Problems and results on linear hypergraphs

Jason Long

Trinity College
Department of Pure Mathematics and Mathematical Statistics
University of Cambridge

This dissertation is submitted for the degree of Doctor of Philosophy.

March 2019

Declaration

This dissertation is the result of my own work and includes nothing which is the outcome of work done in collaboration except as detailed below and specified in the text. It is not substantially the same as any that I have submitted, or, is being concurrently submitted for a degree or diploma or other qualification at the University of Cambridge or any other University or similar institution except as declared here and specified in the text. I further state that no substantial part of my dissertation has already been submitted, or, is being concurrently submitted for any such degree, diploma or other qualification at the University of Cambridge or any other University of similar institution except as declared here and specified in the text.

Jason Long
March 2019

- Chapter 2 is based on recent joint work with W. T. Gowers [36].
- Chapter 3 is based on recent joint work with W. T. Gowers [37].
- Chapter 5 is based on joint work with W. T. Gowers [35], and is to appear in Combinatorics, Probability and Computing.
- Chapter 6 is based on joint work with M. Bucic, B. Lidický and A. Z. Wagner [16], and is to appear in the Journal of Combinatorial Theory (Series A).
- Chapter 7 is based on joint work with A. Z. Wagner and is to appear in the European Journal of Combinatorics.
- Chapter 8 is based on joint work with B. Narayanan and is published in the Journal of Combinatorics.

Problems and results on linear hypergraphs

Jason Long

Abstract

In this thesis, we tackle several problems involving the study of 3-uniform, linear hypergraphs satisfying some additional structural constraint.

We begin with a problem of Hrushovski concerning Latin squares satisfying a partial associativity condition. From an $n \times n$ Latin square A one can define a binary operation $\circ : [n] \times [n] \to [n]$, and $\circ$ is associative if and only if A is a group multiplication table. Hrushovski asked whether, if $\circ$ is only associative a positive proportion of the time, A must still in some sense be close to a group multiplication table. This problem manifests a well-studied combinatorial theme, in which a local structural constraint is relaxed (first to a '99%' version and then to a '1%' version) and the global consequences of the relaxed constraints are analysed. We show that the partial associativity condition is sufficient to deduce powerful global information, allowing us to find within A a large subset with group-like structure. Since Latin squares can be regarded as 3-uniform, linear hypergraphs, and the partial associativity condition can be formulated in terms of the count of a particular subhypergraph, we are able to apply purely combinatorial methods to a problem that touches algebra, model theory and geometric group theory.

We then take this problem further. A condition due to Thomsen provides a combinatorial constraint which, if satisfied by the Latin square A, proves that A is in fact the multiplication table of an abelian group. It is then natural to ask whether a relaxed version of this result is also attainable, and by extending our methods we are able to prove a result of this flavour. Since the combinatorial obstructions to commutativity of $\circ$ are far more complex than those for associativity, topological complications arise that are not present in the earlier work.

We also study a problem of Loh concerning sequences of triples of integers from $[n]$ satisfying a certain 'increasing' property. Loh studied the maximum length of such a sequence, improving a trivial upper bound of n^2 to $n^2/\exp(\log^* n)$ using the triangle removal lemma and conjecturing that a natural construction of length $n^{3/2}$ is best possible. We provide the first power-type improvement to the upper bound, showing that there exists $\epsilon > 0$ such that the length is bounded by $n^{2-\epsilon}$. By viewing the triples as edges in a 3-uniform hypergraph, the increasing property shows that the hypergraph is linear and provides

further restrictions in terms of forbidden subhypergraphs. By considering this formulation, we provide links to various important open problems including the Brown–Erdős–Sós conjecture.

Finally, we present a collection of shorter results. In work connecting to the earlier chapters, we resolve the Brown–Erdős–Sós conjecture in the context of hypergraphs with a group structure, and show moreover that subsets of group multiplication tables exhibit local density far beyond what can be hoped for in general. In work less closely connected to the main theme of the thesis, we also answer a question of Leader, Milićević and Tan concerning partitions of boxes, consider a problem on projective cubes in $\mathbb{Z}_{2^n}$, and resolve a conjecture concerning a diffusion process on graphs.

To Maria and Derek, my beloved parents.

Acknowledgements

I must begin by thanking my supervisor, Tim Gowers, who has taught me everything I know about mathematical research. I am very grateful for his endless patience, guidance and support, and for the time and effort that he invested in our projects. I could not have wished for a better supervisor.

I would also like to thank my collaborators, in particular Adam Wagner and Bhargav Narayanan, who are valued friends. I also thank Imre Leader for his advice, and for interesting mathematical discussions.

I would like to thank my parents, Maria and Derek, for their love and encouragement. Without their support I would never have got this far.

Finally, I would like to thank Stacey for her love and companionship. Thank you for reducing the number of errors in this thesis, and for helping me draw some epic figures. But most of all thank you for being there for me and keeping my spirits up throughout this journey.

Contents

Chapter 1

Introduction

We begin with a brief overview of the contents of this dissertation and give a few words of introduction about each broad topic. We provide more substantial technical introductions at the beginning of each chapter.

The work is divided into two parts. The first part, containing Chapters 2, 3, 4 and 5, concerns problems roughly unified under the theme of linear, 3-uniform hypergraphs. This is the primary focus of the dissertation. In the second part we present several shorter chapters on different topics: in Chapter 6 we study a question about partitioning discrete cubes, in Chapter 7 we consider an extremal problem on set systems, and in Chapter 8 we resolve a conjecture concerning a certain graph 'diffusion' process.

1.1 Linear hypergraphs

Recall that a k-uniform hypergraph H is described by a vertex set $V(H)$ and a set $E(H) \subset \binom{V(H)}{k}$ of subsets of $V(H)$ of size k representing the edges of H. We say that H is *linear* if two distinct edges of $E(H)$ share at most one vertex.

In the first four chapters of this dissertation, the primary objects under consideration are linear, 3-uniform hypergraphs. We consider two problems with the same general profile: given that a 3-uniform hypergraph H satisfies some particular combinatorial constraint, what can we deduce about the global structure of H? Despite the underlying similarity, the problems will concern very different combinatorial constraints, resulting in very different information about H. Nevertheless, some surprising connections arise in our work, leading to a number of natural conjectures with far-reaching implications.

1.1.1 Partial associativity in Latin squares

A Latin square A is a labelling of an $n \times n$ grid from the set $[n] = \{1, \ldots, n\}$ such that every label appears precisely once in every row and column. We can represent A as a 3-uniform hypergraph H_A on $3n$ vertices, which correspond to the rows, columns and labels of the Latin square, where the edge set $E(H_A)$ is given by the set of triples (r, c, l) such that A contains label l at the point contained in row r and column c. This 3-uniform hypergraph is linear because of the constraints imposed on the labelling of A.

One way to obtain a Latin square is to take a group G of order n and write down its multiplication table. In fact, the group structure is more restrictive than necessary; we can drop the associativity requirement from G and we will still obtain a Latin square. A natural question is then the following: if G satisfies some partial associativity condition, in which we might insist, for example, that many (rather than all) triples from G satisfy the associative law, is the resulting Latin square close in some sense to a group multiplication table?

Results of this general form are common in additive combinatorics. Another example can be motivated by observing that among all subsets of $\mathbb{Z}$ of size n, those containing the maximum number of additive quadruples $x_1 + x_2 = x_3 + x_4$ are in fact arithmetic progressions, and thus highly structured. So one might naturally wonder whether arithmetic structure can still be found if the number of additive quadruples is allowed to be only within a constant factor of the maximum. This question is answered by the Balog–Szemerédi–Gowers theorem [7], which states that subsets of $[n]$ containing within a constant factor of the maximum possible number of additive quadruples contain a large subset with small doubling. When combined with Freiman's theorem [29], we obtain the desired approximate arithmetic structure. The theory of approximate groups provides yet another example, in which group properties are relaxed and questions are asked about the structure that remains [14].

In our work, we will formulate the partial associativity condition in a combinatorial way, representing Latin squares as linear hypergraphs in order to translate the whole question into a problem on hypergraphs. Associativity is measured by the count of a particular subhypergraph, the cuboctahedron, which will therefore play a central role in our work.

We will show that a Latin square satisfying the partial associativity condition does contain a large subset with group-like structure. However, the structure that we obtain is that of a 'rough approximate group' (a notion we discuss in detail in Chapter 2) rather than a group itself, and it has been recently been shown by Green [40] this is the most that one can hope for (in a sense described more precisely in Chapter 2). The work explores interesting connections between combinatorics and algebra, and we draw on tools from geometric group theory to complement well-known combinatorial methods.

1.1.2 Latin squares and abelian groups

Just as a certain associativity condition imposes a group structure on a Latin square, a number of slightly stronger conditions can be used to impose an abelian group structure instead. In Chapter 3, we extend our methods to study relaxed versions of these conditions. Our main result in this chapter will be to find an abelian analogue of the results from Chapter 2 – in other words, starting from a relaxed version of one of these stronger conditions, can we find a large subset of our Latin square that has a 'rough approximate abelian group' structure?

There are numerous motivations for pursuing this objective. Firstly, a condition due to Thomsen [85] provides a completely combinatorial condition for a Latin square to be the multiplication table of an abelian group, providing an excellent starting point for our investigation. Moreover, the concepts drawn from geometric group theory in Chapter 2 generalise in an interesting way to the new setting – obstructions to associativity and commutativity combine to introduce topological aspects to the arguments. Lastly, it seems conceivable that much more can be said about rough approximate abelian groups than their non-abelian counterparts – we will discuss this in more detail in Chapter 3.

1.1.3 Brown–Erdős–Sós in group multiplication tables

A famous open problem in extremal combinatorics is the Brown–Erdős–Sós conjecture [15]. This conjecture concerns the emergence of certain structures in dense, linear hypergraphs.

Specifically, the conjecture states that for any $k \geq 3$, if a 3-uniform hypergraph H does not contain a set of $k+3$ vertices inducing at least k edges then $|E(H)| = o(n^2)$. If true, the result is best possible in general, in the sense that the number of vertices $k+3$ cannot be reduced. Although the conjecture is stated for arbitrary 3-uniform hypergraphs, it is easily seen that we may restrict attention to linear, 3-uniform hypergraphs and Solymosi thus observed [75] that we may formulate the conjecture in terms of Latin squares.

The case $k = 3$ is the well-known Ruzsa–Szemerédi theorem [69] – a famous application of the triangle removal lemma. Despite receiving significant attention over the last four decades, this conjecture remains open for all $k \geq 4$.

Following on from our earlier work, it is natural to consider the same question in the context of group multiplication tables. As we have seen, the extra group structure imposes significant restrictions beyond those satisfied by a typical Latin square, and perhaps these restrictions may help us to find local structure.

Indeed this turns out to be the case, and in fact much more is true in the context of groups. In Chapter 4 we show that for any k, any dense subset of a sufficiently large group multiplication table contains a set of $\mathcal{O}(\sqrt{k})$ vertices inducing at least k edges, going far beyond what may be expected in general – the group structure enforces high local density.

The proof applies various powerful arithmetic results, including multidimensional variants of Szemerédi's theorem and the density Hales–Jewett theorem.

1.1.4 Increasing sequences of integer triples

We finish this part of the dissertation with a chapter concerning a rather different style of problem. Although the main object under consideration will still be a linear, 3-uniform hypergraph H satisfying some further restrictions, our objective will not be to find any kind of arithmetic structure but instead to prove a sparseness result showing that H cannot contain many edges.

The problem originates with Loh [56], who proposed it as an alternative formulation for a problem in Ramsey theory. However, our work in this dissertation is focussed on linear hypergraphs and since Loh's formulation is interesting in its own right we will not return to the Ramsey variant.

Loh's question is as follows. Given a sequence $T = (t_i)$ of triples with integer entries we say that T is 2-increasing if for all $i < j$ the triple t_i is less than t_j in at least two coordinates. What is the maximum length of a 2-increasing sequence of triples with entries from $[n]$?

It is easy to see that such a sequence T corresponds to the edge set of a linear hypergraph, and therefore the maximum length is at most n^2. Making progress beyond this trivial observation seems surprisingly difficult. Loh [56] was able to improve the upper bound to $n^2/\exp(\log^* n)$, where $\log^*$ is the iterated logarithm, by using the triangle removal lemma. However, Loh conjectures that the maximum length is $n^{3/2}$, and provides a matching lower bound by construction.

In Chapter 5, we prove that the maximum length of such a sequence is in fact at most $n^{2-\epsilon}$, for some explicit $\epsilon > 0$. The proof showcases an entirely different set of techniques that can be brought to bear on linear, 3-uniform hypergraphs – there are few ideas in common between this proof and those of earlier chapters.

Beyond this contribution to Loh's problem, we propose a number of natural generalisations and variants. In one of these generalisations, we replace the notion of 2-increasing with an alternative that we call '2-comparable', and we prove that the lower bound constructions can be slightly improved in this variant. In fact, it transpires that this 2-comparable variant is equivalent to a different, widely-studied problem known as 'Stein's tripod packing problem' [80]. Our constructions provide the best known lower bounds, improving on a sequence of results on the topic – more details about this problem are given in Chapter 5.

Finally, we explore some connections between Loh's problem and its variants and the aforementioned Brown–Erdős–Sós conjecture. By formulating the 2-increasing and 2-comparable conditions in terms of forbidden subhypergraphs, we show that improved upper bounds would follow from the resolution of certain cases of the Brown–Erdős–Sós conjecture. Along the way, we uncover some surprising links with the work in Chapter 3; the subhypergraphs forbidden by the 2-comparable condition are intimately connected with those arising in the Thomsen condition.

1.2 Other results

In the second part of this dissertation we include a number of shorter results. We give a brief overview of the topic and our contributions here, and provide more details in the introductions to the relevant chapters.

1.2.1 Partitions of discrete boxes

By the *d-dimensional discrete cube* we mean the set $[N]^d$ (for some positive integer N). By a *sub-box*, we mean a subset defined by the Cartesian product of d subsets of $[N]$, which need not be intervals. If all these subsets are proper we call the sub-box *proper*, and if the subsets are all intervals then we call the sub-box a *brick*. In Chapter 6, we study problems relating to partitions of the d-dimensional discrete cube (for large N) into proper sub-boxes.

A beautiful result of Alon, Bohman, Holzman and Kleitman [3] states that the number of proper sub-boxes required to partition the d-dimensional discrete cube is at least 2^d. The argument is probabilistic and difficult to generalise. Leader, Milićević and Tan [54] asked whether a partition of the d-dimensional discrete cube of odd volume into proper sub-boxes of odd volume in fact requires at least 3^d boxes.

In this work, we show that just 2.93^d odd boxes suffice, although for bricks (of odd volume) 3^d are necessary.

We then extend the problem, taking the restriction introduced by Leader, Milićević and Tan a step further. The 'odd volume' restriction forces each axis parallel line through discrete cube to intersect at least three members of the partition (by parity). We describe a *k-piercing* restriction in which every axis parallel line must instead intersect at least k members. For example, the partition that simply divides each dimension into k intervals provides a k-piercing partition of order k^d. We prove that k-piercing partitions can be surprisingly small – in particular, for any fixed k we can find k-piercing partitions of size bounded by kc^d for a fixed constant c.

Finally, inspired by the 'magical' proof of Alon, Bohman, Holzman and Kleitman which dispatched the original question but seems to resist all generalisation, we finish with a number of conjectures and open problems on this topic.

1.2.2 Projective cubes in $\mathbb{Z}_{2^n}$

A typical problem in extremal combinatorics is to characterise set families over X that contain the minimum number of a particular substructure. The Boolean lattice $\mathbb{Z}_2^n$ is perhaps the most traditional choice for the set X in such problems. From the perspective of our work in this area, a relevant example is a theorem of Samotij [71] which states that families in the Boolean lattice which contain the minimum number of chains must have a very specific layered structure.

