A uniqueness result for a simple superlinear eigenvalue problem

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Abstract

We study the eigenvalue problem for a superlinear convolution operator in the special case of bilinear constitutive laws and establish the existence and uniqueness of a one-parameter family of nonlinear eigenfunctions under a topological shape constraint. Our proof uses a nonlinear change of scalar parameters and applies Krein-Rutmann arguments to a linear substitute problem. We also present numerical simulations and discuss the asymptotics of two limiting cases.

Contents

1   Introduction

Nonlinear analogues to linear eigenvalue problems arise in many branches of the sciences and often model coherent structures in spatially extended dynamical systems. Examples include traveling waves for Hamiltonian lattices (see below), breathers in discrete systems [FW98, Wei99], nonlocal aggregation models [BHW13, HO15], and chimera states of the Kuramoto equation [OMT08, Ome18],

Assuming a one-dimensional continuum space variable, the simplest scalar equation

$$\sigma \, u = a * f(u) \tag{1}$$

involves a localized convolution kernel a and a superlinear function f with $f(0) = 0$. The problem is to find solution pairs $(\sigma, \, u)$ consisting of an eigenvalue $\sigma \in \mathbb{R}$ and a nontrivial eigenfunction $u : \mathbb{R} \to \mathbb{R}$ which might be of periodic or homoclinic type.

The existence of solutions to nonlinear eigenvalue problems such as (1) can be established by different approaches and this has already been done for special choices or certain classes of a and f. Prominent candidates are the Crandall-Rabinowitz theory [CR71], local bifurcation analysis [Ome13, ST19], and taylor made fixed point arguments [BHW13]. Further methods that have been successfully applied in the context of lattice waves include critical point techniques [Pan05], constrained optimization [FW94, FV99, Her11, HM20], and asymptotic methods [MA94, IJ05, Jam12].

The uniqueness problem, however, is notoriously difficult in the superlinear case and we are not aware of any such result except for local uniqueness proofs as given in [FP99, HM19b] for certain asymptotic regimes. In the sublinear case we refer to [Rab71], and for 1-homogeneous nonlinearities

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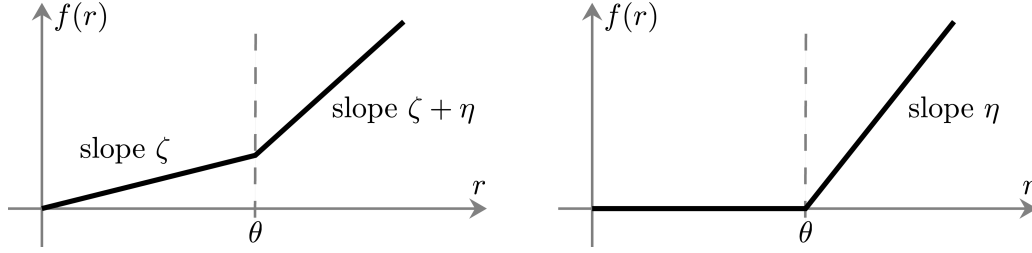
Figure 1: The bilinear function f from (2) as studied in this paper in the superlinear parameter regime (3). The general case (*left panel*) can be traced back to the special case $\zeta = 0$ (*right panel*) according to Proposition 6.

f there exists generalizations of the Krein-Rutmann theorem, see for instance [Mah07, Ara18]. In the scalar case, however, such functions f are necessarily linear.

In this paper we restrict our considerations to bilinear functions f which are piecewise linear with two slopes. This simplifying assumption allows us to transform (1) into the linear eigenvalue problem for a modified convolution operator with cut-off parameter ξ. We further suppose that both the kernel a and the eigenfunction u are nonnegative, even, and unimodal. This shape constraint provides an existence and uniqueness result for the linear substitute problem thanks to a variant of the Krein-Rutmann theorem. Moreover, it ensures that the nonlinear relation between σ and ξ is bijective.

Assumptions and Results As illustrated Figure 1, we consider functions f that are piecewise linear on the interval $[0, \infty)$. More precisely, we set

$$f(r) := \begin{cases} \zeta\, r & \text{for} \quad r \in [0, \theta]\,, \\ (\zeta + \eta)\,(r - \theta) + \zeta\,\theta & \text{for} \quad r \in [\theta, \infty)\,, \end{cases} \tag{2}$$

and assume that the three free parameters ζ, θ, η comply with the superlinearity condition

$$\zeta \geq 0\,, \qquad \theta > 0\,, \qquad \eta > 0\,. \tag{3}$$

It is also essential for our uniqueness result that the kernel $a : \mathbb{R} \to \mathbb{R}$ is integrable and belongs to the cone

$$\mathcal{U} = \left\{ u \in \mathsf{L}^2(\mathbb{R}) \;:\; u(x) = u(-x) \;\text{ and }\; u(x) \geq u(\tilde{x}) \geq 0 \;\text{ for almost all }\; 0 < x < \tilde{x} \right\},$$

which contains all bounded L^2-functions that are even, nonnegative, and unimodal. We suppose further that a is differentiable and strictly unimodal according to

$$a'(x) < 0 \quad \text{for} \quad x > 0\,, \tag{4}$$

because this simplifies the presentation. However, we always discuss how more general results can be obtained by relaxing the precise formulation of theorems or enhancing the technical arguments in their proofs.
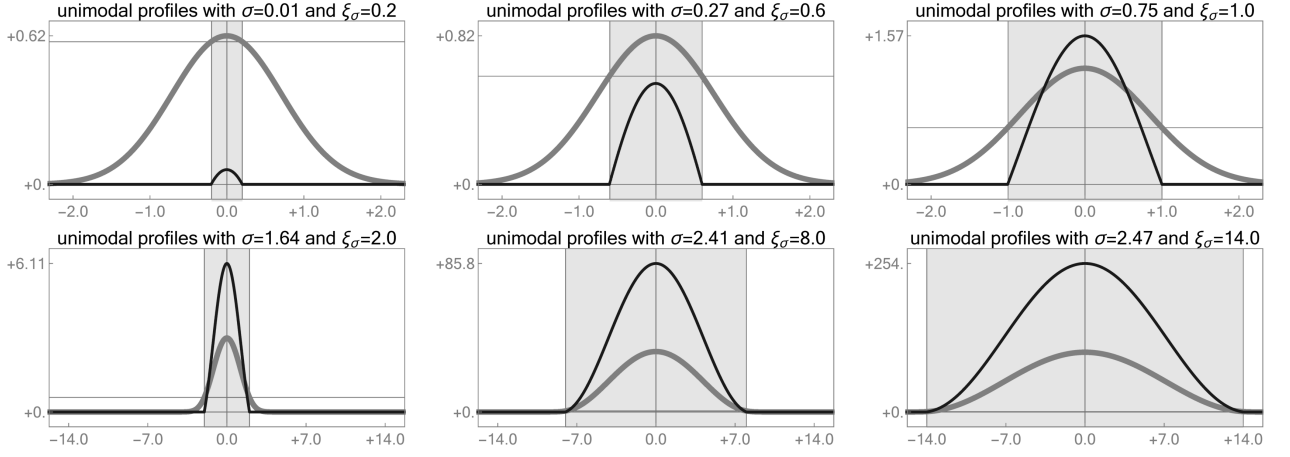
Assumption 1 (properties of the convolution kernel). *The convolution kernel a belongs to $\mathcal{U}$ and $\mathsf{L}^1(\mathbb{R})$, is normalized by*
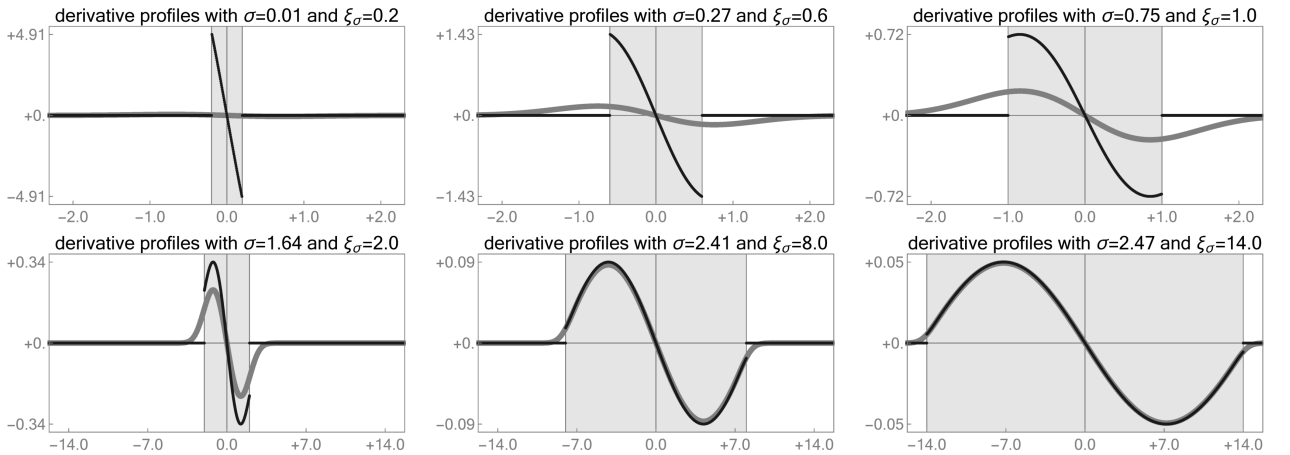
$$\int\limits_{-\infty}^{+\infty} a(x)\,\mathrm{d}x = 1\,,$$

and satisfies additionally (4).

