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Abstract

Multilateral comparison of outcomes drawn from multiple groups pervade the social sciences and measurement of their variability, usually involving functions of respective group location and scale parameters, is of intrinsic interest. However, such approaches frequently mask more fundamental differences that more comprehensive examination of relative group distributional structures reveal. Indeed, in categorical data contexts, location and scale based techniques are no longer feasible without artificial and questionable cardinalization of categories. Here, Ginis' Transvariation measure is extended and employed in providing quantitative and visual multilateral comparison tools in discrete, continuous, categorical, univariate or multivariate settings which are particularly useful in paradigms where cardinal measure is absent. Two applications, one analyzing Eurozone cohesion in terms of the convergence or divergence of constituent nations income distributions, the other, drawn from a study of aging, health and income inequality in China, exemplify their use in a continuous and categorical data environment.

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On Unit Free Assessment of The Extent of Multilateral Distributional Variation

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Summary Multilateral comparison of outcomes drawn from multiple groups pervade the social sciences and measurement of their variability, usually involving functions of respective group location and scale parameters, is of intrinsic interest. However, such approaches frequently mask more fundamental differences that more comprehensive examination of relative group distributional structures reveal. Indeed, in categorical data contexts, location and scale based techniques are no longer feasible without artificial and questionable cardinalization of categories. Here, Ginis' Transvariation measure is extended and employed in providing quantitative and visual multilateral comparison tools in discrete, continuous, categorical, univariate or multivariate settings which are particularly useful in paradigms where cardinal measure is absent. Two applications, one analyzing Eurozone cohesion in terms of the convergence or divergence of constituent nations income distributions, the other, drawn from a study of aging, health and income inequality in China, exemplify their use in a continuous and categorical data environment.

1. INTRODUCTION

Following early concerns about the measurement of aggregate differences (Dalton 1920, Gini 1921) and the path breaking work of Fisher (1932, 1935), multilateral comparisons of grouped outcomes have become ubiquitous in the empirical sciences[1] rendering unit free measurement of their collective variation of intrinsic interest. Unit free measures are

[1]Outcomes of two or more distinct groups are compared and contrasted in Equality of Opportunity, Mobility and wellbeing literatures (e.g. Blackorby and Donaldson 1978: Arrow, Bowles and Durlauf 2000; Herrnstein and Murray 1994; Peragine, Palmisano and Brunori 2014; Roemer 1998; Weymark 2003). The financial returns of a collection of portfolios are compared on a combined mean-variance basis (Markowitz 1952; Bali, Brown and Demirtas 2013; Banz 1981; Basu 1983; Jegadeesh 1990). Within and between firm and industry wage inequalities have been explored in the Industrial Organization and Labour literatures (Abowd et al. 2018, Card et. al. 2018, Song et. al. 2019). In treatment effect, event and matching study and policy evaluation literatures (Angrist and Krueger 2001) assessment is based upon comparisons of conditional means across outcome states. Recent developments in the measurement and analysis of

preferred because they are comparable across different entities. Generally, studies of relative variation within and between groups employ standard summary statistics of location (means and medians) and dispersion (variances and ranges) in various combinations in three basic approaches: range measures, average distance from a central value measures and average distance between all possible pairs measures. Range (largest less the smallest number) or interquartile range measures divided by a location parameter are examples of the first unit free approach, the coefficient of variation or Theil's entropic measures (Theil 1967, Maasoumi 1986, Maasoumi, Racine and Stengos 2007) exemplify average difference from the average approaches, and the Gini Coefficient (Gini 1921, Yitzhaki 1983, Chakravarty 1988) is an example of the third approach. Each has its pros and cons. Range measures are easily computed and capture the potential span of differences but fail to reflect the extent of bilateral differences between groups within the interior of the collection – they are not subgroup decomposable so that a subgroups impact on overall variability cannot always be established. The second group, in accounting for the difference from the average of each element reflects the totality of differences much better and, like the ANOVA technique, they are usually subgroup decomposable. However, Sen (1995) and Yitzhaki (2003) argue that measures of average absolute differences such as the Gini, capture more of the totality of differences than difference from mean-based measures. Unfortunately, when analyzing subgroup impacts, Gini-type measures are not subgroup decomposable (Bourguignon 1979) except in exceptional circumstances (Mookherjee and Shorrocks 1982).

A common problem with these approaches, highlighted in the contexts of treatment effects and growth and convergence models (Carneiro et. al. 2002, 2003, Durlauf and Quah 2002), is that, in confining analyses to subsets of conditional moments, important information concerning differences in moments beyond those subsets is ignored and can thus be misleading. Somewhat trivially, in a collection of distributions with identical means, difference in means tests have zero power against more general distributional differences such as differences in variances. In essence, employing just means and variances creates a "veil of ignorance" that is only countervailed by comparing subgroup distributions in their entirety across their complete range. Moreover, such analyses are not feasible in ordinal environments encountered for example in subjective wellbeing measurement literatures without arbitrary assignment of cardinal scales to ordinal categories. Unfortunately, arbitrary scale assignment is not a solution because of the scale dependency problem (Schroder and Yitzhaki 2017, Liddell and Kruschke 2018, Bond and Lang 2019) and, since objects like the range, coefficient of variation and Gini coefficients are monotone scale dependent, this issue carries over to inequality measurement.

Here, in answer to these concerns, measures of distributional differentness or inequality are introduced. These measures compare a collection of distributions across their complete range of variation. Specifically, Gini's Bilateral Distributional Transvariation (Gini 1916, 1959) is extended to multilateral environments in generating three new general measures, together with their respective asymptotically normal standard errors, which are distributional analogues of the aforementioned three basic measures of variation in collections of numbers. The measures, which focus on relative distributional differences in collections of discrete, continuous, categorical and potentially multivariate distributions, are in the respective forms of a Multilateral Transvariation (MGT) statistic, a distribu-

subjective wellbeing (Kahneman and Krueger 2006) involve multilateral comparisons of groups in the context of ordinal categorical data.

tional coefficient of variation (DCV) and a Distributional Gini (DisGini) coefficient. All come in population weighted and unweighted forms.

MGT is a generalization to K distributions of Gini's Bilateral Transvariation, originally introduced in Anderson, Linton and Thomas (2017), here its sampling distribution is provided. Like its Range counterpart, it is simple to compute and provides a measure of the extremes of variation of the collection of distributions but gives no sense of the extent of bilateral distributional differences or aggregate differences from the "average" distribution.

DCV, like its Coefficient of Variation counterpart, measures aggregated differences of distributions from an average distribution and is particularly useful for studying convergence and divergence issues in collections of distributions. Its development prompted a new concept of universal convergence/divergence whereby all groups in the collection are converging/diverging in concert which can be usefully visualized in a radar chart and tests for which are provided.

DisGini measures the totality of bilateral similarities or differences in a collection of distributions and is the most comprehensive measure of distributional differences. Conceptually, it is based upon an extension of the between group means component of the threefold sub-group decomposition of the Gini (Bourgignon 1979). While the Gini coefficient between-group component captures between group inequalities in terms of differences between sub-group means, DisGini captures between-group dissimilarities in terms of the totality of sub-group distributional differences.

In the following, Section 2 introduces the three new instruments for assessing multilateral distributional differences. Estimators for these measures along with their distributional properties are derived in Section 3. Section 4 reports the main results of two exemplifying applications. The first, a study of the progress of the Eurozone income distribution, addresses the question of increasing commonality in the income distributions of the Eurozone's constituent nations. The second, in exemplifying the efficacy of the techniques in ordered categorical data contexts, examines the progress of health-income inequalities over the aging process. Some conclusions are drawn in Section 5.

2. MULTILATERAL TRANSVARIATION

2.1. MGT: Generalizing Gini's Transvariation measure

In his original bilateral transvariation measure GT, Gini (1916, 1959) provided a measure of the difference between two distributions[2] which, for two distributions $f_i(x), f_j(x)$ whose support[3] is confined to $\mathbb{R}^+$, can be defined, following Anderson, Linton and Thomas (2017), as follows:

$$\begin{aligned}\mathrm{GT}_{ij} &= \frac{1}{2}\int_0^\infty |f_i(x) - f_j(x)|\, dx = \\ &= \frac{1}{2}\int_0^\infty [\max(f_i(x), f_j(x)) - \min(f_i(x), f_j(x))]\, dx. \end{aligned} \tag{2.1}$$

[2] See Pittau and Zelli (2017) for an overview of Gini's original concepts of transvariation.

[3] Since translation to discrete and categorical paradigms is straightforward, discussion is confined to the continuous paradigm for brevity purposes.

