 1/r$ and the proof is complete. □

Remark 5.1.23. We note that Lemma 5.1.22 shows that $\mathbb{S}(\mathbf{G})$ is open. Indeed, for every element L in $\mathbb{S}(\mathbf{G})$, either: $\|\mathbf{G}^L\|_{H^\infty} = 0$, in which case $\mathbf{G} = 0$ and so $\mathbb{S}(\mathbf{G}) = \mathcal{B}(Y, U)$; or, $\|\mathbf{G}^L\|_{H^\infty} \neq 0$ and the ball of radius $1/\|\mathbf{G}^L\|_{H^\infty}$ centred at L is contained in $\mathbb{S}(\mathbf{G})$. ◊

5.1.3 The impulse response and convolution operator

We now discuss convolution operators and causality. We also define what we mean by an impulse response and present a preliminary result which, as previously mentioned, will be key in order to prove the main results of this chapter. With the definition of the truncation of a sequence given in the notation section in mind, we begin by giving the following definition.

Definition 5.1.24. *Let $\mathcal{F} : U^{\mathbb{Z}_+} \to Y^{\mathbb{Z}_+}$.*

(i) *We say that $\mathcal{F}$ is* causal *if $\mathcal{F}$ is linear and $\pi_t\mathcal{F} = \pi_t\mathcal{F}\pi_t$ for every $t \in \mathbb{Z}_+$.*

(ii) *We say that $\mathcal{F}$ is a* convolution operator *if there exists $\mathcal{V} : \mathbb{Z}_+ \to \mathcal{L}(U, Y)$ such that $\mathcal{F}(u) = \mathcal{V} * u$ for all $u \in U^{\mathbb{Z}_+}$, where*

$$(\mathcal{V} * u)(t) := \sum_{j=0}^{t} \mathcal{V}(t-j)u(j) \quad \forall\, u \in U^{\mathbb{Z}_+}, \forall\, t \in \mathbb{Z}_+.$$

The following lemma asserts that all convolution operators are causal.

Lemma 5.1.25. *If $\mathcal{F} : U^{\mathbb{Z}_+} \to Y^{\mathbb{Z}_+}$ is a convolution operator, then $\mathcal{F}$ is causal.*

Whilst Lemma 5.1.25 is a generally known result, it is difficult to find an explicit reference. We hence provide a proof in order to preserve completeness.

Proof of Lemma 5.1.25. Let $\mathcal{V} : \mathbb{Z}_+ \to \mathcal{L}(U, Y)$ be such that $\mathcal{F}(\cdot) = \mathcal{V} * \cdot$. The linearity of $\mathcal{F}$ follows from the linearity of $\mathcal{V}(t)$ for all $t \in \mathbb{Z}_+$. Hence, to prove the result it suffices to show that $\pi_\tau\mathcal{F} = \pi_\tau\mathcal{F}\pi_\tau$ for every $\tau \in \mathbb{Z}_+$. To see this, let $\tau, t \in \mathbb{Z}_+$ and $u \in U^{\mathbb{Z}_+}$ be arbitrary. We note that if $t > \tau$, then

$$\left(\pi_\tau \left(\mathcal{F}(\pi_\tau u)\right)\right)(t) = 0 = \left(\pi_\tau \left(\mathcal{F}u\right)\right)(t).$$

Moreover, if $t \leq \tau$, then

$$\begin{aligned}(\pi_\tau(\mathcal{F}(\pi_\tau u)))(t) &= (\mathcal{F}(\pi_\tau u))(t) \\ &= \sum_{j=0}^{t} \mathcal{V}(t-j)(\pi_\tau u)(j) \\ &= \sum_{j=0}^{t} \mathcal{V}(t-j)u(j) \\ &= (\mathcal{F}u)(t) \\ &= (\pi_\tau(\mathcal{F}u))(t),\end{aligned}$$

which therefore completes the proof. □

We now define the impulse response and convolution operator associated with (5.3).

Definition 5.1.26. *Let $\Sigma \in \mathbb{L}$.*

(i) *We denote the* impulse response *of* (5.3) *(or of Σ) by $\mathcal{G} : \mathbb{Z}_+ \to \mathcal{L}(U, Y)$, that is,*

$$\mathcal{G}(t) := \begin{cases} D, & \text{if } t = 0, \\ CA^{t-1}B, & \text{if } t \in \mathbb{N}. \end{cases}$$

(ii) *We define the* convolution operator *corresponding to* (5.3) *(or of Σ) as*

$$\mathcal{H} : U^{\mathbb{Z}_+} \to Y^{\mathbb{Z}_+}, \quad \mathcal{H}(u) := \mathcal{G} * u \quad \forall\, u \in U^{\mathbb{Z}_+}.$$

The following lemma asserts that the H^∞-norm of a transfer function is equal to the ℓ^2-induced operator norm of the associated convolution operator. The result is well-known in the literature (see, for example, [23, 96]) and so we shall not provide a proof.

Lemma 5.1.27. *Let $\Sigma \in \mathbb{L}$ be exponentially stable. Then the restriction of $\mathcal{H}$ to $\ell^2(\mathbb{Z}_+, U)$ is in $\mathcal{L}(\ell^2(\mathbb{Z}_+, U), \ell^2(\mathbb{Z}_+, Y))$ and $\|\mathcal{H}\|_{2,2} = \|\mathbf{G}\|_{H^\infty}$, where we denote by $\|\cdot\|_{2,2}$ the ℓ^2-induced operator norm.*

The next lemma is the aforementioned key result of this section, underpinning future development.

Lemma 5.1.28. *Let $\Sigma \in \mathbb{L}$ be exponentially stable. Then, there exist $c_1, c_2, c_3 > 0$ such that, for every $(u, v, x, y) \in \mathcal{B}^{\text{lin}}$,*

$$\|\pi_t x\|_{\ell^2} \leq c_1\|x(0)\|_X + c_2\|\pi_{t-1}u\|_{\ell^2} + c_3\|\pi_{t-1}v\|_{\ell^2} \quad \forall\, t \in \mathbb{N}, \tag{5.6}$$

and

$$\|\pi_t y\|_{\ell^2} \leq c_1\|x(0)\|_X + \|\mathbf{G}\|_{H^\infty}\|\pi_t u\|_{\ell^2} + c_3\|\pi_t v\|_{\ell^2} \quad \forall\, t \in \mathbb{Z}_+. \tag{5.7}$$

Proof. Let $(u, v, x, y) \in \mathcal{B}^{\text{lin}}$. To begin with, we note that $y(0) = Cx(0) + Du(0) + D_{\text{e}}v(0)$ and

$$y(t) = CA^t x(0) + \sum_{k=0}^{t-1} CA^{t-1-k}\left(Bu(k) + B_{\text{e}}v(k)\right) + Du(t) + D_{\text{e}}v(t) \quad \forall\, t \in \mathbb{N}.$$

By denoting the impulse response of the system with $B = B_{\mathrm{e}}$ and $D = D_{\mathrm{e}}$ by $\mathcal{G}_{\mathrm{e}}$ (see Definition 5.1.26), we thus see that, for all $t \in \mathbb{N}$,

$$\begin{aligned} y(t) &= CA^t x(0) + \sum_{k=0}^{t-1} \mathcal{G}(t-k)u(k) + Du(t) + \sum_{k=0}^{t-1} \mathcal{G}_{\mathrm{e}}(t-k)v(k) + D_{\mathrm{e}}v(t) \\ &= CA^t x(0) + \sum_{k=0}^{t} \mathcal{G}(t-k)u(k) + \sum_{k=0}^{t} \mathcal{G}_{\mathrm{e}}(t-k)v(k). \end{aligned} \tag{5.8}$$

By further denoting the corresponding convolution operator to $\mathcal{G}_{\mathrm{e}}$ by $\mathcal{H}_{\mathrm{e}}$ (see again Definition 5.1.26), and by defining $g \in Y^{\mathbb{Z}_+}$ by

$$g(t) := CA^t x(0) \quad \forall\, t \in \mathbb{Z}_+,$$

we see that (5.8) implies

$$y(t) = g(t) + (\mathcal{H}u)(t) + (\mathcal{H}_{\mathrm{e}}v)(t) \quad \forall\, t \in \mathbb{Z}_+.$$

We now utilise Lemma 5.1.25, to yield that

$$\pi_t y = \pi_t g + \pi_t \mathcal{H}(\pi_t u) + \pi_t \mathcal{H}_{\mathrm{e}}(\pi_t v) \quad \forall\, t \in \mathbb{Z}_+. \tag{5.9}$$

To obtain (5.7), we make the following estimates. First, for all $t \in \mathbb{Z}_+$,

$$\|\pi_t g\|_{\ell^2} = \left(\sum_{k=0}^{t} \|CA^k x(0)\|_Y^2 \right)^{\frac{1}{2}} \le \|C\| \left(\sum_{k=0}^{\infty} \|A^k\|^2 \right)^{\frac{1}{2}} \|x(0)\|_X \le c_1 \|x(0)\|_X,$$

where $c_1 > 0$ exists by the exponential stability of Σ. With this in mind, we take norms of (5.9) to yield, for all $t \in \mathbb{Z}_+$,

$$\begin{aligned} \|\pi_t y\|_{\ell^2} &\le \|\pi_t g\|_{\ell^2} + \|\pi_t \mathcal{H}(\pi_t u)\|_{\ell^2} + \|\pi_t \mathcal{H}_{\mathrm{e}}(\pi_t v)\|_{\ell^2} \\ &\le c_1 \|x(0)\|_X + \|\mathcal{H}(\pi_t u)\|_{\ell^2} + \|\mathcal{H}_{\mathrm{e}}(\pi_t v)\|_{\ell^2} \\ &\le c_1 \|x(0)\|_X + \|\mathcal{H}\|_{2,2} \|\pi_t u\|_{\ell^2} + c_3 \|\pi_t v\|_{\ell^2}, \end{aligned}$$

where $c_3 := \|\mathcal{H}_{\mathrm{e}}\|_{2,2}$ and is finite by the exponential stability of Σ. We now utilise an application of Lemma 5.1.27 to yield (5.7). In an entirely similar manner, one can easily show that (5.6) holds. In the interest of avoiding repetition, we omit the details. $\square$

5.1.4 The Lur'e system and the initial-value problem

The nonlinear control systems considered in this chapter are given by the interconnection of (5.3) with the nonlinear feedback $u = f(y + w)$ for some $f : Y \to U$, where $w \in Y^{\mathbb{Z}_+}$ is an output disturbance (see Figure 5.1).
Namely, we study

$$\left. \begin{aligned} x^+ &= Ax + Bu + B_{\mathrm{e}}v\,, \\ y &= Cx + Du + D_{\mathrm{e}}v\,, \\ u &= f(y + w)\,, \end{aligned} \right\} \tag{5.10}$$

where $\Sigma \in \mathbb{L}$, $v \in V^{\mathbb{Z}_+}$, $w \in Y^{\mathbb{Z}_+}$ and $f : Y \to U$.

For some commentary relevant to (5.10), we refer the reader to Remark 2.1.30.

We now proceed to give a series of definitions and preliminary results regarding (5.10). To begin with, similar to the behaviour of (5.3), we define the behaviour of (5.10).

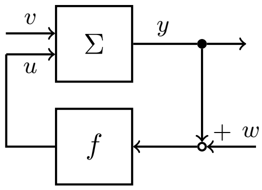

Figure 5.1: *Block diagram of the feedback interconnection of* (5.3) *with* $u = f(y + w)$.

Definition 5.1.29. *Let* $\Sigma \in \mathbb{L}$ *and* $f : Y \to U$. *We define the* behaviour *of* (5.10) *as*

$$\mathcal{B}_f(\Sigma) := \Big\{(v, w, x, y) \in V^{\mathbb{Z}_+} \times Y^{\mathbb{Z}_+} \times X^{\mathbb{Z}_+} \times Y^{\mathbb{Z}_+} : (v, w, x, y) \text{ satisfies } (5.10)\Big\} ,$$

and, when the context is clear, we will suppress this to just $\mathcal{B}$.

The following lemma contains an important (although trivial to prove) property of (the behaviour of) (5.10), which will be crucial throughout this chapter.

Lemma 5.1.30. *Let* $\Sigma \in \mathbb{L}$ *and* $f : Y \to U$.

$$(v, w, x, y) \in \mathcal{B} \quad \Longrightarrow \quad (\Lambda_\tau v, \Lambda_\tau w, \Lambda_\tau x, \Lambda_\tau y) \in \mathcal{B} \quad \forall\, \tau \in \mathbb{Z}_+ .$$

The following definition explains what we mean when we talk about the initial-value problem associated with (5.10).

Definition 5.1.31. *Let* $\Sigma \in \mathbb{L}$ *and* $f : Y \to U$. *Associated with* (5.10) *is the following* initial-value problem (IVP)

$$\left.\begin{aligned} x^+ &= Ax + Bu + B_{\rm e}v\,, \quad x(0) = x^0 \in X\,,\\ y &= Cx + Du + D_{\rm e}v\,,\\ u &= f(y + w)\,, \end{aligned}\right\} \tag{5.11}$$

where $v \in V^{\mathbb{Z}_+}$, $w \in Y^{\mathbb{Z}_+}$. *For a given* $x^0 \in X$, $v \in V^{\mathbb{Z}_+}$ *and* $w \in Y^{\mathbb{Z}_+}$, *we say that* $(x, y) \in X^{\mathbb{Z}_+} \times Y^{\mathbb{Z}_+}$ *is a* solution to the IVP (5.11) *if* $x(0) = x^0$ *and* $(v, w, x, y) \in \mathcal{B}$.

The next two results present sufficient conditions for when, given an initial condition and inputs, solutions to the IVP (5.11) exist and when there is at most one solution. We shall not provide a proof of these results, and shall instead refer the reader to the proofs of Propositions 2.1.36 and 2.1.37. Although they are in the finite-dimensional setting, that property is not utilised and so the proofs apply here also.

Proposition 5.1.32. *Let* $\Sigma \in \mathbb{L}$, $f : Y \to U$, $x^0 \in X$, $v \in V^{\mathbb{Z}_+}$ *and* $w \in Y^{\mathbb{Z}_+}$. *The following statements hold.*

(i) *If the map* $I - Df$ *is surjective, then there exists a solution to the IVP* (5.11).

(ii) *If the map* $I - Df$ *is injective, then there is at most one solution to the IVP* (5.11).

Proposition 5.1.33. *Let* $\Sigma \in \mathbb{L}$, $f : Y \to U$, $x^0 \in X$, $v \in V^{\mathbb{Z}_+}$, $w \in Y^{\mathbb{Z}_+}$ *and* $K \in \mathbb{A}(D)$. *The following statements hold.*

(i) *If the map* $I - D^K(f - K)$ *is surjective, then there exists a solution to the IVP* (5.11).

(ii) *If the map* $I - D^K(f - K)$ *is injective, then there is at most one solution to the IVP* (5.11).

We conclude this section by recalling Example 2.1.38. There, two systems are presented (in the finite-dimensional setting) that satisfy that $I - Df$ is not surjective and not injective, respectively. In the first example, for certain initial conditions and inputs, the corresponding IVP has no solutions. In the second illustration, again for certain initial conditions and inputs, the IVP has more than one solution. This whence tells us that the conclusions of statement (i) and (ii) of Proposition 5.1.32 need not hold if the respective assumptions fail.

5.2 Input-to-state stability properties

This section concerns input-to-state stability properties of systems of the form (5.10). In particular, we consider incremental stability notions (see, for example, [5, 47]). We split the section in two, and begin in the first part by defining relevant stability notions and by presenting the main incremental stability result (see Theorem 5.2.5) of this chapter. In the latter part of this section, we give various corollaries of this result, which have assumptions in terms of positive realness conditions.

5.2.1 Incremental input-to-state stability

In what follows, we shall abuse notation and, when the context is clear, consider an element ξ of a vector space as a constant function mapping $t \mapsto \xi$ for all $t \in \mathbb{Z}_+$.

Definition 5.2.1. *Let* $\Sigma \in \mathbb{L}$ *and* $f : Y \to U$.

(i) *A quadruple* $(v^{\mathrm{e}}, w^{\mathrm{e}}, x^{\mathrm{e}}, y^{\mathrm{e}}) \in V \times Y \times X \times Y$ *is called an* equilibrium quadruple *of* (5.10) *if* $(v^{\mathrm{e}}, w^{\mathrm{e}}, x^{\mathrm{e}}, y^{\mathrm{e}}) \in \mathcal{B}$.

(ii) *An equilibrium quadruple* $(v^{\mathrm{e}}, w^{\mathrm{e}}, x^{\mathrm{e}}, y^{\mathrm{e}})$ *is said to be* exponentially input-to-state stable (ISS) *if there exist* $c > 0$ *and* $a \in (0,1)$ *such that, for all* $(v, w, x, y) \in \mathcal{B}$,

$$\|x(t) - x^{\mathrm{e}}\|_X \leq c \left(a^t \|x(0) - x^{\mathrm{e}}\|_X + \max_{s \in \underline{t-1}} \left\| \begin{pmatrix} v(s) \\ w(s) \end{pmatrix} - \begin{pmatrix} v^{\mathrm{e}} \\ w^{\mathrm{e}} \end{pmatrix} \right\|_{V \times Y} \right) \quad \forall\, t \in \mathbb{N}. \tag{5.12}$$

(iii) *An equilibrium quadruple* $(v^{\mathrm{e}}, w^{\mathrm{e}}, x^{\mathrm{e}}, y^{\mathrm{e}})$ *is said to be* exponentially input-to-state/output stable (ISOS) *if there exist* $c > 0$ *and* $a \in (0,1)$ *such that, for all* $(v, w, x, y) \in \mathcal{B}$, (5.12) *holds and*

$$\|y(t) - y^{\mathrm{e}}\|_Y \leq c \left(a^t \|x(0) - x^{\mathrm{e}}\|_X + \max_{s \in \underline{t}} \left\| \begin{pmatrix} v(s) \\ w(s) \end{pmatrix} - \begin{pmatrix} v^{\mathrm{e}} \\ w^{\mathrm{e}} \end{pmatrix} \right\|_{V \times Y} \right) \quad \forall\, t \in \mathbb{Z}_+.$$

(iv) *We say that* (5.10) *is* exponentially ISS (respectively, ISOS) *if* $(0,0,0,0)$ *is an exponentially ISS (respectively, ISOS) equilibrium quadruple of* (5.10).

Remark 5.2.2. An interesting question is whether or not ISOS of (5.10) is a consequence of ISS. Of course, in the case where the feedthrough is zero, ISOS is trivially implied by ISS. The system given in Example 2.2.6 can be used here as an example of a system which is ISS but not ISOS. This hence gives credence to the justification of our investigation of output stability. Moreover, this justification is furthered by the fact that the aforementioned example is in the finite-dimensional setting. ◊

We now continue with giving stability notions that are attached to (5.10).

Definition 5.2.3. *Let $\Sigma \in \mathbb{L}$, $f : Y \to U$, and let S, S_1, S_2 be non-empty subsets of Y.*

(i) *We define the* sub-behaviour *of* (5.10) *on S by*

$$\mathcal{B}_f^S(\Sigma) := \{(v, w, x, y) \in \mathcal{B}_f(\Sigma) \,:\, y(t) + w(t) \in S \;\forall\, t \in \mathbb{Z}_+\}\,,$$

and, when the context is clear, will suppress this to simply $\mathcal{B}^S$.

(ii) *We say that* (5.10) *is* exponentially incrementally input-to-state stable (δISS) with respect to S_1 and S_2 *if there exist $c > 0$ and $a \in (0, 1)$ such that, for all $(v_1, w_1, x_1, y_1) \in \mathcal{B}^{S_1}$, all $(v_2, w_2, x_2, y_2) \in \mathcal{B}^{S_2}$ and all $t \in \mathbb{Z}_+$,*

$$\|x_1(t) - x_2(t)\|_X \le c\left(a^t\|x_1(0) - x_2(0)\|_X + \max_{s\in\underline{t-1}}\left\|\begin{pmatrix}v_1(s)\\w_1(s)\end{pmatrix} - \begin{pmatrix}v_2(s)\\w_2(s)\end{pmatrix}\right\|_{V\times Y}\right) \tag{5.13}$$

(iii) *We say that* (5.10) *is* exponentially incrementally input-to-state/output stable (δISOS) with respect to S_1 and S_2 *if there exist $c > 0$ and $a \in (0, 1)$ such that, for all $(v_1, w_1, x_1, y_1) \in \mathcal{B}^{S_1}$, all $(v_2, w_2, x_2, y_2) \in \mathcal{B}^{S_2}$ and all $t \in \mathbb{Z}_+$,* (5.13) *holds and*

$$\|y_1(t) - y_2(t)\|_Y \le c\left(a^t\|x_1(0) - x_2(0)\|_X + \max_{s\in\underline{t}}\left\|\begin{pmatrix}v_1(s)\\w_1(s)\end{pmatrix} - \begin{pmatrix}v_2(s)\\w_2(s)\end{pmatrix}\right\|_{V\times Y}\right)$$

(iv) *We say that* (5.10) *is* exponentially δISS (respectively, δISOS) *if $S_1 = S_2 = Y$ in statement (ii) (respectively, statement (iii)).*

Remark 5.2.4. In the case that $f(0) = 0$, if (5.10) is δISS (respectively, δISOS) with respect to $S_1 := Y$ and $S_2 := \{0\}$, then, trivially, (5.10) is also ISS (respectively, ISOS). Moreover, exponential δISOS with respect to S_1 and S_2 implies exponential δISS with respect to the same sets. However, Example 2.2.6 (as discussed in Remark 5.2.2) shows that the converse implication is false. ◊

By recalling the definition of a weighted ℓ^2-space given in the notation section of this thesis, we are now ready to present the aforementioned main theorem of this chapter.

Theorem 5.2.5. *Let $\Sigma \in \mathbb{L}$ be stabilisable and detectable, $f : Y \to U$, and let $S_1, S_2 \subseteq Y$ be non-empty. Assume that $r > 0$ and $K \in \mathcal{L}(Y, U)$ satisfy $\mathbb{B}(K, r) \subseteq \mathbb{S}(\mathbf{G})$ and that there exists $\delta \in (0, r)$ such that*

$$\|f(\xi) - f(\zeta) - K(\xi - \zeta)\|_U \le (r - \delta)\|\xi - \zeta\|_Y \quad \forall\, \xi \in S_1,\ \forall\, \zeta \in S_2\,. \tag{5.14}$$

Then the following hold.

(i) *There exist constants* $a > 0$, $b > 0$ *and* $\omega > 1$ *such that, for all* $(v_1, w_1, x_1, y_1) \in \mathcal{B}^{S_1}$, $(v_2, w_2, x_2, y_2) \in \mathcal{B}^{S_2}$, *and all* $\rho \in [1, \omega]$,

$$\|\pi_t(x_1 - x_2)\|_{\ell^2_\rho} \leq a \left(\|x_1(0) - x_2(0)\|_X + \left\| \pi_{t-1} \begin{pmatrix} v_1 - v_2 \\ w_1 - w_2 \end{pmatrix} \right\|_{\ell^2_\rho} \right) \quad \forall\, t \in \mathbb{N}\,, \tag{5.15}$$

and

$$\|\pi_t(y_1 - y_2)\|_{\ell^2_\rho} \leq b \left(\|x_1(0) - x_2(0)\|_X + \left\| \pi_{t} \begin{pmatrix} v_1 - v_2 \\ w_1 - w_2 \end{pmatrix} \right\|_{\ell^2_\rho} \right) \quad \forall t \in \mathbb{Z}_+\,. \tag{5.16}$$

(ii) *For* $q \in [2, \infty]$, *there exist constants* $c > 0$, $d > 0$ *and* $\theta \in (0, 1)$ *such that, for all* $(v_1, w_1, x_1, y_1) \in \mathcal{B}^{S_1}$ *and* $(v_2, w_2, x_2, y_2) \in \mathcal{B}^{S_2}$,

$$\|x_1(t) - x_2(t)\|_X \leq c \left(\theta^t \|x_1(0) - x_2(0)\|_X + \left\| \pi_{t-1} \begin{pmatrix} v_1 - v_2 \\ w_1 - w_2 \end{pmatrix} \right\|_{\ell^q} \right) \quad \forall\, t \in \mathbb{N}\,, \tag{5.17}$$

and

$$\|y_1(t) - y_2(t)\|_Y \leq d \left(\theta^t \|x_1(0) - x_2(0)\|_X + \left\| \pi_{t} \begin{pmatrix} v_1 - v_2 \\ w_1 - w_2 \end{pmatrix} \right\|_{\ell^q} \right) \quad \forall\, t \in \mathbb{Z}_+\,. \tag{5.18}$$

Here c *and* d *depend on* q, *but* θ *does not.*

Before proving Theorem 5.2.5, we note that, by taking $q = \infty$, the following immediate consequence of Theorem 5.2.5 holds.

Corollary 5.2.6. *Under the assumptions of Theorem 5.2.5, the Lur'e system* (5.10) *is exponentially* δ*ISOS with respect to* S_1 *and* S_2. *In particular, the following statements hold.*

(i) *If* (5.14) *holds with* $S_1 = S_2 = Y$, *then* (5.10) *is exponentially* δ*ISOS.*

(ii) *If* $(v^{\mathrm{e}}, w^{\mathrm{e}}, x^{\mathrm{e}}, y^{\mathrm{e}})$ *is an equilibrium quadruple of* (5.10) *and* (5.14) *holds with* $S_1 = Y$ *and* $S_2 = \{y^{\mathrm{e}} + w^{\mathrm{e}}\}$, *then* $(v^{\mathrm{e}}, w^{\mathrm{e}}, x^{\mathrm{e}}, y^{\mathrm{e}})$ *is an exponentially ISOS equilibrium quadruple of* (5.10).

Proof of Theorem 5.2.5. The proof uses a combination of small-gain and exponential weighting arguments. Since Σ is stabilisable and detectable, it follows from Lemma 5.1.20 that Σ^K is as well. Moreover, since $\mathbf{G}^K \in H^\infty(\mathcal{L}(U, Y))$, an application of Lemma 5.1.11 yields that A^K is exponentially stable, with exponential growth constant $\mu \in (0, 1)$. Let $\alpha \in (\mu, 1)$. Lemma 5.1.12 gives that $\mathbf{G}^K \in H^\infty_\alpha(\mathcal{L}(U, Y))$. Let us consider $\mathbf{G}^K$ on the closed annulus $\mathcal{A} := \{z \in \mathbb{C} : \beta \leq |z| \leq 1\}$, where $\beta \in (\alpha, 1)$. Owing to the continuity of $\mathbf{G}^K$ on $\mathbb{E}_\alpha$, $\mathbf{G}^K$ is uniformly continuous on $\mathcal{A}$. Thus, there exists $\gamma \in (0, 1 - \beta)$ such that, for all $z_1, z_2 \in \mathcal{A}$ with $|z_1 - z_2| < \gamma$,

$$\|\mathbf{G}^K(z_1) - \mathbf{G}^K(z_2)\| < \frac{1}{r - \delta/2} - \frac{1}{r}.$$

Since, from Lemma 5.1.22, $\|\mathbf{G}^K\|_{H^\infty} \le 1/r$, the above inequality implies that, for all $z \in \mathbb{C}$ with $1-\gamma < |z| \le 1$,

$$\begin{aligned}\|\mathbf{G}^K(z)\| &\le \|\mathbf{G}^K(z) - \mathbf{G}^K(z^*)\| + \|\mathbf{G}^K(z^*)\| \\ &< \frac{1}{r-\delta/2} - \frac{1}{r} + \frac{1}{r} \\ &= \frac{1}{r-\delta/2},\end{aligned}$$

where $z^* = z/|z|$. Here, we have used that, for such a z, $z, z^* \in \mathcal{A}$ and $|z - z^*| < \gamma$. Consequently, since $\|\mathbf{G}^K\|_{H^\infty} \le 1/r$, for $\gamma^* \in (1-\gamma, 1)$,

$$\sup_{z \in \mathbb{E}_{\gamma^*}} \|\mathbf{G}^K(z)\| \le \frac{1}{r-\delta/2} < \frac{1}{r-\delta}\,. \tag{5.19}$$

To prove statement (i), set $\omega := 1/\gamma^* > 1$ and let $\rho \in [1,\omega]$. We define $\mathbf{H}(z) := \mathbf{G}^K(z/\rho)$ to obtain that, from (5.19),

$$\|\mathbf{H}\|_{H^\infty} = \sup_{z \in \mathbb{E}} \|\mathbf{H}(z)\| \le \sup_{z \in \mathbb{E}_{\gamma^*}} \|\mathbf{G}^K(z)\| < \frac{1}{r-\delta}\,. \tag{5.20}$$

By the choice of ρ, we have that

$$\rho\mu < \frac{\mu}{1-\gamma} < \frac{\mu}{\beta} < 1\,,$$

which implies that ρA^K is exponentially stable.
Let $(v_1, w_1, x_1, y_1) \in \mathcal{B}^{S_1}$ and $(v_2, w_2, x_2, y_2) \in \mathcal{B}^{S_2}$. By Lemma 5.1.17, it follows that, for $i \in \{1,2\}$,

$$\begin{aligned}x_i^+ &= A^K x_i + B^K\big(f(y_i + w_i) - K y_i\big) + (B_\mathrm{e} + B^K K D_\mathrm{e}) v_i\,, \\ y_i &= C^K x_i + D^K\big(f(y_i + w_i) - K y_i\big) + (I - DK)^{-1} D_\mathrm{e} v_i\,.\end{aligned}$$

Forming the differences then gives

$$\begin{aligned}(x_1 - x_2)^+ = A^K(x_1 - x_2) + B^K\,(&f(y_1 + w_1) - f(y_2 + w_2) \\ &-K(y_1 + w_1 - y_2 - w_2)) + \eta\,,\end{aligned} \tag{5.21}$$

and

$$\begin{aligned}y_1 - y_2 = C^K(x_1 - x_2) + D^K\,(&f(y_1 + w_1) - f(y_2 + w_2) \\ &-K(y_1 + w_1 - y_2 - w_2)) + \nu\,,\end{aligned} \tag{5.22}$$

where

$$\begin{aligned}\eta &:= (B_\mathrm{e} + B^K K D_\mathrm{e})(v_1 - v_2) + B^K K(w_1 - w_2) \quad \text{and} \\ \nu &:= (I - DK)^{-1} D_\mathrm{e}(v_1 - v_2) + D^K K(w_1 - w_2)\,.\end{aligned}$$

Now, by recalling (5.1) and that $\rho \le \omega$, it is easy to check that there exists $\kappa > 0$ which is independent of ρ, (v_1, w_1, x_1, y_1) and (v_2, w_2, x_2, y_2), and is such that

$$\left\| \begin{pmatrix} \rho\eta(t) \\ \nu(t) \end{pmatrix} \right\|_{X \times Y} \le \kappa \left\| \begin{pmatrix} (v_1 - v_2)(t) \\ (w_1 - w_2)(t) \end{pmatrix} \right\|_{V \times Y} \quad \forall\, t \in \mathbb{Z}_+\,. \tag{5.23}$$

By defining $\tilde{u}_\rho(t) = \rho^t \tilde{u}(t)$ for all $t \in \mathbb{Z}_+$ and all sequences $\tilde{u}$, we see that (5.21) and (5.22) yield

$$\left.\begin{aligned} (x_1 - x_2)^+_\rho &= \rho A^K (x_1 - x_2)_\rho + \rho B^K g_\rho + \rho \eta_\rho \,, \\ (y_1 - y_2)_\rho &= C^K (x_1 - x_2)_\rho + D^K g_\rho + \nu_\rho \,, \end{aligned}\right\} \tag{5.24}$$

where

$$g(t) := f(y_1(t) + w_1(t)) - f(y_2(t) + w_2(t)) - K(y_1(t) + w_1(t) - y_2(t) - w_2(t)) \quad \forall\, t \in \mathbb{Z}_+ \,.$$

We thus obtain that the quadruple

$$\left(g_\rho, \begin{pmatrix} \rho \eta_\rho \\ \nu_\rho \end{pmatrix}, (x_1 - x_2)_\rho, (y_1 - y_2)_\rho \right)$$

is in the behaviour of the linear system given by $\big(\rho A^K, \rho B^K, \begin{pmatrix} I & 0 \end{pmatrix}, C^K, D^K, \begin{pmatrix} 0 & I \end{pmatrix}\big) \in \mathbb{L}$ where $V = X \times Y$. Therefore, since ρA^K is exponentially stable, an application of Lemma 5.1.28 to system (5.24) yields the existence of constants c_1, c_2, c_3 (independent of (v_1, w_1, x_1, y_1) and (v_2, w_2, x_2, y_2)) such that, for all $t \in \mathbb{N}$,

$$\|\pi_t (x_1 - x_2)_\rho\|_{\ell^2} \le c_1 \|(x_1 - x_2)(0)\|_X + c_2 \|\pi_{t-1} g_\rho\|_{\ell^2} + c_3 \left\| \pi_{t-1} \begin{pmatrix} \rho \eta_\rho \\ \nu_\rho \end{pmatrix} \right\|_{\ell^2} , \tag{5.25}$$

and that, for all $t \in \mathbb{Z}_+$,

$$\|\pi_t (y_1 - y_2)_\rho\|_{\ell^2} \le c_1 \|(x_1 - x_2)(0)\|_X + \|\mathbf{H}\|_{H^\infty} \|\pi_t g_\rho\|_{\ell^2} + c_3 \left\| \pi_t \begin{pmatrix} \rho \eta_\rho \\ \nu_\rho \end{pmatrix} \right\|_{\ell^2} . \tag{5.26}$$

Now, since A^K is exponentially stable with exponential growth constant $\mu \in (0,1)$ and since $\rho \le \omega = 1/\gamma^*$ where $\gamma^* \in (\mu, 1)$, it is easy to check that c_1, c_2, and c_3 are bounded above by positive constants c_4, c_5, and c_6, respectively, which are independent of ρ, (v_1, w_1, x_1, y_1) and (v_2, w_2, x_2, y_2). With this in mind, we now use the previous two estimates to prove that statement (i) holds. To this end, we first note that, by the definition of g and assumption (5.14), we have, for all $t \in \mathbb{Z}_+$,

$$\begin{aligned} \|g_\rho(t)\|_U &= \rho^t \|f(y_1(t) + w_1(t)) - f(y_2(t) + w_2(t)) - K(y_1(t) + w_1(t) - y_2(t) - w_2(t))\|_U \\ &\le (r - \delta) \rho^t \|(y_1 + w_1 - y_2 - w_2)(t)\|_Y \\ &= (r - \delta) \|(y_1 + w_1 - y_2 - w_2)_\rho(t)\|_Y \,, \end{aligned}$$

which in turn implies that

$$\|\pi_t g_\rho\|_{\ell^2} \le (r - \delta) \|\pi_t (y_1 - y_2)_\rho\|_{\ell^2} + (r - \delta) \|\pi_t (w_1 - w_2)_\rho\|_{\ell^2} \quad \forall\, t \in \mathbb{Z}_+ \,. \tag{5.27}$$

Substituting (5.20), (5.23) and (5.27) into (5.26) and rearranging, subsequently yields

$$\begin{aligned} \|\pi_t (y_1 - y_2)\|_{\ell^2_\rho} &= \|\pi_t (y_1 - y_2)_\rho\|_{\ell^2} \\ &\le c_7 \|(x_1 - x_2)_\rho(0)\|_X + c_8 \left\| \pi_t \begin{pmatrix} (v_1 - v_2)_\rho \\ (w_1 - w_2)_\rho \end{pmatrix} \right\|_{\ell^2} \quad \forall\, t \in \mathbb{Z}_+ \,, \end{aligned} \tag{5.28}$$

where

$$c_7 := \frac{c_4}{1 - (r - \delta) \|\mathbf{H}\|_{H^\infty}} > 0, \quad c_8 := \frac{\kappa c_6 + (r - \delta) \|\mathbf{H}\|_{H^\infty}}{1 - (r - \delta) \|\mathbf{H}\|_{H^\infty}} > 0 \,.$$

Inequality (5.16) now follows from (5.28) with $b := \max\{c_7, c_8\}$. Similarly, by substituting (5.27) and (5.28) into (5.25), we see that, for all $t \in \mathbb{N}$,

$$\begin{aligned}\|\pi_t(x_1 - x_2)\|_{\ell^2_\rho} &= \|\pi_t(x_1 - x_2)_\rho\|_{\ell^2} \\ &\le c_4\|(x_1 - x_2)(0)\|_X + c_6 \left\| \pi_{t-1} \begin{pmatrix} \rho\eta_\rho \\ \nu_\rho \end{pmatrix} \right\|_{\ell^2} \\ &\quad + c_5(r-\delta)\left(\|\pi_{t-1}(y_1 - y_2)_\rho\|_{\ell^2} + \|\pi_{t-1}(w_1 - w_2)_\rho\|_{\ell^2}\right) \\ &\le c_9\|(x_1 - x_2)(0)\|_X + c_{10} \left\| \pi_{t-1} \begin{pmatrix} (v_1 - v_2)_\rho \\ (w_1 - w_2)_\rho \end{pmatrix} \right\|_{\ell^2},\end{aligned}$$

where $c_9 := c_4 + c_5(r-\delta)c_7 > 0$ and $c_{10} := c_6\kappa + c_5(r-\delta)(c_8 + 1) > 0$. We now set $a := \max\{c_9, c_{10}\}$ so that the above inequality gives (5.15). Since a and b are independent of ρ, (v_1, w_1, x_1, y_1) and (v_2, w_2, x_2, y_2), we have thus proven statement (i).

We proceed to prove statement (ii). To this end, we let $\rho \in (1, \omega]$ and note that, since ρA^K is exponentially stable, we have that

$$c_{11} := \sup_{t \in \mathbb{Z}_+} \|(\rho A^K)^t\| < \infty\,.$$

Moreover, let $(v_1, w_1, x_1, y_1) \in \mathcal{B}^{S_1}$ and $(v_2, w_2, x_2, y_2) \in \mathcal{B}^{S_2}$. By applying the variation-of-parameters formula to (5.24), we see that

$$\begin{aligned}(x_1 - x_2)_\rho(t) &= (\rho A^K)^t (x_1 - x_2)(0) + \sum_{k=0}^{t-1} (\rho A^K)^{t-1-k} \rho B^K g_\rho(k) \\ &\quad + \sum_{k=0}^{t-1} (\rho A^K)^{t-1-k} \rho \eta_\rho(k) \quad \forall\, t \in \mathbb{N}\,.\end{aligned}$$

If we then apply the triangle inequality and Hölder's inequality we yield, for all $t \in \mathbb{N}$,

$$\begin{aligned}\|(x_1 - x_2)_\rho(t)\|_X &\le c_{11}\|(x_1 - x_2)(0)\|_X + \|\rho B^K\| \sum_{k=0}^{t-1} \|(\rho A^K)^{t-1-k}\| \|g_\rho(k)\|_U \\ &\quad + \rho \sum_{k=0}^{t-1} \|(\rho A^K)^{t-1-k}\| \|\eta_\rho(k)\|_U \\ &\le c_{11}\|(x_1 - x_2)(0)\|_X + \|\rho B^K\| \left(\sum_{k=0}^{t-1} \|(\rho A^K)^k\|^2 \right)^{1/2} \|\pi_{t-1} g_\rho\|_{\ell^2} \\ &\quad + \rho \left(\sum_{k=0}^{t-1} \|(\rho A^K)^k\|^2 \right)^{1/2} \|\pi_{t-1}\eta_\rho\|_{\ell^2}\,.\end{aligned}$$

Substituting (5.27), and then (5.28) and (5.23), into the above estimate gives

$$\begin{aligned}\|(x_1 - x_2)_\rho(t)\|_X &\le c_{11}\|(x_1 - x_2)(0)\|_X + \rho \left(\sum_{k=0}^{t-1} \|(\rho A^K)^k\|^2 \right)^{1/2} \|\pi_{t-1}\eta_\rho\|_{\ell^2} \\ &\quad + (r-\delta)\|\rho B^K\| \left(\sum_{k=0}^{t-1} \|(\rho A^K)^k\|^2 \right)^{1/2} \left(\|\pi_{t-1}(y_1 - y_2)_\rho\|_{\ell^2} + \|\pi_{t-1}(w_1 - w_2)_\rho\|_{\ell^2}\right) \\ &\le c_{12}\|(x_1 - x_2)(0)\|_X + c_{13} \left\| \pi_{t-1} \begin{pmatrix} (v_1 - v_2)_\rho \\ (w_1 - w_2)_\rho \end{pmatrix} \right\|_{\ell^2} \quad \forall\, t \in \mathbb{N}, \qquad (5.29)\end{aligned}$$

where

$$c_{12} := c_{11} + c_7(r-\delta)\|\rho B^K\| \left(\sum_{k=0}^{\infty} \|(\rho A^K)^k\|^2\right)^{1/2},$$

and

$$c_{13} := \kappa \left(\sum_{k=0}^{\infty} \|(\rho A^K)^k\|^2\right)^{1/2} + (c_8+1)(r-\delta)\|\rho B^K\| \left(\sum_{k=0}^{\infty} \|(\rho A^K)^k\|^2\right)^{1/2}.$$

We note that estimate (5.29) can be written as

$$\|(x_1-x_2)(t)\|_X \le c_{12}\rho^{-t}\|(x_1-x_2)(0)\|_X + c_{13}\rho^{-t}\left\|\pi_{t-1}\begin{pmatrix}(v_1-v_2)_\rho\\(w_1-w_2)_\rho\end{pmatrix}\right\|_{\ell^2} \quad \forall\, t\in\mathbb{N}. \tag{5.30}$$

If $q\in(2,\infty)$, then there exists $p\in(1,\infty)$ such that $2/q+1/p=1$. Whence, by using Hölder's inequality again, we see that

$$\begin{aligned}\left\|\pi_t\begin{pmatrix}(v_1-v_2)_\rho\\(w_1-w_2)_\rho\end{pmatrix}\right\|_{\ell^2}^2 &= \sum_{k=0}^{t}\rho^{2k}\left\|\begin{pmatrix}(v_1-v_2)(k)\\(w_1-w_2)(k)\end{pmatrix}\right\|_{V\times Y}^2\\ &\le \left(\sum_{k=0}^{t}\rho^{2kp}\right)^{1/p}\left\|\pi_t\begin{pmatrix}v_1-v_2\\w_1-w_2\end{pmatrix}\right\|_{\ell^q}^2 \quad \forall\, t\in\mathbb{Z}_+,\end{aligned}$$

which then implies that

$$\begin{aligned}\left\|\pi_t\begin{pmatrix}(v_1-v_2)_\rho\\(w_1-w_2)_\rho\end{pmatrix}\right\|_{\ell^2}^2 &= \left(\frac{\rho^{2p(t+1)}-1}{\rho^{2p}-1}\right)^{1/p}\left\|\pi_t\begin{pmatrix}v_1-v_2\\w_1-w_2\end{pmatrix}\right\|_{\ell^q}^2\\ &\le \rho^{2t}\left(\frac{\rho^2}{(\rho^{2p}-1)^{1/p}}\right)\left\|\pi_t\begin{pmatrix}v_1-v_2\\w_1-w_2\end{pmatrix}\right\|_{\ell^q}^2 \quad \forall\, t\in\mathbb{Z}_+.\end{aligned}$$

If $q=2$, then since $\rho>1$,

$$\begin{aligned}\left\|\pi_t\begin{pmatrix}(v_1-v_2)_\rho\\(w_1-w_2)_\rho\end{pmatrix}\right\|_{\ell^2} &= \left(\sum_{k=0}^{t}\rho^{2k}\left\|\begin{pmatrix}(v_1-v_2)(k)\\(w_1-w_2)(k)\end{pmatrix}\right\|_{V\times Y}^2\right)^{1/2}\\ &\le \rho^t\left\|\pi_t\begin{pmatrix}v_1-v_2\\w_1-w_2\end{pmatrix}\right\|_{\ell^2} \quad \forall\, t\in\mathbb{Z}_+.\end{aligned}$$

Finally, if $q=\infty$,

$$\begin{aligned}\left\|\pi_t\begin{pmatrix}(v_1-v_2)_\rho\\(w_1-w_2)_\rho\end{pmatrix}\right\|_{\ell^2} &= \left(\sum_{k=0}^{t}\rho^{2k}\left\|\begin{pmatrix}(v_1-v_2)(k)\\(w_1-w_2)(k)\end{pmatrix}\right\|_{V\times Y}^2\right)^{1/2}\\ &\le \left(\sum_{k=0}^{t}\rho^{2k}\right)^{1/2}\max_{s\in\underline{t}}\left\|\begin{pmatrix}(v_1-v_2)(s)\\(w_1-w_2)(s)\end{pmatrix}\right\|_{V\times Y}\\ &= \left(\frac{\rho^{2(t+1)}-1}{\rho^2-1}\right)^{1/2}\max_{s\in\underline{t}}\left\|\begin{pmatrix}(v_1-v_2)(s)\\(w_1-w_2)(s)\end{pmatrix}\right\|_{V\times Y}\\ &\le \rho^t\left(\frac{\rho}{(\rho^2-1)^{1/2}}\right)\left\|\pi_t\begin{pmatrix}v_1-v_2\\w_1-w_2\end{pmatrix}\right\|_{\ell^\infty} \quad \forall\, t\in\mathbb{Z}_+.\end{aligned}$$

Therefore, we have shown that for every $q \in [2, \infty]$, there exists a positive constant c_{14} (independent of (v_1, w_1, x_1, y_1) and (v_2, w_2, x_2, y_2)) such that

$$\left\| \pi_t \begin{pmatrix} (v_1 - v_2)_\rho \\ (w_1 - w_2)_\rho \end{pmatrix} \right\|_{\ell^2} \leq \rho^t c_{14} \left\| \pi_t \begin{pmatrix} v_1 - v_2 \\ w_1 - w_2 \end{pmatrix} \right\|_{\ell^q} \quad \forall\, t \in \mathbb{Z}_+ ,$$

which, when substituted into (5.30), gives that, for any $q \in [2, \infty]$,

$$\begin{aligned} \|(x_1 - x_2)(t)\|_X &\leq c_{12}\rho^{-t}\|(x_1 - x_2)(0)\|_X + c_{13}\rho^{-1}c_{14} \left\| \pi_{t-1} \begin{pmatrix} v_1 - v_2 \\ w_1 - w_2 \end{pmatrix} \right\|_{\ell^q} \\ &\leq c_{12}\rho^{-t}\|(x_1 - x_2)(0)\|_X + c_{13}c_{14} \left\| \pi_{t-1} \begin{pmatrix} v_1 - v_2 \\ w_1 - w_2 \end{pmatrix} \right\|_{\ell^q} \quad \forall\, t \in \mathbb{N}. \end{aligned}$$

Setting $c := \max\{c_{12}, c_{13}c_{14}\}$ and $\theta = \rho^{-1}$ then yields (5.17).
It remains to establish (5.18). To this end, we note that

$$\|D^K\| \leq \|\mathbf{G}^K\|_{H^\infty} \leq \frac{1}{r} < \frac{1}{r - \delta} . \tag{5.31}$$

Whence, appealing to (5.14) and (5.22),

$$\begin{aligned} \|(y_1 - y_2)(t)\|_Y \leq &\|C^K\|\|(x_1 - x_2)(t)\|_X \\ &+ \|D^K\|(r - \delta)\left(\|(y_1 - y_2)(t)\|_Y + \|(w_1 - w_2)(t)\|_Y\right) \\ &+ \|\nu(t)\|_Y \quad \forall\, t \in \mathbb{Z}_+ . \end{aligned} \tag{5.32}$$

Define

$$d_1 := \frac{1}{1 - (r - \delta)\|D^K\|} ,$$

which is positive by (5.31). Substituting (5.17) into (5.32) and setting $d_2 := d_1\|C^K\|c$ and $d_3 := d_1\left(\|C^K\|c + \kappa + (r - \delta)\|D^K\|\right)$, we see that

$$\|(y_1 - y_2)(t)\|_Y \leq d_2\theta^t\|(x_1 - x_2)(0)\|_X + d_3 \left\| \pi_t \begin{pmatrix} (v_1 - v_2)_\rho \\ (w_1 - w_2)_\rho \end{pmatrix} \right\|_{\ell^q} \quad \forall\, t \in \mathbb{Z}_+ .$$

Finally, by setting $d := \max\{d_2, d_3\}$, we complete the proof. □

Remark 5.2.7. (i) By inspecting the above proof, we are able to see that Theorem 5.2.5 holds true if X, Y, U and V are real spaces, provided that the complex ball condition $\mathbb{B}_c(K, r) \subseteq \mathbb{S}_c(\mathbf{G})$ holds, with $\mathbb{B}_c(K, r) := \{M \in \mathcal{L}(Y_c, U_c) : \|M - K\| < r\}$ and

$$\mathbb{S}_c(\mathbf{G}) := \{M \in \mathcal{L}(Y_c, U_c) : I - DM \text{ is invertible and } \mathbf{G}^M \in H^\infty(\mathcal{L}(U_c, Y_c))\},$$

where Y_c and U_c denote the complexifications of Y and U, respectively. The same can be said of the rest of the results in the subsequent sections.

(ii) For later purposes, it will be useful to consider Theorem 5.2.5 in the (rather degenerate) situation wherein $\mathbf{G} = 0$. If $\mathbf{G} = 0$, then $\mathbf{G}^K = 0$ for all $K \in \mathcal{L}(Y, U)$ and by Lemma 5.1.22, it follows that $\mathbb{B}(K, r) \subseteq \mathbb{S}(\mathbf{G})$ for all $r > 0$. Consequently, in the case wherein $\mathbf{G} = 0$, the conclusions of Theorem 5.2.5 hold, provided that there exists $K \in \mathcal{L}(Y, U)$ such that

$$\sup_{(\xi,\zeta)\in S_1\times S_2,\, \xi\neq\zeta} \frac{\|f(\xi) - f(\zeta) - K(\xi - \zeta)\|_U}{\|\xi - \zeta\|_Y} < \infty. \tag{5.33}$$

◊

The following result is another consequence of Theorem 5.2.5, and will be especially useful in Section 5.5 and Chapter 6.

Corollary 5.2.8. *Let $\Sigma \in \mathbb{L}$ be stabilisable and detectable and $f : Y \to U$. Assume that $r > 0$ and $K \in \mathcal{L}(Y,U)$ satisfy $\mathbb{B}(K,r) \subseteq \mathbb{S}(\mathbf{G})$ and that there exists $\delta \in (0,r)$ such that*

$$\|f(\xi) - K\xi\|_U \leq (r-\delta)\|\xi\|_Y \quad \forall\, \xi \in Y. \tag{5.34}$$

Then the following statements hold.

(i) *There exist constants $a > 0$, $b > 0$ and $\omega > 1$ such that, for all $(v,w,x,y) \in \mathcal{B}$ and $\rho \in [1,\omega]$,*

$$\|\pi_t x\|_{\ell^2_\rho} \leq a\left(\|x(0)\|_X + \left\|\pi_{t-1}\begin{pmatrix} v \\ w \end{pmatrix}\right\|_{\ell^2_\rho}\right) \quad \forall\, t \in \mathbb{N}, \tag{5.35}$$

and

$$\|\pi_t y\|_{\ell^2_\rho} \leq b\left(\|x(0)\|_X + \left\|\pi_{t}\begin{pmatrix} v \\ w \end{pmatrix}\right\|_{\ell^2_\rho}\right) \quad \forall\, t \in \mathbb{Z}_+. \tag{5.36}$$

(ii) *For $q \in [2,\infty]$, there exist constants $c > 0$, $d > 0$ and $\theta \in (0,1)$ such that, for all $(v,w,x,y) \in \mathcal{B}$,*

$$\|x(t)\|_X \leq c\left(\theta^t\|x(0)\|_X + \left\|\pi_{t-1}\begin{pmatrix} v \\ w \end{pmatrix}\right\|_{\ell^q}\right) \quad \forall\, t \in \mathbb{N}, \tag{5.37}$$

and

$$\|y(t)\|_Y \leq d\left(\theta^t\|x(0)\|_X + \left\|\pi_{t}\begin{pmatrix} v \\ w \end{pmatrix}\right\|_{\ell^q}\right) \quad \forall\, t \in \mathbb{Z}_+. \tag{5.38}$$

Here c and d depend on q, but θ does not.