A prototypical example for Assumption 1 is the Gaussian

$$a_1(x) := \exp\left(-x^2\right)/\sqrt{\pi} \tag{5}$$

Figure 2: Numerical profiles u_σ (gray) and $f(u_\sigma)$ (black) for several values of σ, the regular kernel (5), and nonlinearity parameters $\zeta = 0$, $\theta = 0.6$, $\eta = 2.5$. The gray box represents the interval I_{ξ_σ} which is defined by $u_\sigma(\pm\xi_\sigma) = \theta$. Notice the different plot regions for top and bottom row.

Figure 3: Effective derivative profile $v_\sigma = \chi_{\xi_\sigma} \cdot u'_\sigma$ (gray) as well as $a * v_\sigma = \eta^{-1} \sigma u'_\sigma$ (black) for the nonlinear eigenfunctions from Figure 2. The Krein-Rutman eigenfunction provided by Proposition 4 for ξ_σ is proportional to v_σ.

while the unimodal tent map

$$a_2(x) := \max\{0, 1 - |x|\} \tag{6}$$

and the indicator function

$$a_3(x) := \begin{cases} 1 & \text{for } |x| \leq \frac{1}{2} \\ 0 & \text{else} \end{cases} \tag{7}$$

are compactly supported and violate (4). Both kernels are nonetheless interesting since they are naturally related to traveling waves in Fermi-Pasta-Ulam-Tsingou chains and certain discrete scalar conservation laws, see [HM19a, Her12] and references therein. For the kernel (6), the eigenvalue problem (1) with bilinear function (2) has already been solved in [TV14] by a combination of Fourier and numerical methods, see the more detailed comments at the end of §3.

Our findings for the nonlinearity (2) and kernels as in Assumption 1 can be summarized as follows.

Main Result 2 (existence and uniqueness under shape constraint)**.**

1. *There exists a unique one-parameter family of nonlinear eigenfunction $u_\sigma \in \mathcal{U}$ parametrized by $\zeta < \sigma < \zeta + \eta$.*

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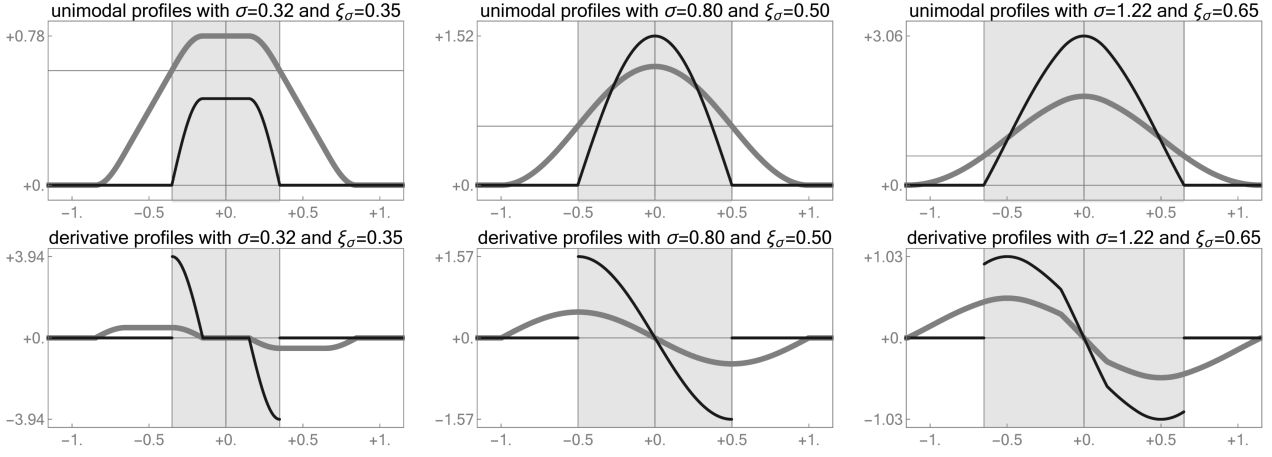
Figure 4: Numerical simulations for the degenerate kernel (4) and nonlinearity parameters as in Figure 2. The statement of some results must be modified in this case since the strict unimodality condition (4) is not satisfied.

2. *Formal asymptotic expansions suggest the following limit behavior:*

   (a) *The eigenfunctions u_σ converge as $\sigma \searrow \zeta$ to a nontrivial limit profile $u_\zeta \in \mathcal{U}$ with*

$$u_\zeta(x) \le \theta \quad \text{for all} \quad x\,, \qquad u_\zeta(0) = \theta\,,$$

   but the details are different for $\zeta = 0$ and $\zeta > 0$.

   (b) *There is no limit $u_{\zeta+\eta}$ due to $\lim_{\sigma \nearrow \zeta+\eta} \|u_\sigma\|_2 = \infty$.*

The rigorous part of our main result is proven in Proposition 7 while the nonrigorous asymptotic analysis is presented in the appendix. We also refer to Figures 2, 3, and 4 for numerical simulations with $\zeta = 0$ and emphasize that our results do not exclude the existence of further eigenfunctions outside the cone $\mathcal{U}$. In addition to the trivial nonuniqueness due to the shift invariance of (1), we expect the existence of families of periodic solutions. Moreover, for rapidly decaying kernels one might think about multi-bump solutions, which can be imagined as superposition of finitely many and well separated single-bump solutions having unimodal profile.

Proof strategy and organization of paper   The central ideas for the proof of our main results can be sketched as follows. Assuming smoothness of u we deduce $u' \in \mathcal{N}$, where

$$\mathcal{N} := \left\{ v \in \mathsf{L}^2(\mathbb{R}) \, : \, v(x) = -v(-x) \text{ and } v(x) \le 0 \text{ for almost all } x > 0 \right\},$$

contains all odd functions that attain nonpositive values on $(0, \infty)$ (usually, elements of $-\mathcal{N}$ and $\mathcal{N}$ are called positive and negative, respectively). Moreover, differentiating (1) we find

$$\sigma\, u' = a * \big(\zeta u' + (\eta - \zeta)\, \chi_\xi \cdot u' \big) \tag{8}$$

with

$$I_\xi := [-\xi, \xi]\,, \qquad \chi_\xi(x) := \begin{cases} 1 & \text{for } x \in I_\xi\,, \\ 0 & \text{else}\,, \end{cases}$$

where ξ is the value at which u attains the value θ. This reads

$$\theta = u(\xi) = \int_\xi^\infty u'(x)\, \mathrm{d}x\,. \tag{9}$$

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In the special case of $\zeta = 0$, the necessary condition (8) reads

$$\lambda\, u' = a * v \tag{10}$$

and involves the effective derivative profile

$$v = \chi_\xi \cdot u' \tag{11}$$

as well as

$$\lambda = \sigma/\eta\,. \tag{12}$$

Formula (10) implies the linear eigenvalue problem

$$\lambda\, v = \mathcal{A}_\xi v\,, \tag{13}$$

where

$$\mathcal{A}_\xi v := \chi_\xi \cdot \big(a * (\chi_\xi \cdot v)\big) \tag{14}$$

is a compact and symmetric operator, which exhibits nice invariance properties. In particular, adapting the strong version of the Krein-Rutmann theorem we can show that the eigenspace to the largest eigenvalue of $\mathcal{A}_\xi$ is simple and generated by an element of $\mathcal{N}$, see Proposition 4. This observation yields in combination with (9) the existence and uniqueness of a family of solutions to (1) which is naturally parametrized by ξ and can be computed efficiently. Moreover, a closer inspection of the linear eigenvalue problem (13) as carried out in Proposition 5 reveals that the mapping $\xi \mapsto \lambda \mapsto \sigma$ is invertible so that ξ can be replaced by σ. This change of parameters is the nonlinear part of our approach.

Another observation extends the existence and uniqueness result to the general case $\zeta > 0$. More precisely, we identify in Proposition 6 a modified kernel $\tilde{a} \in \mathcal{U}$ with $\int_\mathbb{R} \tilde{a}(x)\, \mathrm{d}x = 1$ such that the implication

$$w - \mu\, a * w = a * g \qquad \Longleftrightarrow \qquad w = (1 - \mu)^{-1}\, \tilde{a} * g \tag{15}$$

holds with $\mu = \sigma^{-1}\zeta$. This enables us to transform the general case $\zeta > 0$ into the special case $\zeta = 0$ by replacing the kernel a with $\tilde{a}$ and changing σ. A similar argument applies on the level of the derivatives because (8) can be written as $\tilde{\lambda}\, u' = \tilde{a} * (\chi_\xi \cdot u')$.