Since $0 \leq \int_0^\infty |f_i(x) - f_j(x)| dx \leq 2$, pre-multiplying by 0.5 yields a statistic that will be 0 when the two distributions are identical and 1 when they have mutually exclusive support.[4] Note that, by definition, $\mathrm{GT}_{ij} = \mathrm{GT}_{ji}$, furthermore it has a one to one relationship with distributional overlap OV_{ij} measuring the extent of commonality between the two distributions (Anderson, Linton and Whang 2012), which is given by:

$$\mathrm{OV}_{ij} = \int_0^\infty \min(f_i(x), f_j(x))\, dx. \tag{2.2}$$

Essentially GT = 1-OV.

Generalizing equation (2.1) to K distributions indexed $k = 1, \ldots, K$, suggests contemplating a Multilateral Gini Transvariation measure (MGT), defined as follows:

$$\mathrm{MGT} = \frac{1}{K}\int_0^\infty [\max(f_1(x), f_2(x), \ldots, f_K(x)) - \min(f_1(x), f_2(x), \ldots, f_K(x))]\, dx. \tag{2.3}$$

As in the bilateral comparison, when the distributions have mutually exclusive support MGT = 1, when the distributions are identical MGT = 0.

A weighted version of MGT, MGT-W is also possible, and has the form

$$\text{MGT-W} = \int_0^\infty [\max(w_1 f_1(x), w_2 f_2(x), ., w_K f_K(x)) - \min(w_1 f_1(x), w_2 f_2(x), ., w_K f_K(x))]\, dx, \tag{2.4}$$

where w_k are possible weights associated to the distributions f_k, $k = 1, \ldots, K$. When the K distributions are regarded as subgroups of an overall distribution, w_k are the proportions associated with each density function.

One problem with the multilateral transvariation measure is its maximum-minimum nature. Like the range statistic for a collection of numbers which does not reflect differences in objects in the mid range, the MGT does not reflect the many bi-lateral functional differences and similarities camouflaged by just considering extreme density values. Indeed, it is in essence the distributional analogue of the relative range measure of a collection of numbers wherein the relative locations of interior and low weight members have little or no impact on its value. An alternative which is in effect an aggregation of all distributional differences from the average distribution or what will be referred to as the Distributional Coefficient of Variation (DCV) is introduced in the next section.

2.2. DCV: A Distributional Coefficient of Variation

The collection of K subgroups indexed $k = 1, \ldots, K$ with respective distributions $f_k(x)$ may be considered in the context of individual distributions being components within a mixture $f(x)$ representing the overall population distribution:

$$f(x) = \sum_{k=1}^{K} w_k f_k(x), \sum_{k=1}^{K} w_k = 1 \text{ and } w_k \geq 0 \text{ for all } k \tag{2.5}$$

[4] The Gini bilateral transvariation can be regarded as the Total Variation distance between two probability measures (it is the normalized ℓ_1-distance between them).

where w_k are weights reflecting the importance of the component within the population. So, for example, $f(x)$ may refer to a societal income distribution with $f_k(x)$ being the income distribution of the *k-th* constituency and w_k its relative population size. Alternatively, from a representative agent or treatment effect perspective, the distributions describing outcomes of particular groups could be compared directly, without reference to their relative importance in the collection, in which case w_k would be set to $1/K$ for all k. Indeed, from a policy application perspective, there is no reason why $f(x)$ should not be defined for policy purposes as some "target" distribution that constituencies should aspire to so that DCV provides a measure of the extent to which the policy has not been achieved.

OV_{ko}, the distributional overlap between the k'th subgroup distribution and the overall mixture is such that:

$$\mathrm{OV}_{ko} = \int_0^\infty \min\left(f_k(x), f(x)\right) dx \tag{2.6}$$

The corresponding subgroup/overall transvariation is related to the overlap measure as follows: $\mathrm{GT}_{ko} = 1 - \mathrm{OV}_{ko}$. Then DCV, the weighted average of subgroup-overall distribution transvariations, may then be written as:

$$\mathrm{DCV} = \frac{1}{(1 - \sum_{k=1}^K w_k^2)} \sum_{k=1}^K w_k GT_{ko} = \frac{1}{(1 - \sum_{k=1}^K w_k^2)} \sum_{k=1}^K w_k (1 - \mathrm{OV}_{ko}). \tag{2.7}$$

Note that when subgroup distributions are identical they will be identical to their weighted sum so that $\mathrm{GT}_{ko} = 0$ for all k and DCV=0. When the subgroups have mutually exclusive support $\mathrm{GT}_{ko} = 1 - w_k$ so that DCV=1.

As with the Sen (1995) and Yitzhaki (2003) critiques of mean deviation measures, DCV still does not reflect the full panoply of distributional differences between groups. However a "Distributional" Gini Coefficient will.

2.3. DisGini: The "Distributional" Gini Coefficient

To fully explore distributional differences consider instead:

$$\mathrm{DisGini} = \frac{1}{\varphi} \sum_{i=1}^K \sum_{j=1}^K 0.5 \int_0^\infty w_i w_j |f_i(x) - f_j(x)| dx = \frac{1}{\varphi} \sum_{i=1}^K \sum_{j=1}^K w_i w_j \mathrm{GT}_{ij} \tag{2.8}$$

Where φ is a scaling parameter. Note the term $\int_0^\infty w_i w_j |f_i(x) - f_j(x)| dx$ may be written as "$w_i w_j 2\mathrm{GT}_{i,j}$" which is twice Gini's Transvariation of sub distributions $f_i(x)$ and $f_j(x)$, multiplied by the product of the respective population shares. Given the relationship (2.2) between GT and the overlap measure OV, (2.8) may be written as:

$$\mathrm{DisGini} = \frac{1}{\varphi} \sum_{i=1}^K \sum_{j=1}^K w_i w_j (1 - \mathrm{OV}_{ij})$$

Which, letting c be a K element column vector of ones, may be written in matrix form:

$$\frac{1}{\varphi}c'\begin{bmatrix} 0 & w_1w_2(1-OV_{12}) & \dots & w_1w_K(1-OV_{1K}) \\ w_2w_1(1-OV_{21}) & 0 & \dots & w_2w_K(1-OV_{2K}) \\ \vdots & \vdots & \ddots & \vdots \\ w_Kw_1(1-OV_{K1}) & w_Kw_2(1-OV_{K2}) & \dots & 0 \end{bmatrix}c \tag{2.9}$$

Consider a typical element $w_iw_j(1-\mathrm{OV}_{ij})$, when $i=j$ the element will be zero, also when $f_i(x)=f_j(x)$ for all x (i.e. subgroups i and j have identical distributions), the term will be 0. It follows that when all subgroups have identical distributions, expression (2.9) will be 0 since all of the elements are non-negative this will constitute a lower bound for DisGini.

Now consider the situation where all of the respective subgroup income distributions have mutually exclusive support, i.e. the subgroups are completely segmented so that for all $i \neq j$ and a given x, $f_i(x) \geq 0 \Rightarrow f_j(x)=0$ and $f_j(x) \geq 0 \Rightarrow f_i(x)=0$. This corresponds to the mixture distribution situation where there is no distributional overlap between any constituency pairing, thus Gini's Transvariation would be at a maximum value of 1.

In this case (2.8) may be written:

$$\frac{1}{\varphi}c'\begin{bmatrix} 0 & w_1w_2 & \dots & w_1w_K \\ w_2w_1 & 0 & \dots & w_2w_K \\ \vdots & \vdots & \ddots & \vdots \\ w_Kw_1 & w_Kw_2 & \dots & 0 \end{bmatrix}c = \frac{1}{\varphi}\sum_{k=1}^{K} w_k(1-w_k) = \frac{1-\sum_{k=1}^{K} w_k^2}{\varphi}$$

If the scaling parameter φ is set to $(1-\sum_{k=1}^{K} w_k^2)$ then DisGini will always fall in the interval [0,1] and be equal to 1 when there is complete distributional inequality in terms of complete segmentation of the constituency distributions. It follows that DisGini may finally be written as:

$$\mathrm{DisGini} = \frac{1}{(1-\sum_{k=1}^{K} w_k^2)}\sum_{i=1}^{K}\sum_{j=1}^{K} w_iw_j(1-\mathrm{OV}_{ij}) = \frac{1}{(1-\sum_{k=1}^{K} w_k^2)}\sum_{i=1}^{K}\sum_{j=1}^{K} w_iw_j\mathrm{GT}_{ij}. \tag{2.10}$$

If comparison of the distributions without subgroup weighting is desired, as in the aforementioned representative agent type scenarios, simply set $w_i = \frac{1}{K}$ for all $i=1,\dots,K$.