The set $\mathbb{Z}_{2^n}$ also admits a natural layered structure. We may take the odd integers as the first layer, the integers congruent to 2 modulo 4 as the second layer, those congruent to 4 modulo 8 as the third and so on. Samotij and Sudakov [72] asked whether among subsets of $\mathbb{Z}_{2^n}$ of given size M, the sets that minimize the number of Schur triples (triples (a, b, c) such that $a + b = c$) are those obtained by filling up the largest layers consecutively.

We say that B_d is a projective cube of dimension d if there are numbers $a_1, a_2, \ldots, a_d$ such that

$$B_d = \left\{ \sum_{i \in I} a_i \,\middle|\, \emptyset \neq I \subseteq [d] \right\}.$$

In Chapter 7, we work in $\mathbb{Z}_{2^n}$ rather than the Boolean lattice. Motivated by the above conjecture of Samotij and Sudakov, we show that several statements analagous to those concerning chains in the Boolean lattice hold in $\mathbb{Z}_{2^n}$ if one replaces the notion of chains by that of projective cubes. We also prove the first non-trivial case of the Samotij–Sudakov conjecture, and provide a number of open problems and conjectures.

1.2.3 Diffusion on graphs

In Chapter 8 we study a graph process introduced by Duffy, Lidbetter, Messinger and Nowakowski [23] called *diffusion*. This process is closely related to a class of single-player games known as *chip-firing processes*.

In chip-firing, we start with a graph G on vertex set $V = \{v_i\}$, and a collection $\{w(v_i)\}$ of stacks of chips on each vertex ($w(v_i)$ is the number of chips on vertex v_i). If a vertex has at least as many chips as neighbours in G we may *fire* the vertex, by passing exactly one chip from that vertex to each neighbour. This simple process leads to remarkable complexity of behaviour, as illustrated by the canonical example of chip-firing: the abelian sandpile model [11].

In diffusion on a graph, we do not individually 'fire' vertices. Instead, we evolve through successive time steps, and at each new step each vertex simultaneously fires one chip to each of its neighbours with fewer chips. This firing rule may result in negative labels, since a vertex may have a smaller number of chips than of neighbours with fewer chips.

As a result diffusion, unlike the parallel chip-firing game, is not obviously periodic. Nevertheless, in 2016, Duffy, Lidbetter, Messinger and Nowakowski conjectured that diffusion is always eventually periodic, and moreover, that the process eventually has period either 1 or 2. We verify this conjecture.

Chapter 2

Partial associativity and rough approximate groups

This chapter is based on recent joint work with W. T. Gowers [36].

2.1 Introduction

The following statement is a known result in additive combinatorics. Let n be a prime and write $\mathbb{Z}_n$ for $\mathbb{Z}/n\mathbb{Z}$. Let $A \subset \mathbb{Z}_n$ and let $\phi : A \to \mathbb{Z}_n$ be a map such that the number of quadruples $(a, b, c, d) \in A^4$ with $a + b = c + d$ and $\phi(a) + \phi(b) = \phi(c) + \phi(d)$ is at least αn^3. Then there is a subset $A' \subset A$ of size at least βn, where β depends on α only, such that $\phi(a) + \phi(b) = \phi(c) + \phi(d)$ whenever $a, b, c, d \in A'$ and $a + b = c + d$. A map with this last property is called a *Freiman homomorphism*, so this result is saying that a map that obeys the condition for a Freiman homomorphism a constant fraction of the time can be restricted to a dense set that obeys the condition all the time. One can then go further and show that ϕ agrees on a further dense subset with the restriction of a 'linear-like' function, which gives a global structural characterization of functions that satisfy the initial local conditions.

There are several results of this general flavour, and the purpose of this chapter is to prove another one. Here our starting point is a binary operation $\circ$ defined on a finite set X. We assume that it is a bijection in each variable separately, and that there exists a constant $c > 0$, independent of the size of X, such that the number of triples $(x, y, z) \in X^3$ with $x \circ (y \circ z) = (x \circ y) \circ z$ is at least $c|X|^3$. It is easy to see that if $c = 1$ then these conditions are equivalent to the group axioms, so it is natural to ask whether if a binary operation has this property for some smaller c, then there must be some underlying group structure that 'explains' the prevalence of associative triples. This question appears to have been known to various people – we heard about it from Emmanuel Breuillard, who attributed it to Ehud Hrushovski, and an essentially equivalent question arose out of work we ourselves had been doing – but it does not seem to have appeared in print.

The '99% case' was dealt with by Elad Levi [55], who proved that if c is close to 1, then there must be a group G of size approximately equal to $|X|$ and an injection $\phi : X \to G$ such that $\phi(x \circ y) = \phi(x)\phi(y)$ for almost all pairs $x, y \in X^2$. In other words, the multiplication table of $\circ$ agrees almost everywhere with a group operation. In this chapter we look at the '1% case' – that is, the case where c is a small fixed constant. We also weaken the hypothesis in a small way by considering binary operations that are only partially defined: this has no significant effect on our arguments, but it is convenient when discussing examples not to have to worry about whether they are defined everywhere. In the discussion that follows, we shall often use the word 'operation' to mean 'partial binary operation'.

An easy way to create an operation with many associative triples is to take the operation $\circ$ on a group G and turn it into a partial binary operation by restricting it to a dense subset $X \subset G^2$. This is not guaranteed to work, as there are not necessarily $c|G|^3$ triples $(x, y, z) \in G^3$ such that all of $(x, y), (y, z), (x, y \circ z)$ and $(x \circ y, z)$ belong to X. However, in many cases, such as when A is a random subset, it does. More generally, given any operation with many associative triples, one can find restrictions that still have many associative triples.

Another method is to take a subset A of a group G and restrict the group operation $\circ$ to all pairs $(a, b) \in A^2$ such that $a \circ b \in A$. Again, this is not guaranteed to work, but if A is an *approximate subgroup*, which roughly speaking means that it is closed '1% of the time' (we shall discuss this condition in more detail in a moment), then this gives another source of examples.

A third method is based on structures that are approximately groups in a metric sense. For concreteness, we discuss a specific example. Let $\delta > 0$ and let X be a maximal δ-separated subset of $\mathrm{SO}(3)$. Now define a partial binary operation as follows. Let $\theta > 0$ be a suitable absolute constant (as opposed to δ, which is comparable to $|X|^{-1/3}$) and then for $x, y, z \in X$ let $x \circ y = z$ if and only if $d(xy, z) \leq \theta\delta$. It is possible to show that however X is chosen there will necessarily be many associative triples – this argument is omitted from this thesis, but can be found in an appendix to the paper version [36]. However, there is no obvious way of passing to a subset of X^2 where the operation is isomorphic to a restriction of a group operation. Indeed we conjectured that there was no such subset, and that conjecture has been proved by Ben Green [40].

This example shows that a natural conjecture – that a partially associative binary operation agrees on a substantial set of pairs with a group operation – is false. However, the example has a suggestive structure that suggests an appropriate weakening of the conjecture. Our main result will be that if an operation has many associative triples (and is injective in each variable separately), then it agrees on a large set of pairs with a restriction of a small perturbation of the binary operation on a metric group.

The next theorem is not in fact our main theorem, but a consequence of it. However, to state the main theorem requires one more definition, so we shall state this result first. Loosely speaking, it says that the multiplication table of a partial binary operation with many associative triples must be approximately isomorphic to part of the multiplication table of a metric group G. The precise statement is as follows.

Theorem 2.1. *Let $c > 0$, let X be a finite set and let $\circ$ be a partially defined binary operation on X that is injective in each variable separately. Let E denote the subset of X^2 on which $\circ$ is defined. Suppose that there are at least $\epsilon|X|^3$ triples $(x, y, z) \in X^3$ such that $x \circ (y \circ z) = (x \circ y) \circ z$ (where this means in particular that all expressions and subexpressions are defined). Then for every positive integer b there exist $\delta(\epsilon, b) \geq \epsilon^{b^{26b}}$, a subset $A \subset E$ of density at least δ, a metric group G, and maps $\phi : X \to G$, $\psi : X \to G$ and $\omega : X \to G$, such that the images $\phi(X), \psi(X)$ and $\omega(X)$ are 1-separated, and $d(\phi(x)\psi(y), \omega(z)) \leq b^{-1}$ for every $(x, y, z) \in X^3$ such that $(x, y) \in A$ and $x \circ y = z$.*

2.1.1 Quasigroups, the quadrangle condition, torsors, and our main theorem.

Our main result will have the same conclusion as that of Theorem 2.1 but a hypothesis that is both weaker and in some ways more natural. It arises out of the following simple question: suppose that an $n \times n$ grid is filled with labels. Under what conditions is this labelled grid the multiplication table of some group?

We can ask the question more formally as follows. Suppose we are given three sets X, Y and Z with $|X| = |Y| = n$, and a function $f : X \times Y \to Z$. Under what conditions does there exist a group G of order n and bijections

$\phi : X \to G, \psi : Y \to G$ and $\omega : Z \to G$ such that for every $(x, y) \in X \times Y$ we have $\phi(x)\psi(y) = \omega(f(x, y))$?

To discuss this, we use the following vocabulary. We call the elements of Z *labels*, sets of the form $\{x\} \times Y$ *columns* and sets of the form $X \times \{y\}$ *rows*. If $f(x, y) = z$, we say that z is the label in position (x, y). A very obvious necessary condition is that Z should also have cardinality n. Another is that each label occurs exactly once in each row and each column.

A labelling of an $n \times n$ grid that satisfies these two conditions is known as a *Latin square*. If we think of the labelled grid as the multiplication table of the binary operation f, then it has the property that for each $x \in X$ the function $y \mapsto f(x, y)$ is a bijection from Y to Z, and for each $y \in Y$ the function $x \mapsto f(x, y)$ is a bijection from X to Z. If we identify the sets X, Y and Z (using arbitrary bijections) and write $x \circ y$ instead of $f(x, y)$, then we have a set X with a binary operation $\circ$ with the property that for every $a, b \in X$ the equations $a \circ x = b$ and $x \circ a = b$ have unique solutions. Such an algebraic structure is called a *quasigroup*. (Thus, quasigroups and Latin squares are essentially the same.)

The question now becomes the following: when is a quasigroup a group? Equivalently, when is a Latin square the multiplication table of a group?

It is important to clarify exactly what this question is asking. When we are presented with the Latin square, we are *not* given any correspondences between rows, columns and labels. Rather, we are given an arrangement of labels and asked to *find* correspondences in such a way that the resulting binary operation is a group operation.

Suppose that x_1, x_2, y_1, y_2 are elements of a group G and $x_1y_1 = a, x_2y_1 = b$ and $x_1y_2 = c$. Then $x_2y_2 = ba^{-1}c$. This simple observation tells us that if a group multiplication table ever contains a configuration of the following form,

$$\begin{array}{cccc} c & d & & \\ a & b & & \\ & & c & d' \\ & & a & b \end{array}$$

then $d = d'$. This condition is called the *quadrangle condition*. To put it a different way, we can define a ternary *rectangle completion* operation on the set of labels by mapping (a, b, c) to d whenever there exists a rectangle with labels a, b, c, d such that a is in the same row as b and the same column as c. If the Latin square is a group multiplication table, then this ternary operation is well-defined.

It turns out that the converse is true as well: a Latin square that satisfies the quadrangle condition is the multiplication table of a group. This is a well-known observation of Brandt [13]. Since the proof is short, we give it here.

Proposition 2.2. *Every Latin square that satisfies the quadrangle condition is the multiplication table of a group.*

Proof. Choose an arbitrary row R and column C and define a binary operation $\circ$ on the set of labels as follows. Given labels a and b, find where a appears in row R and where b appears in column C, and then let $a \circ b = c$, where c is the label of the point in the same column as a and the same row as b. The label of the point where R and C intersect is then an identity for $\circ$, and the Latin square condition implies that every element has both a left and a right inverse. It remains to check associativity. To do this, consider the following picture, which is of a portion of the Latin square, chosen to demonstrate that $(a \circ b) \circ c = a \circ (b \circ c)$. We write d for $a \circ b$, f for $d \circ c$, g for $b \circ c$, h for $a \circ g$, and e for the identity.

$$\begin{array}{cccc} g & h & & \\ c & & g & f \\ b & d & & \\ e & a & b & d \end{array}$$

For associativity we need f to equal h. But this follows from the quadrangle condition, since included in the above diagram are the points

$$\begin{array}{cccc} g & h & & \\ & & g & f \\ b & d & & \\ & & b & d \end{array}$$

Thus, the set of labels has an associative binary operation with an identity such that every element has a left and a right inverse, and we are done. □

A notable feature of the above argument is the arbitrary choice of the row R and the column C, and hence the arbitrary choice of which label would serve as the identity element. It shows that if we are presented just with the labelled grid and not with any correspondences between rows, columns and labels, then there is no way of telling which label corresponds to the identity. Another way of expressing this observation is to say that if G is a group and x is any element of G, then we can form a group G_x with identity element x by taking the binary operation $a \circ b = ax^{-1}b$, which is derived from the rectangle-completion operation discussed above.

If one wishes to avoid the artificiality of choosing an arbitrary element to be the identity, one can do so by working with an algebraic structure known as a *torsor*, which can be thought of as a group 'except that we do not know which element is the identity'. The formal definition of a torsor is that it is a set X with a ternary operation τ, where $\tau(x, y, z)$ should be thought of as $xy^{-1}z$, which has the following two properties.

- $\tau(x, x, y) = \tau(y, x, x) = y$ for every $x, y \in X$;
- $\tau(x, y, \tau(z, u, v)) = \tau(\tau(x, y, z), u, v)$ for every $x, y, z, u, v \in X$.

The relationship between groups and torsors is closely analogous to the relationship between vector spaces and affine spaces, and the ternary map is also closely analogous to the (partially defined) map $(a, b, c) \mapsto a - b + c$ that often appears in additive combinatorics when one has a set A with additive structure that is not 'centred on zero'.

Let us now turn our attention to *partial* Latin squares – that is, to grids that are partially labelled in such a way that no label occurs more than once in any row or column. We can define a partial Latin square formally as a quintuple (X, Y, Z, A, ϕ), where X, Y, Z are finite sets, $A \subset X \times Y$, and $\phi : A \to Z$ is a function such that if $\phi(a, b_1) = \phi(a, b_2)$ then $b_1 = b_2$, and if $\phi(a_1, b) = \phi(a_2, b)$, then $a_1 = a_2$. Given a partial Latin square (X, Y, Z, A, ϕ) with $|X| = |Y| = |Z| = n$, we will sometimes abuse notation and say that A is an $n \times n$ partial Latin square (or simply that A is a partial Latin square). If (X, Y, Z, A, ϕ) is a partial Latin square and $B \subset A$, we may also refer to the partial Latin square $(X, Y, Z, B, \phi|_B)$ as B, calling it simply a subset of A (if it is clear from context that both objects are partial Latin squares).

With the above observations in mind, it is natural to formulate a torsor version of the question about binary operations with many associative triples. For reasons that will become clear in the next section, let us call a pair of identically labelled rectangles a *cuboctahedron*. (We allow degeneracies in the definition – for example, the two rectangles might be equal, or the rectangles themselves might each consist of two points repeated twice.) That is, a cuboctahedron is a configuration that looks like this (where we have chosen the example to emphasize that there is no ordering on X or Y, so all we care about is the relations 'is in the same column as', 'is in the same row as', and 'has the same label as').

$$\begin{array}{cccccc}
c & & d & & & \\
& a & & & b & \\
a & & b & & & \\
& & & & & \\
& c & & & d &
\end{array}$$

If $|X| = |Y| = |Z| = n$, then the maximum number of cuboctahedra there can be is n^5. To see this, note that the number of rectangles is n^4, and if one wishes to find another rectangle with the same labelling, then there are at most n choices for the first vertex (since its label is determined) and at most one choice for each remaining vertex (since their labels are determined, as well as at least one of their row and column). So the obvious hypothesis to consider is that the number of cuboctahedra is at least cn^5, where $c > 0$ is a constant independent of n.

We now show that the multiplication table of a binary operation with many associative triples contains many cuboctahedra.