2 Linear eigenvalue problem for the effective derivative profile

In this section we study the eigenvalue problem of the operator $\mathcal{A}_\xi$ from (14) in the space $\mathsf{L}^2_\mathrm{odd}(\mathbb{R})$ which contains all functions that are square-integrable and odd. Notice that the representation formulas

$$(\mathcal{A}_\xi v)(x) = \int\limits_{-\xi}^{+\xi} a(x - y)v(y)\, \mathrm{d}y = \int\limits_{0}^{\xi} \big(a(x - y) - a(x + y)\big)v(y)\, \mathrm{d}y \qquad \text{for} \quad |x| \leq \xi \tag{16}$$

and

$$(\mathcal{A}_\xi v)(x) = 0 \qquad \text{for} \quad |x| > \xi$$

hold for any $v \in \mathsf{L}^2_\mathrm{odd}(\mathbb{R})$.

Lemma 3 (properties of the modified convolution operator)**.** *For any $\xi \in (0, \infty)$, the operator $\mathcal{A}_\xi$, maps $\mathsf{L}^2(\mathbb{R})$ into itself, respects the even-odd parity, and is both compact and self-adjoint. Moreover, the convex cones $\mathcal{N}$ and $\mathcal{U}$ are invariant under the action of $\mathcal{A}_\xi$.*

Proof. The first assertions can be derived from (14) using standard arguments for convolution and multiplication operators. Moreover, since a is even and unimodal, the invariance of $\mathcal{N}$ and $\mathcal{U}$ follows from (16) and the analogous formula in $\mathsf{L}^2_\mathrm{even}(\mathbb{R})$. $\qquad\square$

Linear uniqueness result   As first main auxiliary result we prove that the largest eigenvalue of $\mathcal{A}_\xi$ is simple and spanned by a unique normalized eigenfunction in $\mathcal{N}$. Such uniqueness results are usually inferred from the strong version of the Krein-Rutmann theorem but the classical formulation requires an order preserving operator that maps the cone of positive (or negative) elements in its interiors, see for instance [DL90, appendix to chapter IIX]. In our case, however, the cone $\mathcal{N}$ has no inner points and the compact support of χ_ξ implies for every $v \in \mathcal{N}$ that $\mathcal{A}_\xi v$ belongs to the topological L^2-boundary of $\mathcal{N}$.

The assertions of the Krein-Rutmann Theorem hold notwithstanding. The crucial idea is that any eigenfunction that corresponds to the largest eigenvalue of $\mathcal{A}_\xi$ must belong to the smaller cone

$$\widetilde{\mathcal{N}}_\xi := \left\{ v \in \mathcal{N} \cap \mathsf{C}^1(I_\xi) \; : \; v'(0) < 0 \ \text{ and } \ v(x) < 0 \ \text{ for } \ 0 < x \le \xi \ \text{ but } \ v(x) = 0 \ \text{ for } \ x > \xi \right\},$$

and this observation finally enables us to adapt classical arguments. A similar strategy has been used in [BFH14], which combines a variant of the operator $\mathcal{A}_\xi$ with nonlinear fixed point arguments to prove the existence (but not the uniqueness) of unimodal and compactly supported solutions ϱ to the equation

$$\varrho \, \partial_x \big(g(\varrho) - a * \varrho \big) = 0 \,,$$

where g is a given increasing monomial. This equation describes stationary solutions to a nonlinear biological aggregation model and the link to Krein-Rutmann arguments is that $v = \varrho'$ satisfies the linearized equation $g'(\varrho) \, v = a * v$ on the compact support of ϱ.

Proposition 4 (variant of the Krein-Rutmann theorem). *For any $0 < \xi < \infty$, the largest eigenvalue λ_ξ of $\mathcal{A}_\xi$ in $\mathsf{L}^2_{\mathrm{odd}}(\mathbb{R})$ is simple and the corresponding eigenspace is spanned by a unique eigenfunction $v_\xi \in \widetilde{\mathcal{N}}_\xi$ with $\|v_\xi\|_2 = 1$.*

Proof. <u>*Variational setting*</u>: The operator $\mathcal{A}_\xi$ admits only real eigenvalues thanks to Lemma 3. It is also the Gâteaux derivative of the functional

$$\mathcal{F}_\xi(v) := \tfrac{1}{2} \langle v, \, \mathcal{A}_\xi v \rangle = \tfrac{1}{2} \langle \chi_\xi \cdot v, \, a * (\chi_\xi \cdot v) \rangle \,,$$

which is well-defined on $\mathsf{L}^2_{\mathrm{odd}}(\mathbb{R})$ since Young's inequality implies

$$2 \big| \mathcal{F}_\xi(v) \big| \le \big\| \chi_\xi \cdot v \big\|_2 \big\| a * (\chi_\xi \cdot v) \big\|_2 \le \|a\|_1 \|\chi_\xi\|_\infty^2 \|v\|_2^2 \le \|v\|_2^2 \,. \tag{17}$$

Moreover, the largest eigenvalue λ_ξ of $\mathcal{A}_\xi$ can be characterized variationally via

$$\lambda_\xi = \max \big\{ 2\mathcal{F}_\xi(v) \; : \; v \in \mathsf{L}^2_{\mathrm{odd}}(\mathbb{R}), \, \|v\|_2 = 1 \big\} \,, \tag{18}$$

where any maximizer corresponds to a normalized eigenfunction to λ_ξ and vice versa.

<u>*Existence of eigenfunction in $\mathcal{N}$*</u>: Any $v \in \mathsf{L}^2_{\mathrm{odd}}(\mathbb{R})$ admits a unique and disjoint-support splitting

$$v = v_- - v_+ \qquad \text{with} \qquad v_-, \, v_+ \in \mathcal{N} \qquad \text{and} \qquad \|v\|_2^2 = \|v_-\|_2^2 + \|v_+\|_2^2 \,, \tag{19}$$

and using Assumption 1, Lemma 3, as well as (14) we verify

$$\mathcal{F}_\xi(v) = \tfrac{1}{2} \langle v_-, \, \mathcal{A}_\xi v_- \rangle + \tfrac{1}{2} \langle v_+, \, \mathcal{A}_\xi v_+ \rangle - \langle v_-, \, \mathcal{A}_\xi v_+ \rangle \le \mathcal{F}_\xi(v_+) + \mathcal{F}_\xi(v_-) \,. \tag{20}$$

Now suppose that $v \in \mathsf{L}^2_{\mathrm{odd}}(\mathbb{R})$ is a maximizer for (18) with $v_- \ne 0$ and $v_+ \ne 0$. We then have

$$\mathcal{F}_\xi(v_- / \|v_-\|_2) \le \mathcal{F}_\xi(v) \,, \qquad \mathcal{F}_\xi(v_+ / \|v_+\|_2) \le \mathcal{F}_\xi(v) \tag{21}$$

while the homogeneity of $\mathcal{F}_\xi$ along with (20) yields

$$\mathcal{F}_\xi(v) \le \|v_-\|_2^2 \, \mathcal{F}_\xi\big(v_- / \|v_-\|_2 \big) + \|v_+\|_2^2 \, \mathcal{F}_\xi\big(v_+ / \|v_+\|_2 \big) \,.$$
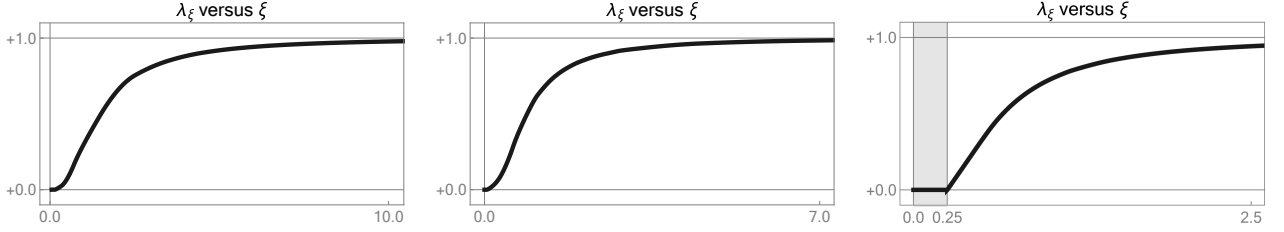
Figure 5: Numerical values of the Krein-Rutmann eigenvalue λ_ξ from Proposition 5 for the kernels (5) (*left*), (6) (*center*), and (7) (*right*). Notice the constant region in the last example (gray box), which stems from the plateau in the kernel.

The combination of the two latter results with (19) and the normalization condition $\|v\|_2^2 = 1$ implies equality in both parts of (21). We conclude that v_- and v_+ are eigenfunctions to the maximal eigenvalue λ_ξ that belong to $\mathcal{N}$.