By noting that the Transvariation and Overlap of two multivariate distributions is given by:

$$\int \sum |f(x,y)-g(x,y)|dx \text{ and } \int \sum \min(f(x,y)-g(x,y))dx$$

respectively, where integration is over all continuous variables x and summation is over all discrete variables y, the foregoing formulae are readily extended to multivariate situations. Furthermore, by replacing $f_i(x)$ with $F_i^h(x)$ where $F_i^h(x) = \int_0^x F_i^{(h-1)}(z)dz$ in (2.3), (2.7) or (2.8) and adjusting the normalizing parameter accordingly, multilateral variation of higher order integrals of distribution functions could be contemplated reflecting the classic stochastic dominance criteria for more restrictive wellbeing structures (see Anderson, Post and Whang, 2020). All of which are matters for future research.

These indices provide a complete ordering of collections of distributions with respect to their differentness, as such they can be shown to satisfy some popular axioms in the inequality literature (Sen, 1995). When applied to the groups as subjects Anonymity, Scale and Translation Invariance, Normalization and Replication Invariance axioms are all satisfied by these indices. When sub-distributions are posited to be the atomistic equivalents of the sub-distributions employed in Duclos, Esteban and Ray (2004) and subjected to the same transformations, they comply with the polarization axioms posed therein.

It should also be noted that although the Gini coefficient has problems with negative values (see Manero 2017), the discussion was confined to distributions defined on the positive orthant. However, the measures proposed here are not subject to this difficulty and are well defined on all support types.

Finally, it is of interest to understand how the DisGini coefficient is affected by the expansion of the number of groups under consideration, Appendix A demonstrates that DisGini will increase or diminish as it is exceeded by or exceeds the weighted sum of the new group's transvariations with respect to the existing groups in the analysis.

3. ESTIMATION AND DISTRIBUTION THEORY

3.1. Estimation and standard errors of MGT

Non-parametric estimation of MGT facilitates analysis of the collection of distributions over their full range revealing the extent of their similarity and differentness without reliance on the limited purview of summary statistics or visual perceptions.

In the case of discrete and categorical variables, estimation of category membership probabilities and their sampling distributions is straightforward following Rao (1973). Suppose there are C categories Γ_c indexed $c = 1, \ldots, C$ with a C vector of category membership probabilities p with typical element p_c and let x be an T vector of independent observations with typical element x_i so that $p_c = \Pr(x_i \in \Gamma_c)$. Then, $\hat{p}_c$, the estimate of p_c, may be obtained by letting $z_{i,c} = 1$ when $x_i \in \Gamma_c$ and 0 otherwise, so $\widehat{p}_{c,k} = \frac{1}{T}\sum_{i=1}^{T} z_{i,c}$. In this case the vector $\hat{p}$ is asymptotically normal with large sample variance equal to $1/T$ times $V = \operatorname{diag}(p_1, \ldots, p_C) - pp^{\intercal}$. We then let

$$\widehat{\theta}_{KT} = \frac{1}{K}\sum_{c=1}^{C}\left(\max\left\{\hat{p}_{c,1}, \hat{p}_{c,2}, \ldots, \hat{p}_{c,K}\right\} - \min\left\{\hat{p}_{c,1}, \hat{p}_{c,2}, \ldots, \hat{p}_{c,K}\right\}\right).$$

We present now the estimator for the case where X_k are continuously distributed with Lebesgue density f_k, $k = 1, \ldots, K$, with common support $\mathbb{R}$. Suppose that we observe independent random samples from the k^{th} population X_{kt}, $t = 1, \ldots, T_k$. We define the kernel estimates:

$$\widehat{f}_k(x) = \frac{1}{T_k}\sum_{h=1}^{T_k}\mathbb{K}_b(x - X_{k,h}), \quad k = 1, \ldots, K, \tag{3.11}$$

where $\mathbb{K}$ is a (potentially d dimensioned multivariate) kernel with $\mathbb{K}_b(.) = \mathbb{K}(./b)/b^d$, where b is a positive bandwidth sequence. We then estimate the unweighted multilateral transvariation index θ_K =MGT defined above,

$$\widehat{\theta}_{KT} = \frac{1}{K}\left(\int \max\left\{\widehat{f}_1(x), \widehat{f}_2(x), \ldots, \widehat{f_K}(x)\right\}dx - \int \min\left(\widehat{f}_1(x), \widehat{f}_2(x), \ldots, \widehat{f_K}(x)\right)dx\right).$$

The integral is computed by numerical quadrature routines.

The theory for $\widehat{\theta}_{KT}$ follows closely the analysis in Anderson, Linton and Whang (2012). We present the theory for the case where the contact sets

$$C_{i,j} = \left\{x \in \mathbb{R}^d : f_i(x) = f_j(x) > 0\right\},$$

all have Lebesgue measure zero. In practice, this is perhaps the most useful case. The main case where the contact set is not of measure zero is the case where the densities are equal, which is a hypothesis of interest; however, in that case there are other tests available. For simplicity of presentation we suppose that $T_k = T$ for $k = 1, \ldots, K$.

Let $\lambda = (\lambda_1, \ldots, \lambda_d)^\top$ denote a vector of nonnegative integer constants. For such vector, we define $|\lambda| = \sum_{i=1}^d \lambda_i$ and, for any function $h(x) : \mathbb{R}^d \to \mathbb{R}$, $D^\lambda h(x) = \partial^{|\lambda|}/(\partial x_1^{\lambda_1} \cdots \partial x_d^{\lambda_d})(h(x))$, where $x = (x_1, \ldots, x_d)^\top$ and $x^\lambda = \prod_{j=1}^d x_j^{\lambda_j}$.

Assumptions

(A1) *K is a sth order kernel function having support in the closed ball of radius* $1/2$ *centered at zero, symmetric around zero, integrates to 1, and s-times continuously differentiable on the interior of its support, where s is an integer that satisfies $s > d$.*

(A2) *The densities f_k, $k = 1, \ldots, K$ are strictly positive*, *bounded and absolutely continuous with respect to Lebesgue measure and s-times continuously differentiable with uniformly bounded derivatives. (ii) For all $\lambda \in \mathbb{N}^d$ with $0 \leq |\lambda| \leq s$, $\int |D^\lambda f_k(x)| dx < \infty$, $k = 1, \ldots, K$.*

(A3) *The bandwidth satisfies: (i) $nb^{2s} \to 0$, (ii) $nb^{2d} \to \infty$* and *(iii) $nb^d / (\log n) \to \infty$.*

(A4) $\{X_{ki} : i \geq 1, k = 1, \ldots, K\}$ *are i.i.d. with support $\mathbb{R}^d \times \cdots \times \mathbb{R}^d$.*

Define the sets $CK_{i,*}$ and $CK^{i,*}$:

$$\begin{aligned} CK_{i,*} &= \{x : f_i(x) < f_j(x), \text{ for all } j = 1, ..., K, j \neq i\} \\ CK^{i,*} &= \{x : f_i(x) > f_j(x), \text{ for all } j = 1, ..., K, j \neq i\}. \end{aligned}$$

Let $p_{kU} = \Pr\left(X_k \in CK^{i,*}\right)$ and $p_{kL} = \Pr\left(X_k \in CK_{i,*}\right)$, and note that $CK_{i,*} \cap CK^{i,*} = \emptyset$ so that $p_{kUL} = \Pr\left(X_k \in CK^{i,*} \cap CK_{i,*}\right) = 0$, and define the positive scalar

$$v_{KT} = \frac{1}{K^2} \sum_{k=1}^K \{p_{kU}(1 - p_{kU}) + p_{kL}(1 - p_{kL}) + 2 p_{kU} p_{kL}\}.$$

Theorem 1. *Suppose that Assumptions A1-A4 hold. Then, we have:*

$$\sqrt{T}\left(\widehat{\theta}_{KT} - \theta_{KT}\right) \Longrightarrow N(0, v_{KT}).$$

The limiting variance can be consistently estimated by

$$\widehat{v}_{KT} = \frac{1}{K^2} \frac{1}{T} \sum_{k=1}^K \{\widehat{p_{kU}}(1 - \widehat{p_{kU}}) + \widehat{p_{kL}}(1 - \widehat{p_{kL}}) + 2\widehat{p_{kU}}\widehat{p_{kL}}\}.$$

We may construct a $1 - \alpha$ asymptotic coverage confidence interval for θ_{KT} as $\widehat{\theta}_{KT} \pm z_{\alpha/2}\sqrt{\widehat{v}_{KT}/T}$.

