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Searching for Results: Optimal Platform Design in a Network Setting

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Searching for Results: Optimal Platform Design in a Network Setting*

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Abstract

Large online platforms, like Airbnb or Amazon Marketplace, increasingly direct users to internal search engines that limit the number of sellers consumers observe. We show that such behaviour is consistent with profit maximisation. To do so, we model buyer-seller interactions as a series bipartite graphs, which are each realised with a probability chosen by the platform owner. Prominent players disproportionately increase competition, which decreases prices. To maximise profit, the platform owner ensures that buyers only observe a consistent number of sellers in every state of the world realised with positive probability. When products are vertically differentiated, the platform owner biases observation towards high-quality products, but doing so reduces prices, and, as a result, the optimal number of sellers in the network. The extent to which platforms in different markets highlight high-quality products and the number of sellers their search processes show is a function of both quality dispersion and substitutability.

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1 Introduction

Many economic interactions occur in settings where sellers are only able to sell to a subset of buyers. This stratification might occur, for example, because: sellers are unable to supply some consumers due to geographic constraints or due to consumer preferences (see, for example, Spiegler, 2006); consumers might be uncertain which firms are active in a market, as in Janssen and Rasmusen (2002); . In such settings, a seller potentially faces different levels of competition for each individual consumer that they can supply - some buyers could be supplied by a large number of sellers, while others may only be able to buy goods from a single seller.

Buyers online platforms generally only observe a subset of sellers in any given market. Ringer and Skiera (2016) find that 99.9% of potential buyers of LCD TV sets only observe sixteen of a possible 1,124 products on a German price comparison website. Similarly, Kim, Albuquerque and Bronnenberg (2010) examine the camcorder market on Amazon and find that the median search set for consumers of these products was eleven out of a total of more than ninety camcorders available on the platform.

By choosing a search environment, owners of large online platforms choose which buyers observe which sellers, and how many sellers effectively compete with one another. Consumers do not tend to engage as much with results lower down on a search page or products not on the first page of search results (Smith and Brynjolfsson, 2001, Baye et al, 2009 and Baye et al 2016) and almost all modern-day online platforms use ordered search with limited results per page as the dominant method of navigating products in a market.

While there are behavioural explanations (e.g. rational inattention, see Hefti and Heinke, 2015) for the similarity in the search environments of online platforms, we show that presenting users with a fixed number of sellers is consistent with profit maximisation on a monopoly platform. While limiting number of products consumers observe reduces total sales, the platform is willing to forego some buyer-seller matches in order to increase prices.

We examine a case where search is decentralised in the sense that each buyer observes each seller with some independent probability, but each observation probability is itself randomly determined. We find that a platform owner prefers a distribution of observation probabilities to a symmetric mean-preserving spread of that distribution; the case where some sellers are randomly more prominent than others is costly. A prominent seller in this setting faces more competition in expectation and therefore sets a relatively low price, lowering prices across the network.

This insight, combined with the observation that the platform owner has an incentive to avoid settings where consumers observe a large amount of products is profit increasing for the platform, implies that an environment where consumers observe the same number of sellers on average is always optimal, as this maximises seller price for a given number of buyer-seller interactions. Real-life search environments generate precisely this set-up as they regulate the number of sellers observed by buyers, such that sellers know how many competitors they face in a given market.

We also analyse the case where goods are vertically differentiated. We find that the platform has the incentive to bias consumer observation towards high-quality sellers. Doing so, however, increases the probability that sellers compete with a high-quality seller, which generates an incentive to reduce the number of sellers in the market.

Markets in which quality dispersion is high tend to result in higher profits, as the platform is able to increase the probability that high-quality sellers are observed. Whether the number of sellers observed by buyers is higher or lower than markets with low-quality depends on product substitutability. If products are not that substitutable, then the competition effects associated with a high-quality seller being observed dominate, and so fewer sellers are observed on average when quality dispersion is high.

However, when products are highly substitutable, there are few competitors in the market to start with, and the cost of missing out on sellers near the top of the quality distribution is in expectation higher when quality is highly dispersed, so the platform shows more sellers to the buyers in this setting compared to the case where seller quality is more similar.

Our analysis therefore generates testable predictions about the structure of the search environment on different platforms, which we summarise in the following diagram:

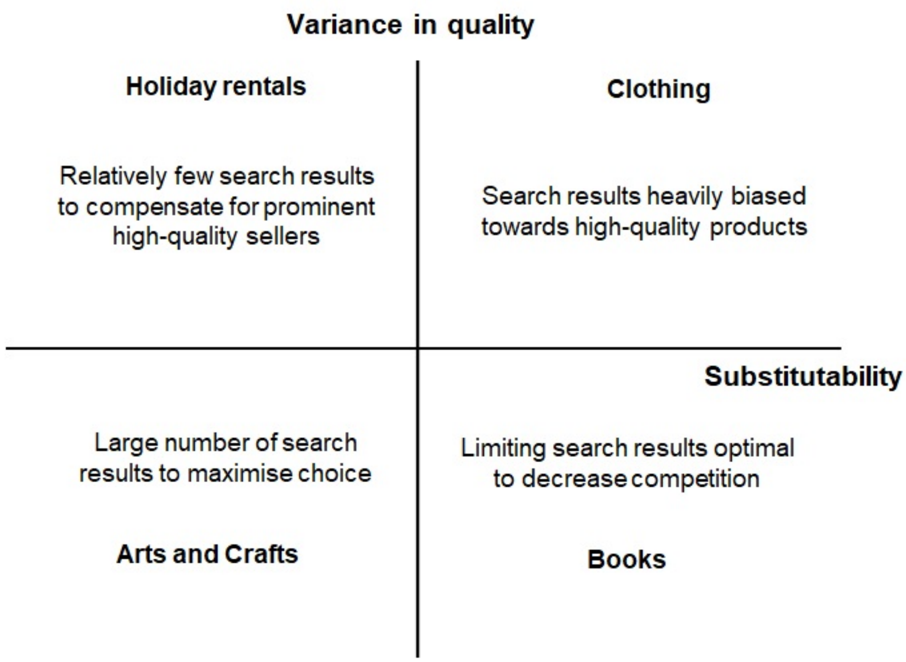

Figure 1: The optimal number of sellers on a platform depends on the market(s) they operate in. Bold text indicates an example market that has the relevant characteristics.

We model the online platform environment as a network that connects sellers with buyers who could potentially demand goods from them. We characterise the equilibrium price setting behaviour of sellers competing for consumer segments (i.e. groups of consumers who share some characteristic like age or location) in a potentially large-scale stochastic network. The network is stochastic in the sense that we assume that there is some probability of a given buyer observing a given seller and that the actual network is only realised after price setting has taken place.

We examine both the case in which the observation network is randomly determined by nature, which we call "decentralised search" and the case where the network is determined by the platform owner directly. In both cases, the observation network generates a competition network with links between sellers that measure the extent to which two sellers compete with one another for consumers, which in turn determines equilibrium prices.

More specifically, we find that a seller's price is linked to their Bonacich centrality in a network connecting sellers and consumer segments who observe them with some probability. This result is consistent with the literature on strategic interaction in networks (e.g. Ballester, Calvò-Armengol and Zenou, 2006), as well as more recent work applying price or quantity competition to network environment (see, for example, Elliott and Galleotti, 2019). When sellers are symmetric in terms of the substitutability of their goods and their own-price elasticity, we find that a seller's price is falling in their centrality.

Our finding that prices are decreasing in seller centrality in turn determine the profit maximising network structure from the platform's perspective. The notion of a seller being highly prominent corresponds precisely with them being more central in the network, and a node being more central than average has a disproportionate impact on the centrality of other nodes. Sellers in direct competition with the prominent seller, which drives a fall in prices, which then produces a feedback effect: sellers in competition with sellers with lower prices must decrease their price in order to compete effectively, as prices are strategic complements (Bulow, Geanakoplos, and Klemperer, 1985).

The remainder of the results described above stem from this feedback effect inherent within the equilibrium action of each seller. Search environments that reduce the probability that one seller is particularly prominent increase aggregate profits. When a seller has higher quality, the increases the probability that they are observed by increasing the number of total sellers buyers observe, but doing so increases competition, which implies that the platform owner has an incentive to reduce the number of sellers that are observed.

Our analysis more generally indicates that monopoly platforms have an incentive to lower competition on the platform by changing its structure. Given that the optimal structure from a consumer surplus perspective is one in which each buyer observes each seller with a probability of one, the fact that platforms may choose to only allow consumers to observe a fraction of the total number of goods on offer is harmful to welfare. This suggests that competition authorities should examine intra-platform competition, in addition to inter-platform competition. Regulating the internal structure of the networks that underpin large online platforms may reduce the extent to which consumers are harmed by the formation of monopolistic platforms.

2 Literature review

In general, the literature that models buyer-seller interactions in a network setting has focused on cases where buyers bargain with sellers they are connected to (Kranton and Minehart, 2001, Corominas-Bosch, 2004 and Polanski, 2007) over a single, indivisible good. This approach seems particularly relevant in relatively "thin" markets, populated by a small amount of buyers and sellers, and where the goods being sold are discrete. In thin networks with discrete goods, individual buyers and sellers can make bilateral agreements with one another easily.

However, many real-world cases of networks connecting buyers and sellers are not thin markets. For example, online platforms, such as Amazon and Airbnb have a very large number of users, with sellers interacting with a large number of buyers at any one time. This would make bilateral bargaining between users difficult, and in general prices on these platforms are not bilaterally negotiated or set by the owner of the platform itself. A natural assumption in a thick market is that sellers choose a single price which is the same for each potential buyer. Our starting framework is therefore more akin to the networks literature spawned by Ballester, Calvò-Armengol and Zenou (2006), and developed in a IO setting in work such as Elliott and Galleotti (2019) and Bimpikis, Ehsani and Ilkiliç (2018).

In terms of equilibrium characterisation, our finding that a seller's price in equilibrium is decreasing in their Bonacich centrality in a seller-only network is consistent with Ballester, Calvò-Armengol and Zenou (2006), which finds that if direct effects are sufficiently small, the equilibrium action of each player in a network is proportional to their Bonacich centrality.

The price setting behaviour of sellers are strategic complements in this setting, which might suggest, as in Bramoullé, Kranton and D'Amours (2014), that the more central a seller in the network, the higher their price.

Bimpikis, Ehsani and Ilkiliç (2018) examine networked quantity competition. Firms compete for multiple markets, and each firm has a non-separable cost function, such that the quantity produced in one market affects the marginal cost of production in another. In this setting, it can be shown that quantities in that model are proportional to their Bonacich centrality with a negative decay factor.

Elliott and Galeotti (2019) show that in a Hotelling environment in which sellers compete on price and are differentiated by location, a seller j's price is determined not only by the sellers with whom directly j competes, but also those sellers in other markets that compete with j's competitors. This result is a corollary of our more general characterisation of the price equilibrium of the sellers on the platform.

The link we find between pricing and centrality is consistent with earlier work in industrial organisation, which examines the role "captive" buyers have on optimal pricing. Ireland (1993) and McAfee (1994) consider a framework in which sellers of a homogeneous good compete for consumers and have "independent reach" - the fact that a consumer observes a firm does not affect the probability that the consumer observes another firm. The unique mixed equilibrium of this game is one in which the lowest price within each seller's strategy set is the same, but the strategy of the seller with the largest proportion of captive consumers contains the highest maximum price. De Francesco and Salvadori (2013) and Armstrong and Vickers (2019) extend this analysis to cases where firms have different capacities and only the largest firm has captive customers respectively.

Our results in seller prominence in the decentralised search setting differ from earlier work on prominence such as Armstrong, Vickers and Zhou (2009), which examines the case where consumers engage in costly search to learn the price and match value of a series of products.

If one product is more prominent than another, then it is observed first, with buyers choosing whether to buy it or search for other products. They find that the prominent firm sets a lower price than other sellers, who set a higher price than in the case where matching is random, increasing profits. In contrast, our analysis finds that prominence causes prices to fall, because prominence increases the expected intensity of competition, which has a disproportionately large effect on prices.

By choosing the probability that different networks are realised, the platform owner has the ability to intervene in order to influence the actions of the sellers. There is a growing literature on intervention in networks, though much of this literature does not consider the question of network design directly.

An early example of a central planner intervening in a network on which players interact is Ballester, Calvò-Armengol and Zenou (2006), which uses their characterisation of equilibrium to identify the key player in the network, the removal of whom would allow a central planner to reduce total activity the most with the removal of a single player.

Birge, Candogan and Chen (2018) construct a model in which firms are connected to buyers on a network controlled by a platform and choose their price in order to attract buyers with different valuations of a single good. They characterise the effect of network structure on the profitability of commissions and subscriptions from a platform's perspective, finding that revenue loss is potentially unbounded when all sellers are charged either a common commission or the same subscription fee regardless of their location in the network.

In an industrial organisation setting, Cominetti, Correa and Stier-Moses (2009) and Chawla and Roughgarden (2009) compare the efficiency of competitive equilibria of a game involving Bertrand competition between firms who act as intermediaries who control the flow of some good (for example) by controlling the edges of a network compared with the optimally efficient network flow.

Galeotti, Golub and Goyal (2019) examine a case where a central planner can partly determine (at an exogenously imposed cost) the payoff of players in a network, which in turn determines the equilibrium actions of each player. While there is similarity between this approach and the one utilised here, our intervention in the network is on its design, rather than on the payoff functions of the agents. Furthermore, the cost of intervention in our model arises endogenously from the effect that changing observation probabilities has on increasing competition.

Li (2019) examines the case where a central planner chooses the design of a directed graph on which agents experience local strategic complements. In this context, and with no constraints on the strength of links, they find that all optimal networks are generalized nested split graphs as this maximises the sum of centralities via the feedback effect inherent in this set-up. Our setting involves a bipartite network in which the sellers are linked by their connections with non-active agents, which generates a natural constraint on the direction of links in the network, and actions are decreasing in centrality, leading to the optimal graph (at least in the symmetric case) being one in which centralities are equal, reducing the feedback effect.

Charlson (2020) uses an initial set-up similar to the one here to examine the case where the platform owner suffers from incomplete information, and must design the network without knowing the quality of the sellers on the platform. The platform owner utilises ratings as a way of biasing search results towards high-quality sellers, but in doing so decreases prices in expectation because increasing the prominence of high-quality sellers drives down prices due to an increase in total centrality. As a result of this trade-off, some platforms may prefer random matching to using ratings as a way of biasing search results, to the detriment of consumers.

More broadly, we contribute to the wider literature on competition and platforms. Traditional accounts of platform industrial organisation have focused on competition between platforms (Tirole and Rochet, 2003, Armstrong, 2006 and Tan and Zhou, 2019), our model regards intra-platform competition. There is less analysis in economics relating to intra-platform competition and platform design, with most of the work on the latter relating to information design on platforms (Armstrong and Zhou, 2020 and Elliott and Galleoti, 2020), rather than network design specifically. Within network competition has been examined in a management context (Zhu and Liu, 2018 and Nambisan and Baron, 2019), but these analyses relate to seller development and platform-seller conflict, rather than how platforms shape buyer-seller interactions.

3 Motivating example

Large, online platform owners must design platforms in which many sellers compete for many buyers. If sellers can only set one price, then network design has implications for the nature of competition between sellers for different buyers. If a platform owner can affect which buyers observe which sellers and sellers only set one price, then the platform owner faces a trade-off between more sellers being observed, increasing demand, and the resultant increase in competition.

As an example, suppose the probability that two buyers observe two sellers is strictly between 0 and 1. Then all the possible ex-post market structures (ignoring the case where no buyer observes any sellers) are shown in Figure 2.

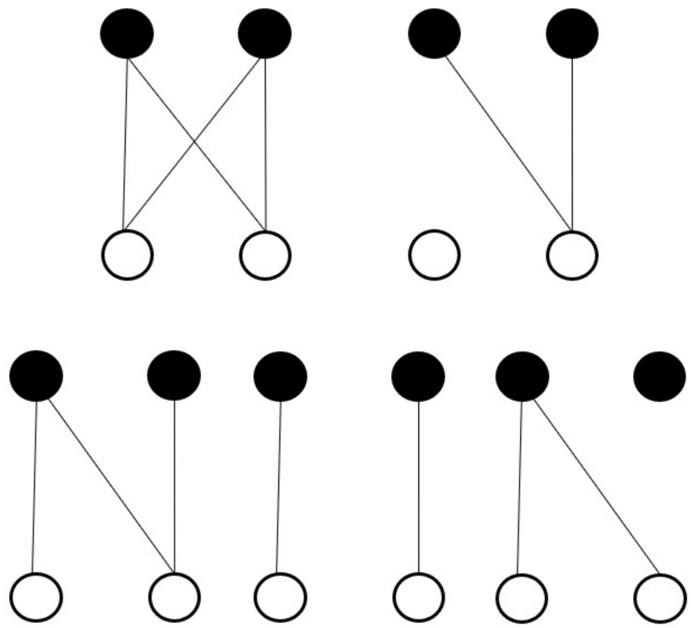

Figure 2: The black nodes represent sellers, the white nodes represent buyers. Assuming sellers and buyers are identical, these configurations represent all possible market types when there are two sellers and two buyers.