Regularity by refined cone analysis: We next show that any eigenfunction $v \in \mathcal{N}$ to λ_ξ belongs in fact to the smaller cone $\widetilde{\mathcal{N}}_\xi$. The strict unimodality and the evenness of a imply

$$a(x - y) - a(x + y) > 0 \qquad \text{for all } 0 < x, y < \xi, \tag{22}$$

so (16) guarantees $(\mathcal{A}_\xi v)(x) < 0$ and hence $v(x) < 0$ for all $0 < x \le \xi$. Moreover, we have

$$(\mathcal{A}_\xi v)'(x) = \int_0^\xi \Big(a'(x - y) - a'(x + y) \Big) v(y) \, \mathrm{d}y \,, \qquad (\mathcal{A}_\xi v)'(0) = -2 \int_0^\xi a'(y) v(y) \, \mathrm{d}y < 0$$

and obtain $v \in \mathsf{BC}^1(I_\xi)$ with $v'(0) < 0$. Here we used $\lambda_\xi > 0$, which holds because (16) and (22) provide $\mathcal{F}_\xi(v) > 0$ for any nontrivial $v \in \mathcal{N}$.

Uniqueness of normalized eigenfunctions: We first show uniqueness within $\mathcal{N}$ as follows. Any two normalized eigenfunctions v_1 and v_2 to λ_ξ lie in $\widetilde{\mathcal{N}}_\xi$ and the properties of this cone imply the existence of a parameter $0 < s < \infty$ such that $v_1 - s\,v_2$ still belongs to $\mathcal{N}$ but no longer to $\widetilde{\mathcal{N}}_\xi$ (because of $v_1'(0) = s\,v_2'(0)$ or $v_1(x) = s\,v_2(x)$ for at least one $0 < x \le \xi$). This, however, means $s = 1$ as well as $v_1 = v_2$ because otherwise $v_1 - s\,v_2$ would be nontrivial eigenfunction in $\mathcal{N}$ that violates the above regularity result. Secondly, suppose there exists a nontrivial and normalized eigenfunction v that belongs to neither $\mathcal{N}$ or $-\mathcal{N}$. Our results derived so far imply that $v_-/\|v_-\|$ and $v_+/\|v_+\|$ are identically maximizers of (18) and hence a contradiction to the assumption on v. In summary, we have shown that the eigenspace to λ_ξ in $\mathsf{L}^2_{\mathrm{odd}}(\mathbb{R})$ is one-dimensional and spanned by a unique normalized function $v_\xi \in \widetilde{\mathcal{N}}_\xi$. $\qquad \square$

Parameter dependence   The second building block for our nonlinar uniqueness result in §3 are the following properties of the Krein-Rutmann quantities. Related numerical simulations are presented in Figure 5.

Proposition 5 (ξ-dependence of λ_ξ and v_ξ)**.** *The map $\xi \mapsto \lambda_\xi$ from Proposition 4 is continuous and strictly increasing with*

$$\lim_{\xi \searrow 0} \lambda_\xi = 0 \,, \qquad \lim_{\xi \nearrow \infty} \lambda_\xi = 1$$

Moreover, the corresponding map $\xi \mapsto v_\xi$ is L^2-continuous.

Proof. Monotonicity: Let $0 < \xi_1 < \xi_2 < \infty$ be fixed. Since v_{ξ_1} from Proposition 4 is supported in I_{ξ_1}, we have

$$\chi_{\xi_1} \cdot v_{\xi_1} = v_{\xi_1} = \chi_{\xi_2} \cdot v_{\xi_1} \,, \qquad \mathcal{F}_{\xi_1}(v_{\xi_1}) = \mathcal{F}_{\xi_2}(v_{\xi_1}) \,,$$

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and the optimization problem (18) ensures the nonstrict monotonicity $\lambda_{\xi_1} \leq \lambda_{\xi_2}$. However, v_{ξ_1} cannot be an eigenfunction of $\mathcal{A}_{\xi_2}$ because the properties of a imply that $\mathcal{A}_{\xi_2} v_{\xi_1}$ has a larger support than $v_{\xi_1} \in \widetilde{\mathcal{N}}_{\xi_1}$. We thus obtain $\lambda_{\xi_1} \neq \lambda_{\xi_2}$.

Continuity and asymptotics for small ξ: Suppose that $(\xi_n)_{n \in \mathbb{N}}$ is a given sequence with

$$\xi_n \to \xi \in (0, \infty)\,, \qquad \lambda_{\xi_n} \to \lambda \in (0, \infty)\,.$$

By weak compactness we can extract a (not relabeled) subsequence such that the normalized eigenfunctions v_{ξ_n} converge weakly to some limit $v \in \mathsf{L}^2_{\mathrm{odd}}(\mathbb{R})$, where the weak closedness of $\mathcal{N}$ ensures $v \in \mathcal{N}$. The properties of convolution operators combined with $\chi_{\xi_n} \to \chi_n$ imply that $\mathcal{A}_{\xi_n} v_{\xi_n}$ converges strongly to $\mathcal{A}_\xi v$ and this guarantees — thanks to the eigenvalue equation for v_{ξ_n} — the strong convergence of v_{ξ_n}. In particular, we obtain

$$\lambda\, v = \mathcal{A}_\xi v\,, \qquad \|v\|_2 = 1\,, \qquad 2\mathcal{F}_\xi(v) = \lambda \tag{23}$$

and hence $\lambda \leq \lambda_\xi$. On the other hand, (18) ensures $\lambda_\xi \leq \lambda$ since

$$2\mathcal{F}_\xi(\tilde{v}) = \lim_{n \to \infty} 2\mathcal{F}_{\xi_n}(\tilde{v}) \leq \lim_{n \to \infty} \lambda_{\xi_n} = \lambda$$

holds for any $\tilde{v} \in \mathsf{L}^2_{\mathrm{odd}}(\mathbb{R})$. We thus get $\lambda = \lambda_\xi$ and Proposition 4 combined with (23) yields $v = v_\xi$. In summary, we have shown for any convergent sequence $(\xi_n)_{n \in \mathbb{N}}$ that λ_ξ is the unique accumulation point of the sequence $(\lambda_{\xi_n})_{n \in \mathbb{N}}$ (which is bounded due to the monotonicity with respect to ξ), and this gives rise to the claimed continuity of λ_ξ. Our arguments also yields the strong convergence $v_{\xi_n} \to v_\xi$ and analogously we derive $\xi_n \to 0$ from $\lambda_{\xi_n} \to 0$ because $\mathcal{A}_{\xi_n} v_{\xi_n} \to 0$ holds along any weakly convergent subsequence.

Asymptotics for large ξ: Direct computations for the piecewise constant and compactly supported test function $\tilde{v}_\xi$ with

$$\tilde{v}_\xi(x) = -(2\xi)^{-1/2} \operatorname{sgn}(x)\, \chi_\xi(x)$$

yield $\|\tilde{v}_\xi\|_2 = 1$ as well as

$$\mathcal{F}_\xi(\tilde{v}_\xi) = \frac{1}{4\xi} \int\limits_{-\xi}^{+\xi} \int\limits_{-\xi}^{+\xi} a(x-y) \operatorname{sgn}(x) \operatorname{sgn}(y)\, \mathrm{d}y\, \mathrm{d}x = \frac{1}{2\xi} \int\limits_0^\xi \int\limits_0^\xi \big(a(x-y) - a(x+y)\big)\, \mathrm{d}y\, \mathrm{d}x$$

$$= \frac{1}{2\xi} \int\limits_0^{2\xi} \int\limits_0^{+t} \big(a(s) - a(t)\big)\, \mathrm{d}s\, \mathrm{d}t = \int\limits_0^{2\xi} a(z)\frac{2\xi - 2z}{2\xi}\, \mathrm{d}z\,.$$

Using (18) and the Dominated Convergence Theorem we therefore get

$$\liminf_{\xi \to \infty} \lambda_\xi \geq 2 \lim_{\xi \to \infty} \mathcal{F}_\xi(\tilde{v}_\xi) = 2 \int\limits_0^\infty a(z)\, \mathrm{d}z = 1$$

thanks to Assumption 1. On the other hand, the Young estimate (17) applied to v_ξ ensures

$$\limsup_{\xi \to \infty} \lambda_\xi = 2 \limsup_{\xi \to \infty} \mathcal{F}_\xi(v_\xi) \leq 1\,,$$

and the proof is complete. $\qquad\qquad\square$

Approximation The unique normalized Krein-Rutmann eigenfunction as provided by Proposition 5 can be computed as limit of the sequence $(v_n)_{n \in \mathbb{N}} \subset \mathcal{N}$ with

$$v_n := \frac{\mathcal{A}_\xi^n v_0}{\big\|\mathcal{A}_\xi^n v_0\big\|_2}$$
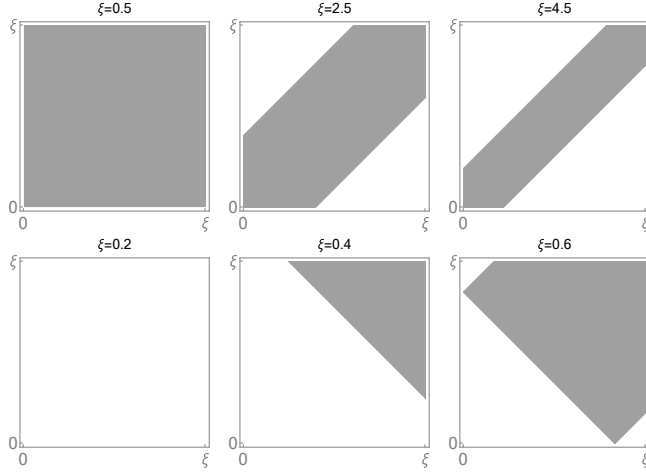
Figure 6: Support of the integral kernel corresponding to $\mathcal{A}_\xi$, see (16), for several values of ξ and the degenerate convolution kernels (6) (*top row*) and (7) (*bottom row*). Since condition (22) is not satisfied for some values of ξ, the proof and/or the precise statement of Proposition 4 needs to be modified as discussed in the text. See also Figure 5 for the corresponding Krein-Rutmann eigenvalues.