The distributional properties of MGTW can be derived as above by working with $w_k \widehat{f}_k(x)$ in place of $\widehat{f}_k(x)$ and modifying $g_{KT}(K)$ accordingly as in (2.4).

3.2. Estimation and standard error of DCV

As the weighted average of K subgroup distribution - overall distribution transvariations, the Distributional Coefficient of Variation (DCV) can be estimated as:

$$\widehat{\theta}_{DCV} = \frac{1}{(1 - \sum_{k=1}^{K} w_k^2)} \sum_{k=1}^{K} w_k \left\{ 1 - \int_a^b \min\left(\widehat{f}_k(x), \widehat{f}(x) \right) dx \right\}, \tag{3.12}$$

where $\widehat{f}_k(x)$ are kernel estimates of $f_k(x)$, $k = 1, \ldots, K$, w_k's are known weights, and $\widehat{f}(x)$ is the kernel estimate of the overall distribution $f(x) = \sum_{k=1}^{K} w_k f_k(x)$, $\sum_{k=1}^{K} w_k = 1$ and $w_k \geq 0$ for all k (or it could be some pre-specified overall target distribution).

Maintaining the assumptions of the previous MGT analysis, in this case define the sets $CK_{k,O}$ and $CK^{k,O}$:

$$CK_{k,O} = \{x : f_k(x) < f(x)\}, \; k = 1, \ldots, K$$
$$CK^{k,O} = \{x : f_k(x) > f(x)\}, \; k = 1, \ldots, K.$$

Let $p_{kU} = \Pr\left(X_k \in CK^{k,O}\right)$ and $p_{kL} = \Pr\left(X_k \in CK_{k,O}\right)$, and define the positive scalar

$$v_{DCV} = \frac{1}{(1 - \sum_{k=1}^{K} w_k^2)^2} \sum_{k=1}^{K} w_k^2 \left(p_{kU}(1 - p_{kU}) + p_{kL}(1 - p_{kL}) + 2 p_{kU} p_{kL} \right).$$

Then in a similar fashion to Theorem 1 above, it may be shown that:

$$\sqrt{T} \left(\widehat{\theta}_{DCV} - \theta_{DCV} \right) \Longrightarrow N(0, v_{DCV}).$$

The limiting variance can be estimated and the asymptotic coverage confidence interval can be computed as above. The representative agent (equally weighted) case can be considered by setting $w_k = 1/K$ for all k.

3.3. Estimation and standard errors of DisGini

We estimate the Distributional Gini Index (DisGini or DG) over K distributions by:

$$\widehat{\theta}_{DG} = \frac{1}{(1 - \sum_{k=1}^{K} w_k^2)} \sum_{i=1}^{K} \sum_{j=1}^{K} w_i w_j \left\{ 1 - \int_a^b \min\left(\widehat{f}_i(x), \widehat{f}_j(x) \right) dx \right\}, \tag{3.13}$$

where $\widehat{f}_k(x)$ are kernel estimates of $f_k(x)$, $k = 1, \ldots, K$, and the w_k's are known weights.

Define the pairwise sets $C_{i,j}$ $i, j = 1, .., K$ $i \neq j$ as:

$$C_{i,j} = \{x : f_i(x) < f_j(x)\}$$

and let

$$v_{DG} = \sum_{i=1}^{K}\sum_{j>i} w_i^2 w_j^2 \left(\Pr\left(X_i{\in}C_{ij}\right) - \Pr\left(X_i{\in}C_{ij}\right)^2\right) +$$
$$+\, 2\sum_{i=1}^{K}\sum_{j>i}\sum_{k>j>i} w_i^2 w_j w_k \left(\Pr\left(X_i{\in}C_{ij}{\cap}C_{ik}\right) - \Pr\left(X_i{\in}C_{ij}\right)\Pr\left(X_i{\in}C_{ik}\right)\right).$$

THEOREM 2. *Suppose that Assumptions A1-A4 hold. Then, we have:*

$$\sqrt{T}\left(\widehat{\theta}_{DG} - \theta_{DG}\right) \Longrightarrow N(0, v_{DG}).$$

The limiting variance may be consistently estimated by replacing the population quantities by their sample analogues.

4. AN EMPIRICAL EXAMPLE

4.1. Household income distributions in the Eurozone

The efficacy of the new techniques is first illustrated in a study of the 21st century evolution of household income inequality in the Eurozone. Milanovic (2011) noted that growing divergence between constituencies within a federation can be a catalyst for the deterioration of its cohesion and the recent rise of economic nationalism in Europe has given cause for concern regarding the European Unions coherence (Krastev 2014, Webber 2018, Lindberg 2019). Formed to promote commonality of wellbeing among its constituents, there is interest in seeing whether the European nations household income distributions are converging. The growth and convergence literature suggests that variation of average incomes across constituencies is of interest since it speaks directly to the question of whether the distribution of income across economies is becoming more or less equitable (Quah 1993). However, deterioration of cohesiveness has much to do with the extent to which economic wellbeing differs across constituencies, the sense in which such differences are perceived by agents within those constituencies and the relative importance of those constituencies. In this context, cohesiveness is more than just a matter of whether or not constituencies have similar average incomes, it is more a matter of whether or not they have common income distributions.

When member nations are equally unequal with relatively similar income levels and distributions, there is a commonality of situation among member constituents which promotes cohesion, whereas a more divisive and alienated situation arises when such inequalities and income levels are not so equally shared in a more segmented society. The cohesiveness of a union of economies is therefore related to the extent to which its respective nation income distributions are segmenting or converging. The new measures are employed to address these distinctions within nations in the Eurozone.

Viewed as an entity, the overall Eurozone household income distribution $f(x)$ is a mixture of the household income distributions $f_k(x)\, k = 1, .., K$ of its K constituent nations where the weights w_k correspond to relative population sizes (see equation 2.5).

Stochastic processes are frequently used to rationalize distributional structures and Gibrat's Law of Proportional Effects and some of its modifications (Gabaix 1999, Reed 2001) have been foundational in providing a theoretical rationale for expecting increasing income inequality. The Law posits that household incomes in subgroup k follow a stochastic process which, in its simplest form in period t, has the form:

$$x_{k,t} = (1 + \delta_{k,t})\, x_{k,t-1}$$

where $\delta_{k,t}$ is a random variable with mean δ_k (which is small relative to one in absolute value) and variance σ_k^2. The law predicts that, given a starting value x_0 and letting $X = \ln(x)$, after T periods X_{kT} will have a mean equal to $X_0 + T\left(\delta_k + 0.5\sigma_k^2\right)$ and variance equal to $T\sigma_k^2$, respectively i.e. log income variation that grows through time. Following Modigliani and Brumberg (1954), classical economic models of income (Hall 1978) use this idea to predict increasingly unequal income distributions (Battistin, Blundell, and Lewbel 2009, Blundell and Preston 1998, Browning and Lusardi 1996). When applied to the $k = 1, \ldots, K$ constituent societies in the Eurozone, clearly different configurations of pairs (δ_k, σ_k^2) for $k = 1, \ldots, K$ will yield collections of distributions that could be converging or diverging, segmenting or increasingly overlapping, becoming more or less equal in distribution.

Multilateral comparisons, as presented in equations (2.4), (2.7), and (2.8), can be estimated using $w_k = 1/K$ or w_k proportional to the population size of nation k. The first case resembles the unweighted inequality between nations in which each country is taken as the unit of observation, disregarding its size. This unweighted version of the measures can be construed as a representative agent model, recording the juxtaposition of nation income distributions directly without respect to their relative importance or impact in the overall income distribution. In the second case each country is weighted by its population. The unit of observation is then a person instead of a country. This weighted version gives insight into distributional differences of the Eurozone as an entity, with small populations given low weight and large populations high weight.

The data source is the European Union Survey on Income and Living Conditions (EU-SILC).[5] To analyze the evolution of the Euro area income distribution over time, four temporally equi-spaced waves, 2006, 2009, 2012 and 2015 were chosen. Since data for Malta are only available from the 2008 wave, this country is excluded from analysis leaving 18 Euro zone countries. Income is the total household net disposable annual income (in thousands Euro) obtained by aggregation of all income sources from all household members net of direct taxes and social contributions.[6] Assuming consumption economies of scale in cohabitation, incomes are age and size-adjusted using the modified-OECD equivalence scale. Given significant disparities in the cost of living between countries, the PPP index for the household final consumption expenditure is used to adjust household incomes.

As an entity, the Eurozone had overall household income Gini coefficients of 0.305, 0.313, 0.317, 0.335 for the years 2006, 2009, 2012 and 2015 respectively, suggesting ever increasing household income disparities in the area over the period.