Proof. From (5.34), we obtain that $f(0) = 0$, which in turn implies that $(0,0,0,0)$ is an equilibrium quadruple. By subsequently defining $S_1 := Y$ and $S_2 := \{0\}$, we see that (5.14) is satisfied. Therefore, since the rest of the hypotheses of Theorem 5.2.5 are also satisfied, we may apply the theorem to yield the result. ☐

5.2.2 Positive real assumptions

We now present several corollaries to Theorem 5.2.5 which have assumptions in terms of "positive realness". We therefore begin by giving the following definition of a positive real operator. We recall that, for a self-adjoint operator $M \in \mathcal{L}(U)$, we use the notation $M \geq 0$ to mean that $\langle \xi, M\xi\rangle_U \geq 0$ for all $\xi \in U$.

Definition 5.2.9. *Let $\alpha \in (0,1]$.*

(i) *We denote by $H^*_\alpha(\mathcal{L}(U,Y))$ the set of functions $\mathbf{H} : \mathbb{E}_\alpha \to \mathcal{L}(U,Y)$ that are holomorphic on $\mathbb{E}_\alpha$, with the exception of isolated singularities, that is, poles and essential singularities. We always assume that removable singularities have been removed via holomorphic extension. For convenience, we set $H^*(\mathcal{L}(U,Y)) := H^*_1(\mathcal{L}(U,Y))$.*

(ii) *For* $\mathbf{H} \in H^*_\alpha(\mathcal{L}(U))$*, we define* $\Sigma_\mathbf{H} \subseteq \mathbb{E}_\alpha$ *to be the set of isolated singularities of* $\mathbf{H}$*. The function* $\mathbf{H}$ *is said to be* positive real *if*

$$\mathbf{H}(z) + \mathbf{H}(z)^* \geq 0 \quad \forall\, z \in \mathbb{E}\backslash\Sigma_\mathbf{H}. \tag{5.39}$$

Remark 5.2.10. We note that an equivalent statement to (5.39) is the assertion that

$$\mathrm{Re}\,\langle \mathbf{H}(z)u, u\rangle_U \geq 0 \quad \forall\, u \in U, \forall\, z \in \mathbb{E}\backslash\Sigma_\mathbf{H}\,. \tag{5.40}$$

In the sequel, we will interchange between (5.39) and (5.40) depending on which is the most useful representation in the current situation. ◊

Before coming to the aforementioned corollaries of Theorem 5.2.5, we first require the use of two lemmas. The first lemma shows that if a function is positive real then it must be holomorphic on $\mathbb{E} \cup \{\infty\}$. The continuous-time analogue is given in [46, Proposition 3.3], however, there the region of analyticity does not include $\{\infty\}$.

Lemma 5.2.11. *Let* $\mathbf{H} \in H^*(\mathcal{L}(U))$ *be positive real. Then* $\mathbf{H}$ *does not have any singularities in* $\mathbb{E} \cup \{\infty\}$.

Proof. We begin by showing that $\Sigma_\mathbf{H} \cap \mathbb{E} = \emptyset$. To do so, we act analogously to that done in the proof of [46, Proposition 3.3]. To this end, we seek a contradiction and so we suppose that $\Sigma_\mathbf{H} \cap \mathbb{E} \neq \emptyset$ and let $z_0 \in \Sigma_\mathbf{H} \cap \mathbb{E}$. From this, we deduce the existence of a punctured open disc $\Delta := \{z \in \mathbb{C} : |z - z_0| < \varepsilon,\, z \neq z_0\}$ centred at z_0 and of radius $\varepsilon > 0$ such that $\mathbf{H}$ has Laurent expansion

$$\mathbf{H}(z) = \sum_{-\infty}^{\infty} H_j (z - z_0)^j \quad \forall\, z \in \Delta,$$

where $H_j \in \mathcal{L}(U)$ for all $j \in \mathbb{Z}$ (see, for example, [43, 1.10.2, p.25]). For $u \in U$, we define

$$J_u := \{j > 0 : \langle H_{-j}u, u\rangle_U \neq 0\}.$$

Let $v \in U$ be such that $J_v \neq \emptyset$. Such a $v \in U$ does exist, because otherwise [46, Lemma 2.1] would yield that $H_{-j} = 0$ for every $j > 1$ and so z_0 would not be a singularity, thus yielding a contradiction. Continuing, define $h \in H^*(\mathbb{C})$ by $h(z) = \langle \mathbf{H}(z)v, v\rangle_U$ for all $z \in \mathbb{E}$. If J_v is infinite, then h has an essential singularity at z_0 and so, using the Casorati-Weierstrass theorem (see, for example, [97, Theorem 10.21]), there exists $z^* \in \Delta$ such that

$$\mathrm{Re}\langle \mathbf{H}(z^*)u, u\rangle_U = \mathrm{Re}\, h(z^*) < 0,$$

contradicting the positive realness of $\mathbf{H}$.

We therefore assume that J_v is finite and set $k := \max J_v$. In this case, h has a pole of order k at z_0 and so h can be written as

$$h(z) = \frac{h_0 + g(z)}{(z - z_0)^k} \quad \forall\, z \in \Delta,$$

where $h_0 \neq 0$, g is holomorphic on $\Delta \cup \{z_0\}$ and $g(z_0) = 0$. For sufficiently small $r > 0$, we have

$$h(z_0 + re^{i\theta}) = r^{-k} e^{-ik\theta}(h_0 + g(z_0 + re^{i\theta})) \quad \forall\, \theta \in (-\pi, \pi].$$

Let $\theta_0 \in (-\pi, \pi]$ be such that $\operatorname{Re}(e^{-ik\theta_0} h_0) < 0$ and note that, from the fact that $g(z_0) = 0$, for sufficiently small $r > 0$,

$$\operatorname{Re}\langle \mathbf{H}(z_0 + re^{i\theta_0})v, v\rangle = \operatorname{Re} h(z_0 + re^{i\theta_0}) < 0.$$

This contradicts the positive realness of $\mathbf{H}$, consequently showing that $\mathbf{H}$ does not have any singularities in $\mathbb{E}$.
Finally, we show that $\mathbf{H}$ is holomorphic at infinity. To this end, we define $\tilde{\mathbf{H}} : \mathbb{D}\backslash\{0\} \to \mathcal{L}(U, Y)$ by

$$\tilde{\mathbf{H}}(z) := \mathbf{H}(1/z) \quad \forall\, z \in \mathbb{D}\backslash\{0\},$$

and note that $\operatorname{Re}\langle \tilde{\mathbf{H}}(z)u, u\rangle \geq 0$ for all $u \in U$ and $z \in \mathbb{D}\backslash\{0\}$. Thus, by supposing that $0 \in \Sigma_{\tilde{\mathbf{H}}}$ and by following the same method as above, we obtain a contradiction, completing the proof. □

We shall also require the following technical lemma, which is a direct consequence of Lemma 5.2.11 and [46, Corollary 2.3].

Lemma 5.2.12. *Let $\mathbf{H} \in H^*(\mathcal{L}(U))$ be positive real. Then, $I + \mathbf{H}(z)$ is invertible for every $z \in \mathbb{E}$ and*

$$\|(I - \mathbf{H})(I + \mathbf{H})^{-1}\|_{H^\infty} \leq 1.$$

We now present the first corollary to Theorem 5.2.5, which is reminiscent of the circle criterion.

Corollary 5.2.13. *Let $\Sigma \in \mathbb{L}$ be stabilisable and detectable, $f : Y \to U$, $S \subseteq Y$ be non-empty and $K_1, K_2 \in \mathcal{L}(Y, U)$ with $K_1 \in \mathbb{A}(D)$. If $(I - K_2\mathbf{G})(I - K_1\mathbf{G})^{-1}$ is positive real and there exists $\varepsilon > 0$ such that*

$$\operatorname{Re}\left\langle f(\zeta+\xi) - f(\xi) - K_1\zeta, f(\zeta+\xi) - f(\xi) - K_2\zeta\right\rangle_U \leq -\varepsilon\|\zeta\|_Y^2 \quad \forall \zeta \in Y, \forall \xi \in S\,, \tag{5.41}$$

then statements (i) and (ii) of Theorem 5.2.5 with $S_1 = Y$, $S_2 = S$ hold.

The following proof is in part inspired by a method outlined in the proof of [46, Theorem 6.8].

Proof of Corollary 5.2.13. We define

$$L := (K_1 - K_2)/2, \qquad M := (K_1 + K_2)/2\,,$$

and rewrite (5.41) so that

$$\begin{aligned} -\varepsilon\|\zeta\|_Y^2 &\geq \operatorname{Re}\left\langle f(\zeta+\xi) - f(\xi) - (L+M)\zeta, f(\zeta+\xi) - f(\xi) + (L-M)\zeta\right\rangle_U \\ &= -\|L\zeta\|_U^2 + \|f(\zeta+\xi) - f(\xi) - M\zeta\|_U^2 \quad \forall\, \zeta \in Y, \forall\, \xi \in S\,. \end{aligned} \tag{5.42}$$

We deduce from (5.42) that $\|L\zeta\|_U \geq \sqrt{\varepsilon}\|\zeta\|_Y$ for all $\zeta \in Y$, which in turn implies that

$$\|L^*L\zeta\|_Y\|\zeta\|_Y \geq |\left\langle L^*L\zeta, \zeta\right\rangle_Y| = \|L\zeta\|_U^2 \geq \varepsilon\|\zeta\|_Y^2 \quad \forall\, \zeta \in Y.$$

Hence L^*L is bounded away from 0 and, by combining this with the self-adjointness of L^*L, we have that L^*L is invertible.We define $L^\# := (L^*L)^{-1}L^*$ and let $Q := LL^\#$. It is clear that $Q^2 = LL^\#LL^\# = Q$ and, since L has a left inverse, $\operatorname{im} L$ is closed. Thus,

$$\operatorname{im} L = (\ker L^*)^\perp = (\ker L^\#)^\perp,$$

where the superscript $\perp$ denotes the orthogonal complement. Therefore, Q is the orthogonal projection onto $(\ker L^{\#})^{\perp}$ along $\ker L^{\#}$. Utilising this with (5.42) gives that

$$\begin{aligned}\|(f\circ L^{\#})(\zeta+\xi)-(f\circ L^{\#})(\xi)-ML^{\#}\zeta\|_U^2 &\le \|Q\zeta\|_U^2-\varepsilon\|L^{\#}\zeta\|_Y^2\\ &\le \|\zeta\|_Y^2-\varepsilon\|L^{\#}\zeta\|_Y^2 \quad \forall\,\zeta\in Y,\,\forall\,\xi\in S\,. \end{aligned}\tag{5.43}$$

Moreover, since $L^{\#}$ is bounded away from 0 on $\operatorname{im} L$, there exists $\nu>0$ such that

$$\|L^{\#}\zeta\|_Y \ge \nu\|\zeta\|_Y \quad \forall\,\zeta\in \operatorname{im} L\,.$$

Hence, since $L^{\#}Q=L^{\#}$, (5.43) yields

$$\begin{aligned}&\|(f\circ L^{\#})(\zeta+\xi)-(f\circ L^{\#})(\xi)-ML^{\#}\zeta\|_U^2\\ &\qquad=\|(f\circ L^{\#})(Q\zeta+Q\xi)-(f\circ L^{\#})(Q\xi)-ML^{\#}Q\zeta\|_U^2\\ &\qquad\le \|Q\zeta\|_U^2-\varepsilon\nu^2\|Q\zeta\|_U^2 \quad \forall\,\zeta\in Y,\,\forall\,\xi\in S\,,\end{aligned}$$

and so

$$\|(f\circ L^{\#})(\zeta+\xi)-(f\circ L^{\#})(\xi)-ML^{\#}\zeta\|_U \le \sqrt{1-\varepsilon\nu^2}\|\zeta\|_Y \quad \forall\,\zeta\in Y,\,\forall\,\xi\in S\,. \tag{5.44}$$

Next, on the one hand we compute that

$$(I-K_2\mathbf{G})(I-K_1\mathbf{G})^{-1}=(I-K_1\mathbf{G}+2L\mathbf{G})(I-K_1\mathbf{G})^{-1}=I+2L\mathbf{G}(I-K_1\mathbf{G})^{-1}\,, \tag{5.45}$$

and, on the other, that

$$(I-K_2\mathbf{G})(I-K_1\mathbf{G})^{-1}=(I-2M\mathbf{G}+K_1\mathbf{G})(I-K_1\mathbf{G})^{-1}\,. \tag{5.46}$$

By invoking the positive realness of $(I-K_2\mathbf{G})(I-K_1\mathbf{G})^{-1}$ along with Lemma 5.2.12, the expressions (5.45) and (5.46) yield that

$$\begin{aligned}1&\ge \left\|2L\mathbf{G}(I-K_1\mathbf{G})^{-1}\,\left(I+(I-2M\mathbf{G}+K_1\mathbf{G})(I-K_1\mathbf{G})^{-1}\right)^{-1}\right\|_{H^\infty}\\ &=\|2L\mathbf{G}(2I-2M\mathbf{G})^{-1}\|_{H^\infty}\\ &=\|L\mathbf{G}(I-M\mathbf{G})^{-1}\|_{H^\infty}\,.\end{aligned}\tag{5.47}$$

In addition, by recalling Definition 5.1.18, we evidently see that

$$L\mathbf{G}(I-M\mathbf{G})^{-1}=L\mathbf{G}(I-ML^{\#}L\mathbf{G})^{-1}=(L\mathbf{G})^{ML^{\#}}\,,$$

and thus, by Lemma 5.1.22, (5.47) obtains that

$$\mathbb{B}(ML^{\#},1)\subseteq \mathbb{S}(L\mathbf{G})\,. \tag{5.48}$$

Finally, let $(v,w,x,y)\in\mathcal{B}$ and note that, since $I=L^{\#}L$,

$$\begin{aligned}x^+&=Ax+B(f\circ L^{\#})(Ly+Lw)+B_{\mathrm{e}}v,\\ Ly&=LCx+LD(f\circ L^{\#})(Ly+Lw)+LD_{\mathrm{e}}v,\end{aligned}$$

which shows that (v,Lw,x,Ly) satisfies (5.10) with non-linearity $f\circ L^{\#}$ and linear component given by $(A,B,B_{\mathrm{e}},LC,LD,LD_{\mathrm{e}})$. Moreover, since Σ is stabilisable and detectable, it is clear, by the left invertibility of L, that $(A,B,B_{\mathrm{e}},LC,LD,LD_{\mathrm{e}})$ is also stabilisable and detectable. By combining this with (5.44) and (5.48), we see that the hypotheses of Theorem 5.2.5 are satisfied, and so the left-invertibility of L completes the proof. □

Remark 5.2.14. An inspection of the proof of Corollary 5.2.13 shows that if instead the estimate given in (5.41) holds for all $\zeta \in S_1$ and for all $\xi \in S$, where S_1 is a non-empty subset of Y, then statements (i) and (ii) of Theorem 5.2.5 with $S_2 = S$ hold, provided that $K_1 - K_2$ is left invertible. $\lozenge$

To conclude this section, we present two more corollaries of Theorem 5.2.5 that present slightly different assumptions than that given in Corollary 5.2.13. Namely, the two results, in order to obtain the same conclusions, weaken the sector condition given by (5.41), but in doing so, strengthen the notion of positive realness that we assert. With this in mind, we make the following definition.

Definition 5.2.15. *Let $\alpha \in (0, 1]$.*

(i) *We say $\mathbf{H} \in H^*_\alpha(\mathcal{L}(U))$ is* strictly positive real *if $\alpha \in (0, 1)$ and there exists $\varepsilon \in (\alpha, 1)$ such that the function $z \mapsto \mathbf{H}(\varepsilon z)$ is positive real.*

(ii) *We say a positive real function $\mathbf{H} \in H^*_\alpha(\mathcal{L}(U))$ is* strongly positive real, *if there exists $\delta > 0$ such that*

$$\mathbf{H}(z) + \mathbf{H}(z)^* - \delta I_U \geq 0 \quad \forall\, z \in \mathbb{E}, \tag{5.49}$$

where I_U is the identity operator on U.

Remark 5.2.16. An equivalent condition to strong positive realness is that there exists $\delta > 0$ such that

$$\operatorname{Re} \langle \mathbf{H}(z)u, u\rangle_U \geq \delta \|u\|^2_U \quad \forall\, u \in U, \forall\, z \in \mathbb{E}\,. \tag{5.50}$$

Similar to (standard) positive realness, we will interchange between (5.49) and (5.50) depending on which is the most useful representation in the current setting. $\lozenge$

Before presenting the final two corollaries of this section, we first provide the following two lemmas.

Lemma 5.2.17. *Let $\alpha \in (0, 1)$ and $\mathbf{H} \in H^*_\alpha(\mathcal{L}(U))$ be strictly positive real. If*

$$\lim_{|z|\to\infty} \operatorname{Re}\langle \mathbf{H}(z)u, u\rangle_U > 0 \quad \forall\, u \in U\backslash\{0\}, \tag{5.51}$$

then

$$\operatorname{Re}\langle \mathbf{H}(z)u, u\rangle_U > 0 \quad \forall\, z \in \mathbb{E}_\varepsilon, \forall\, u \in U\backslash\{0\},$$

where $\varepsilon \in (\alpha, 1)$ is such that the function $z \mapsto \mathbf{H}(\varepsilon z)$ is positive real.

Proof. We shall employ a contradiction argument. To this end, suppose that there exist $z^* \in \mathbb{E}_\varepsilon$ and $u \in U\backslash\{0\}$ such that $\operatorname{Re}\langle \mathbf{H}(z^*)u, u\rangle_U = 0$. Define $\tilde{\mathbf{H}} : \mathbb{D}_{1/\varepsilon} \to \mathcal{L}(U)$ by

$$\tilde{\mathbf{H}}(z) := \begin{cases} \mathbf{H}(1/z), & \text{if } z \in \mathbb{D}_{1/\varepsilon}\backslash\{0\}, \\ \lim_{|w|\to\infty} \mathbf{H}(w), & \text{if } z = 0, \end{cases}$$

where the limit is in the uniform operator topology and exists by Lemma 5.2.11. Note that the function $z \mapsto e^{-\langle \tilde{\mathbf{H}}(z)u,u\rangle_U}$ is holomorphic on $\mathbb{D}_{1/\varepsilon}$. Moreover,

$$|e^{-\langle \tilde{\mathbf{H}}(z)u,u\rangle_U}| = e^{-\operatorname{Re}\langle \tilde{\mathbf{H}}(z)u,u\rangle_U} \quad \forall\, z \in \mathbb{D}_{1/\varepsilon},$$

which implies, in particular, that

$$|e^{-\langle \tilde{\mathbf{H}}(1/z^*)u,u\rangle_U}| = 1 \geq e^{-\operatorname{Re}\langle \tilde{\mathbf{H}}(z)u,u\rangle_U} \quad \forall z \in \mathbb{D}_{1/\varepsilon}.$$

Therefore, an application of the maximum modulus principle applied to the function $z \mapsto e^{-\langle \tilde{\mathbf{H}}(z)u,u\rangle_U}$ yields that $\operatorname{Re}\langle \tilde{\mathbf{H}}(z)u,u\rangle_U = 0$ for all $z \in \mathbb{D}_{1/\varepsilon}$. However, by (5.51), we have

$$\lim_{|z|\to 0} \operatorname{Re}\langle \tilde{\mathbf{H}}(z)u,u\rangle_U > 0.$$

This is a contradiction, and thus the proof is complete. □

Lemma 5.2.18. *Let $\alpha \in (0,1)$ and $\mathbf{H} \in H^*_\alpha(\mathcal{L}(U))$ be strictly positive real. If there exists $\tilde{z} \in \operatorname{clos}(\mathbb{E}) \cup \{\infty\}$ such that*

$$\inf_{\|u\|_U=1} \operatorname{Re}\langle \mathbf{H}(\tilde{z})u,u\rangle_U > 0, \tag{5.52}$$

then there exists $\delta > 0$ such that

$$\operatorname{Re}\langle \mathbf{H}(z)u,u\rangle_U \geq \delta\|u\|_U^2 \quad \forall\, u \in U, \forall\, z \in \operatorname{clos}(\mathbb{E}) \cup \{\infty\}. \tag{5.53}$$

In particular, $\mathbf{H}$ is strongly positive real.

Proof. To prove the result, we shall utilise a contradiction argument. To this end, suppose that the claim is false. From this we deduce the existence of a sequence $(z_k)_{k\in\mathbb{N}} \subseteq \operatorname{clos}(\mathbb{E}) \cup \{\infty\}$ such that

$$\inf_{\|u\|_U=1} \operatorname{Re}\langle \mathbf{H}(z_k)u,u\rangle_U \to 0 \text{ as } k \to \infty.$$

Without loss of generality, we may assume that $(z_k)_{k\in\mathbb{N}}$ converges to $\lambda \in \operatorname{clos}(\mathbb{E}) \cup \{\infty\}$. Now, we claim that the function $h : \operatorname{clos}(\mathbb{E}) \cup \{\infty\} \to \mathbb{R}_+$, defined by

$$h(z) := \inf_{\|u\|_U=1} \operatorname{Re}\langle \mathbf{H}(z)u,u\rangle_U \quad \forall\, z \in \operatorname{clos}(\mathbb{E}) \cup \{\infty\},$$

is continuous. We omit the verification of this claim, since the proof is identical, *mutatis mutandis*, to that done in [46, pp.56-57]. Continuing with the proof of the lemma, the continuity of h implies that

$$\inf_{\|u\|_U=1} \operatorname{Re}\langle \mathbf{H}(\lambda)u,u\rangle_U = 0.$$

By denoting the square root of a self-adjoint positive semi-definite operator M by $M^{1/2}$ (see, for example, [87, Proposition 3.2.11, p.92]), the above equality yields that

$$\inf_{\|u\|_U=1} \|(\mathbf{H}(\lambda) + \mathbf{H}(\lambda)^*)^{1/2}u\|_U^2 = 0,$$

which in turn gives that

$$\inf_{\|u\|_U=1} \|(\mathbf{H}(\lambda) + \mathbf{H}(\lambda)^*)u\|_U^2 = 0.$$

Therefore, we have shown that $0 \in \sigma_{\mathrm{ap}}(\mathbf{H}(\lambda) + \mathbf{H}(\lambda)^*)$, where $\sigma_{\mathrm{ap}}(\mathbf{H}(\lambda) + \mathbf{H}(\lambda)^*)$ denotes the set of approximate eigenvalues of $\mathbf{H}(\lambda) + \mathbf{H}(\lambda)^*$, that is, numbers $\eta \in \mathbb{C}$

satisfying $\inf_{\|u\|_U=1} \|(\mathbf{H}(\lambda) + \mathbf{H}(\lambda)^* - \eta I)u\|_U = 0$. An application of [74, Theorem 4.4] then gives that $0 \in \sigma_{\text{ap}}(\mathbf{H}(z) + \mathbf{H}(z)^*)$ for all $z \in \text{clos}(\mathbb{E}) \cup \{\infty\}$. (Technically, [74, Theorem 4.4] holds for positive real operators defined on the open right half-plane, instead of $\mathbb{E}$. An inspection of the proof shows that it extends very easily to the discrete-time case and so, in the interest of brevity, we do not provide an explicit proof.) In particular, we see that $0 \in \sigma_{\text{ap}}(\mathbf{H}(\tilde{z}) + \mathbf{H}(\tilde{z})^*)$, which contradicts (5.52), thus completing the proof. □

Remark 5.2.19. We note that if $\mathbf{H}$ satisfies (5.52) for some $\tilde{z} \in \mathbb{E} \cup \{\infty\}$, but is only positive real, and not strictly positive real, then the conclusions of Lemma 5.2.18 need not hold. Indeed, for a counterexample consider the function $z \mapsto 1/(z-1)+1/2$, which has real part given by $(|z|^2-1)/(2|z-1|^2)$. It is clear that the function is positive real, but is not strongly positive real since the real part is zero on the unit circle (except at $z = 1$) and tends towards zero as it approaches the unit circle. ◊

We are now ready to present the final two corollaries of this section.

Corollary 5.2.20. *Let $\Sigma \in \mathbb{L}$ be stabilisable and detectable, $f : Y \to U$, $S, S_1 \subseteq Y$ be non-empty and $K_1, K_2 \in \mathcal{L}(Y, U)$ with $K_1 \in \mathbb{A}(D)$. If $K_1 - K_2$ is left invertible, $\mathbf{H} := (I - K_2\mathbf{G})(I - K_1\mathbf{G})^{-1} \in H^\infty(\mathcal{L}(U))$, $\mathbf{H}$ is strongly positive real and*

$$\text{Re}\left\langle f(\zeta+\xi) - f(\xi) - K_1\zeta, f(\zeta+\xi) - f(\xi) - K_2\zeta\right\rangle_U \leq 0 \quad \forall\, \zeta \in S_1,\ \forall\, \xi \in S\,, \quad (5.54)$$

then statements (i) and (ii) of Theorem 5.2.5 with $S_2 = S$ hold.

In order to prove Corollary 5.2.20, we follow a similar method to that shown in [46, Theorem 6.11].

Proof of Corollary 5.2.20. Define $M := K_2 - K_1$ and note that, by assumption,

$$I - M\mathbf{G}(I - K_1\mathbf{G})^{-1} = (I - (M + K_1)\mathbf{G})(I - K_1\mathbf{G})^{-1} = \mathbf{H} \in H^\infty(\mathcal{L}(U)).$$

Since M is left-invertible, we have that $\mathbf{G}(I - K_1\mathbf{G})^{-1} \in H^\infty(\mathcal{L}(U, Y))$, or, equivalently, that $K_1 \in \mathbb{S}(\mathbf{G})$. Lemma 5.1.22 implies that $\mathbb{S}(\mathbf{G})$ is open (see Remark 5.1.23) and so there exists $\rho^* > 0$ such that $K_1 - \rho M \in \mathbb{S}(\mathbf{G})$ for all $\rho \in [0, \rho^*]$. We define

$$\mathbf{H}_\rho := (I - (K_2 + \rho M)\mathbf{G})(I - (K_1 - \rho M)\mathbf{G})^{-1} \quad \forall\, \rho \in [0, \rho^*],$$

and claim that the map

$$[0, \rho^*] \to H^\infty(\mathcal{L}(U)), \quad \rho \mapsto \mathbf{H}_\rho, \quad (5.55)$$

is continuous. In order to verify this claim, we shall only prove that $\rho \mapsto (I - (K_1 - \rho M)\mathbf{G})^{-1}$ is continuous and leave the rest to the reader. To this end, let $\rho \in [0, \rho^*]$ and $(\rho_n)_{n\in\mathbb{N}} \subseteq [0, \rho^*]$ be such that $\rho_n \to \rho$ as $n \to \infty$. For ease of notation, define $T_n := I - (K_1 - \rho_n M)\mathbf{G}$ and $T := I - (K_1 - \rho M)\mathbf{G}$. Note that, for all $n \in \mathbb{Z}_+$,

$$\|T_n^{-1} - T^{-1}\| = \|T_n^{-1}(T - T_n)T^{-1}\|.$$

Since $T_n \to T$ as $n \to \infty$, if we can show that $\sup_{n\in\mathbb{Z}_+} \|T_n^{-1}\|_{H^\infty} < \infty$, then the claim follows. To this end, we define $P_n := T^{-1}(T_n - T)$ so that $P_n \to 0$ as $n \to \infty$. As a consequence, there exists $N \in \mathbb{Z}_+$ such that $\|P_n\| < 1$ for all $n \in \overline{N}$. This in turn implies that $(I + P_n)^{-1} = \sum_{k=0}^{\infty}(-P_n)^k$ for all $n > N$. Therefore,

$$\|(I + P_n)^{-1}\| \leq \frac{1}{1 - \|P_n\|} \to 1, \text{ as } n \to \infty.$$

Finally, we note that $T_n^{-1} = (I + P_n)^{-1}(I + P_n)T_n^{-1} = (I + P_n)^{-1}T^{-1}$ for all $n \in \mathbb{Z}_+$ and so $\sup_{n\in\mathbb{Z}_+} \|T_n^{-1}\|_{H^\infty} < \infty$, completing the proof of the claim.
Continuing again with the proof, by using the continuity of (5.55) along with the strong positive realness of $\mathbf{H}_0 = \mathbf{H}$, we obtain that there exists $\rho^\dagger \in (0, \rho^*]$ such that

$$\mathrm{Re}\langle \mathbf{H}_\rho(z)u, u\rangle_U \geq 0 \quad \forall\, u \in U, \forall\, z \in \mathbb{E}, \forall\, \rho \in [0, \rho^\dagger].$$

Now, we note that the left invertibility of $K_1 - K_2$ implies that $(K_1 - \rho M) - (K_2 + \rho M)$ is left invertible. Whence, from Remark 5.2.14, if we can show that for all $\rho \in (0, \rho^\dagger]$ there exists $\varepsilon > 0$ such that

$$\left.\begin{aligned} \mathrm{Re}\langle f(\zeta + \xi) - f(\xi) - (K_1 - \rho M)\zeta, f(\zeta + \xi) - f(\xi) - (K_2 + \rho M)\zeta\rangle_U \\ \leq -\varepsilon\|\zeta\|_Y^2 \quad \forall\, \zeta \in S_1, \forall\, \xi \in S, \end{aligned}\right\} \tag{5.56}$$

then the result follows from Corollary 5.2.13. Indeed, this is what we shall now do. To this end, by using (5.54), we have, for all $\rho \in (0, \rho^\dagger]$, all $\zeta \in S_1$ and all $\xi \in S$,

$$\begin{aligned} &\mathrm{Re}\langle f(\zeta + \xi) - f(\xi) - (K_1 - \rho M)\zeta, f(\zeta + \xi) - f(\xi) - (K_2 + \rho M)\zeta\rangle_U \\ &\quad = \mathrm{Re}\langle f(\zeta + \xi) - f(\xi) - K_1\zeta, f(\zeta + \xi) - f(\xi) - (K_2 + \rho M)\zeta\rangle_U \\ &\qquad + \mathrm{Re}\langle \rho M\zeta, f(\zeta + \xi) - f(\xi) - (K_2 + \rho M)\zeta\rangle_U \\ &\quad = \mathrm{Re}\langle f(\zeta + \xi) - f(\xi) - K_1\zeta, f(\zeta + \xi) - f(\xi) - K_2\zeta\rangle_U + \mathrm{Re}\langle \rho M\zeta, -\rho M\zeta\rangle_U \\ &\qquad + \mathrm{Re}\langle f(\zeta + \xi) - f(\xi) - K_1\zeta, -\rho M\zeta\rangle_U + \mathrm{Re}\langle \rho M\zeta, f(\zeta + \xi) - f(\xi) - K_2\zeta\rangle_U \\ &\quad \leq 0 - \rho\|M\zeta\|_U^2 - \rho^2\|M\zeta\|_U^2 \\ &\quad = -\rho(\rho + 1)\|M\zeta\|_U^2. \end{aligned}$$

By the left-invertibility of M, there exists $\mu > 0$ such that $\|M\zeta\|_U^2 \geq \mu\|\zeta\|_Y^2$ for all $\zeta \in Y$ and so we see that (5.56) holds with $\varepsilon := \mu\rho(\rho + 1) > 0$, completing the proof. □

Corollary 5.2.21. *Let $\Sigma \in \mathbb{L}$ be stabilisable and detectable, $\alpha \in (0, 1)$, $f : Y \to U$, $S, S_1 \subseteq Y$ be non-empty and $K_1, K_2 \in \mathcal{L}(Y, U)$ with $K_1 \in \mathbb{A}(D)$. Assume that $K_1 - K_2$ is left invertible, $\mathbf{H} := (I - K_2\mathbf{G})(I - K_1\mathbf{G})^{-1} \in H_\alpha^*(\mathcal{L}(U))$ is strictly positive real and there exists $\tilde{z} \in \mathrm{clos}(\mathbb{E}) \cup \{\infty\}$ such that* (5.52) *holds. If* (5.54) *holds, then statements (i) and (ii) of Theorem 5.2.5 with $S_2 = S$ hold*

Proof. The result is immediately obtained by combining Lemma 5.2.18 with Corollary 5.2.20. □

Remark 5.2.22. Due to the strict positive realness assumption used in the statement of Corollary 5.2.21, we may perceive this result to be 'closest', of the three corollaries presented, to the circle-criterion (see, for example, [66, Theorem 7.1, p.265]). ◊

5.3 Convergence properties

In this section, we provide two applications of Theorem 5.2.5. Indeed, we begin by determining conditions for when (5.10) exhibits a notion of convergence known as the converging-input converging-state property [15] (see also Section 2.3). We then present assumptions that guarantee that asymptotically almost periodic inputs generate asymptotically almost periodic state and output trajectories (see, for example, [1, 14, 21] and Appendix C).

5.3.1 The converging-input converging-state property

We begin with the following definition.

Definition 5.3.1. *Let $\Sigma \in \mathbb{L}$ and $f : Y \to U$.*

(i) *We say that* (5.10) *has the* converging-input converging-state (CICS) property *if, for every $v^\infty \in V$ and $w^\infty \in Y$, there exists $x^\infty \in X$ such that, for every $(v, w, x, y) \in \mathcal{B}$ with $\lim_{t\to\infty} v(t) = v^\infty$ and $\lim_{t\to\infty} w(t) = w^\infty$, we have that $\lim_{t\to\infty} x(t) = x^\infty$.*

(ii) *We say that* (5.10) *has the* converging-input converging-state/output (CICSO) property *if, for every $v^\infty \in V$ and $w^\infty \in Y$, there exists $x^\infty \in X$ and $y^\infty \in Y$ such that, for every $(v, w, x, y) \in \mathcal{B}$ with $\lim_{t\to\infty} v(t) = v^\infty$ and $\lim_{t\to\infty} w(t) = w^\infty$, we have that $\lim_{t\to\infty} x(t) = x^\infty$ and $\lim_{t\to\infty} y(t) = y^\infty$.*

Remark 5.3.2. (i) We note that some authors (see, for example, [113]) use the term CICS for the special case wherein $v^\infty = 0$, $w^\infty = 0$ and $x^\infty = 0$.

(ii) In the case that $D = 0$, the CICS property trivially implies the CICSO property. In the nonzero feedthrough situation however, the CICS property does not necessarily imply the CICSO property and there exist systems which exhibit the former but not the latter. Indeed, even in the finite-dimensional case this is true, and an example is presented in Example 2.3.14.

(iii) If w in (5.10) is perceived to be an output disturbance to the system, then convergence of w is not an assumption which will be generically satisfied. Therefore, in what follows, we shall not only investigate the conditions that guarantee the CICS property, but will also develop a result for bounded but not necessarily convergent w. ◊

Before coming to a result that guarantees the CICS property for (5.10), we first present some relevant definitions and results.

Definition 5.3.3. *Let $\Sigma \in \mathbb{L}$ be stabilisable and detectable, $f : Y \to U$ and $K \in \mathbb{S}(\mathbf{G})$. We define*

$$F_K : Y \to Y, \quad F_K(\xi) := \xi - \mathbf{G}^K(1)(f(\xi) - K\xi)\,. \tag{5.57}$$

To facilitate the proofs of the main results in this subsection, it is useful to state the following two lemmas.

Lemma 5.3.4. *Let $\Sigma \in \mathbb{L}$ be stabilisable and detectable, $f : Y \to U$, $S \subseteq Y$ be non-empty, $r > 0$ and $K \in \mathcal{L}(Y, U)$. Assume that $\mathbb{B}(K, r) \subseteq \mathbb{S}(\mathbf{G})$ and*

$$\|f(\xi + \zeta) - f(\zeta) - K\xi\|_U < r\|\xi\|_Y \quad \forall\, \zeta \in S,\ \forall\, \xi \in Y\backslash\{0\}. \tag{5.58}$$

Then the following statements hold.

(i) $\#F_K^{-1}(\xi) = 1$ *for all* $\xi \in Y$ *such that* $F_K^{-1}(\xi) \cap S \neq \emptyset$.

If there exists $\delta > 0$ *such that* f *and* K *satisfy* (5.14) *with* $S_1 = S_2 = Y$, *then*

(ii) F_K *is globally Lipschitz and bijective;*

(iii) *the inverse* F_K^{-1} *is globally Lipschitz.*

Remark 5.3.5. In the case that $\|\mathbf{G}^K\|_{H^\infty} = 0$, F_K is the identity map which is trivially globally Lipschitz and bijective. ◊

Proof of Lemma 5.3.4. We begin with the assumption that $\mathbf{G}^K(1) \neq 0$, since otherwise the result is trivial. We first prove statement (i) and so we let $\xi \in Y$ be such that $F_K^{-1}(\xi) \cap S \neq \emptyset$. Furthermore, let $\xi_1 \in F_K^{-1}(\xi) \cap S$ and $\xi_2 \in F_K^{-1}(\xi)$. We seek to show that $\xi_1 = \xi_2$. To this end, note that

$$\begin{aligned}\|\xi_2 - \xi_1\|_Y &= \|F_K(\xi_2) + \mathbf{G}^K(1)(f(\xi_2) - K\xi_2) - F_K(\xi_1) - \mathbf{G}^K(1)(f(\xi_1) - K\xi_1)\|_Y \\ &\leq \|\mathbf{G}^K\|_{H^\infty}\|f(\xi_2) - f(\xi_1) - K(\xi_2 - \xi_1)\|_U.\end{aligned}$$

Seeking a contradiction, suppose that $\xi_1 \neq \xi_2$. Then, by (5.58) and an application of Lemma 5.1.22, we obtain that

$$\|\xi_2 - \xi_1\|_Y < r\|\mathbf{G}^K\|_{H^\infty}\|\xi_2 - \xi_1\|_Y \leq \|\xi_2 - \xi_1\|_Y,$$

which is a contradiction. Hence, $\xi_1 = \xi_2$ and $\#F_K^{-1}(\xi) = 1$. Turning our attention now to proving statement (ii), since there exists $\delta > 0$ such that f and K satisfy (5.14) with $S_1 = S_2 = Y$, we obtain that, for all $\xi, \zeta \in Y$,

$$\begin{aligned}\|F_K(\xi) - F_K(\zeta)\|_Y &\leq \|\xi - \zeta\|_Y + \|\mathbf{G}^K(1)\|\|f(\xi) - f(\zeta) - K(\xi - \zeta)\|_U \\ &\leq (1 + (r - \delta)\|\mathbf{G}^K(1)\|)\|\xi - \zeta\|_Y.\end{aligned}$$

This shows that F_K is globally Lipschitz with Lipschitz constant $(1 + (r - \delta)\|\mathbf{G}^K(1)\|)$. We now proceed to show surjectivity of F_K. To that end, observe that the map $Y \to Y$, $\xi \mapsto \mathbf{G}^K(1)(f(\xi) - K\xi)$ is a contraction. Indeed, by (5.14) with $S_1 = S_2 = Y$,

$$\|\mathbf{G}^K(1)(f(\xi) - K\xi) - \mathbf{G}^K(1)(f(\zeta) - K\zeta)\|_Y \leq (r - \delta)\|\mathbf{G}^K(1)\|\|\xi - \zeta\|_Y \quad \forall\, \zeta, \xi \in Y\,,$$

and, furthermore, $(r - \delta)\|\mathbf{G}^K(1)\| < r\|\mathbf{G}^K\|_{H^\infty} \leq 1$. Fix $\eta \in Y$ and define the map $h_\eta : Y \to Y$ by

$$h_\eta(\xi) := \xi - F_K(\xi) + \eta \quad \forall\, \xi \in Y\,.$$

We note that h_ζ is also a contraction since,

$$\begin{aligned}\|h_\eta(\xi) - h_\eta(\zeta)\|_Y &= \|\mathbf{G}^K(1)(f(\xi) - K\xi) - \mathbf{G}^K(1)(f(\zeta) - K\zeta)\|_Y \\ &\leq (r - \delta)\|\mathbf{G}^K(1)\|\|\xi - \zeta\|_Y \quad \forall\, \xi, \zeta \in Y.\end{aligned}$$

Hence, by the contraction mapping theorem, there exists a (unique) fixed point of h_η, that is, there exists $\xi^* \in Y$ such that $h_\eta(\xi^*) = \xi^*$. This is equivalent to

$$\xi^* - F_K(\xi^*) + \eta = \xi^*,$$

and so $F_K(\xi^*) = \eta$, showing that F_K is surjective. As for injectivity of F_K, this follows immediately from statement (i) since (5.14) holds with $S_1 = S_2 = Y$ and F_K is surjective. Finally, let us now prove statement (iii). To this end, let $\xi, \zeta \in Y$ and note that, since F_K is surjective, there exist $\eta_1, \eta_2 \in Y$ such that

$$F_K(\eta_1) = \xi, \quad F_K(\eta_2) = \zeta.$$

Now, by the definition of F_K and by (5.14) with $S_1 = S_2 = Y$,

$$\begin{aligned}\|\eta_1 - \eta_2\|_Y &= \|F_K(\eta_1) + \mathbf{G}^K(1)(f(\eta_1) - K\eta_1) - F_K(\eta_2) - \mathbf{G}^K(1)(f(\eta_2) - K\eta_2)\|_Y \\ &\leq \|\xi - \zeta\|_Y + \|\mathbf{G}^K(1)\|\|f(\eta_1) - f(\eta_2) - K(\eta_1 - \eta_2)\|_U \\ &\leq \|\xi - \zeta\|_Y + (r - \delta)\|\mathbf{G}^K(1)\|\|\eta_1 - \eta_2\|_Y .\end{aligned}$$

Moreover, $F_K^{-1}(\xi) = \eta_1$, $F_K^{-1}(\zeta) = \eta_2$, $(r-\delta)\|\mathbf{G}^K(1)\| < 1$, and so we conclude that

$$\|F_K^{-1}(\xi) - F_K^{-1}(\zeta)\|_Y \leq \frac{1}{1 - (r-\delta)\|\mathbf{G}^K(1)\|}\|\xi - \zeta\|_Y ,$$

completing the proof. □

For the next result, recall that $(v^{\mathrm{e}}, w^{\mathrm{e}}, x^{\mathrm{e}}, y^{\mathrm{e}}) \in V \times Y \times X \times Y$ is an equilibrium quadruple of the Lur'e system (5.10), if $(v^{\mathrm{e}}, w^{\mathrm{e}}, x^{\mathrm{e}}, y^{\mathrm{e}}) \in \mathcal{B}$.

Lemma 5.3.6. *Let $\Sigma \in \mathbb{L}$ be stabilisable and detectable, $f : Y \to U$, $v^\infty \in V$ and $w^\infty \in Y$. Assume that $K \in \mathbb{S}(\mathbf{G})$ and*

$$\mathcal{T}_K := F_K^{-1}\begin{pmatrix} C^K(I - A^K)^{-1}\left(B_{\mathrm{e}} + B^K K D_{\mathrm{e}}\right)v^\infty \\ \qquad + (I - DK)^{-1}D_{\mathrm{e}}v^\infty + \left(I + \mathbf{G}^K(1)K\right)w^\infty \end{pmatrix} \tag{5.59}$$

is nonempty. Let $z^\infty \in \mathcal{T}_K$ and define $y^\infty := z^\infty - w^\infty$ and

$$x^\infty := (I - A^K)^{-1}\left(B^K(f(z^\infty) - K(z^\infty - w^\infty)) + (B_{\mathrm{e}} + B^K K D_{\mathrm{e}})v^\infty\right) . \tag{5.60}$$

Then

$$y^\infty = C^K x^\infty + D^K(f(z^\infty) - K(z^\infty - w^\infty)) + (I - DK)^{-1}D_{\mathrm{e}}v^\infty , \tag{5.61}$$

and $(v^\infty, w^\infty, x^\infty, y^\infty)$ is an equilibrium quadruple of the Lur'e system (5.10).

We shall not prove the previous lemma since it can be done so identically, *mutatis mutandis*, to the proof of Proposition 2.3.6. Moreover, we also refer the reader to Remark 2.3.7, since the comments there can analogously be applied to Lemma 5.3.6. Indeed, the remark outlines motiviation for (5.60) and (5.61) and also discusses simple situations for clarity.

The following theorem gives sufficient conditions for when (5.10) exhibits the CICSO property.

Theorem 5.3.7. *Let $\Sigma \in \mathbb{L}$ be stabilisable and detectable, $f : Y \to U$, $S \subseteq Y$ be non-empty, $r > 0$, $K \in \mathcal{L}(Y, U)$, $v^\infty \in V$ and $w^\infty \in Y$. Assume that $\mathbb{B}(K, r) \subseteq \mathbb{S}(\mathbf{G})$, $\mathcal{T}_K \cap S \neq \emptyset$, where $\mathcal{T}_K$ is given by* (5.59)*, and there exists $\delta \in (0, r)$ such that* (5.14) *holds with $S_1 = Y$ and $S_2 = S$. Then $\#\mathcal{T}_K = 1$ and, writing $z^\infty \in \mathcal{T}_K$, $(v^\infty, w^\infty, x^\infty, y^\infty)$ is an equilibrium quadruple of* (5.10) *where $y^\infty := z^\infty - w^\infty$ and x^∞ is as in* (5.60).

Furthermore, there exist $c > 0$ *and* $a \in (0,1)$ *such that, for all* $(v, w, x, y) \in \mathcal{B}$ *and all* $t \in \mathbb{Z}_+$*,*

$$\left\| \begin{pmatrix} x(t) - x^\infty \\ y(t) - y^\infty \end{pmatrix} \right\|_{X \times Y} \leq c \left(a^t \|x(0) - x^\infty\|_X + a^{\lceil t/2 \rceil} \max_{s \in \underline{\lfloor t/2 \rfloor}} \left\| \begin{pmatrix} v(s) \\ w(s) \end{pmatrix} - \begin{pmatrix} v^\infty \\ w^\infty \end{pmatrix} \right\|_{V \times Y} \right.$$
$$\left. + \max_{s \in \underline{\lceil t/2 \rceil}} \left\| \begin{pmatrix} (\Lambda_{\lfloor t/2 \rfloor} v)(s) \\ (\Lambda_{\lfloor t/2 \rfloor} w)(s) \end{pmatrix} - \begin{pmatrix} v^\infty \\ w^\infty \end{pmatrix} \right\|_{V \times Y} \right). \tag{5.62}$$

In particular, for all $(v, w, x, y) \in \mathcal{B}$ *with* $\lim_{t \to \infty} v(t) = v^\infty$ *and* $\lim_{t \to \infty} w(t) = w^\infty$*, we have* $\lim_{t \to \infty} x(t) = x^\infty$ *and* $\lim_{t \to \infty} y(t) = y^\infty$*.*

Before proving Theorem 5.3.7, we provide some commentary.

Remark 5.3.8. (i) Under the hypotheses of Theorem 5.3.7, we note that x^∞ and y^∞ given by (5.60) and (5.61) do not depend on the choice of K. Indeed, if $K_1, K_2 \in \mathbb{S}(\mathbf{G})$, $v^\infty \in V$, $w^\infty \in Y$, Σ is stabilisable and detectable, $f : Y \to Y$, and each of the relevant assumptions regarding K_1 and K_1 respectively hold, then it can be shown directly, or as a consequence of Theorem 5.3.7, that $x_1^\infty = x_2^\infty$ and $y_1^\infty = y_2^\infty$, where

$$x_l^\infty := (I - A^{K_l})^{-1} \left(B^{K_l} (f(y_l^\infty + w^\infty) - K_l y_l^\infty) + (B_\mathrm{e} + B^{K_l} K_l D_\mathrm{e}) v^\infty \right),$$
$$y_l^\infty = C^{K_l} x^\infty + D^{K_l} (f(y_l^\infty + w^\infty) - K_l y_l^\infty) + (I - D K_l)^{-1} D_\mathrm{e} v^\infty,$$

for $l \in \{1, 2\}$. By "directly", we mean that it can be shown without the use of Theorem 5.3.7. We shall not explicitly show this since the argument given in the proof of Lemma 2.3.19 can be used for this infinite-dimensional setting.

(ii) Assumption (5.14) with $S_1 = S_2 = Y$ may be rewritten as

$$\sup_{\substack{\zeta, \xi \in Y \\ \xi \neq 0}} \frac{\|f(\xi + \zeta) - f(\zeta) - K\xi\|_U}{\|\xi\|_Y} < r, \tag{5.63}$$

which trivially implies (5.58) with $S = Y$, and is itself equivalent to the function $\xi \mapsto f(\xi) - K\xi$ being globally Lipschitz with Lipschitz constant smaller than r. Furthermore, by acting in a similar manner to that shown in the proof of Proposition 2.3.10, we can deduce that (5.63) implies that $I - D^K(f - K)$ is bijective. By then combining this with Proposition 5.1.33, we see that, for given $x^0 \in X$, $v \in V^{\mathbb{Z}_+}$ and $w \in Y^{\mathbb{Z}_+}$, the IVP (5.11) has a unique solution.

(iii) Recalling the notion of equi-convergence from Definiton 2.3.15, the convergence property provided by Theorem 5.3.7 is uniform in the following sense: given a set of inputs $\mathcal{V} \subseteq V^{\mathbb{Z}_+} \times Y^{\mathbb{Z}_+}$ which is equi-convergent to (v^∞, w^∞) and $\kappa > 0$, the set of solutions

$$\left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in X^{\mathbb{Z}_+} \times Y^{\mathbb{Z}_+} : \exists \begin{pmatrix} v \\ w \end{pmatrix} \in \mathcal{V} \text{ s.t. } (v, w, x, y) \in \mathcal{B} \right.$$
$$\left. \text{and } \|x(0)\|_X + \sup_{t \in \mathbb{Z}_+} \left\| \begin{pmatrix} v(t) \\ w(t) \end{pmatrix} \right\|_{V \times Y} \leq \kappa \right\},$$

is equi-convergent to (x^∞, y^∞). ◊

Proof of Theorem 5.3.7. First, statement (i) of Lemma 5.3.4 yields that $\#\mathcal{T}_K = 1$. Moreover, an application of Lemma 5.3.6 gives that $(v^\infty, w^\infty, x^\infty, y^\infty)$ is an equilibrium quadruple of (5.10) and, since $y^\infty + w^\infty \in S$, we have that $(v^\infty, w^\infty, x^\infty, y^\infty) \in \mathcal{B}^S$. We invoke statement (ii) of Theorem 5.2.5, with $q = \infty$, to obtain $\tilde{c} > 0$ and $a \in (0,1)$ such that, for all $(v, w, x, y) \in \mathcal{B}$ and all $t \in \mathbb{Z}_+$,

$$\left\| \begin{pmatrix} x(t) - x^\infty \\ y(t) - y^\infty \end{pmatrix} \right\|_{X\times Y} \le \tilde{c}a^t \|x(0) - x^\infty\|_X + \tilde{c} \max_{s\in \underline{t}} \left\| \begin{pmatrix} v(s) \\ w(s) \end{pmatrix} - \begin{pmatrix} v^\infty \\ w^\infty \end{pmatrix} \right\|_{V\times Y} . \tag{5.64}$$

Let $(v, w, x, y) \in \mathcal{B}$ and fix $t \in \mathbb{Z}_+$. Note that (5.64) holds for

$$(\Lambda_{\lfloor t/2\rfloor} v, \Lambda_{\lfloor t/2\rfloor} w, \Lambda_{\lfloor t/2\rfloor} x, \Lambda_{\lfloor t/2\rfloor} y) \in \mathcal{B} ,$$

from the time-invariance property given by Lemma 5.1.30. In light of the identity $\lceil t/2\rceil + \lfloor t/2\rfloor = t$, it follows from (5.64) that

$$\begin{aligned} \left\| \begin{pmatrix} x(t) - x^\infty \\ y(t) - y^\infty \end{pmatrix} \right\|_{X\times Y} &= \left\| \begin{pmatrix} (\Lambda_{\lfloor t/2\rfloor} x)(\lceil t/2\rceil) - x^\infty \\ (\Lambda_{\lfloor t/2\rfloor} y)(\lceil t/2\rceil) - y^\infty \end{pmatrix} \right\|_{X\times Y} \\ &\le \tilde{c}a^{\lceil t/2\rceil} \|x(\lfloor t/2\rfloor) - x^\infty\|_X + \tilde{c} \max_{s\in \underline{\lceil t/2\rceil}} \left\| \begin{pmatrix} \Lambda_{\lfloor t/2\rfloor} v(s) \\ \Lambda_{\lfloor t/2\rfloor} w(s) \end{pmatrix} - \begin{pmatrix} v^\infty \\ w^\infty \end{pmatrix} \right\|_{V\times Y} . \end{aligned}$$

Appealing to (5.64) again, and also using that $a^{\lceil t/2\rceil} a^{\lfloor t/2\rfloor} = a^t$, yields

$$\begin{aligned} \left\| \begin{pmatrix} x(t) - x^\infty \\ y(t) - y^\infty \end{pmatrix} \right\|_{X\times Y} \le& \tilde{c}^2 a^t \|x(0) - x^\infty\|_X + \tilde{c}^2 a^{\lceil t/2\rceil} \max_{s\in \underline{\lfloor t/2\rfloor}} \left\| \begin{pmatrix} v(s) \\ w(s) \end{pmatrix} - \begin{pmatrix} v^\infty \\ w^\infty \end{pmatrix} \right\|_{V\times Y} \\ &+ \tilde{c} \max_{s\in \underline{\lceil t/2\rceil}} \left\| \begin{pmatrix} \Lambda_{\lfloor t/2\rfloor} v(s) \\ \Lambda_{\lfloor t/2\rfloor} w(s) \end{pmatrix} - \begin{pmatrix} v^\infty \\ w^\infty \end{pmatrix} \right\|_{V\times Y} . \end{aligned}$$

Therefore, by defining $c := \max\{\tilde{c}, \tilde{c}^2\}$, we obtain that (5.62) holds, whence completing the proof. □

Remark 5.3.9. By inspecting the above proof we see that in the situation where $\mathbf{G} = 0$ and (5.33) holds with $S_1 = Y$ and $S_2 = S$ for some $K \in \mathcal{L}(Y, U)$, then the conclusions of Theorem 5.3.7 remain valid (see Statement (ii) of Remark 5.2.7 and also Remark 5.3.5). ◊

The following corollary provides succinct hypotheses for when (5.10) exhibits the CI-CSO property.