and arbitrary $v_0 \in \mathcal{N}$. Moreover, using $\mathcal{F}_\xi(-\operatorname{sgn}|v|) \geq \mathcal{F}_\xi(v)$ we deduce that λ_ξ is not only the largest eigenvalue but also the spectral radius of $\mathcal{A}_\xi$ and hence the exponential convergence of v_n as $n \to \infty$. This approximation scheme is often called Power Method and a straight forward discretization (fine but equidistant spatial grid and Riemann sums instead of integrals) has been used to produce the numerical results in this paper. Alternatively, one can write the sequence as

$$\lambda_n v_{n+1} = \mathcal{A}_\xi v_n \,, \qquad \lambda_n = \|\mathcal{A}_\xi v_n\|_2 \,,$$

where λ_n will converge as $n \to \infty$ to the Krein-Rutmann eigenvalue λ_ξ. A similar *improvement dynamics* can also be applied to the nonlinear eigenvalue problem (1), but the convergence is more subtle due to the lack of uniquness results for general functions f. We refer to [HM20] for numerical examples and a more detailed discussion of the analytical properties.

Generalizations   We finally discuss the case of less regular kernels a. The crucial ingredient to the regularity step $v \in \mathcal{N} \Rightarrow \mathcal{A}_\xi v \in \widetilde{\mathcal{N}}_\xi$ in the proof of Proposition 4 is (22). This condition hinges on the strict unimodality of a as in (4) and is not satisfied for all $a \in \mathcal{U}$, see Figure 6 for an illustration. For kernels like the tent map (6), a recursive argument reveals the implication $v \in \mathcal{N} \Rightarrow \mathcal{A}_\xi^n v \in \widetilde{\mathcal{N}}_\xi$ for all sufficiently large $n \in \mathbb{N}$, and this guarantees that all assertions in Proposition 4 remain valid since the Krein-Rutmann eigenfunction still belongs to $\widetilde{\mathcal{N}}_\xi$. This, however, is no longer true for the kernel (7). Instead, we have to distinghuish between the following three parameter regimes:

1. $0 < \xi \leq \frac{1}{4}$ implies that $\mathcal{A}_\xi v$ vanishes for any $v \in \mathsf{L}^2_{\mathrm{odd}}(\mathbb{R})$, so $\mathcal{A}_\xi$ restricted to $\mathsf{L}^2_{\mathrm{odd}}(\mathbb{R})$ is actually the trivial operator and we have $\lambda_\xi = 0$, see the third panel in Figure 5.

2. In the case of $\frac{1}{4} < \xi < \frac{1}{2}$, there exists a unique and normalized Krein-Rutmann eigenfunction v_ξ, which does not belong to $\widetilde{\mathcal{N}}_\xi$ but to a modified subcone of $\mathcal{N}$ with

   $$v'\left(\tfrac{1}{2} - \xi\right) < 0\,, \qquad v(x) < 0 \ \text{for} \ \tfrac{1}{2} - \xi < x \leq \xi\,, \qquad v(x) = 0 \ \text{for} \ 0 < x < \tfrac{1}{2} - \xi \ \text{and} \ x > \xi\,,$$

   see the first column in Figure 4.

3. For $\xi \geq \frac{1}{2}$, we find again $v_\xi \in \widetilde{\mathcal{N}}_\xi$.

In summary, if a is given by (7), then Proposition 4 must be formulated more carefully and the analogue to Proposition 5 stipulates the one-to-one correspondence between $\xi \in (\frac{1}{4}, \infty)$ and $\lambda_\xi \in (0, 1)$. Similar statements apply to any other kernel $a \in \mathcal{U}$ that admits a constant plateau near $x = 0$.

3   Nonlinear eigenfunctions with unimodal profile

We are now able to prove our main result from §1 in two steps. First we show that the general case with given kernel a and $\zeta > 0$ is equivalent to the special case $\zeta = 0$ for a modified kernel $\tilde{a}$.

Proposition 6 (transformation to special case)**.** *A function u solves (1) with $\sigma \in (\zeta, \zeta + \eta)$ if and only the eigenvalue equation*

$$\tilde{\sigma}\, u = \tilde{a} * \tilde{f}(u) \tag{24}$$

is satisfied with

$$\tilde{\sigma} := \sigma - \zeta = (1 - \mu)\,\sigma \in (0, \eta)\,, \qquad \mu := \sigma^{-1}\zeta \in (0, 1)\,.$$

Here, the transformed kernel

$$\tilde{a} := (1 - \mu)\, a * \left(1 + \mu\, a + \mu^2\, a * a + \mu^3 a * a * a + \dots\right)\,, \tag{25}$$

is well-defined, depends on σ, and complies with Assumption 1, while the simplified nonlinearity

$$\tilde{f}(r) := \begin{cases} 0 & \text{for} \quad r \in [0, \theta]\,, \\ \eta\,(r - \theta) & \text{for} \quad r \in [\theta, \infty)\,, \end{cases}$$

represents (2) with $\zeta = 0$.

Proof. <u>*Linear auxiliary operator and modified kernel*</u>: The Young estimate

$$\|a * w\|_2 \le \|a\|_1\,\|w\|_2 = \|w\|_2$$

combined with $0 < \mu < 1$ implies that the linear operator

$$\mathcal{L} : \mathsf{L}^2(\mathbb{R}) \to \mathsf{L}^2(\mathbb{R})\,, \qquad \mathcal{L}w := w - \mu\, a * w$$

is continuously invertible. The Neumann formula

$$\mathcal{L}^{-1}w = \breve{a} * w\,, \qquad \breve{a} := 1 + \mu\, a + \mu^2\, a * a + \mu^3\, a * a * a * \dots$$

reveals $\bar{a} \in \mathcal{U}$ thanks to Lemma 3, which in turn ensures that $\mathcal{L}$ respects the even-odd parity and mediates the implication (15). We further have

$$\tilde{a} = (1 - \mu)\, a * \breve{a} = (1 - \mu)\,\breve{a} * a \in \mathcal{U}$$

and compute

$$\int_{\mathbb{R}} \tilde{a}(x)\,\mathrm{d}x = \frac{1}{1 - \mu}\sum_{k=0}^{\infty}\mu^k = 1$$

as well as

$$\tilde{a}' := (1 - \mu)\,\breve{a} * a'\,.$$

The latter formula provides via

$$\tilde{a}'(x) < 0 \qquad \text{for all } x > 0$$

the strict unimodality of $\tilde{a}$.

   <u>*Transformation of the nonlinear problem*</u>: Let (σ, u) be a given solution to (1). Our definitions imply

$$\tilde{\sigma}\big(u - \mu\, a * u\big) = (1 - \mu)\big(\sigma\, u - \zeta\, a * u\big) = (1 - \mu)\big(a * \big(f(u) - \zeta\, u\big)\big) = (1 - \mu)\, a * \tilde{f}(u)$$

and hence the validity of (24) thanks to the existence of $\mathcal{L}^{-1}$. Similarly, the reverse implication follows by applying $\mathcal{L}$ to both sides of (24). $\qquad\square$

In the second step we finally establish our existence and uniqueness result for nonlinear eigenfunctions.

Proposition 7 (nonlinear existence and uniqueness result). *Equation* (1) *admits for any* $\sigma \in (\zeta, \zeta+\eta)$ *a unique solution* $u_\sigma \in \mathcal{U}$.

Proof. <u>*Preliminaries*</u>: Within this proof, σ is fixed. In view of Proposition 6, we can assume $\theta = 0$ with $\sigma \in (0, \eta)$ because otherwise we replace a by $\tilde{a}$ and σ by $\tilde{\sigma}$. Moreover, Proposition 5 guarantees that

$$\lambda_{\xi_\sigma} = \eta^{-1}\sigma \tag{26}$$

holds for precisely one cut-off parameter $\xi_\sigma \in (0, \infty)$.