Consider the nature of total profits and competition if prices were set **after** the realisation of a network structure. Market structures on the bottom and to the right of the diagram exhibit less competition, but profits are lost because the buyers will be assumed throughout to purchase at least some goods from any seller they observe. This issue is reduced in market structures above and to the left of the diagram; however, there, the two sellers are in more direct competition with one another, which reduces profit due to both sellers setting a lower price.

The trade-off highlighted by this simple example is one that the platform owner faces when designing the network. The framework examined here involves a platform owner that chooses the probability that each possible network between a fixed number buyers and sellers is realised. We identify the extent to which changing these probabilities affects competition and prices, and characterise the profit-maximising probability vector.

4 Model

Sellers, buyers and the platform owner

Suppose there is a finite set, B, whose elements are "consumer segments", in the sense that they are a finite mass of consumers who are assumed to share some trait, such as geographical location, age demographic, occupation, etc. We use n to denote the number of consumer segments.

Similarly, let S be a finite set of sellers, where $|S| = m$. Sellers each sell a single type of completely divisible good, and each seller's good is an imperfect substitute for each of the goods.

Sellers and buyers interact on a platform, with each buyer observing a subset of S. These observations generate a network $G_i = (B \cup S)$. We assume that the graph generating process is stochastic in the sense that there is a probability $\theta_i \in [0,1]$ that a graph G_i is generated for every possible $m - n$ bipartite graph, and hence $\sum_i \theta_i = 1$. Let $\boldsymbol{\theta}$ denote a vector whose ith entry is θ_i.

Consider a simple example of the above set-up, with three sellers (X, Y and Z) and three consumer segments ($1, 2$ and 3). Suppose there are two graphs which can be realised with some positive probability are: (1) the complete graph, in which each buyer observes each seller and; (2) the graph depicted on the right of Figure 3, in which sellers Y and Z compete for consumer segment 2 but the other two segments are captive. Assume both of these graphs are realised with equal probability.

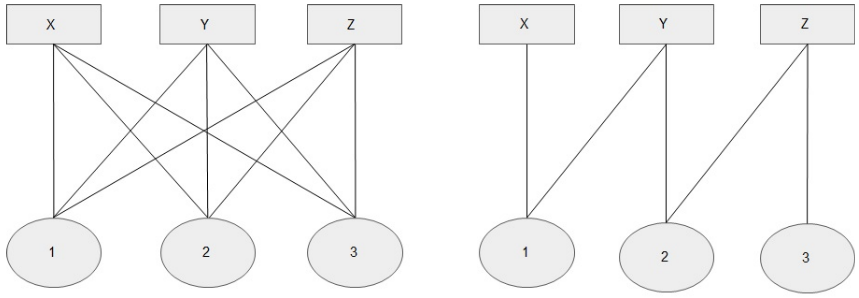

Figure 3: A case where two graphs can be realised.

Let $\boldsymbol{p}$ denote a $m \times 1$ vector whose jth entry is $p_j \in \mathbb{R}_+$, the price of j's good. We assume that sellers set prices prior to the realisation of the links in the network, but with full knowledge of the vector $\boldsymbol{\theta}$. We will assume that if a consumer segment i observes a seller j, then their demand function, $x_{ij}(.) : \mathbb{R}^m \to \mathbb{R}_+$, for product j can be expressed as follows:[1]:

$$x_{ij}(\boldsymbol{p}) = a\gamma_j - ap_j + \sum_{k \neq j} c(p_k - \gamma_k).$$

where a, γ_j, c are all strictly positive scalars. The parameter γ_j can be thought of as a measure of the quality of the seller j. We will assume throughout that a is large enough such that $x_{ij} > 0$ for each observed good: this restriction will be discussed in more detail below. As $c_{ik} > 0$ for all $i, k \neq i$, each product is a gross substitute for every other product.

Note that the demand function above is an ex-post demand function, in that it is generated after the realisation of the network structure is realised, and hence after the sellers set prices. Define i's ex-ante demand function for a good j as follows:

$$\mathrm{E}[x_{ij}(\boldsymbol{p}; \boldsymbol{\theta})] = \sum_{\tau} \theta_\tau \mu_{ij}(G_\tau)(a\gamma_j - ap_j + \sum_{k=1}^{m} \mu_{ik}(G_\tau) c_{jk}(p_k - \gamma_k)),$$

where $\mu_{ij}(G_\tau)$ is a function such that if $E_{ij} \in G_\tau$ and 0 otherwise. Seller j's expected aggregate demand function is then defined:

[1]We show that this assumption is an approximation of the linear demand curve that is generated from a quadratic, quasi-linear demand curve in the first section of the Appendix.

$$\mathrm{E}[x_j(\boldsymbol{p};\boldsymbol{\theta})] = \sum_{i=1}^{n} \mathrm{E}[x_{ij}].$$

A seller, j, is assumed not to be able to price discriminate across buyers, and hence sets a single price $p_j \in R_+$. Sellers compete with one another on price, and set prices simultaneously. Therefore, each seller's maximisation problem can be expressed:

$$\max_{p_j} \mathrm{E}[\pi_j(p_j,\ p_{\_j};\ \boldsymbol{\theta})].$$

Let $\Gamma(\boldsymbol{\theta})$ represent the simultaneous move m-player game played on a network G with payoffs as specified above and strategy spaces R_+.

The platform owner has the following profit function:

$$\mathrm{E}[\pi_P(\boldsymbol{p};\boldsymbol{\theta})] = \chi \sum_{j=1}^{m} \mathrm{E}[\pi_j(\boldsymbol{p};\boldsymbol{\theta})]$$

where $0 < \chi < 1$.

The search environment

We will consider two search environments:

1. The **decentralised search** environment, in which a consumer segment, i, observes a seller j with probability $0 \leq w_{ij} \leq 1$. The observation probabilities are themselves stochastically determined in the following sense. Each w_{ij} is the realisation of a random variable $\tilde{w}_{ij}$ according to the symmetric probability distribution, Λ, which is bounded such that $0 \leq w_{ij} \leq 1$ and has mean υ. The random variables $\tilde{w}_{ij}$ are independently and identically distributed

2. The **centralised search** environment, in which the platform owner chooses the probability vector $\boldsymbol{\theta}$, optimising their above profit function.

In the decentralised case, nature determines the observation probabilities. This in turn generates the probability vector $\boldsymbol{\theta}$. In the centralised case, the platform owner chooses this probability vector directly in order to solve the following maximisation problem:

$$\max_{\boldsymbol{\theta}} \mathrm{E}[\pi_P(\boldsymbol{p};\boldsymbol{\theta})]$$

which we will assume is subject to the constraint that $\theta_i \in [0,1]$, $\sum_i \theta_i = 1$ and:

$$\beta_j(\boldsymbol{\theta}) := \sum_i \sum_\tau \theta_\tau[\mu_{ij}(G_\tau)] > 0 \; \forall j.$$

We therefore assume that each seller is observed by at least one buyer with a strictly positive probability. This assumption ensures that the centrality of each seller in the network is defined for any proposed solution to the above maximisation problem, and seems conceptually legitimate as we do not explicitly model the entry decision either from the seller or platform side.

In both search cases, $\boldsymbol{\theta}$ is common knowledge, and prices are hence set after the realisation of $\boldsymbol{\theta}$, but, as stated previously, prior to the realisation of the actual observation network.

5 Equilibrium characterisation

We characterise the equilibrium price setting behaviour for a given vector of graph probabilities $\boldsymbol{\theta}$. Define $\alpha_j(\boldsymbol{\theta})$ and $\beta_j(\boldsymbol{\theta})$ as follows:

$$\alpha_j(\boldsymbol{\theta}) := \sum_i \sum_\tau \theta_\tau[\mu_{ij}(G_\tau)(a\gamma_j - c\sum_k \mu_{ik}(G_\tau)\gamma_k)],$$

and:

$$\beta_j(\boldsymbol{\theta}) := \sum_i \sum_\tau \theta_\tau \mu_{ij}(G_\tau).$$

Thus, $\alpha_j(\boldsymbol{\theta})$ represents the expected value of the intercept of the aggregate demand function of j, and β_j represents the expected aggregate price sensitivity. It is thus possible to write the profit function above as follows:

$$\mathrm{E}[\pi_j(\boldsymbol{p};\boldsymbol{\theta})] = p_j(\alpha_j - a\beta_j p_j + \sum_{k=1}^{m} \hat{c}_{jk} p_k), \tag{1}$$

where $\hat{c}_{jk} = \sum_\tau \theta_\tau (\sum_i \sum_k \mu_{ik}(G_\tau) c_{ik})$, which is therefore a measure of the strength of the connection between j and k because it measures the weighted link between the sellers and shared buyers. Rescaling the profit function above by $1/b\beta_i$ and multiplying by $\frac{1}{2}$ generates the following:

$$\mathrm{E}[\tilde{\pi}_j(\boldsymbol{p};\boldsymbol{\theta})] = p_j(\boldsymbol{\theta})\tilde{\alpha}_j(\boldsymbol{\theta}) - \frac{1}{2}p_j^2(\boldsymbol{\theta}) + \sum_{k=1}^{m} \tilde{c}_{jk}(\boldsymbol{\theta}) p_j(\boldsymbol{\theta}) p_k(\boldsymbol{\theta})$$

where $\tilde{\alpha}_j(\boldsymbol{\theta}) = \frac{\frac{1}{2}\alpha_j(\boldsymbol{\theta})}{\beta_j(\boldsymbol{\theta})}$ and $\tilde{c}_{jk}(\boldsymbol{\theta}) = \frac{\frac{1}{2}\hat{c}_{jk}(\boldsymbol{\theta})}{\beta_j(\boldsymbol{\theta})}$. The maximisation problem $\max_{p_i} \tilde{\pi}_i(\boldsymbol{p};\boldsymbol{\theta})$ has the same set of first-order conditions as the one that involves maximising a vector containing the profit functions in (1). This transformation yields a *competition network*, $G_S(\boldsymbol{\theta})$, which is a projection of $G(\boldsymbol{\theta})$, where the edge between sellers j and k has the weight $\tilde{c}_{jk}$. The competition network of the probability vector $\boldsymbol{\theta}$ that generates the two graphs in Figure 3 with equal probability is shown in Figure 4 below.

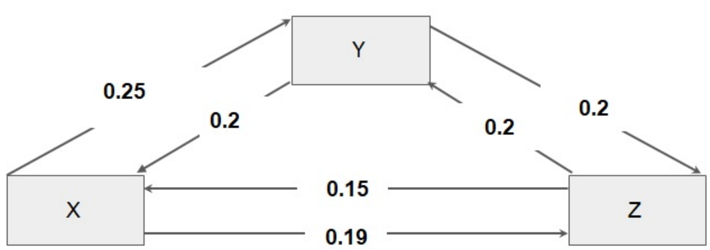

Figure 4: Transforming the network G into the competition network G_S. Here it is assumed that $b = 1$ and $c = 0.25$ for each buyer and $\gamma_j = 1$ for all j.

Define $R_S(\boldsymbol{\theta})$ as a symmetric zero diagonal matrix of a network $G_S(\boldsymbol{\theta})$ with elements $\widetilde{c}_{ij}(\boldsymbol{\theta})$.

Let $\tilde{\boldsymbol{\alpha}}(\boldsymbol{\theta})$ represent a $m \times 1$ vector with element j $\tilde{\alpha}_j$. To ensure that an equilibrium of the game is unique, it is necessary to ensure that demand is positive for all sellers.

Let γ_l denote the smallest element of the vector $\boldsymbol{\gamma}$. Throughout, we make the following assumption:

$$(A1) : a > nc\frac{\sum_{j \neq l} \gamma_j}{\gamma_l}.$$

Define $C_{\tilde{\alpha}}(\boldsymbol{\theta}) = [\boldsymbol{I} - R_S(\boldsymbol{\theta})]^{-1}\tilde{\boldsymbol{\alpha}}(\boldsymbol{\theta})$, which is the weighted transformed Bonacich centrality measure of the network G_S. The following Proposition, which characterises the equilibrium price vector, then holds:

Proposition 1. *If (A1) holds, then the game $\Gamma(\boldsymbol{\theta})$ has a unique Nash equilibrium in pure strategies, which is the equilibrium price vector:*

$$\boldsymbol{p}^*(\boldsymbol{\theta}) = \boldsymbol{C}_{\tilde{\boldsymbol{\alpha}}}(\boldsymbol{\theta}).$$

There exists a unique Nash equilibrium price vector that is equal to the Bonacich centrality of the sellers in the network $G_S(\boldsymbol{\theta})$ multiplied by $\tilde{\boldsymbol{\alpha}}(\boldsymbol{\theta})$.

The assumption (A1) provides a restriction on each $\tilde{c}_{ij}$, which is measures the substitutability of the model, relative to the effect own price has on demand, which is captured by a. Specifically, (A1) guarantees both that (a): $x_{ij}(\boldsymbol{\theta}) > 0$ for all i, j pairs and (b) that $L = \boldsymbol{I} - \lambda R_S(\boldsymbol{\theta})$ is strictly diagonally dominant for all $\boldsymbol{\theta}$, which implies that L is also positive definite. Jointly, these two facts guarantee that the Nash equilibrium of the game both exists and is unique for any graph structure.

Define $\boldsymbol{\gamma}$ as an $m \times 1$ vector with jth element γ_j. Proposition 1 implies the following result:

Corollary 1. *The unique pure-strategy Nash equilibrium of* $\Gamma(\boldsymbol{\theta})$ *is:*

$$\boldsymbol{p}^*(\boldsymbol{\theta}) = \boldsymbol{\gamma} - \frac{1}{2}C(\boldsymbol{\theta})\boldsymbol{\gamma}.$$

Hence, a seller's equilibrium price is decreasing in their centrality in $G_S(\boldsymbol{\theta})$. Sellers who are connected to more isolated consumer segments (particularly those who are captive) face relatively less competition than sellers who are largely connected to segments with different goods to choose from, and therefore are able to set a higher price in equilibrium than other sellers. The above expression implies seller j's price is increasing in γ_j but is decreasing in every other element of the vector $\boldsymbol{\gamma}$.

Returning to the example in Figures 3 and 4, Proposition 1 suggests that the centrality of each seller and their price is as follows for the parameter values specified in Figure 4:

	Centrality	**Price**
X	1.268	3.66
Y	1.272	3.64
Z	1.217	3.91

Seller Z has a uniquely connected segment with probability 0.5 and therefore has lower centrality in the competition network than X or Y. As a result, Z's price is higher than either X or Y's. Y's price is the centrality is the highest because they compete with both X and Z in the graph where every consumers does not observe every seller.

6 Decentralised search

The preceding analysis shows that the platform owner faces a trade-off between increasing sales on the one hand and reducing competition on the other. We now examine the profit maximising graph structure, taking that minimises the level of competition for a given level of expected sales.

The observation probabilities can be thought of as a measure of "prominence" in the sense that they capture the likelihood that the seller is observed by a given buyer. Seller j's prominence in the network potentially increases profits as a result of increasing the probability of sales, but at the same time it imposes a cost on the rest of the network by increasing competition, reducing prices of every seller, including for the more prominent seller.

As the number of sellers and the centrality of those sellers in the network G_S increases, increasing w_{ij} has an increasingly large effect on prices. To see this, note that the centrality vector in G_S can be expressed:

$$C(\boldsymbol{\theta})\mathbf{1} = \sum_{k=0}^{\infty} R_S^k(\boldsymbol{\theta})\mathbf{1}.$$

It follows that $\frac{\partial^2 C_k(\boldsymbol{\theta})}{\partial^2 w_{ij}} > 0$. Recall that prices are falling in the centrality of the sellers in this setting. Hence, increasing w_{ij} imposes a cost upon the platform owner because the centrality measure has a feedback effect such that increasing an observation probability w_{ij} (weakly) reduces j's price, which reduces every other seller's price, which then reduces j's price and so on. This feedback effect, which is a feature of the Bonacich centrality measure, is increasing as the centralities of the sellers in G_S become larger.