In the light of concerns regarding European disintegration, questions arise as to the extent to which such inequalities are equally shared across its various nations, which prompts and investigation into the juxtaposition of the income distributions of the Eurozone's constituent nations.[7]

[5]Version estatCROS 2019ki9, released in May 2019. EU-SILC is a harmonized household-level survey that is a collection of annual national surveys of socio-economic conditions of individuals and households in EU countries.

[6]The income reference period refers to the previous year, consequently analysis with EU-SILC files actually refers to 2005-2014.

[7]An alternative approach would consider a transnational decomposition based upon latent household

Table 1 reports the unweighted Multilateral Gini Transvariation (MGT), the Distributional Coefficient of Variation (DCV), and the Distributional Gini Coefficient (DisGini). The income densities $f_k(x)$ are kernel estimated using the Sheather and Jones (1991) bandwidth as smoothing parameter, and take into account the weighting scheme of the EU-SILC survey. The measures can yield insights into the progress of distributional inequalities over the era, tending toward 0 as distributions converge and tending toward 1 as they segment or diverge. All the unweighted indices record a decline over the whole period with respect to 2006. Thus, in a representative agent view of the world, similar to that pursued in the sigma convergence literature wherein nations are equally weighted (Quah, 1993), the multilateral results present significant evidence of nation income distribution convergence.

Table 1. Unweighted MGT, DCV and DisGini coefficients - Nation group analysis.

Year	MGT	DCV	DisGini
2006	0.135	0.270	0.386
	(0.004)	(0.001)	(0.003)
2009	0.111	0.226	0.341
	(0.004)	(0.001)	(0.003)
2012	0.106	0.234	0.326
	(0.003)	(0.001)	(0.003)
2015	0.107	0.250	0.341
	(0.004)	(0.001)	(0.003)

Note: Asymptotic standard errors are in brackets.

Table 2 reports the weighted Multilateral Gini Transvariation (MGT-W), the Distributional Coefficient of Variation (DCV-W), and the Distributional Gini Coefficient (DisGini-W). Looking at the patterns under population-weighted version of the statistics quite different stories emerge. The population-weighted indices, after a slight dip in 2009, show a significant increase, indicating increasing distributional divergence in terms of increasingly segmented nations.

Taken together, the weighted and unweighted versions of the statistics reveal that lesser populated nations of the Eurozone are exhibiting a convergence pattern whereas nations with larger populations appear to be segmenting.

A further insight on the extent to which each country is converging to, or diverging from, the Eurozone norm is given by GT_{ko}, $k = 1, ..K$, the bilateral transvariations between each country k and the overall Eurozone distribution. These magnitudes can be visualized in a radar chart whose spokes are the respective country/overall distribution transvariations. Figure 1 reports the corresponding radar chart, a decomposed distributional coefficient of variation as it were.

The center of the chart corresponds to zero transvariation where all subgroups have identical distributions. The closer is a point on a nation's spoke to the periphery, the higher is the transvariation of that nation's income distribution with respect to the whole

income classes that transcend nation boundaries. These latent classes can be identified by a semiparametric mixture distribution analysis (see Anderson, Pittau, Zelli and Thomas 2018).

Table 2. Population weighted MGT, DCV and DisGini coefficients - Nation group analysis.

Year	MGT-W	DCV-W	DisGini-W
2006	0.291	0.107	0.237
	(0.005)	(0.002)	(0.003)
2009	0.279	0.103	0.222
	(0.005)	(0.002)	(0.003)
2012	0.323	0.135	0.282
	(0.005)	(0.002)	(0.003)
2015	0.349	0.173	0.361
	(0.005)	(0.002)	(0.003)

Note: Asymptotic standard errors are in brackets.

Eurozone distribution (a value equal to 1 means complete segmentation, i.e. the two distribution are far apart). The points have been colour coded by year, so that intuitively nations with green dots (year 2015) nearer the centre than black dots (year 2006) are converging to the Eurozone distribution over the observation period, whereas nations with green dots outside of the black dots are diverging from the Eurozone distribution.

The bilateral nation-overall transvariations range from 0.03 for Italy to 0.68 for Slovakia in the year 2006. The pattern of this bilateral index shows a process of convergence toward the EuroArea distribution for Eastern European countries (notably low population countries) and significant divergence from the Eurozone distribution for Spain, Finland, France and Greece. Figures in appendix show the evolution of the income distributions of constituent nations and their overlapping with respect to the Eurozone distribution in 2006 and in 2015.

Summing up, what emerges is a collection of distributions that result in a Eurozone with an increasingly unequal overall income distribution comprised of an increasingly similar (i.e. convergent) collection of unweighted distributions that, when population weighted, become divergent as a collection.

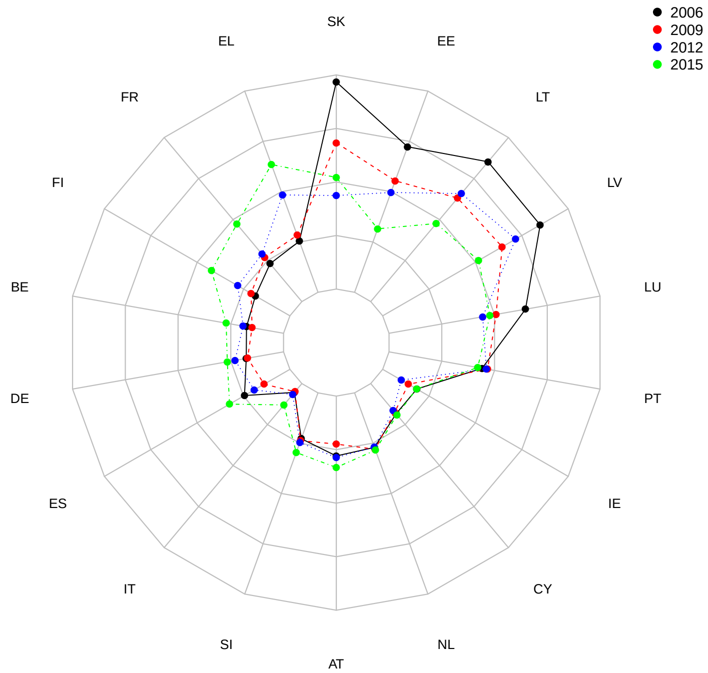

Figure 1. Radar chart of bilateral transvariation of each country with respect to Eurozone. The center of the wheel corresponds complete overlapping with the Eurozone distribution. Moving to the periphery reflects less commonality with the Eurozone. Countries are clockwise ordered starting with the largest positive difference between 2006 and 2015 (indicating convergence) and ending with the largest negative difference (indicating divergence).

4.2. Health-income inequalities and the ageing process in China

To exemplify the use of DisGini in situations where only categorical data is available, age related inequalities in health and incomes in China are examined. The world wide prevalence of aging populations has stimulated interest in the aging process and its connection with wellbeing. For elderly populations, health, income and aging is inextricably interlinked. In this regard indeed, Anand (2004) argues that health should have primacy over consumption in the wellbeing calculus. Welfare programs, in providing support for the elderly and the poor especially in terms of their health outcomes, are also integral to the process. Given its aging population, its unprecedented economic growth and its recently developed welfare program Dibao, China is of particular interest in this respect. Anderson and Fu (2020) study health and income wellbeing in China's older population groups and the impact that Dibao may have had on them. The categorical nature of self reported health status presents a particular challenge in this regard with respect to quantifying wellbeing levels and inequalities, the Distributional Gini provides a solution.

Gao (2017) presents an extensive analysis of the impact of Dibao on work and welfare, and, along with Kakwani (2019), produces evidence of poor targeting, i.e. assistance does not always appear to be reaching those for which the program was defined. However, little has been done to examine the health - income inequalities and the impact that Dibao may have had on inequalities in those dimensions, especially with regard to the elderly.

Here, employing survey data drawn from the China Health and Retirement Longitudinal Study (CHARLS) 2013 follow up to a 2011 baseline study, age group based inequalities in health and incomes are examined. Groups based upon gender, urban/rural location, and Dibao recipient were established. Respondents who were at least 45 years of age were asked to categorize their health as poor, fair, good, very good, excellent and placed in income quintiles (adult equivalized incomes based upon the square root rule were used). The sample was partitioned into age groups 45-50, 50-60, 60-70 and over 70 with respective sample sizes of 1823, 5396, 4782 and 2872 yielding an overall sample size of 14873.