Lemma 2.3. *Let X be a set of size n and let $\circ$ be a partially defined binary operation on X that is injective in each variable separately and for which there are at least ϵn^3 triples $(x, y, z) \in X^3$ with $x \circ (y \circ z) = (x \circ y) \circ z$. Then the multiplication table of $\circ$ contains at least $\epsilon^4 n^5$ cuboctahedra.*

Proof. For each $b \in X$, let W_b be the set of (a, c) such that $a \circ (b \circ c) = (a \circ b) \circ c$. Then the average size of $|W_b|$ is at least ϵn^2. Writing ϵ_b for the density of W_b in X^2, an easy Cauchy-Schwarz argument tells us that W_b contains at least $\epsilon_b^4 n^4$ quadruples (a_0, a_1, c_0, c_1) such that all four points (a_i, c_j) belong to W_b. Therefore, by Jensen's inequality, the average number of such quadruples in W_b is at least $\epsilon^4 n^4$. Each such quadruple yields a diagram of the following form.

$$\begin{array}{cccccc}
g_1 & f_{01} & f_{11} & & & \\
g_0 & f_{00} & f_{10} & & & \\
c_1 & & & g_1 & f_{01} & f_{11} \\
c_0 & & & g_0 & f_{00} & f_{10} \\
b & d_0 & d_1 & & & \\
& & & & & \\
\circ & a_0 & a_1 & b & d_0 & d_1
\end{array}$$

where the left column and bottom row say which elements are being multiplied together. The associativity of the triples (a_i, b, c_j) is used to prove that $a_i \circ (b \circ c_j) = (a_i \circ b) \circ c_j = f_{ij}$, and the result is that each quadruple of triples gives us a cuboctahedron. Note that from the cuboctahedron we can reconstruct the pairs (a_0, d_0) and (a_1, d_1) from looking at which columns are used, and since the equation $a_0 x = d_0$ has a unique solution, we can reconstruct b. Therefore, distinct b give rise to distinct cuboctahedra, and putting all this together implies that there are at least $\epsilon^4 n^5$ cuboctahedra, as claimed. $\square$

Thus, the hypothesis that we wish to consider is a weakening of the hypothesis of Theorem 2.1. Our main result is that we can obtain the same conclusion.

Theorem 2.4. *Let X, Y, Z be sets of size n, let $E \subset X \times Y$, and let $\lambda : E \to Z$ be a partial Latin square with at least ϵn^5 cuboctahedra. Then for every positive integer b there exist a subset $A \subset E$ of density at least $\epsilon^{b^{25b}}$, a metric group G, and maps $\phi : X \to G$, $\psi : Y \to G$ and $\omega : Z \to G$, such that the images $\phi(X), \psi(Y)$ and $\omega(Z)$ are 1-separated, and $d(\phi(x)\psi(y), \omega(z)) \leq b^{-1}$ for every $(x, y, z) \in X \times Y \times Z$ such that $(x, y) \in A$ and $\lambda(x, y) = z$.*

Theorem 2.1 follows immediately by applying Lemma 2.3 followed by Theorem 2.4.

The following combinatorial statement is of independent interest. It is a consequence of Theorem 2.4, but we prove it directly on the way to proving Theorem 2.4. Before we state it, we observe that the definition we gave earlier of the quadrangle condition, which we defined for Latin squares, applies verbatim to partial Latin squares.

Theorem 2.5. *Let X, Y and Z be sets of size n, let $A \subset X \times Y$, and let $\phi : A \to Z$ be a partial Latin square with at least ϵn^5 cuboctahedra. Then there is a subset $B \subset A$ of size at least αn^2, where $\alpha = \alpha(\epsilon) > 0$, that satisfies the quadrangle condition.*

One might at first think that Theorem 2.5 (with a suitable bound) would imply not just Theorem 2.4, but even a stronger result where H is a k-approximate subgroup rather than an (ϵ, k)-approximate subgroup. However, while a Latin square that satisfies the quadrangle condition must be the multiplication table of a group, a partial Latin square is not necessarily part of the multiplication table of a group: indeed, the example mentioned earlier of approximate multiplication on a δ-net of SO(3) is a counterexample. (This is significantly easier to prove than Green's result, which says that one cannot even restrict it to a dense set that is isomorphic to part of a group multiplication table.) More elementary counterexamples can be obtained by observing that if a group multiplication table ever contains the following configuration,

$$\begin{array}{ccc} e & d & \\ & f & c \\ a & & b \end{array}$$

then $ab^{-1}cd^{-1}ef^{-1}$ is equal to the identity, so any five of the labels determine the sixth. Thus, in a group multiplication table we have not only the quadrangle condition but also a natural ‘pair of 6-cycles’ generalization, which states that in a configuration such as the following, f must equal f'.

$$\begin{array}{cccccc} e & d & & & & \\ & f & c & & & \\ a & & b & & & \\ & & & e & d & \\ & & & & f' & c \\ & & & a & & b \end{array}$$

Note that that configuration itself contains no non-degenerate cuboctahedra, so it satisfies the quadrangle condition even if $f \neq f'$.

What we therefore need to do in order to prove Theorem 2.4 is find a subset of the partial Latin square that satisfies a generalized quadrangle condition that applies to all configurations up to a certain size. Exactly what those configurations are is the topic of §2.1.3 below.

2.1.2 The linear hypergraphs picture.

There turn out to be two other equivalent ways of describing partial Latin squares and configurations that live inside them, both of which will be extremely convenient at certain points in the argument. The first, which we shall discuss in this section, is to associate with a partial Latin square (X, Y, Z, A, ϕ) the tripartite 3-uniform hypergraph that consists of all triples $(x, y, z) \in X \times Y \times Z$ such that $(x, y) \in A$ and $\phi(x, y) = z$. Given any pair (x, y) there is obviously at most one z such that $\phi(x, y) = z$, so each pair (x, y) is contained in at most one face (x, y, z) of the hypergraph. But we can say the same for the other two pairs of coordinates, since ϕ is injective in each variable separately. For instance, each pair (x, z) is part of at most one face, since there is at most one y such that $\phi(x, y) = z$. A 3-uniform hypergraph with this property is called *linear*.

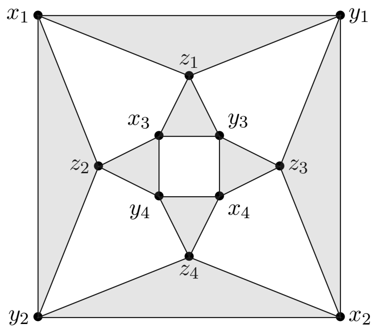


Figure 2.1: A cuboctahedron in hypergraph form. The triangles are the faces of the hypergraph.

A cuboctahedron in a partial Latin square corresponds to a hypergraph with vertices

$$x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4, z_1, z_2, z_3, z_4$$

and faces

$$x_1y_1z_1, x_1y_2z_2, x_2y_1z_3, x_2y_2z_4, x_3y_3z_1, x_3y_4z_2, x_4y_3z_3, x_4y_4z_4.$$

The vertices z_1, z_2, z_3, z_4 correspond to the labels a, b, c, d in our earlier description of a cuboctahedron. This hypergraph is illustrated in Figure 2.1.

It is now clear why we call it a cuboctahedron: it is (isomorphic to) the 3-uniform hypergraph one obtains from a cuboctahedron by taking its triangular faces.

In hypergraph terms, the statement that a partial Latin square satisfies the quadrangle condition is the statement that it does not contain a copy of a hypergraph that is like the cuboctahedron except that two faces that should meet at a vertex have been pulled apart. For example, if we take the cuboctahedron above and replace the face $x_4y_4z_4$ by a face $x_4y_4z_4'$ with $z_4' \neq z_4$, then we obtain a forbidden configuration. As can be seen from our visual representation in Figure 2.2, the two faces that have been pulled apart are no longer fixed in place but become 'flaps': for this reason we call the forbidden configuration a *flappy cuboctahedron*. Occasionally we will want to talk about a configuration that is either a cuboctahedron or a flappy cuboctahedron. We shall call such a configuration a *potential cuboctahedron*.

At this point we must make an important remark, which is that because we are talking about tripartite hypergraphs, the vertex pulled apart to make the flap can belong to any one of three different vertex classes, and each one corresponds to a different configuration in a Latin square. If it belongs to the

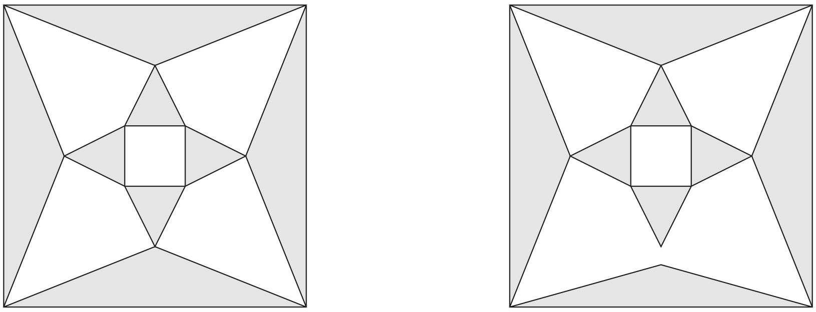


Figure 2.2: An actual cuboctahedron and a flappy cuboctahedron.

class Z of labels, then we obtain the configuration we obtained before, but if it belongs to the class X of columns or the class Y of rows, then we obtain different configurations that we also need to forbid. These are illustrated below: the first is a label-flappy cuboctahedron, the second a column-flappy cuboctahedron, and the third a row-flappy cuboctahedron. In each case we have a cuboctahedron except that one element has the wrong label, column, or row.

$$\begin{array}{cccc} c & d & & \\ a & b & & \\ & & c & d' \\ & & a & b \end{array} \qquad \begin{array}{ccccc} c & d & & & \\ a & b & & & \\ & & c & & d \\ & & a & b & \end{array} \qquad \begin{array}{cccc} c & d & & \\ a & b & & \\ & & & \\ & & & d \\ & & c & \\ & & a & b \end{array}$$

We shall say that a partial Latin square satisfies the label/column/row quadrangle condition if it contains no label/column/row-flappy cuboctahedra, respectively. This distinction is not important for Latin squares, since all three conditions are equivalent, but for partial Latin squares they are genuinely distinct.

From now on, when we say that a partial Latin square satisfies the *quadrangle condition*, we shall mean that it satisfies the label-, column-, and row-quadrangle conditions. When we stated Theorem 2.5 earlier, we had formulated only the label-quadrangle condition, but the result holds for the full quadrangle condition. Indeed, the stronger result follows from the weaker one, since by symmetry Theorem 2.5 implies the same result for the row- and column-quadrangle conditions, and applying all three results one after another gives the result for the full condition. Sequentially applying these results is possible only because applying one of these results leaves us in a position to imply the next. In particular, if we start with ϵn^5 cuboctahedra and apply Theorem 2.5 we obtain a partial Latin square with $\epsilon' n^5$ cuboctahedra. This follows because each of the row/column/label-quadrangle conditions implies (in a dense partial Latin square) the existence of many cuboctahedra – see, for instance, the proof of Proposition 2.25. While this idea of repeatedly applying a weaker result for each choice of row/column/label is indeed used (see Theorem 2.26), in practice it turns out to be more efficient not to directly apply Theorem 2.5 itself but instead to sequentially apply a key lemma in the proof of Theorem 2.5.

The following statement is a precise version of this stronger result, which generalizes Theorem 2.5 to yield a subset that satisfies the full quadrangle condition. The proof will appear in Section 2.5.3.

Theorem 2.6. *Let A be a tripartite linear 3-uniform hypergraph with vertex sets X, Y and Z of size n and suppose that A contains at least ϵn^5 cuboctahedra. Then there is a subhypergraph B of A with at least αn^2 faces, where $\alpha = \alpha(\epsilon) = \epsilon^{2^{453}}$, that contains no flappy cuboctahedra.*

Before we finish this subsection, we introduce two further definitions that will play important roles in our later proofs. The first we have already met in the cases $r = 2$, where it is a labelled rectangle, and $r = 3$, where it is a labelled 6-cycle of the kind discussed at the end of the previous subsection.

Definition 2.7. *A* $2r$-cycle *in a partially labelled grid consists of $2r$ points forming a* row-column cycle*. In other words, we have $2r$ points that alternate between sharing rows and columns, with no restriction on the labels.*

We use this name because if we disregard the labelling and think of the resulting subset of the grid as a bipartite adjacency matrix, then the above definition is just the usual definition a $2r$-cycle in the corresponding bipartite graph.

If we look at $2r$-cycles in the setting of tripartite 3-uniform hypergraphs, we are forming $2r$-cycles of faces, where two faces are joined if they share a vertex either in X or in Y. However, whereas in a labelled grid, the set of labels is somewhat different from the set of rows or columns, from a hypergraph perspective, it is unnatural to pick out one vertex class as 'special'. We shall therefore give a different name to hypergraphs that are the obvious generalization of $2r$-cycles where the two vertex sets in which the faces are joined do not have to be X and Y.

Definition 2.8. *Let H be a tripartite 3-uniform hypergraph. A* $2r$-petalled flower*, or* $2r$-PF *is a cycle of $2r$ faces such that each face shares one vertex with* *the next face and a different vertex with the previous face, and such that the third vertex always comes from the same vertex class. We refer to the $2r$ vertices of degree 2 in the $2r$-PF as the* inner vertices *and to the $2r$ vertices of degree 1 as* petals. *We shall sometimes refer to* PFs *when the number of faces is not to be specified.*

Thus, a $2r$-cycle in the labelled grid corresponds to a $2r$-PF in which the petals come from the class corresponding to the label coordinate.

2.1.3 Spherical hypergraphs.

We shall now describe the class of configurations that we shall use for our generalized quadrangle condition.

Let us call a triangulation of the sphere *kaleidoscopic* if there is a proper 3-colouring of the vertices of the triangles. Note that for such a triangulation, each vertex belongs to an even number of faces, since the faces form a polygon and the vertices of the polygon have to alternate between two different colours.

Given a kaleidoscopic triangulation T, we can form a tripartite linear hypergraph H as follows. The vertices of H are the edges of the triangulation, and the faces of H are the triples of edges that bound the triangles of T. We call a hypergraph that can be obtained in this way *spherical*.

The cuboctahedron is an example of a spherical hypergraph. To obtain it, start with an octahedron, the faces of which we can think of as a triangulation of the sphere, and colour the vertices according to which of the three antipodal pairs they belong to. Then we can construct the corresponding hypergraph geometrically by replacing each face F of the octahedron by the triangle whose vertices are the midpoints of the edges of F. Those triangles together form the hypergraph we call the cuboctahedron.

Later we shall discuss triangulated surfaces in more detail, and in particular the relevance of van Kampen diagrams to our results, at which point it will become clear why spherical hypergraphs are a natural class of hypergraphs to consider. But for now we state a strengthening of Theorem 2.6.

Define a *flappy* spherical hypergraph to be a hypergraph obtained from a spherical hypergraph by changing one vertex of one of its faces, or equivalently a hypergraph that becomes spherical when two of its vertices are identified.

Theorem 2.9. *Let A be a tripartite linear 3-uniform hypergraph with vertex sets X, Y and Z of size n, and suppose that H contains at least ϵn^5 cuboctahedra.*

Then for any positive integer b there is a subhypergraph of A with at least αn^2 faces, where $\alpha = \alpha(\epsilon, b) = \epsilon^{b^{25b}}$, that contains no flappy spherical hypergraphs with b faces or fewer.

Although this theorem is not the headline result of this chapter, it is the mathematical heart of it, and is in that sense our main result. Once we have proved it, obtaining a metric group structure turns out to be straightforward.

2.1.4 The organization of the rest of the chapter.

The proof of Theorem 2.9 has several stages. Recall that one way of describing the (label) quadrangle condition is to say that the ternary operation that maps a triple (a, b, c) of labels to a label d if there is a rectangle with labelling $\begin{matrix} c & d \\ a & b \end{matrix}$ is well-defined. In §2.2 we aim for a target that is weaker than this in one respect and stronger in another. The stronger respect is that we ask not just for information about the rectangle-completion operation but about a more general $2r$-cycle-completion operation. The weaker respect is that we do not ask for this operation to be well-defined, but merely that it should be C-valued for some bounded C. Let us define this formally.