Corollary 5.3.10. *Let $\Sigma \in \mathbb{L}$ be stabilisable and detectable, $f : Y \to U$, and $r > 0$ and $K \in \mathcal{L}(Y, U)$ be such that $\mathbb{B}(K, r) \subseteq \mathbb{S}(\mathbf{G})$. If either:*

(i) *there exists $\delta \in (0, r)$ such that* (5.14) *holds with $S_1 = S_2 = Y$, or;*

(ii) *$\mathbf{G} = 0$ and* (5.33) *holds with $S_1 = S_2 = Y$,*

then (5.10) *has the CICSO property.*

Proof. Let $v^\infty \in V$ and $w^\infty \in Y$. In either case, Lemma 5.3.4 implies that F_K is bijective. Therefore, $\mathcal{T}_K \neq \emptyset$, where $\mathcal{T}_K$ is given by (5.59). An application of Thereom 5.3.7 (see also Remark 5.3.9) then yields that $\lim_{t\to\infty} x(t) = x^\infty$ and $\lim_{t\to\infty} y(t) = y^\infty$ for all $(v, w, x, y) \in \mathcal{B}$ with $\lim_{t\to\infty} v(t) = v^\infty$ and $\lim_{t\to\infty} w(t) = w^\infty$, and where $y^\infty + w^\infty = z^\infty \in \mathcal{T}_K$ and x^∞ is given by (5.60). Thus, (5.10) has the CICSO property. □

As previously mentioned in Remark 5.3.2, if w in (5.10) is considered to be an output disturbance, then it may be unreasonable to expect convergence of w. The next result is an immediate corollary to Theorem 5.3.7 and yields that asymptotic 'closeness' of the state and output of (5.10) to the equilibrium components x^∞ and y^∞, respectively, is linearly bounded by $\|w\|_{\ell^\infty}$.

Corollary 5.3.11. *Under the assumptions of Theorem 5.3.7 with $w^\infty = 0$, for all $(v, w, x, y) \in \mathcal{B}$ with $\lim_{t\to\infty} v(t) = v^\infty$ and bounded w,*

$$\limsup_{t\to\infty} \Big(\|x(t) - x^\infty\|_X + \|y(t) - y^\infty\|_Y \Big) \leq c \limsup_{t\to\infty} \|w(t)\|_Y,$$

where $c > 0$ is as in (5.62).

Proof. The claim follows from (5.62), Lemma 5.1.30, and a standard time-invariance argument. □

To conclude this subsection, we briefly comment on the steady-state gain maps associated with (5.10). In the interest of avoiding repetition, we refer the reader to Section 2.3.3 for the motivation of these maps.

Definition 5.3.12. *Let $\Sigma \in \mathbb{L}$ and $f : Y \to U$. If* (5.10) *has the CICSO property, then the maps*

$$\Gamma_{\rm is} : V \times Y \to X, \quad (v^\infty, w^\infty) \mapsto x^\infty,$$

and

$$\Gamma_{\rm io} : V \times Y \to Y, \quad (v^\infty, w^\infty) \mapsto y^\infty,$$

are well-defined. As in the finite-dimensional setting, we call Γ_{is} the input-to-state steady-state (ISSS) gain *and Γ_{io} the* input-to-output steady-state (IOSS) gain.

Under the assumptions of Theorem 5.3.7 with $S = Y$, Lemma 5.3.4 gives that F_K^{-1} exists and is globally Lipschitz. When these assumptions are imposed, we shall denote by G_K the map

$$V \times Y \ni (\xi_1, \xi_2) \mapsto F_K^{-1} \begin{pmatrix} C^K (I - A^K)^{-1} \left(B_{\rm e} + B^K K D_{\rm e}\right) \xi_1 \\ \qquad + (I - DK)^{-1} D_{\rm e} \xi_1 + \left(I + \mathbf{G}^K(1) K\right) \xi_2 \end{pmatrix} \in Y$$

We now present the following result concerning the steady-state gain maps. For brevity, we shall not prove the result since it is straightforward to do so.

Corollary 5.3.13. *Under the assumptions of Theorem 5.3.7 with $S = Y$, for all $(\xi, \zeta) \in V \times Y$,*

$$\Gamma_{\rm is}(\xi_1, \xi_2) = (I - A^K)^{-1} (B^K (f - K) G_K(\xi_1, \xi_2) + B^K K \xi_2 + (B_{\rm e} + B^K K D_{\rm e}) \xi_1),$$

and

$$\Gamma_{\rm io}(\xi_1, \xi_2) = G_K(\xi_1, \xi_2) - \xi_2.$$

Furthermore, $\Gamma_{\rm is}$ and $\Gamma_{\rm io}$ are globally Lipschitz.

5.3.2 The response to asymptotically almost periodic inputs

We now apply Theorem 5.2.5 to obtain a result that guarantees that asymptotically almost periodic inputs generate asymptotically almost periodic state and output trajectories of (5.10). Almost periodic functions were used in Chapter 3 and a thorough presentation is given in Appendix C. Therefore, in what follows, we will not define almost and asymptotically almost periodic functions or give much relevant theory, and instead refer the reader to the aforementioned parts of this thesis. For background reading, we also refer the reader to literature such as [1, 14, 17, 21, 22, 52].

What we will recall here, however, is that $AP(Z,X)$ denotes the space of almost periodic functions mapping $Z \to X$, where $Z = \mathbb{Z}$ or $\mathbb{Z}_+$; $P(v^{\mathrm{ap}},\varepsilon)$ denotes the set of ε-periods of an almost periodic function v^{ap}, where $\varepsilon > 0$; the space of asymptotically almost periodic functions $\mathbb{Z}_+ \to X$ is denoted by $AAP(\mathbb{Z}_+,X)$; and $c_0(\mathbb{Z}_+,X)$ denotes the space of all functions $\mathbb{Z}_+ \to X$ that converge to zero. Moreover, we recall that every $v^{\mathrm{ap}} \in AP(\mathbb{Z}_+,X)$ has a unique extension $v^{\mathrm{ap}}_{\mathrm{e}} \in AP(\mathbb{Z},X)$ such that

$$v^{\mathrm{ap}}_{\mathrm{e}}(t) = v^{\mathrm{ap}}(t) \quad \forall\, t \in \mathbb{Z}_+ \quad \text{and} \quad \sup_{t\in\mathbb{Z}} \|v^{\mathrm{ap}}_{\mathrm{e}}(t)\|_X = \sup_{t\in\mathbb{Z}_+} \|v^{\mathrm{ap}}(t)\|_X.$$

Indeed, as done in Appendix C, by following an idea in [12, Remark on p.318], we define $v^{\mathrm{ap}}_{\mathrm{e}}$ by

$$v^{\mathrm{ap}}_{\mathrm{e}}(t) = \lim_{k\to\infty} v^{\mathrm{ap}}(t+\tau_k) \quad \forall\, t \in \mathbb{Z}, \tag{5.65}$$

where, for each $k \in \mathbb{N}$, $\tau_k \in P(v^{\mathrm{ap}},1/k)$ and $\tau_k \nearrow \infty$ as $k \to \infty$. In Appendix C, this function is shown to be well-defined and having all of the previously mentioned properties. As a consequence, we obtain the following, which is Theorem C.1.35 in the appendix.

Theorem 5.3.14. *The map $AP(\mathbb{R}_+,X) \to AP(\mathbb{R},X)$ which maps v^{ap} to $v^{\mathrm{ap}}_{\mathrm{e}}$, where $v^{\mathrm{ap}}_{\mathrm{e}}$ is given by* (5.65)*, is an isometric isomorphism.*

The following is a corollary of Theorem 5.2.5 and is the main result of this subsection.

Corollary 5.3.15. *Let $v^{\mathrm{ap}} \in AP(\mathbb{Z}_+,V)$ and $w^{\mathrm{ap}} \in AP(\mathbb{Z}_+,Y)$. If the assumptions of Theorem 5.2.5 hold with $S_1 = S_2 = Y$, then the following statements are true.*

(i) *There exists a unique pair $(x^{\mathrm{ap}},y^{\mathrm{ap}}) \in AP(\mathbb{Z}_+,X) \times AP(\mathbb{Z}_+,Y)$ such that $(v^{\mathrm{ap}}, w^{\mathrm{ap}},x^{\mathrm{ap}},y^{\mathrm{ap}}) \in \mathcal{B}$ and, for all $(v,w,x,y) \in \mathcal{B}$ such that $v - v^{\mathrm{ap}} \in c_0(\mathbb{Z}_+,V)$ and $w - w^{\mathrm{ap}} \in c_0(\mathbb{Z}_+,Y)$,*

$$\lim_{t\to\infty} \|x(t) - x^{\mathrm{ap}}(t)\|_X = 0 = \lim_{t\to\infty} \|y(t) - y^{\mathrm{ap}}(t)\|_Y. \tag{5.66}$$

Furthermore, for all $\varepsilon > 0$, there exists $\delta > 0$ such that $P(v^{\mathrm{ap}},\delta) \cap P(w^{\mathrm{ap}},\delta) \subseteq P(x^{\mathrm{ap}},\varepsilon) \cap P(y^{\mathrm{ap}},\varepsilon)$. In particular, if v^{ap} and w^{ap} are τ-periodic, then x^{ap} and y^{ap} are τ-periodic.

(ii) *The pair of almost periodic extensions $(x^{\mathrm{ap}}_{\mathrm{e}},y^{\mathrm{ap}}_{\mathrm{e}})$ of $(x^{\mathrm{ap}},y^{\mathrm{ap}})$ to $\mathbb{Z}$, is the unique bounded pair of functions which satisfy*

$$\left.\begin{aligned} z_1(t+1) &= Az_1(t) + Bf(z_2(t) + w^{\mathrm{ap}}_{\mathrm{e}}(t)) + B_{\mathrm{e}}v^{\mathrm{ap}}_{\mathrm{e}}(t), \\ z_2(t) &= Cz_1(t) + Df(z_2(t) + w^{\mathrm{ap}}_{\mathrm{e}}(t)) + D_{\mathrm{e}}v^{\mathrm{ap}}_{\mathrm{e}}(t), \end{aligned}\right\} \quad \forall\, t \in \mathbb{Z}. \tag{5.67}$$

Remark 5.3.16. Corollary 5.3.15 shows that, under the assumptions of Theorem 5.2.5 with $S_1 = S_2 = Y$, asymptotically almost periodic inputs generate asymptotically almost periodic state and output trajectories. $\Diamond$

Proof of Corollary 5.3.15. We shall begin with the proof of statement (i). An application of Lemma 5.3.4 gives that F_K is bijective and, consequently, an application of Lemma 5.3.6 yields that, for every $v^* \in V$ and $w^* \in Y$, there exist (unique) $x^* \in X$ and $y^* \in Y$ such that (v^*, w^*, x^*, y^*) is an equilibrium quadruple. We now fix such a quadruple (v^*, w^*, x^*, y^*). Moreover, we note that since (5.14) holds with $S_1 = S_2 = Y$, then $I - D^K(f - K)$ is bijective (see Remark 5.3.8) and so Proposition 5.1.33 yields that, for all $x^0 \in X$, all $v \in V^{\mathbb{Z}_+}$ and all $w \in Y^{\mathbb{Z}_+}$, the IVP (5.11) has a unique solution. With this in mind, we let $(v^{\mathrm{ap}}, w^{\mathrm{ap}}, x, y) \in \mathcal{B}$. We also note that, since v^{ap} and w^{ap} are almost periodic, v^{ap} and w^{ap} are bounded. Now, since the hypotheses of Theorem 5.2.5 hold, we see that the conclusions of statement (ii) of that result hold with $q = \infty$. Hence, it follows that there exist $c, d > 0$ and $\theta \in (0, 1)$ such that (5.17) and (5.18) hold for all (v_1, w_1, x_1, y_1) and $(v_2, w_2, x_2, y_2) \in \mathcal{B}$. In particular, we may use (5.17) and (5.18) with the trajectories (v^*, w^*, x^*, y^*) and $(v^{\mathrm{ap}}, w^{\mathrm{ap}}, x, y)$. By now utilising the boundedness of (v^*, w^*, x^*, y^*) and v^{ap} and w^{ap}, it is clear that there exists $\mu > 0$ such that

$$\|x\|_{\ell^\infty} + \|y\|_{\ell^\infty} \leq \mu, \tag{5.68}$$

whence showing that x and y are bounded. Moreover, since v^{ap} and w^{ap} are almost periodic, by Lemma C.1.37, we may obtain a sequence $(\tau_k)_{k\in\mathbb{N}} \subseteq \mathbb{Z}_+$ such that

$$\tau_k \in P\left(v^{\mathrm{ap}}, \frac{1}{k}\right) \cap P\left(w^{\mathrm{ap}}, \frac{1}{k}\right) \quad \forall\, k \in \mathbb{N} \quad \text{and} \quad \tau_k \to \infty \text{ as } k \to \infty. \tag{5.69}$$

Inspired by an argument from the proof of [5, Proposition 4.4], we claim that $(\Lambda_{\tau_k} x)_{k\in\mathbb{N}}$ and $(\Lambda_{\tau_k} y)_{k\in\mathbb{N}}$ are Cauchy sequences in $\ell^\infty(\mathbb{Z}_+, X)$ and $\ell^\infty(\mathbb{Z}_+, Y)$, respectively. To see this, we let $\varepsilon > 0$ and let $k, l \in \mathbb{N}$ be sufficiently large so that

$$2\mu\theta^{\tau_k}, 2\mu\theta^{\tau_l} \leq \frac{\varepsilon}{2(c+d)} \quad \text{and} \quad 2\left(\frac{1}{k} + \frac{1}{l}\right) \leq \frac{\varepsilon}{2(c+d)},$$

and, without loss of generality, assume that $\tau_l \geq \tau_k$. Then, for all $t \in \mathbb{Z}_+$,

$$\begin{aligned}
\sup_{s\in\underline{\tau_k}} &\left\| \begin{pmatrix} v^{\mathrm{ap}}(s+t) - v^{\mathrm{ap}}(s+t+\tau_l-\tau_k) \\ w^{\mathrm{ap}}(s+t) - w^{\mathrm{ap}}(s+t+\tau_l-\tau_k) \end{pmatrix} \right\|_{V\times Y} \\
&\leq \sup_{s\in\underline{\tau_k}} \left\| \begin{pmatrix} v^{\mathrm{ap}}(s+t) - v^{\mathrm{ap}}(s+t+\tau_l) \\ w^{\mathrm{ap}}(s+t) - w^{\mathrm{ap}}(s+t+\tau_l) \end{pmatrix} \right\|_{V\times Y} \\
&\quad + \sup_{s\in\underline{\tau_k}} \left\| \begin{pmatrix} v^{\mathrm{ap}}(s+t+\tau_l) - v^{\mathrm{ap}}(s+t+\tau_l-\tau_k) \\ w^{\mathrm{ap}}(s+t+\tau_l) - w^{\mathrm{ap}}(s+t+\tau_l-\tau_k) \end{pmatrix} \right\|_{V\times Y} \\
&\leq 2\left(\frac{1}{l} + \frac{1}{k}\right)
\end{aligned}$$

Hence, by combining this with Lemma 5.1.30 and (5.17), (5.18) and (5.68), we obtain that, for all $t \in \mathbb{Z}_+$,

$$\begin{aligned}\left\| \begin{pmatrix} (\Lambda_{\tau_k} x)(t) - (\Lambda_{\tau_l} x)(t) \\ (\Lambda_{\tau_k} y)(t) - (\Lambda_{\tau_l} y)(t) \end{pmatrix} \right\|_{X \times Y} &= \left\| \begin{pmatrix} (\Lambda_t x)(\tau_k) - (\Lambda_{t+\tau_l-\tau_k} x)(\tau_k) \\ (\Lambda_t y)(\tau_k) - (\Lambda_{t+\tau_l-\tau_k} y)(\tau_k) \end{pmatrix} \right\|_{X \times Y} \\ &\leq (c+d) \left(\theta^{\tau_k} \| x(t) - x(t+\tau_l - \tau_k) \|_X + 2 \left(\frac{1}{l} + \frac{1}{k} \right) \right) \\ &\leq (c+d) \left(\theta^{\tau_k} 2\mu + 2 \left(\frac{1}{l} + \frac{1}{k} \right) \right) \\ &\leq \varepsilon. \end{aligned}$$

Whence, we have shown that $(\Lambda_{\tau_k} x)_{k\in\mathbb{N}}$ and $(\Lambda_{\tau_k} y)_{k\in\mathbb{N}}$ are Cauchy sequences in $\ell^\infty(\mathbb{Z}_+, X)$ and $\ell^\infty(\mathbb{Z}_+, Y)$, respectively. They therefore converge to functions $x^{\mathrm{ap}} \in \ell^\infty(\mathbb{Z}_+, X)$ and $y^{\mathrm{ap}} \in \ell^\infty(\mathbb{Z}_+, Y)$, respectively.
In order to show that $(v^{\mathrm{ap}}, w^{\mathrm{ap}}, x^{\mathrm{ap}}, y^{\mathrm{ap}}) \in \mathcal{B}$, we note that, by once again applying Lemma 5.1.30, for all $k \in \mathbb{Z}_+$ and $t \in \mathbb{Z}_+$,

$$\begin{aligned}(\Lambda_{\tau_k} x)(t+1) &= A(\Lambda_{\tau_k} x)(t) + Bf((\Lambda_{\tau_k} y)(t) + (\Lambda_{\tau_k} w^{\mathrm{ap}})(t)) + B_{\mathrm{e}}(\Lambda_{\tau_k} v^{\mathrm{ap}})(t), \\ (\Lambda_{\tau_k} y)(t) &= C(\Lambda_{\tau_k} x)(t) + Df((\Lambda_{\tau_k} y)(t) + (\Lambda_{\tau_k} w^{\mathrm{ap}})(t)) + D_{\mathrm{e}}(\Lambda_{\tau_k} v^{\mathrm{ap}})(t).\end{aligned}$$

Since f is continuous, we recall (5.69) and take the limit in the above equality as $k \to \infty$, to obtain that $(v^{\mathrm{ap}}, w^{\mathrm{ap}}, x^{\mathrm{ap}}, y^{\mathrm{ap}}) \in \mathcal{B}$.
Our attention now turns to showing that $x^{\mathrm{ap}} \in AP(\mathbb{Z}_+, X)$ and $y^{\mathrm{ap}} \in AP(\mathbb{Z}_+, Y)$. To this end, we fix $\varepsilon > 0$ and let $\delta := \varepsilon/(2(c+d))$ and let $\tau \in P(v^{\mathrm{ap}}, \delta) \cap P(w^{\mathrm{ap}}, \delta)$, which exists by Lemma C.1.37. Then, by combining Lemma 5.1.30 with (5.17) and (5.18), we see that, for all $t \in \mathbb{Z}_+$ and all $k \in \mathbb{N}$,

$$\begin{aligned}&\left\| \begin{pmatrix} (\Lambda_{\tau_k} x)(t) - (\Lambda_{\tau_k} x)(t+\tau) \\ (\Lambda_{\tau_k} y)(t) - (\Lambda_{\tau_k} y)(t+\tau) \end{pmatrix} \right\|_{X \times Y} = \left\| \begin{pmatrix} (\Lambda_t x)(\tau_k) - (\Lambda_{t+\tau} x)(\tau_k) \\ (\Lambda_t y)(\tau_k) - (\Lambda_{t+\tau} y)(\tau_k) \end{pmatrix} \right\|_{X \times Y} \\ &\leq (c+d) \left(\theta^{\tau_k} \| x(t) - x(t+\tau) \|_X + \sup_{s \in \underline{\tau_k}} \left\| \begin{pmatrix} v^{\mathrm{ap}}(s+t) - v^{\mathrm{ap}}(s+t+\tau) \\ w^{\mathrm{ap}}(s+t) - w^{\mathrm{ap}}(s+t+\tau) \end{pmatrix} \right\|_{V \times Y} \right) \\ &\leq (c+d) \left(\theta^{\tau_k} \| x(t) - x(t+\tau) \|_X + 2\delta \right). \end{aligned} \tag{5.70}$$

Since $(\tau_k)_{k\in\mathbb{N}}$ converges to ∞ as $k \to \infty$, (5.70) hence yields

$$\left\| \begin{pmatrix} x^{\mathrm{ap}}(t) - x^{\mathrm{ap}}(t+\tau) \\ y^{\mathrm{ap}}(t) - y^{\mathrm{ap}}(t+\tau) \end{pmatrix} \right\|_{X \times Y} \leq \varepsilon \quad \forall \, t \in \mathbb{Z}_+,$$

showing that $\tau \in P(x^{\mathrm{ap}}, \varepsilon) \cap P(y^{\mathrm{ap}}, \varepsilon)$. It thus follows that $P(v^{\mathrm{ap}}, \delta) \cap P(w^{\mathrm{ap}}, \delta) \subseteq P(x^{\mathrm{ap}}, \varepsilon) \cap P(y^{\mathrm{ap}}, \varepsilon)$, and, since $P(v^{\mathrm{ap}}, \delta) \cap P(w^{\mathrm{ap}}, \delta)$ is relatively dense in $\mathbb{Z}_+$ (see Defintion C.1.29) by Lemma C.1.37, we obtain that $P(x^{\mathrm{ap}}, \varepsilon) \cap P(y^{\mathrm{ap}}, \varepsilon)$ is also relatively dense in $\mathbb{Z}_+$. We have therefore shown that $x^{\mathrm{ap}} \in AP(\mathbb{Z}_+, X)$ and $y^{\mathrm{ap}} \in AP(\mathbb{Z}_+, Y)$.
To prove that (5.66) holds, let $(v, w, x, y) \in \mathcal{B}$ be such that $v - v^{\mathrm{ap}} \in c_0(\mathbb{Z}_+, V)$ and $w - w^{\mathrm{ap}} \in c_0(\mathbb{Z}_+, Y)$, and define $\tilde{v} := v - v^{\mathrm{ap}}$ and $\tilde{w} := w - w^{\mathrm{ap}}$. By utilising more applications of Lemma 5.1.30, (5.17) and (5.18), we yield, for all $t \in \mathbb{Z}_+$,

$$\begin{aligned}\left\| \begin{pmatrix} x(t) - x^{\mathrm{ap}}(t) \\ y(t) - y^{\mathrm{ap}}(t) \end{pmatrix} \right\|_{X \times Y} &= \left\| \begin{pmatrix} (\Lambda_{\lfloor t/2 \rfloor} x)(\lceil t/2 \rceil) - (\Lambda_{\lfloor t/2 \rfloor} x^{\mathrm{ap}})(\lceil t/2 \rceil) \\ (\Lambda_{\lfloor t/2 \rfloor} y)(\lceil t/2 \rceil) - (\Lambda_{\lfloor t/2 \rfloor} y^{\mathrm{ap}})(\lceil t/2 \rceil) \end{pmatrix} \right\|_{X \times Y} \\ &\leq (c+d) \left(\theta^{\lceil t/2 \rceil} \| x(\lfloor t/2 \rfloor) - x^{\mathrm{ap}}(\lfloor t/2 \rfloor) \|_X + \left\| \pi_{\lceil t/2 \rceil} \begin{pmatrix} (\Lambda_{\lfloor t/2 \rfloor} \tilde{v}) \\ (\Lambda_{\lfloor t/2 \rfloor} \tilde{w}) \end{pmatrix} \right\|_{\ell^\infty} \right). \end{aligned}$$

By estimating the term $\|x(\lfloor t/2 \rfloor) - x^{\mathrm{ap}}(\lfloor t/2 \rfloor)\|$ in the above inequality via (5.17), we deduce that, for all $t \in \mathbb{Z}_+$,

$$\left\| \begin{pmatrix} x(t) - x^{\mathrm{ap}}(t) \\ y(t) - y^{\mathrm{ap}}(t) \end{pmatrix} \right\|_{X \times Y} \leq (c+d)\theta^{\lceil t/2 \rceil} c \left(\theta^{\lfloor t/2 \rfloor} \|x(0) - x^{\mathrm{ap}}(0)\|_X + \left\| \pi_{\lfloor t/2 \rfloor} \begin{pmatrix} \tilde{v} \\ \tilde{w} \end{pmatrix} \right\|_{\ell^\infty} \right) + (c+d) \left\| \pi_{\lceil t/2 \rceil} \begin{pmatrix} (\Lambda_{\lfloor t/2 \rfloor} \tilde{v}) \\ (\Lambda_{\lfloor t/2 \rfloor} \tilde{w}) \end{pmatrix} \right\|_{\ell^\infty}.$$

Finally, the right hand side of the above inequality converges to 0 as $t \to \infty$, showing that (5.66) holds. To conclude the proof of statement (i), we combine (5.66) with Lemma C.1.39 to yield that $(x^{\mathrm{ap}}, y^{\mathrm{ap}})$ is the unique almost periodic pair such that $(v^{\mathrm{ap}}, w^{\mathrm{ap}}, x^{\mathrm{ap}}, y^{\mathrm{ap}}) \in \mathcal{B}$ and, for all $(v, w, x, y) \in \mathcal{B}$ such that $v - v^{\mathrm{ap}} \in c_0(\mathbb{Z}_+, V)$ and $w - w^{\mathrm{ap}} \in c_0(\mathbb{Z}_+, Y)$, (5.66) holds.

We shall now prove statement (ii). We begin by noting that the almost periodic extensions $x_{\mathrm{e}}^{\mathrm{ap}}$ and $y_{\mathrm{e}}^{\mathrm{ap}}$ of x^{ap} and y^{ap} to $\mathbb{Z}$, respectively, are bounded. To see that $(x_{\mathrm{e}}^{\mathrm{ap}}, y_{\mathrm{e}}^{\mathrm{ap}})$ satisfies (5.67), we highlight that since $\Lambda_1 x_{\mathrm{e}}^{\mathrm{ap}}$ is the almost periodic extension of $\Lambda_1 x^{\mathrm{ap}}$, we deduce that $\Lambda_1 x_{\mathrm{e}}^{\mathrm{ap}}$ is also the almost periodic extension of $Ax^{\mathrm{ap}} + Bf(y^{\mathrm{ap}} + w^{\mathrm{ap}}) + B_{\mathrm{e}} v^{\mathrm{ap}}$. Moreover, since f is (globally) Lipschitz, $f(y_{\mathrm{e}}^{\mathrm{ap}} + w_{\mathrm{e}}^{\mathrm{ap}})$ is almost periodic and, consequently, $Ax_{\mathrm{e}}^{\mathrm{ap}} + Bf(y_{\mathrm{e}}^{\mathrm{ap}} + w_{\mathrm{e}}^{\mathrm{ap}}) + B_{\mathrm{e}} v_{\mathrm{e}}^{\mathrm{ap}}$ is also an almost periodic extension of $Ax^{\mathrm{ap}} + Bf(y^{\mathrm{ap}} + w^{\mathrm{ap}}) + B_{\mathrm{e}} v^{\mathrm{ap}}$. Therefore, Theorem 5.3.14 gives that

$$\Lambda_1 x_{\mathrm{e}}^{\mathrm{ap}} = Ax_{\mathrm{e}}^{\mathrm{ap}} + Bf(y_{\mathrm{e}}^{\mathrm{ap}} + w_{\mathrm{e}}^{\mathrm{ap}}) + B_{\mathrm{e}} v_{\mathrm{e}}^{\mathrm{ap}}.$$

Similarly, it is straight forward to show that

$$y_{\mathrm{e}}^{\mathrm{ap}} = Cx_{\mathrm{e}}^{\mathrm{ap}} + Df(y_{\mathrm{e}}^{\mathrm{ap}} + w_{\mathrm{e}}^{\mathrm{ap}}) + D_{\mathrm{e}} v_{\mathrm{e}}^{\mathrm{ap}},$$

which thus yields that $(x_{\mathrm{e}}^{\mathrm{ap}}, y_{\mathrm{e}}^{\mathrm{ap}})$ satisfies (5.67).

To prove that $(x_{\mathrm{e}}^{\mathrm{ap}}, y_{\mathrm{e}}^{\mathrm{ap}})$ is the unique pair which satisfies (5.67) on $\mathbb{Z}$, let $(\tilde{x}, \tilde{y}) \in X^{\mathbb{Z}} \times Y^{\mathbb{Z}}$ be another bounded pair satisfying (5.67). Furthermore, let $M > 0$ be such that

$$\|x^{\mathrm{ap}}\|_{\ell^\infty} + \|\tilde{x}\|_{\ell^\infty} \leq M.$$

We fix $\varepsilon > 0$ and $t \in \mathbb{Z}$, and we also choose $\tau \in \mathbb{Z}$ such that $\tau \leq t$ and

$$\theta^{t-\tau} \leq \frac{\varepsilon}{2M(c+d)}.$$

Now, since, from Lemma 5.1.30, the restrictions of $(\Lambda_\tau v_{\mathrm{e}}^{\mathrm{ap}}, \Lambda_\tau w_{\mathrm{e}}^{\mathrm{ap}}, \Lambda_\tau x_{\mathrm{e}}^{\mathrm{ap}}, \Lambda_\tau y_{\mathrm{e}}^{\mathrm{ap}})$ and $(\Lambda_\tau v_{\mathrm{e}}^{\mathrm{ap}}, \Lambda_\tau w_{\mathrm{e}}^{\mathrm{ap}}, \Lambda_\tau \tilde{x}, \Lambda_\tau \tilde{y})$ to $\mathbb{Z}_+$ are in $\mathcal{B}$, (5.17) and (5.18) then yield that

$$\begin{aligned} \left\| \begin{pmatrix} x_{\mathrm{e}}^{\mathrm{ap}}(t) - \tilde{x}(t) \\ y_{\mathrm{e}}^{\mathrm{ap}}(t) - \tilde{y}(t) \end{pmatrix} \right\|_{X \times Y} &= \left\| \begin{pmatrix} (\Lambda_\tau x_{\mathrm{e}}^{\mathrm{ap}})(t-\tau) + (\Lambda_\tau \tilde{x})(t-\tau) \\ (\Lambda_\tau y_{\mathrm{e}}^{\mathrm{ap}})(t-\tau) + (\Lambda_\tau \tilde{y})(t-\tau) \end{pmatrix} \right\|_{X \times Y} \\ &\leq (c+d)\theta^{t-\tau} \|x^{\mathrm{ap}}(\tau) - \tilde{x}(\tau)\|_X \\ &\leq (c+d)\theta^{t-\tau} 2M \\ &\leq \varepsilon. \end{aligned}$$

Therefore, since ε was arbitrary, we obtain that $x_{\mathrm{e}}^{\mathrm{ap}}(t) = \tilde{x}(t)$ and $y_{\mathrm{e}}^{\mathrm{ap}}(t) = \tilde{y}(t)$ and, since t was also arbitrary, it follows that $x_{\mathrm{e}}^{\mathrm{ap}} = \tilde{x}$ and $y_{\mathrm{e}}^{\mathrm{ap}} = \tilde{y}$, completing the proof. $\square$

5.4 Application to the four-block problem

In the following, we demonstrate how the results of earlier sections apply to the related class of so-called 'four-block' Lur'e systems which are informally described by the block diagram arrangement in Figure 5.2, where $\Sigma = (A, B, B_e, C, D, D_e) \in \mathbb{L}$ and the signal y is given by

$$y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

We highlight that the output space is split in two, and only one part is utilised for feedback purposes. The four-block system we shall investigate here, is an infinite-dimensional version of the finite-dimensional four-block forced discrete-time Lur'e system seen in Section 2.4 (see (2.76)). Similar to that situation, the key to developing analogous results of the prior sections for the four-block Lur'e system, will be to rewrite the system in terms of one that is of the form (5.10).

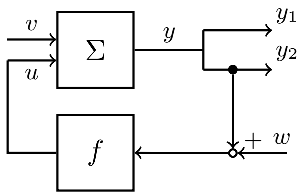

Figure 5.2: *Block diagram of a four-block forced Lur'e system: the feedback interconnection of a linear system specified by Σ and the static nonlinearity f, with output disturbance w.*

We shall now lay foundation so that we may explicitly describe the four-block Lur'e system of interest to us. Moreover, we shall also give some preliminary results regarding the said system. We comment that we shall be brief in doing this, since the finite-dimensional versions of these can be found in Section 2.4. Indeed, we shall only prove results, and go into particular detail, when the infinite-dimensional setting causes us to do so.

We begin by assuming throughout this section, that the output space Y is of the form

$$Y = Y_1 \times Y_2,$$

where Y_1 and Y_2 are complex Hilbert spaces.

Definition 5.4.1. *For $i = 1, 2$, we define*

$$P_i : Y \to Y_i, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \mapsto y_i.$$

With the previous in mind, we shall now explicitly give the four-block forced discrete-time Lur'e system in the infinite-dimensional setting, informally described by Figure 5.2:

$$\left.\begin{aligned} x^+ &= Ax + Bu + B_e v\,, \\ y &= Cx + Du + D_e v\,, \\ u &= f(P_i y + w)\,, \end{aligned}\right\} \tag{5.71}$$

where $\Sigma \in \mathbb{L}$, $v \in V^{\mathbb{Z}_+}$, $w \in (Y_i)^{\mathbb{Z}_+}$, $f : Y_i \to U$ and $i = 1, 2$.

By multiplying the output equation of (5.71) by P_i, we see that the decomposition of the output space $Y = Y_1 \times Y_2$ induces two linear systems, viz.

$$(A, B, B_{\mathrm{e}}, P_iC, P_iD, P_iD_{\mathrm{e}}), \quad i = 1, 2. \tag{5.72}$$

For the rest of this section, we consider $i \in \{1, 2\}$ to be a fixed quantity.

We now make the following definitions associated with (5.71).

Definition 5.4.2. *Let $\Sigma \in \mathbb{L}$ and $f : Y_i \to U$.*

(i) *We denote by Σ_i the components of the subsystem of* (5.71) *(generated by applying P_i to the output equation), that is,*

$$\Sigma_i := (A, B, B_{\mathrm{e}}, P_iC, P_iD, P_iD_{\mathrm{e}}).$$

(ii) *We denote the behaviour of* (5.71) *by $\tilde{\mathcal{B}}$, that is,*

$$\tilde{\mathcal{B}} := \left\{(v, w, x, y) \in V^{\mathbb{Z}_+} \times (Y_i)^{\mathbb{Z}_+} \times X^{\mathbb{Z}_+} \times Y^{\mathbb{Z}_+} : (v, w, x, y) \text{ satisfies } (5.71)\right\}.$$

(iii) *For $S \subseteq Y_i$, we set*

$$\tilde{\mathcal{B}}^S := \{(v, w, x, y) \in \tilde{\mathcal{B}} \, : \, P_iy(t) + w(t) \in S, \; \forall\, t \in \mathbb{Z}_+\}.$$

Remark 5.4.3. It was explained in Remark 2.4.5 how the finite-dimensional four-block forced discrete-time Lur'e system can be written in the form of the finite-dimensional version of (5.10). Identically, (in the infinite-dimensional setting) (5.71) can be expressed in the form of (5.10). For brevity, we do not do this here, and instead refer the reader to Remark 2.4.5. ♢

The following two propositions provide relations between the overall system (5.71) to the subsystem (5.72) once feedback has been applied. We shall not provide a proof of these results since the proofs of Proposition 2.4.6 and Proposition 2.4.7 can be applied to each of the following, respectively.

Proposition 5.4.4. *Let $\Sigma \in \mathbb{L}$, $K_i \in \mathcal{L}(Y_i, U)$ and $K := K_iP_i$. Then $K_i \in \mathbb{A}(P_iD)$ if, and only if, $K \in \mathbb{A}(D)$, and in which case*

$$A^{K_i} = A^K, \quad B^{K_i} = B^K, \quad (P_iC)^{K_i} = P_iC^K, \quad (P_iD)^{K_i} = P_iD^K$$

and

$$(P_i\mathbf{G})^{K_i} = P_i\mathbf{G}^K.$$

Proposition 5.4.5. *Let $\Sigma \in \mathbb{L}$, $K_i \in \mathcal{L}(Y_i, U)$ and $K := K_iP_i$. The following statements hold.*

(i) *Σ is stabilisable if Σ_i is stabilisable.*

(ii) *Σ is detectable if Σ_i is detectable.*

(iii) *If Σ_i is stabilisable and detectable, then $K_i \in \mathbb{S}(P_i\mathbf{G})$ if, and only if, $K \in \mathbb{S}(\mathbf{G})$.*

One may also associate to (5.71), an initial-value problem. In the interest of not repeating ourselves, we do not do this, and instead refer the reader to Section 2.4 where this is done in the finite-dimensional setting (see Definition 2.4.8, Proposition 2.4.9 and Corollary 2.4.10).

Our attention now turns towards proving that the stability and convergence properties of earlier sections apply analogously to (5.71), provided that the linear system Σ_i and nonlinearity f satisfy the relevant assumptions. To this end, the following is an analogous result to Theorem 5.2.5.

Corollary 5.4.6. *Let $\Sigma \in \mathbb{L}$, $f : Y_i \to U$, and let $S_1, S_2 \subseteq Y_i$ be non-empty. Assume that Σ_i is stabilisable and detectable, $r > 0$ and $K_i \in \mathcal{L}(Y_i, U)$ satisfy $\mathbb{B}(K_i, r) \subseteq \mathbb{S}(P_i\mathbf{G})$, and that there exists $\delta \in (0, r)$ such that*

$$\|f(\xi) - f(\zeta) - K_i(\xi - \zeta)\|_U \leq (r - \delta)\|\xi - \zeta\|_{Y_i} \quad \forall\, \xi \in S_1,\ \forall\, \zeta \in S_2\,. \tag{5.73}$$

Then the conclusions of Theorem 5.2.5 hold for the Lur'e system (5.71)*.*

Proof. In the following, we shall only prove that statement (i) of Theorem 5.2.5 holds for (5.71), since the proof of statement (ii) is similar. Let us consider the Lur'e system

$$\left.\begin{aligned} x^+ &= Ax + Bf(P_iy + w) + B_\mathrm{e}v\,, \\ P_iy &= P_iCx + P_iDf(P_iy + w) + P_iD_\mathrm{e}v\,, \end{aligned}\right\} \tag{5.74}$$

which is obtained from (5.71) by applying P_i to the output equation. Note that the Lur'e system (5.74) is of the form (5.10) with Y, C, D, D_e and y replaced by Y_i, P_iC, P_iD, P_iD_e and P_iy, respectively.
By hypothesis, the conclusions of Theorem 5.2.5 apply to (5.74) and so there exist constants $a > 0$, $b > 0$ and $\omega > 1$ such that, for all $(v_1, w_1, x_1, y_1) \in \tilde{\mathcal{B}}^{S_1}$, $(v_2, w_2, x_2, y_2) \in \tilde{\mathcal{B}}^{S_2}$, and all $\rho \in [1, \omega]$, we have

$$\|\pi_t(x_1 - x_2)\|_{\ell^2_\rho} \leq a\left(\|x_1(0) - x_2(0)\|_X + \left\|\pi_{t-1}\begin{pmatrix} v_1 - v_2 \\ w_1 - w_2 \end{pmatrix}\right\|_{\ell^2_\rho}\right) \quad \forall\, t \in \mathbb{N}\,, \tag{5.75}$$

and

$$\|\pi_t(P_iy_1 - P_iy_2)\|_{\ell^2_\rho} \leq b\left(\|x_1(0) - x_2(0)\|_X + \left\|\pi_t\begin{pmatrix} v_1 - v_2 \\ w_1 - w_2 \end{pmatrix}\right\|_{\ell^2_\rho}\right) \quad \forall\, t \in \mathbb{Z}_+\,. \tag{5.76}$$

It remains to establish that an estimate of the form (5.76) holds for the difference $y_1 - y_2$. For which purpose, fix $(v_1, w_1, x_1, y_1) \in \tilde{\mathcal{B}}^{S_1}$ and $(v_2, w_2, x_2, y_2) \in \tilde{\mathcal{B}}^{S_2}$. Since $K_i \in \mathbb{A}(P_iD)$, Lemma 5.4.4 gives that $K := K_iP_i \in \mathbb{A}(D)$. Hence, by using Lemma 5.1.17 and (5.71), we have that the difference $y_1 - y_2$ satisfies

$$\left.\begin{aligned} y_1 - y_2 = &C^K(x_1 - x_2) \\ &+ D^K\left(f(P_iy_1 + w_1) - f(P_iy_2 + w_2) - K_i(P_iy_1 - P_iy_2)\right) \\ &+ (I - DK)^{-1}D_\mathrm{e}(v_1 - v_2)\,. \end{aligned}\right\} \tag{5.77}$$

Estimating (5.77) by invoking (5.73), (5.75) and (5.76) gives

$$\begin{aligned}\|\pi_t(y_1-y_2)\|_{\ell^2_\rho} &\le \|C^K\|\|\pi_t(x_1-x_2)\|_{\ell^2_\rho} + (r-\delta)\|D^K\|\|\pi_t(P_iy_1-P_iy_2+w_1-w_2)\|_{\ell^2_\rho} \\ &\quad + \|(I-DK)^{-1}D_{\mathrm{e}}\|\|\pi_t(v_1-v_2)\|_{\ell^2_\rho} + \|D^KK_i\|\|\pi_t(w_1-w_2)\|_{\ell^2_\rho} \\ &\le a\|C^K\|\left(\|x_1(0)-x_2(0)\|_X + \left\|\pi_t\begin{pmatrix}v_1-v_2\\ w_1-w_2\end{pmatrix}\right\|_{\ell^2_\rho}\right) \\ &\quad + b(r-\delta)\|D^K\|\left(\|x_1(0)-x_2(0)\|_X + \left\|\pi_t\begin{pmatrix}v_1-v_2\\ w_1-w_2\end{pmatrix}\right\|_{\ell^2_\rho}\right) \\ &\quad + \zeta\left\|\pi_t\begin{pmatrix}v_1-v_2\\ w_1-w_2\end{pmatrix}\right\|_{\ell^2_\rho},\end{aligned}$$

for some $\zeta > 0$ independent of (v_1, w_1, x_1, y_1) and (v_2, w_2, x_2, y_2). Thus, by setting $\tilde{b} := a\|C^K\| + b(r-\delta)\|D^K\| + \zeta$, it follows that

$$\|\pi_t(y_1-y_2)\|_{\ell^2_\rho} \le \tilde{b}\Big(\|x_1(0)-x_2(0)\|_X + \left\|\pi_t\begin{pmatrix}v_1-v_2\\ w_1-w_2\end{pmatrix}\right\|_{\ell^2_\rho}\Big) \quad \forall\, t \in \mathbb{Z}_+\,,$$

completing the proof. □

We close the current section with the following important remark.

Remark 5.4.7. We make clear that the various other results presented in Sections 5.2 and 5.3 for the forced Lur'e system (5.10), also have obvious extensions to the four-block setting considered here, namely system (5.71). For brevity and to avoid repetition, we do not give formal statements of these results, and instead leave the details to the reader. ◊

5.5 Application to sampled-data systems

In this section, we provide an application of Theorem 5.2.5 (or more precisely Corollary 5.2.8) in the form of an ISS result for a class of forced, infinite-dimensional sampled-data control systems (see, for example, [65, 73, 75]). Briefly, in sampled-data control, a continuous-time system is controlled by a discrete-time controller, via the use of sample and hold operations. This discrete-time controller may be thought of, for example, as a processor of a digital computer.

Before coming to the aforementioned result, we need to explicitly give the sampled-data system that we are interested in. To this end, we begin this section by making a series of relevant definitions and assumptions.

Definition 5.5.1. (i) *A family* $(\mathbf{T}(t))_{t\ge 0}$ *of bounded linear operators mapping* $X \to X$ *is a* strongly continuous semigroup *on* X *if* $\mathbf{T}(t+s) = \mathbf{T}(t)\mathbf{T}(s)$ *for all* $t, s \ge 0$, $\mathbf{T}(0) = I$, *and*

$$\lim_{t\to 0^+} \|\mathbf{T}(t)x - x\|_X = 0 \quad \forall\, x \in X.$$

(ii) *The* generator A *of a strongly continous semigroup* $(\mathbf{T}(t))_{t\geq 0}$ *is defined by*

$$Ax = \lim_{t\to 0^+} \frac{1}{t}(\mathbf{T}(t) - I)x,$$

for all $x \in X$ *for which the limit exists. The domain of* A *is the set in* X *for which the limit exists.*

(iii) *We denote the* exponential growth constant *of a strongly continuous semigroup* $(\mathbf{T}(t))_{t\geq 0}$ *by* $\omega(\mathbf{T})$*, that is,*

$$\omega(\mathbf{T}) := \lim_{t\to\infty} \frac{1}{t} \ln \|\mathbf{T}(t)\|,$$

which is finite by, for example, [26, Theorem 2.1.6].

(iv) *A strongly continuous semigroup* $(\mathbf{T}(t))_{t\geq 0}$ *is* exponentially stable *if there exist* $M \geq 1$ *and* $\omega < 0$ *such that*

$$\|\mathbf{T}(t)\| \leq Me^{\omega t} \quad \forall\, t \geq 0.$$

From [26, Theorem 2.1.6], exponential stability is equivalent to the condition $\omega(\mathbf{T}) < 0$ *holding.*

For brevity, we do not provide a complete theory regarding strongly continuous semigroups and their generators. We instead refer the reader to sources such as [26, 86, 118]. What we will mention here, however, is the following which is a combination of [86, Theorem 2.3, p.6] with [86, Theorem 1.1, p.76].

Lemma 5.5.2. (i) *Let* $(\mathbf{T}(t))_{t\geq 0}$ *and* $(\mathbf{S}(t))_{t\geq 0}$ *be strongly continuous semigroups with generators* A *and* B*, respectively. If* $A = B$*, then* $\mathbf{T}(t) = \mathbf{S}(t)$ *for all* $t \geq 0$*.*

(ii) *Let* $(\mathbf{T}(t))_{t\geq 0}$ *be a strongly continuous semigroup with generator* A *satisfying* $\|\mathbf{T}(t)\| \leq Me^{\omega t}$ *for all* $t \geq 0$*, for some* $M \geq 1$ *and* $\omega \in \mathbb{R}$*. If* $B \in \mathcal{L}(X)$*, then* $A + B$ *is the generator of a strongly continuous semigroup* $(\mathbf{S}(t))_{t\geq 0}$ *such that*

$$\|\mathbf{S}(t)\| \leq Me^{(\omega + M\|B\|)t} \quad \forall\, t \geq 0.$$

We now fix A to be the generator of a strongly continuous semigroup on X, denoted by $(\mathbf{T}(t))_{t\geq 0}$, and fix $B \in \mathcal{L}(U, X)$ and $C \in \mathcal{L}(X, Y)$. We consider the following continuous-time, infinite-dimensional linear system

$$\left.\begin{aligned} \dot{x} &= Ax + Bu + v, \quad x(0) = x^0 \in X, \\ y &= Cx\,. \end{aligned}\right\} \tag{5.78}$$

As usual, x and y in (5.78) are respectively called the state and output, and u and v are inputs, with the former being available for feedback purposes.

Throughout this section, we assume that

- X, U and Y are Hilbert spaces, with U and Y finite-dimensional;
- the pair (A, B) is *(exponentially) stabilisable*, that is, there exists $F \in \mathcal{L}(X, U)$ such that the strongly continuous semigroup generated by $A+BF$ is exponentially stable;

- the pair (C, A) is *(exponentially) detectable*, that is, there exists $H \in \mathcal{L}(Y, X)$ such that the strongly continuous semigroup generated by $A + HC$ is exponentially stable.

We make the following definitions associated with the linear system (5.78).

Definition 5.5.3. (i) *We denote by* $\mathbf{H}$ *the* transfer function *of* (5.78)*, that is,* $\mathbf{H}(s) = C(sI - A)^{-1}B$.

(ii) *For* $K \in \mathcal{L}(Y, U)$*, we define* $\mathbf{H}^K := \mathbf{H}(I - K\mathbf{H})^{-1}$ *and we denote the* set of stabilising feedback operators *by*

$$\mathbb{S}^{\mathrm{c}}(\mathbf{H}) := \{K \in \mathcal{L}(Y, U) : \mathbf{H}^K \text{ is a bounded and holomorphic function on } \mathbb{C}_0\},$$

where the superscript 'c' *indicates the continuous-time setting.*

In addition to the previous definitions, in order to define the sampled-data system that we are interested in, we also require the use of two particular operators, namely the sampling and hold operators.

Definition 5.5.4. *Let* $\tau \in \mathbb{Z}_+$.

(i) *We define the* sampling operator *(with respect to* τ*)* $\mathcal{S} : C(\mathbb{R}_+, Y) \to Y^{\mathbb{Z}_+}$ *by*

$$(\mathcal{S}y)(k) := y(k\tau) \quad \forall\, y \in C(\mathbb{R}_+, Y),\ \forall\, k \in \mathbb{Z}_+\,.$$

(ii) *We define the (zero order)* hold operator *(with respect to* τ*)* $\mathcal{H}$ *by*

$$(\mathcal{H}u)(t) := u(k) \quad \forall\, u \in U^{\mathbb{Z}_+},\ \forall\, t \in [k\tau, (k+1)\tau)\,,$$

which maps $U^{\mathbb{Z}_+}$ *into the set of step-functions mapping* $\mathbb{R}_+$ *to* U.

We are now in a position to define the aforementioned sampled-data Lur'e system. Indeed, we shall consider the forced sampled-data Lur'e system arising from the feedback interconnection of (5.78) with the nonlinear sampled-data output feedback control

$$u = \mathcal{H}(f(\mathcal{S}(y) + w))\,, \tag{5.79}$$

where $w \in Y^{\mathbb{Z}_+}$ is an output disturbance and $f : Y \to U$ with $f(0) = 0$. Thus, for given $\tau \in \mathbb{Z}_+$, $x^0 \in X$, $v \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+, X)$ and $w \in Y^{\mathbb{Z}_+}$, we consider the initial-value problem

$$\dot{x} = Ax + B\mathcal{H}(f(\mathcal{S}(Cx) + w)) + v, \quad x(0) = x^0 \in X\,, \tag{5.80}$$

see Figure 5.3.