<u>*Existence and construction*</u>: By integration, there exists a unique function $\tilde{u}_\sigma \in \mathcal{U}$ with

$$\tilde{u}'_\sigma(x) = v_{\xi_\sigma}(x) \quad \text{for} \quad x \in \mathbb{R}, \qquad \tilde{u}_\sigma(x) = 0 \quad \text{for} \quad |x| \geq \xi_\sigma, \tag{27}$$

where $v_{\xi_\sigma} \in \widetilde{\mathcal{N}_\xi}$ is the normalized and compactly supported Krein-Rutmann eigenfunction from Proposition 4. Since $\tilde{u}_\sigma$ is strictly positive for $|x| < \xi_\sigma$, the function $a * \tilde{u}_\sigma$ attains a positive value at $x = \xi_\sigma$. We thus define $u_\sigma \in \mathcal{U}$ by

$$u_\sigma := \tau_\sigma \, a * \tilde{u}_\sigma, \qquad \tau_\sigma := \frac{\theta}{(a * \tilde{u}_\sigma)(\xi)} \tag{28}$$

and observe that this guarantees

$$u_\sigma(\pm\xi_\sigma) = \theta, \qquad \theta \leq u_\sigma(x) \quad \text{for} \quad |x| < \xi_\sigma, \qquad 0 \leq u_\sigma(x) \leq \theta \quad \text{for} \quad |x| > \xi_\sigma$$

due to the unimodality of u_σ. Since we also have

$$u'_\sigma(x) = \tau_\sigma \left(a * \tilde{u}'_\sigma\right)(x) = \tau_\sigma \left(a * v_{\xi_\sigma}\right)(x) = \tau_\sigma \lambda_{\xi_\sigma} v_{\xi_\sigma}(x) = \tau_\sigma \lambda_{\xi_\sigma} \tilde{u}'_\sigma(x)$$

for all $|x| \leq \xi_\sigma$ (but not for $|x| > \xi_\sigma$), we find

$$u_\sigma(x) = \theta + \tau_\sigma \lambda_{\xi_\sigma} \tilde{u}_\sigma(x) \quad \text{for} \quad |x| \leq \xi_\sigma.$$

In particular, we have

$$f(u_\sigma(x)) = \tau_\sigma \lambda_{\xi_\sigma} \eta \, \tilde{u}_\sigma(x) \qquad \text{for all} \quad x \in \mathbb{R}$$

thanks to (2) with $\zeta = 0$, so

$$a * f(u_\sigma) = a * \left(\tau_\sigma \lambda_{\xi_\sigma} \eta \, \tilde{u}_\sigma\right) = \lambda_{\xi_\sigma} \eta \, u_\sigma = \sigma \, u_\sigma$$

follows from our definition of u_σ in (28).

<u>*Uniqueness*</u>: Now suppose that $0 \neq u \in \mathcal{U}$ solves (1) for $\zeta = 0$. This gives $0 < u(0) = \|u\|_\infty$ (otherwise $f(u)$ would vanish identically) and there exists $\xi \in (0, \infty)$ with $u(\xi) = \theta$. By differentiating (1) with respect to x — and using the notations (11), (12) — we establish the formulas (10) and (13), so Propositions 4 and 5 provide

$$\xi = \xi_\sigma \qquad \text{and} \qquad v = c \, v_{\xi_\sigma}$$

for some factor $c > 0$. Combining this with (27) and (28) we conclude that the derivatives of u and u_σ are proportional, and the consistency relations

$$\int\limits_{\xi_\sigma}^{\infty} u'(x)\,\mathrm{d}x = u(\xi_\sigma) = \theta = u_\sigma(\xi_\sigma) = \int\limits_{\xi_\sigma}^{\infty} u'_\sigma(x)\,\mathrm{d}x$$

give $u = u_\sigma$. $\qquad\square$
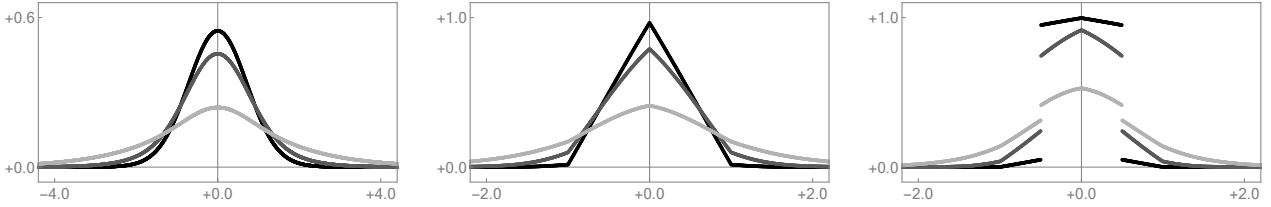
Figure 7: The transformed kernel (25) for the Gaussian (*left*), the tent map (*middle*), and the indicator function (*right*) from (5), (6), and (7). The three curves correspond to $\mu = 0.1$ (black), $\mu = 0.5$ (dark gray), $\mu = 0.9$ (light gray).

Comments   Notice that the transformed kernel $\tilde{a}$ in (25) depends also on ζ and satisfies the analogue to the crucial condition (22) even if the kernel a is not strictly unimodal due to a compact support or constant plateaus, see Figure 7 for an illustration. Consequently, all results in this section cover for $\zeta > 0$ the kernels (6) and (7) as well, and our comments at the end of §2 on how to generalize Propositions 4 and 5 are relevant for $\zeta = 0$ only.

The transformed kernel $\tilde{a}$ also features prominently in [TV14], which studies the bilinear eigenvalue problem for the tent map kernel by different methods. In our notations, the main ideas can be described as follows. The function $\tilde{w}$ with

$$\tilde{w}(x) = \frac{\tilde{f}\big(u(x)\big)}{\tilde{\sigma}} = \frac{\eta\,\chi_\xi(x)\,\big(u(x) - \theta\big)}{\sigma - \zeta}\,,$$

represents the inhomogeneity, is continuous (especially at $x = \pm\xi$), and satisfies

$$w'(x) = \frac{\eta\,\chi_\xi(x)\,u'(x)}{\sigma - \zeta} = \frac{\eta\,v(x)}{\sigma - \zeta}$$

for all $x \in \mathbb{R}$. The linear problem for the effective derivative profile (11) can hence be written analogously to (13) as

$$v(x) - \frac{\sigma - \zeta}{\eta} \int_{-\xi}^{+\xi} \tilde{a}(x - y) v(y)\,\mathrm{d}y = 0 \qquad \text{for} \quad -\xi \le x \le +\xi \tag{29}$$

and [TV14] solves this equation under the consistency relation $\int_0^\xi v(x)\,\mathrm{d}x = \theta$ by combining two ingredients. The fixed point problem (29) is solved with parameter ξ by Wiener-Hopf methods. This requires detailed information on the Fourier transform of $\tilde{a}$ and direct computations reveal

$$\widehat{\tilde{a}}(k) = \frac{\sigma\,\widehat{a}(k)}{\sigma - \zeta\,\widehat{a}(k)} = \frac{\sigma}{\eta}\left(1 - \frac{(\zeta + \eta)\,\widehat{a}(k)\,k^2 - \sigma\,k^2}{\zeta\,\widehat{a}(k)\,k^2 - \sigma\,k^2}\right)$$

as well as $\widehat{a}(k)\,k^2 = 4\sin^2\left(k/2\right)$ for the kernel (6). Moreover, the value of ξ is determined numerically by means of a scalar nonlinear equation equivalent to (26). This approach works well since $\widehat{a}$ is a nice known function and provides power series expressions for v and u, which can be used to derive intricate but almost explicit approximation formulas.

We finally mention that nonnegative eigenfunctions (being unimodal or not) cannot exist for $\sigma \ge \zeta + \eta$ since (1) implies

$$\sigma\big\|u\big\|_2 \le \big\|a * f(u)\big\|_2 \le \big\|a\big\|_1\,\big\|f(u)\big\|_2 \le (\zeta + \eta)\big\|u\big\|_2\,,$$

where the last estimate is actually strict for any nontrivial u. Similarly, using

$$\sigma \int_{\mathbb{R}} u(x)\,\mathrm{d}x = \int_{\mathbb{R}} f\big(u(x)\big)\,\mathrm{d}x$$

we can disprove the existence of nonnegative eigenfunctions for $\sigma \leq \zeta$ under the additional assumption $u \in \mathsf{L}^1(\mathbb{R})$. Notice that the eigenfunction from Proposition 7 are integrable due to (24) and because $\tilde{f}(u_\sigma)$ is always compactly supported. More precisely, u_σ decays as fast as a or the transformed kernel $\tilde{a}$ for $\zeta = 0$ and $\zeta > 0$, respectively. A similar argument has been applied in [HM20], which proves the existence and the localization (but not the uniquness) of nonlinear eigenfunctions for more general functions f in a nonlinear variational setting.