Definition 2.10. *Let (X, Y, Z, A, ϕ) be a partial Latin square. We say that* the $2r$-cycle-completion operation is C-well-defined *if the following condition holds. For every choice of $(z_1, \ldots, z_{2r-1}) \in Z^{2r-1}$ there are at most C elements $z_{2r} \in Z$ for which there exists a $2r$-cycle labelled $(z_1, \ldots, z_r)$.*

For example, if $r = 3$, this says that for any labels a, b, c, d, e there are at most C possible labels f such that there is a 6-cycle

$$\begin{matrix} a & b & \\ & c & d \\ f & & e \end{matrix}$$

The proof of this first main step is related to, but significantly more complicated than, a proof by the first author of the Balog-Szemerédi theorem. Using a combination of two different dependent-random-selection arguments, we pass to a subset of A such that every cycle of length $2r$ shares its petal vertices with a large number of configurations H_{2r} of a certain kind, where 'large' means 'within a constant of the trivial maximum'.

At this point a simple but slightly surprising phenomenon is crucial, which is that for many configurations, the trivial upper bound for their number when a labelling is imposed on their petal vertices is the same as the trivial upper bound when all but one of the labels are specified. For example, the maximum number of rectangles labelled $\begin{matrix} c & d \\ a & b \end{matrix}$ is n, which is also the maximum number of rectangles labelled $\begin{matrix} c & * \\ a & b \end{matrix}$, where the asterisk can have an arbitrary label.

Thanks to this, if we have a set of labels $(z_1, \ldots, z_{2r-1})$ for which there are C choices of label z_{2r} such that $(z_1, \ldots, z_{2r})$ labels some $2r$-cycle, and for each one the number of configurations H_{2r} that share the petal vertices with the corresponding $2r$-PF is at least c times the trivial maximum, then C must be at most c^{-1}.

There remains the task of getting from C-well-defined operations to well-defined (or 1-well-defined) operations. In fact, we want more. The statement that the $2r$-cycle-completion operation is well-defined is equivalent to the hypergraph statement that there are no copies of the flappy spherical hypergraph one obtains by taking two $2r$-PFs, with inner vertices coming from the row and column vertex sets, and gluing them together in the obvious way on all but one of their petal vertices. However, we want to prove that there are no flappy spherical hypergraphs of any kind at all up to size k.

In §2.3, we use the C-well-defined property to prove an upgraded version of the first part of the argument, showing that we can pass to a further subset of our partial Latin square in which every $2r$-PF shares its petal vertices with a large number of configurations H'_{2r}. The configuration H'_{2r} is significantly simpler and easy to work with than the original configuration H_{2r}.

In §2.4 we explain how to reinterpret linear hypergraphs as 2-dimensional simplicial complexes, and the important subhypergraphs such as cuboctahedra and flappy cuboctahedra as triangulated surfaces. This language is much more convenient for the next stage of the proof, though for the purposes of this introduction we continue the discussion in terms of hypergraphs.

In §2.5, we use the fact that $2r$-PFs share their vertices with many copies of the configuration H'_{2r} to describe a 'popular replacement' process. That is, given a single flappy spherical hypergraph H, one repeatedly cuts out a $2r$-PF and replaces it by an H'_{2r}, keeping track very carefully of the number of ways of doing the entire process. We end up with a large number of copies of some flappy spherical hypergraph H' that is much more complicated than H and shares the two vertices of degree 1. Indeed, the number of copies is within some constant c of its trivial maximum. We now define an auxiliary graph by joining two vertices if they form the degree 1 vertices of some copy of H'. If any vertex in the auxiliary graph has degree d, then the number of copies we obtain from the popular replacement process is at least dc times the trivial maximum, and therefore $d \leq c^{-1}$. That last step may at first look wrong because the first trivial maximum is for copies of H' with two vertices fixed, whereas the second is for copies of H with just one vertex fixed, but again there is a slight surprise and these two trivial maxima are the same.

This shows that the graph has bounded degree, which implies that it contains a large independent set. But an independent set corresponds to a hypergraph where no two vertices form the flaps of a copy of H. That is, the hypergraph contains no flappy Hs and Theorem 2.9 is established.

In §2.6 we show that Theorem 2.4 is a straightforward consequence of Theorem 2.9. It is followed by a brief section with concluding remarks and questions.

In an appendix to this chapter, we explain the notion of a rough approximate group, which is an approximation in a metric sense to that of an approximate group, and we show that we can obtain a rough approximate group from the conclusion of Theorem 2.4.

2.2 Obtaining C-well-defined completion operations

In this section, we show how, starting with a partial Latin square A with at least ϵn^5 cuboctahedra, we can find a dense subset and a constant $C = C(k, \epsilon)$ such that the $2r$-cycle completion operation is C-well-defined for all $r \leq k$. We begin with a well-known bound for the number of $2r$-cycles in a bipartite graph, which will underlie many of the calculations throughout this section.

Lemma 2.11. *Let A be a subset of the $n \times n$ grid of density α. Then A contains at least $\alpha^{2r} n^{2r}$ and at most $\alpha^r n^{2r}$ distinct labelled $2r$-cycles.*

Proof. We may view A as a bipartite graph with vertex sets X and Y of size n and αn^2 edges. Let $\lambda_1, \ldots, \lambda_n$ be the singular values of the adjacency matrix of this graph. Then the number of $2r$-cycles is equal to $\sum_i \lambda_i^{2r}$. But the largest singular value is at least αn, so this sum is at least $\alpha^{2r} n^{2r}$.

For the upper bound we observe that the number of $2r$-cycles can be counted by summing, over all (ordered) r-tuples $(x_1, \ldots, x_r) \in A^r$, the indicator that there is a $2r$-cycle $x_1 y_1 \ldots x_r y_r$. This sum is clearly at most $|A|^r = \alpha^r n^{2r}$, since that is the number of ways of choosing $(x_1, \ldots, x_r)$. $\square$

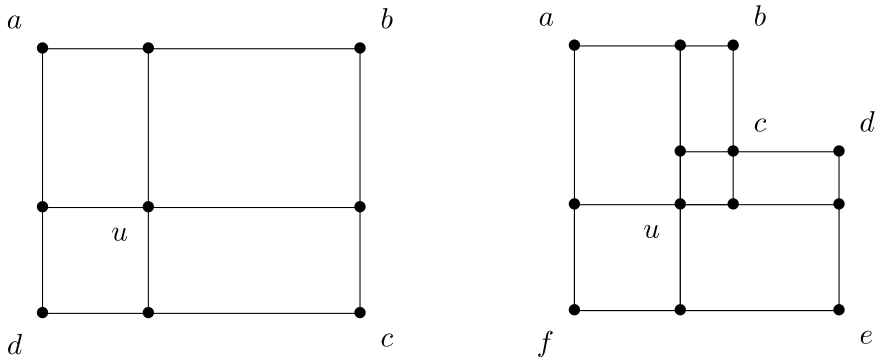


Figure 2.3: An ϵ-popular point decomposition of a rectangle (a, b, c, d) and a 6-cycle (a, b, c, d, e, f). All rectangles with opposite corners consisting of u and a vertex from the cycle are ϵ-popular.

The lower bound on the cuboctahedron count in A requires that the labelling of a random rectangle is repeated, on average, many times. This motivates the following definition.

Definition 2.12. *Given a partial Latin square A, a $2r$-cycle θ-*popular *in A if the labelling of the cycle occurs at least θn times in A.*

Note that the trivial maximum for the number of occurrences of a given labelling is n, since once one has chosen which of at most n points to choose with the first label, the condition that no label is repeated in any row or column implies that rest of the $2r$-cycle is determined by the labelling.

The first step towards obtaining the decompositions we need is a dependent random selection that ensures that almost all $2r$-cycles can be decomposed into popular rectangles in many ways. The decomposition we use at this stage will be referred to as the *point decomposition*.

Definition 2.13. *Given a $2r$-cycle $C = x_1y_1 \ldots x_ry_r$ in a partial Latin square A, a* point decomposition *of C in A is a collection of $2r$ rectangles, all belonging to A and all sharing a point u, with the corners opposite to u being the x_i and y_i. We call the point decomposition ϵ-*popular *if each of the $2r$ rectangles is ϵ-popular in A.*

Point decompositions for a rectangle and a 6-cycle are shown in Figure 2.3.

Lemma 2.14. *Let $0 < \epsilon, \delta < \frac{1}{100}$ and let $k > 1$ be a fixed integer. Given a partial $n \times n$ Latin square A containing at least ϵn^5 cuboctahedra, we can find a subset $B_1 \subset A$ of density $\beta_1 \geq \epsilon/2$ such that for each $2 \leq r \leq k$, a proportion at least $1 - \delta$ of $2r$-cycles in B_1 have at least $\delta\epsilon^{4k}n^2$ different $\epsilon/2$-popular point decompositions in A.*

Proof. We define a graph G with vertex set given by $[n]^2$ corresponding to the cells of the $n \times n$ grid, and edges given by joining x to y if the rectangle with opposite corners x and y has all its vertices in A and is $\epsilon/2$-popular.

Let X be the number of edges in G and Y be the number of non-edges. An edge in G can be associated to a set of at least $\epsilon n/2$ (and at most n) cuboctahedra, by combining the rectangle corresponding to the edge with one of the other rectangles with the same labelling. Similarly, a non-edge in G can be associated to a set of less than $\epsilon n/2$ cuboctahedra. In such a way, all cuboctahedra of A are accounted for. Therefore

$$Xn + Y\epsilon n/2 \geq \epsilon n^5$$

$$\Rightarrow Xn + \epsilon n^5/2 \geq \epsilon n^5$$

so G has average degree at least ϵn^2.

Let $\eta = \eta(\delta, k) = \delta\epsilon^{4k}$. A $2r$-cycle has at least ηn^2 different $\epsilon/2$-popular point decompositions in A if the common neighbourhood (in G) of the $2r$ corner vertices has size at least ηn^2.

We choose a vertex v in G uniformly at random, and let $N(v)$ be the neighbourhood of v in G. This is our dependent random selection. It remains to prove that it works with positive probability.

Let $C = x_1 y_1 \dots x_r y_r$ be a given $2r$-cycle in A. Let $N(C)$ be the set of vertices in G that are joined to all of $x_1, \dots, y_r$. We shall say that C is *bad* if $|N(C)| < \eta n^2$. If C is bad, we have that

$$\mathbb{P}(C \subset N(v)) = \frac{|N(C)|}{n^2} < \eta.$$

Let Z_r count the number of bad $2r$-cycles in $N(v)$. We have $\mathbb{E}Z_r \leq \eta n^{2r}$. Let $Z = \sum_{r=2}^{k} n^{-2r} Z_r$. Then

$$\mathbb{E}Z \leq \sum_{r=2}^{k} \eta \leq k\eta.$$

Our lower bound on the average degree of G also gives us that

$$\mathbb{E}(|N(v)|) \geq \epsilon n^2.$$

In particular, we have

$$\mathbb{E}\Big(|N(v)|n^{-2} - \epsilon/2 - \epsilon Z(2k\eta)^{-1}\Big) \geq 0$$

so there is a choice of vertex v such that $|N(v)|n^{-2} \geq \epsilon/2$ and $|N(v)|n^{-2} \geq \epsilon\eta^{-1}Z/2k$. The first inequality gives us that the total count, X_r, of $2r$-cycles in $N(v)$ is at least $(\epsilon/2)^{2r}n^{2r}$, while the second inequality implies that $Z \leq 2k\eta\epsilon^{-1}$. So for each $2 \leq r \leq k$,

$$Z_r n^{-2r} \leq 2k\eta\epsilon^{-1} \leq 2k\eta\epsilon^{-1}(\epsilon/2)^{-2r}n^{-2r}X_r,$$

which implies that

$$Z_r \leq k\eta(\epsilon/2)^{-(2r+1)}X_r.$$

Therefore, letting $B_1 = N(v)$ for this choice of v, we have $\beta_1 n^2 = |N(v)| \geq \epsilon n^2/2$ and the proportion of $2r$-cycles in $N(v)$ which are bad is at most

$$k\eta(\epsilon/2)^{-(2r+1)} \leq \delta.$$

□

Using Lemma 2.14 we may pass to a dense subset B_1 of A such that almost all $2r$-cycles have many (within a constant factor of the trivial maximum) popular point decompositions in A. However, for our purposes the 'almost all' is not sufficient, and we need to use a more complicated decomposition to boost Lemma 2.14 into an 'all' statement.

The following definition introduces these more complex decompositions.

Definition 2.15. *Let X be a partial Latin square. Given a $2r$-cycle $C = x_1y_1\dots x_ry_r$, a* ring decomposition *of C in X consists of a second $2r$-cycle $C' = x'_1y'_1\dots x'_ry'_r$ in X such that all the points of all the rectangles with opposite corner pairs either (x_i, x'_i) or (y_i, y'_i) belong to X. If C' and all the rectangles are ϵ-popular, we call the collection of all the rectangles together with C' an* ϵ-popular ring decomposition *of C. An* ϵ-popular full decomposition *of C is a $2r$-cycle C' together with ϵ-popular point decompositions of C' and the $2r$ rectangles just defined.*

A ring decomposition of a 4-cycle is shown in Figure 2.4 and a full decomposition is shown in Figure 2.5.

Remark 2.16. It will be important to keep track of the order (in n) of the trivial maxima for the number of ring decompositions and full decompositions of a

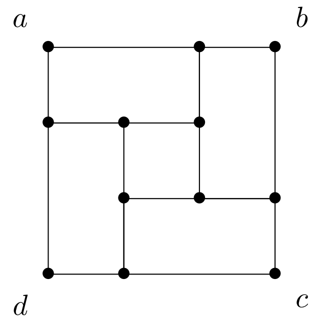


Figure 2.4: A ring decomposition of a 4-cycle (a, b, c, d).

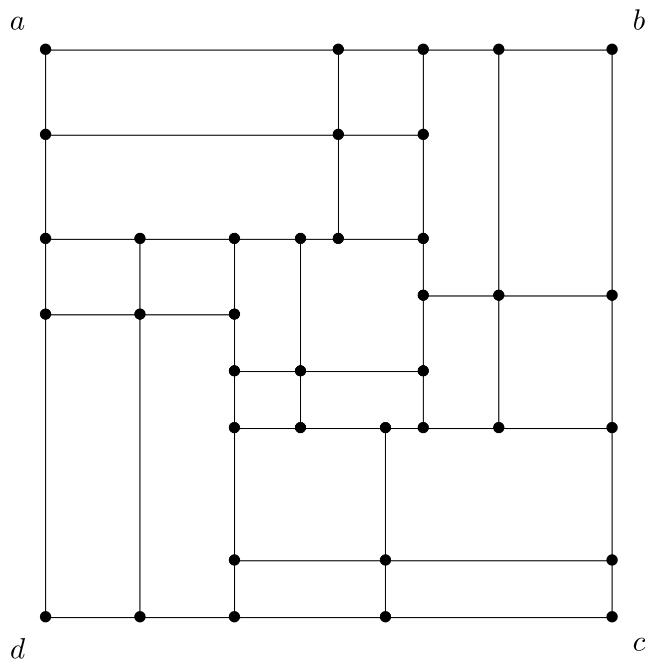


Figure 2.5: A full decomposition of the 4-cycle (a, b, c, d). If the decomposition is ϵ-popular then each of the 20 small rectangles in the figure is ϵ-popular.

$2r$-cycle in a dense subset of an $n \times n$ Latin square. The number of ring decompositions is at most n^{2r}, since a ring decomposition of a $2r$-cycle C is uniquely defined by a $2r$-cycle C'. In a full decomposition, C' and all the rectangles in the ring decomposition are given point decompositions, each of which can be chosen in at most n^2 ways. So the number of full decompositions is at most $n^{2r+2(2r+1)} = n^{6r+2}$.

Our next step is to pass to a subset B_2 of B_1 such that all $2r$-cycles in B_2 have within a constant factor of the trivial maximum number of ring decompositions. Since almost all $2r$-cycles in B_1 have popular point decompositions, we will then be able to pass to a further subset B_3 of B_2 so that all $2r$-cycles in B_3 have within a constant factor of the trivial maximum number of ϵ-popular full decompositions.