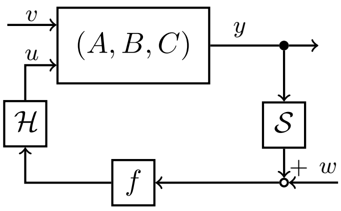

Figure 5.3: *Block diagram illustrating the sampled-data Lur'e system* (5.80).

We say that a continuous function $x : \mathbb{R}_+ \to X$ is a (mild) solution to (5.80) if x satisfies $x(0) = x^0$ and

$$\begin{aligned} x(k\tau + t) =& \mathbf{T}(t)x(k\tau) + \int_0^t \mathbf{T}(t-s)Bf(Cx(k\tau) + w(k))ds \\ &+ \int_0^t \mathbf{T}(t-s)v(k\tau + s)ds \quad \forall\, t \in (0,\tau],\ \forall\, k \in \mathbb{Z}_+ . \end{aligned} \tag{5.81}$$

It is clear that, for all $\tau \in \mathbb{Z}_+$, $x^0 \in X$, $v \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+, X)$ and $w \in Y^{\mathbb{Z}_+}$, there exists a unique solution of (5.80). Note that if $x^0 = 0$, $v = 0$ and $w = 0$, then 0 is a solution of (5.80), as $f(0) = 0$.

We now define what we mean when we say that (5.80) is exponentially input-to-state stable.

Definition 5.5.5. *For a given $\tau \in \mathbb{Z}_+$, the sampled-data Lur'e system* (5.80) *is said to be* exponentially input-to-state stable (ISS) *if there exist constants $c, \gamma > 0$ such that, for all initial states $x^0 \in X$, all inputs $v \in L^\infty_{loc}(\mathbb{R}_+, X)$ and all output disturbances $w \in Y^{\mathbb{Z}_+}$, the solution x of* (5.80) *satisfies*

$$\|x(k\tau + t)\| \le c\left(e^{-\gamma(k\tau+t)}\|x^0\| + \|v\|_{L^\infty([0,k\tau+t])} + \|\pi_k w\|_{l^\infty}\right) \quad \forall\, t \in (0,\tau],\ \forall\, k \in \mathbb{Z}_+ .$$

The following theorem is the main result of this section and gives a sufficient condition for exponential ISS of (5.80).

Theorem 5.5.6. *Assume that $K \in \mathbb{S}^c(\mathbf{H})$ and*

$$\|f(\xi) - K\xi\| \le r\|\xi\| \quad \forall\, \xi \in Y, \tag{5.82}$$

where $r < 1/\sup_{s\in\mathbb{C}_0} \|\mathbf{H}^K(s)\|$. Then there exists $\tau^* > 0$ such that (5.80) *is exponentially ISS for all $\tau \in (0, \tau^*)$.*

Before proving Theorem 5.5.6, we give some commentary in the form of a remark.

Remark 5.5.7. We note that under the assumptions of Theorem 5.5.6, it follows from [47, Theorem 4.1] that the continuous-time Lur'e system

$$\dot{x} = Ax + Bf(Cx) + v, \quad x(0) = x^0 \in X\,, \tag{5.83}$$

is exponentially ISS. Theorem 5.5.6 shows that exponential ISS is inherited by the sample-hold discretization (5.79) of the continuous-time system (5.83), provided the sampling period is sufficiently small. ◊

To facilitate the proof of Theorem 5.5.6, we state the following definitions and technical lemma.

Definition 5.5.8. (i) *For $\tau > 0$, we set*

$$A_\tau := \mathbf{T}(\tau) \in \mathcal{L}(X), \quad B_\tau := \int_0^\tau \mathbf{T}(s)Bds \in \mathcal{L}(U, X).$$

Moreover, for $L \in \mathcal{L}(Y, U)$, we define

$$A_\tau^L = (A_\tau)^L := A_\tau + B_\tau LC\,.$$

(See Definition 5.1.15.)

(ii) *For $L \in \mathcal{L}(Y,U)$ and $r > 0$, we let*

$$\mathbb{B}^{\mathrm{cl}}(L,r) := \big\{ M \in \mathcal{L}(Y,U) \,:\, \|M - L\| \leq r \big\},$$

denote the closed ball of radius r, centred at L.

Lemma 5.5.9. *Let $r > 0$ and $K \in \mathcal{L}(Y,U)$ and assume that $\mathbb{B}^{\mathrm{cl}}(K,r) \subseteq \mathbb{S}^{\mathrm{c}}(\mathbf{H})$. Then there exists $\tau^* > 0$ such that, for all $L \in \mathbb{B}^{\mathrm{cl}}(K,r)$ and every $\tau \in (0,\tau^*)$, the operator A_τ^L is (discrete-time) exponentially stable (i.e. stable in the sense of Definition 5.1.1).*

To avoid disruption of the flow of the presentation, the proof of Lemma 5.5.9 is placed at the end of this section.

Proof of Theorem 5.5.6. Let $\rho \in \mathbb{R}$ be such that $r < \rho < 1/\sup_{s \in \mathbb{C}_0} \|\mathbf{H}^K(s)\|$. Then, by [46, Proposition 5.6], $\mathbb{B}^{\mathrm{cl}}(K,\rho) \subseteq \mathbb{S}^{\mathrm{c}}(\mathbf{H})$. Consequently, by Lemma 5.5.9, there exists $\tau^* > 0$ such that, for all $L \in \mathbb{B}^{\mathrm{cl}}(K,\rho)$ and every $\tau \in (0,\tau^*)$, the operator A_τ^L is exponentially stable. Now, let $\tau \in (0,\tau^*)$ be fixed. Moreover, let $x^0 \in X$, $v \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+, X)$, $w \in Y^{\mathbb{Z}_+}$ and let x be a solution of (5.80). Then, for every $k \in \mathbb{Z}_+$ and all $t \in (0,\tau]$, x satisfies (5.81). Letting $t = \tau$ in (5.81) and changing variables, it follows that, for every $k \in \mathbb{Z}_+$,

$$x((k+1)\tau) = \mathbf{T}(\tau)x(k\tau) + \int_0^\tau \mathbf{T}(s)Bds f(Cx(k\tau) + w(k)) + \int_0^\tau \mathbf{T}(s)v((k+1)\tau - s)ds.$$

Setting $x_k := x(k\tau)$, $w_k := w(k)$ and $v_k := \int_0^\tau \mathbf{T}(s)v((k+1)\tau - s)ds$ for all $k \in \mathbb{Z}_+$, we see that (v_k, w_k, x_k) satisfies the following discrete-time system

$$\begin{aligned} x_{k+1} &= A_\tau x_k + B_\tau f(Cx_k + w_k) + v_k, \quad x(0) = x^0 \in X, \\ y_k &= Cx_k. \end{aligned}$$

Therefore, statement (ii) of Theorem 5.2.5 (or more precisely of Corollary 5.2.8) yields the existence of constants $c_1 > 0$ and $\theta \in (0,1)$ (independent of v, w and x) such that, for all $k \in \mathbb{N}$,

$$\|x_k\|_X \leq c_1 \left(\theta^k \|x^0\|_X + \max_{\sigma \in \underline{k-1}} \left\| \begin{pmatrix} v_\sigma \\ w_\sigma \end{pmatrix} \right\|_{X \times Y} \right). \tag{5.84}$$

Let

$$\mu := \tau \sup_{s \in [0,\tau]} \|\mathbf{T}(s)\|,$$

and note that, for all $k \in \mathbb{Z}_+$,

$$\|v_k\|_X \leq \mu \, \|v\|_{L^\infty([k\tau,(k+1)\tau])} \,.$$

Hence, there exists $c_2 > 0$ (again independent of v, w and x) such that

$$\|x_k\|_X \leq c_2 \left(\theta^k \|x^0\|_X + \mu \, \|v\|_{L^\infty([0,k\tau])} + \|\pi_{k-1} w\|_{l^\infty} \right) \quad \forall \, k \in \mathbb{Z}_+ \,. \tag{5.85}$$

It remains to use the discrete-time estimate (5.85) to bound the state x over all times. To this end, note that for all $k \in \mathbb{Z}_+$ and all $t \in (0,\tau]$, (5.81) gives that

$$x(k\tau + t) = \mathbf{T}(t)x_k + \int_0^t \mathbf{T}(s)Bds f(Cx(k\tau) + w(k)) + \int_0^t \mathbf{T}(t-s)v(k\tau + s)ds \,. \tag{5.86}$$

Appealing to (5.82), we estimate, for all $k \in \mathbb{Z}_+$ and all $t \in (0, \tau]$,

$$\left\| \int_0^t \mathbf{T}(s) ds B f(Cx(k\tau) + w(k)) \right\| \leq \mu \|B\| (r + \|K\|)(\|C\| \|x_k\| + \|w_k\|). \tag{5.87}$$

Moreover, for all $k \in \mathbb{Z}_+$ and all $t \in (0, \tau]$,

$$\left\| \int_0^t \mathbf{T}(t-s) B v(k\tau + s) ds \right\| \leq \mu \|B\| \|v\|_{L^\infty([k\tau, k\tau+t])}. \tag{5.88}$$

Taking norms in (5.86) and substituting in (5.87) and (5.88) yields that, for all $k \in \mathbb{Z}_+$ and all $t \in (0, \tau]$,

$$\begin{aligned} \|x(k\tau + t)\| \leq & \Big(\|\mathbf{T}(t)\| + \mu \|B\| (r + \|K\|) \|C\| \Big) \|x_k\| + \mu \|B\| \, \|v\|_{L^\infty([k\tau, k\tau+t])} \\ & + \mu \|B\| (r + \|K\|) \|w_k\|. \end{aligned}$$

The claim now follows in light of the above inequality and (5.85). □

Proof of Lemma 5.5.9. The proof is a refinement of that of [75, Theorem 3.1]. For $F \in \mathcal{L}(Y, U)$, we let $\mathbf{T}_F$ denote the strongly continuous semigroup generated by $A + BFC$ (see Lemma 5.5.2). By hypothesis, $\mathbb{B}^{\mathrm{cl}}(K, r) \subseteq \mathbb{S}^{\mathrm{c}}(\mathbf{H})$, (A, B) is stabilisable and (C, A) is detectable, and so, by [26, Theorem 7.32], for each $F \in \mathbb{B}^{\mathrm{cl}}(K, r)$, there exist $\omega_F < 0$ and $M_F \geq 1$ such that $\|\mathbf{T}_F(t)\| \leq M_F e^{\omega_F t}$ for all $t \geq 0$. We seek to show that there exist $\omega \in (-\infty, 0)$ and $M \in [1, \infty)$ such that

$$\|\mathbf{T}_F(t)\| \leq M e^{\omega t} \quad \forall\, t \geq 0, \ \forall\, F \in \mathbb{B}^{\mathrm{cl}}(K, r) . \tag{5.89}$$

To this end, note that for each $F \in \mathbb{B}^{\mathrm{cl}}(K, r)$, there exists $\varepsilon_F > 0$ such that

$$\omega_F + M_F \|B(L - F)C\| \leq \frac{\omega_F}{2} \quad \forall\, L \in \mathbb{B}(F, \varepsilon_F)$$

and thus, by Lemma 5.5.2,

$$\|\mathbf{T}_L(t)\| \leq M_F e^{(\omega_F/2)t} \quad \forall\, t \geq 0, \forall\, L \in \mathbb{B}(F, \varepsilon_F).$$

Here we have used the fact that $A + BLC = (A + BFC) + B(L - F)C$. Now, the balls $\mathbb{B}(F, \varepsilon_F)$ form an open cover of $\mathbb{B}^{\mathrm{cl}}(K, r)$ and, since U and Y are finite dimensional, $\mathbb{B}^{\mathrm{cl}}(K, r)$ is compact. Hence, there exist $F_1, \ldots, F_n \in \mathbb{B}^{\mathrm{cl}}(K, r)$, $\varepsilon_1, \ldots, \varepsilon_n \in (0, \infty)$ and $\omega_1, \ldots, \omega_n \in (-\infty, 0)$ such that $\mathbb{B}^{\mathrm{cl}}(K, r) \subseteq \cup_{i=1}^n \mathbb{B}(F_i, \varepsilon_i)$ and

$$\|\mathbf{T}_L(t)\| \leq M_{F_i} e^{\omega_i t} \quad \forall\, t \geq 0, \forall\, L \in \mathbb{B}(F_i, \varepsilon_i), \forall\, i \in \{1, \ldots, n\}.$$

By setting $M := \max\{M_{F_1}, \ldots, M_{F_n}\}$ and $\omega := \max\{\omega_1, \ldots, \omega_n\}$, it follows that (5.89) holds.

Next, we claim that for all $\varepsilon > 0$, there exists $T > 0$ such that

$$\|FC(I - \mathbf{T}_F(t))\| < \varepsilon \quad \forall\, F \in \mathbb{B}^{\mathrm{cl}}(K, r), \ \forall\, t \in [0, T] . \tag{5.90}$$

To prove (5.90), we will show that for all $\varepsilon > 0$ and all $F \in \mathbb{B}^{\mathrm{cl}}(K, r)$, there exist $r_F > 0$ and $T_F > 0$ such that

$$\|LC(I - \mathbf{T}_L(t))\| < \varepsilon \quad \forall\, t \leq T_F, \quad \forall\, L \in \mathbb{B}(F, r_F)$$

and then use another compactness argument. To this end, fix $\varepsilon > 0$ and let $F \in \mathbb{B}^{\mathrm{cl}}(K, r)$. Since U and Y are finite dimensional and C is bounded, it follows that $FC \in \mathcal{L}(X, U)$ is a compact operator. Furthermore, as X is a Hilbert space, [86, Corollary 10.6, p.41] yields that $\mathbf{T}_F^*$ is a strongly continuous semigroup, and thus

$$\lim_{t\to 0}(I - \mathbf{T}_F(t))^* x = 0 \quad \forall\, x \in X\,.$$

Therefore, we invoke [75, Lemma 2.1] to yield that

$$\lim_{t\to 0}\|FC(I - \mathbf{T}_F(t))\| = 0\,.$$

Consequently, we let $\tilde{T}_F > 0$ be such that

$$\|FC(I - \mathbf{T}_F(t))\| < \frac{\varepsilon}{3} \quad \forall\, t \in [0, \tilde{T}_F]\,. \tag{5.91}$$

We now choose $r_F > 0$ such that

$$\|L - F\|\|C\|M < \frac{\varepsilon}{6} \quad \forall\, L \in \mathbb{B}(F, r_F). \tag{5.92}$$

We apply [86, Corollary 1.3, p.78] and recall (5.89) to obtain, for all $L \in \mathbb{B}(F, r_F)$ and all $t \geq 0$,

$$\|\mathbf{T}_L(t) - \mathbf{T}_F(t)\| \leq Me^{\omega t}(e^{M\|B\|\|L-F\|\|C\|t} - 1) \leq Me^{\omega t}(e^{r_F M\|B\|\|C\|t} - 1),$$

which in turn implies that

$$\|LC\|\|\mathbf{T}_L(t) - \mathbf{T}_F(t)\| \leq (\|F\| + r_F)\|C\|Me^{\omega t}(e^{r_F M\|B\|\|C\|t} - 1).$$

We then let $\hat{T}_F > 0$ be such that

$$\|LC\|\|\mathbf{T}_L(t) - \mathbf{T}_F(t)\| < \frac{\varepsilon}{6} \quad \forall\, t \in [0, \hat{T}_F],\, \forall\, L \in \mathbb{B}(F, r_F)\,. \tag{5.93}$$

Setting $T_F := \min\{\tilde{T}_F, \hat{T}_F\}$, it follows from (5.89), (5.91), (5.92) and (5.93) that, for all $t \in [0, T_F]$ and all $L \in \mathbb{B}(F, r_F)$,

$$\begin{aligned}\|LC(I - \mathbf{T}_L(t))\| &\leq \|LC - FC\| + \|LC\mathbf{T}_L(t) - FC\mathbf{T}_F(t)\| + \|FC(I - \mathbf{T}_F(t))\| \\ &< \frac{\varepsilon}{6} + \|LC\|\|\mathbf{T}_L(t) - \mathbf{T}_F(t)\| + \|L - F\|\|C\|\|\mathbf{T}_F(t)\| + \frac{\varepsilon}{3} \\ &< \varepsilon.\end{aligned}$$

Hence, for all $\varepsilon > 0$ and for all $F \in \mathbb{B}^{\mathrm{cl}}(K, r)$, there exists $r_F > 0$ and $T_F > 0$ such that

$$\|LC(I - \mathbf{T}_L(t))\| < \varepsilon \quad \forall\, t \in [0, T_F], \quad \forall\, L \in \mathbb{B}(F, r_F).$$

A compactness argument similar to that establishing (5.89) can now be used to prove that for all $\varepsilon > 0$, there exists $T > 0$ such that (5.90) holds.
Finally, we seek to use (5.89) and (5.90) to yield the existence of $\tau^* > 0$ such that A_τ^L is (discrete-time) exponentially stable for all $L \in \mathbb{B}^{\mathrm{cl}}(K, r)$ and every $\tau \in (0, \tau^*)$.

To that end, fix $L \in \mathbb{B}^{\mathrm{cl}}(K,r)$. The variation-of-parameters formula for perturbed semigroups [86, Equation (1.2), p.77] gives, for all $\tau \geq 0$ and all $x \in X$,

$$\begin{aligned} A_\tau x + B_\tau LCx &= \mathbf{T}(\tau)x + \int_0^\tau \mathbf{T}(s)BdsLCx \\ &= \mathbf{T}(\tau)x + \int_0^\tau \mathbf{T}(\tau-s)BLC(I-\mathbf{T}_L(s))xds \\ &\quad + \int_0^\tau \mathbf{T}(\tau-s)BLC\mathbf{T}_L(s)xds \\ &= \mathbf{T}_L(\tau)x + P_\tau x, \end{aligned} \tag{5.94}$$

where $P_\tau x := \int_0^\tau \mathbf{T}(\tau-s)BLC(I-\mathbf{T}_L(s))xds$ for all $x \in X$.
As in the proof of [75, Theorem 3.1], let us introduce a new norm on X given by

$$|x| := \sup_{t\geq 0} \|\mathbf{T}_L(t)x\|e^{-\omega t} \quad \forall\, x \in X,$$

where recall $\omega < 0$ is as in (5.89). We note that, from (5.89) and the fact that $\mathbf{T}_L(0) = I$,

$$\|x\|_X \leq |x| \leq M\|x\|_X \quad \forall\, x \in X. \tag{5.95}$$

For all $x \in X$ and all $t \geq 0$, we have

$$\begin{aligned} |\mathbf{T}_L(t)x| &= \sup_{s\geq 0} \|\mathbf{T}_L(s)\mathbf{T}_L(t)x\|e^{-\omega s} \\ &= \sup_{s\geq 0} \|\mathbf{T}_L(s+t)x\|e^{-\omega(s+t)}e^{\omega t} \\ &\leq \sup_{s\geq 0} \|\mathbf{T}_L(s)x\|e^{-\omega s}e^{\omega t}. \end{aligned}$$

Therefore,

$$|\mathbf{T}_L(t)x| \leq e^{\omega t}|x| \quad \forall\, t \geq 0, \forall\, x \in X. \tag{5.96}$$

For $G \in \mathcal{L}(X)$, let $|G|$ denote the operator norm of G induced by the new norm, that is,

$$|G| = \sup_{\substack{x\in X \\ x\neq 0}} \frac{|Gx|}{|x|}\,.$$

Combining (5.94) with (5.95), (5.96) and the inequality

$$e^{\omega\tau} \leq 1 + \omega\tau e^{\omega\tau} \quad \forall\, \tau \in \mathbb{R}_+,$$

we obtain that

$$\begin{aligned} |A_\tau + B_\tau LC| &\leq e^{\omega\tau} + M\|P_\tau\| \\ &\leq 1 + \omega\tau + (\omega(-1+e^{\omega\tau}) + h(\tau))\,\tau \quad \forall\, \tau \in \mathbb{R}_+\,, \end{aligned}$$

where

$$h(\tau) := M \sup_{s\in[0,\tau]} \|\mathbf{T}(\tau-s)BLC(I-\mathbf{T}_L(s))\|\,.$$

Combining this with (5.90) shows that, for fixed $\delta \in (0,-\omega)$, there exists $\tau^* > 0$ (independent of $L \in \mathbb{B}^{\mathrm{cl}}(K,r)$) such that

$$|A_\tau + B_\tau LC| < 1 + (\omega+\delta)\tau < 1 \quad \forall\, \tau \in (0,\tau^*). \tag{5.97}$$

Finally, by invoking (5.95), we obtain that, for all $\tau \in (0, \tau^*)$, all $n \in \mathbb{Z}_+$ and all $x \in X$,

$$\|(A_\tau + B_\tau LC)^n x\| \leq |(A_\tau + B_\tau LC)^n x| \leq |A_\tau + B_\tau LC|^n |x| \leq M|A_\tau + B_\tau LC|^n \|x\|.$$

In light of (5.97), the above inequality yields the exponential stability of A^L_τ for all $L \in \mathbb{B}^{\mathrm{cl}}(K, r)$ and all $\tau \in (0, \tau^*)$, completing the proof. □

5.6 Examples

We now give a detailed discussion of two examples. Before coming to these however, we highlight to the reader that the numerical simulations seen in both examples were created Dr. Chris Guiver (University of Bath) and appear in our co-authored paper [40].

Example 5.6.1. Consider the following controlled and observed heat equation describing the temperature evolution in a unit rod

$$\left.\begin{aligned} z_t(\xi, t) &= z_{\xi\xi}(\xi, t) + 2\chi_{[1/2,1]}(\xi)u(t) + b_{\mathrm{e}}(\xi)v(t), \\ z_\xi(0, t) &= z_\xi(1, t) = 0, \ z(\xi, 0) = z^0, \\ y(t) &= 2\int_0^{\frac{1}{2}} z(t, \zeta)\, d\zeta, \end{aligned}\right\} \quad \xi \in (0, 1), \ t > 0\,. \tag{5.98}$$

Here $z(\xi, t)$ denotes the temperature of the rod at position ξ and time t, $z^0 \in L^2(0, 1)$ is the initial temperature distribution, $\chi_{[1/2,1]}$ is the indicator function of the interval $[1/2, 1]$ and $b_{\mathrm{e}} \in L^2(0, 1)$. Further, u and v are inputs and y is the output (or observation). It is shown in [26, Example 4.3.11] that (5.98) (with $v = 0$) may be written in the form (5.78), with state-space $X = L^2(0, 1)$, A the Laplacian with zero Neumann boundary conditions, and bounded B and C operators. Furthermore, (A, B) is stabilisable and (C, A) is detectable by [26, Example 5.2.8]. The transfer function $\mathbf{H}$ from u to y is given by

$$\mathbf{H}(s) = \frac{2\tanh(\sqrt{s}/2)}{s\sqrt{s}},$$

where we have again used [26, Example 4.3.11], which has a simple pole at $s = 0$ and so (5.98) is neither exponentially nor input-output stable. To illustrate the sampled-data control results of Section 5.5, we consider the following problem: find conditions which are sufficient for the sampled-data system given by (5.98) and the feedback (5.79) to be exponentially ISS.

Writing $\mathbf{L}(s) := s\mathbf{H}(s)$ enables us to exploit the results of [78] to compute stabilising gains for $\mathbf{H}$. For which purpose, we note that (see again [26, Example 4.3.11]) $\mathbf{L}(0) = 1 > 0$ and that $\mathbf{L}$ is bounded and holomorphic on $\{s \in \mathbb{C} : \mathrm{Re}\,(s) > \alpha\}$ for every $\alpha > -\pi^2$. Setting

$$\lambda := 2\sup_{\omega \in \mathbb{R}} \left| \mathrm{Re}\,\frac{\mathbf{L}(i\omega) - \mathbf{L}(0)}{i\omega} \right| > 0\,,$$

an application of [78, Lemma 3.1 and Corollary 3.4] yields that

$$\sup_{s \in \mathbb{C}_0} |\mathbf{H}^{-k}(s)| = 1/k \quad \forall\, k \in (0, 1/\lambda)\,.$$

In light of [46, Proposition 5.6], it follows that

$$\mathbb{B}(-k,k) \subseteq \mathbb{S}^{\mathrm{c}}(\mathbf{H}) \quad \forall\, k \in (0, 1/\lambda)\,,$$

whence, for all $\rho \in (0,1)$,

$$\mathbb{B}^{\mathrm{cl}}(-k,\rho k) \subseteq \mathbb{B}(-k,k) \subseteq \mathbb{S}^{\mathrm{c}}(\mathbf{H}) \quad \forall\, k \in (0, 1/\lambda)\,. \tag{5.99}$$

Consequently, Theorem 5.5.6 ensures that, for all $k \in (0, 1/\lambda)$ and all $\rho \in (0,1)$, if $f : \mathbb{R} \to \mathbb{R}$ is such that

$$|f(\theta) + k\theta| \leq \rho k |\theta| \quad \forall\, \theta \in \mathbb{R}\,, \tag{5.100}$$

then there exists $\tau^* > 0$ such that the sampled-data feedback interconnection of (5.98) and (5.79) is exponentially ISS for all sampling/hold periods $\tau \in (0, \tau^*)$.
For a numerical illustration, we first compute numerically that $\lambda \approx 0.3707$, so that $5/2 < 1/\lambda$. We take $k = 1$, $\rho = 0.95$ and

$$f : \mathbb{R} \to \mathbb{R}, \quad f(\theta) = -\theta + 0.95\big(1 - \mathrm{e}^{-|\theta|}\big)\sin(3\theta)\,, \tag{5.101}$$

so that (5.99) and (5.100) both hold. Note that f is Lipschitz, but has Lipschitz constant bigger than one. Figure 5.4 illustrates the sector condition (5.100).

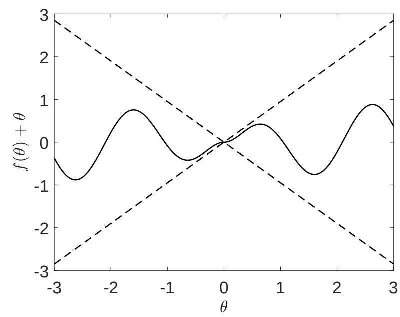

Figure 5.4: *Graph of $f(\theta) + \theta$ against θ, for f in* (5.101). *The dashed lines have gradient* ± 0.95.

We set $b_{\mathrm{e}} = \chi_{[1/4,1/2]}$, and define the constant and periodic input $v_1 = 3$ and $v_2(t) = 3\sin(2t)$ for all $t \geq 0$, respectively. Certain simulations use the initial temperature distribution $z^0(\xi) = e^{-|\xi - 1/2|^2}$. We simulate the closed-loop feedback system (5.98) and (5.79) by performing a semi-discretization in space using a finite-element method with 31 elements. In the interest of brevity, we refer the reader to [40, Appendix 2] for the details concerning this. Figures 5.5(a)–5.5(d) show plots of $\|x(t)\|$ against t in the following situations described in Table 5.1. Simulation data are also listed in Table 5.1.
In Figure 5.5(a), we see the exponential stability property of the unforced ($v = 0$) sampled-data feedback system — a consequence of exponential ISS. In Figures 5.5(a) and 5.5(c), we see the ISS property — the state is bounded in the presence of persistent inputs and, as might be expected, $\|x(t)\|$ increases as $\|v\|_{L^\infty(0,t)}$ increases.
To conclude the example, we comment that although taking $\tau = 0.25$ appears to "work", in the sense that the numerical results agree with what the theory predicts, in fact the constant τ^*, the existence of which is guaranteed by Theorem 5.5.6, could be either smaller or larger than 0.25. Determining the maximal τ^* analytically or numerically is a difficult open problem. It seems that Figure 5.5(d) shows divergence when $\tau = 2.5$, indicating that $\tau^* < 2.5$. ◊

Figure	Initial condition $z(\cdot, 0)$	Input v	Sampling/hold period τ
5.5(a)	$j^2 z^0$	0	0.25
5.5(b)	0	jv_1	0.25
5.5(c)	0	jv_2	0.25
5.5(d)	z^0	0	$1, 2, 2.5$

Table 5.1: *Model parameters used in the numerical simulations in Example 5.6.1. Here* $j \in \{1, 2, 3\}$.

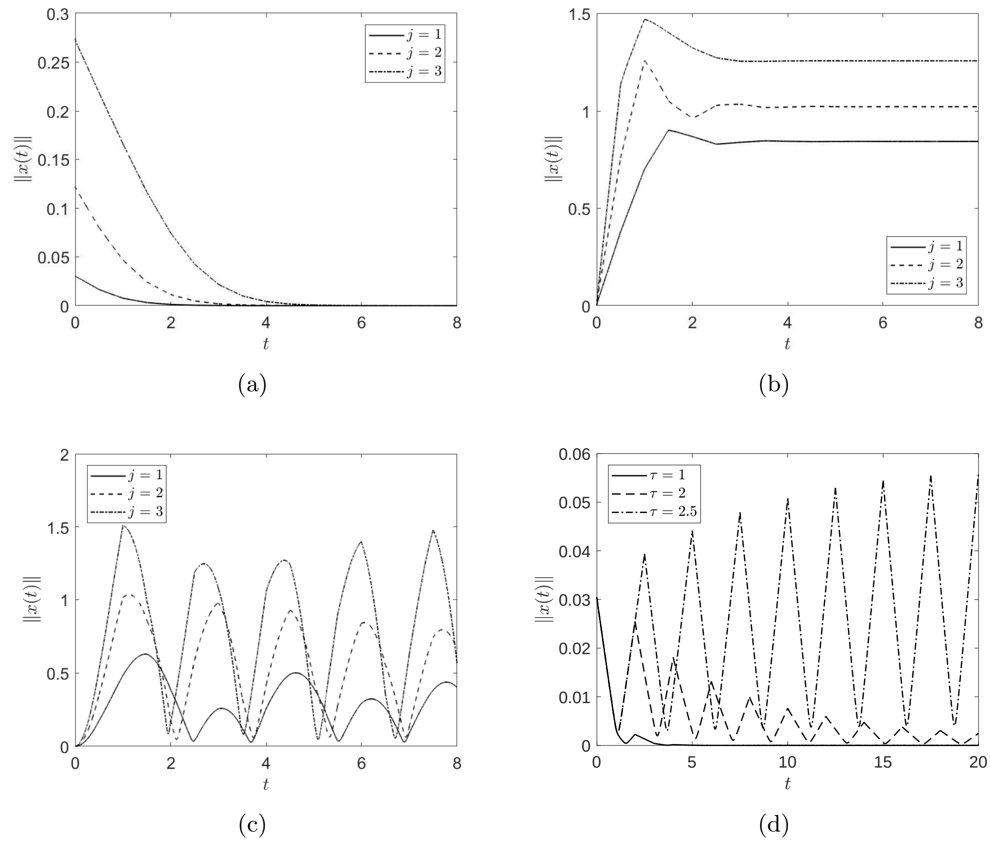

Figure 5.5: *Numerical simulation of the sampled-data feedback interconnection of* (5.98) *and* (5.79) *from Example 5.6.1.*

Example 5.6.2. We consider a forced Integral Projection Model (IPM) for the monocarpic plant Platte thistle (*Cirsium canescens*), based on the model presented in [18, 95]. For a recent overview of IPMs we refer the reader to [82]. Platte thistle is a perennial plant native to central North America. The IPM describes the distribution of plant size, according to the natural logarithm of the crown diameter in mm. Following [18], we assume that the continuous variable of natural logarithm of crown diameter takes minimum and maximum values given by $m_1 = -0.5$ and $m_2 = 3.5$ (so that roughly $e^{m_1} = 0.6$mm and $e^{m_2} = 33$mm), respectively, and that the time-steps correspond to years. Incorporating an additive input, the model is

$$\left.\begin{aligned} \eta(t+1,\xi) &= \int_{m_1}^{m_2} p(\xi,\zeta)\eta(t,\zeta)\,d\zeta + b(\xi)h(c^*\eta(t,\xi))c^*\eta(t,\xi) \\ &\quad + b_{\mathrm{e}}(\xi)v(t), \\ \eta(0,\xi) &= \eta_0(\xi) \end{aligned}\right\} \quad \begin{aligned} &\forall\, t \in \mathbb{Z}_+ \\ &\text{a.e. } \xi \in [m_1, m_2]\,, \end{aligned} \tag{5.102}$$

where $\eta(t,\cdot)$ denotes the distribution of plant size at time-step t, with initial distribution $\eta_0 \in L^1([m_1, m_2])$. In the following, our aim is to write (5.102) in the form of a forced, infinite-dimensional Lur'e system (5.10) with the natural state space $X = L^1(\Omega)$, where $\Omega := [m_1, m_2]$. Before doing this, we provide some commentary on the model (5.102).
The first term on the right hand side of the difference equation in (5.102) models survival and growth of existing plants. Here $p(\xi,\zeta)$ denotes the probability of an individual of size ζ surviving to one of size ξ in one time-step, and is assumed in [18, 95] to have the structure

$$p(\xi,\zeta) = s(\zeta)(1 - f_{\mathrm{p}}(\zeta))g(\xi,\zeta) \quad \forall\, \xi,\zeta \in \Omega\,, \tag{5.103}$$

where $s(\zeta)$ is the survival probability of an individual of size ζ, $f_{\mathrm{p}}(\zeta)$ is the probability that an individual of size ζ flowers, and $g(\xi,\zeta)$ is the probability of an individual of size ζ growing to size ξ, each over one time-step. We take s, f_{p} and g as in [18, Table 2]. The term $1 - f_{\mathrm{p}}$ appears on the right-hand side of (5.103) as flowering is fatal to Platte thistle, that is, it is monocarpic.
The second term on the right hand side of the difference equation in (5.102) models reproduction and recruitment into the population. In particular, $b \in X$ denotes the distribution of offspring plant size, c^*z equals the total number of new seeds recruited into the population by the distribution $z \in X$ in one time-step, and is given by

$$c^*z = \int_{\Omega} s(\theta)f_{\mathrm{p}}(\theta)S(\theta)z(\theta)d\theta \quad \forall\, z \in X\,.$$

In addition to the terms in (5.103), $S(\theta)$ denotes the number of seeds produced on average by a plant of size θ. We take $b = J$, where J is as in [18, Table 2], and the function S is given in [18, Table 2]. We have $c^* \in X^*$ as $\theta \mapsto s(\theta)f_{\mathrm{p}}(\theta)S(\theta) \in L^\infty(\Omega)$. The function h in (5.102) denotes the probability of seed germination, and is a nonlinear function of the total number of seeds produced, and so seeks to model density-dependence in the seed germination probability. As such, it is assumed to be non-increasing, representing competition or crowding affects at higher seed abundances. Two situations are explored in [18]: first, h is constant with value 0.067, and; second, h is defined by $h(s) = s^{-0.33}$. We note that there is uncertainty in modelling nonlinear terms for Platte thistle, see [32], and in order to demonstrate different settings where the incremental condition (5.14) holds, we shall choose a different h below.
The third term term on the right hand side of the difference equation in (5.102) is an additive input, which may be the arrival of new plants via planned replanting schemes, or accidental movement. We assume that $b_{\mathrm{e}} \in X$ and $v \in (\mathbb{R}_+)^{\mathbb{Z}_+}$, which capture the distribution and magnitude, respectively.
We define the integral operator $A : X \to X$ by

$$(Ax)(\cdot) = \int_{\Omega} p(\cdot,\zeta)x(\zeta)d\zeta \quad \forall\, x \in X\,,$$

and impose the ecologically reasonable assumption

$$\sup_{\zeta\in\Omega}\int_\Omega p(\xi,\zeta)\,d\xi < 1\,,$$

which corresponds to some positive level of mortality in the population at all sizes. An application of [70, Theorem 1, p. 173] yields that $\|A\| < 1$.
Combining the above, and setting $x(t) = \eta(t,\cdot)$ for all $t \in \mathbb{Z}_+$, we see that (5.102) may be written as a forced Lur'e system,

$$x^+ = Ax + bf(c^*x) + b_{\mathrm{e}}v\,, \tag{5.104}$$

on the state-space $X = L^1(\Omega)$, and with $U = V = Y = \mathbb{R}$. Here $f(s) := h(s)s$ for $s \geq 0$ and we extend h and f to all of $\mathbb{R}$ by setting $h(s) = f(s) = 0$ for $s \in (-\infty, 0)$. The extension is to ensure that f is defined on the whole of Y, so that the results of the chapter are applicable.
We seek to apply Theorem 5.2.5, Theorem 5.3.7 and Corollary 5.3.15 to (5.104) to infer various (incremental) stability and convergence notions. To simulate (5.104) we use a finite-element approximation. Once again, in the interest of brevity, we refer the reader to [40, Appendix 3] for the details of this.
The property $\|A\| < 1$ implies that $\mathbb{B}(0,r) \subseteq \mathbb{S}(\mathbf{G})$ for all $r \in (0, 1/\|\mathbf{G}\|_{H^\infty})$, where $\mathbf{G}(z) = c^*(zI - A)^{-1}b$. Moreover, by [36, Proposition 3.1], we have that $\|\mathbf{G}\|_{H^\infty} = \mathbf{G}(1)$, and we compute numerically that $\mathbf{G}(1) \approx 43.8$. We propose a negative sigmoid type function for $h : \mathbb{R} \to \mathbb{R}_+$, namely

$$h(s) = \begin{cases} 0 & s < 0\,, \\ \dfrac{\rho_1}{1 + e^{\rho_2(s-\rho_3)}} & s \geq 0\,, \end{cases} \tag{5.105}$$

where ρ_1, ρ_2 and ρ_3 are positive parameters. Broadly, ρ_1 captures the probability of germination at low abundance, ρ_2 determines the rate of transition and ρ_3 the value at which the transition occurs. Figure 5.6 contains plots of h for several parameter values. Since h is nonnegative-valued and nonincreasing, it follows that there exists $\delta > 0$ such

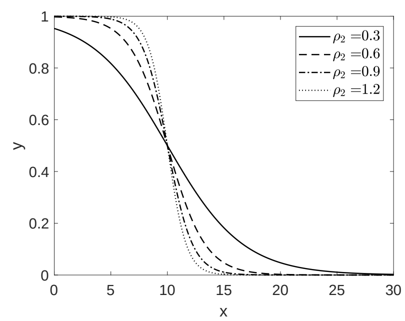

Figure 5.6: *Graph of function h in (5.105) for nonnegative arguments. Here $\rho_1 = 1$ and $\rho_3 = 10$, and ρ_2 varies.*

that $s \mapsto f(s) = h(s)s$ satisfies (5.14) with $K = 0$, $S_1 = \mathbb{R}$, $S_2 = \{0\}$ and $r = 1/\mathbf{G}(1)$ whenever

$$h(0) = \frac{\rho_1}{1 + \mathrm{e}^{-\rho_2\rho_3}} < \frac{1}{\mathbf{G}(1)}\,. \tag{5.106}$$

If (5.106) holds, then Theorem 5.2.5 yields that (5.104) is exponentially ISS. If the inequality

$$\sup_{s>0} |f'(s)| < \frac{1}{\mathbf{G}(1)}\,, \tag{5.107}$$

holds, then f satisfies (5.14) with $K = 0$, $S_1 = S_2 = \mathbb{R}$, and hence Theorem 5.2.5 yields that (5.104) is exponentially δISS. In this case, it also follows from Theorem 5.3.7 that (5.104) has the CICS property. Moreover, the inequality (5.107) is sufficient for the hypotheses of Corollary 5.3.15 to hold, which ensures that (5.104) admits an asymptotically almost periodic trajectory when subject to asymptotically almost periodic inputs. Numerical simulations are plotted in Figure 5.7. Throughout, we take

$$\rho_1 = \frac{0.9}{\mathbf{G}(1)}, \quad \rho_2 = 2, \quad \rho_3 = 20\,,$$

and with these parameter values it can be shown that f satisfies (5.107). In each of panels (a), (b) and (d) of Figure 5.7, three lines are plotted, which each show the norm of a state trajectory as time varies. For each panel, the three different choices of initial conditions and inputs given in Table 5.2 give rise to the three state trajectories. In the inner and outer subpanels of panel (c), each of the three lines, plots the norm of the difference of two inputs as time varies and the norm of the difference of the two corresponding state trajectories as time varies, respectively. Once again, the details of the initial conditions and inputs are given in Table 5.2. We now briefly give some commentary of the panels of Figure 5.7. In panel (a), we see that in each of the three cases of initial conditions, a 0 input implies exponential convergence of the corresponding state towards 0, as guaranteed by Theorem 5.2.5. In panel (b), we see an ISS-property of state trajectories corresponding to zero initial conditions, and in panel (c), we observe a characteristic of the incremental ISS notion (again guaranteed by Theorem 5.2.5). Finally, panel (d) shows that we obtain asymptotically periodic responses to periodic forcing, thus illustrating Corollary 5.3.15. We note that panel (d) shows that, asymptotically, the responses to the same input are identical and do not depend on the initial conditions, thereby illustrating a typical aspect of ISS. $\diamondsuit$

Figure	Initial condition $x(0)$	Input v
5.7(a)	Random s.t. $\\|x(0)\\| = 4j^2$	0
5.7(b)	0	$v(t)$ random perturbation of $4j^2$
5.7(c)	Random s.t. $\\|x^1(0)\\|, \\|x^2(0)\\| = 4j^2$	$v^1(t)$, $v^2(t)$ convergent to $4j^2$
5.7(d)	Random s.t. $\\|x(0)\\| = 4j^2$	$v(t) = 4\sin(2\pi t/12)$

Table 5.2: *Model parameters used in the numerical simulations in Example 5.6.2, where* $j \in \{1, 2, 3\}$.

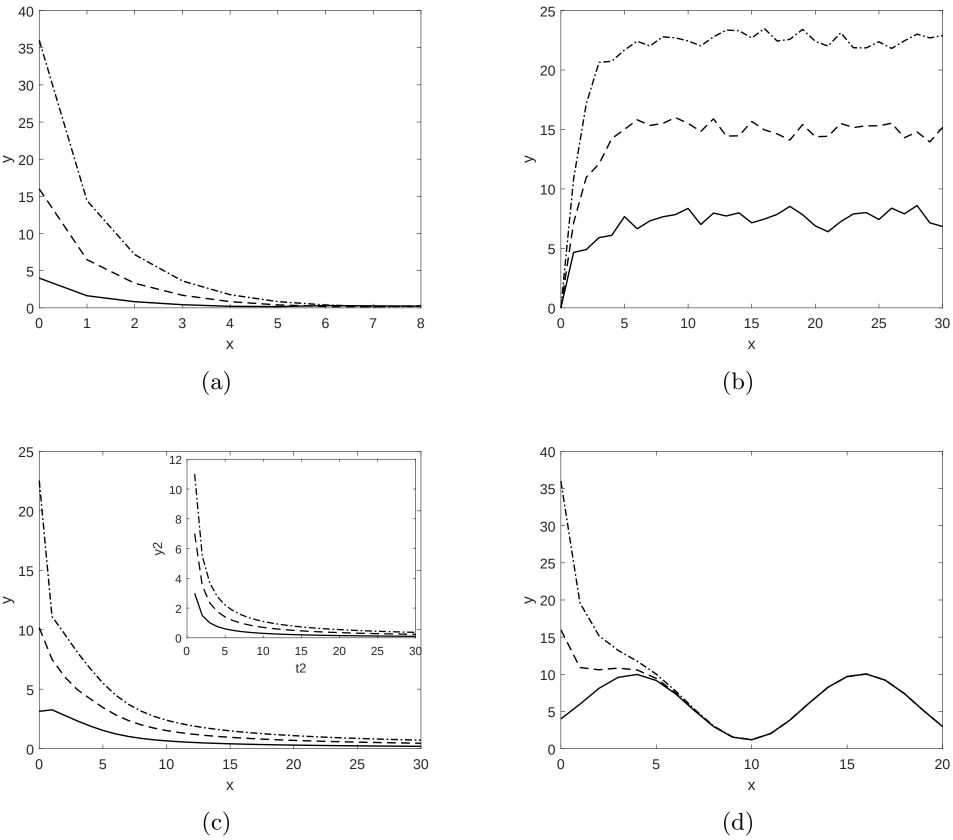

Figure 5.7: *Numerical simulation of the IPM* (5.102) *from Example 5.6.2. In each panel the solid, dashed and dashed-dotted lines correspond to $j = 1, 2, 3$, respectively. The x-axes of all panels correspond to time, and the y-axes of panels (a), (b) and (d) correspond to $\|x(t)\|$. The y-axes for the inner and outer graphs of panel (c) represent $|v^1(t) - v^2(t)|$ and $\|x^1(t) - x^2(t)\|$, respectively. For commentary, see the main body of text.*

5.7 Notes and references

In the main result of this chapter, Theorem 5.2.5, we made use of exponential weighting and small gain arguments. We comment that these methods are not new and refer the reader to works such as [59] and the references therein for further details. The result, and indeed much of the chapter, is based off the work achieved in [40] (of which the author of this thesis co-authored), and may be considered to be a discrete-time analogue of the main contributions of [47], with a few exceptions. Indeed, in [47], the effect of (asymptotically) almost periodic inputs on the state and output trajectories of the relevant system is not considered, and, of course, neither are sampled-data systems.

To the best of the author's knowledge, there is no existing literature that overlaps with any significance with the present chapter. For background literature regarding the stability results of the current chapter, we refer the reader to [5] for a suite of Lyapunov methods for finite-dimensional, continuous-time nonlinear control systems. Moreover, we also refer to works such as [55, 83, 88, 91] for ISS investigations in the infinite-dimensional setting.

The inspiration of the results of Section 5.2.2 orginate from the papers [46, 74]. In [74], the invertibility of positive real operators is investigated, and in [46], characterisations and relationships of different notions of positive realness are presented. Both of these papers are in the infinite-dimensional setting and both define positive realness in the continuous-time format, that is, functions are defined on the open right-half plane. In addition, a result closely related to the corollaries seen in Section 5.2.2 is [57, Theorem 4.5], which deduces ISS of infinite-dimensional continuous-time Lur'e systems.

Some noteworthy papers with results similar to Corollary 5.3.15 that can be found in the literature are [103, 105, 128]. Since a discussion of these papers is given in the Notes and References section of Chapter 3, we refer the reader to Chapter 3 for more details. We shall simply state here that the methods of proofs used in the papers are entirely different to the proof of Corollary 5.3.15, and the continuous-time systems considered are much more restrictive.

Chapter 6

Sampled-data integral control of linear infinite-dimensional systems with input nonlinearities

Low-gain integral control is a well-studied technique that seeks to conclude that the output of a stable linear system converges to a predescribed constant reference signal. It is very well-known (see, for example, [29, 84]) that for a stable, continuous-time linear system in the finite-dimensional setting, by applying an integrator with sufficiently small gain parameter and closing the feedback loop, the corresponding output does indeed converge to the prescribed reference value, so long as the eigenvalues of the transfer function at 0 have positive real part. Much work has been attributed to the investigation of low-gain integral control for more general systems. Indeed, we refer the reader to works such as [23, 25, 65, 80, 89, 90] for integral control results in the infinite-dimensional setting, discrete-time setting and sampled-data setting. In much of the literature (see, for example, [35, 76, 79]), the assumptions imposed to obtain tracking of the output to a reference signal, limit the system to be in the single-input single-output setting. However, recently, in [48], low-gain integral control is investigated for multi-input multi-output, linear, continuous-time, finite-dimensional systems with input nonlinearities. In particular, the authors consider a low-gain integral controller that includes an antiwindup component, and they show that it achieves global exponential tracking for all feasible reference signals, so long as the controller has small enough integrator gain and the nonlinearity exhibits a Lipschitz-like assumption. The inclusion of the so-called antiwindup component of the controller makes it feasible for multi-input multi-output systems. Indeed, the antiwindup component of the controller is implemented to reduce, or even remove, integrator windup [8]. Briefly, integrator windup is a potentially destabilising effect on the tracking of multi-input multi-output integral control systems that have input saturation, and occurs due to controllers being based upon linear assumptions. The theory of antiwindup control, which seeks to remove this effect, is a much researched area. We refer the reader to works such as [13, 37, 120] and the references therein for background reading.

A problem not addressed in the literature is that of sampled-data low-gain integral control of linear infinite-dimensional multi-input multi-output systems with input nonlinearities. Briefly, in sampled-data control, a continuous-time system is controlled by a discrete-time controller, via the use of sample and hold operations. This discrete-

time controller may be thought of, for example, as a processor of a digital computer. The main result of this chapter, Theorem 6.3.5, provides a stability criterion for the sampled-data feedback interconnection of a well-posed linear system (see, for example, [118]) with input nonlinearities, with a discrete-time low-gain integral controller which contains an antiwindup component. Our sampled-data model includes continuous as well as discrete-time external disturbances. In the absence of these disturbances, Theorem 6.3.5 guarantees asymptotic tracking. To prove Theorem 6.3.5, we first extend the finite-dimensional continuous-time results of [48] to infinite-dimensional discrete-time systems that are interconnected with a low-gain integral controller with an antiwindup component, have potentially nonzero feedthrough and include external disturbances, that is, under analogous assumptions of those in [48, Theorem 4], we derive a stability estimate of the difference between the output and reference signal which, in the absence of disturbances, guarantees exponentially fast asymptotic tracking. We comment that the work of this chapter is based of that presented in [42].
The layout of the chapter is as follows. We begin in Section 6.1 by giving some useful preliminary definitions and results. Following this, in Section 6.2, we provide the previously descibed low-gain integral control result which is a discrete-time generalisation of the work of [48]. Then, in Section 6.3, we apply this to obtain the main contribution of this chapter, that is, a sampled-data low-gain integral control result for well-posed infinite-dimensional systems with input nonlinearities.

As in the previous chapter, we assume that the norm on the cartesian product of two normed spaces W_1 and W_2 is given by (5.1). We note that, if W_1 and W_2 are Hilbert spaces, then $W_1 \times W_2$ is itself a Hilbert space when equipped with the inner-product

$$\left\langle \begin{pmatrix} \xi_1 \\ \zeta_1 \end{pmatrix}, \begin{pmatrix} \xi_2 \\ \zeta_2 \end{pmatrix} \right\rangle_{W_1 \times W_2} := \langle \xi_1, \xi_2 \rangle_{W_1} + \langle \zeta_1, \zeta_2 \rangle_{W_2} \quad \forall\, \xi_1, \xi_2 \in W_1,\ \forall\, \zeta_1, \zeta_2 \in W_2.$$

6.1 Preliminaries

We begin this section by recalling the following definitions.

Definition 6.1.1. *Let $\alpha \in \mathbb{R}$ and W be a Hilbert space.*

(i) *We define the* exponentially weighted L^2-space $L^2_\alpha(\mathbb{R}_+, W)$ *by*

$$L^2_\alpha(\mathbb{R}_+, W) := \left\{ w \in L^2_{loc}(\mathbb{R}_+, W) : e_{-\alpha} w \in L^2(\mathbb{R}_+, W) \right\},$$

with norm $\|w\|_{L^2_\alpha} := \|e_{-\alpha} w\|_{L^2}$ and where, for $\beta \in \mathbb{R}$, $e_\beta(t) := e^{\beta t}$ for all $t \in \mathbb{R}_+$.

(ii) *We recall the definition of the Hardy space*

$$H^2(\mathbb{C}_\alpha, W) := \left\{ h : \mathbb{C}_\alpha \to W \ : \ h \text{ is holomorphic and } \sup_{x > \alpha} \left(\int_{-\infty}^{\infty} \|h(x+iy)\|_W^2 dy \right) < \infty \right\}.$$

We shall also give the definition of discrete- and continuous-time exponential convergence.

Definition 6.1.2. *Let W be a normed space. For $u \in W^{\mathbb{Z}_+}$ and $\tilde{u} \in W$, we say that $u(k) \to \tilde{u}$ as $k \to \infty$* (discrete-time) exponentially *if there exists $\delta > 0$ such that $k \mapsto e^{\delta k}\|u(k) - \tilde{u}\|_W$ is bounded. Similarly, for $v : \mathbb{R}_+ \to W$ and $\tilde{v} \in W$, we say that $v(t) \to \tilde{v}$ as $t \to \infty$* (continuous-time) exponentially *if there exists $\delta > 0$ such that $t \mapsto e^{\delta t}\|v(t) - \tilde{v}\|_W$ is bounded.*

We now concern ourselves with the invertibility of triangular operator matrices.