The preceding analysis implies the following result. Suppose $\tilde{w}_{ij} \sim \Lambda_1$ and let Λ_2 be a mean-preserving spread of Λ_1 such that when $\tilde{w}_{ij}^{'} \sim \Lambda_2$ constructed in the following way:

$$\tilde{w}_{ij}^{'} = \tilde{w}_{ij} + \epsilon_{ij}$$

where ϵ_{ij} is symmetrically distributed and has mean 0, and is bounded such $0 \leq \tilde{w}_{ij}^{'} \leq 1$. Let $\tilde{\boldsymbol{\theta}}_k$ denote the random probability vector generated by the distribution Λ_k. Then the following result holds:

Theorem 1. *Suppose $\gamma_j = \gamma$ $\forall j$ and $c > 0$. Then, $E[p_j(\tilde{\boldsymbol{\theta}}_1)] > E[p_j(\tilde{\boldsymbol{\theta}}_2)]$ and $E[\pi_P(\tilde{\boldsymbol{\theta}}_1)] > E[\pi_P(\tilde{\boldsymbol{\theta}}_2)]$.*

The expected number of matches (i.e. $\mathrm{E}[\sum_i \sum_j \tilde{w}_{ij}]$) is the same for both probability distributions. Hence, any differences in expected profit between the two are the result of differences in the expected price level.

As the quality vector $\boldsymbol{\gamma}$ is independent of centrality in this case, the expected price level can be denoted:

$$\mathrm{E}[\boldsymbol{p}^*(\tilde{\boldsymbol{\theta}})] = \boldsymbol{\gamma} - \frac{1}{2}\mathrm{E}[C(\tilde{\boldsymbol{\theta}})]\boldsymbol{\gamma}.$$

Suppose that $\tilde{w}_{ij} \sim \Lambda_1$ and $\tilde{w}_{ij}' \sim \Lambda_2$. Recalling that $\frac{\partial^2 C_k}{\partial^2 w_{ij}} > 0$ and that the $\tilde{w}_{ij}$s are independent of one another, then it must be the case that:

$$\mathrm{E}[C_j(\tilde{\boldsymbol{\theta}}_2)] > \mathrm{E}[C_j(\tilde{\boldsymbol{\theta}}_1)] \quad \forall j.$$

Profit is increasing and concave in price if $p_j^* \in [0, \frac{1}{2}\gamma]$, which is true for any realisation of $\boldsymbol{\theta}$. The above inequality implies that expected prices are lower in the case where each $\tilde{w}_{ij}$ is distributed according to the mean preserving contraction $\tilde{\boldsymbol{\theta}}_2$. Intuitively, this result is driven by the fact that high realisations of an observation probability $\tilde{w}_{ij}$ result in a disproportionately low price compared to low realisations of $\tilde{w}_{ij}$.

Furthermore, as observation probabilities are independent of each other, if they have a distribution of $\tilde{\boldsymbol{\theta}}_2$ it results in a higher probability that two or more sellers are prominent for a large number (or all) of the consumer segments. We refer the case where there is relatively intense competition between a subset of the sellers on the network as one in which competition is **concentrated**. There being a-higher-than-average probability that two sellers compete with one another drives their own prices down, which propagates across the network. The effect of concentrated competition is depicted in Figure 5.

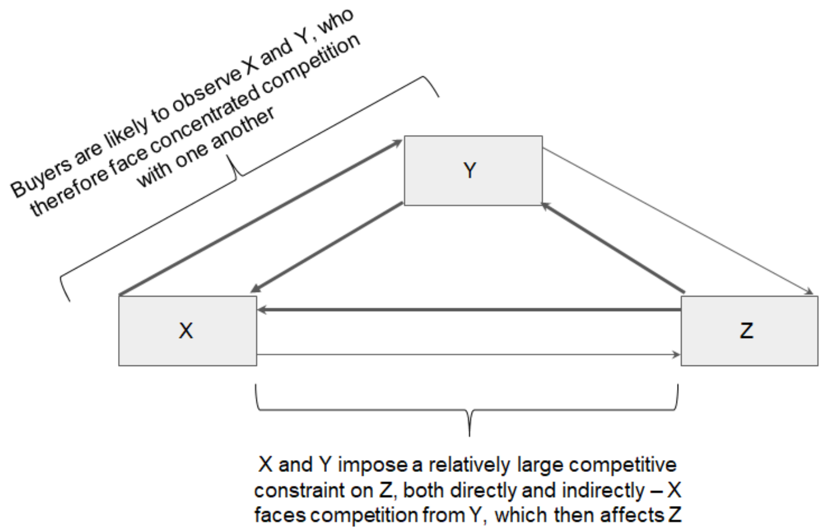

Figure 5: The prominence of X and Y results in concentrated competition between the two sellers.

Let $\Lambda_D(\upsilon)$ denote the degenerate distribution where $w_{ij} = \upsilon < 1$ for each i, j with probability 1. Theorem 1 implies the following result:

Proposition 2. *For any symmetric, continuous distribution $\Lambda \neq \Lambda_D(\upsilon)$ with mean υ and corresponding probability vector, $\tilde{\boldsymbol{\theta}}$, $E[\pi_P(\boldsymbol{\theta}_D)] > E[\pi_P(\tilde{\boldsymbol{\theta}})]$.*

From the platform owner's perspective, $\Lambda_D(\upsilon)$ is the optimal probability distribution of all symmetric, continuous distributions with mean υ. Such a probability distribution yields a bipartite, Erdos-Renyi graph with nodes m, n and link probability υ.

However, the corresponding probability vector, $\boldsymbol{\theta_D}$, generated in the case where each observation probability is equal to υ results in there being some positive probability of states in which each segment observes a large proportion of or all of the sellers in the network. For example, the complete network is realised with probability $\upsilon^{mn} > 0$ when the probability vector is $\boldsymbol{\theta_D}$.

A positive probability of the realisation of high-competition states are costly to the platform because they increase the sum of the edges emanating from most if not all sellers. As a result, these states have a relatively large effect on the centrality of each seller in the competition graph G_S. It follows that the platform owner would prefer to avoid placing any probability of the realisation of such outcomes.

7 The centralised search environment

Consumer surplus

We first characterise the networks that maximise consumer surplus. Define the expected consumer surplus of consumer segment i for a given equilibrium price vector $\boldsymbol{p^*}$ and demand vector $\boldsymbol{x_i^*}$ is as follows:

$$\mathrm{E}[\mathrm{CS}_i(\boldsymbol{x_i^*};\boldsymbol{p}^*)] = \frac{1}{2}\sum_{\tau}\theta_\tau[\sum_{j=1}^{m}\mu_{ij}(G_\tau)x_{ij}^*(G_\tau)(\gamma_j + \sum_{k=1}^{m}\mu_{ik}(G_\tau)\frac{c}{b}(p_k^* - \gamma_k) - p_j^*)].$$

Define $\mathrm{CS}(\boldsymbol{x^*};\boldsymbol{p}^*) := \sum_i \mathrm{CS}_i(\boldsymbol{x_i^*};\boldsymbol{p}^*)$, where $\boldsymbol{x^*}$ is an $m \times n$ matrix whose ijth component is $x_{ij}^*(\boldsymbol{p^*})$. As $I - \lambda R_S$ is diagonally dominant by (A1), it is clear from the above expression that the expected value of each $CS_i(\boldsymbol{x_i^*};\boldsymbol{p}^*)$ is falling in p^*. It is also straightforward to show that expected consumer surplus, ceteris paribus, is increasing in the expected number of connections in G a buyer has. Clearly then, CS is a function of $\boldsymbol{\theta}$ and can be written $\mathrm{CS}(\boldsymbol{\theta})$. Define G_c as the complete graph, in which each consumer segment observes each seller. Let $\boldsymbol{\theta}_c$ denote the probability vector in which the complete graph G_c is yielded with probability $\theta_c = 1$. The following proposition holds:

Proposition 3. *For any probability vector* $\boldsymbol{\theta} \neq \boldsymbol{\theta}_c$, $E[CS(\boldsymbol{\theta}_c)] > E[CS(\boldsymbol{\theta})]$.

The centrality of each agent is at its maximum for a given number of buyers and sellers when the network is complete. Intuitively, when the network is complete, each buyer is competed for by each seller. A complete network maximises competition, which reduces the equilibrium price level of each seller. This result is consistent with Bimpikis, Ehsani and Ilkiliç (2018), who find that a complete network maximises consumer welfare in Cournot setting if buyers' demand functions are homogeneous.[2]

Hence, despite the fact that in a Cournot model the sellers' actions are strategic substitutes, while in a Bertrand setting they are strategic complements, the driving logic in both cases is that competition reduces prices and a complete network maximises competition.

Profit maximising graphs and hidden products

Consider first the case where $c = 0$. If c is zero, then that each seller's product is not a substitute for the other goods in the market, which implies that each interaction effect parameter linking the two sellers in G_S is equal to zero for any θ. The following Proposition then holds:

Proposition 4. *Suppose $c = 0$. For any probability vector $\boldsymbol{\theta} \neq \boldsymbol{\theta}_c$, $E[\pi_P(\boldsymbol{\theta}_c)] > E[\pi_P(\boldsymbol{\theta})]$.*

When goods are non-substitutable, sellers are not in competition with one another. Prices are therefore set at the monopoly level. The platform owner then always has an incentive to increase the probability that the complete network is realised, as such a network is always more profitable than any graph in which at least one segment does not observe at least one seller. It follows that the complete network maximises seller profit, which in turn maximises the platform owner's profit.

However, despite the assumption here that demand from each consumer segment is strictly

[2] Strictly, the model of Bimpikis, Ehsani and Ilkiliç (2018) is one where firms compete for "markets" rather than consumers. They find that if markets are of the same size, which would be equivalent to homogeneous buyers in this model.

positive for any good they observe, the complete network does not necessarily maximise the platform's profit. The platform owner has an incentive to reduce the probability that sellers are observed in order to increase prices. In decreasing observation probabilities, the platform owner potentially (assuming a is sufficiently large) reduces demand. In order to maximise profit then, the platform owner must choose a network structure that maximises the expected number of sellers each consumer segment observes while accounting for the constraint that increasing observability decreases prices.

Proposition 5 formalises the above intuition:

Proposition 5. *For all $\boldsymbol{\gamma}$, there exists a $\bar{c} \in \mathbb{R}_+$ such that if $c > \bar{c}$ then in any solution to the platform owner's maximisation problem, $\boldsymbol{\theta^*}$, $\theta_c^* < 1$.*

Proposition 5 highlights the platform owner's trade-off with respect to network design. Increasing the probability that each of the seller's is observed increases profits as sales are increasing in the probability that each buyer observes each seller. At the same time, increasing sales increases competition, reducing prices. If goods are sufficiently substitutable, then the platform owner is willing to forgo some potential sales in order to increase prices.

Profit-maximing graphs with no vertical differentiation

Suppose $m \geq 2$ and $\gamma_j = \gamma$ for all j. We consider a probability vector $\boldsymbol{\theta}$ generates a competition graph in which $C_i(\theta) > C_j(\theta)$ for at least one i, j pair and show that such a probability vector can never be a solution to the platform owner's maximisation problem.

We consider the following reallocation of probabilities. Let $C_j(\boldsymbol{\theta})$ be (jointly one of) the smallest component(s) of the vector $\boldsymbol{C(\theta)1_m}$. Take a graph G which is realised with probability $\theta_G > 0$ when the probability vector is $\boldsymbol{\theta}$. Define G_{jk} as a graph which is the result of performing a neighbourhood switch between two sellers j and k in G, such that for any i where $E_{ij} \in G$ and $E_{ik} \notin G$, $E_{ij} \notin G_{jk}$ and vice versa. Such a switch is depicted in Figure 6.

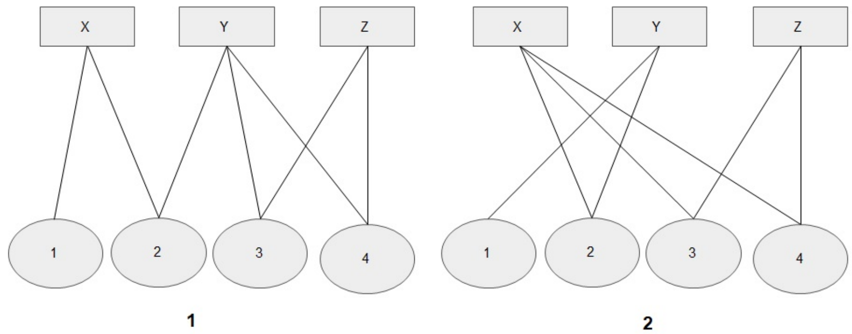

Figure 6: A neighbourhood switch between sellers X and Y.

Define a vector $\boldsymbol{\theta}_{jk}$ where the probability that the graph G_{jk} is realised is equal to θ_G, the probability that the graph G was realised in the probability vector $\boldsymbol{\theta}$. Now define another probability vector, $\hat{\boldsymbol{\theta}}_j$, as follows:

$$\hat{\boldsymbol{\theta}}_j := (1-(m-1)\varepsilon)\boldsymbol{\theta} + \sum_k \varepsilon\boldsymbol{\theta}_{jk},$$

where $\varepsilon \in \mathbb{R}_+$ is arbitrarily small.

We show that $\pi_P(\hat{\boldsymbol{\theta}}_j) > \pi_P(\boldsymbol{\theta})$ when $C_j(\theta) > C_k(\theta)$. To see this, we consider the effect of a single neighbourhood switch between j and k where $C_k(\theta) > C_j(\theta)$, holding the prices of sellers other than j and k constant.

Such a switch between j and k results in an increase in k's centrality and a decrease in j. However, as a result of the fact that $\tilde{c}_{ij}$ is convex in β_i for all $j \neq i$ and because $C_i(\theta)$ is convex in $\tilde{c}_{ij}$, the decrease in j's centrality must be larger than the increase in k's centrality. This in turn implies that k's price falls less than j's price increases.

As a result of the additional fact that profits are increasing and concave in prices below the monopoly price (which is implied by (A1)), it follows that the proposed switch will result in an increase in the total profits the platform receives from j and k. The same logic applies for any seller $C_l(\boldsymbol{\theta}) > C_j(\boldsymbol{\theta})$ and if $C_l(\boldsymbol{\theta}) = C_j(\boldsymbol{\theta})$, then the proposed reallocation has no direct effect on prices.

Now, consider the additional second-order effect of prices changing as a result of each neighbourhood switch. A change in j and k centrality affects the centrality of every other seller in the network, which also affects prices and demand. Recall that the centrality of a seller i in G_S, as per Bonacich (1972), can be written as follows:

$$C_i(\boldsymbol{\theta}) = 1 + \sum_{i \neq j} \tilde{c}_{ij} C_j(\boldsymbol{\theta}).$$

As the centrality of j is (weakly) less than every other seller, and the decrease in the centrality of k associated with a neighbourhood switch with j is (again, weakly) larger than the increase in j's centrality implies that the spillover effects associated a change from $\boldsymbol{\theta}$ to $\hat{\boldsymbol{\theta}}_j$ are profit increasing. Hence, the second-order effect of the proposed set of neighbourhood switches is positive. Theorem 2 summarises these results:

Theorem 2. *Suppose $\gamma_j = \gamma$ for all j and $m \geq 2$. Any solution, $\boldsymbol{\theta}^*$, to the platform owner's maximisation problem, induces a seller-only graph $G_S(\boldsymbol{\theta}^*)$ such that $C_j(\boldsymbol{\theta}^*) = C_k(\boldsymbol{\theta}^*)$ for all j, k pairs.*

The optimal seller-only graph structure is one in which each seller is as central as every other seller. If this is not the case, then the platform owner can always find a marginal re-allocation that increases the expected number of consumer segments observing the higher priced seller and increases prices across the network.