We need a technical lemma to achieve the first step of this process.

Lemma 2.17. *Let k be a positive integer. Let G be a bipartite graph with vertex classes X, Y of size n and edge density δ, with $0 < \delta < \frac{1}{100}$. Then we can pass to subsets $X' \subset X$ and $Y' \subset Y$, each of size at least $\delta^2 n/16$, such that the edge density in $G' = G|_{X' \times Y'}$ is at least $\delta/4$ and for any $2 \leq r \leq k$ and any choice of r vertices $x_1, \ldots, x_r \in X'$ and $y_1, \ldots, y_r \in Y'$ we have at least $\delta^{5k^2+4k} n^{2r}$ choices of vertices $u_1, \ldots, u_r, v_1, \ldots, v_r$ in G with $u_i v_j \in E(G)$, $x_i u_i \in E(G)$ and $y_i v_i \in E(G)$ for each $i, j \in \{1, \ldots, r\}$.*

Proof. Let us begin by discarding all vertices from X of degree smaller than $\delta n/2$. This leaves a set $X_1 \subset X$ of size at least $\delta n/2$.

Let $c_1 = \delta^{2k}$ and $c_2 = \delta^{5k}$. We will use a dependent random selection argument that allows us to pass to a subset $X_2 \subset X_1$ of size at least $(\delta^2/8)n$ with the property that for a $(1 - c_1)$ proportion of choices $(x_1, \ldots, x_{k+1})$ from X_2 we have at least $c_2 n$ vertices in the shared neighbourhood $\Gamma(x_1, \ldots, x_{k+1}) \subset Y$.

We do this by picking a vertex $y \in Y$ at random and considering $\Gamma(y)$. Observe that

$$\mathbb{E}(|\Gamma(y)|) \geq \delta |X_1|/2 \geq \delta^2 n/4.$$

Let us call a $(k+1)$-tuple $(x_1, \ldots, x_{k+1})$ *bad* if $|\Gamma(x_1, \ldots, x_{k+1})| < c_2 n$. Let B be the number of bad tuples in $\Gamma(y)$. The probability that a given bad $(k+1)$-tuple belongs to $\Gamma(y)$ is less than c_2, since for this to happen y must be picked from $\Gamma(x_1, \ldots, x_{k+1})$. Therefore

$$\mathbb{E}(B) < c_2 n^{k+1}$$

and so

$$\mathbb{E}(c_1|\Gamma(y)|^{k+1} - c_1(\delta^2/8)^{k+1}n^{k+1} - B) > (c_1(\delta^2/4)^{k+1} - c_1(\delta^2/8)^{k+1} - c_2)n^{k+1}.$$

Since $c_2 = c_1\delta^{3k}$, this expectation is positive and so there is some choice of y for which both $c_1|\Gamma(y)|^{k+1} \geq c_1(\delta^2/8)^{k+1}n^{k+1}$ and $c_1|\Gamma(y)|^{k+1} \geq B$. These inequalities imply that $|\Gamma(y)| \geq (\delta^2/8)n$ and that at most a proportion c_1 of the $(k+1)$-tuples from $\Gamma(y)$ are bad. So we may take $X_2 = \Gamma(y)$.

Now we let X_3 be the subset of X_2 consisting of all vertices $x_1 \in X_2$ with the property that for a proportion $(1-2c_1)$ of the choices of $x_2, \ldots, x_{k+1} \in X_2$, the shared neighbourhood $\Gamma(x_1, \ldots, x_{k+1}) \subset Y$ contains at least $c_2 n$ vertices. Since $|\Gamma(x_1, \ldots, x_{k+1})| \geq c_2 n$ for at least a proportion $(1-c_1)$ of all $(k+1)$-tuples, $|X_3| \geq |X_2|/2 \geq \delta^2 n/16$.

Since each vertex in X_3 has at least $\delta n/2$ neighbours in Y, the number of edges from Y to X_3 is at least $\delta n|X_3|/2$. We now pass to the subset $Y_1 \subset Y$ that consists of all vertices with at least $\delta|X_3|/4$ edges into X_3. We note that $|Y_1| \geq \delta n/4$.

Now let $x_1, \ldots, x_k$ be chosen from X_3 and $y_1, \ldots, y_k$ from Y_1. Let $A_1, \ldots, A_k$ be the neighbourhoods of the y_i in X_3 – note that $|A_i| \geq \delta|X_3|/4$. Let $T = A_1 \times \cdots \times A_k$ and note that it has cardinality at least $(\delta|X_3|/4)^k \geq (\delta|X_2|/8)^k$.

By the choice of X_3, we know that the number of choices of $u_1, \ldots, u_k \in X_2$ such that $|\Gamma(x_i, u_1, \ldots, u_k)| < c_2 n$ is at most $2c_1|X_2|^k$ for each $i = 1, \ldots, k$. Letting $c_1 = \delta^{2k}$ so that $2c_1 k < (\delta/8)^k/2$ and noting that $|T| = (\delta|X_2|/8)^k$, we see that there must be at least $(\delta|X_2|/8)^k/2$ choices of $(a_1, \ldots, a_k) \in T$ such that $|\Gamma(x_i, a_1, \ldots, a_k)| \geq c_2 n$ for each $i = 1, \ldots, k$. Observe that for any such choice of $(a_1, \ldots, a_k)$ and for any choice of $b_i \in \Gamma(x_i, a_1, \ldots, a_k)$ we get a complete bipartite graph between the a_i and the b_i as well as the edges $x_i b_i$ and $y_i a_i$ for each i.

The number of choices of the a_i and b_i from the above paragraph is at least

$$\begin{aligned}\Big((\delta|X_2|/8)^k/2\Big) \times \Big((c_2 n)^k\Big) &\geq (\delta^3/64)^k(\delta^{5k})^k n^{2k} \\ &\geq \delta^{5k^2+4k} n^{2k}.\end{aligned}$$

Observe that the subgraph induced by the x_i, y_j, a_k and b_l contains a $2r$-cycle $a_1 b_1 \ldots a_r b_r$ as well as the edges $x_i a_i$ and $y_i b_i$ for each i. Moreover, the edge density in $X_3 \times Y_1$ is at least $\delta/4$, so taking $X' = X_3$ and $Y' = Y_1$, the result follows. $\square$

Remark 2.18. It is well known that given a dense bipartite graph G, we may pass to a dense subgraph H such that any two vertices of H are joined by many P_3s in G. Lemma 2.17 shows that a considerable generalization of this statement is available for relatively little extra effort: given any fixed bipartite graph H' with t special vertices $v_1, \ldots, v_t$ such that the shortest path from any v_i to any v_j has length at least 3, we may pass to a dense subgraph H of G such that for any $u_1, \ldots, u_t$ the number of isomorphic copies $\phi(H')$ of H' in H with $\phi(v_i) = u_i$ for all i is within a constant of the trivial maximum. The P_3 statement is the special case where H' is a path of length 3 and v_1 and v_2 are its endpoints. (A similar observation was made in a blog post of Tao [84], but he was content to discuss just the special case he needed, and he left the proof as an exercise for the reader.)

As an immediate corollary we obtain the following result, which will soon be applied in order to help guarantee the presence of many ring decompositions.

Lemma 2.19. *Let $k > 1$ be a positive integer. Let G be a bipartite graph with vertex classes X, Y of size n and edge density δ, with $0 < \delta < \frac{1}{100}$. Then we can pass to subsets $X' \subset X$ and $Y' \subset Y$, each of size at least $\delta^2 n/16$, such that the edge density in $G' = G|_{X' \times Y'}$ is at least $\delta/4$ and for any $2 \leq r \leq k$ and any choice of r vertices $x_1, \ldots, x_r \in X'$ and $y_1, \ldots, y_r \in Y'$ we have at least $\delta^{7k^2} n^{2r}$ choices of $2r$-cycle $u_1 v_1 \ldots u_r v_r$ in G with $x_i u_i \in E(G)$ and $y_i v_i \in E(G)$ for each $i = 1, \ldots, r$.*

Proof. The result follows by applying Lemma 2.17, and noting that the complete bipartite graph on $r + r$ vertices contains a $2r$-cycle. $\square$

When viewed as a statement about subsets of the grid, Lemma 2.19 states that we may pass to a dense subset $B_2 \subset B_1$ such that all $2r$-cycles in B_2 have many ring decompositions in B_1. We must now pass to a further subset in which all $2r$-cycles have many popular full decompositions.

In the statement of the following lemma we shall assume that we are given some property of cycles, and cycles that have that property will be called 'good'. The reason we work at this level of abstraction is partly that we can, and partly that we shall apply the lemma twice, with different definitions of 'good' each time.

Lemma 2.20. *Let $0 < \beta, \delta, \gamma < \frac{1}{100}$ and $k > 1$. Let B be a subset of an $n \times n$ grid of density at least β with the property that for each $2 \leq r \leq k$ at least a proportion $1 - \delta$ of $2r$-cycles in B are good. If $\delta \leq \beta^{9k^2}$ then we can find a subset*

B' of B with density $\beta' \geq \beta^8$ with the property that any $2r$-cycle in B' has at least $\beta^{8k^2} n^{2r}$ different ring decompositions into good cycles in B.

Proof. Recall that a ring decomposition of a cycle C involves a paired cycle C', which we shall refer to as the *back face*, and $2r$ rectangles between these cycles, which we shall refer to as the *side faces*. We shall call a ring decomposition of a cycle C *good* if the cycle C' making up the back face and all the rectangles involved in the side faces are good.

We shall call a $2r$-cycle C *indecomposable* if it has fewer than $\beta^{7k^2} n^{2r}$ good ring decompositions. We shall say that a $2r$-cycle is *bad on the back face* if it is indecomposable and at least one third of its ring decompositions have a bad (ie not good) cycle on the back face, and *bad on the side faces* if it is indecomposable and at least one third of its decompositions have a bad rectangle on a side face.

In parallel with the subset B of the Latin square, we shall also consider the corresponding bipartite graph G in which the rows and columns form the vertex sets and the points of B form the edges.

We begin by applying Lemma 2.19. This allows us to pass to a subset B' of B of density at least $(\beta^2/16)^2(\beta/4) \geq \beta^7$ with the property that each $2r$-cycle in B' has at least $\beta^{7k^2} n^{2r}$ ring decompositions in B. Since $\beta^{8k^2} \leq \beta^{7k^2}/3$, any $2r$-cycle in B' which is indecomposable is either bad on the back face or bad on the side faces.

Consider a given $2r$-cycle $C = x_1 y_1 \dots x_r y_r$ in B'. Suppose that C is bad on the back face. Then there are at least $\beta^{7k^2} n^{2r}/3$ bad $2r$-cycles in B. But only a proportion δ of all $2r$-cycles in B are bad, and the maximum possible number of $2r$-cycles in B is $\beta^r n^{2r}$. So if $\delta < \beta^{7k^2}$ then we have a contradiction.

Therefore no $2r$-cycles are bad on the back face (for any $2 \leq r \leq k$), and so all indecomposable $2r$-cycles are bad on a side face. If for each r there are no more than $\beta^7 n^2/4k^2$ vertex disjoint indecomposable $2r$-cycles, then discarding all points from a maximal vertex-disjoint set of indecomposable cycles we discard at most $\beta^7 n^2/2$ points, leaving a set of density at least $\beta^7/2 \geq \beta^8$ with no indecomposable cycles (and so we are done).

Thus, for some r it must be possible to find at least $\beta^7 n^2/4k^2$ vertex disjoint indecomposable $2r$-cycles. Since there are no cycles bad on the back face, all these cycles are bad on a side face. This means that each of these $2r$-cycles has at least $\beta^{7k^2} n^{2r}/3$ ring decompositions involving a bad rectangle as a side face. Each bad rectangle can belong to at most n^{2r-2} ring decompositions, so we get at least $\beta^{7k^2} n^2/3$ bad rectangles sharing a vertex with each of these indecomposable cycles. This gives us at least

$$\left(\beta^{7k^2}n^2/3\right)\left(\beta^7 n^2/4k^2\right) > \beta^{9k^2}n^4$$

bad rectangles in B.

But the number of bad rectangles is at most $\delta\beta^2 n^4$, so if $\delta \le \beta^{9k^2}$ then we have a contradiction. $\square$

By applying Lemmas 2.14, 2.19 and 2.20 we will be able to pass to a dense subset B of A in which all $2r$-cycles have many popular full decompositions. This will still not be sufficient for our later purposes, which will require obtaining ϵ-popular ring decompositions. So before we fill in the details, we shall give more technical lemmas that will help us with this objective.

Lemma 2.21. *Let A be a partial Latin square and let B be a subset of A. Suppose that every $2r$-cycle in B has at least γn^{6r+2} different ϵ-popular full decompositions in A. Then for every $(a_1,\dots,a_{2r-1})$ the number of a_{2r} such that $(a_1,\dots,a_{2r})$ is a labelling of some $2r$-cycle in B is at most $\epsilon^{-10r}\gamma^{-1}$.*

Proof. Suppose that we have a tuple $(a_1,\dots,a_{2r-1})$ such that the set $\{x_i\}$ of possible labelling completions has size at least K. For each completion we can find γn^{6r+2} ϵ-popular full decompositions.

Let us think about a typical one of these decompositions as follows. (For the discussion that follows, it may well help to look back at Figure 2.5.) We begin with a $2r$-cycle C with points $x_1, y_1, \dots, x_r, y_r$, where x_i has label a_{2i-1} when $1 \le i \le k$, y_i has label a_{2i} when $1 \le i \le k-1$, and we do not know about the label attached to the point y_r. (It is important to be clear that the x_i and y_i are elements of $[n]^2$ and not of $[n]$ in this discussion.)

Next, we have another $2r$-cycle C' with points $x'_1, y'_1, \dots, x'_r, y'_r$. However, it is 'reflected', in the sense that whereas x_i and y_i are in the same row, x'_i and y'_i are in the same column, and whereas y_i and x_{i+1} are in the same column, y'_i and x'_{i+1} are in the same row.

Now we complete the cycles C and C' to a ring decomposition by adding in $2r$ points $u_1, v_1, \dots, u_r, v_r$, where u_i is in the row that contains x_i and y_i and the column that contains x'_i and y'_i, and v_i is in the column that contains y_i and x_{i+1} and the row that contains y'_i and x'_{i+1}. (The points u_i and v_i do not form a $2r$-cycle.)

The rectangles of this ring decomposition are given by $R_i = (x_i, u_i, x'_i, v_{i-1})$ and $S_i = (y_i, v_i, y'_i, u_i)$. To form a point decomposition, we now add points p_i

and q_i, and form the four rectangles that have a vertex in R_i and the opposite vertex at p_i, and the four rectangles that have a vertex in S_i and the opposite vertex at q_i. As well as the point p_i, we have to add four more points to R_i in order to complete the decomposition into four rectangles. Of these four points, let r_i and s_i be the ones in the same row and the same column as x_i; we shall not bother giving names to the other two. Similarly, let w_i and z_i be the points in the same column and row as y_i that are part of the decomposition of S_i into four rectangles.

Now let us consider a certain subset of the (variable set of) points of the full decomposition. We shall take the points u_i and v_i, the points r_i and s_i, and the points w_i and z_i with $1 \leq i \leq r-1$. We shall also take the two points from the point decomposition of C' that are in the same row and column as x_1', and the two points from the decomposition of the rectangle S_r that are in the same row and column as y_r'. This makes a total of $6r+2$ points, so by the pigeonhole principle we can find some choice of labellings of these $6r+2$ points that occurs at least $K\gamma$ times amongst the set of ϵ-popular full decompositions of $2r$-cycles C for which the points $x_1, y_1, \dots, x_r$ are labelled $a_1, \dots, a_{2r-1}$.

Observe that a full decomposition of a given cycle is uniquely determined by the way it is labelled, since once a point has been specified, any other point in the same row or column is then determined by its label. Observe also that since each rectangle in an ϵ-popular full decomposition is ϵ-popular, given three labels of any rectangle there are at most $1/\epsilon$ different choices of label for the fourth, since otherwise there would be more than n rectangles that shared three labels, which is impossible.