Lemma 6.1.3. *Let $(L_1, L_2, L_3, L_4) \in \mathcal{L}(W_1) \times \mathcal{L}(W_2, W_1) \times \mathcal{L}(W_1, W_2) \times \mathcal{L}(W_2)$, where W_1 and W_2 are normed spaces. If $L, M \in \mathcal{L}(W_1 \times W_2)$ are defined by*

$$L := \begin{pmatrix} L_1 & L_2 \\ 0 & L_4 \end{pmatrix} \quad \text{and} \quad M := \begin{pmatrix} L_1 & 0 \\ L_3 & L_4 \end{pmatrix},$$

then

$$\sigma(L) \subseteq \sigma(L_1) \cup \sigma(L_4) \quad \text{and} \quad \sigma(M) \subseteq \sigma(L_1) \cup \sigma(L_4).$$

Lemma 6.1.3 is a well-known result, and for a proof we refer the reader to the proof of [10, Lemma 3], for example. Interestingly, in the finite-dimensional setting, it is known that the spectrum of a triangular block matrix is equal to the union of the spectra of the diagonal matrices. The following example proves that this is not true in the infinite-dimensional case, and so we cannot expect more from Lemma 6.1.3, without any extra assumptions.

Example 6.1.4. Let $W_1 = W_2 = \ell^\infty(\mathbb{Z}_+, \mathbb{R})$. Define $(L_1, L_2, L_4) \in \mathcal{L}(W_1) \times \mathcal{L}(W_2, W_1) \times \mathcal{L}(W_2)$ by

$$(L_1 w)(t) := \begin{cases} 0, & \text{if } t = 0 \\ w(t-1), & \text{if } t \in \mathbb{N} \end{cases}, \quad (L_2 w)(t) := \begin{cases} w(0), & \text{if } t = 0 \\ 0, & \text{if } t \in \mathbb{N} \end{cases},$$

and $(L_4 w)(t) := w(t+1)$ for all $t \in \mathbb{Z}_+$, and all $w \in \ell^\infty(\mathbb{Z}_+, \mathbb{R})$. It is easily seen that L_1 is not surjective and L_4 is not injective. However, we claim that

$$L := \begin{pmatrix} L_1 & L_2 \\ 0 & L_4 \end{pmatrix}$$

is invertible, which we shall now show. To this end, let $u, v, \tilde{u}, \tilde{v} \in \ell^\infty(\mathbb{Z}_+, \mathbb{R})$ and assume that

$$L \begin{pmatrix} u \\ v \end{pmatrix} = L \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix}.$$

As a consequence of this, $L_4 v = L_4 \tilde{v}$ and

$$L_1 u + L_2 v = L_1 \tilde{u} + L_2 \tilde{v}. \tag{6.1}$$

From this, we see that $u(t-1) = \tilde{u}(t-1)$ for all $t \in \mathbb{N}$, that is, $u = \tilde{u}$. Moreover, since $L_4 v = L_4 \tilde{v}$, we see that $v(t+1) = \tilde{v}(t+1)$ for all $t \in \mathbb{Z}_+$. By combining this with (6.1), we see that also $v(0) = \tilde{v}(0)$ and so $v = \tilde{v}$.

As for surjectivity, let $u, v \in \ell^\infty(\mathbb{Z}_+, \mathbb{R})$. Define $\tilde{u}, \tilde{v} \in \ell^\infty(\mathbb{Z}_+, \mathbb{R})$ by $\tilde{u}(t) := u(t+1)$ for all $t \in \mathbb{Z}_+$ and

$$\tilde{v}(t) := \begin{cases} u(0), & \text{if } t = 0 \\ v(t-1), & \text{if } t \in \mathbb{N}. \end{cases}$$

It is then easily checked that

$$L \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix},$$

which proves that L is surjective. ◊

6.2 Low-gain integral control in discrete-time

In this section, we give a low-gain integral control result for infinite-dimensional, discrete-time linear systems with input nonlinearities and external disturbances. We proceed to outline the mathematical formulation. Throughout this section, we let X and V be complex Banach spaces and Y and U be complex Hilbert spaces. We consider the discrete-time system given by

$$\left.\begin{aligned} x^+ &= Ax + B\phi(u) + B_\mathrm{e}v, \quad x(0) = x^0 \in X, \\ y &= Cx + D\phi(u) + D_\mathrm{e}v, \end{aligned}\right\} \tag{6.2}$$

where

$$(A, B, B_\mathrm{e}, C, D, D_\mathrm{e}) \in \mathcal{L}(X) \times \mathcal{L}(U, X) \times \mathcal{L}(V, X) \times \mathcal{L}(X, Y) \times \mathcal{L}(U, Y) \times \mathcal{L}(V, Y),$$

$u \in U^{\mathbb{Z}_+}$, $v \in V^{\mathbb{Z}_+}$, $\phi : U \to U$ and A is exponentially stable (see Definition 5.1.1). Keeping with the theme of this thesis, x in (6.2) will be called the state, y the output, and u and v forcing, inputs or disturbances. For ease of notation, we shall employ notation used in the previous chapter (see Definition 5.1.8). To this end, we set

$$\mathbb{L} = \mathcal{L}(X) \times \mathcal{L}(U, X) \times \mathcal{L}(V, X) \times \mathcal{L}(X, Y) \times \mathcal{L}(U, Y) \times \mathcal{L}(V, Y),$$

and, for $(A, B, B_\mathrm{e}, C, D, D_\mathrm{e}) \in \mathbb{L}$, we define

$$\Sigma = (A, B, B_\mathrm{e}, C, D, D_\mathrm{e}).$$

Moreover, we denote the transfer function of (6.2) by $\mathbf{G}$ (see Definition 5.1.10), that is,

$$\mathbf{G}(z) = C(zI - A)^{-1}B + D,$$

for all $z \in \mathbb{C}$ for which this makes sense.

Remark 6.2.1. For $\Sigma \in \mathbb{L}$, the generality of the operators B_e and D_e means that (6.2) encompasses many (even seemingly more general) systems. We shall not go into detail here, since this has already been discussed in previous chapters (see Remark 2.1.10). ♢

We aim to apply low-gain integral control to (6.2) so that the output, when $v = 0$, converges to some constant reference signal $r \in Y$. With this in mind, we give the following definitions.

Definition 6.2.2. *Let $\Sigma \in \mathbb{L}$ be such that A is exponentially stable and let $\phi : U \to U$.*

(i) *We say that $r \in Y$ is* feasible *(with respect to* (6.2)*), if the set*

$$U^r := \{w \in U : \mathbf{G}(1)\phi(w) = r\},$$

is non-empty.

(ii) *We say that a set $R \subseteq Y$ is* feasible *(with respect to* (6.2)*) if every $r \in R$ is feasible.*

Some immediate implications concerning feasibility are presented in the following remark.

Remark 6.2.3. Let $\Sigma \in \mathbb{L}$ be such that A is exponentially stable and let $\phi : U \to U$. If $r \in Y$ is feasible, then by taking $w \in U^r$ and setting $u(t) := w$ for all $t \in \mathbb{Z}_+$, it is easily seen that, for $(x, y) \in X^{\mathbb{Z}_+} \times Y^{\mathbb{Z}_+}$ satisfying (6.2) with $v = 0$, it follows that $y(t) \to r$ as $t \to \infty$. $\Diamond$

Given $\Sigma \in \mathbb{L}$ such that A is exponentially stable, $\phi : U \to U$, $R \subseteq Y$ feasible, $r \in R$ and $u^r \in U^r$, consider the control law

$$u^+ = u + gK(r - y) - g\Gamma(u - \phi(u) - u^r + \phi(u^r)) + w, \quad u(0) = u^0 \in U, \tag{6.3}$$

where $g > 0$, $K \in \mathcal{L}(Y, U)$ and $\Gamma \in \mathcal{L}(U)$ are design parameters and $w \in U^{\mathbb{Z}_+}$ is a disturbance.

Remark 6.2.4. We denote the term $g\Gamma(u - \phi(u) - u^r + \phi(u^r))$ in (6.3) as the *anti-windup component of the controller* [48]. As already mentioned, it seeks to mitigate against *integrator windup* when under input saturation. For more details on this, we refer the reader to the preamble given at the start of the present chapter. Finally, we highlight that if $\Gamma = 0$ or $\phi = \mathrm{id}$, then (6.3) becomes 'standard' integral control (subject to a disturbance w). $\Diamond$

Our focus in this section is the feedback interconnection of (6.2) and (6.3), which yields the closed-loop system

$$\left.\begin{aligned} x^+ &= Ax + B\phi(u) + B_{\mathrm{e}}v, \quad x(0) = x^0 \in X, \\ y &= Cx + D\phi(u) + D_{\mathrm{e}}v, \\ u^+ &= u + gK(r - y) - g\Gamma(u - \phi(u) - u^r + \phi(u^r)) + w, \quad u(0) = u^0 \in U. \end{aligned}\right\} \tag{6.4}$$

The following result is the main theorem of this section and provides a low-gain integral control result for (6.4).

Theorem 6.2.5. *Let $\Sigma \in \mathbb{L}$ be such that A is exponentially stable, $\phi : U \to U$, $K \in \mathcal{L}(Y, U)$, $\Gamma \in \mathcal{L}(U)$ be such that $I - \Gamma$ is exponentially stable, and $R \subseteq Y$ feasible. Assume that there exists $L > 0$ such that*

$$\|\phi(\xi + \zeta) - \phi(\zeta)\|_U \le L\|\xi\|_U \quad \forall\, \xi \in U, \forall\, \zeta \in \bigcup_{r \in R} U^r, \tag{6.5}$$

and

$$\sup_{z \in \mathbb{E}} \|(zI - (I - \Gamma))^{-1}\| \|\Gamma - K\mathbf{G}(1)\| < 1/L. \tag{6.6}$$

Then, there exists $g^* \in (0, 1]$ such that, for all $g \in (0, g^*)$, there exist constants $c, d > 0$ and $\theta \in (0, 1)$ such that, for all $r \in R$, $u^r \in U^r$ and $(u, w, v, x, y) \in U^{\mathbb{Z}_+} \times U^{\mathbb{Z}_+} \times V^{\mathbb{Z}_+} \times X^{\mathbb{Z}_+} \times Y^{\mathbb{Z}_+}$ satisfying (6.4),

$$\left\| \begin{pmatrix} x(t) - (I - A)^{-1}B\phi(u^r) \\ u(t) - u^r \end{pmatrix} \right\| \le c\left(\theta^t \left\| \begin{pmatrix} x^0 - (I - A)^{-1}B\phi(u^r) \\ u^0 - u^r \end{pmatrix} \right\| + \max_{s \in \underline{t-1}} \left\| \begin{pmatrix} v(s) \\ w(s) \end{pmatrix} \right\| \right) \quad \forall\, t \in \mathbb{N}, \tag{6.7}$$

$$\|y(t) - r\| \le d\left(\theta^t \left\| \begin{pmatrix} x^0 - (I - A)^{-1}B\phi(u^r) \\ u^0 - u^r \end{pmatrix} \right\| + \max_{s \in \underline{t}} \left\| \begin{pmatrix} v(s) \\ w(s) \end{pmatrix} \right\| \right) \quad \forall\, t \in \mathbb{Z}_+. \tag{6.8}$$

Before proving Theorem 6.2.5, we provide some commentary in the following remark.

Remark 6.2.6. (i) We note that Theorem 6.2.5 guarantees that, under zero forcing, $u(t) \to u^r$, $x(t) \to (I - A)^{-1}B\phi(u^r)$ and $y(t) \to r$ as $t \to \infty$ and each with exponential rate of convergence - see Definition 6.1.2.

(ii) Assumption (6.5) is evidently satisfied if ϕ is globally Lipschitz with Lipschitz constant L. Moreover, if there exist distinct $u_1, u_2 \in U$ such that $\phi(u_1) = u_1$, $\phi(u_2) = u_2$ and at least one of u_1 and u_2 is in U^r for some $r \in R$, then an immediate consequence of (6.5) is that $L \geq 1$.

(iii) The assumptions of Theorem 6.2.5 imply that any $u^r \in U^r$ that satisfies $u^r = \phi(u^r)$ is in fact unique. To see this, suppose that $w^r \in U^r$ also satisfies $w^r = \phi(w^r)$. From (6.5) we see that

$$\|u^r - w^r\|_U = \|\phi(u^r) - \phi(w^r)\|_U \leq L\|u^r - w^r\|_U,$$

which in turn yields that either $u^r = w^r$ or $L \geq 1$. In the latter case, (6.6) gives that

$$\|\Gamma - K\mathbf{G}(1)\| < \frac{1}{\sup_{z\in\mathbb{E}} \|(zI - (I - \Gamma))^{-1}\|}.$$

Therefore, from Lemma 5.1.22 (see also Remark 5.1.23), $\Gamma - K\mathbf{G}(1) \in \mathbb{S}(\mathbf{H})$, where $\mathbf{H}(z) = (zI - (I - \Gamma))^{-1}$ for all $z \in \mathbb{E}$. Since $I - \Gamma$ is exponentially stable, Lemma 5.1.21 gives that

$$\sigma(I - K\mathbf{G}(1)) = \sigma((I - \Gamma) + (\Gamma - K\mathbf{G}(1))) \subseteq \mathbb{D},$$

which in turn implies that $K\mathbf{G}(1)$ is invertible. Finally, by definition of $u^r, w^r \in U^r$,

$$K\mathbf{G}(1)u^r = K\mathbf{G}(1)\phi(u^r) = Kr = K\mathbf{G}(1)\phi(w^r) = K\mathbf{G}(1)w^r,$$

which shows that $u^r = w^r$.

(iv) Theorem 6.2.5 remains valid if the spaces V, U, X and Y are real, provided that the estimate (6.6) holds for the canonical extensions of Γ and $K\mathbf{G}(1)$ to the complexifications U^{c} and Y^{c} of U and Y, respectively.

(v) In the situation wherein $U = \mathbb{R}^m$ and $Y = \mathbb{R}^p$ for some $m, p \in \mathbb{N}$, if $L \geq 1$, then part (iii) of this remark shows that $K\mathbf{G}(1)$ is invertible. From this, we deduce that $\operatorname{rk}\mathbf{G}(1) = m$ is a necessary condition for the assumptions of Theorem 6.2.5 to hold.

(vi) The assumption (6.6) is trivially satisfied for any $L > 0$, if we choose K as a left inverse of $\mathbf{G}(1)$ and $\Gamma = I$. Such a choice naturally requires knowledge of $\mathbf{G}(1)$ to be implemented, although the condition (6.6) carries robustness with respect to parametric uncertainty in $\mathbf{G}(1)$. ◊

In order to prove Theorem 6.2.5, we first require the following lemma, which is a discrete-time version of [48, Lemma 6], extended to the case of potentially nonzero feedthrough and to the infinite-dimensional setting.

Lemma 6.2.7. *Let $\Sigma \in \mathbb{L}$, $K \in \mathcal{L}(Y,U)$ and $\Gamma \in \mathcal{B}(U)$. Assume that A and $I-\Gamma$ are exponentially stable. For $g>0$, we define*

$$\tilde{A} := \begin{pmatrix} A & 0 \\ -gKC & I-g\Gamma \end{pmatrix}, \quad \tilde{B} := \begin{pmatrix} B \\ g(\Gamma - KD) \end{pmatrix}, \quad \tilde{C} := \begin{pmatrix} 0 & I \end{pmatrix}, \tag{6.9}$$

and denote by $\tilde{\mathbf{G}}$ the transfer function of $(\tilde{A},\tilde{B},\tilde{C})$, that is, $\tilde{\mathbf{G}}(z) = \tilde{C}(zI-\tilde{A})^{-1}\tilde{B}$. The following the statements hold.

(i) *For each $g \in (0,1]$, $\tilde{A}$ is exponentially stable.*

(ii) *For all $\varepsilon > 0$, there exists $g^* \in (0,1]$ such that, for all $g \in (0,g^*)$,*

$$\|\tilde{\mathbf{G}}\|_{H^\infty} \le \varepsilon + \sup_{z\in\mathbb{E}} \|(zI-(I-\Gamma))^{-1}\|\|\Gamma - K\mathbf{G}(1)\|. \tag{6.10}$$

The following proof is inspired by the proof of [48, Lemma 6].

Proof of Lemma 6.2.7. To prove statement (i), fix $g \in (0,1]$. We begin with an application of Lemma 6.1.3 to obtain that $\sigma(\tilde{A}) \subseteq \sigma(A) \cup \sigma(I-g\Gamma)$. Thus, if we show that $I-g\Gamma$ is exponentially stable, the exponential stability of A will yield the exponential stability of $\tilde{A}$. Indeed, this is what we shall now show. For which purpose, note that, for $\lambda \in \mathbb{C}$,

$$\lambda I - (I-g\Gamma) = (\lambda-1)I + g\Gamma = g\left(\frac{\lambda-1}{g}I + \Gamma\right) = g\left(\left(\frac{\lambda-1}{g}+1\right)I - (I-\Gamma)\right).$$

Hence, $\lambda \in \sigma(I-g\Gamma)$ if, and only if, $((\lambda-1)/g+1) \in \sigma(I-\Gamma)$. By combining this with the fact that $I-\Gamma$ is exponentially stable, we see that if $\lambda \in \sigma(I-g\Gamma)$, then $((\lambda-1)/g+1) \in \mathbb{D}$. This in turn implies that $(\lambda-1)/g \in \mathbb{B}_{\mathbb{C}}(-1,1)$. For such a λ, it is easily checked that since $g \le 1$, then $\lambda - 1 \in \mathbb{B}_{\mathbb{C}}(-1,1)$ and hence $\lambda \in \mathbb{D}$. We have thus shown that $\sigma(I-g\Gamma) \subseteq \mathbb{D}$, i.e. that $I-g\Gamma$ is exponentially stable. As previously mentioned, from Lemma 6.1.3, the exponential stability of A and $I-g\Gamma$ yields the exponential stability of $\tilde{A}$.
For statement (ii), fix $\varepsilon > 0$ and note that, for all $z \in \mathbb{E}$ and all $g \in (0,1]$,

$$\begin{aligned}\tilde{\mathbf{G}}(z) &= \begin{pmatrix} 0 & I \end{pmatrix} \begin{pmatrix} zI-A & 0 \\ gKC & zI-(I-g\Gamma) \end{pmatrix}^{-1} \begin{pmatrix} B \\ g(\Gamma-KD) \end{pmatrix} \\ &= \begin{pmatrix} 0 & I \end{pmatrix} \begin{pmatrix} (zI-A)^{-1} & 0 \\ -(zI-(I-g\Gamma))^{-1}gKC(zI-A)^{-1} & (zI-(I-g\Gamma))^{-1} \end{pmatrix} \begin{pmatrix} B \\ g(\Gamma-KD) \end{pmatrix} \\ &= g(zI-(I-g\Gamma))^{-1}(\Gamma - K\mathbf{G}(z)).\end{aligned}$$

We write $\tilde{\mathbf{G}} = \mathbf{H}_1 + \mathbf{H}_2$, where, for all $z \in \mathbb{E}$ and all $g \in (0,1]$,

$$\begin{aligned}\mathbf{H}_1(z) &:= g(zI-(I-g\Gamma))^{-1}K(\mathbf{G}(1)-\mathbf{G}(z)), \\ \mathbf{H}_2(z) &:= -g(zI-(I-g\Gamma))^{-1}(K\mathbf{G}(1)-\Gamma).\end{aligned}$$

We highlight that for all $g \in (0,1]$, we have $\mathbf{H}_1, \mathbf{H}_2 \in H^\infty(\mathcal{L}(U))$. We claim that there exists $g^* \in (0,1]$ such that

$$\|\mathbf{H}_1\|_{H^\infty} \le \varepsilon \quad \forall\, g \in (0,g^*), \tag{6.11}$$

and

$$\|\mathbf{H}_2\|_{H^\infty} \leq \sup_{z \in \mathbb{E}} \|(zI - (I - \Gamma))^{-1}\| \|\Gamma - K\mathbf{G}(1)\| \quad \forall\, g \in (0,1]. \tag{6.12}$$

Since $\tilde{\mathbf{G}} = \mathbf{H}_1 + \mathbf{H}_2$, the desired estimate (6.10) follows from the conjunction of (6.11) and (6.12). Before proving these equations, we record that

$$\frac{z-1}{g} + 1 \in \mathbb{E} \quad \forall\, z \in \mathbb{E}, \forall\, g \in (0,1], \tag{6.13}$$

which follows from the estimates

$$\left|\frac{z-1}{g} + 1\right| \geq \left|\frac{z-1}{g} + \frac{1}{g}\right| - \left|\frac{1}{g} - 1\right| = \frac{|z|}{g} - \frac{1}{g} + 1 > 1 \quad \forall\, z \in \mathbb{E}, \forall\, g \in (0,1].$$

Consequently, from the exponential stability of $I - \Gamma$, Lemma 5.1.11 yields that

$$\sup_{z \in \mathbb{E}} \left\| \left(\left(\frac{z-1}{g} + 1 \right) I - (I - \Gamma) \right)^{-1} \right\| = M_1 < \infty \quad \forall\, g \in (0,1]. \tag{6.14}$$

To establish (6.11), we express $\mathbf{H}_1$ as

$$\mathbf{H}_1(z) = g \left(I + \frac{g}{z-1}\Gamma \right)^{-1} K \frac{\mathbf{G}(1) - \mathbf{G}(z)}{z-1} \quad \forall\, z \in \mathbb{E}, \forall\, g \in (0,1]. \tag{6.15}$$

We claim that there exists $M > 0$ such that

$$\left\| \left(I + \frac{g}{z-1}\Gamma \right)^{-1} \right\| \leq M \quad \forall\, z \in \mathbb{E}, \forall\, g \in (0,1]. \tag{6.16}$$

To show this, fix $\rho > 1$ and let $g \in (0,1]$ and $z \in \mathbb{E}$. If $|z-1| \leq \rho\|\Gamma\|g$, then, from (6.14),

$$\begin{aligned} \left\| \left(I + \frac{g}{z-1}\Gamma \right)^{-1} \right\| &= \frac{|z-1|}{g} \left\| \left(\left(\frac{z-1}{g} + 1 \right) I - (I - \Gamma) \right)^{-1} \right\| \\ &\leq \rho\|\Gamma\| M_1. \end{aligned}$$

If instead $|z-1| \geq \rho\|\Gamma\|g$, then

$$\left\| \frac{g}{z-1}\Gamma \right\| = g\frac{\|\Gamma\|}{|z-1|} \leq \frac{1}{\rho} < 1.$$

Whence, from an application of Lemma 5.1.6,

$$\begin{aligned} \left\| \left(I + \frac{g}{z-1}\Gamma \right)^{-1} \right\| &= \left\| \sum_{j=0}^{\infty} (-\frac{g}{z-1}\Gamma)^j \right\| \\ &\leq \sum_{j=0}^{\infty} \left\| \frac{g}{z-1}\Gamma \right\|^j \\ &\leq \frac{\rho}{\rho - 1} \\ &=: M_2. \end{aligned}$$

Setting $M := \max\{\rho\|\Gamma\|M_1, M_2\}$ gives (6.16). A combination of (6.15) and (6.16) yields that

$$\|\mathbf{H}_1(z)\| \leq gM\|\mathbf{J}\|_{H^\infty} \quad \forall\, z \in \mathbb{E},\, \forall\, g \in (0,1], \tag{6.17}$$

where

$$\mathbf{J}(z) := \begin{cases} K\frac{\mathbf{G}(z)-\mathbf{G}(1)}{z-1}, & \text{if } z \neq 1 \\ K\mathbf{G}'(1), & \text{if } z = 1. \end{cases}$$

The bound (6.11) now follows from (6.17) by taking $g^* := \min\{1, \varepsilon/(M\|\mathbf{J}\|_{H^\infty})\}$.
To establish (6.12), we note that

$$\begin{aligned} \|\mathbf{H}_2(z)\| &\leq \left\| \left(\frac{z-1}{g}I + \Gamma\right)^{-1} \right\| \|\Gamma - K\mathbf{G}(1)\| \\ &= \left\| \left(\left(\frac{z-1}{g} + 1\right) I - (I - \Gamma) \right)^{-1} \right\| \|\Gamma - K\mathbf{G}(1)\| \\ &\leq \sup_{\lambda \in \mathbb{E}} \|(\lambda I - (I-\Gamma))^{-1}\| \|\Gamma - K\mathbf{G}(1)\| \quad \forall\, z \in \mathbb{E},\, \forall\, g \in (0,1], \end{aligned}$$

where the final inequality invokes (6.13). This then yields (6.12) and completes the proof. □

We are now in a position to prove Theorem 6.2.5.

Proof of Theorem 6.2.5. To begin with, we note that (6.6) implies that we may choose $\varepsilon > 0$ so that

$$\varepsilon + \sup_{z \in \mathbb{E}} \|(zI - (I-\Gamma))^{-1}\| \|\Gamma - K\mathbf{G}(1)\| < 1/L. \tag{6.18}$$

We may apply Lemma 6.2.7 to obtain the existence of $g^* \in (0,1]$ such that, for all $g \in (0, g^*)$, it follows that (6.10) holds. With this in mind, we let $r \in R$, $u^r \in U^r$, $g \in (0, g^*)$, and $(u, w, v, x, y) \in U^{\mathbb{Z}_+} \times U^{\mathbb{Z}_+} \times V^{\mathbb{Z}_+} \times X^{\mathbb{Z}_+} \times Y^{\mathbb{Z}_+}$ satisfy (6.4). Define

$$\tilde{x} := x - (I-A)^{-1}B\phi(u^r), \quad \tilde{y} := y - r \quad \text{and} \quad \tilde{u} := u - u^r.$$

In follows that

$$\begin{aligned} \tilde{x}^+ &= Ax + B\phi(u) + B_{\mathrm{e}}v - (I-A)^{-1}B\phi(u^r) \\ &= A\tilde{x} - (I-A)(I-A)^{-1}B\phi(u^r) + B\phi(u) + B_{\mathrm{e}}v \\ &= A\tilde{x} + B(\phi(\tilde{u} + u^r) - \phi(u^r)) + B_{\mathrm{e}}v \\ &= A\tilde{x} + B\phi_{u^r}(\tilde{u}) + B_{\mathrm{e}}v, \end{aligned} \tag{6.19}$$

where $\phi_{u^r}(\tilde{u}) = \phi(\tilde{u} + u^r) - \phi(u^r)$. Furthermore, noting that $r = \mathbf{G}(1)\phi(u^r)$, we see that

$$\begin{aligned} \tilde{y} &= Cx + D\phi(u) + D_{\mathrm{e}}v - r \\ &= C\tilde{x} + C(I-A)^{-1}B\phi(u^r) + D\phi(u) + D_{\mathrm{e}}v - \mathbf{G}(1)\phi(u^r) \\ &= C\tilde{x} + D\phi_{u^r}(\tilde{u}) + D_{\mathrm{e}}v. \end{aligned} \tag{6.20}$$

By combining this with (6.19), we obtain that

$$\begin{aligned}\tilde{u}^{+} &= u + gK(r-y) - g\Gamma(u - \phi(u) - u^{r} + \phi(u^{r})) + w - u^{r}\\ &= \tilde{u} - gKC\tilde{x} - gKD\phi_{u^{r}}(\tilde{u}) - gKD_{\mathrm{e}}v - g\Gamma(u - \phi(u) - u^{r} + \phi(u^{r})) + w\\ &= (I - g\Gamma)\tilde{u} - gKC\tilde{x} + g(\Gamma - KD)\phi_{u^{r}}(\tilde{u}) - gKD_{\mathrm{e}}v + w. \qquad (6.21)\end{aligned}$$

We may re-write (6.19) and (6.21) as

$$\begin{aligned}\begin{pmatrix}\tilde{x}\\ \tilde{u}\end{pmatrix}^{+} &= \begin{pmatrix}A & 0\\ -gKC & I - g\Gamma\end{pmatrix}\begin{pmatrix}\tilde{x}\\ \tilde{u}\end{pmatrix} + \begin{pmatrix}B\\ g(\Gamma - KD)\end{pmatrix}\phi_{u^{r}}\left(\begin{pmatrix}0 & I\end{pmatrix}\begin{pmatrix}\tilde{x}\\ \tilde{u}\end{pmatrix}\right)\\ &\quad + \begin{pmatrix}B_{\mathrm{e}}v\\ -gKD_{\mathrm{e}}v + w\end{pmatrix}.\end{aligned}$$

By defining $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$ as in (6.9), this becomes

$$\begin{pmatrix}\tilde{x}\\ \tilde{u}\end{pmatrix}^{+} = \tilde{A}\begin{pmatrix}\tilde{x}\\ \tilde{u}\end{pmatrix} + \tilde{B}\phi_{u^{r}}\left(\tilde{C}\begin{pmatrix}\tilde{x}\\ \tilde{u}\end{pmatrix}\right) + \begin{pmatrix}B_{\mathrm{e}} & 0\\ -gKD_{\mathrm{e}} & I\end{pmatrix}\begin{pmatrix}v\\ w\end{pmatrix}. \qquad (6.22)$$

The estimate (6.7) follows from an application of Corollary 5.2.8 to the system (6.22). We thus shall verify that the relevant hypotheses are satisfied. To this end, (6.18) combined with (6.10) yields that

$$\|\tilde{\mathbf{G}}\|_{H^{\infty}} < 1/L,$$

where $\tilde{\mathbf{G}}$ the transfer function of $(\tilde{A}, \tilde{B}, \tilde{C})$. Finally, since, from (6.5), $\phi_{u^{r}}$ satisfies

$$\|\phi_{u^{r}}(\xi)\|_{U} \leq L\|\xi\|_{U} \quad \forall\, \xi \in U,$$

we see that the hypotheses of Corollary 5.2.8 are satisfied, which in turn then yields the existence of constants $c, d > 0$ and $\theta \in (0,1)$ such that (6.7) holds. By thoroughly inspecting the proof of Theorem 5.2.5, we are able to deduce that c, d and θ are independent of $\phi_{u^{r}}$, and hence are therefore independent of u, w, v, x, y, r and u^{r}. As for (6.8), we shall use (6.5) and (6.20) to obtain that, for all $t \in \mathbb{Z}_{+}$,

$$\begin{aligned}\|y(t) - r\| &\leq \|C\|\|x(t) - (I-A)^{-1}B\phi(u^{r})\| + \|D\|\|\phi(u(t)) - \phi(u^{r})\| + \|D_{\mathrm{e}}\|\|v(t)\|\\ &\leq 2(\|C\| + L\|D\|)\left\|\begin{pmatrix}x(t) - (I-A)^{-1}B\phi(u^{r})\\ u(t) - u^{r}\end{pmatrix}\right\| + \|D_{\mathrm{e}}\|\left\|\begin{pmatrix}v(t)\\ w(t)\end{pmatrix}\right\|,\end{aligned}$$

and then combine this with (6.7). This yields that (6.8) holds with $d := 2(\|C\| + L\|D\|)c + \|D_{\mathrm{e}}\|$, thus completing the proof. □

6.3 Sampled-data integral control

In this section, we apply Theorem 6.2.5 in the context of sampled-data low-gain integral control of well-posed linear systems. In the interest of brevity, we will be brief in our setup, since there is much literature concerning these systems. Indeed, we refer the reader to the references [56, 101, 102, 116, 117, 118, 122, 123, 124] for more details.

Throughout, we consider an L^2-well-posed system with state space X, input space $U \times U_{\mathrm{e}}$, output space Y (all Hilbert spaces), generating operators $(A, (B B_{\mathrm{e}}), C)$ and transfer function $(\mathbf{H}\ \mathbf{H}_{\mathrm{e}})$. Control inputs will act through B, whilst external disturbances will act through B_{e}. By definition, A is the generator of a strongly continuous semigroup $\mathbf{T}$ on X (see Definition 5.5.1), $(B\ B_{\mathrm{e}}) \in \mathcal{L}(U \times U_{\mathrm{e}}, X_{-1})$ and $C \in \mathcal{L}(X_1, Y)$, where X_1 is the domain of A endowed with the graph norm $\|x\|_1 := \|x\| + \|Ax\|$, and X_{-1} is the completion of X with respect to the norm $\|x\|_{-1} := \|(\beta I - A)^{-1}x\|$, where β is in the resolvent set of A. We recall here that the choice of β is unimportant, since a different choice leads to equivalent norms. It is clear that $X_1 \subset X \subset X_{-1}$ and that the canonical injections are dense. It is well-known that the semigroup $\mathbf{T}$ restricts to a strongly continuous semigroup on X_1 and extends to a strongly continuous semigroup on X_{-1}, with the exponential growth constants being the same on all three spaces X_1, X and X_{-1}. It is also true that the generator of the restricted semigroup is a restriction of A, and the generator of the extended semigroup is an extension of A. We shall use the same symbols, $\mathbf{T}$ and A, for the restriction and extension of the semigroups and their generators, respectively.

The operator $(B\ \ B_{\mathrm{e}})$ is an admissible control operator, that is, for every $t \geq 0$ there exists $b_t \geq 0$ such that

$$\left\| \int_0^t \mathbf{T}(t-s)(B\ B_{\mathrm{e}})v(s)ds \right\| \leq b_t \|v\|_{L^2} \quad \forall\, v \in L^2([0,t], U \times U_{\mathrm{e}}),$$

and C is an admissible observation operator, that is, for every $t \geq 0$ there exists $c_t \geq 0$ such that

$$\int_0^t \|C\mathbf{T}(t)\xi\|^2 dt \leq c_t \|\xi\|^2 \quad \forall\, \xi \in X_1.$$

We highlight that the admissibility of $(B\ B_{\mathrm{e}})$ implies that $(sI - A)^{-1}(B\ B_{\mathrm{e}}) \in \mathcal{L}(U \times U_{\mathrm{e}}, X)$ for every s in the resolvent set of A. We define the Λ-*extension of* C as

$$C_\Lambda \xi := \lim_{\lambda \to \infty} C\lambda(\lambda I - A)^{-1}\xi \quad \forall\, \xi \in \mathrm{dom}(C_\Lambda),$$

where $\mathrm{dom}(C_\Lambda)$ is the set of all $\xi \in X$ such that the limit exists. We note that $X_1 \subseteq \mathrm{dom}(C_\Lambda)$. Moreover, for all $\xi \in X$, it follows that $\mathbf{T}(t)\xi \in \mathrm{dom}(C_\Lambda)$ for almost all $t \geq 0$, and $C_\Lambda \mathbf{T}\xi \in L^2_\alpha(\mathbb{R}_+, Y)$ for all $\alpha > \omega(\mathbf{T})$, where recall $\omega(\mathbf{T})$ denotes the exponential growth constant of $\mathbf{T}$ (see Definition 5.5.1). Furthermore, the transfer function is such that

$$(\mathbf{H}\ \ \mathbf{H}_{\mathrm{e}}) \in \{g : \mathbb{C}_\alpha \to \mathcal{L}(U \times U_{\mathrm{e}}, Y) : g \text{ is holomorphic and bounded}\} \quad \forall\, \alpha > \omega(\mathbf{T}).$$

Finally, throughout this section, we assume that $\mathbf{T}$ is exponentially stable (see again Definition 5.5.1).

For given $\phi : U \to U$ globally Lipschitz, we shall consider the continuous-time system of the form

$$\left.\begin{aligned} &\dot{x} = Ax + B\phi(w) + B_{\mathrm{e}}v_1, \quad x(0) = x^0 \in X, \\ &y = C_\Lambda\left(x - (\eta I - A)^{-1}B(\phi(w) + B_{\mathrm{e}}v_1)\right) + \mathbf{H}(\eta)\phi(w) + \mathbf{H}_{\mathrm{e}}(\eta)v_1), \end{aligned}\right\} \tag{6.23}$$

where $w \in L^2_{\mathrm{loc}}(\mathbb{R}_+, U)$ is a to-be-determined control, $v_1 \in L^2_{\mathrm{loc}}(\mathbb{R}_+, U_{\mathrm{e}})$ denotes a forcing or disturbance term, and $\mathrm{Re}\,\eta > \omega(\mathbf{T})$.

Remark 6.3.1. (i) We interpret the differential equation in (6.23) in the larger space X_{-1}.

(ii) The expression (6.23) is a generalisation of the familiar

$$\dot{x} = Ax + B\phi(w) + B_{\mathrm{e}}v_1, \quad x(0) = x^0, \quad y = Cx + D\phi(w) + D_{\mathrm{e}}v_1,$$

for finite-dimensional control systems to well-posed infinite-dimensional systems. $\Diamond$

In order to discuss the sampled-data system of interest to us, we now recall the definitions of (zero-order) sample- and hold- operators. To this end, let us first fix $\tau > 0$, the sampling period. The (zero-order) hold operator $\mathcal{H}$ is defined as

$$(\mathcal{H}u)(t) := u(k) \quad \forall\, t \in [k\tau, (k+1)\tau),\, \forall\, u \in U^{\mathbb{Z}_+},$$

which maps $U^{\mathbb{Z}_+}$ into the set of U-valued step-functions (of step length τ) defined on $[0, \infty)$. Furthermore, we let $a \in L^2([0, \tau], \mathbb{R})$ be such that

$$\text{(i)} \int_0^\tau a(t)dt = 1 \quad \text{and} \quad \text{(ii)} \int_0^\tau a(t)\mathbf{T}(t)xdt \in X_1 \quad \forall\, x \in X. \tag{6.24}$$

Remark 6.3.2. We comment that (ii) holds in (6.24) if a is piecewise absolutely continuous, which follows from integration by parts and the fact that $\int_0^s \mathbf{T}(t)\, xdt = A^{-1}(\mathbf{T}(s) - I)x$ is a continuous X_1-valued function of s for every $x \in X$ (see [118, Theorem 4.4.4]). A trivial example of a function a satisfying (6.24) is the constant function $a(t) = 1/\tau$ for all $t \in [0, \tau]$. $\Diamond$

For $a \in L^2([0, \tau], \mathbb{R})$ satisfying (6.24), $\mathcal{S} : L^2_{\mathrm{loc}}(\mathbb{R}_+, Y) \to Y^{\mathbb{Z}_+}$ denotes the (generalised) sampling operator and is defined by

$$(\mathcal{S}y)(k) := \int_0^\tau a(t)y(k\tau + t)dt \quad \forall\, y \in L^2_{\mathrm{loc}}(\mathbb{R}_+, Y),\, \forall\, k \in \mathbb{Z}_+.$$

Finally, we say that $r \in Y$ is feasible (with respect to (6.23)), if the set

$$U^r := \{w \in U : \mathbf{H}(0)\phi(w) = r\},$$

is non-empty. A subset $R \subseteq Y$ is said to be feasible (with respect to (6.23)) if every $r \in R$ is feasible.

We are now in a position to give the discrete-time integral control law of interest to us. Indeed, given $R \subseteq Y$ feasible, $r \in R$ and $u^r \in U^r$, consider the discrete-time integral control law

$$u^+ = u + gK(r - \mathcal{S}y) - g\Gamma(u - \phi(u) - u^r + \phi(u^r)) + v_2, \quad u(0) = u^0 \in U, \tag{6.25}$$

where y is the output of (6.23), and $g > 0$, $K \in \mathcal{L}(Y, U)$ and $\Gamma \in \mathcal{L}(U)$ are design parameters. The signal $v_2 \in U^{\mathbb{Z}_+}$ is a disturbance.

The feedback interconnection of (6.23) and the control law (6.25) via

$$w = \mathcal{H}u,$$

yields the closed-loop system

$$\dot{x} = Ax + B\phi(\mathcal{H}u) + B_{\mathrm{e}}v_1, \quad x(0) = x^0 \in X, \tag{6.26a}$$

$$y = C_\Lambda\big(x - (\eta I - A)^{-1}B(\phi(\mathcal{H}u) + B_{\mathrm{e}}v_1)\big) + \mathbf{H}(\eta)\phi(\mathcal{H}u) + \mathbf{H}_{\mathrm{e}}(\eta)v_1, \tag{6.26b}$$

$$u^+ = u + gK(r - \mathcal{S}y) - g\Gamma(u - \phi(u) - u^r + \phi(u^r)) + v_2, \quad u(0) = u^0 \in U. \tag{6.26c}$$

Definition 6.3.3. *For given $v_1 \in L^2_{\mathrm{loc}}(\mathbb{R}_+, U_{\mathrm{e}})$, $v_2 \in U^{\mathbb{Z}_+}$, $x^0 \in X$ and $u^0 \in U$, we say that $(x, y, u) \in C(\mathbb{R}_+, X) \times L^2_{\mathrm{loc}}(\mathbb{R}_+, Y) \times U^{\mathbb{Z}_+}$ is a solution of* (6.26) *if*

$$x(t) = \mathbf{T}(t)x^0 + \int_0^t \mathbf{T}(t-s)\,(B\phi((\mathcal{H}u)(s)) + B_{\mathrm{e}}v_1(s))\,ds \quad \forall\, t \geq 0,$$

and (6.26b) *and* (6.26c) *are satisfied.*

Remark 6.3.4. For all $v_1 \in L^2_{\mathrm{loc}}(\mathbb{R}_+, U_{\mathrm{e}})$, $v_2 \in U^{\mathbb{Z}_+}$, $x^0 \in X$ and $u^0 \in U$, there exists a unique solution to (6.26). Indeed, we define $(x, y, u) \in C(\mathbb{R}_+, X) \times L^2_{\mathrm{loc}}(\mathbb{R}_+, Y) \times U^{\mathbb{Z}_+}$ iteratively as follows. For $t \in [0, \tau]$, define

$$x(t) = \mathbf{T}(t)x^0 + \int_0^t \mathbf{T}(t-s)\,\big(B\phi(u^0) + B_{\mathrm{e}}v_1(s)\big)\,ds,$$

that is, x is the solution of the well-posed system with input $(\phi(u^0)\ v_1)$ and initial condition x^0 on the interval $[0, \tau]$. From this, we then define y on $[0, \tau]$ to be the corresponding output of this system, and hence satisfies (6.26b) almost everywhere on the interval $[0, \tau]$. Furthermore, we define

$$\begin{aligned} u(1) &= u^0 + gK\left(r - \int_0^\tau a(t)y(t)dt\right) - g\Gamma(u^0 - \phi(u^0) - u^r + \phi(u^r)) + v_2(0) \\ &= u^0 + gK\left(r - (\mathcal{S}y)(0)\right) - g\Gamma(u^0 - \phi(u^0) - u^r + \phi(u^r)) + v_2(0). \end{aligned}$$

For $t \in [\tau, 2\tau]$, we define

$$\begin{aligned} x(t) &= \mathbf{T}(t-\tau)x(\tau) + \int_\tau^t \mathbf{T}(t-s)\,(B\phi(u(1)) + B_{\mathrm{e}}v_1(s))\,ds \\ &= \mathbf{T}(t-\tau)\left(\mathbf{T}(\tau)x^0 + \int_0^\tau \mathbf{T}(\tau-s)\,\big(B\phi(u^0) + B_{\mathrm{e}}v_1(s)\big)\,ds\right) \\ &\quad + \int_\tau^t \mathbf{T}(t-s)\,(B\phi(u(1)) + B_{\mathrm{e}}v_1(s))\,ds \\ &= \mathbf{T}(t)x^0 + \int_0^t \mathbf{T}(t-s)\,(B\phi((\mathcal{H}u)(s)) + B_{\mathrm{e}}v_1(s))\,ds. \end{aligned}$$

By repeating this process, it is clear that the resulting triple $(x, y, u) \in C(\mathbb{R}_+, X) \times L^2_{\mathrm{loc}}(\mathbb{R}_+, Y) \times U^{\mathbb{Z}_+}$, is a solution of (6.26). The uniqueness of (x, y, u) can be shown via similar methods. Finally, we note that, by the previous construction, the sampled-data system (6.26) is causal. ♢

We denote by $PC(\mathbb{R}_+, U)$ the set of piecewise continuous functions defined on $\mathbb{R}_+$ and with values in U. With this in mind, we now present the main theorem of this section, which is a stability result for the sampled-data low-gain integral control system (6.26).

Theorem 6.3.5. *Let $\omega(\mathbf{T}) < 0$, $\phi : U \to U$, $K \in \mathcal{L}(Y, U)$, $\Gamma \in \mathcal{L}(U)$, $R \subseteq Y$ be feasible and let H denote the input-output operator associated with $\mathbf{H}$. Assume that $I - \Gamma$ is discrete-time exponentially stable, ϕ is globally Lipschitz continuous with Lipschitz constant $L > 0$ and*

$$\sup_{z \in \mathbb{E}} \|(zI - (I - \Gamma))^{-1}\| \|\Gamma - K\mathbf{H}(0)\| < 1/L. \tag{6.27}$$

Then there exists $g^* \in (0,1]$ *such that, for all* $g \in (0,g^*)$, *there exist constants* $c_1, c_2, \gamma > 0$ *and* $\theta \in (0,1)$ *such that, for all* $r \in R$, $u^r \in U^r$, *all* $(x^0,u^0) \in X \times U$ *and all* $v_1 \in L^\infty_{\text{loc}}(\mathbb{R}_+, U_{\text{e}})$, $v_2 \in U^{\mathbb{Z}_+}$, *the solution* (x,y,u) *of* (6.26) *satisfies the following estimates.*

$$\|u(k)-u^r\| \le c_1 \left(\theta^k \left\| \begin{pmatrix} x^0 + A^{-1}B\phi(u^r) \\ u^0 - u^r \end{pmatrix} \right\| + \|v_1\|_{L^\infty([0,k\tau],U_{\text{e}})} + \|\pi_{k-1}v_2\|_{\ell^\infty}\right) \quad \forall\, k \in \mathbb{N}, \tag{6.28}$$

and

$$\|x(k\tau+t) + A^{-1}B\phi(u^r)\| \le c_2 \left(e^{-\gamma(k\tau+t)} \left\| \begin{pmatrix} x^0 + A^{-1}B\phi(u^r) \\ u^0 - u^r \end{pmatrix} \right\| + \|v_1\|_{L^\infty([0,k\tau+t],U_{\text{e}})} + \|\pi_k v_2\|_{\ell^\infty}\right) \quad \forall\, k \in \mathbb{Z}_+,\ \forall\, t \in [0,\tau). \tag{6.29}$$

In particular, if $v_1 = 0$ *and* $v_2 = 0$, *then* $u(k) \to u^r$ *as* $k \to \infty$ *and* $x(t) \to -A^{-1}B\phi(u^r)$ *as* $t \to \infty$, *and the rates of convergence are discrete- and continuous-time exponential, respectively. Furthermore, again if* $v_1 = 0$ *and* $v_2 = 0$, *then the following statements hold.*

(i) *There exists* $\varepsilon > 0$ *such that* $r - y \in L^2_{-\varepsilon}(\mathbb{R}_+, Y)$.

(ii) *Under the additional assumptions that*

$$\lim_{t\to\infty} (Hf)(t) = 0 \quad \forall\, f \in PC(\mathbb{R}_+,U) \cap L^2(\mathbb{R}_+,U) \text{ with } \lim_{t\to\infty} f(t) = 0, \tag{6.30}$$

and, for some $t_0 \ge 0$,

$$\mathbf{T}(t_0)(Ax^0 + B\phi(u^r)) \in X, \tag{6.31}$$

$y(t) \to r$ *as* $t \to \infty$.

Remark 6.3.6. (i) We note that the conclusions of Theorem 6.3.5 hold for any sampling period $\tau > 0$.

(ii) If U and Y are finite-dimensional and H is of the form

$$(Hu)(t) = \int_0^t \mu(ds)u(t-s) \quad \forall\, t \ge 0, \forall\, u \in L^1_{\text{loc}}(\mathbb{R}_+, \mathbb{C}^m),$$

where μ is a $\mathbb{C}^{p\times m}$-valued Borel measure on $\mathbb{R}_+$ and $m, p \in \mathbb{N}$ (that is, the impulse response of H is a matrix-valued Borel measure on $\mathbb{R}_+$), then it can be shown that (6.30) holds (see [64, Lemma 6.2.4]).

(iii) If $\mathbf{T}$ is an analytic semigroup, then $\mathbf{T}(t)$ maps X to X_1 for all $t > 0$. Hence, in this case, for all $t_0 > 0$, all $x^0 \in X$, all $r \in R$ and all $u^r \in U^r$,

$$\mathbf{T}(t_0)(Ax^0 + B\phi(u^r)) = A\mathbf{T}(t_0)(x^0 + A^{-1}B\phi(u^r)) \in X,$$

that is, (6.31) holds. ◊

The remainder of this section is dedicated to proving Theorem 6.3.5. To that end, we begin with the following lemma.

Lemma 6.3.7. *The operator $M : X \to X_1$ defined by*

$$Mx := \int_0^\tau a(t)\mathbf{T}(t)x dt \quad \forall\, x \in X,$$

is in $\mathcal{L}(X, X_1)$.

From property (ii) in (6.24), $Mx \in X_1$ for all $x \in X$. It is straightforward to show that M is closed, and hence $M \in \mathcal{L}(X, X_1)$ by the closed-graph theorem. Indeed, for an explicit proof, we refer the reader to the proof of [64, Lemma 7.2.1].

In addition to the previous, in order to facilitate the proof of Theorem 6.3.5, we shall also define the operators

$$A_\tau := \mathbf{T}(\tau), \quad B_\tau := \int_0^\tau \mathbf{T}(s)B ds, \quad C_\tau := CM, \quad D_\tau := CMA^{-1}B + \mathbf{H}(0). \quad (6.32)$$

We see that $A_\tau \in \mathcal{L}(X)$ and, since $\omega(\mathbf{T}) < 0$, A_τ is discrete-time exponentially stable (see Definition 5.1.1). Moreover,

$$B_\tau = (\mathbf{T}(\tau) - I)A^{-1}B \in \mathcal{L}(U, X),$$

and by rearranging this, we see that

$$(I - A_\tau)^{-1}B_\tau = -A^{-1}B. \quad (6.33)$$

By the boundedness of M guaranteed by Lemma 6.3.7, it follows that $C_\tau \in \mathcal{L}(X, Y)$ and $D_\tau \in \mathcal{L}(U, Y)$. We denote the transfer function of the discrete-time system given by the operators $(A_\tau, B_\tau, C_\tau, D_\tau)$ by $\mathbf{G}_\tau$.

The final result that we shall use to prove Theorem 6.3.5 is the next lemma, which is a generalisation of [65, Proposition 3.1].

Lemma 6.3.8. *Let $\omega(\mathbf{T}) < 0$, $\phi : U \to U$, $K \in \mathcal{L}(Y, U)$, $\Gamma \in \mathcal{L}(U)$, $g > 0$, $R \subseteq Y$ feasible, $r \in R$, $u^r \in U^r$ and let $(A_\tau, B_\tau, C_\tau, D_\tau)$ be given as in (6.32). Furthermore, let $(x^0, u^0) \in X \times U$, $v_1 \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+, U_\mathrm{e})$, $v_2 \in U^{\mathbb{Z}_+}$, and let (x, y, u) be the solution of (6.26). Then*

$$\left.\begin{aligned} \xi^+ &= A_\tau \xi + B_\tau \phi(u) + \eta_1, \\ \zeta &= C_\tau \xi + D_\tau \phi(u) + \eta_2, \end{aligned}\right\} \quad (6.34)$$

where, for all $k \in \mathbb{Z}_+$, $\xi(k) := x(k\tau)$, $\zeta(k) = (\mathcal{S}y)(k)$,

$$\eta_1(k) := \int_0^\tau \mathbf{T}(\tau - s)B_\mathrm{e} v_1(s + k\tau)ds, \quad (6.35)$$

and

$$\eta_2(k) := \int_0^\tau a(t)C_\Lambda \left(\int_0^t \mathbf{T}(t - s)B_\mathrm{e} v_1(s + k\tau))ds + A^{-1}B_\mathrm{e} v_1(k\tau + t) \right) dt + \int_0^\tau a(t)\mathbf{H}_\mathrm{e}(0)v_1(k\tau + t)dt. \quad (6.36)$$

The following proof is a generalisation of the proof of [65, Proposition 3.1].