Theorem 2 does not fully characterise the optimal solution to the platform owner's problem. Instead, it provides a condition under which a graph G_S is the result of the platform owner's maximisation problem. However, it is possible to use the result in Theorem 2 to map the optimal set of competition graphs onto a. Again noting that $C_i(\theta) = 1 + \sum_{i \neq j} \tilde{c}_{ij} C_j(\theta)$ the Theorem implies the following corollary:

Corollary 2. *Suppose $\gamma_j = \gamma$ for all j. Any solution, $\boldsymbol{\theta}^*$, to the platform owner's maximisation problem, induces a seller-only graph $G_S(\boldsymbol{\theta}^*)$ such that $\sum_{i \neq j} \tilde{c}_{ij} = \tilde{c} \in \mathbb{R}_+$.*

For any solution to the platform owner's optimisation problem, it must be that the sum of the links a seller i has to every other seller must be equal to the same sum for another seller j. If this does not hold, it cannot be that the centralities of the sellers are equal.

Define $\sigma_{iy}(G)$ as the number of buyers for which seller i faces competition from exactly $y \in \{0, 1..., m-1\}$ sellers in the graph G. Then we can write $\sum_{i \neq j} \tilde{c}_{ij}(\theta)$ as follows:

$$\sum_{i \neq j} \tilde{c}_{ij}(\theta) = \frac{c \sum_y \sum_\tau \theta_\tau \sigma_{iy}(\tau) y}{a\beta_i}.$$

Recall that β_i is the expected number of consumer segments that observe i. Hence, $\sum_{i \neq j} \tilde{c}_{ij}(\theta)$ is equal to the expected average number of competitors i faces. Corollary 2 therefore implies that in any solution to the platform owner's maximisation problem, the average number of competitors each seller faces when active (i.e. when they are observed by at least one consumer segment) is the same for each seller.

Let $\varphi_i(G)$ denote the number of sellers consumer segment i observes in the graph G. Corollary 2 also pins down the average number of sellers buyers observe in the optimal solution, which is simply $\frac{a}{c}\tilde{c}(\boldsymbol{\theta}^*) + 1 := \hat{\varphi}(\boldsymbol{\theta}^*)$. We can show that each consumer segment's number of observations should be centered closely around this average in any optimal solution, as Theorem 3 makes clear:

Theorem 3. *Suppose $\gamma_j = \gamma$ for all j. For any solution to the platform owner's problem, $\boldsymbol{\theta}^*$, it must be the case that if $\theta^*_G > 0$ for some graph G then $\varphi_i(G) = \lfloor \hat{\varphi}(\boldsymbol{\theta}^*) \rfloor$ or $\varphi_i(G) = \lceil \hat{\varphi}(\boldsymbol{\theta}^*) \rceil$ for all i.*

A network G in which a consumer segment i observes $\hat{\varphi}(\boldsymbol{\theta}^*) + k$ (where $k > 1$) sellers has a disproportionately negative effect on profits compared with the otherwise identical network G' in which i observes $\hat{\varphi}(\boldsymbol{\theta}^*) - k$ sellers. The reason for this is that in G each of the $\hat{\varphi}(\boldsymbol{\theta}^*) + k$ sellers competes with $\hat{\varphi}(\boldsymbol{\theta}^*) + k - 1$ other sellers. Hence, the sum of links generated by i's observation in any network is convex in the number of sellers observed.

The above analysis suggests that a graph (or weighted average of two graphs) in which i

observes the average number of sellers is more profitable than a weighted mix of graphs G and $G^{'}$. In the proof of Theorem 3, we show that in the case where there is probability of graphs such as G and $G^{'}$ being generated, it is always possible to find a reallocation of probabilities such that (a) consumer segment i (and every other consumer segment) observes the same number of sellers in expectation and (b) prices increase.

Note that Theorems 2 and 3 do not show there is generally a unique vector $\boldsymbol{\theta^*}$ that solves the platform owner's maximisation problem. As consumer preferences and sellers are identical in this set up, for any solution, $\boldsymbol{\theta_1^*}$, where $\theta_c^* \neq 1$, there exists a vector, $\boldsymbol{\theta_2^*}$, where the expected number of sellers segments observe is the same, and prices are the same as in $\boldsymbol{\theta_1^*}$. This implies that $\pi_P(\boldsymbol{\theta_1^*}) = \pi_P(\boldsymbol{\theta_2^*})$.

However, we can show that while generally there is not a unique solution to the platform owner's problem, the following Proposition implies that there is a unique price vector associated with any solution to the platform owner's problem:

Proposition 6. *Suppose $\gamma_j = \gamma$ for all j. For any two solutions, $\boldsymbol{\theta_1^*}$ and $\boldsymbol{\theta_2^*}$,to the platform owner's maximisation problem, it must be the case that $\boldsymbol{p}^*(\boldsymbol{\theta_1^*}) = \boldsymbol{p}^*(\boldsymbol{\theta_2^*})$.*

We show in the proof of Proposition 6 that when preferences and seller prices are identical, the platform owner's maximisation problem amounts to choosing the overall expected number of sellers observed by consumer segments. We show that there is a unique solution to this problem, which in turn implies that the price vector for any solution to the original profit maximisation problem must generate the same price vector as another solution to that problem.

The previous discussion implies that in any optimal solution to the platform owner's problem $\boldsymbol{\theta^*}$:

- Seller prices are all equal to some price p^*;
- sellers either face $\lfloor \hat{\varphi}(\boldsymbol{\theta^*}) \rfloor - 1$ or $\lceil \hat{\varphi}(\boldsymbol{\theta^*}) \rceil - 1$ competitors for any graph G where $\theta_G^* > 0$;
- the probability that j encounters $\lceil \hat{\varphi}(\boldsymbol{\theta^*}) \rceil - 1$ competitors is the same for all j.

The platform owner maximises aggregate profit by ensuring that segments observe a fixed number of sellers that is tightly constrained around a mean determined by innate demand for the sellers' good and how substitutable those goods are.

8 Vertical differentiation

In the previous section, we considered the case in which goods are horizontally differentiated, but are of the same quality. We now consider the case where products may differ in quality, which in the model corresponds to the case where $\gamma_j > \gamma_k$ for at least one pair of sellers j and k.

To illustrate the effect of vertical differentiation on optimal network design, we first examine the case where the platform owner sets an optimal graph structure θ^* and there is a marginal increase in the quality of a single good j. Let $\boldsymbol{\gamma}$ denote a $m \times 1$ vector of seller qualities. Then the following Proposition holds:

Proposition 7. *Consider any optimal probability vector $\boldsymbol{\theta}^*(\boldsymbol{\gamma})$. The following statements hold: (i) $\frac{\partial \beta_j(\boldsymbol{\theta}^*;\boldsymbol{\gamma})}{\partial \gamma_j} \geq 0$ with the inequality strict if $\boldsymbol{\theta}^* \neq \boldsymbol{\theta_c}$; (ii) $\frac{\partial \beta_i(\boldsymbol{\theta}^*;\boldsymbol{\gamma})}{\partial \gamma_j} \leq 0 \; \forall i \neq j$; and (iii) $\frac{\partial \hat{\varphi}(\boldsymbol{\theta}^*;\boldsymbol{\gamma})}{\partial \gamma_j} \leq 0$ with the inequality strict if $\boldsymbol{\theta}^* \neq \boldsymbol{\theta_c}$.*

If the quality of j's product increases, then the platform owner has an incentive to increase the probability that j is observed, assuming that j is not observed with probability 1. This is because j's demand is increasing in γ_j for a given $\boldsymbol{p}(\boldsymbol{\theta}^*)$.

At the same time, an increase in γ_j leads to a reduction in the demand for the products of sellers directly competing with j. Hence, if a seller i competes with j in a graph realised with positive probability in the vector $\boldsymbol{\theta}^*$, then the platform owner has an incentive to reduce the probability that i is observed in order to reduce the interaction between i and j. Reducing the total probability that i is observed has the effect of increasing i's price and demand, increasing profits.

The fact that i's demand is falling in γ_j implies the platform owner has an incentive to reduce the total number of sellers who compete with j. As the probability that j is observed increases, this implies that the expected number of sellers that consumer segments observe decreases in the case where $\boldsymbol{\theta^*} \neq \boldsymbol{\theta_c}$. When $\boldsymbol{\theta^*} = \boldsymbol{\theta_c}$, then it is possible that substitutability is sufficiently low such that the complete network being realised with probability 1 is still optimal even with the increase in γ_j.

We show an example of the effect of an increase in γ_j in Figure 7.

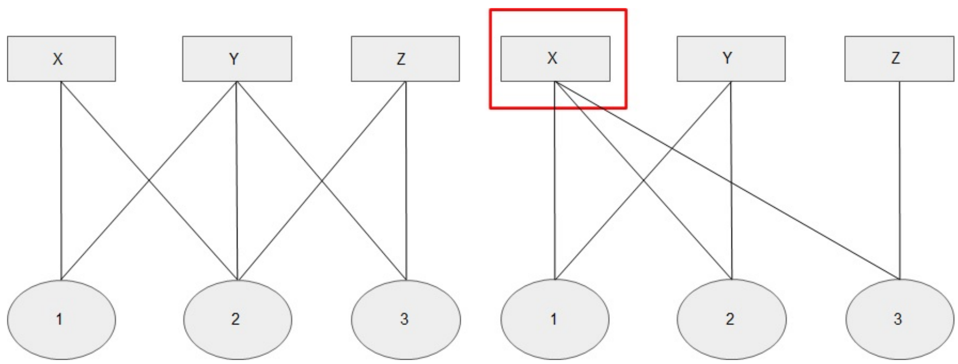

Figure 7: As γ_X increases, the platform owner increases the probability that the graph on the right-hand side is realised increases and reduces the probability that the graph on the left-hand side is realised decreases.

Now, consider the more general case where each γ_j can differ from one another. To generate a distribution of quality vectors, suppose that each $\tilde{\gamma}_j \sim \Phi$, where Φ is a symmetric and bounded probability distribution, such that the realisation of $\tilde{\gamma}_j$'s value, $\gamma_j > 0$ and $\mathrm{E}[\tilde{\gamma}_j] = \bar{\gamma}$ for all j. Suppose that the platform owner sets the vector $\boldsymbol{\theta}$ after the realisation of $\boldsymbol{\gamma}$.

Let $\tilde{\boldsymbol{\gamma}}$ denote the random quality vector associated with the case where each $\tilde{\gamma}_j \sim \Phi_i$. Suppose that if $\tilde{\gamma}_j \sim \Phi_1$ it is bounded such that $\tilde{\gamma}_j \sim [\gamma_L, \gamma_H]$. Now define Φ_2 such that when $\tilde{\gamma}_j \sim \Phi_2$, $\tilde{\gamma}_j$ can be decomposed in the following way:

$$\tilde{\gamma}_j = \tilde{\gamma}_j^{'} + \varepsilon_j,$$

where $\tilde{\gamma}_j^{'} \sim \Phi_1$ and ε_j is distributed symmetrically with mean 0 and is bounded such that

$\varepsilon_j \sim [\varepsilon_L, \varepsilon_H]$. We examine the ex-ante (i.e prior to the realisation of $\tilde{\gamma}$) profits and expected number of sellers observed by each segment when qualities are have a distribution of Φ_i in the following theorem:

Theorem 4. *i) For any value of c, $E[\pi_P(\theta^*)|\tilde{\gamma} \sim \Phi_2] \geq E[\pi_P(\theta^*)|\tilde{\gamma} \sim \Phi_1]$ and ii) $\exists c_T \in \mathbb{R}_+$ such that if $c \leq c_T$, $E[\hat{\varphi}(\theta^*)|\tilde{\gamma} \sim \Phi_1] \geq E[\hat{\varphi}(\theta^*)|\tilde{\gamma} \sim \Phi_2]$ and if $c > c_T$, $E[\hat{\varphi}(\theta^*)|\tilde{\gamma} \sim \Phi_2] > E[\hat{\varphi}(\theta^*)|\tilde{\gamma} \sim \Phi_1]$.*

To illustrate the results in Theorem 4, we consider a more limited case where under Φ_1 each $\tilde{\gamma}_j = \bar{\gamma}$ with probability 1. Consider first the claim relating to profit. If c is sufficiently small (e.g. equal to zero), then in expectation the optimal probability vector for either distribution will be such that $\theta^*_c = 1$. In this case, expected profit is the same under both distributions.

However, in the case where the platform owner restricts the number of sellers consumers observe, they are able to bias consumer observation towards high-quality products. In the case where Φ_1 results in each seller having the same quality with probability 1, this is clearly not possible, whereas the mean-preserving spread Φ_2 generates some high-quality and low-quality players in expectation. Thus, when c is sufficiently high, $\mathrm{E}[\pi_P(\theta^*)|\tilde{\gamma} \sim \Phi_2] > \mathrm{E}[\pi_P(\theta^*)|\tilde{\gamma} \sim \Phi_1]$ due to consumers being more likely to observe high-quality sellers. We depict this result in Figure 8.

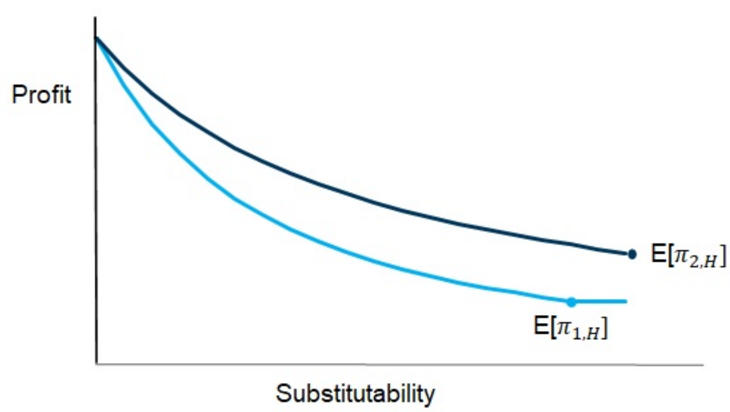

Figure 8: The expected profit when $\tilde{\gamma} \sim \Phi_2$ is weakly larger than $\tilde{\gamma} \sim \Phi_1$ for all c. $\mathrm{E}[\pi_{i,H}]$ is the expected profit associated with the seller with the highest quality being a monopolist when the distribution is Φ_i.

Now consider the second result in Theorem 4. As c increases, the number of sellers observed by consumers reduces for either distribution of qualities. However, the expected loss of a segment observing fewer sellers to platform profit is increasing more slowly in the case where is no vertical differentiation. The reason for this is that as c becomes large, the expected quality of a seller that the platform owner is marginally willing to exclude in the case where quality is dispersed becomes greater than the mean quality level, $\bar{\gamma}$.

The platform owner is less willing to exclude such high-quality sellers from being observed. Hence, when c is sufficiently large, the optimal number of sellers a segment observes is, in expectation, greater for the vertically differentiated case compared to the case where product quality is the same. This is shown in Figure 9.

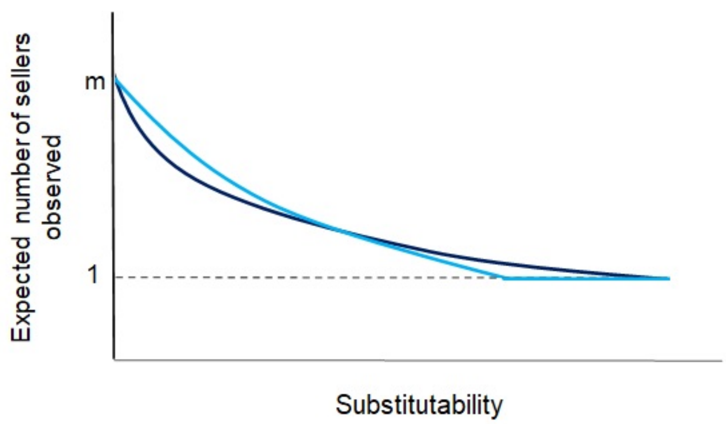

Figure 9: Dark blue line denotes the case where $\tilde{\gamma} \sim \Phi_2$, light blue where $\tilde{\gamma} \sim \Phi_1$.

9 Discussion

When a platform owner's revenue is a proportion of the profits of sellers on their platform, they have an incentive to reduce the probability that buyers observe sellers. While ensuring that a consumer segment observes a seller increases sales, it produces a cost because it reduces prices through increased competition. If product substitutability is sufficiently high, the platform owner reduces observability to maintain high prices, reducing consumer welfare.

The result that platform owners have an incentive to reduce the number of sellers observed by consumers is consistent with the observed behaviour of online platforms. A number of empirical studies (Ringer and Skiera, 2016 and Kim, Albuquerque and Bronnenberg, 2010) highlight that consumers on online platforms only observe a small subset of the total products on offer. Anecdotal evidence suggests that searching for products with a large number of results on platforms like Amazon.com does not return the entirety of the products relevant to that search.[3]

Our results explain some other observed behaviour of real-world platforms. Almost all large platforms that link buyer-sellers use a search environment that displays a consistent number of results to each consumer segment. Theorem 3 explains the observed structure of the search process on real-world platforms. Consumers, at least by default, observe a relatively small number of sellers for any given search, with more results shown on pages they have to click through to observe. Empirical evidence (Baye et al, 2009, Smith and Brynjolfsson, 2001 and Baye et al 2015) suggests that relatively few consumers click onto the second page of search results, and as such constructing the search process in this way yields a competition structure similar to the one predicted by Theorem 3.