Two exercises were performed using the unweighted formulation which treats all groups equally which is as it should be in a representative agent situation which looks at the health and income inequality risks facing a randomly selected member from each group. One formulation separately identified Dibao recipients as a separate group within each category, the other formulation did not separately identify Dibao recipients[8] (see Table 3). What is observed as part of the aging process is significantly increasing inequality in the joint distribution of health and income in post retirement years. When Dibao recipients are separately identified, distributional inequalities increase uniformly across age groups which, from considerations regarding augmenting of groups in section **??**, indicates inequalities suffered by those groups are on average even greater than those endured by non-Dibao recipients, suggesting that targeting may well not be as bad as has been claimed.

Of particular interest from an aging perspective is the radar chart (Figure 2) that shows

[8]The two formulations refer to the same sample. Initially individuals are stratified by four age class. For each age-class, in the first formulation individuals are grouped in four clusters based on their residential area (urban or rural) and their sex (male and female). In the second formulation, instead, for each age-class the initial four groups are further split in Dibao recipients or not, a binary variable in the survey. The class shares of Dibao recipients (youngest to oldest) were respectively 0.0910, 0.0804, 0.1041 and 0.1496.

Table 3. Distributional Gini coefficients. Age group analysis when Dibao recipients are separately identified and when they are not.

Age groups	DisGini (Dibao Recipients Separately Identified)	DisGini (Dibao Recipients Not Identified)
45–50	0.3645	0.2170
	(0.0019)	(0.0024)
50–60	0.2625	0.1890
	(0.0010)	(0.0013)
60–70	0.3145	0.2839
	(0.0011)	(0.0016)
>70	0.3943	0.3103
	(0.0015)	(0.0021)

Note: Asymptotic standard errors in brackets.

the bilateral transvariation of each group with respect to the overall distribution (GT_{ko}), by age class. The polygon formed by joining the points of spokes is a representation of the aggregated extent of differences of the groups from the "average". Consider two age groups, A and B, if the polygon representing the inequality measure in age class A, is everywhere inside the corresponding polygon for B, then A corresponds to an universal, unequivocal and comprehensive reduction of inequality over B in the sense that all subgroup distributions are closer to the mean distribution in A than they are in B. Here, the 50-60 year olds polygon is completely inside the 60-70 year olds polygon which in turn is completely inside over 70 year olds polygon reflecting a universal increase in health and income wellbeing inequality over the aging process in later life for every category.

In a wellbeing measurement setting it suggests the idea of a comprehensive reduction in inequality with all groups converging to the overall norm. More generally it implies the notion of universal convergence/divergence amongst a collection of groups. This may be examined statistically by noting that the respective vectors of estimated spokes in A and B, $\widehat{GT}_O^A$ and $\widehat{GT}_O^B$, are respectively asymptotically distributed

$$\mathrm{N} \sim \left(GT_O^A, \frac{diag(GT_O^A) - GT_O^A \cdot GT_O^{A'}}{T} \right) \quad \text{and} \quad \mathrm{N} \sim \left(GT_O^B, \frac{diag(GT_O^B) - GT_O^B \cdot GT_O^{B'}}{T} \right)$$

and testing the joint hypothesis:

$$\mathrm{H}_0 : GT_O^A - GT_O^B > 0, \ \text{ against } \ \mathrm{H}_1 : GT_O^A - GT_O^B \leq 0$$

or vice versa using the Maximum Modulus Distribution (Stoline and Ury 1979).

This is verified in Table 4 which fails to reject the hypothesis that older age group polygons lay outside younger age group polygons for successive over 50's age groups. Older age groups clearly suffer increasing health and income inequalities with the aging process.

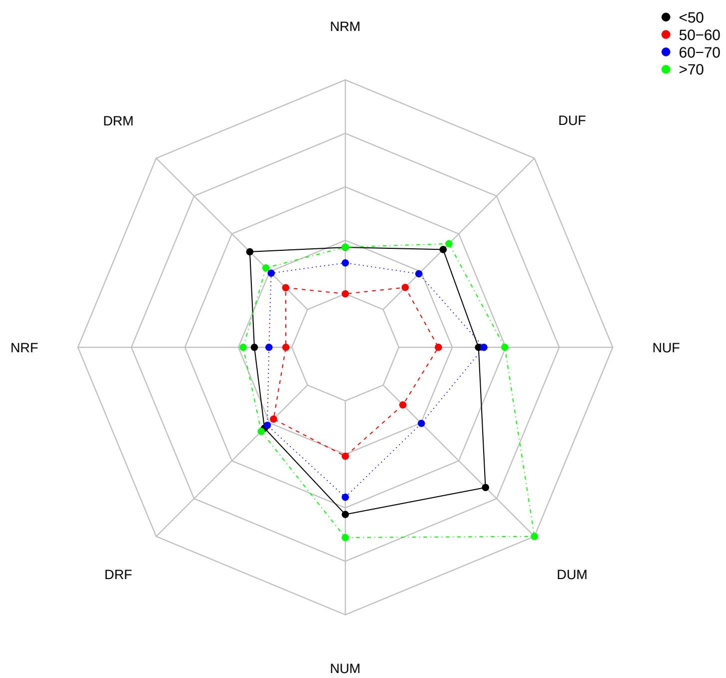

Figure 2. Radar chart of bilateral transvariation of each group with respect to the overall distribution (GT_{ko}), by age class. The center of the wheel corresponds to the minimum value of GT_{ko}, that is the maximum overlapping. Moving to the periphery reflects more dissimilarity with respect to the overall distribution. Subgroups are identified by a three-letter acronym of their labels. The first letter indicates Dibao recipient (D) or not recipient (N). The second letter indicates urban(U) or rural (R). The third letter indicates female (F) or male (M).

Table 4. Stoline-Ury maximum modulus statistics (SUMMS) for spoke changes for successive age classes. Subgroups are identified by a three-letter acronym of their labels. The first letter indicates Dibao recipient (D) or not recipient (N). The second letter indicates urban(U) or rural (R). The third letter indicates female (F) or male (M).

	45–50 vs 50–60			50–60 vs 60–70			60–70 vs >70		
Groups	diff	se	SUMMS	diff	se	SUMMS	diff	se	SUMMS
NRM	0.079	0.015	5.306	-0.111	0.015	7.310	-0.020	0.016	1.243
DRM	0.001	0.007	0.182	-0.028	0.008	3.698	-0.037	0.010	3.873
NRF	0.135	0.016	8.513	-0.043	0.015	2.801	-0.123	0.016	7.568
DRF	0.006	0.008	0.761	-0.029	0.009	3.440	-0.015	0.010	1.552
NUM	0.001	0.013	0.099	-0.055	0.013	4.130	-0.077	0.015	5.208
DUM	0.019	0.005	3.541	0.000	0.004	0.066	-0.001	0.004	0.331
NUF	0.025	0.013	1.923	-0.043	0.013	3.228	-0.013	0.014	0.923
DUF	0.033	0.006	5.354	0.000	0.004	0.050	-0.010	0.005	2.067

Note: Maximum modulus 5% critical value 2.8.

5. CONCLUSIONS

When comparing collections of groups, simple first and second order moment multilateral comparisons can overlook substantive differences between groups that a more comprehensive multilateral distributional comparison can reveal. Here, some new tools for the multilateral comparison of many distributions in univariate or multivariate, discrete and continuous, weighted and unweighted environments have been introduced. Based on extensions of Ginis' Transvariation Measure, new Multilateral Transvariation measures and more comprehensive Gini-like distributional difference measures, together with their asymptotic distributions, have been developed, namely the Multilateral Gini Transvariation (MGT), the Distributional Coefficient of Variation (DCV), and the Distributional Gini Coefficient (DisGini). The Distributional Coefficient of Variation is a scaled weighted sum of subgroup vs overall distribution transvariations, the magnitude of which can be represented as a polygon within a radar chart which in turn has prompted definition of the notion of comprehensive inequality reduction (increase), the consequence of all subgroups converging to (diverging from) the overall distribution. Assessing distributional differences in categorical - non cardinal environments is particularly challenging and these techniques have been shown to overcome these challenges in these situations. The measures have been exemplified in applications which study national household income distributions in the Eurozone in the 21^{st} century and income and health inequalities and the aging process in China.