Proof of Lemma 6.3.8. We begin by noting that we may, without loss of generality, take $\eta = 0$ in (6.26), which we do so. For $k \in \mathbb{Z}_+$, we note that

$$\begin{aligned}
\xi(k+1) &= x((k+1)\tau) \\
&= \mathbf{T}(\tau)x(k\tau) + \int_{k\tau}^{(k+1)\tau} \mathbf{T}((k+1)\tau - s)(B\phi(u(k)) + B_{\mathrm{e}}v_1(s))ds \\
&= A_\tau \xi(k) + \int_0^\tau \mathbf{T}(\tau - s)(B\phi(u(k)) + B_{\mathrm{e}}v_1(s + k\tau))ds \\
&= A_\tau + B_\tau \phi(u(k)) + \eta_1(k),
\end{aligned}$$

which gives the state equation of (6.34). As for the output equation, let us first note that, for all $k \in \mathbb{Z}_+$ and $t \in [0, \tau)$,

$$\begin{aligned}
y(k\tau + t) &= C_\Lambda \left(x(k\tau + t) + A^{-1}B\phi(u(k)) + A^{-1}B_{\mathrm{e}}v_1(k\tau + t)\right) + \mathbf{H}(0)\phi(u(k)) \\
&\quad + \mathbf{H}_{\mathrm{e}}(0)v_1(k\tau + t) \\
&= C_\Lambda \Bigg(\mathbf{T}(t)x(k\tau) + \int_{k\tau}^{k\tau+t} \mathbf{T}(k\tau + t - s)(B\phi(u(k)) + B_{\mathrm{e}}v_1(s))ds \\
&\qquad + A^{-1}B\phi(u(k)) + A^{-1}B_{\mathrm{e}}v_1(k\tau + t)\Bigg) \\
&\quad + \mathbf{H}(0)\phi(u(k) + \mathbf{H}_{\mathrm{e}}(0)v_1(k\tau + t).
\end{aligned}$$

By using a change of variables, this becomes, for all $k \in \mathbb{Z}_+$ and $t \in [0, \tau)$,

$$\begin{aligned}
y(k\tau + t) = C_\Lambda \Bigg(&\mathbf{T}(t)x(k\tau) + \int_0^t \mathbf{T}(t - s)(B\phi(u(k)) + B_{\mathrm{e}}v_1(s + k\tau))ds \\
&+ A^{-1}B\phi(u(k)) + A^{-1}B_{\mathrm{e}}v_1(k\tau + t)\Bigg) \\
&+ \mathbf{H}(0)\phi(u(k) + \mathbf{H}_{\mathrm{e}}(0)v_1(k\tau + t).
\end{aligned}$$

Since

$$\int_0^t \mathbf{T}(t - s)B\phi(u(k))ds = A^{-1}(\mathbf{T}(t) - I)B\phi(u(k)) \quad \forall\, k \in \mathbb{Z}_+, \forall\, t \in [0, \tau),$$

we see that, for all $k \in \mathbb{Z}_+$ and $t \in [0, \tau)$,

$$\begin{aligned}
y(k\tau + t) = C_\Lambda \Bigg(&\mathbf{T}(t)x(k\tau) + \mathbf{T}(t)A^{-1}B\phi(u(k)) + \int_0^t \mathbf{T}(t - s)B_{\mathrm{e}}v_1(s + k\tau)ds \\
&+ A^{-1}B_{\mathrm{e}}v_1(k\tau + t)\Bigg) + \mathbf{H}(0)\phi(u(k) + \mathbf{H}_{\mathrm{e}}(0)v_1(k\tau + t).
\end{aligned}$$

Consequently, for all $k \in \mathbb{Z}_+$,

$$\begin{aligned}
\zeta(k) = \int_0^\tau a(t)C_\Lambda \Bigg(&\mathbf{T}(t)x(k\tau) + \mathbf{T}(t)A^{-1}B\phi(u(k)) + \int_0^t \mathbf{T}(t - s)B_{\mathrm{e}}v_1(s + k\tau)ds \\
&+ A^{-1}B_{\mathrm{e}}v_1(k\tau + t)\Bigg) dt + \int_0^\tau a(t)(\mathbf{H}(0)\phi(u(k)) + \mathbf{H}_{\mathrm{e}}(0)v_1(k\tau + t))dt.
\end{aligned}$$

If we now recall that, for all $\xi \in X$, $\mathbf{T}(t)\xi \in \mathrm{dom}(C_\Lambda)$ for almost all $t \geq 0$, we see that, for all $k \in \mathbb{Z}_+$,

$$\begin{aligned}\zeta(k) &= \int_0^\tau a(t)C_\Lambda \mathbf{T}(t)(x(k\tau) + A^{-1}B\phi(u(k)))dt \\ &\quad + \int_0^\tau a(t)C_\Lambda \left(\int_0^t \mathbf{T}(t-s)B_{\mathrm{e}}v_1(s+k\tau))ds + A^{-1}B_{\mathrm{e}}v_1(k\tau+t)\right)dt \\ &\quad + \int_0^\tau a(t)(\mathbf{H}(0)\phi(u(k)) + \mathbf{H}_{\mathrm{e}}(0)v_1(k\tau+t))dt. \end{aligned} \tag{6.37}$$

We pause here and note that

$$CMz = \int_0^\tau a(t)C_\Lambda \mathbf{T}(t)z dt \quad \forall\, z \in X. \tag{6.38}$$

This is straightforward to prove given the density of X_1 in X and that C is admissible. (For an explicit proof, see the proof of [64, Equation (7.20)].) By combining (6.37) with (6.38), (6.24), Lemma 6.3.7 and (6.32), we obtain that, for all $k \in \mathbb{Z}_+$,

$$\begin{aligned}\zeta(k) &= CMx(k\tau) + CMA^{-1}B\phi(u(k)) + \mathbf{H}(0)\phi(u(k)) + \eta_2(k) \\ &= C_\tau\xi(k) + D_\tau\phi(u(k)) + \eta_2(k).\end{aligned}$$

We have therefore shown that the output equation of (6.34) holds, which completes the proof. □

We are now in a position to prove Theorem 6.3.5.

Proof of Theorem 6.3.5. The first part of the current proof is to apply to the discrete-time system (6.34) the integral control result Theorem 6.2.5. In order to do so, we first write (6.34) as

$$\left.\begin{aligned}\xi^+ &= A_\tau\xi + B_\tau\phi(u) + \begin{pmatrix} I & 0\end{pmatrix}\begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix}, \\ \zeta &= C_\tau\xi + D_\tau\phi(u) + \begin{pmatrix} 0 & I\end{pmatrix}\begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix},\end{aligned}\right\} \tag{6.39}$$

where $(A_\tau, B_\tau, C_\tau, D_\tau)$ are as in (6.32), and then check that the relevant hypotheses are satisfied. Indeed, A_τ is discrete-time exponentially stable since $\omega(\mathbf{T}) < 0$, and it is clear that (6.5) holds by the global Lipschitz property of ϕ. Thus, we only need to verify the feasibility of the set R with respect to (6.39) and that

$$\sup_{z\in\mathbb{E}} \|(zI - (I-\Gamma))^{-1}\|\|\Gamma - K\mathbf{G}_\tau(1)\| < 1/L, \tag{6.40}$$

which is simply (6.6) in the context of (6.39). From an application of [65, Proposition 3.1], we yield that $\mathbf{G}_\tau(1) = \mathbf{H}(0)$. Hence, R is feasible with respect to (6.39), and (6.40) holds from (6.27). We may now apply Theorem 6.2.5 to yield the existence of $g^* \in (0,1]$ such that, for all $g \in (0, g^*)$, there exist constants $c, d > 0$ and $\theta \in (0,1)$ such that (6.7) and (6.8) both hold for all $r \in R$, $u^r \in U^r$ and $(u, w, v, x, y) \in U^{\mathbb{Z}_+} \times U^{\mathbb{Z}_+} \times V^{\mathbb{Z}_+} \times X^{\mathbb{Z}_+} \times Y^{\mathbb{Z}_+}$ satisfying (6.39).

We now fix $g \in (0, g^*)$ and let c, d, θ be such described constants. Moreover, let $r \in R$, $u^r \in U^r$, $(x^0, u^0) \in X \times U$ and $v_1 \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+, U_{\mathrm{e}})$, $v_2 \in U^{\mathbb{Z}_+}$ be given, and let (x, y, u)

denote the solution of (6.26). We utilise an application of Lemma 6.3.8 to obtain that (6.39) holds, where, for all $k \in \mathbb{Z}_+$, $\xi(k) := x(k\tau)$, $\zeta(k) = (\mathcal{S}y)(k)$, $\eta_1(k)$ is given by (6.35) and $\eta_2(k)$ is given by (6.36). From an application of Lemma 6.3.8 and the previous application of Theorem 6.2.5,

$$\left\| \begin{pmatrix} \xi(k) - (I - A_\tau)^{-1} B_\tau \phi(u^r) \\ u(k) - u^r \end{pmatrix} \right\| \leq c \left(\theta^k \left\| \begin{pmatrix} x^0 - (I - A_\tau)^{-1} B_\tau \phi(u^r) \\ u^0 - u^r \end{pmatrix} \right\| + \max_{s \in \underline{k-1}} \left\| \begin{pmatrix} \eta_1(s) \\ \eta_2(s) \\ v_2(s) \end{pmatrix} \right\| \right) \quad \forall\, k \in \mathbb{N}, \tag{6.41}$$

where, for all $k \in \mathbb{Z}_+$, $\xi(k) := x(k\tau)$, $\zeta(k) = (\mathcal{S}y)(k)$, and $\eta_1(k)$ and $\eta_2(k)$ are given by (6.35) and (6.36), respectively. To establish (6.28), we give two bounds of η_1 and η_2 in terms of v_1. First, the admissibility of B_e yields the existence of $\kappa > 0$ (independent of v_1) such that

$$\left\| \int_0^t \mathbf{T}(t-s) B_\mathrm{e} v_1(k\tau + s)\, ds \right\| \leq \kappa \|v_1\|_{L^2([k\tau, k\tau + t], U_\mathrm{e})} \leq \kappa \sqrt{\tau} \|v_1\|_{L^\infty([k\tau, k\tau+t], U_\mathrm{e})} \quad \forall\, k \in \mathbb{Z}_+,\ \forall\, t \in [0, \tau)\,. \tag{6.42}$$

Whence

$$\|\pi_{k-1} \eta_1\|_{\ell^\infty} \leq \kappa \sqrt{\tau} \|v_1\|_{L^\infty([0,k\tau], U_\mathrm{e})} \quad \forall\, k \in \mathbb{N}. \tag{6.43}$$

We now seek to prove the existence of $d_1 > 0$ (independent of v_1) such that

$$\|\pi_{k-1} \eta_2\|_{\ell^\infty} \leq d_1 \|v_1\|_{L^\infty([0,k\tau], U_\mathrm{e})} \quad \forall\, k \in \mathbb{N}. \tag{6.44}$$

To this end, by writing

$$\tilde{w}(t) := C_\Lambda \left(\int_0^t \mathbf{T}(t-s) B_\mathrm{e} v_1(s + k\tau)) ds + A^{-1} B_\mathrm{e} v_1(k\tau + t) \right) + \mathbf{H}_\mathrm{e}(0) v_1(k\tau + t),$$

we see that $\tilde{w}$ is the output of an exponentially stable well-posed system with zero initial condition. Hence, there exists $d_2 > 0$ such that

$$\|\tilde{w}\|_{L^2([0,\tau])} \leq d_2 \|v_1\|_{L^2([k\tau,(k+1)\tau])} \leq d_2 \sqrt{\tau} \|v_1\|_{L^\infty([k\tau,(k+1)\tau])}.$$

By use of Hölder's inequality, we then obtain from (6.36) that, for all $k \in \mathbb{Z}_+$,

$$\|\eta_2(k)\| \leq \|a\|_{L^2([0,\tau])} \|\tilde{w}\|_{L^2([0,\tau])} \leq d_1 \|v_1\|_{L^\infty([k\tau,(k+1)\tau])},$$

where $d_1 := d_2 \sqrt{\tau} \|a\|_{L^2([0,\tau])} > 0$. It is thus evident that (6.44) holds. Continuing with the proof, by combining (6.41) with (6.43) and (6.44), we see that

$$\left\| \begin{pmatrix} \xi(k) - (I - A_\tau)^{-1} B_\tau \phi(u^r) \\ u(k) - u^r \end{pmatrix} \right\| \leq c_1 \left(\theta^k \left\| \begin{pmatrix} x^0 - (I - A_\tau)^{-1} B_\tau \phi(u^r) \\ u^0 - u^r \end{pmatrix} \right\| + \|v_1\|_{L^\infty([0,k\tau], U_\mathrm{e})} + \|\pi_{k-1} v_2\|_{\ell^\infty} \right) \quad \forall\, k \in \mathbb{N}, \tag{6.45}$$

where $c_1 := \sqrt{3}c(\kappa\sqrt{\tau} + d_1 + 1) > 0$. As a consequence of this, (6.28) holds.
We proceed to derive (6.29). To this end, we note that, for each $k \in \mathbb{N}$ and $t \in [0, \tau)$,

$$\begin{aligned} x(k\tau + t) &= \mathbf{T}(t)x(k\tau) + \int_{k\tau}^{k\tau+t} \mathbf{T}(k\tau + t - s)(B\phi(u(k)) + B_{\mathrm{e}}v_1(s))ds \\ &= \mathbf{T}(t)x(k\tau) + \int_0^t \mathbf{T}(s)(B\phi(u(k))ds + \int_0^t \mathbf{T}(t-s)B_{\mathrm{e}}v_1(k\tau + s))ds \\ &= \mathbf{T}(t)x(k\tau) + (\mathbf{T}(t) - I)A^{-1}B\phi(u(k)) + \int_0^t \mathbf{T}(t-s)B_{\mathrm{e}}v_1(k\tau + s))ds. \end{aligned} \tag{6.46}$$

Therefore, taking norms in (6.46) and invoking (6.42), we see that

$$\begin{aligned} &\|x(k\tau + t) + A^{-1}B\phi(u^r)\| \le \|\mathbf{T}(t)\|\|x(k\tau) + A^{-1}B\phi(u(k))\| \\ &\quad + \|A^{-1}B(\phi(u^r) - \phi(u(k)))\| + \kappa\sqrt{\tau}\|v_1\|_{L^\infty([k\tau,k\tau+t],U_{\mathrm{e}})} \quad \forall\, k \in \mathbb{Z}_+,\, \forall\, t \in [0, \tau). \end{aligned}$$

With this in mind, if we define the positive constants μ_1, μ_2 by

$$\mu_1 := \sup_{t\in[0,\tau]} \|\mathbf{T}(t)\| < \infty \quad \text{and} \quad \mu_2 := \|A^{-1}B\| < \infty,$$

and recall the Lipschitz property of ϕ, we obtain that

$$\begin{aligned} \|x(k\tau + t) + A^{-1}B\phi(u^r)\| &\le \mu_1\|x(k\tau) + A^{-1}B\phi(u^r)\| + \mu_2 L(1 + \mu_1)\|u^r - u(k)\| \\ &\quad + \kappa\sqrt{\tau}\|v_1\|_{L^\infty([k\tau,k\tau+t],U_{\mathrm{e}})} \quad \forall\, k \in \mathbb{Z}_+,\, \forall\, t \in [0, \tau). \end{aligned} \tag{6.47}$$

Set $\gamma := -(\ln\theta)/\tau$ and $\mu_3 := e^{\gamma\tau}$. We then have that

$$\theta^k = \mu_3 e^{-\gamma(k+1)\tau} \le \mu_3 e^{-\gamma(k\tau+t)} \quad \forall\, k \in \mathbb{Z}_+,\ \forall\, t \in [0, \tau)\,. \tag{6.48}$$

Combining (6.45) with (6.33), (6.47) and (6.48) gives (6.29), where c_2 is a positive constant independent of r, u^r, x^0, u^0, v_1 and v_2.
We next prove statement (i). To this end, let $v_1 = 0$ and $v_2 = 0$, let $\rho \in (\theta, 1)$ and let $\delta_1 := -(\ln\rho)/\tau > 0$. We compute that

$$\begin{aligned} \int_0^\infty \|e^{\delta_1 t}(\mathcal{H}(\phi(u))(t) - \phi(u^r))\|^2 dt &= \sum_{k=0}^\infty \int_{k\tau}^{(k+1)\tau} \|e^{\delta_1 t}(\phi(u(k)) - \phi(u^r))\|^2 dt \\ &= \left(\int_0^\tau e^{2\delta_1 t} dt\right) \sum_{k=0}^\infty e^{2\delta_1 k\tau}\|\phi(u(k)) - \phi(u^r)\|^2 \\ &= \frac{1}{2\delta_1}(e^{2\delta_1\tau} - 1) \sum_{k=0}^\infty \rho^{-2k}\|\phi(u(k)) - \phi(u^r)\|^2 \\ &\le b\frac{1}{2\delta_1}(e^{2\delta_1\tau} - 1) \sum_{k=0}^\infty (\rho^{-1}\theta)^{2k} \\ &< \infty, \end{aligned}$$

where $b > 0$ is guaranteed to exist by (6.28) and the Lipschitz property of ϕ. Hence,

$$\mathcal{H}(\phi(u)) - \phi(u^r) \in L^2_{-\delta_1}(\mathbb{R}_+, U). \tag{6.49}$$

Continuing, we have that

$$\begin{aligned} y(t) - r &= C_\Lambda \mathbf{T}(t)x^0 + H(\mathcal{H}(\phi(u)))(t) - \mathbf{H}(0)\phi(u^r) \\ &= C_\Lambda \mathbf{T}(t)x^0 + H(\mathcal{H}(\phi(u)) - \phi(u^r))(t) + H(\phi(u^r))(t) - \mathbf{H}(0)\phi(u^r) \text{ a.e. } t \geq 0, \end{aligned} \tag{6.50}$$

where we view $\phi(u^r)$ as a constant function in $L^2_{\mathrm{loc}}(\mathbb{R}_+, U)$, and have used that $r = \mathbf{H}(0)\phi(u^r)$. In addition, by noting that the Laplace transform of $H(\phi(u^r)) - \mathbf{H}(0)\phi(u^r)$ is

$$\frac{1}{s}\left(\mathbf{H}(s) - \mathbf{H}(0)\right)\phi(u^r) \quad \forall\, s \in \mathbb{C}_{\omega(\mathbf{T})},$$

we obtain, by exponential stability, the existence of $\delta_2 > 0$ such that

$$s \mapsto \frac{1}{s}\left(\mathbf{H}(s) - \mathbf{H}(0)\right)\phi(u^r) \in H^2(\mathbb{C}_{-\delta_2}, Y).$$

The Payley-Wiener theorem (see, for example, [118, Theorem 10.3.4]) then gives that

$$H(\phi(u^r)) - \mathbf{H}(0)\phi(u^r) \in L^2_{-\delta_2}(\mathbb{R}_+, Y). \tag{6.51}$$

Furthermore, using (6.49) and the fact that $H \in \mathcal{L}(L^2_\alpha(\mathbb{R}_+, U), L^2_\alpha(\mathbb{R}_+, Y))$ for every $\alpha > \omega(\mathbf{T})$, we see that

$$H(\mathcal{H}(\phi(u)) - \phi(u^r)) \in L^2_{-\delta_3}(\mathbb{R}_+, Y), \tag{6.52}$$

for some $\delta_3 \in (0, -\omega(\mathbf{T}))$. Upon recalling that $C_\Lambda \mathbf{T}x^0 \in L^2_\alpha(\mathbb{R}_+, Y)$ for all $\alpha > \omega(\mathbf{T})$, if we combine (6.51) and (6.52) with (6.50), we arrive at $y - r \in L^2_{-\varepsilon}(\mathbb{R}_+, Y)$ where $\varepsilon := \min\{\delta_2, \delta_3\}$, hence giving statement (i).
To prove statement (ii), we now additionally assume that (6.30) and (6.31) hold. We first note that, by (6.50), we have

$$y - r = e_1 + e_2,$$

where

$$e_1 := C_\Lambda \mathbf{T}x^0 + H(\phi(u^r)) - \mathbf{H}(0)\phi(u^r) \quad \text{and} \quad e_2 := H(\mathcal{H}(\phi(u)) - \phi(u^r)).$$

Using [118, Statement (iii) of Theorem 4.4.2], we obtain that the Laplace transform of e_1, which we denote by $\hat{e}_1$, is given by

$$\hat{e}_1(s) = C(sI - A)^{-1}x^0 + \frac{1}{s}(\mathbf{H}(s) - \mathbf{H}(0))\phi(u^r) \quad \forall\, s \in \mathbb{C}_{\omega(\mathbf{T})}. \tag{6.53}$$

Moreover, [118, Theorem 4.6.7] yields that

$$\frac{1}{s}(\mathbf{H}(s) - \mathbf{H}(0))\phi(u^r) = C(sI - A)^{-1}A^{-1}B\phi(u^r) \quad \forall\, s \in \mathbb{C}_{\omega(\mathbf{T})}.$$

Combining this with (6.53) gives

$$\hat{e}_1(s) = C(sI - A)^{-1}A^{-1}(Ax_0 + B\phi(u^r)) \quad \forall s \in \mathbb{C}_{\omega(\mathbf{T})}.$$

Consequently,

$$e_1(t) = C_\Lambda \mathbf{T}(t)A^{-1}(Ax_0 + B\phi(u^r)) = CA^{-1}\mathbf{T}(t - t_0)\mathbf{T}(t_0)(Ax_0 + B\phi(u^r)) \quad \forall\, t \geq t_0.$$

Combining (6.31) with the exponential stability of $\mathbf{T}$, we see that $\lim_{t\to\infty} e_1(t) = 0$. Finally, we invoke (6.30) to yield that $\lim_{t\to\infty} e_2(t) = 0$, hence completing the proof. □

6.4 Notes and references

The papers from the literature that are the most relevant to the main results of this chapter, are [48] and [65]. We now proceed to discuss similarities and differences. To begin with, as already commented on, the content of Section 6.2 is a discrete-time analogue of the finite-dimensional, continuous-time work achieved in [48]. In particular, Theorem 6.2.5 and Lemma 6.2.7 are infinite-dimensional discrete-time generalisations of [48, Theorem 4] and [48, Lemma 6], respectively. As for Section 6.3, the main result, Theorem 6.3.5, is partly inspired by [65, Theorem 3.2]. In [65], an adaptive control law is considered in feedback, via sample- and hold- operations, with a well-posed linear system. We comment that the system investigated in [65] does not involve input nonlinearities, nor external disturbances, which majorly differs from the system (6.26) under consideration presently. The inclusion of the external disturbances and the input nonlinearities, complicates the analysis. For more comparison on this, we refer the reader to the proofs of [65, Proposition 3.1 and Theorem 3.2] and Theorem 6.3.5. Due to the differences in the systems under consideration, we comment that [65, Theorem 3.2] does not provide explicit stability estimates as we do in (6.28) and (6.29). Thus, the most similar part of Theorem 6.3.5 to [65, Theorem 3.2] concerns the output convergence and the additional assumptions utilised to achieve it. It is this from where we draw our inspiration. Indeed, the final part of the proof of Theorem 6.3.5 follows a method employed in the proof of [65, Theorem 3.2].

Appendices

Appendix A

Discrete-time ISS Lyapunov functions

In this appendix, we give the proof of Proposition 2.2.9, which asserts that the existence of an ISS-Lyapunov function (see Definition 2.2.8) is sufficient for (2.17) being ISS (see Definition 2.2.5).

In order to prove Proposition 2.2.9, we generalise the proof of [106, Proposition 5.2.4] to our current setting. To this end, we begin by first highlighting an equivalent definition of ISS. Indeed, let $\Sigma \in \mathbb{L}$ and $f : \mathbb{R}^p \to \mathbb{R}^m$, and note that $\max\{a, b\} \leq a+b \leq 2\max\{a, b\}$ for all $a, b \in \mathbb{R}$. It is therefore clear that (2.17) being ISS is equivalent to the existence of $\psi \in \mathcal{KL}$ and $\phi \in \mathcal{K}$ such that

$$\|x(t)\| \leq \max\left\{\psi\left(\|x(0)\|, t\right), \phi\left(\max_{s \in \underline{t-1}} \left\|\begin{pmatrix} v(s) \\ w(s) \end{pmatrix}\right\|\right)\right\} \quad \forall\, t \in \mathbb{N},\ \forall\, (v, w, x, y) \in \mathcal{B}. \tag{A.1}$$

The following proof is a generalisation of [106, Proposition 5.2.4].

Proof of Proposition 2.2.9. Let $V : \mathbb{R}^n \to \mathbb{R}_+$ be an ISS-Lyapunov function of (2.17) and let $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathcal{K}_\infty$ be such that (2.25) holds and (2.26) holds for all $(v, w, x, y) \in \mathcal{B}$ and all $t \in \mathbb{N}$. Note that, from Lemma 2.1.2, $\alpha_1^{-1}, \alpha_2^{-1} \in \mathcal{K}_\infty$. Hence, from (2.25),

$$\|\xi\| \leq \alpha_1^{-1}(V(\xi)) \quad \text{and} \quad \alpha_2^{-1}(V(\xi)) \leq \|\xi\| \quad \forall\, \xi \in \mathbb{R}^n. \tag{A.2}$$

By combining this with the equivalent definition of ISS given by (A.1), in order to prove the proposition it suffices to show that there exist $\psi \in \mathcal{KL}$ and $\phi \in \mathcal{K}$ such that

$$V(x(t+1)) \leq \max\left\{\psi(V(x(0)), t+1), \phi\left(\max_{s \in \underline{t}} \left\|\begin{pmatrix} v(s) \\ w(s) \end{pmatrix}\right\|\right)\right\} \quad \forall\, t \in \mathbb{Z}_+,\ \forall\, (v, w, x, y) \in \mathcal{B}, \tag{A.3}$$

and that ψ and ϕ only depend on α_1, α_2, α_3 and α_4. Indeed, this is what we shall do. To this end, note that, from (2.26) and (A.2), for all $(v, w, x, y) \in \mathcal{B}$ and all $t \in \mathbb{Z}_+$,

$$V(x(t+1)) - V(x(t)) \leq -\alpha_3(\alpha_2^{-1}(V(x(t)))) + \alpha_4\left(\max_{s \in \underline{t}} \left\|\begin{pmatrix} v(s) \\ w(s) \end{pmatrix}\right\|\right). \tag{A.4}$$

An application of [106, Lemma 5.1.6] gives the existence of $\alpha \in \mathcal{K}_\infty$ such that $\alpha(s) \leq \alpha_3(\alpha_2^{-1}(s))$ for all $s \in \mathbb{R}_+$, and $\mathrm{id} - \alpha \in \mathcal{K}_\infty$. Moreover, [106, Lemma 5.1.7] gives the existence of $\rho, \phi \in \mathcal{K}_\infty$ such that $\rho(s) < s$ for all $s > 0$, and $(\mathrm{id} - \alpha)(s_1) + \alpha_4(s_2) \leq \max\{\rho(s_1), \phi(s_2)\}$ for all $s_1, s_2 \geq 0$. Hence, from (A.4), we see that, for all $(v, w, x, y) \in \mathcal{B}$ and all $t \in \mathbb{N}$,

$$
\begin{aligned}
V(x(t+1)) &\leq (\mathrm{id} - \alpha)(V(x(t)) + \alpha_4 \left(\max_{s \in \underline{t}} \left\| \begin{pmatrix} v(s) \\ w(s) \end{pmatrix} \right\| \right) \\
&\leq \max \left\{ \rho(V(x(t))), \phi \left(\max_{s \in \underline{t}} \left\| \begin{pmatrix} v(s) \\ w(s) \end{pmatrix} \right\| \right) \right\} \\
&\leq \max \left\{ \rho \left(\max \left\{ \rho(V(x(t-1))), \phi \left(\max_{s \in \underline{t}} \left\| \begin{pmatrix} v(s) \\ w(s) \end{pmatrix} \right\| \right) \right\} \right), \phi \left(\max_{s \in \underline{t}} \left\| \begin{pmatrix} v(s) \\ w(s) \end{pmatrix} \right\| \right) \right\} \\
&\leq \max \left\{ \rho^2(V(x(t-1))), \phi \left(\max_{s \in \underline{t}} \left\| \begin{pmatrix} v(s) \\ w(s) \end{pmatrix} \right\| \right) \right\} \\
&\;\;\vdots \\
&\leq \max \left\{ \rho^{t+1}(V(x(0))), \phi \left(\max_{s \in \underline{t}} \left\| \begin{pmatrix} v(s) \\ w(s) \end{pmatrix} \right\| \right) \right\}.
\end{aligned}
$$

In view of (A.3), to conclude the proof, it suffices to prove that $\psi : \mathbb{R}_+ \times \mathbb{Z}_+ \to \mathbb{R}_+$, defined by $\psi(s, t) := \rho^t(s)$ for all $(s, t) \in \mathbb{R}_+ \times \mathbb{Z}_+$, is a $\mathcal{KL}$ function. To that end, it is clear that for each fixed $t \in \mathbb{Z}_+$, $\psi(\cdot, t) \in \mathcal{K}$. Moreover, for $s > 0$ fixed, we have $0 < \rho^t(s) < \rho^{t-1}(s) < \cdots < s$, since $\rho(s) < s$. Also, since the sequence $(\rho^t(s))_{t \in \mathbb{Z}_+}$ is bounded below and decreasing, it has a limit. This limit is 0, since otherwise, $\lim_{t\to\infty} \rho^t(s) > \rho(\lim_{t\to\infty} \rho^t(s)) = \lim_{t\to\infty} \rho^t(s)$, which is a contradiction. Therefore, $\psi \in \mathcal{KL}$. Finally, we note that ψ and ϕ depend only upon α_1, α_2, α_3 and α_4. $\square$

Appendix B

A convergence property of asymptotically stable equilibria

In this appendix, we concern ourselves with the discrete-time non-linear control system:

$$x^+ = g(x, v), \quad x(0) = x^0 \in \mathbb{R}^n, \tag{B.1}$$

where $v \in (\mathbb{R}^m)^{\mathbb{Z}_+}$, $g : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is continuous and $n, m \in \mathbb{N}$. For given $x^0 \in \mathbb{R}^n$ and $v \in (\mathbb{R}^m)^{\mathbb{Z}_+}$, we denote by $x(\cdot; x^0, v)$ the unique solution of (B.1). Moreover, for $\zeta \in \mathbb{R}^m$, we will abuse notation and write $x(\cdot; x^0, \zeta) = x(\cdot; x^0, t \mapsto \zeta)$.

The aim of this appendix is to give a discrete-time analogue of [112, Theorem 1]. We used this discrete-time result in the proof of Theorem 2.3.16 in Chapter 2. In order for us to present the theorem, we first give the following definitions associated with (B.1).

Definition B.0.1. (i) *We say that* $(v^\infty, x^\infty) \in \mathbb{R}^m \times \mathbb{R}^n$ *is an* equilibrium pair *of* (B.1)*, if* $x^\infty = g(x^\infty, v^\infty)$.

(ii) *We say that an equilibrium pair* $(v^\infty, x^\infty) \in \mathbb{R}^m \times \mathbb{R}^n$ *is* stable *if for all* $\varepsilon > 0$*, there exists* $\delta > 0$ *such that, for all* $x^0 \in \mathbb{R}^n$ *with* $\|x^0 - x^\infty\| \leq \delta$*, it follows that* $\|x(t; x^0, v^\infty) - x^\infty\| \leq \varepsilon$ *for all* $t \in \mathbb{Z}_+$*. Moreover,* (v^∞, x^∞) *is* attractive *if there exists* $\delta > 0$ *such that, for all* $x^0 \in \mathbb{R}^n$ *with* $\|x^0 - x^\infty\| \leq \delta$,

$$\|x(t; x^0, v^\infty) - x^\infty\| \to 0 \text{ as } t \to \infty.$$

We say that (v^∞, x^∞) *is* asymptotically stable *if it is both stable and attractive.*

(iii) *The* domain of attraction *of an asymptotically stable equilibrium pair* (v^∞, x^∞) *is*

$$\mathcal{O}(v^\infty, x^\infty) := \{x^0 \in \mathbb{R}^n : \|x(t; x^0, v^\infty) - x^\infty\| \to 0 \text{ as } t \to \infty\}.$$

When the context is clear, we shall simply write $\mathcal{O}$ *instead of* $\mathcal{O}(v^\infty, x^\infty)$.

(iv) *For* $\mathcal{C}$*, a compact set in* $\mathbb{R}^n$*, and* $(x^0, v) \in \mathbb{R}^n \times (\mathbb{R}^m)^{\mathbb{Z}_+}$*, we say that* $x(\cdot; x^0, v)$ *is* $\mathcal{C}$-recurrent*, if for all* $T \in \mathbb{Z}_+$*, there exists* $t \in \overline{T}$ *such that* $x(t; x^0, v) \in \mathcal{C}$.

We may now present the aformentioned theorem, which is a discrete-time analogue of [112, Theorem 1].

Theorem B.0.2. *Let $(v^\infty, x^\infty) \in \mathbb{R}^m \times \mathbb{R}^n$ be an asymptotically stable equilibrium pair of* (B.1)*. Furthermore, let $\mathcal{C}$ be a compact set in $\mathcal{O}$, $v \in (\mathbb{R}^m)^{\mathbb{Z}_+}$, $x^0 \in \mathbb{R}^n$ and $x(\cdot; x^0, v)$ a $\mathcal{C}$-recurrent solution of* (B.1)*. If $\lim_{t\to\infty} v(t) = v^\infty$, then $\lim_{t\to\infty} x(t; x^0, v) = x^\infty$.*

To prove the above theorem, we shall proceed analogously to that done in [112]. Since the proof uses several preliminary results, we collect these, and the proof itself, in the following section.

B.1 Proof of Theorem B.0.2

We begin with the following two lemmas, which each provide a useful property of (B.1).

Lemma B.1.1. *Let $T \in \mathbb{Z}_+$. Then for all $\varepsilon > 0$ and all $(x_1^0, v) \in \mathbb{R}^n \times (\mathbb{R}^m)^{\mathbb{Z}_+}$, there exists $\delta > 0$ such that, for all $(x_2^0, w) \in \mathbb{R}^n \times (\mathbb{R}^m)^{\mathbb{Z}_+}$ satisfying $\|x_1^0 - x_2^0\| + \max_{s\in \underline{T}} \|v(s) - w(s)\| \le \delta$,*

$$\|x(t; x_1^0, v) - x(t; x_2^0, w)\| \le \varepsilon \quad \forall\, t \in \underline{T}.$$

Proof. Let $(x_1^0, v) \in \mathbb{R}^n \times (\mathbb{R}^m)^{\mathbb{Z}_+}$ and $\varepsilon > 0$. We shall first show that for all $t \in \mathbb{Z}_+$, (P_t) holds, where (P_t) denotes the property: for every $\tilde{\varepsilon} > 0$, there exists $\delta_{t,\tilde{\varepsilon}} > 0$ such that, for all $(x_2^0, w) \in \mathbb{R}^n \times (\mathbb{R}^m)^{\mathbb{Z}_+}$ satisfying $\|x_1^0 - x_2^0\| + \max_{s\in \underline{t}} \|v(s) - w(s)\| \le \delta_{t,\tilde{\varepsilon}}$,

$$\|x(t; x_1^0, v) - x(t; x_2^0, w)\| \le \tilde{\varepsilon}.$$

To prove that (P_t) holds for all $t \in \mathbb{Z}_+$, we utilise an inductive argument. The case $t = 0$ is trivial and is hence omitted. Now assume that (P_t) holds for some $t \in \mathbb{Z}_+$. Fix $\tilde{\varepsilon} > 0$ and note that, since g is continuous at $(x(t; x_1^0, v), v(t))$, there exists $\tilde{\delta} > 0$ such that

$$\|g(x(t; x_1^0, v), v(t)) - g(\xi, \zeta)\| \le \tilde{\varepsilon}, \tag{B.2}$$

for all $(\xi, \zeta) \in \mathbb{R}^n \times \mathbb{R}^m$ such that $\|x(t; x_1^0, v) - \xi\| + \|v(t) - \zeta\| \le \tilde{\delta}$. Moreover, since (P_t) holds, there exists $\delta_{t,\tilde{\delta}/2} > 0$ such that, for all $(x_2^0, w) \in \mathbb{R}^n \times (\mathbb{R}^m)^{\mathbb{Z}_+}$ satisfying $\|x_1^0 - x_2^0\| + \max_{s\in \underline{t}} \|v(s) - w(s)\| \le \delta_{t,\tilde{\delta}/2}$,

$$\|x(t; x_1^0, v) - x(t; x_2^0, w)\| \le \frac{\tilde{\delta}}{2}.$$

Whence, by setting $\delta_{t+1,\tilde{\varepsilon}} := \min\{\tilde{\delta}/2, \delta_{t,\tilde{\delta}/2}\} > 0$ and by combining this with (B.2), we yield that, for all $(x_2^0, w) \in \mathbb{R}^n \times (\mathbb{R}^m)^{\mathbb{Z}_+}$ satisfying $\|x_1^0 - x_2^0\| + \max_{s\in \underline{t+1}} \|v(s) - w(s)\| \le \delta_{t+1,\tilde{\varepsilon}}$,

$$\|x(t+1; x_1^0, v) - x(t+1; x_2^0, w)\| = \|g(x(t; x_1^0, v), v(t)) - g(x(t; x_2^0, w), w(t))\| \le \tilde{\varepsilon}.$$

Therefore, (P_{t+1}) holds, and via induction, (P_t) holds for all $t \in \mathbb{Z}_+$. By simply setting

$$\delta := \min\{\delta_{0,\varepsilon}, \dots, \delta_{T,\varepsilon}\} > 0,$$

we complete the proof. □

Lemma B.1.2. *Let $(x^0, v) \in \mathbb{R}^n \times (\mathbb{R}^m)^{\mathbb{Z}_+}$. Then*

$$x(t + T; x^0, v) = x(t; x(T; x^0, v), \Lambda_T v) \quad \forall\, t, T \in \mathbb{Z}_+.$$

Proof. Fix $T \in \mathbb{Z}_+$. We shall induct on $t \in \mathbb{Z}_+$. The case $t = 0$ is trivial and is thus omitted. Now assume that $x(t+T; x^0, v) = x(t; x(T; x^0, v), \Lambda_T v)$ for some $t \in \mathbb{Z}_+$. From (B.1), we then have that

$$\begin{aligned} x(t+1+T; x^0, v) &= g(x(t+T; x^0, v), v(t+T)) \\ &= g\left(x(t; x(T; x^0, v), \Lambda_T v), (\Lambda_T v)(t)\right) \\ &= x(t+1; x(T; x^0, v), \Lambda_T v). \end{aligned}$$

The proof is whence complete. □

The following asserts that the domain of attraction is an open set. It is well-known in the continuous-time setting (see, for example, [77, Exercise 5.14]), but it is difficult to find a proof for our discrete-time setting. We thus provide a proof for completeness.

Lemma B.1.3. *Let $(v^\infty, x^\infty) \in \mathbb{R}^m \times \mathbb{R}^n$ be an asymptotically stable equilibrium pair of* (B.1)*. Then $\mathcal{O}$ is an open subset of $\mathbb{R}^n$.*

Proof. We begin by noting that there exists $\delta_1 > 0$ such that, for all $\xi \in \mathbb{R}^n$ satisfying $\|\xi - x^\infty\| < \delta_1$,

$$\|x(t; \xi, v^\infty) - x^\infty\| \to 0 \text{ as } t \to \infty. \tag{B.3}$$

Indeed, this is due to the asymptotic stability of (v^∞, x^∞). Let $x^0 \in \mathcal{O}$. We shall show that there is a open neighbourhood around x^0 which is contained in $\mathcal{O}$. To this end, we begin by highlighting that, since $x^0 \in \mathcal{O}$, there exists $T \in \mathbb{Z}_+$ such that

$$\|x(t+T; x^0, v^\infty) - x^\infty\| < \frac{\delta_1}{2} \quad \forall\, t \in \mathbb{Z}_+. \tag{B.4}$$

Moreover, an application of Lemma B.1.1 yields the existence of $\delta > 0$ such that, for all $\xi \in \mathbb{R}^n$ satisfying $\|\xi - x^0\| < \delta$,

$$\|x(t; x^0, v^\infty) - x(t; \xi, v^\infty)\| < \frac{\delta_1}{2} \quad \forall\, t \in \underline{T}. \tag{B.5}$$

Therefore, for $\xi \in \mathbb{R}^n$ such that $\|\xi - x^0\| < \delta$, a combination of (B.4) and (B.5) yields

$$\|x(T; \xi, v^\infty) - x^\infty\| \le \|x(T; \xi, v^\infty) - x(T; x^0, v^\infty)\| + \|x(T; x^0, v^\infty) - x^\infty\| < \delta_1.$$

By invoking (B.3) and recalling Lemma B.1.2, we see therefore that

$$\|x(t+T; \xi, v^\infty) - x^\infty\| = \|x(t; x(T; \xi, v^\infty), v^\infty) - x^\infty\| \to 0 \text{ as } t \to \infty.$$

Whence, we see that $\xi \in \mathcal{O}$ for all $\xi \in \mathbb{R}^n$ such that $\|\xi - x^0\| < \delta$, completing the proof. □

We now present a discrete-time analogue of [112, Lemma II.1].

Lemma B.1.4. *Let $(v^\infty, x^\infty) \in \mathbb{R}^m \times \mathbb{R}^n$ be an asymptotically stable equilibrium pair of* (B.1)*. Then for every compact set $\mathcal{C} \subseteq \mathcal{O}$, every $T \in \mathbb{Z}_+$ and every $\varepsilon > 0$, there exists $\delta > 0$ such that, for all $(x^0, v) \in \mathcal{C} \times (\mathbb{R}^m)^{\mathbb{Z}_+}$ satisfying* $\max_{s \in \underline{T}} \|v(s) - v^\infty\| \le \delta$,

$$\|x(t; x^0, v) - x(t; x^0, v^\infty)\| \le \varepsilon \quad \forall\, t \in \underline{T}.$$

Proof. Let $\mathcal{C} \subseteq \mathcal{O}$ be a compact set, $T \in \mathbb{Z}_+$ and $\varepsilon > 0$. We begin with an application of Lemma B.1.1 to yield, for each $\xi \in \mathcal{C}$, the existence of $\delta_\xi > 0$ such that, for all $(x^0, v) \in \mathbb{R}^n \times (\mathbb{R}^m)^{\mathbb{Z}_+}$ satisfying $\|x^0 - \xi\| + \max_{s \in \underline{T}} \|v(s) - v^\infty\| \leq 2\delta_\xi$,

$$\|x(t; x^0, v) - x(t; \xi, v^\infty)\| \leq \frac{\varepsilon}{2} \quad \forall\, t \in \underline{T}. \tag{B.6}$$

It is clear that the family of open balls $\{\mathbb{B}(\xi, \delta_\xi) : \xi \in \mathcal{C}\}$ forms an open cover of $\mathcal{C}$. By the compactness of $\mathcal{C}$, there exist $\xi_1, \ldots, \xi_k \in \mathcal{C}$, for some $k \in \mathbb{N}$, such that $\cup_{i=1}^k \mathbb{B}(\xi_i, \delta_{\xi_i})$ is a finite subcover of $\mathcal{C}$. We subsequently define $\delta := \min\{\delta_{\xi_1}, \ldots, \delta_{\xi_k}\} > 0$. To conclude the proof, let $(x^0, v) \in \mathcal{C} \times (\mathbb{R}^m)^{\mathbb{Z}_+}$ be such that $\max_{s \in \underline{T}} \|v(s) - v^\infty\| \leq \delta$. Then, since $x^0 \in \mathbb{B}(\xi_i, \delta_{\xi_i})$ for some $i \in \{1, \ldots, k\}$, we obtain, from (B.6), that, for all $t \in \underline{T}$,

$$\begin{aligned}\|x(t; x^0, v) - x(t; x^0, v^\infty)\| &\leq \|x(t; x^0, v) - x(t; \xi_i, v^\infty)\| + \|x(t; \xi_i, v^\infty) - x(t; x^0, v^\infty)\| \\ &\leq \varepsilon,\end{aligned}$$

which completes the proof. □

The following result is a discrete-time version of [114, Lemma 5.9.12].

Lemma B.1.5. *Let $(v^\infty, x^\infty) \in \mathbb{R}^m \times \mathbb{R}^n$ be an asymptotically stable equilibrium pair of* (B.1). *Then for all $x^0 \in \mathcal{O}$ and all $\varepsilon > 0$, there exist $\delta > 0$ and $T \in \mathbb{Z}_+$ such that, for all $\xi \in \mathbb{R}^n$ satisfying $\|\xi - x^0\| \leq \delta$,*

$$\|x(t; \xi, v^\infty) - x^\infty\| \leq \varepsilon \quad \forall\, t \in \overline{T}.$$

Proof. Let $x^0 \in \mathcal{O}$ and $\varepsilon > 0$. Since (v^∞, x^∞) is a stable equilibrium, there exists $\delta_1 > 0$ such that, for all $\xi \in \mathbb{R}^n$ satisfying $\|\xi - x^\infty\| \leq \delta_1$,

$$\|x(t; \xi, v^\infty) - x^\infty\| \leq \varepsilon \quad \forall\, t \in \mathbb{Z}_+. \tag{B.7}$$

Moreover, by utilising the asymptotic stability of x^∞ and that $x^0 \in \mathcal{O}$, there exists $T \in \mathbb{Z}_+$ such that

$$\|x(t; x^0, v^\infty) - x^\infty\| \leq \frac{\delta_1}{2} \quad \forall\, t \in \overline{T}. \tag{B.8}$$

Furthermore, an application of Lemma B.1.1 gives the existence of $\delta > 0$ such that, for all $\xi \in \mathbb{R}^n$ satisfying $\|\xi - x^0\| \leq \delta$,

$$\|x(t; \xi, v^\infty) - x(t; x^0, v^\infty)\| \leq \frac{\delta_1}{2} \quad \forall\, t \in \underline{T}.$$

Hence, by combining this with (B.8), we obtain that, for all $\xi \in \mathbb{R}^n$ satisfying $\|\xi - x^0\| \leq \delta$,

$$\|x(T; \xi, v^\infty) - x^\infty\| \leq \delta_1.$$

By recalling (B.7), we see that this implies, for all $\xi \in \mathbb{R}^n$ satisfying $\|\xi - x^0\| \leq \delta$,

$$\|x(t; x(T; \xi, v^\infty), v^\infty) - x^\infty\| \leq \varepsilon \quad \forall\, t \in \mathbb{Z}_+.$$

Finally, from Lemma B.1.2, this is equivalent to

$$\|x(t; \xi, v^\infty) - x^\infty\| \leq \varepsilon \quad \forall\, t \in \overline{T},$$

which completes the proof. □

Our attention now turns towards discretising [112, Lemma II.2], which we do so now.

Lemma B.1.6. *Let* $(v^\infty, x^\infty) \in \mathbb{R}^m \times \mathbb{R}^n$ *be an asymptotically stable equilibrium pair of* (B.1)*. Then for all compact* $\mathcal{C} \subseteq \mathcal{O}$ *and all* $\varepsilon > 0$*, there exists* $T \in \mathbb{Z}_+$ *such that*

$$\|x(T; x^0, v^\infty) - x^\infty\| \leq \varepsilon \quad \forall\, x^0 \in \mathcal{C}.$$

Proof. Let $\mathcal{C} \subseteq \mathcal{O}$ be compact and let $\varepsilon > 0$. For each $\xi \in \mathcal{C}$, we can apply Lemma B.1.5 to obtain the existence of δ_ξ and $T_\xi \in \mathbb{Z}_+$ such that, for all $\zeta \in \mathbb{R}^n$ satisfying $\|\xi - \zeta\| \leq \delta_\xi$,

$$\|x(t; \zeta, v^\infty) - x^\infty\| \leq \varepsilon \quad \forall\, t \in \overline{T_\xi}. \tag{B.9}$$

The compactness of $\mathcal{C}$ then yields the existence of $\xi_1, \dots, \xi_k \in \mathcal{C}$, for some $k \in \mathbb{N}$, such that $\cup_{i=1}^k \mathbb{B}(\xi_i, \delta_{\xi_i})$ is a finite subcover of $\mathcal{C}$. We subsequently define $T := \max\{T_{\xi_1}, \dots, T_{\xi_k}\} \in \mathbb{Z}_+$. Therefore, for each $x^0 \in \mathcal{C}$, it follows that $x^0 \in \mathbb{B}(\xi_i, \delta_{\xi_i})$ for some $i \in \{1, \dots, k\}$, and so, by (B.9),

$$\|x(t; x^0, v^\infty) - x^\infty\| \leq \varepsilon \quad \forall\, t \in \overline{T_{\xi_i}}.$$

By using the definition of T, this implies that

$$\|x(T; x^0, v^\infty) - x^\infty\| \leq \varepsilon,$$

hence completing the proof. □

The following is inspired by [112, Lemma II.3].

Lemma B.1.7. *Let* $(v^\infty, x^\infty) \in \mathbb{R}^m \times \mathbb{R}^n$ *be an asymptotically stable equilibrium pair of* (B.1)*. Then for every compact* $\mathcal{C} \subseteq \mathcal{O}$ *and every* $\varepsilon > 0$*, there exist* $T \in \mathbb{Z}_+$ *and* $\delta > 0$ *such that, for every* $(x^0, v) \in \mathcal{C} \times (\mathbb{R}^m)^{\mathbb{Z}_+}$ *satisfying* $\max_{s \in \underline{T}} \|v(s) - v^\infty\| \leq \delta$,

$$\|x(t; x^0, v) - x(t; x^0, v^\infty)\| \leq \varepsilon \quad \forall\, t \in \underline{T}, \tag{B.10}$$

and

$$\|x(T; x^0, v) - x^\infty\| \leq \varepsilon. \tag{B.11}$$

Proof. Let $\mathcal{C} \subseteq \mathcal{O}$ be compact and $\varepsilon > 0$. An application of Lemma B.1.6 then yields the existence of $T \in \mathbb{Z}_+$ such that

$$\|x(T; x^0, v^\infty) - x^\infty\| \leq \frac{\varepsilon}{2} \quad \forall\, x^0 \in \mathcal{C}. \tag{B.12}$$

Furthermore, by Lemma B.1.4, there exists $\delta > 0$ such that, for every $(x^0, v) \in \mathcal{C} \times (\mathbb{R}^m)^{\mathbb{Z}_+}$ satisfying $\max_{s \in \underline{T}} \|v(s) - v^\infty\| \leq \delta$,

$$\|x(t; x^0, v) - x(t; x^0, v^\infty)\| \leq \frac{\varepsilon}{2} \quad \forall\, t \in \underline{T}. \tag{B.13}$$

Let $(x^0, v) \in \mathcal{C} \times (\mathbb{R}^m)^{\mathbb{Z}_+}$ be such that $\max_{s \in \underline{T}} \|v(s) - v^\infty\| \leq \delta$. From (B.13) we trivially obtain (B.10). Moreover, by combining (B.13) with (B.12), we yield

$$\|x(T; x^0, v) - x^\infty\| \leq \|x(T; x^0, v) - x(T; x^0, v^\infty)\| + \|x(T; x^0, v^\infty) - x^\infty\| \leq \varepsilon,$$

whence giving (B.11) and completing the proof. □

The following is a stability result and will be key in proving Theorem B.0.2.