Given differences in technical ability and ability to process information online, it might be expected that different segments were displayed a different numbers of sellers, something that does not appear to happen on most platforms. Our results show that one reason why platforms may show a consistent number of sellers to all consumer segments is that doing so minimises the level of competition for a given number of consumer-seller links, maximising prices.

Proposition 7 gives an account of the incentives platforms have in the case where there is a high-quality product in the market. The platform owner has an incentive to increase

[3] For example, searching the words "economics textbooks" into Amazon generates 20 pages of results, far fewer than the 60,000+ results that the platform claims to have available.

the probability that this seller is observed, which is consistent with platforms highlighting particular products. Perhaps less obviously, our model shows that highlighting such products increases competition, and thus incentivises the platform owner to reduce the number of sellers observed in expectation.

While platforms tend to be quite consistent in displaying a fixed subset of sellers to consumers, the number of sellers shown differs from platform to platform. Our analysis shows that the extent to which high-quality products should be showcased, the number of results displayed and the probability that a given seller is observed more generally depends on both the substitutability of products on the platform and the variation in quality.

Our analysis allows us to characterise the optimal number of sellers displayed by different types of platforms, as shown in Figure 10.

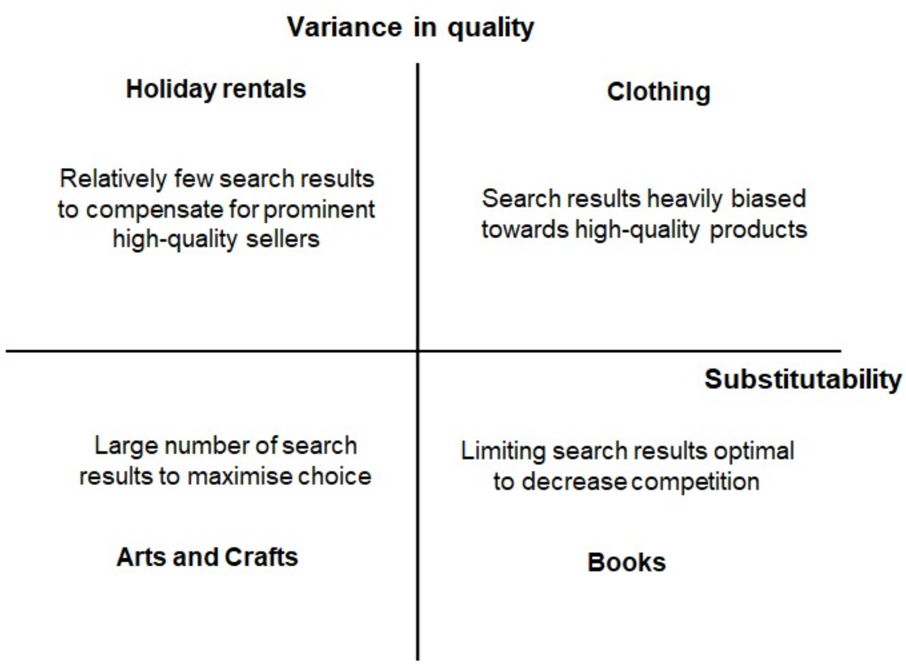

Figure 10: The optimal number of sellers on a platform depends on the market(s) they operate in. Bold text indicates an example market that has the relevant characteristics.

The optimal number of sellers in the market depends on both substitutability and variance in quality. As substitutability increases, the number of sellers in all markets reduce, but the reduction is less fast in the case where quality variance is high.

When substitutability is high, we find that platforms where products are less differentiated by quality will display relatively fewer sellers, as not showing high-quality products at all times loses less revenue than in markets where quality is highly dispersed. When substitutability is low, more quality dispersion implies that higher quality products will be displayed more, increasing the platform owner's incentive to reduce the probability that these products will be observed.

Useful future empirical work would examine the extent to which real-world platforms act in the manner predicted in the model. It would be particularly worthwhile analysing in detail the extent to which platforms from different sectors highlight particular products and the number of products displayed by the platform.

There are a few of potentially important issues left unaddressed by our analysis. We have assumed that marginal costs are zero. If sellers are heterogeneous with respect to their marginal cost, then the platform owner may have an incentive to increase the prominence of low cost sellers in the network at the expense of sellers with higher marginal costs. This would imply that some form of paid prominence could be profit maximising from the platform owner's perspective, as Armstrong and Zhou (2011) point out.

More broadly, the framework here could be used to examine the effect of entry, exit and mergers have in different parts of networked markets. For example, the Bonacich centrality vector is informative of which sellers impose the most competition on the network. A regulator or central wishing to maintain low prices would pay particular attention to such a player when performing merger control or which firms to bail out during recessions.

10 Conclusion

We analyse the case where consumers only observe a subset of sellers on a platform, which can be thought of as a bipartite observation network. The probability that an observation network is realised is determined either by nature or by the owner of the platform. Prices are set prior to the realisation of the network, but the realisation probabilities are common knowledge.

We find there is a unique, interior pure strategy equilibrium where each seller's price is falling in their Bonacich centrality in a sellers-only network that is strategically equivalent to the original bipartite network. The more central a player in this "competition" network, the more competition they face, and the lower their price.

Using the characterisation of equilibrium, it is possible to see how changes in network structure affect prices. In the decentralised search case, where observation probabilities are independent and set by nature, we find that a type of symmetric mean-preserving spread of some distribution of observation probabilities decreases profits compared to the case where probabilities are distributed according to the original distribution.

Due to the feedback effect inherent to actions determined by the sellers' Bonacich centralities in the network, a seller being more likely to be prominent results in an increase in competition that is larger than the corresponding decrease in competition associated with a reduction in prominence. Prominent sellers are more likely in the case where observation probabilities are more dispersed, and such sellers disproportionately increase competition, decreasing prices.

At the same time, our analysis of the decentralised case draws attention to the disproportionately large cost to platform profits associated with there being a positive probability that high competition states, in which consumers observe most or all sellers, are realised. This observation shapes the platform owner's incentives in the case where they can choose the observation network.

Turning to the centralised search case, at a high level we find that while consumer surplus is maximised in the case where the complete network is realised with probability 1, the platform owner has an incentive to "hide" products from consumers if substitutability is large enough.

We find that in the case where there is no vertical differentiation, the optimal seller graph is one in which each seller has the same centrality. We show that this implies that in any profit-maximising observation probability vector, the expected number of sellers observed by consumers is as close to the average number of sellers observed across all possible networks. This minimises competition for a given number of expected buyer-seller links, increasing profits.

When products are vertically differentiated, the platform owner has an incentive to increase the probability that sellers of higher quality are observed. This increases the effective competition faced by other sellers in the network, which reduces prices. To reduce the significance of this effect, the platform owner has an incentive to reduce the total number of sellers observed by consumers.

If seller quality is random, the platform owner prefers a mean-preserving spread of some distribution of quality over the original distribution. The reason for this is that the platform owner can generate more profit by biasing observation towards high-quality sellers. If products are not that substitutable, then the platform hides more sellers when quality dispersion is high, as doing so alleviates the competition effect discussed above. When product substitutability is high, fewer products are visible in either case, and the platform is less willing to hide the highest quality products when quality dispersion is high, leading to more products being optimally observed in expectation.

As platforms have an incentive to reduce competition in order to increase prices, our analysis suggests that competition authorities would be well-advised to take seriously attempts by platforms to control intra-platform competition. Regulation, insofar as it has been directed at online platforms, has tended to focus on competition between platforms. As online platforms become more established and dominant, this kind of competition becomes less relevant, and the incentives to increase prices by tweaking search algorithms or the use of private information will become increasingly important.

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11 Appendix

The demand function

We show that the linear demand curve for the game $\Gamma(\boldsymbol{\theta})$ is a simplified form of the one generated by the following demand system. Let y_i denote i's demand for a numeraire good. Suppose that i has the following quasi-linear, quadratic utility function:

$$u_i(\boldsymbol{x_i}) = \sum_{j=1}^{m} \gamma_j x_{ij} - \sum_{j=1}^{m} \kappa x_{ij}^2 - \sum_{j=1}^{m} x_{ij} (\sum_{k=1}^{m} \rho x_{ik}) + y_i$$

where $\kappa, \rho \in \mathbb{R}_+$. Suppose that each buyer has an income of l. Assuming l is sufficiently large and $a > \bar{a}$, then demand for each product is positive. Define the $m \times m$ matrix $\boldsymbol{\kappa}$ as follows:

$$\boldsymbol{\kappa} = \begin{matrix} 1 & \rho & \dots & \rho \\ \rho & \ddots & \rho & \vdots \\ \vdots & \rho & \ddots & \vdots \\ \rho & \dots & \rho & 1 \end{matrix}$$

Then, as discussed in Amir, Erikson and Jin (2015), the demand vector $\boldsymbol{x}_i$ can be written:

$$\boldsymbol{x_i} = \boldsymbol{\kappa}^{-1}(\boldsymbol{\gamma} - \boldsymbol{p}).$$

Hence, for any consumer segment i and any seller j, the intercept term of the i's demand for j's product is some constant, a, multiplied by γ and their own price sensitivity term is also equal to a.

Proof of Proposition 1

First, note that assumption (A1) guarantees that: (a) $\tilde{\alpha}_j(\boldsymbol{\theta}) > 0 \ \forall j$ and (b) $[\boldsymbol{I} - \lambda R_S(\theta)]$ is positive definite for all $\boldsymbol{\theta}$. To see the former claim, recall that:

$$\alpha_j(\boldsymbol{\theta}) := \sum_i \sum_\tau \theta_\tau[\mu_{ij}(G_\tau)(a\gamma_j - c\sum_k \mu_{ik}(G_\tau)\gamma_k)].$$

As $\beta_i > 0$, it follows that $\alpha_j(\boldsymbol{\theta}) > 0$, which implies that $\tilde{\alpha}_j(\boldsymbol{\theta}) > 0$. As $c\gamma_k > 0$, then for a given vector $\boldsymbol{\gamma}$, and given values for the parameters c and a, $\alpha_j(\boldsymbol{\theta})$ is lowest when $\boldsymbol{\theta} = \boldsymbol{\theta}_c$, where $\boldsymbol{\theta}_c$ denotes the probability vector where $\theta_c = 1$, where θ_c is the probability that the complete bipartite graph G_c is realised. It is also clear that when $\boldsymbol{\theta} = \boldsymbol{\theta_c}$, $\alpha_l(\boldsymbol{\theta}_c) < \alpha_j(\boldsymbol{\theta})$ $\forall i \neq l$. When (A1) holds, $\alpha_l(\boldsymbol{\theta}_c) > 0$, which implies $\tilde{\alpha}_l(\boldsymbol{\theta}) > 0$ for all $\boldsymbol{\theta}$.

Now note that when (A1) holds, it must be the case that $a > n(m-1)c$, as $\frac{\sum_{j\neq l}\gamma_j}{\gamma_l} \geq m-1$. This immediately implies that the $\boldsymbol{I} - \lambda R_S(\boldsymbol{\theta}_c)$ is positive definite, as it is strictly diagonally dominant. Let $\sigma_{ij}(\boldsymbol{\theta})$ denote the ijth component of $\boldsymbol{I} - \lambda R_S(\boldsymbol{\theta})$. The following trivially holds:

$$|\sum_{j\neq i} \sigma_{ij}(\boldsymbol{\theta}_c)| \geq |\sum_{j\neq i} \sigma_{ij}(\boldsymbol{\theta})| \;\; \forall i, \boldsymbol{\theta} \neq \boldsymbol{\theta}_c.$$

Hence, if the matrix $\boldsymbol{I} - \lambda R_S(\boldsymbol{\theta}_c)$ is diagonally dominant, then for any $\boldsymbol{\theta}$, the matrix $\boldsymbol{I} - \lambda R_S(\boldsymbol{\theta})$ is also diagonally dominant.

The first result that $\tilde{\alpha}_j(\boldsymbol{\theta}) > 0 \; \forall j$ guarantees that there exists a price, p'_l, such that $x_{il}(\boldsymbol{p}; \boldsymbol{\theta}_c) > 0$. Since consumer segments have identical preferences, this holds for all i, and hence at any optimal solution it must be the case that: (a) $p^*_j(\boldsymbol{\theta}) > 0 \; \forall j$ and (b) $x_{il}(\boldsymbol{p^*}; \boldsymbol{\theta}) > 0$.

It can be readily shown that the first-order condition (and therefore the resulting optimisation problem) for the payoff vector associated with the payoff described in (1) is equivalent to the first-order condition of the payoff vector associated with the original payoff function. The first-order condition of the payoff vector with individual components described in (1) is as follows:

$$\tilde{\boldsymbol{\alpha}} = [\boldsymbol{I} - \lambda R_S(\boldsymbol{\theta})]\boldsymbol{p}(\boldsymbol{\theta}).$$

As the matrix $\boldsymbol{I} - \lambda R_S(\boldsymbol{\theta})$ is positive definite, it is non-singular and the above first-order condition has a solution, which is denoted $\boldsymbol{p^*}(\boldsymbol{\theta})$. Rearranging this first-order condition leads to the expression in Proposition 1.

The first-order condition above yields a unique, interior solution. As shown above, (A1) guarantees that $p_j = 0$ cannot be a solution for any seller j's maximisation problem, as there exists a p_j such that $x_{ij}(\boldsymbol{p};\boldsymbol{\theta}) > 0$. Hence, there exists a $\varepsilon > 0$ such that $p_j = \varepsilon$ generates a strictly positive level of demand $x_{ij} > 0$. This would yield a strictly positive level of profit, which implies that j would have an incentive to deviate at the proposed equilibrium.

Proof of Corollary 1

Note that $\frac{\frac{1}{2}\alpha_j}{\beta_j} = \frac{1}{2}\gamma$ in this setting and define $\boldsymbol{\gamma}$ as a $m \times 1$ vector whose jth element is equal to γ_j. Then the right-hand side of the expression in Proposition 1 can be rewritten as follows:

$$C_\alpha(\boldsymbol{\theta}) = \frac{1}{2}\sum_{k=0}^{\infty} R_S^k(\boldsymbol{\theta})\boldsymbol{\gamma} - \sum_{k=1}^{\infty} R_{\bar{S}}^k(\boldsymbol{\theta})\boldsymbol{\gamma}$$

thus:

$$C_\alpha(G_S,\lambda) = \frac{1}{2}\boldsymbol{\gamma} + \frac{1}{2}\boldsymbol{\gamma}^{\boldsymbol{T}}\sum_{k=0}^{\infty} R_S^k(\boldsymbol{\theta}) - \sum_{k=0}^{\infty} R_S^k(\boldsymbol{\theta})\boldsymbol{\gamma}$$

$$C_\alpha(G_S,\lambda) = \frac{1}{2}\boldsymbol{\gamma} - \frac{1}{2}\boldsymbol{\gamma}^T\sum_{k=0}^{\infty} R_S^k(\boldsymbol{\theta}),$$

which implies:

$$C_\alpha(\boldsymbol{\theta}) = \frac{1}{2}\boldsymbol{\gamma} - \frac{1}{2}\boldsymbol{\gamma}^{\boldsymbol{T}}C(\boldsymbol{\theta}).$$

Proof of Theorem 1

First, note that the expected price of each seller can be written as follows:

$$\mathrm{E}[p(\boldsymbol{\theta},\boldsymbol{\gamma})] = \boldsymbol{\gamma} - \frac{1}{2}\mathrm{E}[C(\boldsymbol{\theta})]\boldsymbol{\gamma}.$$

Recall that $\tilde{w}_{ij}^{''} = \tilde{w}_{ij}^{'} + \epsilon_{ij}$, where ϵ_{ij} is a symmetric random variable. Recall also that:

$$C(\boldsymbol{\theta})\mathbf{1} = \sum_{k=0}^{\infty} R_S^k \mathbf{1}.$$

It follows then that $\mathrm{E}[C(\tilde{\boldsymbol{\theta}}_1)]$ is an increasing function of $\mathrm{E}[\tilde{w}_{ij}^{''}]$. This in turn implies that $\mathrm{E}[C(\tilde{\boldsymbol{\theta}}_2)]$ is a increasing function of $\mathrm{E}[\epsilon_{ij}^k]$ for each $k \geq 1$ for some i, j pair. Noting that ϵ_{ij} is symmetric by definition, it must be the case that $\mathrm{E}[\epsilon_{ij}^k] = 0$ when k is odd. Furthermore, $\mathrm{E}[\epsilon_{ij}^k] > 0$ when k is even.