Our aim now is use this observation to show that once the labellings of the $6r+2$ points specified earlier are given, the number of possible labellings of the remaining points is at most ϵ^{-10r}. Since we know that it is also at least $K\gamma$, this will give us our desired upper bound on K.

To do this, we consider the natural closure operation, where three points of a rectangle generate the fourth. The observation implies that if we know the labels at some set of points that generates the entire decomposition, and if there are t other points, then the number of possible ways of completing the labelling is at most ϵ^{-t}. We apply this to the set of $6r+2$ points we have chosen.

Note first that the side faces of the full decomposition, apart from the rectangle containing the unfixed point of C, each contain five points from the set in their point decompositions, and furthermore these five generate the other four. Therefore the closure of the set contains all the points in all the point decompositions of the rectangles $R_1, \dots, R_r$ and $S_1, \dots, S_{r-1}$. These include the points $x'_1, \dots, x'_r$ and $y'_1, \dots, y'_{r-1}$. Since we also have the points in the same row and column as x'_1, we obtain the central point of the back face of the decomposition, and using this we can work round the cycle and obtain all the points in its point decomposition. And now we have five points of the rectangle S_r that generate the others (since they lie along two edges), which shows that the $6r+2$ points we choose generate all the points of the full decomposition. It is not hard to check that a full decomposition contains $18r+1$ points, so we find, as promised, that the number of labellings given the labels at the $6r+2$ points and $2r-1$ of the points of C is at most ϵ^{-10r}, as claimed, and this proves that $K \le \epsilon^{-10r}\gamma$. □

By combining these lemmas we are now at a stage where we can pass to a subset B of A in which for each $2 \le r \le k$ the number of different ways of completing the labelling of a $2r$-cycle in B given $2r-1$ of its labels is bounded. In order to state this concisely, we introduce the following definition.

Definition 2.22. *Let B be an $n \times n$ partial Latin square. Suppose that for any sequence of $2r-1$ labels, the number of different labellings of a $2r$-cycle in B with its first $2r-1$ points labelled using that sequence is always at most C. Then we say that the* $2r$-cycle completion operation in B is C-well-defined.

In particular, if for any three labels a, b, c the number of labels d for which there is a rectangle (thought of as an ordered quadruple of points) in B labelled a, b, c, d is at most C, then the 4-cycle completion operation in B is C-well-defined.

With this definition, we can describe our progress so far as follows.

Theorem 2.23. *Let $0 < \epsilon < 10^{-3}$ and let $k \ge 2$ be a fixed positive integer. Let A be a partial Latin square containing at least ϵn^5 cuboctohedra. Then we can find a subset $B \subset A$ of density $\beta \ge \epsilon^{10}$ with the property that for each $2 \le r \le k$ the $2r$-cycle completion operation in B is ϵ^{-33k^3}-well-defined.*

Proof. We now apply Lemma 2.14 with $\delta = (\epsilon/2)^{9k^2}$. This allows us to pass to a subset $B_1 \subset A$ of density $\beta_1 \ge \epsilon/2$ such that for each $2 \le r \le k$ a proportion at least $1-\delta$ of $2r$-cycles in B_1 have at least

$$(\epsilon/2)^{9k^2}\epsilon^{4k}n^2 \ge \epsilon^{11k^2}n^2$$

different $\epsilon/2$-popular point decompositions.

From here we apply Lemma 2.20, where we take the property 'good' for a $2r$-cycle to mean that the cycle has at least $\epsilon^{11k^2}n^2$ different $\epsilon/2$-popular point decompositions. We can do this since B_1 has density $\beta_1 \geq \epsilon/2$, so $\delta \leq \beta_1^{9k^2}$. The lemma gives us a subset B_2 of B_1 of density (in the original $n \times n$ grid) $\beta_2 \geq \beta_1^8 \geq \epsilon^{10}$ in which every $2r$-cycle in B_2 has at least

$$\begin{aligned}\left((\epsilon/2)^{8k^2} n^{2r}\right)\left(\epsilon^{11k^2} n^2\right)^{2r+1} &\geq \epsilon^{20k^2+20k^3} n^{6r+2} \\ &\geq \epsilon^{30k^3} n^{6r+2}\end{aligned}$$

different $\epsilon/2$-popular full decompositions. (The first bracket on the left is a lower bound for the number of good ring decompositions, and the second is a lower bound for the number of ways of converting each one into an $\epsilon/2$-popular full decomposition.)

This allows us to apply Lemma 2.21 with $\gamma = \epsilon^{30k^3}$, which implies the result with $B = B_2$ (since $(\epsilon/2)^{-10r}\gamma^{-1} \leq (\epsilon/2)^{-10k-30k^3} \leq \epsilon^{-33k^3}$). $\square$

We draw attention here to an analogy with the additive combinatorics result mentioned at the beginning of the chapter, which states that if $\phi : \mathbb{Z}_N \to \mathbb{Z}_N$ is a map such that $\phi(x) + \phi(y) = \phi(z) + \phi(w)$ for a positive proportion of the quadruples $x + y = z + w$, then we can pass to a dense subset $A \subset \mathbb{Z}_N$ such that the restriction of ϕ to A is a Freiman homomorphism. One way of proving this result begins by showing that it is possible to pass to a set A' such that for each w, the number of values that $\phi(x) + \phi(y) - \phi(z)$ can take when $x + y - z = w$ is bounded by some constant C that is independent of N. This first step mirrors what we have achieved thus far. It is then necessary to find a separate argument to pass to a further subset where C is reduced to 1.

We have to do the same here, though at this point the analogy breaks down somewhat, since in the additive problem, Plünnecke's inequality is used, but our setting does not involve an ambient group so we do not appear to have an analogous tool. Thus, while Theorem 2.23 constitutes significant progress towards our positive result, it turns out that we are still quite a long way from reducing C to 1.

2.3 Simplifying the decompositions

Perhaps surprisingly, the first step towards reducing C to 1 will involve abandoning full decompositions. While full decompositions are easy to understand in the grid setting, they are more difficult to visualize in the hypergraph setting, because of the presence of vertices that are contained in more than two faces, which also means that they will correspond not to surfaces but to complexes in which four or more faces can share an edge. For these reasons, they are not a natural tool for what is to come. Instead, we shall use the C-well-defined property to start again, reapplying Lemma 2.20 with the added information. This will allow us to find ring decompositions into popular rectangles (rather than into rectangles with many popular point decompositions), which will greatly simplify the structures we have to consider.

In this section, we shall use Theorem 2.23 as a tool in a 'second pass' through the arguments in Section 2.2. Our first lemma for this section shows that the property of C-well-definedness is sufficient to ensure that almost all of the cycles in B are popular (for a lower threshold of popularity). This is significant because it enables us to repeat the above process but eliminates the need for Lemma 2.14 and point decompositions. We will simply be able to reapply Lemma 2.20 to the subset B with a different meaning for the property 'good': now it will mean 'θ-popular', for some appropriate θ, rather than 'having many popular point decompositions'.

Lemma 2.24. *Let B be an $n \times n$ partial Latin square of density β. Suppose that the $2r$-cycle completion operation in B is C-well-defined. Let δ, θ be such that $\beta^{2r}\delta\theta^{-1} > C$. Then the proportion of $2r$-cycles in B that are not θ-popular is at most δ.*

Proof. By Lemma 2.11, the number of $2r$-cycles in B is at least $\beta^{2r}n^{2r}$. Therefore, given a tuple $(a_1, \ldots, a_{2r-1})$ of labels, the number of $2r$-cycles with first $2r-1$ labels $(a_1, \ldots, a_{2r-1})$ is on average at least $\beta^{2r}n$. However, since the $2r$-cycle completion operation is C-well-defined we have further that the number of different a_{2r} completing a $2r$-cycle labelling $(a_1, \ldots, a_{2r})$ in B is at most C.

If a proportion greater than δ of $2r$-cycles are not θ-popular, then by averaging there must be some $(a_1, \ldots, a_{2r-1})$ such that a proportion greater than δ of $2r$-cycles starting with these labels are not θ-popular. But that means that there must be more than $\beta^{2r}\delta\theta^{-1} > C$ completions which is a contradiction to the assumption that the $2r$-cycle completion operation in B is C-well-defined. □

We are now ready to put together our technical lemmas to prove the following proposition, which will be the main tool in the proof of Theorem 2.9.

Proposition 2.25. *Let $0 < \epsilon < 10^{-3}$ and let A be a partial Latin square containing at least ϵn^5 cuboctohedra. Let $k \geq 100$ be an integer. Then we can find*

$B \subset A$ of density $\beta \geq \epsilon^{80}$ such that for each $r = 2, \ldots, k$ we have that every $2r$-cycle in B has at least $\epsilon^{80k^2} n^{2r}$ different θ-popular ring decompositions in A, where $\theta \geq \epsilon^{35k^3}$. Moreover, the number of cuboctahedra in B is at least $\epsilon^{70k^3} n^5$.

Proof. We begin by applying Theorem 2.23. This allows us to pass to a subset $B_1 \subset A$ of density $\beta_1 \geq \epsilon^{10}$ with the property that for each $2 \leq r \leq k$ the $2r$-cycle completion operation in B is C-well-defined, where $C = \epsilon^{-33k^3}$.

By Lemma 2.24 we see that a proportion greater than $1 - \delta$ of $2r$-cycles (for each $2 \leq r \leq k$) in B are θ-popular for any choice of $\theta < \beta_1^{2k} \delta / C$.

We now apply Lemma 2.20 again, but taking the property 'good' for a $2r$-cycle to mean that the cycle is θ-popular. To do this, we take $\delta = (\beta_1)^{9k^2}$. With this value of δ, we may take some $\theta \geq \epsilon^{20k+90k^2+33k^3} \geq \epsilon^{35k^3}$.

The lemma then gives us a subset B_2 of density $\beta_2 \geq \beta_1^8 \geq \epsilon^{80}$ in which every $2r$-cycle in B_2 has at least $\beta_1^{8k^2} n^{2r} \geq \epsilon^{80k^2} n^{2r}$ many θ-popular ring decompositions in A.

Since B_2 is a subset of B_1, the rectangle completion operation in B_2 is still C-well-defined. By Lemma 2.11 the number of rectangles in B_2 is at least $\beta_2^4 n^4$, and since cuboctahedra are counted by pairs of rectangles with the same labelling, the cuboctahedron count is minimized when the the number of rectangles with each labelling is as balanced as possible (by convexity). For each triple of labels (a, b, c) the number of possible completions d is at most C, so the number of cuboctahedra is at least

$$(\beta_2^4 n / C)^2 n^3 = (\beta_2^8 / C^2) n^5 \geq \epsilon^{70k^3} n^5$$

as required. $\square$

We now observe that a cuboctahedron, which consists of two identically labelled rectangles, still corresponds to a cuboctahedron if we permute the coordinates of the points. Viewing A as a 3-uniform, linear hypergraph we may associate it with a partial Latin square by designating any particular coordinate to represent the 'label coordinate' and the other two to represent the row and column coordinates. The cuboctahedron count in A does not depend on which coordinate we choose.

This observation allows us to repeatedly apply Proposition 2.25, permuting the coordinates at each step. This enables us to make the step from finding decompositions of $2r$-cycles to finding decompositions of $2r$-PFs more generally. This means that we will no longer need to treat the row and column vertex classes differently from the label class, and this added symmetry will be crucial for our popular decomposition argument in Section 2.5.

Theorem 2.26. *Fix $\epsilon \leq 10^{-3}$ and $k \geq 100$. Let A be a 3-uniform, linear hypergraph that contains at least ϵn^5 cuboctahedra. Then there exists a sequence $A = A_0 \supset A_1 \supset \dots$ such that each A_i has density at least $\alpha_i(\epsilon, k)$ and A_i contains at least $\epsilon_i(\epsilon, k)n^5$ cuboctahedra, and for each $r = 2, \dots, k$, every $2r$-PF in A_i is $\theta_i(\epsilon, k)$-popularly decomposable in A_{i-1} in at least $\gamma_i(\epsilon, k)n^{2r}$ different ways. Each of the parameters $\alpha_i, \epsilon_i, \theta_i, \gamma_i$ may be chosen to be at least $\epsilon^{k^{15i}}$.*

Proof. Given a 3-uniform, linear hypergraph B we define three different partially labelled $n \times n$ grids, $B^{(1)}, B^{(2)}$ and $B^{(3)}$. If (x, y, z) is a face of B, then we put the label z in position (x, y) of $B^{(1)}$, y in position (z, x) of $B^{(2)}$, and x in position (y, z) of $B^{(3)}$.

Once we have chosen A_i, we first consider $A_i^{(1)}$. We apply Proposition 2.25 to pass to a dense subset $B_i^{(1)}$ in which all $2r$-cycles have at least $\gamma_i n^{2r}$ different θ_i-popular ring decompositions in $A_i^{(1)}$ for $2 \leq r \leq k$. We then 'rotate coordinates' to obtain the partially labelled grid $B_i^{(2)}$. Since rotation does not change the number of cuboctahedra, we are still in a position to apply Proposition 2.25 (albeit with different parameters) to obtain a subset $C_i^{(2)}$ in which all $2r$-cycles have at least $\gamma_i' n^{2r}$ different θ_i'-popular ring decompositions in $A_i^{(2)}$ for $2 \leq r \leq k$. Finally we rotate again to obtain a set $C_i^{(3)}$, to which we apply Proposition 2.25 again to obtain a subset $D_i^{(3)}$ in which all $2r$-cycles have at least $\gamma_i'' n^{2r}$ different θ_i''-popular ring decompositions in $A_i^{(3)}$ for $2 \leq r \leq k$.

If the density of A_i is α_i and the number of cuboctahedra is $\epsilon_i n^5$, then the density of B_i is at least ϵ_i^{80}. Moreover, the cuboctahedron count of B_i is at least $\epsilon_i^{70k^3} n^5$. Therefore, the density of C_i is at least

$$(\epsilon_i^{70k^3})^{80} \geq \epsilon_i^{2^{13}k^3}$$

and the cuboctahedron count of C_i is at least

$$(\epsilon_i^{70k^3})^{70k^3} \geq \epsilon_i^{2^{13}k^6}.$$

This implies that the density of D_i is at least

$$(\epsilon_i^{2^{13}k^6})^{80} \geq \epsilon_i^{2^{20}k^6} \geq \epsilon_i^{k^{15}}$$

and the cuboctahedron count is at least

$$(\epsilon_i^{2^{13}k^6})^{70k^3} n^5 \geq \epsilon_i^{2^{20}k^9} n^5 \geq \epsilon_i^{k^{15}} n^5.$$

Lastly, we also have

$$\theta_i'' \geq (\epsilon_i^{2^{13}k^6})^{35k^3} \geq \epsilon_i^{2^{19}k^9} \geq \epsilon_i^{k^{15}}$$

and

$$\gamma_i'' \geq (\epsilon_i^{2^{13}k^6})^{80k^2} \geq \epsilon_i^{2^{20}k^8} \geq \epsilon_i^{k^{15}}.$$

Note also that $\gamma_i, \gamma_i' \geq \gamma_i''$ and $\theta_i, \theta_i' \geq \theta_i''$ since the cuboctahedron counts of A_i and B_i are larger than that of C_i.

This gives us a subgraph D_i of A_i which is still dense, and has the property that any $2r$-PF (for $2 \leq r \leq k$) in D_i is popularly decomposable in A_i. We thus let $A_{i+1} = D_i$.

After each step of the inductive construction, the density α_{i+1} is at least $\epsilon_i^{k^{15}}$ and the cuboctahedron count $\epsilon_{i+1}n^5$ is at least $\epsilon_i^{k^{15}}n^5$. The threshold for popularity θ_{i+1} is at least $\epsilon_i^{k^{15}}$, and $\gamma_{i+1} \geq \epsilon_i^{k^{15}}$ also.