Lemma B.1.8. *Let* $(v^\infty, x^\infty) \in \mathbb{R}^m \times \mathbb{R}^n$ *be an asymptotically stable equilibrium pair of* (B.1). *Then for all* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that, for all* $(x^0, v) \in \mathbb{R}^n \times (\mathbb{R}^m)^{\mathbb{Z}_+}$ *satisfying* $\|x^0 - x^\infty\| + \|v - v^\infty\|_{\ell^\infty} \le \delta$,

$$\|x(t; x^0, v) - x^\infty\| \le \varepsilon \quad \forall\, t \in \mathbb{Z}_+. \tag{B.14}$$

Proof. Let $\varepsilon > 0$ and note that, from Lemma B.1.3, there exists $\delta_1 > 0$ such that, for all $x^0 \in \mathbb{R}^n$ satisfying $\|x^0 - x^\infty\| \le \delta_1$, it follows that $x^0 \in \mathcal{O}$. Moreover, by the stability of (v^∞, x^∞), there exists $\delta_2 > 0$ such that, for all $x^0 \in \mathbb{R}^n$ satisfying $\|x^0 - x^\infty\| \le \delta_2$,

$$\|x(t; x^0, v^\infty) - x^\infty\| \le \frac{\varepsilon}{2} \quad \forall\, t \in \mathbb{Z}_+. \tag{B.15}$$

We then define $\delta_3 := \min\{\varepsilon, \delta_1, \delta_2\} > 0$ and let $\mathcal{C} \subseteq \mathcal{O}$ be the closed ball of radius $\delta_3/2$ centred at x^∞. An application of Lemma B.1.7 yields the existence of $T \in \mathbb{Z}_+$ and $\delta_4 > 0$ such that, for all $(x^0, v) \in \mathcal{C} \times (\mathbb{R}^m)^{\mathbb{Z}_+}$ satisfying $\max_{s \in \underline{T}} \|v(s) - v^\infty\| \le \delta_4$,

$$\|x(t; x^0, v) - x(t; x^0, v^\infty)\| \le \frac{\delta_3}{2} \quad \forall\, t \in \underline{T}, \tag{B.16}$$

and

$$\|x(T; x^0, v) - x^\infty\| \le \frac{\delta_3}{2}. \tag{B.17}$$

We now define

$$\delta := \min\left\{\delta_4, \frac{\delta_3}{2}\right\} > 0,$$

and let $(x^0, v) \in \mathbb{R}^n \times (\mathbb{R}^m)^{\mathbb{Z}_+}$ be such that $\|x^0 - x^\infty\| + \|v - v^\infty\|_{\ell^\infty} \le \delta$. With the intent of showing that (B.14) holds, we claim that

$$x((k+1)T; x^0, v) \in \mathcal{C} \quad \forall\, k \in \mathbb{Z}_+, \tag{B.18}$$

and

$$\|x(t; x^0, v) - x^\infty\| \le \varepsilon \quad \forall\, t \in \{kT, \dots, (k+1)T\}, \forall\, k \in \mathbb{Z}_+. \tag{B.19}$$

To show this, we utilise an inductive method. To this end, for the case $k = 0$, we use (B.17) to immediately give that $x(T; x^0, v) \in \mathcal{C}$. Additionally, since $\|x^0 - x^\infty\| \le \delta_3/2 < \delta_2$, we see that (B.15) holds. By combining this with (B.16), we thus yield that

$$\|x(t; x^0, v) - x^\infty\| \le \|x(t; x^0, v) - x(t; x^0, v^\infty)\| + \|x(t; x^0, v^\infty) - x^\infty\| \le \varepsilon \quad \forall\, t \in \underline{T},$$

thus proving the case $k = 0$. We now assume that the claim holds true for arbitrary $k \in \mathbb{Z}_+$. First note that since $\|v - v^\infty\|_{\ell^\infty} \le \delta$, we have that $\|\Lambda_{(k+1)T} v - v^\infty\|_{\ell^\infty} \le \delta$. Moreover, by the induction hypothesis, $x((k+1)T; x^0, v) \in \mathcal{C}$ and so we are able to invoke (B.17) to obtain that

$$\left\|x\left(T; x((k+1)T; x^0, v), \Lambda_{(k+1)T} v\right) - x^\infty\right\| \le \frac{\delta_3}{2},$$

that is, (see Lemma B.1.2)

$$\|x((k+2)T; x^0, v) - x^\infty\| \le \frac{\delta_3}{2},$$

whence giving that $x((k+2)T;x^0,v) \in \mathcal{C}$. Finally, using (B.16),

$$\left\|x\left(t;x\left((k+1)T;x^0,v\right),\Lambda_{(k+1)T}v\right) - x\left(t;x\left((k+1)T;x^0,v\right),v^\infty\right)\right\| \leq \frac{\delta_3}{2} \leq \frac{\varepsilon}{2},$$

for all $t \in \underline{T}$. By once again applying Lemma B.1.2, from the above we deduce that

$$\|x(t+(k+1)T;x^0,v) - x\left(t;x\left((k+1)T;x^0,v\right),v^\infty\right)\| \leq \frac{\varepsilon}{2} \quad \forall\, t \in \underline{T}.$$

We recall that $\xi := x\left((k+1)T;x^0,v\right) \in \mathcal{C}$. Whence, by combining the previous inequality with (B.15), we obtain that, for all $t \in \underline{T}$,

$$\begin{aligned}\|x(t+(k+1)T;x^0,v) - x^\infty\| &\leq \|x(t+(k+1)T;x^0,v) - x\left(t;\xi,v^\infty\right)\| \\ &\quad + \|x\left(t;\xi,v^\infty\right) - x^\infty\| \\ &\leq \varepsilon.\end{aligned}$$

This is equivalent to saying that

$$\|x(t;x^0,v) - x^\infty\| \leq \varepsilon \quad \forall\, t \in \{(k+1)T,\ldots,(k+2)T\}.$$

By induction, we infer that (B.18) and (B.19) hold. To conclude the proof, we note that we have shown that, for all $(x^0,v) \in \mathbb{R}^n \times (\mathbb{R}^m)^{\mathbb{Z}_+}$ satisfying $\|x^0-x^\infty\|+\|v-v^\infty\|_{\ell^\infty} \leq \delta$, (B.14) holds, completing the proof. □

We now give the final preliminary result that we shall use in order to prove Theorem B.0.2. The result is a discrete-time version of [112, Proposition II.4].

Lemma B.1.9. *Let $(v^\infty,x^\infty) \in \mathbb{R}^m \times \mathbb{R}^n$ be an asymptotically stable equilibrium pair of* (B.1)*. Then for every compact set $\mathcal{C} \subseteq \mathcal{O}$ and every $\varepsilon > 0$, there exist $T \in \mathbb{Z}_+$ and $\delta > 0$ such that, for every $(x^0,v) \in \mathcal{C} \times (\mathbb{R}^m)^{\mathbb{Z}_+}$ with $\|v-v^\infty\|_{\ell^\infty} \leq \delta$,*

$$\|x(t;x^0,v) - x^\infty\| \leq \varepsilon \quad \forall\, t \in \overline{T}.$$

Proof. Let $\mathcal{C} \subseteq \mathcal{O}$ be a compact set and $\varepsilon > 0$. An application of Lemma B.1.8 gives the existence of $\delta_1 > 0$ such that, for all $(x^0,v) \in \mathbb{R}^n \times (\mathbb{R}^m)^{\mathbb{Z}_+}$ satisfying $\|x^0-x^\infty\| + \|v-v^\infty\|_{\ell^\infty} \leq \delta_1$, (B.14) holds. Additionally, by invoking Lemma B.1.7, we obtain the existence of $T \in \mathbb{Z}_+$ and $\delta_2 > 0$ such that, for all $(x^0,v) \in \mathcal{C} \times (\mathbb{R}^m)^{\mathbb{Z}_+}$ satisfying $\max_{s\in\underline{T}} \|v(s) - v^\infty\| \leq \delta_2$,

$$\|x(T;x^0,v) - x^\infty\| \leq \frac{\delta_1}{2}. \tag{B.20}$$

We define $\delta := \min\{\delta_1,\delta_2\}/2 > 0$. Furthermore, let $(x^0,v) \in \mathcal{C} \times (\mathbb{R}^m)^{\mathbb{Z}_+}$ be such that $\|v-v^\infty\|_{\ell^\infty} \leq \delta$. This implies that $\|\Lambda_T v - v^\infty\|_{\ell^\infty} \leq \delta$ and so, by recalling (B.20) and invoking (B.14), we obtain that

$$\|x(t;x(T;x^0,v),\Lambda_T v) - x^\infty\| \leq \varepsilon \quad \forall\, t \in \mathbb{Z}_+.$$

Since Lemma B.1.2 gives that $x(t+T;x^0,v) = x(t;x(T;x^0,v),\Lambda_T v)$ for all $t \in \mathbb{Z}_+$, we see that the above inequality implies

$$\|x(t+T;x^0,v) - x^\infty\| \leq \varepsilon \quad \forall\, t \in \mathbb{Z}_+,$$

i.e.

$$\|x(t;x^0,v) - x^\infty\| \leq \varepsilon \quad \forall\, t \in \overline{T},$$

thus completing the proof. □

We are now in a position to give the proof of Theorem B.0.2.

Proof of Theorem B.0.2. Let $\mathcal{C}$ be a compact set in $\mathcal{O}$, $v \in (\mathbb{R}^m)^{\mathbb{Z}_+}$, $x^0 \in \mathbb{R}^n$ and $x(\cdot; x^0, v)$ be a $\mathcal{C}$-recurrent solution of (B.1). Furthermore, assume that $\lim_{t\to\infty} v(t) = v^\infty$. Let $\varepsilon > 0$. To complete the proof, it is sufficient to show that there exists $T \in \mathbb{Z}_+$ such that

$$\|x(t; x^0, v) - x^\infty\| \le \varepsilon \quad \forall\, t \in \overline{T}.$$

To this end, by Lemma B.1.9, there exist $T_0 \in \mathbb{Z}_+$ and $\delta > 0$ such that, for all $(\xi, w) \in \mathcal{C} \times (\mathbb{R}^m)^{\mathbb{Z}_+}$ with $\|w - v^\infty\|_{\ell^\infty} \le \delta$,

$$\|x(t; \xi, w) - x^\infty\| \le \varepsilon \quad \forall\, t \in \overline{T_0}. \tag{B.21}$$

Since $v(t) \to v^\infty$ as $t \to \infty$, there exists $T_1 \in \mathbb{Z}_+$ such that

$$\|v(t) - v^\infty\| \le \delta \quad \forall\, t \in \overline{T_1}.$$

Furthermore, since $x(\cdot; x^0, v)$ is $\mathcal{C}$-recurrent, there exists $T_2 \in \overline{T_1}$ such that $x(T_2; x^0, v) \in \mathcal{C}$. Now, using (B.21) with $\xi = x(T_2; x^0, v)$ and $w = \Lambda_{T_2} v$, we obtain that

$$\|x(t; x(T_2; x^0, v), \Lambda_{T_2} v) - x^\infty\| \le \varepsilon \quad \forall\, t \in \overline{T_0}.$$

Since, from Lemma B.1.2, $x(t + T_2; x^0, v) = x(t; x(T_2; x^0, v), \Lambda_{T_2} v)$ for all $t \in \mathbb{Z}_+$, we see that this implies

$$\|x(t + T_2; x^0, v) - x^\infty\| \le \varepsilon \quad \forall\, t \in \overline{T_0},$$

which is equivalent to

$$\|x(t; x^0, v) - x^\infty\| \le \varepsilon \quad \forall\, t \in \overline{T},$$

where $T := T_0 + T_2$, thus completing the proof. □

Appendix C

Almost periodic functions

Almost periodic functions were first conceived by Harold Bohr in the 1920s [16]. They are a generalisation of the notion of continuous periodic functions and are frequently used in the theory of differential and difference equations (see, for example, [34, 52, 105, 128]). Much work has been attributed to the theory, and several generalisations of the notion have been developed (see, for example, [14]). One such is the notion of almost periodic functions in the sense of Stepanov. In this appendix, we present theory of (Bohr) almost periodic functions and Stepanov almost periodic functions over the time domains $\mathbb{R}_+$, $\mathbb{R}$, $\mathbb{Z}_+$ and $\mathbb{Z}$. As a rule-of-thumb, literature concerning the time domains $\mathbb{R}$ and $\mathbb{Z}$ is easy to find (see, such as, [1, 14, 17, 21, 22]), however it is difficult to find explicit references for the cases of $\mathbb{R}_+$ and $\mathbb{Z}_+$. Thus, in the sequel, where theory is difficult to find, we shall give explicit proofs of results.

We split this appendix in two, with the first section concerning (Bohr) almost periodic functions, and the second comprising theory of Stepanov almost periodic functions.

Throughout this appendix, we let X and Y be Banach spaces and, moreover, we define

$$\left\| \begin{pmatrix} x \\ y \end{pmatrix} \right\|_{X\times Y} := \sqrt{\|x\|_X^2 + \|y\|_Y^2} \quad \forall\, x \in X, \forall\, y \in Y, \tag{C.1}$$

and let

$$R = \mathbb{R}_+ \text{ or } \mathbb{R}, \quad \text{and} \quad Z = \mathbb{Z}_+ \text{ or } \mathbb{Z}.$$

C.1 Bohr almost periodicity

We shall first consider almost periodic functions on the time domains $\mathbb{R}_+$ and $\mathbb{R}$. To this end, we begin with the following definition.

Definition C.1.1. (i) *A set $S \subseteq R$ is said to be* relatively dense *(in R) if there exists $L > 0$ such that*

$$[a, a+L] \cap S \neq \emptyset \quad \forall\, a \in R.$$

(ii) *Let $\varepsilon > 0$. Then $\tau \in R$ is said to be an* ε-period *of a continuous function $f : R \to X$ if*

$$\|f(t) - f(t+\tau)\|_X \leq \varepsilon \quad \forall\, t \in R.$$

We shall denote by $P(f, \varepsilon) \subseteq R$ the set of ε-periods of f.

(iii) *We say that a continuous function $f : R \to X$ is* almost periodic *(in the sense of Bohr) if $P(f,\varepsilon)$ is relatively dense in R for every $\varepsilon > 0$.*

(iv) *We denote the set of almost periodic functions $R \to X$ by $AP(R,X)$.*

Remark C.1.2. (i) An example of a relatively dense set in $\mathbb{R}_+$ is $\mathbb{Z}_+$. Moreover, it is worth noting that the intersection of two relatively dense sets need not be non-empty. Indeed, the sets of even and odd positive integers are both relatively dense in $\mathbb{R}_+$, but clearly have empty intersection.

(ii) Trivially, a continuous periodic function is almost periodic. We hence may think of almost periodicity as a generalisation of the standard notion of periodicity. However, the converse is false. For a counterexample, see [14, p.ix].

(iii) For $\tau \in R$, if an almost periodic function $f : R \to X$ satisfies that $\tau \in P(f,\varepsilon)$ for every $\varepsilon > 0$, then f is τ-periodic. ◊

The following result asserts that almost periodic functions are bounded and uniformly continuous.

Lemma C.1.3. *Let $f \in AP(R,X)$. Then f is bounded and uniformly continuous.*

This is very well-known in the case $R = \mathbb{R}$ (see, for example, [22, Proposition 3.16, p.71]). The half-line case can be proven via a similar method. For completeness, we provide a proof of this situation.

Proof of Lemma C.1.3. Let $f \in AP(R,X)$ and $R = \mathbb{R}_+$. We begin by showing that f is bounded. To this end, by taking $\varepsilon = 1$, the almost periodicity of f guarantees the existence of $\delta > 0$ such that, for all $t_0 \in R$, there exists $\tau \in [t_0, t_0 + \delta]$ which is an ε-period of f. Define

$$c := \sup_{s \in [0,\delta]} \|f(s)\|_X,$$

and note that, in order to prove the boundedness of f, it is sufficient to show that $\|f(t)\|_X \leq 1 + c$ for all $t \in [\delta, \infty)$, which we shall do so now. Let $t \in [\delta, \infty)$ and additionally let $\tau \in [t - \delta, t]$ be an ε-period of f. We note that $t - \tau \in [0, \delta]$, which subsequently yields that

$$\|f(t)\|_X \leq \|f(t-\tau) - f(t)\|_X + \|f(t-\tau)\|_X \leq 1 + c,$$

and shows that f is bounded. Moving on to showing uniform continuity, let $\varepsilon > 0$ and let $\delta_1 > 0$ be such that, for all $t_0 \in R$, there exists $\tau \in [t_0, t_0 + \delta]$ which is an $\varepsilon/3$-period. Since f is continuous, f is uniformly continuous on $[0, \delta_1 + 1]$, and so we let $\delta_2 \in (0,1)$ be such that

$$\|f(t_1) - f(t_2)\|_X \leq \frac{\varepsilon}{3} \quad \forall\, t_1, t_2 \in [0, \delta_1 + 1] \text{ s.t. } |t_1 - t_2| \leq \delta_2. \tag{C.2}$$

We now fix $t_1, t_2 \in R$ such that $|t_1 - t_2| < \delta_2$. If one of t_1 or t_2, say t_2, is in $[0, \delta_1]$, then

$$t_1 \leq |t_1 - t_2| + t_2 \leq \delta_2 + \delta_1 \leq 1 + \delta_1,$$

and, by (C.2), the proof is complete. Thus, we assume that t_1 and t_2 are both strictly larger than δ_1. Further, without loss of generality, we assume that $t_2 \geq t_1$. Let

$\tau \in [t_1 - \delta_1, t_1]$ be an $\varepsilon/3$-period and define $s_1 := t_1 - \tau$ and $s_2 := t_2 - \tau$. It is easily verified that $s_1 \in [0, \delta_1]$ and $s_2 \in [0, \delta_1 + 1]$. Finally, we thus obtain, by (C.2) and the choice of τ, that

$$\|f(t_1) - f(t_2)\|_X \leq \|f(t_1) - f(s_1)\|_X + \|f(s_1) - f(s_2)\|_X + \|f(s_2) - f(t_2)\|_X \leq \varepsilon,$$

which completes the proof. □

The next lemma is a useful property regarding the 'tails' of almost periodic functions.

Lemma C.1.4. *Let $f \in AP(R, X)$. The following statements hold.*

(i) *If $R = \mathbb{R}_+$, then*

$$\sup_{t \geq T} \|f(t)\|_X = \sup_{t \in R} \|f(t)\|_X \quad \forall T \in \mathbb{R}_+.$$

(ii) *If $R = \mathbb{R}$, then*

$$\sup_{t \leq T} \|f(t)\|_X = \sup_{t \in R} \|f(t)\|_X = \sup_{t \geq T} \|f(t)\|_X \quad \forall T \in \mathbb{R}.$$

The result is surely known in the literature (at least in the case that $R = \mathbb{R}$), but it is difficult to locate. We hence prove a proof here.

Proof of Lemma C.1.4. We shall only provide a proof of statement (i), since statement (ii) can be proven in a similar manner. We begin noting that the suprema in question are well-defined and finite by Lemma C.1.3. Fix $T \in R = \mathbb{R}_+$ and note that, trivially,

$$\sup_{t \in R} \|f(t)\|_X \geq \sup_{t \geq T} \|f(t)\|_X.$$

Therefore, in order to prove statement (i), it is sufficient to show that

$$\sup_{t \in R} \|f(t)\|_X \leq \sup_{t \geq T} \|f(t)\|_X + \varepsilon \quad \forall \varepsilon > 0, \tag{C.3}$$

which we do so now. Let $\varepsilon > 0$, $t \in R$ and let $\tau \in \mathbb{R}_+$ be an ε-period of f chosen large enough so that $\tau + t \geq T$. We then see that

$$\|f(t)\|_X \leq \|f(t + \tau) - f(t)\|_X + \|f(t + \tau)\|_X \leq \varepsilon + \sup_{s \geq T} \|f(s)\|_X.$$

Since t was arbitrary, we obtain that (C.3) holds and the proof is complete. □

Lemma C.1.4 can be used to deduce many properties of almost periodic functions. Two such, are outlined in the following remark.

Remark C.1.5. (i) If $f \in AP(\mathbb{R}, X)$ is such that $\lim_{t \in \mathbb{R}, t \to \infty} f(t) = 0$, then Lemma C.1.4 yields that $f = 0$.

(ii) Let $T \in \mathbb{R}$. If $f \in AP(\mathbb{R}, X)$ is such that $f(t) = 0$ for all $t \geq T$, or $f(t) = 0$ for all $t \leq T$, then Lemma C.1.4 gives that $f = 0$. ◊

The next result asserts a relationship between $AP(\mathbb{R}, X)$ and $AP(\mathbb{R}_+, X)$.

Lemma C.1.6. *For every $f \in AP(\mathbb{R}_+, X)$, there exists a unique function $f_\mathrm{e} : \mathbb{R} \to X$ such that $f_\mathrm{e} \in AP(\mathbb{R}, X)$ and $f_\mathrm{e}(t) = f(t)$ for all $t \in \mathbb{R}_+$. Moreover,*

$$\sup_{t\in\mathbb{R}} \|f_\mathrm{e}(t)\|_X = \sup_{t\in\mathbb{R}_+} \|f(t)\|_X, \tag{C.4}$$

and, for every $\varepsilon > 0$, $P(f_\mathrm{e}, \varepsilon) = \{\pm\tau : \tau \in P(f, \varepsilon)\}$.

Before proving this, we give some commentary.

Remark C.1.7. Lemma C.1.6 will be especially useful in asserting basic properties of $AP(\mathbb{R}_+, X)$. Since many of these properties are easy to find in the literature for $AP(\mathbb{R}, X)$, Lemma C.1.6 will allow us to carry these properties through and deduce them for $AP(\mathbb{R}_+, X)$. ♢

Proof of Lemma C.1.6. In the sequel, we act following an idea in [12, Remark on p.318]. To this end, let $f \in AP(\mathbb{R}_+, X)$ and, for each $k \in \mathbb{N}$, let $\tau_k \in P(f, 1/k)$ be such that $\tau_k \nearrow \infty$ as $k \to \infty$. We define $f_\mathrm{e} : \mathbb{R} \to X$ by

$$f_\mathrm{e}(t) = \lim_{k\to\infty} f(t + \tau_k) \quad \forall\, t \in \mathbb{R}. \tag{C.5}$$

We pause here and show that f_e is well-defined. Indeed, note that, for given $t \in \mathbb{R}$,

$$\begin{aligned}\|f(t+\tau_k) - f(t+\tau_l)\|_X &\le \|f(t+\tau_k) - f(t+\tau_k+\tau_l)\|_X \\ &\quad + \|f(t+\tau_k+\tau_l) - f(t+\tau_l)\|_X \\ &\le \frac{1}{l} + \frac{1}{k},\end{aligned}$$

for all $k, l \in \mathbb{N}$ sufficiently large. We hence see that $(f(t+\tau_k))_{k\in\mathbb{N}} \subseteq X$ is a Cauchy sequence, which in turn implies that $f_\mathrm{e}(t)$ is well-defined for every $t \in \mathbb{R}$. Moving on, it is trivial that $f_\mathrm{e}(t) = f(t)$ for all $t \in \mathbb{R}_+$. Moreover, it is straight forward to prove that, for all $\varepsilon > 0$, $P(f_\mathrm{e}, \varepsilon) = \{\pm\tau : \tau \in P(f, \varepsilon)\}$. Thus, in order to prove that $f_\mathrm{e} \in AP(\mathbb{R}, X)$, it suffices to show that f_e is continuous, which we shall do so now. Let $\varepsilon > 0$ and $t \in \mathbb{R}$. Note that, since f is uniformly continuous by Lemma C.1.3, we may obtain the existence of $\delta > 0$ such that

$$\|f(s_1) - f(s_2)\| \le \frac{\varepsilon}{3} \quad \forall\, s_1, s_2 \in \mathbb{R}_+ \text{ s.t. } |s_1 - s_2| \le \delta.$$

Furthermore, for $s \in \mathbb{R}$ such that $|s - t| \le \delta$, note that there exist $K_t, K_s \in \mathbb{N}$ such that

$$\|f_\mathrm{e}(t) - f(t+\tau_k)\|, \|f_\mathrm{e}(s) - f(s+\tau_k)\| \le \frac{\varepsilon}{3},$$

for all $k \ge K_t$ and K_s, respectively. Let $k \ge K_t + K_s$. By combining the previous two inequalities, we yield that

$$\begin{aligned}\|f_\mathrm{e}(t) - f_\mathrm{e}(s)\| &\le \|f_\mathrm{e}(t) - f(t+\tau_k)\| + \|f(t+\tau_k) - f(s+\tau_k)\| + \|f(s+\tau_k) - f_\mathrm{e}(s)\| \\ &\le \varepsilon,\end{aligned}$$

whence giving that f_e is continuous. For the uniqueness of f_e, assume that there exists another almost periodic extension of f to $\mathbb{R}$, and denote this by g. Since $f_\mathrm{e}(t) - g(t) = 0$ for all $t \in \mathbb{R}_+$, and $f_\mathrm{e} - g \in AP(\mathbb{R}, X)$ (see, for example, [1, IX, p.10]), Lemma C.1.4 implies that $f_\mathrm{e} = g$. Finally, (C.4) is immediately obtained from Lemma C.1.4, hence concluding the proof. □

We now give some basic properties of $AP(R,X)$. The case $R = \mathbb{R}$ is known in the literature and we will provide explicit references of these properties as we proceed. For the case $R = \mathbb{R}_+$, we shall deduce the properties by using the known results of the case $R = \mathbb{R}$ along with Lemma C.1.6 (see Remark C.1.7). We begin with the following linearity assertion.

Lemma C.1.8. *If $f, g \in AP(R,X)$, then $f + g$ is in $AP(R,X)$.*

The case that $R = \mathbb{R}$ can be found in, for example, [1, IX, p.10], and so the following proof is for the situation that $R = \mathbb{R}_+$.

Proof of Lemma C.1.8. Let $f, g \in AP(\mathbb{R}_+, X)$. We apply Lemma C.1.6 to obtain (unique) extensions $f_{\mathrm{e}}, g_{\mathrm{e}} \in AP(\mathbb{R}, X)$ of f and g, respectively. We note that $f_{\mathrm{e}} + g_{\mathrm{e}} \in AP(\mathbb{R}, X)$ (again, see such as [1, IX, p.10]) and that, for all $t \in \mathbb{R}_+$,

$$(f_{\mathrm{e}} + g_{\mathrm{e}})(t) = f_{\mathrm{e}}(t) + g_{\mathrm{e}}(t) = f(t) + g(t) = (f + g)(t).$$

Hence $f_{\mathrm{e}} + g_{\mathrm{e}}$ is an extension of $f + g$ and is almost periodic. This trivially implies that $f + g$ is almost periodic, since the restriction of an almost periodic function on $\mathbb{R}$ to $\mathbb{R}_+$ is almost periodic. □

The next lemma is a closedness result for almost periodic functions.

Lemma C.1.9. *If $(f_k)_{k \in \mathbb{N}} \subseteq AP(R,X)$ converges uniformly to $f : R \to X$, then $f \in AP(R,X)$.*

The case $R = \mathbb{R}$ is easily found in the literature: see, for example, [1, V, p.6].

Proof of Lemma C.1.9. As done for the full-line case in the proof of [1, V, p.6], the half-line case is immediately obtained by use of the inequality

$$\begin{aligned}\|f(t+\tau) - f(t)\|_X \leq & \|f(t+\tau) - f_k(t+\tau)\|_X + \|f_k(t+\tau) - f_k(t)\|_X \\ & + \|f_k(t) - f(t)\|_X \quad \forall\, t, \tau \in R, \forall\, k \in \mathbb{Z}_+.\end{aligned}$$

□

As a consequence of Lemmas C.1.3, C.1.8 and C.1.9, and the fact that X is a Banach space, it is easy to deduce the following result.

Corollary C.1.10. *$AP(R,X)$ is a Banach space with resepect to the relevant supremum-norm.*

With this and Lemma C.1.4 in mind, may now "improve" Lemma C.1.6 with the following result, the proof of which is straightforward and so is omitted.

Theorem C.1.11. *The map $AP(\mathbb{R}_+, X) \to AP(\mathbb{R}, X)$ which maps f to f_{e}, where f_{e} is given by* (C.5)*, is an isometric isomorphism.*

Remark C.1.12. We note that an isometric isomporphism is proven to exist between $AP(\mathbb{R}_+, X)$ and $AP(\mathbb{R}, X)$ in [11, Lemma 7]. However, in [11], it is not clear what the explicit formulation of f_{e} is. ◊

For the following result, recall that for the cartesian product of Banach spaces, we assume the norm is given by (C.1) (or something equivalent).

Lemma C.1.13. *Let $f \in AP(R, X)$ and $g \in AP(R, Y)$. Then, for all $\varepsilon > 0$, $P(f, \varepsilon) \cap P(g, \varepsilon)$ is non-empty and relatively dense in R. Furthermore,*

$$AP(R, X \times Y) = AP(R, X) \times AP(R, Y). \tag{C.6}$$

The case that $R = \mathbb{R}$ can be found in, for example, [1, p.10].

Proof of Lemma C.1. We only consider the case $R = \mathbb{R}_+$ since, as previously mentioned, the case $R = \mathbb{R}$ is not difficult to find in the literature. To this end, we let $\varepsilon > 0$ and apply Lemma C.1.6 to obtain (unique) extensions $f_\mathrm{e} \in AP(\mathbb{R}, X)$ and $g_\mathrm{e} \in AP(\mathbb{R}, Y)$ of f and g, respectively. From the full-line case, we obtain that $P(f_\mathrm{e}, \varepsilon) \cap P(g_\mathrm{e}, \varepsilon)$ is non-empty and relatively dense in $\mathbb{R}$ for all $\varepsilon > 0$. Trivially, we yield that $P(f, \varepsilon) \cap P(g, \varepsilon)$ is non-empty and relatively dense in $\mathbb{R}_+$ for all $\varepsilon > 0$. As for showing that (C.6) holds, note that, by the definition of the norm on $X \times Y$ (see (C.1)),

$$\tau \in P(f, \varepsilon) \cap P(g, \varepsilon) \implies \tau \in P\left(\begin{pmatrix} f \\ g \end{pmatrix}, \sqrt{2}\varepsilon\right) \quad \forall\, \varepsilon > 0,$$

and

$$\tau \in P\left(\begin{pmatrix} f \\ g \end{pmatrix}, \varepsilon\right) \implies \tau \in P(f, \varepsilon) \cap P(g, \varepsilon) \quad \forall\, \varepsilon > 0.$$

Therefore, by combining this with the first part of the proof, we obtain that (C.6) is true, and the proof is complete. ☐

We now define what we mean by an asymptotically almost periodic function over $\mathbb{R}_+$.

Definition C.1.14. *We denote by $C_0(\mathbb{R}_+, X)$ the space of continuous functions $f : \mathbb{R}_+ \to X$ such that $\lim_{t\to\infty} f(t) = 0$, and endow it with the supremum-norm. Moreover, we say that a continuous function $f : \mathbb{R}_+ \to X$ is* asymptotically almost periodic *if $f \in AP(\mathbb{R}_+, X) + C_0(\mathbb{R}_+, X)$, and denote the space of asymptotically almost periodic functions $\mathbb{R}_+ \to X$ by $AAP(\mathbb{R}_+, X)$.*

As an immediate consequence of Lemma C.1.4, we obtain the next result.

Lemma C.1.15. *The following statements hold.*

(i) $AP(\mathbb{R}_+, X) \cap C_0(\mathbb{R}_+, X) = \{0\}$.

(ii) *If $f \in AAP(\mathbb{R}_+, X)$, then the decomposition $f = f^{\mathrm{ap}} + f^0$, where $f^{\mathrm{ap}} \in AP(\mathbb{R}_+, X)$ and $f^0 \in C_0(\mathbb{R}_+, X)$, is unique.*

It is well-known (see, for example, [14]) that (Bohr) almost periodicity over $\mathbb{R}$ is equivalent to several other notions. We will now discuss whether or not these equivalences hold in the 'half-line' case. We begin with the following definition.

Definition C.1.16. *We denote by $TP(R, X)$ the space of all X-valued* trigonometric functions *over R endowed with the supremum-norm, that is, functions $p : R \to X$ of the form*

$$p(t) = \sum_{k=0}^{N} a_k e^{i\lambda_k t} \quad \forall\, t \in R,$$

where $N \in \mathbb{Z}_+$ and, for each $k \in \{0, \ldots, N\}$, $a_k \in X$ and $\lambda_k \in \mathbb{R}$.

The following propositon provides a characterisation of the space of almost periodic functions.

Proposition C.1.17. *We have that*

$$AP(R,X) = \operatorname{clos}(TP(R,X)), \tag{C.7}$$

where we take the closure of $TP(R,X)$ in the space of bounded uniformly continuous functions $R \to X$.

The case of $R = \mathbb{R}$ is easily found in the literature (see, for example, [22, Proposition 3.20 and Theorem 4.8]) and so we shall only prove the case $R = \mathbb{R}_+$.

Proof of Proposition C.1.17. Let $R = \mathbb{R}_+$. We begin with the realisation that if $f \in TP(\mathbb{R}_+,X)$, then since f is a linear combination of almost periodic functions, it is almost periodic by Corollary C.1.10. From this same result, we then see that the uniform limit of a sequence of functions in $TP(\mathbb{R}_+,X)$ is almost periodic, hence yielding that $\operatorname{clos}(TP(\mathbb{R}_+,X)) \subseteq AP(\mathbb{R}_+,X)$. As for the reverse inclusion, if $f \in AP(\mathbb{R}_+,X)$, then, by Theorem C.1.11, f_{e} is the unique almost periodic extension of f to $\mathbb{R}$. We hence deduce, by using (C.7) with $R = \mathbb{R}$, that $f_{\mathrm{e}} \in \operatorname{clos}(TP(\mathbb{R},X))$. By letting $(p_k)_{k\in\mathbb{Z}_+} \subseteq TP(\mathbb{R},X)$ be such that

$$f_{\mathrm{e}}(t) = \lim_{k\to\infty} p_k(t) \quad \forall\, t \in \mathbb{R},$$

we see that

$$f(t) = f_{\mathrm{e}}(t) = \lim_{k\to\infty} p_k(t) = \lim_{k\to\infty} q_k(t) \quad \forall\, t \in \mathbb{R}_+,$$

where $(q_k)_{k\in\mathbb{Z}_+} \subseteq TP(\mathbb{R}_+,X)$ are the restrictions of $(p_k)_{k\in\mathbb{Z}_+}$ to $\mathbb{R}_+$. Therefore, $AP(\mathbb{R}_+,X) \subseteq \operatorname{clos}(TP(\mathbb{R}_+,X))$ and the proof is complete. □

We now define what we mean by normality (see again, for example, [14]).

Definition C.1.18. *We say that a continuous function $f : R \to X$ is* normal *if for all sequences $(\sigma_k)_{k\in\mathbb{Z}_+} \subseteq R$, the sequence of translates $(\Lambda_{\sigma_k} f)_{k\in\mathbb{Z}_+}$ has a uniformly convergent subsequence. Moreover, we denote by $N(R,X)$ the set of X-valued normal functions over R.*

It is well-known (see, for example, [22, Theorem 3.2]) that

$$AP(\mathbb{R},X) = N(\mathbb{R},X). \tag{C.8}$$

We note that the notion of normality is often termed "Bochner's definition of almost periodicity", see, for example, [14, p.10]. It is easily seen that this equality does not hold in the half-line case of $R = \mathbb{R}_+$. Indeed, by Lemma C.1.15, any non-constant function in $C_0(\mathbb{R}_+,X)$ provides a counter-example. What is true in the half-line case, however, is that

$$N(\mathbb{R}_+,X) = AAP(\mathbb{R}_+,X). \tag{C.9}$$

Indeed, this is proven in [98, Theorem 3.1].

Our attention now moves towards forming a 'module containment' result for almost periodic vector-valued functions, that is, a generalisation of [34, Theorem 4.5, p.61]. Before coming to this, we first give some preliminaries, starting with the following definition.

Definition C.1.19. *Let $f \in L^1_{\text{loc}}(\mathbb{R}, X)$. The* mean value *of f is defined to be*

$$M(f) := \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} f(t)dt,$$

whenever the limit exists.

From [1, II, p.21], we obtain that the mean value of an almost periodic function exists.

Lemma C.1.20. *The mean value $M(f)$ exists for all $f \in AP(\mathbb{R}, X)$, and, furthermore,*

$$M(f) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T+a}^{T+a} f(t)dt,$$

uniformly with respect to $a \in \mathbb{R}$.

We now present a result which, as a consequence, shall provide us with an alternative definition of the mean value of an almost periodic function.

Lemma C.1.21. *Let $f \in L^1_{\text{loc}}(\mathbb{R}, X)$. Then $\lim_{T\to\infty} \frac{1}{2T}\int_{-T+a}^{T+a} f(t)dt$ exists uniformly with respect to $a \in \mathbb{R}$ if, and only if, $\lim_{T\to\infty} \frac{1}{T}\int_{a}^{T+a} f(t)dt$ exists uniformly with respect to $a \in \mathbb{R}$. Furthermore, when the previous limits exist uniformly with respect to $a \in \mathbb{R}$,*

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-T+a}^{T+a} f(t)dt = \lim_{T\to\infty} \frac{1}{T}\int_{a}^{T+a} f(t)dt. \tag{C.10}$$

Proof. Let us begin by assuming that $c := \lim_{T\to\infty} \frac{1}{2T}\int_{-T+a}^{T+a} f(t)dt \in X$ exists uniformly with respect to $a \in \mathbb{R}$ and then deduce that $\lim_{T\to\infty} \frac{1}{T}\int_{a}^{T+a} f(t)dt$ exists uniformly with respect to $a \in \mathbb{R}$. To prove this, it is sufficient to show that for all $\varepsilon > 0$, there exists $\tilde{T} > 0$ such that

$$\left\| c - \frac{1}{T}\int_{a}^{T+a} f(t)dt \right\|_X \leq \varepsilon \quad \forall\, T \geq \tilde{T},\, \forall\, a \in \mathbb{R}. \tag{C.11}$$

To this end, let $\varepsilon > 0$ and let $\tilde{L} > 0$ be such that

$$\left\| c - \frac{1}{2L}\int_{-L+b}^{L+b} f(t)dt \right\|_X \leq \varepsilon \quad \forall\, L \geq \tilde{L},\, \forall\, b \in \mathbb{R}. \tag{C.12}$$

The existence of $\tilde{L}$ is guaranteed by the hypothesis. Set $\tilde{T} := 2\tilde{L} > 0$. Let $T \geq \tilde{T}$ and $a \in \mathbb{R}$. By setting $L := T/2 \geq \tilde{T}/2 = \tilde{L}$ and $b := a + L \in \mathbb{R}$, (C.12) implies that

$$\left\| c - \frac{1}{T}\int_{a}^{T+a} f(t)dt \right\|_X = \left\| c - \frac{1}{2L}\int_{-L+b}^{L+b} f(t)dt \right\|_X \leq \varepsilon.$$

We have therefore shown that (C.11) holds, thus proving that $\lim_{T\to\infty} \frac{1}{T}\int_{a}^{T+a} f(t)dt$ exists uniformly with respect to $a \in \mathbb{R}$ and equals c.
For the converse implication, we now instead assume that $d := \lim_{T\to\infty} \frac{1}{T}\int_{a}^{T+a} f(t)dt$ exists uniformly with respect to $a \in \mathbb{R}$ and we will prove that, for all $\varepsilon > 0$, there exists $\tilde{T} > 0$ such that

$$\left\| d - \frac{1}{2T}\int_{-T+a}^{T+a} f(t)dt \right\|_X \leq \varepsilon \quad \forall\, T \geq \tilde{T},\, \forall\, a \in \mathbb{R}. \tag{C.13}$$

To see this holds, let $\varepsilon > 0$ and note that by hypothesis, there exists $\tilde{L} > 0$ such that

$$\left\| d - \frac{1}{L}\int_b^{L+b} f(t)dt \right\|_X \leq \varepsilon \quad \forall\, L \geq \tilde{L}, \forall\, b \in \mathbb{R}. \tag{C.14}$$

Set $\tilde{T} := \tilde{L}/2 > 0$ and let $T \geq \tilde{T}$ and $a \in \mathbb{R}$. We set $L := 2T \geq 2\tilde{T} = \tilde{L}$ and $b := a - T \in \mathbb{R}$. From (C.14), we obtain that

$$\left\| d - \frac{1}{2T}\int_{-T+a}^{T+a} f(t)dt \right\|_X = \left\| d - \frac{1}{L}\int_b^{L+b} f(t)dt \right\|_X \leq \varepsilon.$$

Therefore, (C.13) holds, and so $\lim_{T\to\infty} \frac{1}{2T}\int_{-T+a}^{T+a} f(t)dt$ exists uniformly with respect to $a \in \mathbb{R}$ and equals d. □

We now outline some terminology and notation which will prove useful in the sequel.

Definition C.1.22. *Let $f : R \to X$ and $\lambda \in \mathbb{R}$. We define $f_\lambda : R \to X$ by*

$$f_\lambda(t) := f(t)e^{-i\lambda t} \quad \forall\, t \in R.$$

From Lemmas C.1.20 and C.1.21, we obtain the following result regarding the mean value of f_λ when f is almost periodic.

Corollary C.1.23. *Let $f \in AP(\mathbb{R}, X)$ and $\lambda \in \mathbb{R}$. Then $M(f_\lambda)$ exists and*

$$M(f_\lambda) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T+a}^{T+a} f_\lambda(t)dt = \lim_{T\to\infty} \frac{1}{T}\int_a^{T+a} f_\lambda(t)dt,$$

uniformly with respect to $a \in \mathbb{R}$.

Proof. From [1, IX, p.10], we obtain that $f_\lambda \in AP(\mathbb{R}, X)$ since $f \in AP(\mathbb{R}, X)$ and $t \mapsto e^{-i\lambda t}$ is (almost) periodic. The result thus follows immediately from Lemma C.1.20 and Lemma C.1.21. □

Definition C.1.24. *Let $f \in L^1_{\text{loc}}(\mathbb{R}, X)$. We define the set $\Lambda(f) \subseteq \mathbb{R}$ of* characteristic exponents *of f by*

$$\Lambda(f) := \{\lambda \in \mathbb{R} : M(f_\lambda) \neq 0\}.$$

The next lemma asserts that $\Lambda(f)$ is countable if f is almost periodic. The result is given in [1, III, p.22] and so we shall not provide a proof here.

Lemma C.1.25. *Let $f \in AP(\mathbb{R}, X)$. Then $\Lambda(f)$ is countable.*

Associated with $\Lambda(f)$ is the module of f, which we define now.

Definition C.1.26. *Let $f \in L^1_{\text{loc}}(\mathbb{R}, X)$ be such that $\Lambda(f)$ is a countable set. We denote by* $\text{mod}(f)$ *the smallest additive group of real numbers that contains $\Lambda(f)$. That is,* $\text{mod}(f)$ *is the set of all real numbers that are a finite linear combination of the elements of $\Lambda(f)$ with integer coefficients:*

$$\text{mod}(f) = \left\{ \Sigma_{j=1}^N \alpha_j \lambda_j : \alpha_j \in \mathbb{Z},\, \lambda_j \in \Lambda(f),\, N \in \mathbb{N} \right\}.$$

We now present the following module containment result for almost periodic vector-valued functions.

Theorem C.1.27. *Let $f \in AP(\mathbb{R}, X)$ and $g \in AP(\mathbb{R}, Y)$ be such that for all $\varepsilon > 0$, there exists $\delta > 0$ such that $P(f,\delta) \subseteq P(g,\varepsilon)$. We then have that*

$$\mathrm{mod}(f) \supseteq \mathrm{mod}(g).$$

In the case that $X = Y = \mathbb{R}$ or $\mathbb{C}$, the above is a particular case of [34, Theorem 4.5].

Proof of Theorem C.1.27. In [1, p.11], a sequence $(\sigma_k)_{k\in\mathbb{N}} \subseteq \mathbb{R}$ is called *regular* (with respect to f), if $(f(\cdot + \sigma_k))_{k\in\mathbb{N}}$ is uniformly convergent. From [1, X, p.34], we obtain that

$$\zeta_f \subseteq \zeta_g \quad \iff \quad \mathrm{mod}(f) \supseteq \mathrm{mod}(g),$$

where ζ_f and ζ_g denote the sets of all regular sequences of f and g, respectively. Furthermore, [1, XII, p.12] yields that

$$\zeta_f \subseteq \zeta_g \quad \iff \quad \lim_{\xi\downarrow 0} \omega_{f,g}(\xi) = 0,$$

where

$$\omega_{f,g}(\xi) := \sup_{\tau\in P(f,\xi)} \left(\sup_{t\in\mathbb{R}} \|g(t+\tau) - g(t)\|_Y \right) \quad \forall\, \xi > 0.$$

Hence, to complete the proof it suffices to show that $\lim_{\xi\downarrow 0} \omega_{f,g}(\xi) = 0$. Before we do this however, we note that in [1, p.12], it is shown that $\omega_{f,g}$ is nondecreasing and $\lim_{\xi\downarrow 0} \omega_{f,g}(\xi)$ exists. Let $\varepsilon > 0$ and choose $\delta > 0$ such that $P(f,\delta) \subseteq P(g,\varepsilon)$. We then see that

$$\sup_{t\in\mathbb{R}} \|g(t+\tau) - g(t)\| \le \varepsilon \quad \forall\, \tau \in P(f,\delta),$$

which in turn implies that

$$\omega_{f,g}(\delta) = \sup_{\tau\in P(f,\delta)} \left(\sup_{t\in\mathbb{R}} \|g(t+\tau) - g(t)\| \right) \le \varepsilon.$$

Therefore, since $\omega_{f,g}$ is nondecreasing, $\omega_{f,g}(\xi) \le \varepsilon$ for all $\xi \in (0,\delta]$. Since ε was arbitrary, the proof is thus complete. □

Remark C.1.28. With Theorem C.1.11 in mind, by defining the mean value of a function $f \in L^1_{\mathrm{loc}}(\mathbb{R}_+, X)$ by

$$M(f) := \lim_{T\to\infty} \frac{1}{T} \int_0^T f(t)dt,$$

whenever the limit exists, we see, from Corollary C.1.23 (with $a = 0$), that if $f \in AP(\mathbb{R}_+, X)$ and $\lambda \in \mathbb{R}$, then

$$M((f_{\mathrm{e}})_\lambda) = \lim_{T\to\infty} \frac{1}{T} \int_0^T (f_{\mathrm{e}})_\lambda(t)dt = \lim_{T\to\infty} \frac{1}{T} \int_0^T f_\lambda(t)dt = M(f_\lambda).$$

That is, $M(f_\lambda)$ exists and equals $M((f_{\mathrm{e}})_\lambda)$. As a consequence of this,

$$\mathrm{mod}(f) = \mathrm{mod}(f_{\mathrm{e}}) \quad \forall\, f \in AP(\mathbb{R}_+, X). \tag{C.15}$$

Furthermore, by also recalling Lemma C.1.6, we see that a 'half-line' version of Theorem C.1.27 holds. By this, we mean that the conclusions of Theorem C.1.27 remain true if $f \in AP(\mathbb{R}_+, X)$ and $g \in AP(\mathbb{R}_+, Y)$. ◊

For the rest of this section, we concern ourselves with almost periodic functions on $\mathbb{Z}_+$ and $\mathbb{Z}$. In parallel with the cases $\mathbb{R}_+$ and $\mathbb{R}$, the literature concerning almost periodic functions on $\mathbb{Z}$ is easy to find, but for almost periodic functions on $\mathbb{Z}_+$, it is difficult to locate. Therefore, we are reasonable in giving the subsequent presentation. We begin with the following definition.

Definition C.1.29. (i) *A set $S \subseteq Z$ is said to be* relatively dense *(in Z) if there exists $L \in \mathbb{N}$ such that*

$$\{a, \ldots, a + L\} \cap S \neq \emptyset \quad \forall\, a \in Z.$$

(ii) *Let $\varepsilon > 0$. Then $\tau \in Z$ is said to be an ε-*period *of $f : Z \to X$ if*

$$\|f(t) - f(t + \tau)\|_X < \varepsilon \quad \forall\, t \in Z.$$

We denote by $P(f, \varepsilon) \subseteq Z$ the set of ε-periods of f.

(iii) *We say that $f : Z \to X$ is* almost periodic *if $P(f, \varepsilon)$ is relatively dense in Z for every $\varepsilon > 0$ and denote the set of almost periodic functions $Z \to X$ by $AP(Z, X)$.*

Remark C.1.30. Trivially, a periodic function over Z is almost periodic, but the converse is false:

$$v(t) := \sin(\pi\sqrt{2}t) \quad \forall\, t \in Z. \qquad \diamondsuit$$

The methods used to prove the earlier results relevant to the spaces $AP(\mathbb{R}, X)$ and $AP(\mathbb{R}_+, X)$, can be used analogously to obtain similar results for both $AP(\mathbb{Z}, X)$ and $AP(\mathbb{Z}_+, X)$. Because of this, in what follows, we will mostly just state the subsequent results, and only provide explicit proofs where the content is entirely different from the earlier results. We begin with the following two lemmas concerning the suprema of almost periodic sequences.

Lemma C.1.31. *Let $f \in AP(Z, X)$. Then f is bounded.*

Lemma C.1.32. *Let $f \in AP(Z, X)$. The following statements hold.*

(i) *If $Z = \mathbb{Z}_+$, then*

$$\sup_{t \in Z,\, t \geq T} \|f(t)\|_X = \sup_{t \in Z} \|f(t)\|_X \quad \forall\, T \in \mathbb{Z}_+.$$

(ii) *If $Z = \mathbb{Z}$, then*

$$\sup_{t \in Z,\, t \leq T} \|f(t)\|_X = \sup_{t \in Z} \|f(t)\|_X = \sup_{t \in Z,\, t \geq T} \|f(t)\|_X \quad \forall\, T \in \mathbb{Z}.$$

As a consequence of these results, we may act identically to that done in the proof of Lemma C.1.6 to obtain the following.

Corollary C.1.33. *For every $f \in AP(\mathbb{Z}_+, X)$, if we define $f_{\mathrm{e}} : \mathbb{Z} \to X$ by*

$$f_{\mathrm{e}}(t) = \lim_{k \to \infty} f(t + \tau_k) \quad \forall\, t \in \mathbb{Z}, \tag{C.16}$$

where, for each $k \in \mathbb{N}$, $\tau_k \in P(f, 1/k)$ and $\tau_k \nearrow \infty$ as $k \to \infty$, then f_{e} is the unique function such that $f_{\mathrm{e}} \in AP(\mathbb{Z}, X)$ and $f_{\mathrm{e}}(t) = f(t)$ for all $t \in \mathbb{Z}_+$. Moreover,

$$\sup_{t \in \mathbb{Z}} \|f_{\mathrm{e}}(t)\|_X = \sup_{t \in \mathbb{Z}_+} \|f(t)\|_X,$$

and, for every $\varepsilon > 0$, $P(f_{\mathrm{e}}, \varepsilon) = \{\pm\tau : \tau \in P(f, \varepsilon)\}$.

Analogously to the continuous-time setting, we can apply this corollary with the known theory of almost periodic functions defined on $\mathbb{Z}$ (see, for example [52, Appendix B]) to prove the following two results.

Corollary C.1.34. *$AP(Z, X)$ is a Banach space with resepect to the relevant supremum-norm.*

Theorem C.1.35. *The map $AP(\mathbb{Z}_+, X) \to AP(\mathbb{Z}, X)$ which maps f to f_{e}, where f_{e} is given by* (C.16)*, is an isometric isomorphism.*

We now present the following interesting connection between the spaces $AP(Z, X)$ and $AP(R, X)$, where $(R, Z) = (\mathbb{R}_+, \mathbb{Z}_+)$ or $(\mathbb{R}, \mathbb{Z})$.

Theorem C.1.36. *Let $(R, Z) = (\mathbb{R}_+, \mathbb{Z}_+)$ or $(\mathbb{R}, \mathbb{Z})$. Then $f \in AP(Z, X)$ if, and only if, there exists $g \in AP(R, X)$ such that $f(t) = g(t)$ for all $t \in Z$.*

The case that $(R, Z) = (\mathbb{R}, \mathbb{Z})$ is given in [52, Theorem 8 in Appendix B].

Proof of Theorem C.1.36. Let $f \in AP(\mathbb{Z}_+, X)$. Then, from Theorem C.1.35, we obtain the existence of a unique almost periodic extension of f to $\mathbb{Z}$, f_{e}. An application of [52, Theorem 8 in Appendix B] then gives the existence of $h \in AP(\mathbb{R}, X)$ such that $h(t) = f_{\mathrm{e}}(t)$ for all $t \in \mathbb{Z}$. Trivially, since h is almost periodic, the restriction of h to $\mathbb{R}_+$, which we shall denote by g, is almost periodic. Moreover,

$$g(t) = h(t) = f_{\mathrm{e}}(t) = f(t) \quad \forall\, t \in \mathbb{Z}_+,$$

thus proving one of the required implications. We omit the proof of the converse, since it can be proven by reversing the previous argument. □

Theorem C.1.36 is an entirely useful result, since it allows us to assert discrete-time analogues of some of the previously given results with little difficulty. Indeed, as an illustration of this, the following is easily obtained from combining Theorem C.1.36 with Lemma C.1.13.