As each element of the set of random variables $\{\epsilon_{ij}\}$ is independent of every other element of that set, it then follows that $\mathrm{E}[\epsilon_{ij}^y \epsilon_{lk}^z] = 0$ for all $y, z \geq 1$ where either $i \neq l$ or $j \neq k$. Hence, by the definition of $\tilde{w}_{ij}^{''}$, $\mathrm{E}[C(\tilde{\boldsymbol{\theta}}_2)]$ is simply a weakly increasing function of both $\mathrm{E}[\epsilon_{ij}^k]$ and $\mathrm{E}[(\tilde{w}_{ij}^{'})^k]$ for each $k \geq 1$ for all i, j pairs. Given that $\tilde{w}_{ij}^{''} = \tilde{w}_{ij}^{'} + \epsilon_{ij}$, it then follows that:

$$\mathrm{E}[C_j(\tilde{\boldsymbol{\theta}}_2)] > \mathrm{E}[C_j(\tilde{\boldsymbol{\theta}}_1)] \quad \forall j.$$

The above result immediately implies the claim that $\mathrm{E}[p_j(\tilde{\boldsymbol{\theta}}_1)] > \mathrm{E}[p_j(\tilde{\boldsymbol{\theta}}_2)]$.

Now consider the ex-ante profit function of a seller j :

$$\mathrm{E}[\pi_j(\tilde{\boldsymbol{\theta}}_2)] = \mathrm{E}[p_j(\tilde{\boldsymbol{\theta}}_2)\alpha_j] - b\mathrm{E}[\beta_j p_j^2(\tilde{\boldsymbol{\theta}}_2)] + \mathrm{E}[\sum_{k=1}^{m} \hat{c}_{jk} p_j(\tilde{\boldsymbol{\theta}}_2) p_k(\tilde{\boldsymbol{\theta}}_2)].$$

Just for the sake of argument, we first assume that the parameters α_j, β_j, and each $\hat{c}_{jk}$ are independent of the price vector $\boldsymbol{p}$. As $\mathrm{E}[\tilde{w}_{ij}^{'}] = \mathrm{E}[\tilde{w}_{ij}]$ and each element of set $\{\tilde{w}_{ij}\}$ is independent of every other element of that set, it follows that the expectation profit generated by observation probabilities with distribution Λ_2 would be lower than Λ_1. The reason for this is that: (a) $\mathrm{E}[p_j(\tilde{\boldsymbol{\theta}}_2)] < \mathrm{E}[p_j(\tilde{\boldsymbol{\theta}}_1)]$ and (b) (A1) implies that $a \geq (m-1)c$, which in turn implies profit is concave in p_j. Hence, even if $\mathrm{E}[p_j(\tilde{\boldsymbol{\theta}}_2)] = \mathrm{E}[p_j(\tilde{\boldsymbol{\theta}}_1)]$, the following inequality:

$$\mathrm{E}[p_j(\tilde{\boldsymbol{\theta}}_2)]\mathrm{E}[\alpha_j]-b\mathrm{E}[\beta_j]\mathrm{E}[p_j^2(\tilde{\boldsymbol{\theta}}_2)]+\mathrm{E}[C(\tilde{\boldsymbol{\theta}}_2)] < \mathrm{E}[p_j(\tilde{\boldsymbol{\theta}}_1)]\mathrm{E}[\alpha_j]-b\mathrm{E}[\beta_j]\mathrm{E}[p_j^2(\tilde{\boldsymbol{\theta}}_1)]+\mathrm{E}[C(\tilde{\boldsymbol{\theta}}_1)], \quad (2)$$

would still hold because $\mathrm{var}(\epsilon_{ij}) > 0$.

However, the parameters α_j, β_j and each $\hat{c}_{jk}$ are not independent of the realisation of each random observation probability as they are a function of $\tilde{w}_{ij}$ Expected demand in this environment can be written:

$$\mathrm{E}[\tilde{x}_{ij}(\boldsymbol{p}^*)] = \tilde{w}_{ij}(a\gamma - ap_j + c\sum_{j\neq k} w_{ik}(p_k - \gamma))$$

Hence, it follows that:

$$\mathrm{cov}(\tilde{x}_{ij}(\boldsymbol{p}^*), \boldsymbol{p}^*) < 0,$$

which holds both because (A1) implies that demand conditional on i observing j is falling in price and because $\mathrm{cov}(\tilde{w}_{ij}'', \boldsymbol{p}^*) < 0$. Furthermore, $|\mathrm{cov}(\tilde{w}_{ij}'', \boldsymbol{p}^*)| > |\mathrm{cov}(\tilde{w}_{ij}', \boldsymbol{p}^*)|$. A combination of this fact and the inequality in (2) then implies:

$$\mathrm{E}[\pi_j(\tilde{\boldsymbol{\theta}}_2)] > \mathrm{E}[\pi_j(\tilde{\boldsymbol{\theta}}_1)].$$

for all $c > 0$.

Proof of Proposition 2

The proof follows almost immediately from that in Theorem 1; in fact it is just a restatement of that Theorem when $\Lambda_1 = \Lambda_D$.

Proof of Proposition 3

Recalling that $C(\boldsymbol{\theta})\mathbf{1} = \sum_{k=0}^{\infty} R_S^k(\boldsymbol{\theta})\mathbf{1}$ and the expression for the equilibrium price vector, it is clear that the complete network maximises the centrality of each node in G, which then minimises the price vector $\boldsymbol{p}$ for a given m. At the same time, as (A1) holds, $x_{ij} > 0$, for each consumer i and seller j in a complete network, which implies that, holding price constant, i's expected consumer surplus is maximised where $\theta_c = 1$. Hence, consumer surplus is maximises when $\boldsymbol{\theta} = \boldsymbol{\theta_c}$.

Proof of Proposition 4

When $c = 0$, $p_j^* = \frac{1}{2}\gamma$ for all $\boldsymbol{\theta}$ and all j. As (A1) holds, it follows that for any $\boldsymbol{\theta}$ where $\theta_c < 1$, there always exists a reallocation that reduces the probability that some other graph G_i is realised and increases the probability that G_c is realised that increases aggregate demand. As equilibrium prices remain the same in this case, it follows such a reallocation is profit increases, and hence the result holds.

Proof of Proposition 5

Consider the case where $\theta_c = 1$ and let G denote a graph which is defined as follows:

$$G_c - E_{ij} = G$$

for some buyer i and the seller for whom γ_j is the smallest component in the vector $\boldsymbol{\gamma}$. Let $\theta_1 = \theta_G - \theta_c$ and $p_k^*(\theta_c) = p_k^*$. By the envelope theorem (Milgrom and Segal, 2002):

$$\frac{\partial \pi_P}{\partial \theta_1}|_{\theta_c=1} = -ap_j^*(\gamma - p_j^* + 2\sum_{k \neq j} c(p_k^* - \gamma)) + \sum_i \sum_{k \neq i} cp_i^* \frac{\partial p_k^*}{\partial \theta_1}.$$

As $(p_k^* - \gamma) < 0$, it follows that:

$$\frac{\partial^2 \pi_P}{\partial \theta_1 \partial c}|_{\theta_c=1} > 0.$$

Hence, there exists a $\bar{c}$ such that if $c > \bar{c}$, then $\frac{\partial \pi_P}{\partial \theta_1}|_{\theta_c=1} > 0$.

Proof of Theorem 2

To verify the claims in the main text, we first examine the effect of the change from $\boldsymbol{\theta}$ to $\hat{\boldsymbol{\theta}}_{jk}$, which we define as follows:

$$\hat{\boldsymbol{\theta}}_{jk} := (1-\varepsilon)\boldsymbol{\theta} + \varepsilon\boldsymbol{\theta}_{jk},$$

on the sum of the profits of j and k, holding p_i $i \neq j,k$ fixed. Note that in this case, the sum of these profits can be written:

$$\mathrm{E}[\pi_j(\boldsymbol{\theta}) + \pi_k(\boldsymbol{\theta})] = \sum_{j,k}[p_i(\theta)(\alpha_i - b\beta_i p_i(\theta) + \sum_{l\neq i}\hat{c}_{il}p_l(\theta))].$$

As shown in the main text the following inequality holds:

$$p_j(\theta) + p_k(\theta) < p_j(\hat{\boldsymbol{\theta}}_{jk}) + p_k(\hat{\boldsymbol{\theta}}_{jk}).$$

Hence:

$$\hat{c}_{jk}p_k(\theta)) + \hat{c}_{kj}p_j(\theta)) < \hat{c}_{jk}p_k(\hat{\boldsymbol{\theta}}_{jk})) + \hat{c}_{kj}p_j(\hat{\boldsymbol{\theta}}_{jk})).$$

Furthermore, given that $p_j(\boldsymbol{\theta}) < p_k(\boldsymbol{\theta})$, and that $\sum_{j,k} p_i(\boldsymbol{\theta})(\alpha_i - a\beta_i p_i(\boldsymbol{\theta}))$ is increasing and concave in $p_i \in [0, \frac{1}{2}\gamma]$, it follows that the effect of the change from $\boldsymbol{\theta}$ to $\hat{\boldsymbol{\theta}}_{jk}$ on the sum of the profits of j and k, holding p_i $i \neq j,k$ fixed is an increase in platform profits. This then then implies that the direct effect of a change in probability vector from change from $\boldsymbol{\theta}$ to $\hat{\boldsymbol{\theta}}_j$ (i.e. where the proposed set of neighbourhood switches takes place between j and every other seller in the network).

We now turn the second order effects of a the proposed reallocation when $m > 2$. By second-order effects, we refer to the effect of the proposed set of neighbourhood switches between j and every $k \neq i$ has on i's profits. Formally, we compare:

$$\sum_i \pi_i((1-(m-2)\varepsilon)\boldsymbol{\theta} + \sum_{k\neq j}\varepsilon\boldsymbol{\theta}_{jk} - \varepsilon\boldsymbol{\theta}_{ji})$$

with the sum of profits generated by $\boldsymbol{\theta}$, $\sum_i \pi_i(\boldsymbol{\theta})$. Define:

$$\Delta C_i := C_i((1-(m-2)\varepsilon)\boldsymbol{\theta} + \sum_{k\neq j} \varepsilon \boldsymbol{\theta}_{jk} - \varepsilon \boldsymbol{\theta}_{ji}) - C_i(\boldsymbol{\theta})$$

Recall that:

$$C_i(\boldsymbol{\theta}) = 1 + \sum_{l=1}^{m} \tilde{c}_{il} C_l(\boldsymbol{\theta}).$$

Two observations follow the above expression. First, as the direct effect of each neighbourhood switch between j and k increases j's centrality less than it decreases k's, it follows that $\sum_i \Delta C_i(\boldsymbol{\theta}) > 0$. Furthermore, it must also be the case that if $C_i(\boldsymbol{\theta}) \geq C_l(\boldsymbol{\theta})$ then $|\Delta C_i(\boldsymbol{\theta})| \geq |\Delta C_l(\boldsymbol{\theta})|$, $i, l \neq j$.

The two above facts imply that the sum of second-order prices changes is positive and that the prices of more central sellers increase more than the prices of less central players. Again, as $\pi_P(\boldsymbol{\theta})$ is concave and increasing in $p_i \in [0, \frac{1}{2}\gamma]$ for all i, it follows that the sum of the second-order effects of a switch from $\boldsymbol{\theta}$ to $\hat{\boldsymbol{\theta}}_j$ are profit increasing.

The above analysis then jointly implies that $\pi_P(\hat{\boldsymbol{\theta}}_j) > \pi_P(\boldsymbol{\theta})$. This implies the result: for any vector in which there exists a pair of sellers j and k such that $C_j(\boldsymbol{\theta}) > C_k(\boldsymbol{\theta})$, there is always a series of neighhbourhood switches that increases profits. Hence, any solution to the platform owner's maximisation problem must be such that $C_j(\boldsymbol{\theta}) = C_k(\boldsymbol{\theta})$ for all j, k pairs.

Proof of Theorem 3

Consider a proposed profit-maximising vector $\boldsymbol{\theta}$ in which (a) $C_j(\boldsymbol{\theta}) = C_k(\boldsymbol{\theta})$ for all j, k pairs and (b) It is true for at least one segment i that $\varphi_i(\tau) \leq \lfloor \hat{\varphi}(\boldsymbol{\theta}) \rfloor$ or $\varphi_i(\tau) \geq \lceil \hat{\varphi}(\boldsymbol{\theta}) \rceil$, with at least one of the inequalities strict, in at least one graph τ realised with probability $\theta_\tau > 0$. We also rule out that $\varphi_i(\tau) = 0$ for any seller i in any graph τ realised with probability $\theta_\tau > 0$, as this clearly suboptimal.

As each seller and segment is identical in such a graph (prices and preferences are the same across segments and all sellers), it is possible to construct the following probability vector, which yields the same profits as $\boldsymbol{\theta}$. Take a graph τ generated with probability $\theta_\tau > 0$. As before, let τ_{jk} denote the graph generated by a neighbourhood switch between two sellers j and k being performed on the graph τ. The number of potential switches between $i, j \in S$ and $k, l \in B$ is $\frac{m!}{2!(m-2)!} + \frac{n!}{n!(n-2)!} := X$.

Let $\bar{\boldsymbol{\theta}}$ denote the following probability vector. Suppose $\theta_\tau > 0$. Then the probability that τ is realised in $\bar{\boldsymbol{\theta}}$ is $\frac{\theta_\tau}{X+1}$, which is also equal to the realisation probability of each τ_{ij} for $i, j \in S$ where $i \neq j$ and each τ_{kl} for $k, l \in B$, where $k \neq l$. As $C_j(\boldsymbol{\theta}) = C_k(\boldsymbol{\theta})$ and consumers have identical preferences, $\pi_P(\bar{\boldsymbol{\theta}}) = \pi_P(\boldsymbol{\theta})$. The transformation makes it possible to show that $\boldsymbol{\theta}$ is not a solution to the platform owner's profit maximisation problem

We consider first the case where in $\boldsymbol{\theta}$,there exists a graph τ where $\varphi_i(\tau) < \lfloor \hat{\varphi}(\boldsymbol{\theta}) \rfloor$ and a graph $\tau^{'}$ in which $\varphi_j(\tau^{'}) > \lceil \hat{\varphi}(\boldsymbol{\theta}) \rceil$, where i and j may not be the same segment, and then consider afterwards the case where one of these inequalities is not strict.

Under this assumption, when the probability vector is $\bar{\boldsymbol{\theta}}$, there is a strictly positive probability that a graph τ_H will be realised, where $\varphi_i(\tau_H) > \lceil \hat{\varphi}(\boldsymbol{\theta}) \rceil$ and i observes a set of sellers S_H. There is also a strictly positive probability that a graph, $\tau_{H,L}$, is realised, where $\tau_{H,L}$ is "paired" with τ_H in the sense that i observes a set of sellers $S_{H,L} \subset S_H$ and $\varphi_i(\tau_{H,L}) < \lfloor \hat{\varphi}(\boldsymbol{\theta^*}) \rfloor$. Let $\bar{\varphi}_i = \bar{\theta}_H \varphi_i(\tau_H) + \bar{\theta}_L \varphi_i(\tau_{H,L})$,

Suppose $\tau_H^{'}$ is a graph identical to τ_H except that i observes a set of sellers $S_H^{'} \subset S_H$, such that $\varphi_i(\tau_H^{'}) = \lceil \bar{\varphi}_i \rceil$. This is ensured by deleting the edge between i and j, E_{ij}, for at least one seller j where $E_{ij} \in \tau_H$. Let $\tau_H^{''}$ be a graph identical to $\tau_H^{'}$ except that i observes a set of sellers $S_H^{''} \subseteq S_H^{'} \subset S_H$, such that $\varphi_i(\tau_H^{'}) = \lfloor \bar{\varphi}_i \rfloor$, by deleting at most one edge between i and k where $E_{ik} \in \tau_H^{'}$. Note that if $\bar{\varphi}$ is an integer, then $\tau_H^{'}$ and $\tau_H^{''}$ are identical, otherwise $S_H^{''} \subset S_H^{'} \subset S_H$.