Therefore, starting at $A_0 = A$ with ϵn^5 cuboctahedra, we find that for $i \geq 1$ we have

$$\epsilon_i \geq \epsilon^{k^{15i}}.$$

This gives us

$$\alpha_i \geq (\epsilon^{k^{15(i-1)}})^{k^{15}} = \epsilon^{k^{15i}}$$

and similarly $\theta_i \geq \epsilon^{k^{15i}}$ and $\gamma_i \geq \epsilon^{k^{15i}}$.

Therefore every $2r$-PF in A_i is $\epsilon^{k^{15i}}$-popularly decomposable in A_{i-1} in at least $\epsilon^{k^{15i}} n^{2r}$ different ways. $\square$

To close this section, we shall briefly discuss what it means for a $2r$-PF to be $\theta_i(\epsilon, k)$-popularly decomposable in at least $\gamma_i(\epsilon, k)n^{2r}$ different ways, and how this is going to be used in later sections. For this purpose, we need another definition.

Definition 2.27. *Let C be a $2r$-cycle $x_1y_1 \dots x_ry_r$ with x_1 and y_1 sharing a row. A* shattered ring decomposition *of C consists of a $2r$-cycle $x_1'y_1' \dots x_r'y_r'$ with x_1' and y_1' sharing a column, together with rectangles $R_i = x_i''u_ix_i'''v_i$ and $S_i = y_i''w_iy_i'''z_i$ (where u_i shares a row with x_i'' and w_i shares a column with y_i'') such that for each i, x_i and x_i'' have the same label, x_i' and x_i''' have the same*

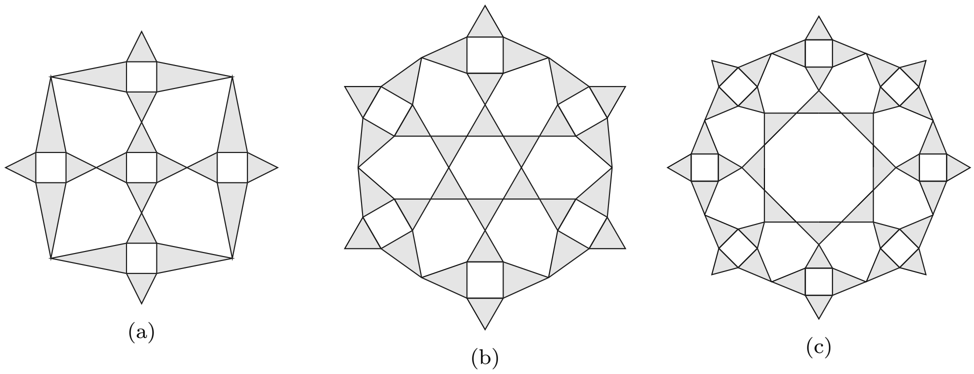


Figure 2.6: A shattered ring decomposition of a 4-PF, 6-PF and 8-PF in the hypergraph representation are depicted in (a), (b) and (c) respectively. In each figure the triangles correspond to faces of the 3-uniform hypergraph.

label, y_i and y_i'' have the same label, y_i' and y_i''' have the same label, u_i and z_i have the same label, and w_i and v_{i+1} have the same label.

The reason for this terminology is that one obtains a shattered ring decomposition if one begins with a ring decomposition and then replaces the back face and all the side faces by other cycles that have the same labellings. The conditions above are precisely the ones that will hold when we do this: a point in one cycle has to have the same label as a point in another cycle if before the 'shattering' they were the same point. Note that to say that a ring decomposition is popular is precisely to say that one can obtain many shattered ring decompositions from it in this way.

Although we have formulated this definition in grid terms, referring to cycles and labels, it has a natural description in hypergraph terms.

Definition 2.28. *Let F be a $2r$-PF. A* shattered ring decomposition *of F consists of a second $2r$-PF F' with petals in the same vertex class, together with $2r$ 4-PFs, each of has a petal equal to a petal of F and its opposite petal equal to the corresponding petal of F', and each of which shares a petal with its predecessor and a petal with its successor, in such a way that the assignment of vertex classes to the inner vertices of each 4-PF is the reflection of the assigment of classes to its predecessor.*

The hypergraph forms of shattered ring decompositions of a 4-PF, 6-PF and 8-PF are shown in Figure 2.6 (with the 4-PF, 6-PF and 8-PF not drawn – their petals will coincide with the degree-1 vertices in the diagrams).

If a $2r$-PF F is $\theta_i(\epsilon, k)$-popularly decomposable in at least $\gamma_i(\epsilon, k)n^{2r}$ different ways, this means that there are at least $\gamma_i(\epsilon, k)n^{2r}$ different ring decompositions of F into PFs that are $\theta_i(\epsilon, k)$-popular. If a $2s$-PF F' is $\theta_i(\epsilon, k)$-popular, this means that there are at least $\theta_i(\epsilon, k)n$ different $2s$-PFs that share all their petals with F'. This gives us the following lemma.

Lemma 2.29. *Let A be a tripartite linear hypergraph with n vertices in each class. Let F be a $2r$-PF which is $\theta_i(\epsilon, k)$-popularly decomposable in A in at least $\gamma_i(\epsilon, k)n^{2r}$ different ways. Then F has at least $\gamma_i \theta_i^{2r+1} n^{4r+1}$ different shattered ring decompositions.*

Proof. As discussed above, there are at least $\gamma_i n^{2r}$ different ring decompositions of F into PFs which are θ_i-popular. Each of these popular PFs can be replaced with one of $\theta_i n$ different PFs sharing petals with the original, giving a total of $(\theta_i n)^{2r+1}$ further choices, from which the result follows. $\square$

Broadly speaking, the arguments in the next section will involve starting with a particular hypergraph H and repeatedly replacing $2r$-PFs in H with shattered ring decompositions. Keeping track of the number of ways these replacements are possible will be achieved using Lemma 2.29.

2.4 The van Kampen picture

So far, we have described partial Latin squares both as labelled subsets of grids and as tripartite linear 3-uniform hypergraphs. For the next stage of the argument, we shall use a third description, which significantly simplifies the arguments and helps us to relate the combinatorial structures we consider to the more group-theoretic conclusions we wish to draw later.

We begin by observing that the definition of a spherical hypergraph given in 2.1.3 can be generalized to triangulations of any surface, and that the surface can have a boundary. As in the spherical case, one takes the edges of the triangulation to be the vertices of the hypergraph, and the triples of edges that bound triangles from the triangulation to be the faces of the hypergraph. Also as in the spherical case, we ask for the triangulation to be kaleidoscopic – that is, we ask for the edges to be properly 3-coloured in such a way that each vertex sees edges of only two colours.

If the surface has a boundary, then the boundary edges of the triangulation will correspond to what we have been calling petal vertices of the hypergraph. A simple example of this is illustrated in Figure 2.7, which shows how a triangulated square corresponds to a 4PF (in this case a row-column 4PF). More generally,

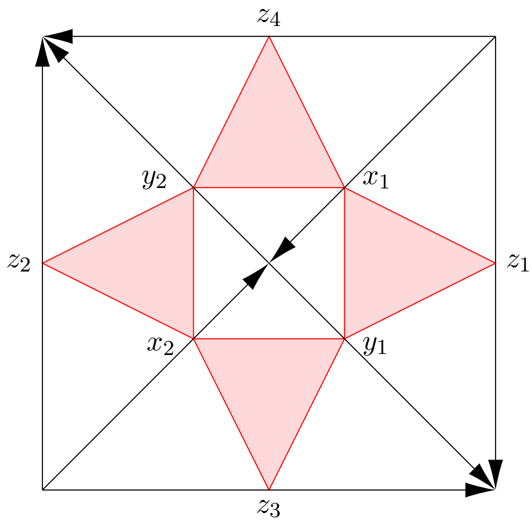


Figure 2.7: A 4-PF and the corresponding triangulated square. The 4-PF is pictured in red, and the triangulated square in black.

the triangulated surface corresponding to a $2r$PF is a $2r$-gon triangulated by joining a point inside it to each of its vertices.

As we discussed in §2.1.3, a cuboctahedron corresponds to the triangulation of the sphere that one obtains from the faces of an octahedron. A flappy cuboctahedron, on the other hand, corresponds to a triangulation of a surface that looks like an octahedron with a 'slit' – a single edge of an octahedron has been split into two edges with the same endpoints – which means that topologically it is a disc. This triangulation of the disc, with the appropriate labelling of edges, is illustrated in Figure 2.8. The orientations of the edges come from the fact that each face represents a relation of the form $x_i y_j = z_k$, which also explains why the triangulation is kaleidoscopic.

In some later parts of the argument, it will be most natural to abandon the grid and hypergraph representations for our Latin square and work entirely with triangulated surfaces. For this purpose we make the following definition.

Definition 2.30. *Given a partial Latin square A, the* van Kampen complex *of A is the simplicial complex K_A built as follows. For each $(x, y) \in A$ labelled with z, take a triangle with its edges labelled x, y and z, and oriented in such a way that the start vertex of the z edge is the start vertex of the x edge and the end vertex of the z edge is the end vertex of the y edge. Then identify all edges that are labelled in the same way, preserving their orientations. Vertices are identified only when this is forced by the identification of edges.*

It is possible to be more explicit about what the vertices are after identifi-

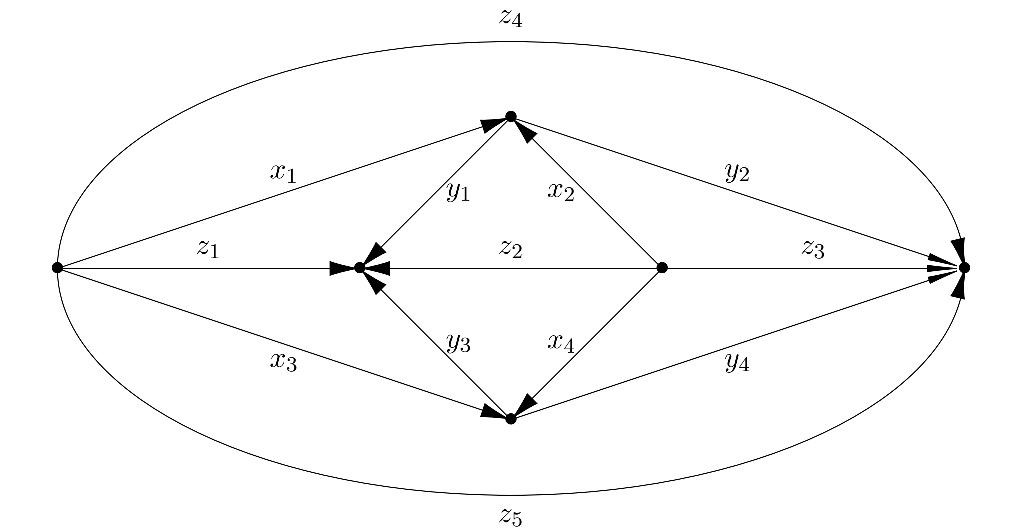


Figure 2.8: A triangulation of the disc that corresponds to a label-flappy cuboctahedron with flaps labelled z_4 and z_5.

cation. Let us denote an element of the Latin square A by a triple xyz, which indicates that label z occurs in position (x, y). Then let us call the edges of the corresponding triangle x, y and z, and the vertices xy, yz and xz (where, for example, the vertex xz is where the edge x meets the edge z). Then if $x_1y_1z_1$ and $x_1y_2z_2$ are elements of A, the corresponding triangles are identified along their edges x_1, so the vertices x_1y_1 and x_2y_2 are identified, as are the vertices x_1z_1 and x_1z_2.

Thus, if we form the bipartite graph mentioned earlier with vertex sets X and Y where $x \in X$ is joined to $y \in Y$ if and only if $xyz \in A$ for some z, then the xy vertices of K_A are the connected components of this graph. Similarly, the yz and xz vertices are the connected components of the two other bipartite graphs constructed in a similar way. Given an element xyz of the Latin square, the edge x joins the component that contains xz to the component that contains xy (in that direction).

For a typical dense partial Latin square we would expect these three bipartite graphs to be connected, so the van Kampen complex has just three vertices, but for a small surface this will not be the case. For example, the bipartite graphs coming from the octahedron each have two components, which correspond to the three antipodal pairs of vertices.

2.5 The popular replacement argument

Let A be a partially labelled grid with many cuboctahedra. The next part of the argument describes how we pass to a dense subset of A in which there are no small, flappy structures. Since the details will get somewhat involved, it will be instructive to begin with the case of flappy cuboctahedra, which will be enough to make the general strategy clear.

For this stage of the argument, it is most natural and convenient to use the van Kampen representation. By interpreting Theorem 2.26 in this framework, we will be able to view our popular replacements as a kind of 'unfixing' process: we start with a fixed triangulated surface, and little by little we 'unfix' vertices in order to convert it into a variable surface, at each stage ensuring that the number of possibilities for the variable surface is within a constant of the trivial maximum, given the points that are still fixed. This idea will be explained in more detail in the next section.

For the rest of this section, when we use the word 'surface' we shall mean 'triangulated surface'. If the surface has a boundary, the *length* of the boundary is the number of edges it contains.

2.5.1 Overview

As we have already mentioned, $2r$-PFs in the hypergraph represention of A correspond to $2r$-gons in the van Kampen complex K_A of A, which are triangulated using $2r$ triangles that each contain a single internal vertex. Given such a collection C of triangular faces, let F_C be the corresponding $2r$-PF in the hypergraph representation. If F_C is θ-popularly decomposable in γn^{2r} different ways, then Lemma 2.29 gives us $\gamma\theta^{2r+1}n^{4r+1}$ different shattered ring decompositions of F_C. Each of these shattered ring decompositions corresponds to a certain triangulated surface whose boundary coincides with the boundary of C. The boundary of C is a $2r$-gon, and the patch of surface corresponding to a shattered ring decomposition of a $2r$-PF consists of an inner $2r$-gon connected to the outer $2r$-gon with $2r$ edges between corresponding vertices, with the whole picture then triangulated by adding a new vertex to the center of each face – this is shown in Figure 2.9 for a 4PF. (The 4PF is not shown, apart from its boundary, which consists of the outer four edges in the diagram.)

The structure of our argument will be as follows. We start with a dense partial Latin square A, represented as a hypergraph. After applying Theorem 2.26 to create our sequence $A = A_0 \supset A_1 \supset \dots$, we shall fix some s and pick a partic-

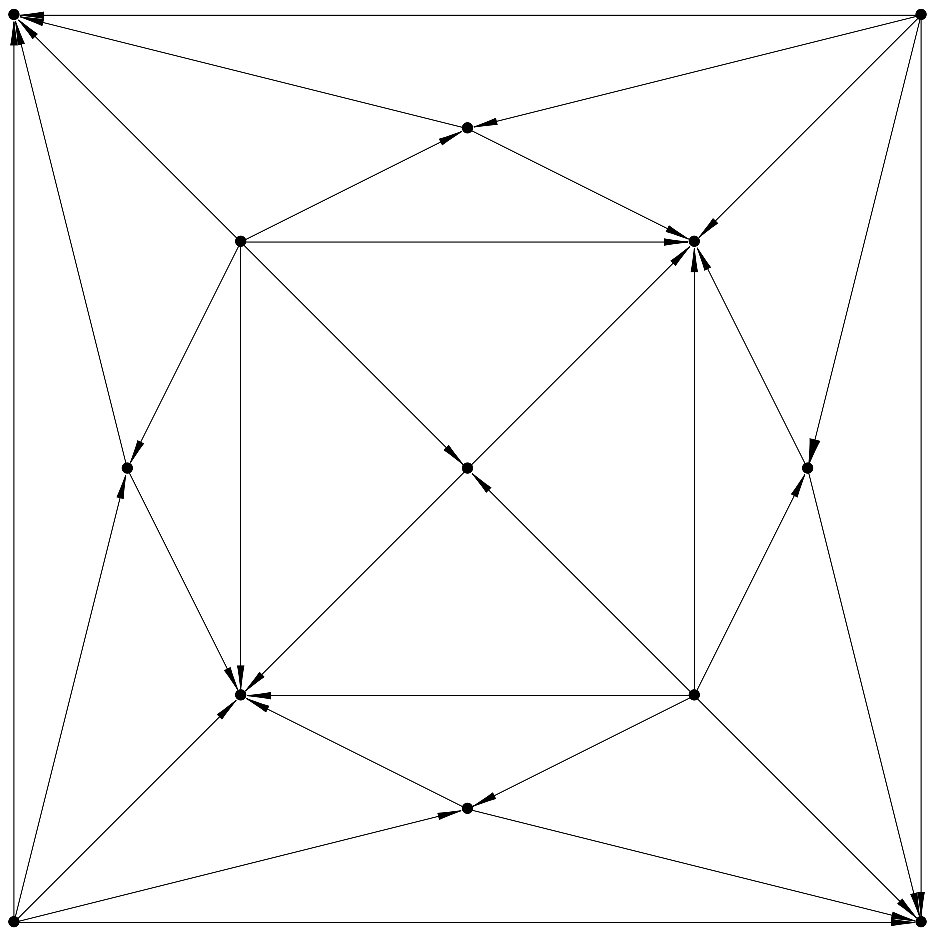


Figure 2.9: The surface corresponding to the shattered ring decomposition of a 4-PF. Since the shattered ring decomposition has four vertices of degree 1, the surface has a boundary of length 4. We have omitted the labels on the edges.

ular small flappy structure H_0 (such as the flappy cuboctahedron) and consider the auxiliary graph on the faces of A_s formed by joining two faces if they form the flaps of a copy of H_0. If the maximum degree of this auxiliary graph is bounded then we may pass to a dense independent set, which corresponds to a dense subhypergraph that avoids any copies of the chosen flappy structure.