A   Formal asymptotic analysis

We characterize the limiting behavior of the nonlinear eigenfunction u_σ from Proposition 7. We always start with the special case $\zeta = 0$ and discuss the necessary modification in the case $\zeta > 0$ afterwards. The natural quantity for the formal asymptotic analysis with $\zeta = 0$ are the compactly supported functions

$$w_\sigma(x) := \sigma^{-1} f(u_\sigma(x)), \tag{30}$$

which determine u_σ via

$$u_\sigma = a * w_\sigma. \tag{31}$$

We rescale w_σ according to

$$\overline{w}_\sigma(\overline{x}) = w(x), \qquad x = \xi_\sigma \, \overline{x},$$

because this allows us to work on the fixed interval $\overline{I} := [-1, +1]$, and notice that

$$\overline{w}_\sigma(\pm 1) = 0.$$

Moreover, the nonlinear eigenvalue equation (1) combined with $\chi_{\xi_\sigma} \cdot u_\sigma = \theta + \lambda_\sigma w_\sigma$ and $\sigma = \eta \, \lambda_\sigma$ yields

$$\theta + \lambda_\sigma \, \overline{w}_\sigma = \overline{\mathcal{A}}_{\xi_\sigma} \overline{w}_\sigma, \tag{32}$$

where the operator

$$(\overline{\mathcal{A}}_\xi \overline{w})(\overline{x}) := \int\limits_{-1}^{+1} \xi \, a(\xi \, \overline{x} - \xi \, \overline{y}) \, \overline{w}(\overline{y}) \, \mathrm{d}\overline{y} \tag{33}$$

is the rescaled counterpart of $\mathcal{A}_\xi$ from (14).

Small eigenvalues for $\zeta = 0$ Proposition 5 shows $\xi_\sigma \approx 0$ and by Taylor expansion of (33) we get
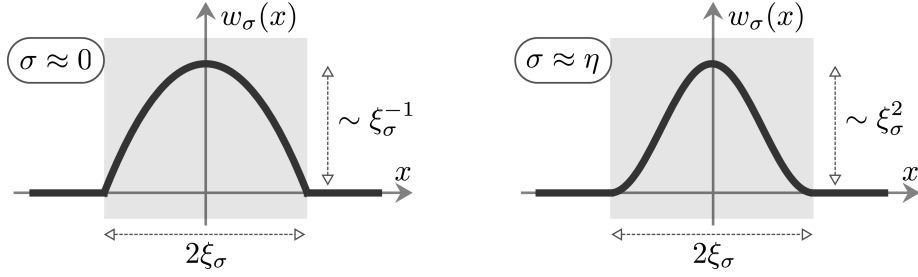
$$(\overline{\mathcal{A}}_\xi \overline{w})(\overline{x}) = \xi \cdot \left(a(0) \int\limits_{-1}^{+1} \overline{w}(\overline{y}) \, \mathrm{d}\overline{y} \right) - \xi^3 \cdot \left(\tfrac{1}{2} \, |a''(0)| \int\limits_{-1}^{+1} \overline{w}(\overline{y}) \overline{y}^2 \, \mathrm{d}\overline{y} \right)$$
$$- \overline{x}^2 \cdot \xi^3 \cdot \left(\tfrac{1}{2} \, |a''(0)| \int\limits_{-1}^{+1} \overline{w}(\overline{y}) \, \mathrm{d}\overline{y} \right) + O(\xi^5).$$

The eigenvalue equation (32) thus implies the approximate identity

$$\overline{w}_\sigma(\overline{x}) \approx c_\sigma \left(1 - \overline{x}^2 \right)$$

for some constant c_σ and any $\overline{x} \in [-1, +1]$. Computing the integrals we find

$$\theta + \lambda_\sigma \, c_\sigma \left(1 - \overline{x}^2 \right) \approx c_\sigma \, \xi_\sigma \left(\tfrac{4}{3} \, a(0) \right) - c_\sigma \, \xi_\sigma^3 \left(\tfrac{2}{15} \, |a''(0)| \right) - \overline{x}^2 \, c_\sigma \, \xi_\sigma^3 \left(\tfrac{2}{3} \, |a''(0)| \right).$$

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Figure 8: Schematic representation of the approximate formulas (34) (*left panel, $\xi_\sigma \sim \sigma^{1/3}$*) and (40) (*right panel, $\xi_\sigma \sim (\eta - \sigma)^{-1}$*), which describe the asymptotics of u_σ in the special case $\zeta = 0$. See also (30) and (31) as well as Conjectures 8 and 10.

and equating the coefficients — first in front of $\bar{x}^2$ and afterwards in front of 1 — we identify the scaling relations

$$\lambda_\sigma \approx \tfrac{2}{3} \left| a''(0) \right| \xi_\sigma^3 \,, \qquad \tfrac{4}{3} \, a(0) \, c_\sigma \, \xi_\sigma \approx \theta \,,$$

where λ_σ satisfies (26). Moreover, the function

$$w_\sigma(x) \approx \frac{3\,\theta}{4\,a(0)\,\xi_\sigma} \, \chi_{\xi_\sigma}(x) \left(1 - \xi_\sigma^{-2} x^2 \right) \tag{34}$$

approaches a Dirac distribution as $\sigma \to 0$, see also Figure 8, so (31) implies that u_σ converges to a multiple of a, where the scaling factor is consistent with $u_\sigma(\xi_\sigma) = \theta$. In summary, we expect the following behavior for small σ.

Conjecture 8 (asymptotics for $\zeta = 0$ and $\sigma \approx 0$). *Suppose that a that is sufficiently smooth at the origin with $a''(0) < 0$. Then, we have*

$$\sigma^{-1/3}\xi_\sigma \xrightarrow{\ \sigma \searrow 0\ } \frac{3^{1/3}}{2^{1/3}\,\eta^{1/3}\,|a''(0)|^{1/3}} \tag{35}$$

as well as

$$\sigma^{1/3}\overline{w}_\sigma(\overline{x}) \xrightarrow{\ \sigma \searrow 0\ } \frac{3^{2/3}\,\eta^{1/3}\,\theta\,|a''(0)|^{1/3}}{2^{5/3}\,a(0)} \, \overline{\chi}(\overline{x}) \left(1 - \overline{x}^2 \right) , \tag{36}$$

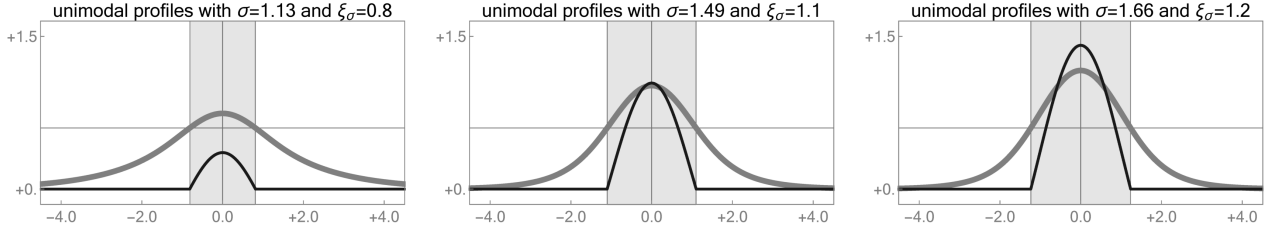*where $\overline{\chi}$ is the indicator function of $\overline{I}$. Moreover,*

$$\lim_{\sigma \to 0} u_\sigma(x) = \frac{\theta\,a(x)}{a(0)} \,, \qquad \lim_{\sigma \to 0} f\big(u_\sigma(x)\big) = 0 \tag{37}$$

holds at least in the sense of pointwise convergence.

We believe that the assertions of Conjecture 8 can be derived by standard arguments but notice that (35) and (36) do not cover the kernels (6) and (7) since these are not smooth and satisfy $a''(0) = -\infty$ and $a''(0) = 0$, respectively. Nonetheless, a modified asymptotic analysis should reveal that (37) is still satisfied. For the tent map kernel, a similar convergence result has been derived in [TV14] for $\zeta > 0$ and $\eta \to \infty$. This limit has much in common with the anticontinuum or high-energy limit studied in [HM19b].

Small eigenvalues for $\zeta > 0$ In this case, we have to replace the kernel a by the transformed kernel from Proposition 6, which we now denote by $\tilde{a}_\sigma$ as it depends on σ, and σ by $\tilde{\sigma} = \sigma - \zeta$. The problem is that $\tilde{a}_\sigma$ does not converge strongly as $\sigma \searrow \zeta$ but its amplitude gets smaller while its effective width approaches ∞, see Figure 7 for an illustration. The precise analysis depends on the singular scaling behavior of $\tilde{a}_\sigma$. For sufficiently nice kernels we can suppose that the limits

$$\kappa_0 := \lim_{\sigma \searrow \zeta} (\sigma - \zeta)^{-1/2}\,\tilde{a}_\sigma(0) \,, \qquad \kappa_2 := \lim_{\sigma \searrow \zeta} (\sigma - \zeta)^{-1}\,\big|\tilde{a}_\sigma''(0)\big| \tag{38}$$

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Figure 9: Numerical simulation with $\zeta = 1.0$, $\theta = 0.6$, $\eta = 2.5$ for the Gaussian kernel (5). The computations are performed with the nonlinear improvement dynamics described in [HM20] and the profiles u_σ (gray) and $\tilde{f}(u_\sigma)$ (black) are shown for several values of σ. Conjecture 9 predicts that u_σ converges as $\sigma \searrow \zeta$ to the constant function with value θ, but this limit is hard to capture numerically.

are well defined, and direct computations for the Gaussian kernel (5) provide the values $\kappa_0 = 1$ as well as $\kappa_2 = 2\,\zeta(3/2)/\sqrt{\pi}$ in terms of the zeta function.