Similarly, define $\tau_{H,L}^{'}$ as a graph identical to $\tau_{H,L}$ except that i observes the set of sellers $S_H^{'}$, such that $\varphi_i(\tau_{H,L}^{'}) = \lceil \bar{\varphi}_i \rceil$. This is ensured by adding an edge between i and j, E_{ij}, for at least one seller j where $E_{ij} \notin \tau$. $\tau_{H,L}^{''}$ is constructed in an analogous way to $\tau_H^{''}$, and hence i

observes a set of sellers $S_H^{''} \subseteq S_H^{'} \subset S_H$.

Let the constant $\eta > 0$ be such that it solves the expression $\eta\lceil\bar{\varphi}_i\rceil + (1-\eta)\lfloor\bar{\varphi}_i\rfloor = \bar{\varphi}_i$. We define the probability vector $\bar{\boldsymbol{\theta}}^{'}$ in the following way. $\bar{\theta}_i^{'} = \bar{\theta}_i$ for all graphs except the probability that τ_H and its pair $\tau_{H,L}$, $\bar{\theta}_H^{'}$ and $\bar{\theta}_L^{'}$, are realised is zero. Instead, $\eta\bar{\theta}_{H'}^{'} + (1-\eta)\bar{\theta}_{H''}^{'} = \bar{\theta}_H$ and $\eta\bar{\theta}_{L'}^{'} + (1-\eta)\bar{\theta}_{L''}^{'} = \bar{\theta}_L$ where $\bar{\theta}_{H'}^{'}$ represents the probability that the graph $\tau_H^{'}$ is realised in the probability vector $\bar{\boldsymbol{\theta}}^{'}$.

It is clear that each segment in expectation observes the same number of sellers in both $\bar{\boldsymbol{\theta}}^{'}$ and $\bar{\boldsymbol{\theta}}$. If there is a difference in profit between the two, it is driven by differences in prices. Suppose that $j \in S_H^{''}$. If it is also the case that $k \in S_H^{''}$ then $\tilde{c}_{jk}(\bar{\boldsymbol{\theta}}^{'}) = \tilde{c}_{jk}(\bar{\boldsymbol{\theta}})$. However, by construction, $S_H^{''} \subseteq S_H^{'} \subset S_H$ and there exists a $l \in S_H$ but $l \notin S_H^{''}$. It follows that $\tilde{c}_{jl}(\bar{\boldsymbol{\theta}}^{'}) < \tilde{c}_{jl}(\bar{\boldsymbol{\theta}})$. This implies that:

$$\sum \tilde{c}_{jl}(\bar{\boldsymbol{\theta}}^{'}) < \tilde{c}_{jl}(\bar{\boldsymbol{\theta}})$$

for at least one j, l pair. It follows that the centrality of j and l are lower in $\bar{\boldsymbol{\theta}}^{'}$ than in $\bar{\boldsymbol{\theta}}$. This implies that prices are higher across the network in $\bar{\boldsymbol{\theta}}^{'}$ than in $\bar{\boldsymbol{\theta}}$, which in turn implies that $\pi_P(\bar{\boldsymbol{\theta}}^{'}) > \pi_P(\bar{\boldsymbol{\theta}})$.

Now consider the case in which there is a positive probability that a graph $\tau^{'}$ where $\varphi_i(\tau^{'}) > \lceil\hat{\varphi}(\boldsymbol{\theta})\rceil$ is realised when the probability vector is $\boldsymbol{\theta}$, but there is no graph with positive realisation probability where $\varphi_j(\tau) < \lfloor\hat{\varphi}(\boldsymbol{\theta})\rfloor$. Given that $\hat{\varphi}(\boldsymbol{\theta})$ is the number of sellers observed in expectation, it must be the case that $\varphi_j(\tau) = \lfloor\hat{\varphi}(\boldsymbol{\theta})\rfloor$ for at least one j and τ pair. It follows that if the graph $\tau^{'}$ is paired with a graph in which i observes exactly $\lfloor\hat{\varphi}(\boldsymbol{\theta})\rfloor$ sellers in the way described above, the vector $\bar{\boldsymbol{\theta}}^{'}$ will still be more profitable for the platform owner than $\boldsymbol{\theta}$.

Suppose there exists a graph τ where $\varphi_i(\tau) < \lfloor\hat{\varphi}(\boldsymbol{\theta})\rfloor$, but for no graph generated with positive probability by the vector $\boldsymbol{\theta}$ is it the case that $\varphi_j(\tau^{'}) > \lceil\hat{\varphi}(\boldsymbol{\theta})\rceil$ for any $j, \tau^{'}$ pair. Mathematically, for this to hold it must be the case that $\hat{\varphi}(\boldsymbol{\theta})$ is not an integer and that

$\varphi_j(\tau^{'}) = \lceil \hat{\varphi}(\boldsymbol{\theta}) \rceil$ and $\theta_{\tau^{'}} > 0$ for at least one $j, \tau^{'}$ pair.

If $\varphi_i(\tau_H) = \lceil \hat{\varphi}(\boldsymbol{\theta}) \rceil$ and $\varphi_i(\tau_{H,L}) < \lfloor \hat{\varphi}(\boldsymbol{\theta^*}) \rfloor$, then the preceding analysis implies that $S_H^{''} \subset S_H^{'} \subset S_H$. Hence, by the same logic as the case where both original inequalities were strict, it must be true that $\vec{\boldsymbol{\theta}}^{'}$ will still be more profitable for the platform owner than $\boldsymbol{\theta}$,even in the case where no segment observes more than $\lceil \hat{\varphi}(\boldsymbol{\theta}) \rceil$ sellers.

Proof of Proposition 6

As Theorem 2 shows $p_i(\theta^*) = p_j(\theta^*)$ for all i, j. When the latter holds, the platform owner's profit function can be stated:

$$\pi_P(\boldsymbol{\theta^*}) = \sum_j p_j(\boldsymbol{\theta^*})\alpha_j(\boldsymbol{\theta^*}) - a\beta_j(\theta^*)p_j^2(\boldsymbol{\theta^*}) + \sum_{k \neq j} \hat{c}_{jk}(\theta^*)p_k^2(\boldsymbol{\theta^*}).$$

As every segment is identical in terms of preferences and prices are equal, it is possible to restate the above profit function as follows:

$$\hat{\pi}_P(\hat{\varphi}) = nb\hat{\varphi}(\gamma\hat{p}(\hat{\varphi}) - \hat{\varphi}\hat{p}^2(\hat{\varphi})) - n\hat{\varphi}c(\hat{\varphi} - 1)[\gamma - \hat{p}^2(\hat{\varphi})],$$

where $\hat{p}(\hat{\varphi})$ is the highest price level that pertains when the average number of sellers observed is $\hat{\varphi}$, which is the result of each segment observing either $\lfloor \hat{\varphi}(\boldsymbol{\theta^*}) \rfloor$ or $\lceil \hat{\varphi}(\boldsymbol{\theta}) \rceil$ sellers in any graph realised with positive probability.

Recall that when $\gamma_j = \gamma$ for all j :

$$p_i^* = \gamma(1 - \frac{1}{2}C_i(G_S(\boldsymbol{\theta}), \lambda)).$$

We can rewrite i's centrality as $\hat{C}_i(\hat{\varphi})$ as the lowest centrality that pertains when the average number of sellers observed is $\hat{\varphi}$. The second derivative of the expression $\hat{p}^2(\hat{\varphi})$ can be written:

$$\frac{\partial^2 \hat{p}^2(\hat{\varphi})}{\partial^2 \hat{\varphi}} = \gamma^2(-\hat{C}_i^{''}(\hat{\varphi}) + \frac{1}{2}(\hat{C}_i^{'}(\hat{\varphi})\hat{C}_i^{'}(\hat{\varphi}) + \hat{C}_i^{''}(\hat{\varphi})\hat{C}_i(\hat{\varphi})).$$

(A1) implies immediately that $1 > \frac{1}{2}\hat{C}_i(\hat{\varphi})$, which in turn implies that:

$$\frac{\partial^2 \hat{p}^2(\hat{\varphi})}{\partial^2 \hat{\varphi}} > 0,$$

as $\hat{C}_i''(\hat{\varphi}) < 0$ and $\hat{C}_i'(\hat{\varphi}) > 0$. It follows that:

$$\frac{\partial^2 \hat{\pi}_P(\hat{\varphi})}{\partial^2 \hat{\varphi}} < 0.$$

The above inequality shows that $\hat{\pi}_P(\hat{\varphi})$ is concave in $\hat{\varphi}$. Restating the platform owner's maximisation problem as follows:

$$\max_{\hat{\varphi}} \hat{\pi}_P(\hat{\varphi})$$

subject to the constraint that $\hat{\varphi} \leq m$. The preceding analysis then implies that there exists a $\hat{\varphi}^* \leq m$ which solves the following first order condition:

$$\frac{\partial \hat{\pi}_P(\hat{\varphi}^*)}{\partial \hat{\varphi}} = 0$$

it uniquely solves the platform's maximisation problem. Otherwise $\hat{\varphi}^* = m$ is the constrained optimum to the platform owner's problem. Either way, there exists a unique $\hat{\varphi}$ that is the solution to the maximisation problem, which immediately implies the result.

Proof of Proposition 7

Let $\boldsymbol{\beta}(\boldsymbol{\theta})$ denote the $m \times 1$ vector whose ith element is $\beta_i(\theta)$. One way of stating the platform owner's maximisation problem is as follows:

$$\max_{\boldsymbol{\beta}} \pi_P(\boldsymbol{\beta}(\boldsymbol{\theta})),$$

subject to the constraints that:

$$\theta_i \geq 0 \ \forall i,$$

and:

$$\beta_i = \beta_i(\boldsymbol{\theta^*}) \ \forall i.$$

For any optimum of the platform owner's problem, it must be the case that:

$$\frac{\partial \pi_P(\boldsymbol{\beta}(\boldsymbol{\theta^*}))}{\partial \beta_i(\boldsymbol{\theta^*})} \frac{\partial \beta_i(\boldsymbol{\theta^*})}{\partial \boldsymbol{\theta}} = 0 \ \forall i.$$

If this equality did not hold, then the platform owner would prefer to change the θ^* such that seller i is observed more or less depending on the above expression's sign. Denote the first-order conditions of the platform owner's maximisation problem as follows:

$$\pi_{\beta_i} := \frac{\partial \pi_P(\boldsymbol{\theta^*}; \gamma_j, \gamma)}{\partial \beta_i(\boldsymbol{\theta^*})} = 0.$$

To understand how π_i changes due to an increase in γ_j, we first consider a case where $\hat{\varphi}(\boldsymbol{\theta^*})$ is an integer. In this case, it possible for the platform owner to increase the probability that j is observed without increasing j's centrality. This holds because it is possible because the following result holds:

$$\sum_{k \neq j} \tilde{c}_{jk}(\theta) = \frac{c \sum_y \sum_\tau \theta_\tau \sigma_{jy}(\tau) y}{b \beta_j}.$$

In the case where y is equal for all realisable outcomes (which is true when $\hat{\varphi}(\boldsymbol{\theta^*})$ is an integer) increasing β_j has the effect of also increasing $\sum_\tau \theta_\tau \sigma_{jy}(\tau) y$ by exactly the same amount, which implies that $\sum_{k \neq j} \tilde{c}_{jk}(\theta)$ remains the same. The same also holds for every other seller: as β_j increases, β_i weakly decreases, but $\sum_\tau \theta_\tau \sigma_{iy}(\tau) y$ decreases by the same proportion.

However, recall that:

$$\boldsymbol{p}^*(\theta) = \boldsymbol{\gamma} - \frac{1}{2}C(G_S(\boldsymbol{\theta}), \lambda)\boldsymbol{\gamma}$$

This implies that $\frac{\partial p_i}{\partial \beta_j} < 0$ when $\beta_j \in [0, n]$, with the inequality strict for one seller. As β_j increases, the number of paths than begin at i and end at j increases of length $l \geq 1$ weakly increases, with the increase becoming strict as $\beta_j \to n$. As the centrality measure is weighted towards shorter paths, it follows that it must be the case that $\frac{\partial^2 p_i}{\partial^2 \beta_j} \leq 0$ for all $i \neq j$.

Now consider the case where $\hat{\varphi}(\boldsymbol{\theta^*})$ is not an integer. In this case, increasing β_j leads to an increase in the sum of seller centralities. If β_j increases such that j is relatively more likely to compete with $\lfloor \hat{\varphi}(\boldsymbol{\theta^*}) \rfloor - 1$ sellers, then directly this increases the relative probability that at least one other seller, i, will compete with $\lceil \hat{\varphi}(\boldsymbol{\theta}) \rceil - 1$ other sellers, increasing i's centrality.

If β_j increases such that the relatively likelihood seller j will compete with $\lfloor \hat{\varphi}(\boldsymbol{\theta^*}) \rfloor - 1$ sellers remains unchanged, this still increases at least one seller's centrality. The reason for this is that as fewer sellers are observed in low competition states in general, which, by the pigeonhole principle, implies that the relatively likelihood that the relative probability of at least one other seller competing with $\lceil \hat{\varphi}(\boldsymbol{\theta}) \rceil - 1$ other sellers.

It is possible that β_j increases such that the relatively likelihood seller j will compete with $\lceil \hat{\varphi}(\boldsymbol{\theta}) \rceil - 1$ increases. However, doing so will necessarily increase j's centrality, and by the proof of Theorem 2 this increase will necessarily result in an increase in to $j's$ centrality which is larger than the sum of the total decreases in other seller's centralities.

Given that $\pi_P(\boldsymbol{\theta})$ is concave and increasing in $p_i \in [0, \frac{1}{2}\gamma]$ for all i, it follows that whether $\hat{\varphi}(\boldsymbol{\theta^*})$ is an integer or not, it must be the case that $\frac{\partial^2 \pi_P(\boldsymbol{\beta}(\boldsymbol{\theta}))}{\partial^2 \beta_i} \leq 0$ for all i. Furthermore, as α_i is weakly decreasing in γ_j for $i \neq j$ and α_j is strictly increasing in γ_j, it is clear that:

$$\frac{\partial \pi_{\beta_i}}{\partial \gamma_j} \leq 0 \; \forall i \neq j$$

and:

$$\frac{\partial \pi_{\beta_j}}{\partial \gamma_j} > 0.$$

Let $\boldsymbol{H}$ denote the Hessian of the platform owner's maximisation problem:

$$H = \begin{bmatrix} \pi_{\beta_1\beta_1} & \cdots & \pi_{\beta_1\beta_m} \\ \vdots & \vdots & \vdots \\ \pi_{\beta_m\beta_1} & \cdots & \pi_{\beta_m\beta_m} \end{bmatrix}.$$

By the implicit function theorem:

$$\begin{bmatrix} \frac{\partial \pi_{\beta_1}}{\partial \gamma_j} \\ \vdots \\ \frac{\partial \pi_{\beta_m}}{\partial \gamma_j} \end{bmatrix} = -H \begin{bmatrix} \frac{\partial \beta_1(\boldsymbol{\theta^*})}{\partial \gamma_j} \\ \vdots \\ \frac{\partial \beta_m(\theta^*)}{\partial \gamma_j} \end{bmatrix}.$$

This then implies that:

$$\frac{\partial \beta_j(\boldsymbol{\theta^*})}{\partial \gamma_j} \geq 0.$$

and:

$$\frac{\partial \beta_i(\boldsymbol{\theta^*})}{\partial \gamma_j} \leq 0.$$

This result immediately implies statements (i) and (ii) in Proposition 7.