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APPENDIX A: DISGINI AND ADDITIONAL GROUPS

The relationship between DisGini_K and DisGini_{K+1} may be understood as follows. Let the original weights be $w_k,\ \ k=1,\ldots,K$ where $\sum_{k=1}^{K} w_k = 1$ and the new weights in the extended collection of groups $w_k^{new},\ \ k=1,\ldots K+1$ where $\sum_{k=1}^{K+1} w_k^{new}=1$ are such that:

$$w_k = \frac{w_k^{new}}{\left(1-w_{K+1}^{new}\right)} = \frac{w_k^{new}}{\theta},\ \ k=1,\ldots,K$$

Let $\varphi^{new} = \sum_{k=1}^{K+1}\left(1-(w_k^{new})^2\right)$, then

$$\text{DisGini}_{K+1} = \frac{1}{\varphi^{new}}\sum_{i=1}^{K+1}\sum_{j=1}^{K+1} w_i^{new}w_j^{new}\text{GT}_{i,j} = \frac{1}{\varphi^{new}}\sum_{i=2}^{K+1}\sum_{j=1}^{i}\int_0^\infty w_i^{new}w_j^{new}|f_i(x)-f_j(x)|dx$$

$$= \frac{1}{\varphi^{new}}\sum_{i=2}^{K}\sum_{j=1}^{i}\int_0^\infty w_i^{new}w_j^{new}|f_i\left(x\right)-f_j\left(x\right)|dx + \frac{1}{\varphi^{new}}\sum_{j=1}^{K}\int_0^\infty w_{K+1}^{new}w_j^{new}|f_{K+1}\left(x\right)-f_j\left(x\right)|dx$$

$$= \frac{\theta^2}{\varphi^{new}}\sum_{i=2}^{K}\sum_{j=1}^{i}\int_0^\infty w_iw_j|f_i\left(x\right)-f_j\left(x\right)|dx + \frac{1}{\varphi^{new}}\sum_{j=1}^{K}\int_0^\infty w_{K+1}^{new}w_j^{new}|f_{K+1}\left(x\right)-f_j\left(x\right)|dx$$

$$= \frac{\theta^2\varphi}{\varphi^{new}}\text{DisGini}_K + \frac{1}{\varphi^{n}}\sum_{j=1}^{K}\int_0^\infty w_{K+1}^{new}w_j^{new}|f_{K+1}\left(x\right)-f_j\left(x\right)|dx$$

$$= \text{DisGini}_K - \left(1-\frac{\theta^2\varphi}{\varphi^{new}}\right)\text{DisGini}_K + \frac{1}{\varphi^{new}}\sum_{j=1}^{K}\int_0^\infty w_{K+1}^{new}w_j^{new}|f_{K+1}\left(x\right)-f_j\left(x\right)|dx$$

$$= \text{DisGini}_K - \left(\frac{\varphi^{new}-\theta^2\varphi}{\varphi^{new}}\right)\text{DisGini}_K + \frac{w_{K+1}^{new}}{\varphi^{n}}\sum_{j=1}^{K}\int_0^\infty w_j^{new}|f_{K+1}\left(x\right)-f_j\left(x\right)|dx$$

$$= \text{DisGini}_K - \left(\frac{w_{K+1}^{new}}{\varphi^{new}}\right)2\left(1-w_{k+1}^{new}\right)\text{DisGini}_K + \frac{w_{K+1}^{new}}{\varphi^{n}}\sum_{j=1}^{K}\int_0^\infty w_j^{new}|f_{K+1}\left(x\right)-f_j\left(x\right)|dx$$

$$= \text{DisGini}_K - \left(\frac{w_{K+1}^{new}}{\varphi^{new}}\right)\left(2\theta\text{DisGini}_K - \sum_{j=1}^{K}\int_0^\infty w_j^{new}|f_{K+1}\left(x\right)-f_j\left(x\right)|dx\right)$$

$$= \text{DisGini}_K - \left(\frac{2\theta w_{k+1}^{new}}{\varphi^{new}}\right)\left(\text{DisGini}_K - \frac{1}{2}\sum_{j=1}^{K}\int_0^\infty w_j^{new}|f_{K+1}\left(x\right)-f_j\left(x\right)|dx\right)$$

APPENDIX B: DERIVATION OF LARGE SAMPLE PROPERTIES

Proof of Theorem 1. We can write

$$\widehat{\theta}_{KT} = \widehat{\theta}_{KTU} - \widehat{\theta}_{KTL}$$

where

$$\widehat{\theta}_{KTU} = \frac{1}{K} \int \max\left(\widehat{f_1}(x), \widehat{f_2}(x), \ldots, \widehat{f_K}(x)\right) dx,$$
$$\widehat{\theta}_{KTL} = \frac{1}{K} \int \min\left(\widehat{f_1}(x), \widehat{f_2}(x), \ldots, \widehat{f_K}(x)\right) dx.$$

We have following the arguments of Anderson, Linton and Whang (2012)

$$\widehat{\theta}_{KTU} - \theta_{KTU} = \frac{1}{g_{KT}(K)} \sum_{k=1}^{K} \int_{CK^{k,*}} \left(\widehat{f}_k(x) - E\left(\widehat{f}_k(x)\right)\right) dx + r_T, \tag{B.1}$$

where r_T is generic notation for a remainder term that is of smaller order in probability (r_T may be different from expression to expression). Similarly,

$$\widehat{\theta}_{KTL} - \theta_{KTL} = \frac{1}{g_{KT}(K)} \sum_{k=1}^{K} \int_{CK_{i,*}} \left(\widehat{f}_k(x) - E\left(\widehat{f}_k(x)\right)\right) dx + r_T. \tag{B.2}$$

Combining (B.1) and (B.2) together, we have

$$\widehat{\theta}_{KT} - \theta_{KT} = \frac{1}{K} \sum_{k=1}^{K} \int_{CK^{i,*}} \left(\widehat{f}_k(x) - E\left(\widehat{f}_k(x)\right)\right) dx - \frac{1}{K} \sum_{k=1}^{K} \int_{CK_{i,*}} \left(\widehat{f}_k(x) - E\left(\widehat{f}_k(x)\right)\right) dx + r_T.$$

The limiting distribution follows.

Proof of Theorem 2. The estimator $\widehat{\theta}_{DG}$ may be written as

$$\widehat{\theta}_{DG} = \frac{1}{K}\left(2K^2 - \sum_{i=1}^{K}\sum_{j=1}^{K} w_i w_j \left\{\int_a^b \min\left(\widehat{f}_i(x), \widehat{f}_j(x)\right)\right\} dx\right).$$

So, for the distributional properties of $\widehat{\theta}_{DG}$ attention can be focussed upon:

$$\widehat{\theta}_{OV} = \sum_{i=1}^{K}\sum_{j=1}^{K} w_i w_j \left\{\int_a^b \min\left(\widehat{f}_i(x), \widehat{f}_j(x)\right) dx\right\} = \sum_{i=1}^{K}\sum_{j=1}^{K} w_i w_j \left\{\widehat{\theta}_{i,j}\right\}, \tag{B.3}$$

where $\widehat{f}_k(x)$ are defined as in (3.11).

Considering the $\widehat{\theta}_{i,j}$, for simplicity assume independent samples of T observations and that the contact sets are of measure 0.

Then

$$\widehat{\theta}_{i,j} - \theta_{i,j} = \int_{C_{i,j}} \left(\widehat{f}_i(x) - E\left(\widehat{f}_i(x)\right)\right) dx + \int_{C_{j.i}} \left(\widehat{f}_j(x) - E\left(\widehat{f}_j(x)\right)\right) dx + r_T$$

and thus

$$\widehat{\theta}_{OV}-\theta_{OV}=\sum_{i=1}^{K}\sum_{j=1}^{K}w_iw_j\left(\widehat{\theta}_{i,j}-\theta_{i,j}\right)=$$
$$=\sum_{i=1}^{K}\sum_{j=1}^{K}w_iw_j\left(\int_{C_{i,j}}\left(\widehat{f}_i\left(x\right)-E\left(\widehat{f}_i\left(x\right)\right)\right)dx+\int_{C_{j,i}}\left(\widehat{f}_j\left(x\right)-E\left(\widehat{f}_j\left(x\right)\right)\right)dx\right)+r_T.$$

Then generally,

$$AVAR\left(\widehat{\theta}_{i,j}\right)=\frac{1}{T}\left(p_{i:ij}\left(1-p_{i:ij}\right)+p_{j:ji}\left(1-p_{j:ji}\right)+2\left(p_{ij}-p_{i:ij}p_{j:ji}\right)\right)$$

$$p_{i:ij}=\Pr\left(X_i\in C_{i,j}\right)\ \text{ and }\ p_{ij}=\Pr\left(X_i\in C_{i,j}\cap X_j\in C_{j,i}\right),$$

which may simplify with independent sampling. However even if X_i and X_j are independent $\widehat{\theta}_{i,j}$ and $\widehat{\theta}_{k,l}$ will be dependent if they have one subscript in common so that $ACOV(\widehat{\theta}_{i,j},\widehat{\theta}_{k,l})\neq 0$ when there is a commonality in subscripts. All such terms need to be considered so that a threefold summation is required involving probabilities of sets of the form:

$$C_{i,j}\cap C_{i,k}=\left\{x:f_i\left(x\right)<\min\left(f_j\left(x\right),f_k\left(x\right)\right)\right\}.$$

APPENDIX C: FIGURES OF INCOME DISTRIBUTION OVERLAPS

Figure C.1. Income distribution of Slovakia and Estonia and their overlap with the Eurozone income distribution: years 2006 and 2015.