In consistency with Proposition 6 we base our analysis of

$$\overline{\tilde{w}}_\sigma(\overline{x}) = \tilde{w}_\sigma(\xi_\sigma \overline{x})\,, \qquad \tilde{w}_\sigma(x) = \frac{\tilde{f}(u_\sigma(x))}{\tilde{\sigma}}\,,$$

which solves an equation similar to (32). The scaling relations encoded by (38) still allow us to Taylor expand $\tilde{a}_\sigma$ even though the corresponding cut-off parameter ξ_σ can no longer assumed to be small. In fact, repeating the arguments from above we find

$$\overline{\tilde{w}}(\overline{x}) = \tilde{c}_\sigma \left(1 - \overline{x}^2\right)$$

as well as

$$\theta + \eta^{-1}(\sigma - \zeta)\,\tilde{c}_\sigma \left(1 - \overline{x}^2\right) \approx \tfrac{4}{3}\,\kappa_0\,\tilde{c}_\sigma\,\xi_\sigma\,(\sigma - \zeta)^{1/2} - \tfrac{2}{15}\,\kappa_2\,\tilde{c}_\sigma\,\xi_\sigma^3\,(\sigma - \zeta) - \overline{x}^2\,\tfrac{2}{3}\,\kappa_2\,\tilde{c}_\sigma\,\xi_\sigma^3\,(\sigma - \zeta)\,.$$

Equating coefficients gives

$$\tfrac{2}{3}\,\kappa_2\,\xi_\sigma^3 \approx \eta^{-1}\,, \qquad \tfrac{4}{3}\,\kappa_0\,\tilde{c}_\sigma\,\xi_\sigma\,(\sigma - \zeta)^{1/2} \approx \theta$$

as well as

$$(\sigma - \zeta)^{1/2}\,\tilde{w}_\sigma \approx \frac{3\,\theta}{4\,\kappa_0\,\xi_\sigma}\,\chi_\xi(x)\left(1 - \xi_\sigma^{-2}\,x^2\right)\,,$$

and in summary we find a limiting behavior that differs considerably from Conjecture 8. In particular, $(\sigma - \zeta)^{1/2}\tilde{w}_\sigma$ converges to a piecewise smooth limit function with integral $\kappa_0^{-1}\theta$, whose convolution with the small and slowly varying function $\tilde{a}_\sigma$ approximates u_σ up to a small prefactor. See also Figure 9.

Conjecture 9 (asymptotics for $\zeta > 0$ and $\sigma \approx \zeta$). *Suppose that the kernel a is sufficiently regular so that (38) is satisfied. Then, we have*

$$\xi_\sigma \xrightarrow{\ \sigma \searrow 0\ } \frac{3^{1/3}}{2^{1/3}\,\eta^{1/3}\,\kappa_2^{1/3}}\,, \qquad (\sigma - \zeta)^{1/2}\overline{w}_\sigma(\overline{x}) \xrightarrow{\ \sigma \searrow 0\ } \frac{3^{2/3}\,\eta^{1/3}\,\theta\,\kappa_2^{1/3}}{2^{5/3}\,\kappa_0}\,\overline{\chi}(\overline{x})\left(1 - \overline{x}^2\right)$$

as well as

$$\lim_{\sigma \to 0} u_\sigma(x) = \theta\,, \qquad \lim_{\sigma \to 0} \tilde{f}\big(u_\sigma(x)\big) = 0$$

in the sense of pointwise convergence.

The different limit behaviors for $\zeta = 0$ and $\zeta > 0$ can also be understood heuristically as follows. For $\zeta = \sigma = 0$, there exists a plethora of nonlinear eigenfunction in $\mathcal{U}$, namely any function $u \in \mathcal{U}$ which satisfies $f(u) \equiv 0$ due to $\|u\|_\infty = u(0) \leq \theta$. In the case of $\zeta > 0$, however, $\sigma = \zeta$ combined with $0 \leq u \leq \theta$ reduces (1) to $u = a * u$, but the only fixed points of the convolution operator are the constant functions.

Large eigenvalues We start again with $\zeta = 0$ but it turns out that our formal asymptotic results cover the case $\zeta > 0$ as well. Since ξ_σ is now large, we restate (33) as

$$\left(\overline{\mathcal{A}_\xi w}\right)(\overline{x}) = \int\limits_{\xi(\overline{x}-1)}^{\xi(\overline{x}+1)} a(y)\,\overline{w}\left(\overline{x} + \xi^{-1}y\right) \mathrm{d}y$$

and employ the formal asymptotic expansion

$$\left(\overline{\mathcal{A}_\xi w}\right)(\overline{x}) \approx \overline{w}(\overline{x}) - m\,\xi^{-2}\,\overline{w}''(x) \int\limits_{\xi(\overline{x}-1)}^{\xi(\overline{x}+1)} a(y)\,\overline{w}\left(\overline{x} + \xi^{-1}y\right) \mathrm{d}y\,.$$

This formula involves

$$m := -\tfrac{1}{2} \int\limits_{-\infty}^{\infty} y^2 a(y)\,\mathrm{d}y$$

and holds for any fixed $-1 < \overline{x} < +1$ thanks to Assumption 1. In combination with (32) we thus obtain the approximate ODE

$$\theta + \lambda_\sigma\,\overline{w}_\sigma \approx \overline{w}_\sigma - \xi_\sigma^{-2}\,m\,\overline{w}''\,,$$

which admits the consistent solution

$$\lambda_\sigma \approx 1 - \xi_\sigma^{-2}\,\pi^2\,m\,, \qquad \overline{w}_\sigma(\overline{x}) \approx d_\sigma\left(1 + \cos\left(\pi\overline{x}\right)\right)\,, \qquad d_\sigma = \xi_\sigma^2 \frac{\theta}{\pi^2\,m}\,. \tag{39}$$

These relation imply

$$w_\sigma(x) \approx \frac{\theta\,\xi_\sigma^2}{\pi^2\,m}\,\chi_{\xi_\sigma}(x)\left(1 + \cos\left(\pi\,\xi_\sigma^{-1}\,x\right)\right) \tag{40}$$

and give rise to the following claim.

Conjecture 10 (asymptotics for $\zeta = 0$ and $\sigma \approx \eta$). *For any kernel a that decays sufficiently fast at infinity, we have*

$$(\eta - \sigma)^{1/2}\xi_\sigma \xrightarrow{\;\sigma \nearrow \eta\;} \pi\,m^{1/2}\,\eta^{1/2}\,, \qquad (\eta - \sigma)\overline{w}_\sigma(\overline{x}) \xrightarrow{\;\sigma \nearrow \eta\;} \eta\,\theta\,\overline{\chi}(\overline{x})\left(1 - \cos\left(\pi\overline{x}\right)\right) \tag{41}$$

and hence $\|u_\sigma\|_p \sim (\eta - \sigma)^{-1/2 - 1/(2p)}$ for any $p \in [1, \infty]$.

The limit $\sigma \nearrow \zeta + \eta$ in case of $\zeta > 0$ involves again the transformed kernel from Proposition 25, which satisfy $\tilde{a}_\sigma \to \tilde{a}_\eta$. We therefore expect that (41) remains true provided that $\sigma - \eta$ is replaced by $\tilde{\sigma} - \eta = \sigma - \zeta - \eta$ and m is computed with $\tilde{a}_\eta$ instead of a.

We further emphasize that the scaling relations (41) are consistent with our numerical simulations in Figures 2 and 3. The analytical justification of Conjecture 10 and the underlying approximation (39) is much harder than the rigorous derivation of (35) and (36) because the Taylor expansion is now applied to $\overline{w}_\sigma$ and requires uniform estimates of $\|\overline{w}_\sigma''\|_p$ for at least one $p \in [1, \infty]$. A similar problem concerns the linear eigenvalue problem from §2. In order to ensure the consistent approximation

$$v_\xi(x) \approx -(\xi)^{-1/2}\,\chi_\xi(x)\,\sin\left(\xi^{-1}\pi x\right)$$

for the normalized eigenfunction from Proposition 4, we have to guarantee that $\|v_\xi''\|_p \le C_p\,\xi^{\frac{1}{p} - \frac{5}{2}}$ holds for all large ξ and some constant C_p independent of ξ. We are, however, not ware of a corresponding reference. Moreover, the limit $\xi \to \infty$ is rather intricate due to a huge number of nearby eigenvalues that discretize the continuous spectrum of the limit operator $\mathcal{A}_\infty w = a * w$.

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