To see that (iii) holds, first assume that $\hat{\varphi}(\boldsymbol{\theta^*})$ is not an integer. Let $\theta^*_\tau > 0$ be the probability of the realisation of a graph τ in which segment i observes $\lceil \hat{\varphi}(\boldsymbol{\theta^*}) \rceil$ sellers, including j and $\theta^*_{\tau'}$ be the probability of the realisation of a graph $\tau^{'}$ identical to τ except that i observes $\lfloor \hat{\varphi}(\boldsymbol{\theta^*}) \rfloor$ sellers, also including j. Define $\theta^{'}_\tau := \theta_\tau - \theta_{\tau'}$ and:

$$\pi_{\theta'_\tau} := \frac{\partial \pi_P(\boldsymbol{\theta^*}; \gamma_j, \gamma)}{\partial \theta'_\tau} = 0.$$

This holds as otherwise it a reallocation such that increasing either θ_τ or $\theta_{\tau'}$ would be profit increasing and $\boldsymbol{\theta^*}$ would not be optimal. As $\frac{\partial^2 C_i(\boldsymbol{\theta})}{\partial^2 \tilde{c}_{jk}} > 0$ for all i, j, k, it follows that:

$$\pi_{\theta'_\tau \theta'_\tau} < 0.$$

Furthermore, as j is observed by the same number of segments in both τ and $\tau^{'}$, it follows that:

$$\pi_{\theta'_\tau \gamma_j} < 0.$$

Then, by the implicit function theorem:

$$\frac{\partial \theta^{'}_\tau}{\partial \gamma_j} = -\frac{\pi_{\theta'_\tau \theta'_\tau}}{\pi_{\theta'_\tau \gamma_j}} < 0,$$

which immediately implies that $\frac{\partial \hat{\varphi}(\boldsymbol{\theta})}{\partial \gamma_j} < 0$, as $\varphi_i(\tau) > \varphi_i(\tau^{'})$.

Now consider the case where $\hat{\varphi}(\boldsymbol{\theta})$ is an integer, then define $\tau^{''}$ as an identical graph to τ that i observes $\hat{\varphi}(\boldsymbol{\theta^*}) - 1$ sellers, including j. Exactly the same proof applies with $\theta^{''}_\tau$ in place of $\theta^{'}_\tau$ if:

$$\pi_{\theta''_\tau} := \frac{\partial \pi_P(\boldsymbol{\theta^*}; \gamma_j, \gamma)}{\partial \theta^{''}_\tau} = 0,$$

which must hold when $\hat{\varphi}(\boldsymbol{\theta^*}) < m$. If $\pi_{\theta''_\tau} < 0$, then $\boldsymbol{\theta^*}$ would not have been optimal because it would have been profit increasing to put some positive probability on $\tau^{''}$ being realised. If $\pi_{\theta''_\tau} > 0$, then it follows that there would be an incentive to put some probability mass on i observing $\hat{\varphi}(\boldsymbol{\theta^*}) + 1$ sellers. Hence, $\frac{\partial \theta^{''}_\tau}{\partial \gamma_j} < 0$ in this case,

However, if $\hat{\varphi}(\boldsymbol{\theta^*}) = m$, such a reallocation would not be possible, and hence $\pi_{\theta''_\tau} > 0$ could be consistent with optimality as the complete network would be a corner solution. Hence, $\frac{\partial \theta^{'}_\tau}{\partial \gamma_j} \leq 0$.

Proof of Theorem 4

First note that, by the envelope theorem, the following result holds:

$$\frac{\partial \mathrm{E}[\pi_P(\boldsymbol{\theta})|\Phi = \Phi_i]}{\partial \gamma_j} = \mathrm{E}[p_j^*(\theta)(\frac{\partial \alpha_j(\theta)}{\partial \gamma_j}) + \sum_i \sum_k \hat{c}_{ik} p_i^*(\theta) \frac{\partial p_k^*(\theta)}{\partial \gamma_j} - \sum_k \hat{c}_{jk} p_k^*(\theta)(\frac{\partial p_j^*(\theta)}{\partial \gamma_j})|\Phi = \Phi_i].$$

Note that $\frac{\partial \alpha_j(\theta)}{\partial \gamma_j}$ is linear in γ_j and increasing in β_j. It is also the case that $\frac{\partial p_k^*(\theta)}{\partial \gamma_j} = -\frac{1}{2}\{R_S\}_{j,k}$ for all $k \neq j$ and $\frac{\partial p_j^*(\theta)}{\partial \gamma_j} = 1 - \frac{1}{2}\{R_S\}_{jj}$, where $\{R_S\}_{ij}$ denotes the ijth component of the matrix R_S. By (A1), it follows that even in the complete network, the sum of the second and third terms of above expression are positive and linear in γ_j. For small changes in γ_j profit is approximately increasing linearly in γ_j. We will show the result holds in the linear case, and then show that for larger changes in γ_j the result must hold.

If $\mathrm{E}[\pi_P(\boldsymbol{\theta})]$ were increasing and linear in γ_j, the following statement holds:

$$\sum_i \mathrm{E}[\beta_i \tilde{\gamma}_i|\Phi = \Phi_j] \geq \sum_i \mathrm{E}[\beta_i \tilde{\gamma}_i|\Phi = \Phi_k] \leftrightarrow \mathrm{E}[\pi_P(\boldsymbol{\theta})|\Phi = \Phi_j] \geq \mathrm{E}[\pi_P(\boldsymbol{\theta})|\Phi = \Phi_k].$$

Proposition 6 indicates that $\beta_i = \beta_i(\tilde{\gamma}_i)$ where $\beta_i(.) : \mathbb{R} \rightarrow \mathbb{R}$ and $\beta_i^{'}(.) \geq 0$. The following result holds:

$$\sum_i \mathrm{E}[\beta_i(\tilde{\gamma}_i)\tilde{\gamma}_i|\Phi = \Phi_2] = \sum_i (\mathrm{E}[\beta_i(\tilde{\gamma}_i)\tilde{\gamma}_i|\Phi = \Phi_1] + \mathrm{E}[\beta_i(\tilde{\gamma}_i)\varepsilon_i]$$

It is clear that $\mathrm{E}[\beta_i(\tilde{\gamma}_i)\varepsilon_i] \geq 0$ as $\mathrm{cov}(\beta_i(\tilde{\gamma}_i), \varepsilon_i) \geq 0$ and $\mathrm{E}[\varepsilon_i] = 0$. Hence:

$$\sum_i \mathrm{E}[\beta_i(\tilde{\gamma}_i)\tilde{\gamma}_i|\Phi = \Phi_2] \geq \sum_i \mathrm{E}[\beta_i(\tilde{\gamma}_i)\tilde{\gamma}_i|\Phi = \Phi_1].$$

As stated above, $\mathrm{E}[\pi_P(\boldsymbol{\theta})]$ is not linear in γ_j. This is because j's price is linearly increasing in γ_j, and hence j's profit is a function of γ_j^2. Given that (A1) holds, $\frac{\partial^2 \mathrm{E}[\pi_P(\boldsymbol{\theta})]}{\partial^2 \gamma_j} > 0$ for all j, which in turn implies that if $\beta_i(\tilde{\gamma}_i) = \beta_j$ that profit under Φ_2 would be larger than Φ_1. This implies that:

$$\sum_i \mathrm{E}[\beta_i \tilde{\gamma}_i | \Phi = \Phi_2] \geq \sum_i \mathrm{E}[\beta_i \tilde{\gamma}_i | \Phi = \Phi_1] \to \mathrm{E}[\pi_P(\boldsymbol{\theta}) | \Phi = \Phi_j] > \mathrm{E}[\pi_P(\boldsymbol{\theta}) | \Phi = \Phi_k]$$

which is the result in (i).

With regards to (ii), we consider first the case where c increases from 0. When $c = 0$, $\theta_c = 1$ for either distribution, as per Proposition 4. Let θ_G denote the realisation probability of a graph, G, in which each segment observes every seller except that i does not observe a seller k and thus observes $m - 1$ sellers. Define $\theta_\tau := \theta_c - \theta_G$. We consider the ex post expression $\frac{\partial \pi_P(\boldsymbol{\theta^*}; \boldsymbol{\gamma})}{\partial \theta'_\tau} = \pi_{\theta_\tau}$.

When c is sufficiently low, $\pi_{\theta_\tau} \geq 0$ when $\boldsymbol{\theta} = \boldsymbol{\theta}_c$, in which case the optimal solution is $\boldsymbol{\theta}^* = \boldsymbol{\theta}_c$. It is clear that:

$$\frac{\partial \pi_{\theta_\tau}}{\partial c} < 0,$$

$$\frac{\partial \pi_{\theta_\tau}}{\partial \gamma_k} < 0,$$

and:

$$\frac{\partial |\pi_{\theta_\tau c}|}{\partial \gamma_j} > 0,$$

for some $j \neq k$. Furthermore, $\frac{\partial |\pi_{\theta_\tau c}|}{\partial \gamma_j}$ is independent of γ_k, as $\boldsymbol{p^* = \gamma - C(G_S, \lambda)\gamma}$. Abusing notation slightly, we can then write π_{θ_τ} as a function of γ_k and $\boldsymbol{\gamma_{\_k}}$, the $(m-1) \times 1$ vector of quality parameters not including k, $\pi_{\theta_\tau}(\gamma_k, \boldsymbol{\gamma_{\_k}})$.

For any $(\gamma_l, \boldsymbol{\gamma_h}) = (\gamma_k, \boldsymbol{\gamma_{\_k}})$, there exists a threshold level of c, $c'(\gamma_l, \boldsymbol{\gamma_h})$ such that if $c \geq c'(\gamma_l, \boldsymbol{\gamma_h})$ then $\pi_{\theta_\tau}(\gamma_l, \boldsymbol{\gamma_h}) \leq 0$, but if $c < c'(\gamma_l, \boldsymbol{\gamma_h})$ then $\pi_{\theta_\tau}(\gamma_k, \boldsymbol{\gamma_{\_k}}) > 0$. Note that, when $c = c^{'}(\gamma_l, \boldsymbol{\gamma_h})$, $\pi_{\theta_\tau}(\gamma_k, \boldsymbol{\gamma_{\_k}}) > 0$ if $\gamma_k > \gamma_l$ and $\boldsymbol{\gamma_{\_k}} \leq \boldsymbol{\gamma_h}$ or $\gamma_k < \gamma_h$ and $\gamma_k \geq \gamma_l$. Let $\hat{\gamma}_H = \gamma_H + \varepsilon_H$ and $\hat{\gamma}_L = \gamma_L + \varepsilon_L$, the lowest and highest possible values the quality of a seller can take when $\tilde{\gamma}_i \sim \Phi_2$. As Φ_2 is symmetric, it must be the case that:

$$\Pr(\tilde{\gamma}_j = \hat{\gamma}_H | \Phi = \Phi_2) = \Pr(\tilde{\gamma}_k = \hat{\gamma}_L | \Phi = \Phi_2) > \Pr(\tilde{\gamma}_j = \hat{\gamma}_H | \Phi = \Phi_1) = \Pr(\tilde{\gamma}_k = \hat{\gamma}_L | \Phi = \Phi_1) = 0.$$

Let $\boldsymbol{\hat{\gamma}_H}$ denote an $(m-1) \times 1$ vector with components all equal to $\hat{\gamma}_H$. When $c = c^{'}(\hat{\gamma}_L, \boldsymbol{\hat{\gamma}_H})$:

$$\Pr(\pi_{\theta_\tau} < 0 | \Phi = \Phi_2) > \Pr(\pi_{\theta_\tau} < 0 | \Phi = \Phi_1) = 0.$$

It follows that:

$$\mathrm{E}[\hat{\varphi}(\boldsymbol{\theta}^*) | c = c^{'}(\hat{\gamma}_L, \boldsymbol{\hat{\gamma}_H}), \Phi = \Phi_2] < m = \mathrm{E}[\hat{\varphi}(\boldsymbol{\theta}^*) | c = c^{'}(\hat{\gamma}_L, \boldsymbol{\hat{\gamma}_H}), \Phi = \Phi_1].$$

Suppose the largest component of $\boldsymbol{\gamma}$ is γ_h. Let G_j denote the graph in which seller h and only h is observed by every consumer segment. Let $\boldsymbol{\theta_H}$ denote the probability vector in which $\theta_h = 1$.

Let γ_s. be the largest component of the vector $\boldsymbol{\gamma_{\_k}}$. Define G_s as a graph which is identical to G_j but where i observes h and seller s, and let θ_s denote the probability that this graph is realised. Let $\theta_d = \theta_s - \theta_h$.

For a given quality vector $\boldsymbol{\gamma}$, when c is sufficiently high, $\pi_{\theta_d} < 0$ when $\boldsymbol{\theta} = \boldsymbol{\theta}_H$ and in this case $\boldsymbol{\theta} = \boldsymbol{\theta}_H$ is an optimal solution to the platform owner's problem. As before, it is clear that:

$$\frac{\partial \pi_{\theta_d}}{\partial c} < 0,$$

$$\frac{\partial \pi_{\theta_d}}{\partial \gamma_s} > 0,$$

and:

$$\frac{\partial |\pi_{\pi_{\theta_d} c}|}{\partial \gamma_k} > 0.$$

For a given γ_h, the incentive to increase θ_d is greatest when $\gamma_s = \gamma_h$. Consider the marginal effect of increasing θ_d from 0 when $\boldsymbol{\theta} = \boldsymbol{\theta}_H$, using the envelope theorem:

$$\frac{\partial \mathrm{E}[\pi_P(\boldsymbol{\theta})]}{\partial \theta_d} = a p_s^*(\boldsymbol{\theta})(\gamma_h - p_s^*(\boldsymbol{\theta})) + \hat{c}_{sh}(p_s^*(\boldsymbol{\theta})\frac{\partial p_h^*(\boldsymbol{\theta})}{\partial \theta_d} - \gamma_h) + \hat{c}_{hs}(p_h^*(\boldsymbol{\theta})\frac{\partial p_s^*(\boldsymbol{\theta})}{\partial \theta_d} - \gamma_h).$$

While $\frac{\partial p_h^*(\boldsymbol{\theta})}{\partial \theta_d} = -\{R_S\}_{hs}\gamma_h$ and $\frac{\partial p_s^*(\boldsymbol{\theta})}{\partial \theta_d} = -\{R_S\}_{sh}\gamma_h$, by (A1) the above expression is increasing in γ_h.

The fact $\frac{\partial \pi_{\theta_d}}{\partial c} < 0$ and we know there exists a values of c such that $\boldsymbol{\theta} = \boldsymbol{\theta}_c$ and that $\boldsymbol{\theta} = \boldsymbol{\theta}_H$. For a given γ_h, and assuming $\gamma_s = \gamma_h$, then there exists a $c''(\gamma_h)$ such that if $\gamma_s = \gamma_h$ and $c \leq c''(\gamma_h)$,then $\pi_{\theta_d} \geq 0$ and $c > c''(\gamma_h)$ then $\pi_{\theta_d} < 0$. The analysis above relating to $\frac{\partial \mathrm{E}[\pi_P(\boldsymbol{\theta})]}{\partial \theta_d}$ directly implies that $c''(\gamma_h)$ is increasing in γ_h.

Suppose $c = c''(\hat{\gamma}_H)$. It follows that:

$$\Pr(\tilde{\gamma}_h = \hat{\gamma}_H|\Phi = \Phi_2)\Pr(\tilde{\gamma}_s = \hat{\gamma}_H|\Phi = \Phi_2) > \Pr(\tilde{\gamma}_h = \hat{\gamma}_H|\Phi = \Phi_1)\Pr(\tilde{\gamma}_s = \hat{\gamma}_H|\Phi = \Phi_1) = 0.$$

It follows that:

$$\Pr(\pi_{\theta_d} < 0|\Phi = \Phi_2) > \Pr(\pi_{\theta_d} < 0|\Phi = \Phi_1) = 0,$$

and thus:

$$\mathrm{E}[\hat{\varphi}(\boldsymbol{\theta^*})|c = c^{''}(\hat{\gamma}_H), \Phi = \Phi_2] > 1 = \mathrm{E}[\hat{\varphi}(\boldsymbol{\theta^*})|c = c^{''}(\hat{\gamma}_H), \Phi = \Phi_1].$$

Now consider the function:

$$\hat{\varphi}^{'}(c) = \mathrm{E}[\hat{\varphi}(\boldsymbol{\theta^*})|c, \Phi = \Phi_2] - \mathrm{E}[\hat{\varphi}(\boldsymbol{\theta^*})|c, \Phi = \Phi_1].$$

The above analysis shows that there exists a $c^{'}$ where $\hat{\varphi}^{'}(c^{'}) < 0$ and a $c^{''}$ where $\hat{\varphi}^{'}(c^{''}) > 0$. Furthermore, $\mathrm{E}[\hat{\varphi}(\boldsymbol{\theta^*})|c, \Phi = \Phi_i]$ is a continuous function, and hence $\hat{\varphi}^{'}(c)$ is as well. Hence, by the intermediate value theorem, there exists a $c_T \in \mathbb{R}$ such that $\hat{\varphi}^{'}(\bar{c}) = 0$. It follows that if $c \leq c_T$, $\hat{\varphi}^{'}(c) \leq 0$ and if $c > c_T$, $\hat{\varphi}^{'}(c) > 0$.