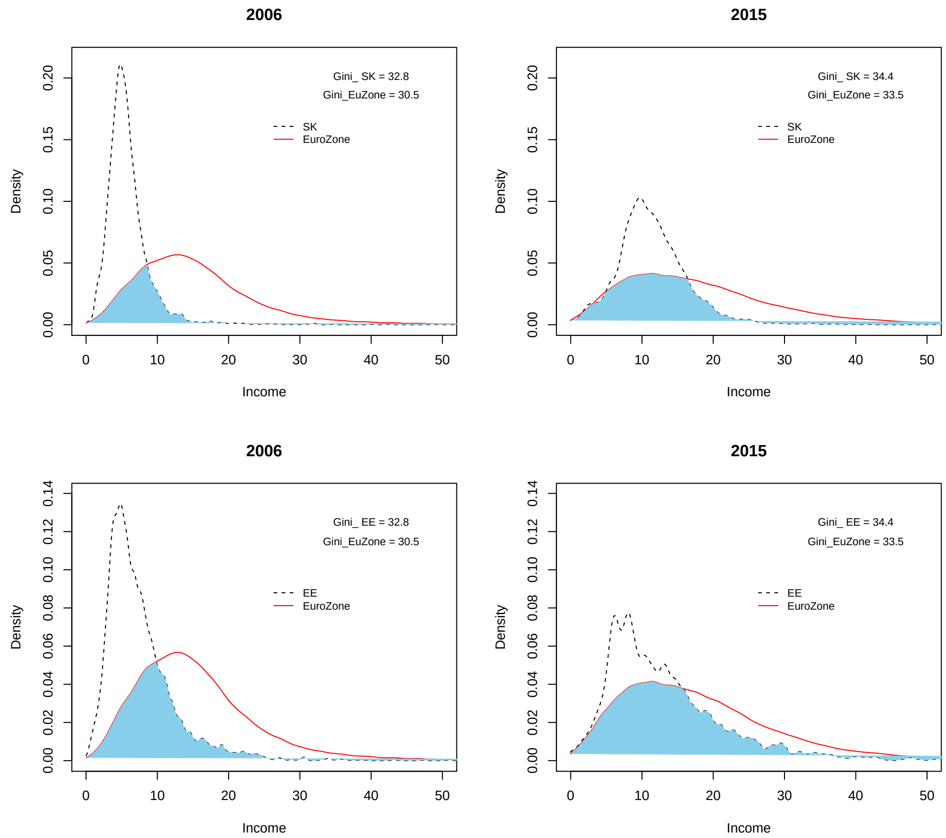

Figure C.2. Income distribution of Latvia and Lithuania and their overlap with the Eurozone income distribution: years 2006 and 2015.

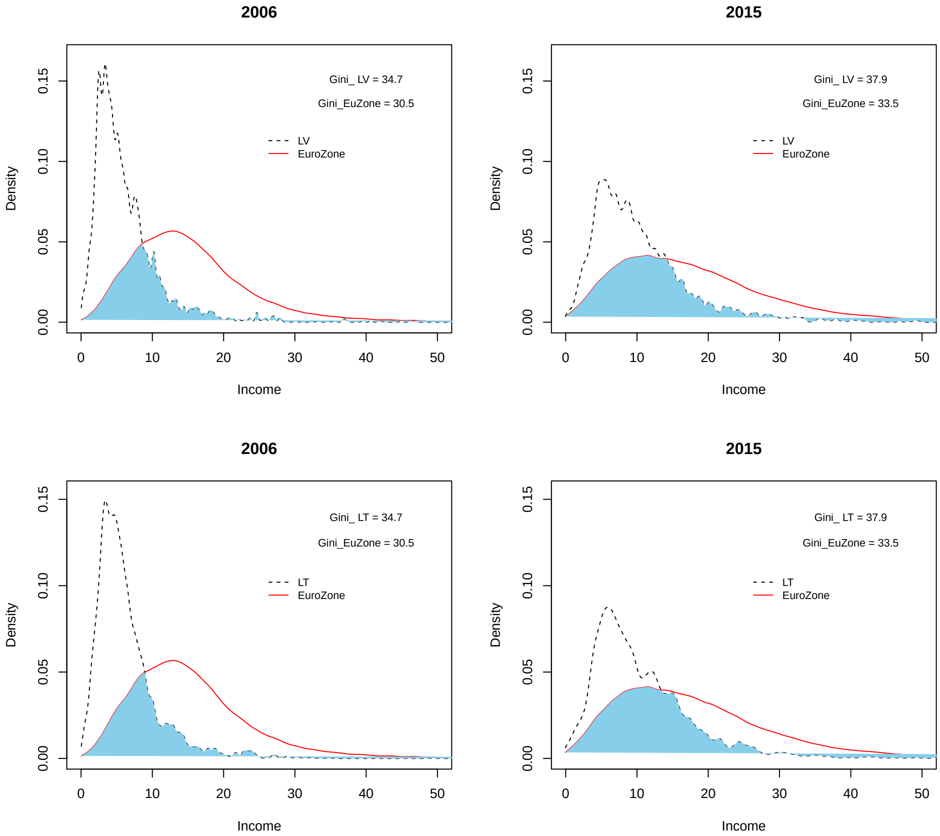

Figure C.3. Income distribution of Greece and Spain and their overlap with the Eurozone income distribution: years 2006 and 2015.

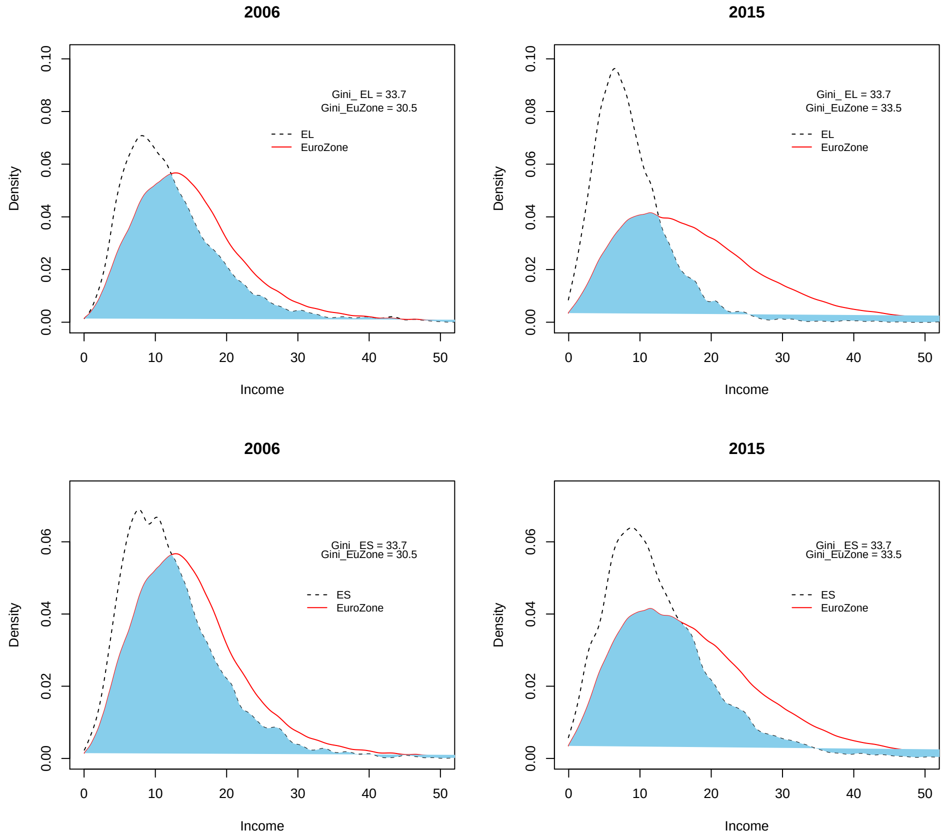

Figure C.4. Income distribution of Finland and France and their overlap with the Eurozone income distribution: years 2006 and 2015.