Lemma C.1.37. *Let $f \in AP(Z, X)$ and $g \in AP(Z, Y)$. Then, for all $\varepsilon > 0$, $P(f, \varepsilon) \cap P(g, \varepsilon)$ is non-empty and relatively dense in Z. Furthermore,*

$$AP(Z, X \times Y) = AP(Z, X) \times AP(Z, Y). \tag{C.17}$$

We now define what we mean by an asymptotically almost periodic function over $\mathbb{Z}_+$.

Definition C.1.38. *We denote by $c_0(\mathbb{Z}_+, X)$ the space of functions $f \in X^{\mathbb{Z}_+}$ such that $\lim_{t\to\infty} f(t) = 0$. Moreover, we say that $f \in X^{\mathbb{Z}_+}$ is* asymptotically almost periodic *if $f \in AP(\mathbb{Z}_+, X) + c_0(\mathbb{Z}_+, X)$, and denote the space of asymptotically almost periodic functions $\mathbb{Z}_+ \to X$ by $AAP(\mathbb{Z}_+, X)$.*

As an immediate consequence of Lemma C.1.32, we obtain the next result.

Lemma C.1.39. *The following statements hold.*

(i) $AP(\mathbb{Z}_+, X) \cap c_0(\mathbb{Z}_+, X) = \{0\}$.

(ii) *If $f \in AAP(\mathbb{Z}_+, X)$, then the decomposition $f = f^{\mathrm{ap}} + f^0$, where $f^{\mathrm{ap}} \in AP(\mathbb{Z}_+, X)$ and $f^0 \in c_0(\mathbb{Z}_+, X)$, is unique.*

As in the continuous-time setting, one could also develop equivalent notions of almost periodic functions $Z \to X$, where $Z = \mathbb{Z}_+$ or $\mathbb{Z}$. For brevity, we shall not do this and instead leave it to the reader.

C.2 Stepanov almost periodicity

We now consider a generalisation of the notion of almost periodic functions, namely, almost periodicity in the sense of Stepanov. Contrary to the last section, we shall not discuss the time domains $\mathbb{Z}_+$ and $\mathbb{Z}$, since Stepanov almost periodicity, as will become apparent, is not relevant in those situations.

Definition C.2.1. *Let $f \in L^1_{\mathrm{loc}}(R, X)$.*

(i) *For $\varepsilon > 0$, we say that $\tau \in R$ is an* ε-period (in the sense of Stepanov) *of f if*

$$\sup_{a\in R} \int_a^{a+1} \|f(s+\tau) - f(s)\|_X ds \leq \varepsilon.$$

We denote by $P_1(f, \varepsilon)$, the set of all ε-periods (in the sense of Stepanov) of f.

(ii) *We say that f is* Stepanov almost periodic *if $P_1(f, \varepsilon)$ is relatively dense in R for every $\varepsilon > 0$, and denote the set of all Stepanov almost periodic functions $R \to X$ by $S^1(R, X)$.*

(iii) *We define $\tilde{f} : R \to L^1([0, 1], X)$ by*

$$(\tilde{f}(t))(s) := f(t + s) \quad \forall\, t \in R, \forall\, s \in [0, 1].$$

The function $\tilde{f}$ is often called the Bochner transform *of f (see, for example, [1]). We note that $\tilde{f}$ is continuous. This is due to the fact that translation operators of L^1 functions are continuous (see, for example, [97, Theorem 9.5, p.182]).*

(iv) *We define the space of uniformly locally integrable functions on R by*

$$UL^1_{\mathrm{loc}}(R, X) := \left\{ f \in L^1_{\mathrm{loc}}(R, X) : \sup_{a\in R} \int_a^{a+1} \|f(s)\|_X ds < \infty \right\},$$

and endow it with the norm

$$\|f\|_S := \sup_{a\in R} \int_a^{a+1} \|f(s)\|_X ds.$$

Remark C.2.2. (i) Stepanov almost periodicity can be thought of as a generalisation of almost periodic functions, since, trivially, $AP(R, X) \subseteq S^1(R, X)$. This is a strict inclusion. Indeed, a trivial example of a Stepanov almost periodic function that is not almost periodic is $f : R \to \mathbb{R}$ defined by

$$f(t) = \begin{cases} 0, & t \in [2k, 2k + 1] \\ 1, & t \in [2k + 1, 2(k + 1)] \end{cases} \quad \forall\, k \in Z.$$

It is clear that f is periodic with period 2. Moreover, since f is not continuous, f is not almost periodic. However, it is easy to show that f is Stepanov almost periodic.

(ii) For every $b > 0$, the functional

$$f \mapsto \sup_{a\in R} \int_a^{a+b} \|f(t)\|_X dt$$

is a norm on $UL^1_{\mathrm{loc}}(R, X)$ equivalent to $\|\cdot\|_S$. ♢

The following is straightforward to prove. Indeed, an infinite-dimensional generalisation of the proof given in [22, pp.38-39] can be applied here.

Lemma C.2.3. *$UL^1_{\text{loc}}(R, X)$ is a Banach space.*

An interesting question is whether or not every continuous function $R \to L^1([0,1], X)$ can be written as the Bochner transform applied to a function in $L^1_{\text{loc}}(R, X)$. The following trivial example answers this question.

Example C.2.4. We let $\lambda \in L^1([0,1], X)$ be non-zero and such that $\lambda(s) = 0$ for almost every $s \in [0, 1/2]$. Subsequently, we define $F \in AP(R, L^1([0,1], X))$ by

$$F(t) := \lambda \quad \forall\, t \in R.$$

Let us now assume that there exists $f \in L^1_{\text{loc}}(R, X)$ such that $\tilde{f} = F$. As a consequence of the fact that $\lambda(s) = 0$ for almost every $s \in [0, 1/2]$, we obtain that, for all $t \in R$,

$$f(t+s) = (\tilde{f}(t))(s) = (F(t))(s) = \lambda(s) = 0 \quad \text{a.e. } s \in [0, 1/2].$$

Since this holds for all $t \in R$, we may consider the above identity for $t = k/2$ for all $k \in \mathbb{Z}_+$ if $R = \mathbb{R}_+$ and all $k \in \mathbb{Z}$ if $R = \mathbb{R}$. From this we deduce that $f(t) = 0$ for almost every $t \in R$. This in turn implies that $F = \tilde{f} = 0$, which is a contradiction. Therefore, no such f exists. ◊

The subsequent result asserts a relationship between Stepanov and Bohr almost periodic functions.

Lemma C.2.5. *Let $f \in L^1_{\text{loc}}(R, X)$. Then*

$$f \in S^1(R, X) \qquad \Longleftrightarrow \qquad \tilde{f} \in AP(R, L^1([0,1], X)),$$

and if $f \in S^1(R, X)$, then, for all $\varepsilon > 0$, $P_1(f, \varepsilon) = P(\tilde{f}, \varepsilon)$.

Lemma C.2.5 is proven in [1, pp.77-78]. Since the proof is straightforward, we provide it here for completeness.

Proof of Lemma C.2.5. The proof follows immediately upon noticing that, for all $a, \tau \in R$,

$$\begin{aligned}\int_a^{a+1} \|f(s+\tau) - f(s)\|_X ds &= \int_0^1 \|f(a+s+\tau) - f(a+s)\|_X ds \\ &= \int_0^1 \|(\tilde{f}(a+\tau))(s) - (\tilde{f}(a))(s)\|_X ds \\ &= \|\tilde{f}(a+\tau) - \tilde{f}(a)\|_{L^1([0,1],X)}.\end{aligned}$$

□

Remark C.2.6. We shall see that Lemma C.2.5 is very useful. This is because it allows us to bring forward some of the knowledge that we already know about (Bohr) almost periodic functions and apply it in the Stepanov sense. ◊

The Bochner transform can be thought of as an isometry.

Lemma C.2.7. *The Bochner transform is a linear operation and*

$$\|f\|_S = \|\tilde{f}\|_{L^\infty} \quad \forall\, f \in L^1_{\text{loc}}(R, X). \tag{C.18}$$

Proof. The linearity claim is easy to prove, and so we omit this and will instead focus on showing that (C.18) holds for all $f \in L^1_{\text{loc}}(R, X)$. To this end, let $f \in L^1_{\text{loc}}(R, X)$ and note that

$$\|f\|_S = \sup_{a\in R} \int_0^1 \|f(s+a)\|_X ds = \sup_{a\in R} \int_0^1 \|(\tilde{f}(a))(s)\|_X ds = \|\tilde{f}\|_{L^\infty},$$

which completes the proof. □

As a consequence of Lemmas C.2.5 and C.2.7, we yield the following.

Lemma C.2.8. *$S^1(R, X)$ is a closed subspace of $UL^1_{\text{loc}}(R, X)$.*

Proof. The fact that $S^1(R, X)$ is a vector space follows easily from Corollary C.1.10 and Lemmas C.2.5 and C.2.7. Moreover, since every almost periodic function is bounded (see Lemma C.1.3), Lemmas C.2.5 and C.2.7 yield that every $f \in S^1(R, X)$ has the property that $f \in UL^1_{\text{loc}}(R, X)$. Finally, for the closedness property, let $(f_k)_{k\in\mathbb{N}}$ be a sequence of functions in $S^1(R, X)$ converging to $f : R \to X$. It is clear that $f \in L^1_{\text{loc}}(R, X)$. Another application of Lemma C.2.7 then yields that

$$\|\tilde{f}_k - \tilde{f}\|_\infty = \|f_k - f\|_S \to 0 \text{ as } k \to \infty.$$

Therefore, Lemma C.1.9 implies that $\tilde{f} \in AP(R, L^1([0,1], X))$, which, by Lemma C.2.5, completes the proof. □

The next result can be thought of as an analogue to Lemma C.1.6.

Lemma C.2.9. *Let $f \in S^1(\mathbb{R}_+, X)$. Then there exists a unique function $f^{\text{e}} \in S^1(\mathbb{R}, X)$ such that $f^{\text{e}}(t) = f(t)$ for almost every $t \in \mathbb{R}_+$. Moreover,*

$$\sup_{a\in\mathbb{R}_+} \int_a^{a+1} \|f(s)\|_X ds = \sup_{a\in\mathbb{R}} \int_a^{a+1} \|f^{\text{e}}(s)\|_X ds, \tag{C.19}$$

and, for every $\varepsilon > 0$, $P_1(f^{\text{e}}, \varepsilon) = \{\pm\tau : \tau \in P_1(f, \varepsilon)\}$.

Remark C.2.10. By combining Lemma C.2.9 with Lemma C.1.6, we see that if $f \in AP(\mathbb{R}_+, X)$, then $f^{\text{e}} = f_{\text{e}}$. ♢

Proof of Lemma C.2.9. For each $k \in \mathbb{N}$, let $\tau_k \in P_1(f, 1/k)$ be such that $\tau_k \nearrow \infty$ as $k \to \infty$. We define $f^{\text{e}} \in L^1_{\text{loc}}(\mathbb{R}, X)$ by

$$f^{\text{e}}(\cdot) = \lim_{k\to\infty} f(\cdot + \tau_k) \tag{C.20}$$

in $L^1([-a, a], X)$ for every $a \in \mathbb{N}$. We claim that f^{e} is well-defined since $(f(\cdot + \tau_k))_{k\in\mathbb{N}}$ is a Cauchy sequence in $L^1([-a, a], X)$ for every $a \in \mathbb{N}$. Indeed, to see this, fix $a \in \mathbb{N}$ and note that

$$\int_0^a \|f(s) - f(s+\tau_k)\|_X ds \leq \sum_{i=1}^a \int_{i-1}^i \|f(s) - f(s+\tau_k)\|_X ds \leq \frac{a}{k}. \tag{C.21}$$

Therefore, $f(\cdot + \tau_k) \to f(\cdot)$ as $k \to \infty$ in $L^1([0,a], X)$. As a consequence of this, $f^{\mathrm{e}}(t) = f(t)$ for almost every $t \in \mathbb{R}_+$. Furthermore, since $\tau_k \nearrow \infty$ as $k \to \infty$, there exists $K \in \mathbb{N}$ such that $\tau_k > a$ for all $k \in \overline{K}$. Hence, for all $k, l \in \overline{K}$,

$$\begin{aligned}\int_{-a}^{0} \|f(s+\tau_k) - f(s+\tau_l)\|_X ds &= \int_{-a}^{0} \|f(s+\tau_l+\tau_k-\tau_l) - f(s+\tau_l)\|_X ds \\ &= \int_{\tau_l - a}^{\tau_l} \|f(s+\tau_k-\tau_l) - f(s)\|_X ds \\ &\leq \int_{\tau_l - a}^{\tau_l} \|f(s+\tau_k-\tau_l) - f(s+\tau_k-\tau_l+\tau_l)\|_X ds \\ &\quad + \int_{\tau_l - a}^{\tau_l} \|f(s+\tau_k) - f(s)\|_X ds.\end{aligned}$$

By again changing variables, we see that the above inequality becomes

$$\begin{aligned}\int_{-a}^{0} \|f(s+\tau_k) - f(s+\tau_l)\|_X ds &\leq \int_{\tau_k - a}^{\tau_k} \|f(s) - f(s+\tau_l)\|_X ds \\ &\quad + \int_{\tau_l - a}^{\tau_l} \|f(s+\tau_k) - f(s)\|_X ds \\ &\leq a\left(\frac{1}{l} + \frac{1}{k}\right).\end{aligned}$$

We have therefore shown that $(f(\cdot + \tau_k))_{k\in\mathbb{N}}$ is a Cauchy sequence in $L^1([-a,0], X)$. By combining this with (C.21), we thus see that $(f(\cdot + \tau_k))_{k\in\mathbb{N}}$ is a Cauchy sequence in $L^1([-a,a], X)$, which hence shows that f^{e} is well-defined. To show that $f^{\mathrm{e}} \in S^1(\mathbb{R}, X)$, we shall use the Bochner transform and Lemma C.2.5. To this end, let $t \in \mathbb{R}$ and note that, for $k \in \mathbb{N}$ large enough,

$$\begin{aligned}\|\widetilde{(f^{\mathrm{e}})}(t) - \tilde{f}(t+\tau_k)\|_{L^1([0,1],X)} &= \int_0^1 \|(\widetilde{(f^{\mathrm{e}})}(t))(s) - (\tilde{f}(t+\tau_k))(s)\|_X ds \\ &= \int_0^1 \|f^{\mathrm{e}}(t+s) - f(t+\tau_k+s)\|_X ds \\ &= \int_t^{t+1} \|f^{\mathrm{e}}(s) - f(s+\tau_k)\|_X ds \\ &= \|f^{\mathrm{e}}(\cdot) - f(\cdot+\tau_k)\|_{L^1([t,t+1],X)},\end{aligned}$$

which converges to zero as $k \to \infty$. Therefore, $\widetilde{(f^{\mathrm{e}})}(t) = \lim_{k\to\infty} \tilde{f}(t+\tau_k)$. Moreover, since $\tilde{f} \in AP(\mathbb{R}_+, L^1([0,1], X))$ and $\tau_k \in P(\tilde{f}, 1/k)$ for all $k \in \mathbb{N}$, we hence obtain, from Lemma C.1.6 (see, in particular, (C.5)), that $\widetilde{(f^{\mathrm{e}})}$ is the unique almost periodic extension of $\tilde{f}$ to $\mathbb{R}$. That is, $\widetilde{(f^{\mathrm{e}})} = (\tilde{f})_{\mathrm{e}}$. By applying Lemma C.2.5 once again, we thus yield that $f^{\mathrm{e}} \in S^1(\mathbb{R}, X)$. As for the uniqueness of f^{e}, assume that there exists another Stepanov almost periodic extension of f to $\mathbb{R}$, and denote this function by g. We therefore see that $g - f^{\mathrm{e}} = 0$ almost everywhere on $\mathbb{R}_+$. This implies that $\tilde{g} - \widetilde{(f^{\mathrm{e}})} = 0$ almost everywhere on $\mathbb{R}_+$. Since $\tilde{g}$ and $\widetilde{(f^{\mathrm{e}})}$ are continuous, we see that $\tilde{g} - \widetilde{(f^{\mathrm{e}})} = 0$ on $\mathbb{R}_+$. As a consequence (see Lemma C.1.4), $\tilde{g} - \widetilde{(f^{\mathrm{e}})} = 0$ on $\mathbb{R}$. Therefore, $g - f^{\mathrm{e}} = 0$ almost everywhere on $\mathbb{R}$ and f^{e} is unique. To show that (C.19) holds, note that since $\widetilde{(f^{\mathrm{e}})}$ is the unique almost periodic extension of $\tilde{f}$ to $\mathbb{R}$, Lemma C.1.6 and Lemma C.2.7 give that

$$\begin{aligned}\sup_{a\in\mathbb{R}_+}\int_a^{a+1}\|f(s)\|_X ds &= \sup_{t\in\mathbb{R}_+}\|\tilde{f}(t)\|_{L^1([0,1],X)}\\ &= \sup_{t\in\mathbb{R}}\|\widetilde{(f^{\mathrm{e}})}(t)\|_{L^1([0,1],X)}\\ &= \sup_{a\in\mathbb{R}}\int_a^{a+1}\|f^{\mathrm{e}}(s)\|_X ds,\end{aligned}$$

which is precisely (C.19). Finally, let $\varepsilon > 0$ and note that, from Lemma C.2.5, from the fact that $\widetilde{(f^{\mathrm{e}})}$ is the unique almost periodic extension of $\tilde{f}$ to $\mathbb{R}$, and from Lemma C.1.6,

$$P_1(f^{\mathrm{e}},\varepsilon) = P(\widetilde{(f^{\mathrm{e}})},\varepsilon) = \{\pm\tau : \tau \in P(\tilde{f},\varepsilon)\} = \{\pm\tau : \tau \in P_1(f,\varepsilon)\},$$

which completes the proof. □

As a consequence of the previous result and Lemma C.2.8, we obtain the following.

Theorem C.2.11. *The map $S^1(\mathbb{R}_+,X) \to S^1(\mathbb{R},X)$ which maps f to f^{e}, where f^{e} is given by* (C.20)*, is an isometric isomorphism.*

In addition to the previous, an inspection of the proof of Lemma C.2.9 leads to the next result, which provides an interesting relationship between taking extensions of (Stepanov) almost periodic functions and taking Bochner transforms: the two operations commute.

Corollary C.2.12. *We have that*

$$(\tilde{f})_{\mathrm{e}} = \widetilde{(f^{\mathrm{e}})} \quad \forall\, f \in S^1(\mathbb{R}_+,X).$$

For the rest of this section, we establish a module containment result for Stepanov almost periodic functions. That is, an analogue of Theorem C.1.27 applied to Stepanov almost periodic functions. To do this, we shall again use the Bochner transform to infer the known results of the previous section. Before doing so, we first present the following preliminary result.

Lemma C.2.13. *Let $f \in L^1_{\mathrm{loc}}(R,X)$, $\lambda \in \mathbb{R}$, $a \in R$ and $b > a$. Then*

$$\left(\int_a^b \tilde{f}(t)e^{-i\lambda t}dt\right)(s) = \int_a^b f(t+s)e^{-i\lambda t}dt \quad \text{a.e. } s \in [0,1]. \tag{C.22}$$

The identity (C.22) is implicitly used in the arguments given in [1, pp.80-81], but is not proven.

Proof of Lemma C.2.13. For $s \in [0,1)$ and $h > 0$ such that $s+h \le 1$, we define an operator $J_{s,h} : L^1([0,1],X) \to X$ by

$$J_{s,h}(g) := \int_s^{s+h} g(t)dt \quad \forall\, g \in L^1([0,1],X).$$

Since $\tilde{f}$ is continuous, we see that $(\tilde{f})_\lambda$, which is defined in Definition C.1.22, is also continuous and hence locally integrable. Therefore, by combining this with the fact that $J_{s,h}$ is a bounded linear operator for all $s \in [0,1)$ and $h > 0$ such that $s + h \leq 1$, [129, Corollary 2, p.134] yields that

$$J_{s,h}\left(\int_a^b \tilde{f}(t)e^{-i\lambda t}dt\right) = \int_a^b J_{s,h}(\tilde{f}(t)e^{-i\lambda t})dt.$$

By recalling the definition of $J_{s,h}$, we then see that, for all $s \in [0,1)$ and $h > 0$ such that $s + h \leq 1$,

$$\int_s^{s+h}\left(\int_a^b \tilde{f}(t)e^{-i\lambda t}dt\right)(\sigma)d\sigma = \int_a^b \left(\int_s^{s+h} (\tilde{f}(t))(\sigma)e^{-i\lambda t}d\sigma\right)dt,$$

which, when recalling the definition of the Bochner transform, becomes

$$\int_s^{s+h}\left(\int_a^b \tilde{f}(t)e^{-i\lambda t}dt\right)(\sigma)d\sigma = \int_a^b \left(\int_s^{s+h} f(t+\sigma)e^{-i\lambda t}d\sigma\right)dt.$$

By Fubini's theorem, we may interchange the integrals on the right-hand side of the above equation to obtain that, for all $s \in [0,1)$ and $h > 0$ such that $s + h \leq 1$,

$$\int_s^{s+h}\left(\int_a^b \tilde{f}(t)e^{-i\lambda t}dt\right)(\sigma)d\sigma = \int_s^{s+h} \left(\int_a^b f(t+\sigma)e^{-i\lambda t}dt\right)d\sigma.$$

If we now divide by h on both sides of the above equality and take the limit as $h \to 0$, an application of [129, Theorem 2, p.134] yields the result. ☐

We now show that the mean value of a Stepanov almost periodic function exists and is closely related to the mean value of its Bochner transform.

Lemma C.2.14. *Let $f \in S^1(\mathbb{R}, X)$ and $\lambda \in \mathbb{R}$. Then $M(f_\lambda)$ exists and is given by*

$$M(f_\lambda) = (M(\tilde{f}_\lambda))(\eta)e^{-i\lambda\eta} \quad a.e.\ \eta \in [0,1]. \tag{C.23}$$

In particular, $(M(\tilde{f}_\lambda))(\eta)e^{-i\lambda\eta}$ does not depend on $\eta \in [0,1]$. Furthermore,

$$M(f_\lambda) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T+a}^{T+a} f(t)e^{-i\lambda t}dt = \lim_{T\to\infty}\frac{1}{T}\int_a^{T+a} f(t)e^{-i\lambda t}dt, \tag{C.24}$$

uniformly with respect to $a \in \mathbb{R}$.

A proof of the above result may be obtained from arguments similar to those presented in [1, pp.80-81]. For completeness, we shall present such a proof.

Proof of Lemma C.2.14. We note that since $\tilde{f} \in AP(\mathbb{R}, L^1([0,1], X))$, Corollary C.1.23 gives that $M(\tilde{f}_\lambda)$ exists and

$$\lim_{T\to\infty}\int_0^1 \left\|(M(\tilde{f}_\lambda))(\eta) - \left(\frac{1}{T}\int_a^{a+T} \tilde{f}_\lambda(t)dt\right)(\eta)\right\| d\eta = 0,$$

uniformly with respect to $a \in \mathbb{R}$. An application of Lemma C.2.13 yields that

$$\lim_{T\to\infty}\int_0^1 \left\|(M(\tilde{f}_\lambda))(\eta) - \frac{1}{T}\int_a^{a+T} f(t+\eta)e^{-i\lambda t}dt\right\| d\eta = 0,$$

uniformly with respect to $a \in \mathbb{R}$. By using a change of variables, we see that this becomes

$$\lim_{T\to\infty} \int_0^1 \left\| (M(\tilde{f}_\lambda))(\eta)e^{-i\lambda\eta} - \frac{1}{T}\int_{a+\eta}^{a+T+\eta} f(t)e^{-i\lambda t}dt \right\| d\eta = 0, \tag{C.25}$$

uniformly with respect to $a \in \mathbb{R}$. Now, $f \in UL^1_{\text{loc}}(\mathbb{R}, X)$ and so $f_\lambda \in UL^1_{\text{loc}}(\mathbb{R}, X)$. Hence, there exists $c > 0$ such that

$$\left\| \int_a^{a+\eta} f(t)e^{-i\lambda t}dt \right\| \leq c \quad \forall\, a \in \mathbb{R}, \forall\, \eta \in [0,1].$$

Therefore, by combining this with (C.25), we see that

$$\lim_{T\to\infty} \int_0^1 \left\| (M(\tilde{f}_\lambda))(\eta)e^{-i\lambda\eta} - \frac{1}{T}\int_{a}^{a+T} f(t)e^{-i\lambda t}dt \right\| d\eta = 0, \tag{C.26}$$

uniformly with respect to $a \in \mathbb{R}$. We now recall (from, for example, [97, Theorem 3.12]) that every Cauchy sequence in $L^1([0,1], X)$ has a subsequence which converges pointwise almost everywhere to its limit. Whence, from (C.26) with $a = 0$, we obtain the existence of a sequence $(T_k)_{k\in\mathbb{N}} \subseteq \mathbb{R}_+$ such that $\lim_{k\to\infty} T_k = \infty$ and

$$\lim_{k\to\infty} \frac{1}{T_k}\int_0^{T_k} f(t)e^{-i\lambda t}dt = (M(\tilde{f}_\lambda))(\eta)e^{-i\lambda\eta} \quad \text{a.e. } \eta \in [0,1].$$

We have thus shown that $M(\tilde{f}_\lambda)(\eta)e^{-i\lambda\eta}$ does not depend on $\eta \in [0,1]$. Therefore, (C.26) gives, uniformly with respect to $a \in \mathbb{R}$,

$$\lim_{T\to\infty} \frac{1}{T}\int_a^{T+a} f(t)e^{-i\lambda t}dt = (M(\tilde{f}_\lambda))(\eta)e^{-i\lambda\eta} \quad \text{a.e. } \eta \in [0,1]. \tag{C.27}$$

An application of Lemma C.1.21 then guarantees that $M(f_\lambda)$ exists and that (C.24) holds. Finally, by combining (C.24) with (C.27), we see that (C.23) holds, thus completing the proof. □

As an immediate consequence of Lemma C.2.14, and by recalling Lemma C.1.25, we obtain the following.

Corollary C.2.15. *Let $f \in S^1(\mathbb{R}, X)$. Then $\Lambda(f) = \Lambda(\tilde{f})$ and $\text{mod}(f) = \text{mod}(\tilde{f})$.*

We now present the aforementioned module containment result for Stepanov almost periodic functions.

Theorem C.2.16. *Let $f \in S^1(\mathbb{R}, X)$ and $g \in S^1(\mathbb{R}, Y)$. If for all $\varepsilon > 0$ there exists $\delta > 0$ such that $P_1(f, \delta) \subseteq P_1(g, \varepsilon)$, then*

$$\text{mod}(f) \supseteq \text{mod}(g).$$

Proof. We begin by noting that, from Lemma C.2.5

$$P_1(f, \xi) = P(\tilde{f}, \xi) \quad \text{and} \quad P_1(g, \xi) = P(\tilde{g}, \xi) \quad \forall\, \xi > 0. \tag{C.28}$$

Hence, we obtain that for all $\varepsilon > 0$ there exists $\delta > 0$ such that $P(\tilde{f}, \delta) \subseteq P(\tilde{g}, \varepsilon)$. An application of Theorem C.1.27 then gives that $\text{mod}(\tilde{f}) \supseteq \text{mod}(\tilde{g})$. Therefore, the proof is complete upon invoking Corollary C.2.15. □

Remark C.2.17. As in Remark C.1.28, let us define the mean value of a function $f \in L^1_{\text{loc}}(\mathbb{R}_+, X)$ by

$$M(f) := \lim_{T\to\infty} \frac{1}{T}\int_0^T f(t)dt,$$

whenever the limit exists. By recalling Theorem C.2.11, an application of Lemma C.2.14 (with $a = 0$) yields, for all $f \in S^1(\mathbb{R}_+, X)$ and $\lambda \in \mathbb{R}$, that $M((f^{\mathrm{e}})_\lambda)$ exists and

$$M((f^{\mathrm{e}})_\lambda) = \lim_{T\to\infty} \frac{1}{T}\int_0^T f^{\mathrm{e}}(t)e^{-i\lambda t}dt = \lim_{T\to\infty} \frac{1}{T}\int_0^T f(t)e^{-i\lambda t}dt = M(f_\lambda).$$

Therefore, $M(f_\lambda)$ exists for all $f \in S^1(\mathbb{R}_+, X)$ and all $\lambda \in \mathbb{R}$, and equals $M((f^{\mathrm{e}})_\lambda)$. From this, we deduce that

$$\operatorname{mod}(f) = \operatorname{mod}(f^{\mathrm{e}}) \quad \forall\, f \in S^1(\mathbb{R}_+, X). \tag{C.29}$$

By also recalling Lemma C.2.9, we see from (C.29) that the conclusions of Theorem C.2.16 hold if $f \in S^1(\mathbb{R}_+, X)$ and $g \in S^1(\mathbb{R}_+, Y)$. ◊

Bibliography

[1] L. Amerio and G. Prouse. *Almost-periodic functions and functional equations.* Van Nostrand Reinhold Co., New York-Toronto, Ont.-Melbourne, 1971.

[2] Z. Aminzare and E. D. Sontag. Contraction methods for nonlinear systems: A brief introduction and some open problems. In *53rd IEEE Conference on Decision and Control*, pages 3835–3847, Dec 2014.

[3] B. D. O. Anderson. A system theory criterion for positive real matrices. *SIAM J. Control*, 5:171–182, 1967.

[4] B. D. O. Anderson and S. Vongpanitlerd. *Network analysis and synthesis: a modern systems theory approach.* Prentice Hall, 1973.

[5] D. Angeli. A Lyapunov approach to the incremental stability properties. *IEEE Trans. Automat. Control*, 47(3):410–421, 2002.

[6] D. Angeli, E. D. Sontag, and Y. Wang. A characterization of integral input-to-state stability. *IEEE Trans. Automat. Control*, 45(6):1082–1097, 2000.

[7] M. Arcak and A. Teel. Input-to-state stability for a class of Lurie systems. *Automatica J. IFAC*, 38(11):1945–1949, 2002.

[8] K. J. Astrom and L. Rundqwist. Integrator windup and how to avoid it. In *American Control Conference*, pages 1693–1698, 1989.

[9] S. Axler. *Linear algebra done right.* Undergraduate Texts in Mathematics. Springer, Cham, third edition, 2015.

[10] M. Barraa and M. Boumazgour. A note on the spectrum of an upper triangular operator matrix. *Proc. Amer. Math. Soc.*, 131(10):3083–3088, 2003.

[11] H. Bart and S. Goldberg. Characterizations of almost periodic strongly continuous groups and semigroups. *Math. Ann.*, 236(2):105–116, 1978.

[12] A. Bellow and V. Losert. The weighted pointwise ergodic theorem and the individual ergodic theorem along subsequences. *Trans. Amer. Math. Soc.*, 288(1):307–345, 1985.

[13] D. S. Bernstein and A. N. Michel. A chronological bibliography on saturating actuators. *Internat. J. Robust Nonlinear Control*, 5(5):375–380, 1995.

[14] A. S. Besicovitch. *Almost periodic functions.* Dover Publications, Inc., New York, 1955.

[15] A. Bill, C. Guiver, H. Logemann, and S. Townley. The converging-input converging-state property for Lur'e systems. *Math. Control Signals Systems*, 29:4, 2017, DOI:10.1007/s00498-016-0184-3.

[16] H. Bohr. Zur theorie der fast periodischen funktionen. *Acta Math.*, 45(1):29–127, 1925. I. Eine verallgemeinerung der theorie der fourierreihen.

[17] H. Bohr. *Almost Periodic Functions*. Chelsea Publishing Company, New York, N.Y., 1947.

[18] J. Briggs, K. Dabbs, M. Holm, J. Lubben, R. Rebarber, B. Tenhumberg, and D. Riser-Espinoza. Structured population dynamics: an introduction to integral modeling. *Math. Mag.*, 83(4):243–257, 2010.

[19] T. Cazenave and A. Haraux. *An introduction to semilinear evolution equations*, volume 13 of *Oxford Lecture Series in Mathematics and its Applications*. The Clarendon Press, Oxford University Press, New York, 1998. Translated from the 1990 French original by Yvan Martel and revised by the authors.

[20] A. Chaillet, A. Y. Pogromsky, and B. S. Rüffer. A razumikhin approach for the incremental stability of delayed nonlinear systems. *Proceedings of 52nd IEEE conference on decision and control (CDC),Florence, Italy*, pages 1596–1601, 2013.

[21] C. Corduneanu. *Almost periodic functions*. Interscience Publishers [John Wiley & Sons], New York-London-Sydney, 1968. With the collaboration of N. Gheorghiu and V. Barbu, Translated from the Romanian by Gitta Bernstein and Eugene Tomer, Interscience Tracts in Pure and Applied Mathematics, No. 22.

[22] C. Corduneanu. *Almost periodic oscillations and waves*. Springer, New York, 2009.

[23] J. J Coughlan. *Absolute stability results for infinite-dimensional discrete-time systems with applications to sampled-data integral control*. PhD Thesis, University of Bath, 2007.

[24] J. J. Coughlan, A. T. Hill, and H. Logemann. The Z-transform and linear multistep stability. *IMA J. Numer. Anal.*, 27(1):45–73, 2007.

[25] J. J. Coughlan and H. Logemann. Absolute stability and integral control for infinite-dimensional discrete-time systems. *Nonlinear Anal.*, 71(10):4769–4789, 2009.

[26] R. F. Curtain and H. Zwart. *An introduction to infinite-dimensional linear systems theory*, volume 21 of *Texts in Applied Mathematics*. Springer-Verlag, New York, 1995.

[27] S. N. Dashkovskiy, D. V. Efimov, and E. D. Sontag. Input-to-state stability and allied system properties. *Autom. Remote Control*, 72(8):1579–1614, 2011.

[28] S. N. Dashkovskiy, B. S. Rüffer, and F. R. Wirth. An ISS small gain theorem for general networks. *Math. Control Signals Systems*, 19(2):93–122, 2007.

[29] E. J. Davison. Multivariable tuning regulators: the feedforward and robust control of a general servomechanism problem. *IEEE Trans. Automatic Control*, 21(1):35–47, 1976.

[30] K. Deimling. *Nonlinear functional analysis.* Springer-Verlag, Berlin, 1985.

[31] C. A. Desoer and M. Vidyasagar. *Feedback systems: input-output properties.* Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London, 1975. Electrical Science.

[32] E. A. Eager, R. Rebarber, and B. Tenhumberg. Choice of density-dependent seedling recruitment function affects predicted transient dynamics: a case study with platte thistle. *Theoretical Ecology*, 5(3):387–401, 2012.

[33] A. F. Filippov. *Differential equations with discontinuous righthand sides*, volume 18 of *Mathematics and its Applications (Soviet Series).* Kluwer Academic Publishers Group, Dordrecht, 1988. Translated from the Russian.

[34] A. M. Fink. *Almost periodic differential equations.* Lecture Notes in Mathematics, Vol. 377. Springer-Verlag, Berlin-New York, 1974.

[35] T. Fliegner, H. Logemann, and E. P. Ryan. Discrete-time low-gain control of linear systems with input/output nonlinearities. *Internat. J. Robust Nonlinear Control*, 11(12):1127–1143, 2001.

[36] D Franco, C. Guiver, H. Logemann, and J. Peran. Boundedness, persistence and stability for classes of forced difference equations arising population ecology. *Mathematical Biology*, 79:1029–1076, 2019.

[37] S. Galeani, S. Tarbouriech, M. Turner, and L. Zaccarian. A tutorial on modern anti-windup design. *Eur. J. Control*, 15(3-4):418–440, 2009.

[38] M. E. Gilmore, C. Guiver, and H. Logemann. Infinite-dimensional Lur'e systems with almost periodic forcing. *Submitted for publication*, 2019.

[39] M. E. Gilmore, C. Guiver, and H. Logemann. Semi-global incremental input-to-state stability of discrete-time Lur'e systems. *Accepted for publication in Systems & Control Letters*, 2019.

[40] M. E. Gilmore, C. Guiver, and H. Logemann. Stability and convergence properties of infinite-dimensional discrete-time Lur'e systems. *International journal of control*, 2019, DOI:10.1080/00207179.2019.1575528.

[41] M. E. Gilmore, C. Guiver, and H. Logemann. Convergence properties for Lur'e systems with Stepanov almost periodic forcing. *Submitted for publication*, 2020.

[42] M. E. Gilmore, C. Guiver, and H. Logemann. Sampled-data integral control of forced infinite-dimensional linear systems with input nonlinearities. *Submitted for publication*, 2020.

[43] I. Gohberg and J. Leiterer. *Holomorphic operator functions of one variable and applications*, volume 192 of *Operator Theory: Advances and Applications.* Birkhäuser Verlag, Basel, 2009. Methods from complex analysis in several variables.

[44] M. Green and D. J. N. Limebeer. *Linear robust control.* Prentice-Hall, 1995.

[45] C. Guiver and H. Logemann. A circle criterion for strong integral input-to-state stability. *Automatica*, 2020, https://doi.org/10.1016/j.automatica.2019.108641.

[46] C. Guiver, H. Logemann, and M. R. Opmeer. Transfer functions of infinite-dimensional systems: positive realness and stabilization. *Math. Control Signals Systems*, 29:20, 2017, https://doi.org/10.1007/s00498-017-0203-z.

[47] C. Guiver, H. Logemann, and M. R. Opmeer. Infinite-dimensional Lur'e systems: input-to-state stability and convergence properties. *SIAM J. Control Optim.*, 57(1):334–365, 2019.

[48] C. Guiver, H. Logemann, and S. Townley. Low-gain integral control for multi-input, multi-output linear systems with input nonlinearities. *IEEE Transactions on Automatic Control*, 62(9):4776–4783, 2017.

[49] W. M. Haddad and D. S. Bernstein. Robust stabilization with positive real uncertainty: beyond the small gain theorem. *Systems Control Lett.*, 17(3):191–208, 1991.

[50] W. M. Haddad and D. S. Bernstein. Explicit construction of quadratic Lyapunov functions for the small gain, positivity, circle, and Popov theorems and their application to robust stability. I. Continuous-time theory. *Internat. J. Robust Nonlinear Control*, 3(4):313–339, 1993.

[51] W. M. Haddad and D. S. Bernstein. Explicit construction of quadratic Lyapunov functions for the small gain, positivity, circle, and Popov theorems and their application to robust stability. II. Discrete-time theory. *Internat. J. Robust Nonlinear Control*, 4(2):249–265, 1994.

[52] A. Halanay and V. Ionescu. *Time-varying discrete linear systems*, volume 68 of *Operator Theory: Advances and Applications*. Birkhäuser Verlag, Basel, 1994. Input-output operators. Riccati equations. Disturbance attenuation.

[53] D. Hinrichsen and A. J. Pritchard. *Mathematical systems theory. I*, volume 48 of *Texts in Applied Mathematics*. Springer-Verlag, Berlin, 2005. Modelling, state space analysis, stability and robustness.

[54] L. Hitz and B. D. O. Anderson. Discrete positive-real functions and their application to system stability. *Proc. Inst. Elec. Engrs.*, 116:153–155, 1969.

[55] B. Jacob, R. Nabiullin, J. R. Partington, and F. L. Schwenninger. Infinite-dimensional input-to-state stability and Orlicz spaces. *SIAM J. Control Optim.*, 56(2):868–889, 2018.

[56] B. Jacob and J. R. Partington. Admissibility of control and observation operators for semigroups: a survey. In *Current trends in operator theory and its applications*, volume 149 of *Oper. Theory Adv. Appl.*, pages 199–221. Birkhäuser, Basel, 2004.

[57] B. Jayawardhana, H. Logemann, and E. P. Ryan. Infinite-dimensional feedback systems: the circle criterion and input-to-state stability. *Commun. Inf. Syst.*, 8(4):413–414, 2008.

[58] B. Jayawardhana, H. Logemann, and E. P. Ryan. Input-to-state stability of differential inclusions with applications to hysteretic and quantized feedback systems. *SIAM J. Control Optim.*, 48(2):1031–1054, 2009.

[59] B. Jayawardhana, H. Logemann, and E. P. Ryan. The circle criterion and input-to-state stability: new perspectives on a classical result. *IEEE Control Syst. Mag.*, 31(4):32–67, 2011.

[60] B. Jayawardhana, E. P. Ryan, and A. R. Teel. Bounded-energy-input convergent-state property of dissipative nonlinear systems: an iISS approach. *IEEE Trans. Automat. Control*, 55(1):159–164, 2010.

[61] Z.-P. Jiang and Y. Wang. Input-to-state stability for discrete-time nonlinear systems. *Automatica J. IFAC*, 37(6):857–869, 2001.

[62] Z.-P. Jiang and Y. Wang. A converse Lyapunov theorem for discrete-time systems with disturbances. *Systems Control Lett.*, 45(1):49–58, 2002.

[63] J. Jouffroy and T. I Fossen. A tutorial on incremental stability analysis using contraction theory. *Modeling, Identification and Control*, 31(3):93–106, 2010.

[64] Z Ke. *Sampled-data control: stabilization, tracking and disturbance rejection.* PhD Thesis, University of Bath, 2008.

[65] Z. Ke, H. Logemann, and S. Townley. Adaptive sampled-data integral control of stable infinite-dimensional linear systems. *Systems Control Lett.*, 58(4):233–240, 2009.

[66] H. K. Khalil. *Nonlinear systems.* 3rd ed., Prentice-Hall, Upper Saddle River NJ, 2002.

[67] K. K. K. Kim and R. D. Braatz. Observer-based output feedback control of discrete-time Luré systems with sector-bounded slope-restricted nonlinearities. *Internat. J. Robust Nonlinear Control*, 24(16):2458–2472, 2014.

[68] N. Kottenstette, M. J. McCourt, M. Xia, V. Gupta, and P. J. Antsaklis. On relationships among passivity, positive realness, and dissipativity in linear systems. *Automatica J. IFAC*, 50(4):1003–1016, 2014.

[69] P. Lancaster and L. Rodman. *Algebraic Riccati equations.* Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1995.

[70] P. D. Lax. *Functional analysis.* Pure and Applied Mathematics (New York). Wiley-Interscience [John Wiley & Sons], New York, 2002.

[71] M. R. Liberzon. Essays on the absolute stability theory. *Automation and Remote Control*, 67(10):1610–1644, 2006.

[72] H. Logemann. Stability and stabilizability of linear infinite-dimensional discrete-time systems. *IMA Journal of Mathematical Control and Information*, 9(3):255–263, 1992.

[73] H. Logemann. Stabilization of well-posed infinite-dimensional systems by dynamic sampled-data feedback. *SIAM J. Control Optim.*, 51(2):1203–1231, 2013.

[74] H. Logemann. Some spectral properties of operator-valued positive-real functions. *Submitted for publication*, 2019.

[75] H. Logemann, R. Rebarber, and S. Townley. Stability of infinite-dimensional sampled-data systems. *Trans. Amer. Math. Soc.*, 355(8):3301–3328, 2003.

[76] H. Logemann and E. P. Ryan. Time-varying and adaptive discrete-time low-gain control of infinite-dimensional linear systems with input nonlinearities. *Math. Control Signals Systems*, 13(4):293–317, 2000.

[77] H. Logemann and E. P. Ryan. *Ordinary differential equations*. Springer Undergraduate Mathematics Series. Springer, London, 2014. Analysis, qualitative theory and control.

[78] H. Logemann, E. P. Ryan, and S. Townley. Integral control of linear systems with actuator nonlinearities: lower bounds for the maximal regulating gain. *IEEE Trans. Automat. Control*, 44(6):1315–1319, 1999.

[79] H. Logemann and S. Townley. Discrete-time low-gain control of uncertain infinite-dimensional systems. *IEEE Trans. Automat. Control*, 42(1):22–37, 1997.

[80] H. Logemann and S. Townley. Low-gain control of uncertain regular linear systems. *SIAM J. Control Optim.*, 35(1):78–116, 1997.

[81] A. I. Lur'e and V. N. Postnikov. Concerning the theory of stability of regulating systems. *Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.]*, 8:246–248, 1944.

[82] C. Merow, J. P. Dahlgren, C. J. E. Metcalf, D. Z. Childs, M. E. K. Evans, E. Jongejans, S. Record, M. Rees, R. Salguero-Gomez, and S. M. McMahon. Advancing population ecology with integral projection models: a practical guide. *Methods in Ecology and Evolution*, 5(2):99–110, 2014.

[83] A. Mironchenko and F. Wirth. Characterizations of input-to-state stability for infinite-dimensional systems. *IEEE Trans. Automat. Control*, 63(6):1602–1617, 2018.

[84] M. Morari. Robust stability of systems with integral control. *IEEE Trans. Automat. Control*, 30(6):574–577, 1985.

[85] P. C. Opdenacker and E. A. Jonckheere. A contraction mapping preserving balanced reduction scheme and its infinity norm error bounds. *IEEE Trans. Circuits and Systems*, 35(2):184–189, 1988.

[86] A. Pazy. *Semigroups of linear operators and applications to partial differential equations*, volume 44 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1983.

[87] G. K. Pedersen. *Analysis now*, volume 118 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1989.

[88] P. Pepe and Z.-P. Jiang. A Lyapunov-Krasovskii methodology for ISS and iISS of time-delay systems. *Systems Control Lett.*, 55(12):1006–1014, 2006.

[89] S. Pohjolainen. Robust controller for systems with exponentially stable strongly continuous semigroups. *J. Math. Anal. Appl.*, 111(2):622–636, 1985.

[90] S. A. Pohjolainen. Robust multivariable PI-controller for infinite-dimensional systems. *IEEE Trans. Automat. Control*, 27(1):17–30, 1982.

[91] C. Prieur and F. Mazenc. ISS-Lyapunov functions for time-varying hyperbolic systems of balance laws. *Math. Control Signals Systems*, 24(1-2):111–134, 2012.

[92] K. M. Przyłuski. The Lyapunov equation and the problem of stability for linear bounded discrete-time systems in Hilbert space. *Appl. Math. Optim.*, 6(2):97–112, 1980.

[93] M. Renardy and R. C. Rogers. *An introduction to partial differential equations*. Springer-Verlag, New York, 1993.

[94] R. T. Rockafellar and R. J.-B. Wets. *Variational analysis*, volume 317 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1998.

[95] K. E. Rose, S. M. Louda, and M. Rees. Demographic and evolutionary impacts of native and invasive insect herbivores on cirsium canescens. *Ecology*, 86(2):453–465, 2005.

[96] M. Rosenblum and J. Rovnyak. *Hardy classes and operator theory*. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 1985. Oxford Science Publications.

[97] W. Rudin. *Real and complex analysis*. McGraw-Hill Book Co., New York, third edition, 1987.

[98] W. M. Ruess and W. H. Summers. Compactness in spaces of vector valued continuous functions and asymptotic almost periodicity. *Math. Nachr.*, 135:7–33, 1988.

[99] B. S. Rüffer, N. van de Wouw, and M. Mueller. Convergent system vs. incremental stability. *Systems Control Lett.*, 62(3):277–285, 2013.

[100] E. P. Ryan and E. D. Sontag. Well-defined steady-state response does not imply CICS. *Systems Control Lett.*, 55(9):707–710, 2006.

[101] D. Salamon. Infinite-dimensional linear systems with unbounded control and observation: a functional analytic approach. *Trans. Amer. Math. Soc.*, 300(2):383–431, 1987.

[102] D. Salamon. Realization theory in Hilbert space. *Math. Systems Theory*, 21(3):147–164, 1989.

[103] I. W. Sandberg. The circle criterion and almost periodic inputs. *IEEE Trans. Circuits Systems I Fund. Theory Appl.*, 47(6):825–829, 2000.

[104] I. W. Sandberg and K. Kramer Johnson. Steady-state errors in nonlinear control systems. *IEEE Trans. Automat. Control*, 37(12):1985–1989, 1992.

[105] I. W. Sandberg and G. J. J. van Zyl. The spectral coefficients of the response of nonlinear systems to asymptotically almost periodic inputs. *IEEE Trans. Circuits Systems I Fund. Theory Appl.*, 48(2):170–176, 2001.

[106] E Sarkans. *Absolute stability and input-to-state stability of Lur'e systems.* PhD Thesis, University of Bath, 2014.

[107] E. Sarkans and H. Logemann. Input-to-state stability of Lur'e systems. *Math. Control Signals Systems*, 27(4):439–465, 2015.

[108] E. Sarkans and H. Logemann. Input-to-state stability of discrete-time Lur'e systems. *SIAM J. Control Optim.*, 54(3):1739–1768, 2016.

[109] E. Sarkans and H. Logemann. Stability of higher-order discrete-time Lur'e systems. *Linear Algebra Appl.*, 506:183–211, 2016.

[110] E. D. Sontag. Smooth stabilization implies coprime factorization. *IEEE Trans. Automat. Control*, 34(4):435–443, 1989.

[111] E. D. Sontag. Comments on integral variants of ISS. *Systems Control Lett.*, 34(1-2):93–100, 1998.

[112] E. D. Sontag. A remark on the converging-input converging-state property. *IEEE Trans. Automat. Control*, 48(2):313–314, 2003.

[113] E. D. Sontag. Input to state stability: basic concepts and results. In *Nonlinear and optimal control theory*, volume 1932 of *Lecture Notes in Math.*, pages 163–220. Springer, Berlin, 2008.

[114] E. D. Sontag. *Mathematical control theory: deterministic finite dimensional systems*, volume 6. Springer Science & Business Media, 2013.

[115] E. D. Sontag and Y. Wang. Notions of input to output stability. *Systems Control Lett.*, 38(4-5):235–248, 1999.

[116] O. J. Staffans. Quadratic optimal control of stable well-posed linear systems. *Trans. Amer. Math. Soc.*, 349(9):3679–3715, 1997.

[117] O. J. Staffans. J-energy preserving well-posed linear systems. volume 11, pages 1361–1378. 2001. Infinite-dimensional systems theory and operator theory (Perpignan, 2000).

[118] O. J. Staffans. *Well-posed linear systems*, volume 103 of *Encyclopedia of Mathematics and its Applications.* Cambridge University Press, Cambridge, 2005.

[119] G. Tao and P. A. Ioannou. Necessary and sufficient conditions for strictly positive real matrices. *IEE Proceedings G - Circuits, Devices and Systems*, 137(5):360–366, 1990.

[120] S. Tarbouriech and M. Turner. Anti-windup design: an overview of some recent advances and open problems. *IET Control Theory Appl.*, 3(1):1–19, 2009.

[121] M. Vidyasagar. *Nonlinear systems analysis*, volume 42 of *Classics in Applied Mathematics*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2002. Reprint of the second (1993) edition.

[122] G. Weiss. Admissibility of unbounded control operators. *SIAM J. Control Optim.*, 27(3):527–545, 1989.

[123] G. Weiss. Admissible observation operators for linear semigroups. *Israel J. Math.*, 65(1):17–43, 1989.

[124] G. Weiss. Regular linear systems with feedback. *Math. Control Signals Systems*, 7(1):23–57, 1994.

[125] J. T. Wen. Time domain and frequency domain conditions for strict positive realness. *IEEE Trans. Automat. Control*, 33(10):988–992, 1988.

[126] H. K. Wimmer. Existence of unmixed solutions of the discrete-time algebraic Riccati equation and a nonstrictly bounded real lemma. *Systems Control Lett.*, 43(2):141–147, 2001.

[127] C. Xiao and D. J. Hill. Generalizations and new proof of the discrete-time positive real lemma and bounded real lemma. *IEEE Trans. Circuits Systems I Fund. Theory Appl.*, 46(6):740–743, 1999.

[128] V. A. Yakubovich. The matrix inequality method in the theory of the stability of nonlinear control systems. Part I: Absolute stability of forced vibrations. *Autom. Remote Control*, 7:905–917, 1964.

[129] K. Yosida. *Functional analysis*. Classics in Mathematics. Springer-Verlag, Berlin, 1995. Reprint of the sixth (1980) edition.

Index