Otherwise, we would like to find a contradiction. We are given a vertex of large degree in the auxiliary graph, which corresponds to a face of A_s that is contained in many different copies of H_0, each with a different 'opposite flap'. Each of these copies corresponds to a copy of a certain surface K_0 with boundary of length 2 in the van Kampen complex K_{A_s}. Given one of these surfaces, we perform our unfixing process. Initially, we say that all edges are *fixed*, meaning that we have specified precisely one copy of K_0. We then find a $2r$-gon in this copy and use the popular decomposability obtained from Theorem 2.26 to replace it with a new, more complicated surface, which we can do in many different ways. However we do the replacement, K_0 turns into a copy of a larger surface K_1 that still has a boundary of length 2. The copies of K_1 thus obtained lie in $A_{s-1} \supset A_s$, and we obtain $\Omega(n^{4r+1})$ of them, the trivial maximum being n^{4r+1}. We say that the internal edges in the chosen $2r$-PF are *unfixed*, since they may differ from copy to copy. Note that the number of fixed edges has decreased.

We may continue this process, choosing at each step a $2r$-PF with some fixed internal edges from K_i and using the popular decomposability to generate a larger collection of copies of a surface K_{i+1} that lies in A_{s-i-1}, with fewer fixed edges. If s is chosen sufficiently large relative to the area of K_0 then we may proceed until we obtain a collection $\mathcal{C}$ of copies of some diagram K_t in which the two boundary edges are fixed but every edge incident to an internal vertex is unfixed. One of the boundary edges corresponds to our initial vertex of high degree in the auxiliary graph. By repeating this process for each choice of neighbour of our chosen vertex from that auxiliary graph, we obtain many different collections of copies of K_t, each of which share one of the two boundary edges. By taking the union of all of these collections, we end up violating the trivial upper bound on the maximum possible number of copies of K_t in a van Kampen complex.

The next sections will expand on the details required for this argument. As promised earlier, we shall begin with a detailed account of the argument when H_0 is the flappy cuboctahedron, and then we shall tackle the necessary generalizations. Before we embark on this it will be necessary to work out the trivial maximum for the number of copies of a given surface with a given set of fixed edges in a dense van Kampen complex. The main task of this section will then be to verify that during the unfixing process, the number of copies we obtain is always within a constant of the appropriate trivial maximum, so that in particular this is the case when we reach the unfixed van Kampen diagram K_t, which is essential for obtaining our desired contradiction.

2.5.2 The maximum number of copies of a partially fixed surface

Let L be a van Kampen complex of a Latin square A. We define a *partially fixed surface* in L to be a triple (K, E, γ), where K is a surface with a kaleidoscopic triangulation, E is a subset of the edges of K, and γ is a homomorphism from E to the 1-skeleton of K that respects the tripartition of the vertices of K. We call the edges in E *fixed* and the other edges *unfixed*. We call a face *unfixed* if it contains at least one unfixed edge.

A *copy of* (K, E, γ) in L is a homomorphism $\phi : K \to L$ that extends γ in the obvious sense. Less formally, it is a copy of K in L for which the images of the fixed edges have to be given by γ. By the *trivial maximum number of copies* of a partially fixed surface (K, E, γ) we mean the maximum possible number of copies of a partially fixed surface (K, E, γ') in a van Kampen complex K_A of a partial Latin square A with column, row and label sets of size n.

Since the trivial maximum does not depend on the complex L or the map γ, we also define an *abstract partially fixed surface* to be just a pair (K, E), where K and E are as above. If no confusion is likely to arise, we shall omit the word 'abstract'. As above, the edges in E will be called fixed.

Lemma 2.31. *Let K be an abstract partially fixed surface obtained by triangulating the disc and fixing the boundary edges. Then the trivial maximum number of copies of K is at most n^{V_I} where V_I is the number of internal vertices – that is, vertices that do not lie on the boundary.*

Proof. The proof is by induction on the number of faces of K. The result is trivial when K is a single face with all three edges fixed. Now suppose that K has at least two faces. Suppose first that there is a face f that has two boundary edges. Then the third edge must be internal. The label of this edge is determined by the labels on the two boundary edges. If we remove the face f and fix its internal edge, then we obtain a surface that still has V_I internal vertices, and hence at most n^{V_I} copies, so we are done.

If K does not have such a face, then we split into two further cases. Suppose first that K has an internal vertex: that is, a vertex that does not lie on the boundary. Then there must be an internal vertex that is joined by an edge to a boundary vertex w. The neighbours of w form a path from its predecessor along the boundary to its successor. Let v be the first internal vertex along this path. Then v is joined to w and to its predecessor, which gives us a face that has one boundary edge and two internal edges. We can choose the label for one of the internal edges in at most n ways, and that determines the label for the other. Having done so, if we remove the face and fix the two internal edges, we obtain a simply connected van Kampen diagram K' with one less internal vertex. For each of the at most n choices of labelling for the newly fixed edges we get at most n^{V_I-1} copies of K', by the inductive hypothesis, so the number of copies of K is at most n^{V_I} as required.

The final case is where K does not have any internal vertices or any faces with two boundary edges. This case cannot in fact occur. Indeed, if it did, then note that the number of vertices would equal the number of boundary edges, and the number of faces would be at most the number of internal edges (since each face would contain at least two internal edges and each internal edge would be contained in two faces). It would follow that $V - E + F \leq 0$, contradicting Euler's formula (which would give $V - E + F = 1$, since we are not counting the external face as a face). □

A simple example that is important for us is a $2r$-gon with a single internal vertex in the middle: if the boundary is fixed, then we are left with at most n possibilities. In the grid picture, this corresponds to the fact that if we know the labels of a $2r$-cycle, then the first point of the cycle (which can be chosen in at most n ways) determines the rest of the cycle if it exists.

An even more important example is where K is taken to be the surface corresponding to the shattered ring decomposition of a $2r$-PF, again with the boundary cycle fixed. This bounds the maximum possible number of surfaces corresponding to shattered ring decompositions of a given $2r$-cycle, since all such diagrams share the boundary of the original $2r$-gon. The number of internal vertices is $4r + 1$, since the opposite $2r$-gon contributes $2r$ vertices, its central vertex contributes one vertex, and each of the $2r$ triangulated rectangles has a further internal vertex in the middle. Thus, Lemma 2.31 gives an upper bound of n^{4r+1} for the number of shattered ring decompositions of a given $2r$-PF. But Lemma 2.29 gives us $\Omega(n^{4r+1})$ such decompositions, so we see again that our machinery from the previous section gives us within a constant factor of the maximum number of such objects.

2.5.3 The flappy cuboctahedron case

Given an $n \times n$ partial Latin square A with ϵn^5 cuboctahedra, the aim of this section is to apply the results of the previous sections in order to pass to a dense subset of our given partial Latin square A in which there are no flappy cuboctahedra. Recall that a partial Latin square can be associated to a 3-uniform hypergraph as described in Section 2.1.2, and so our results will be formulated in these terms.

If B is a tripartite linear 3-uniform hypergraph, we define an auxiliary graph $G(B)$ on the same vertex set as B by joining vertices u and v if there is a flappy cuboctahedron in B with its petals at the vertices u and v. In terms of triangulated surfaces this is telling us that there is an octahedron with one of its edges 'slit' into two, with those boundary edges corresponding to u and v (see for example Figure 2.10 below). In grid terms, what an edge looks like depends on the types of the vertices u and v, but if, for example, they are label vertices corresponding to the labels d and d', then there will be an edge between them if there is a rectangle with labels a, b, c, d and another rectangle with labels a, b, c, d'. As discussed in Section 2.1.4, if we can prove that this auxiliary graph is of bounded degree, then we will be able to pass to a dense independent subset of the vertices and thereby eliminate all flappy cuboctahedra.

Our aim is to achieve this by taking B to be the subgraph A_s (for appropriately chosen s) given to us by Theorem 2.26. The rough idea is that if we fix a vertex x, then each edge xy_i in the auxiliary graph gives rise to a large number of flappy structures, or equivalently van Kampen diagrams with boundary word xy^{-1}, which we build from the initial slit-octahedron van Kampen diagram by unfixing all the interior vertices using popular replacements that are guaranteed by the theorem. If there are too many edges xy_i, this ends up contradicting Lemma 2.31.

We now give the details.

Lemma 2.32. *Let A be a tripartite linear 3-uniform hypergraph on vertex sets X, Y, Z each of size n with at least ϵn^5 cuboctahedra. Then A has a subgraph B of density at least $\epsilon^{2^{400}}$ such that the maximum degree in the graph $G(B)$ is at most $\epsilon^{-2^{450}}$.*

Proof. We shall begin with a flappy cuboctahedron, and associate with it a partially fixed surface (pictured in Figure 2.10) that has all its edges fixed. We shall then build a large collection of different copies of a more complicated partially fixed surface by performing popular replacements. Note that there are

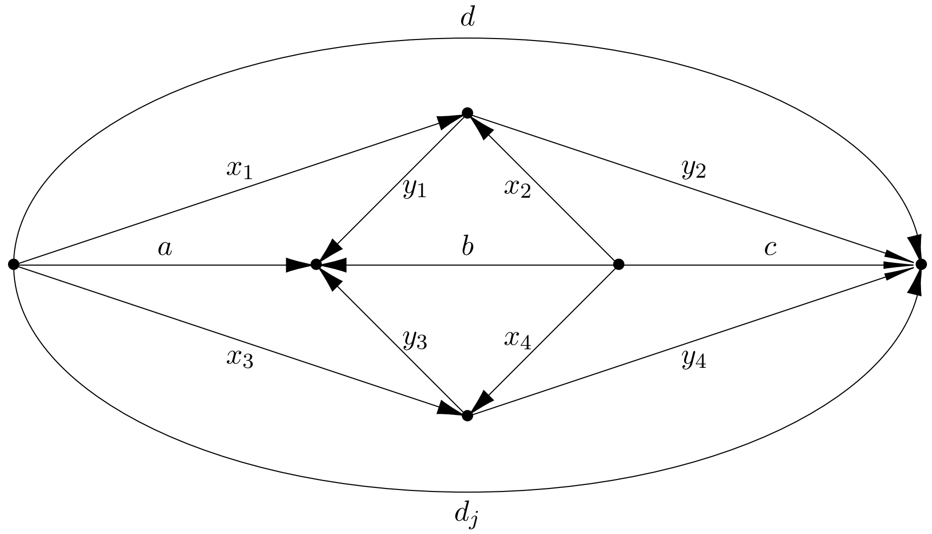


Figure 2.10: A surface corresponding to a label flappy cuboctahedron with flaps labelled d and d_j. The labels x_i correspond to rows in the grid representation, and the labels y_i correspond to columns.

four internal vertices in the van Kampen diagram. Each time we perform a popular replacement, we shall choose an internal vertex and replace the $2r$-gon that contains it by a more complicated surface that has the same boundary (corresponding to a shattered ring decomposition) and has all its internal edges unfixed. Once we have done this, we will obtain a partially fixed surface that has the same boundary as the original one and no fixed internal edges. At each stage of the process, the number of copies of the partially fixed surface will be within a constant of the trivial maximum number.

First we apply Theorem 2.26 with $k = 100$ (we only need $k \geq 4$) to obtain a sequence $A = A_0 \supset A_1 \supset \cdots \supset A_4$ with the property that A_i has density $\alpha_i(\epsilon)$, and for each $r = 2, \ldots, 4$ we have that every $2r$-PF in A_i is $\theta_i(\epsilon)$-popularly decomposable in A_{i-1} in at least $\gamma_i(\epsilon)n^{2r}$ different ways. The parameters $\alpha_i(\epsilon), \theta_i(\epsilon)$ and $\gamma_i(\epsilon)$ are all at least $\epsilon^{100^{15i}} \geq \epsilon^{2^{100i}}$.

Now suppose that the auxiliary graph $G(A_4)$ of A_4 has a vertex d of degree at least M. Without loss of generality, let us assume that the vertex class that contains this vertex is Z. This means that we can find a set $\{d_1, \ldots, d_M\}$ of distinct vertices in Z such that for each j there exists a flappy cuboctahedron in A_4 for which the flaps have label vertices d and d_j.

Let us now fix j and let K_0 be the corresponding surface, which we shall think of as a partially fixed surface for which every edge is fixed and γ is the appropriate inclusion map. It is illustrated in Figure 2.10. We now select a triangulated 4-gon

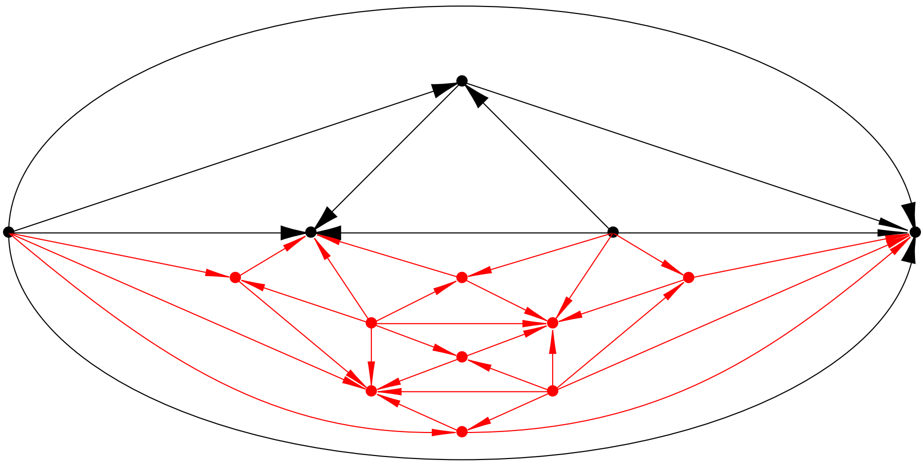


Figure 2.11: The surface K_1 obtained after the first popular replacement in a flappy cuboctahedron. The shattered ring decomposition is represented with the red part of the diagram. All labels have been omitted for simplicity.

in K_0 by choosing some internal vertex and taking the four faces that surround it. For instance, we may select the bottom internal vertex, which is incident to the edges labelled x_3, y_3, x_4 and y_4. This gives us the triangulated 4-gon represented by the four faces in the bottom half of the diagram. We now create a new partially fixed surface K_1 as follows. First we remove this 4-gon from K_0 and replace it by its shattered ring decomposition. Then we declare all the internal edges of the shattered ring decomposition to be unfixed, and the map γ takes the same values as before, but is applied only to the fixed edges. The surface K_1 is illustrated in Figure 2.11, with the unfixed edges in red.

Since the 4-gon is θ_4-popularly decomposable in A_3 in at least $\gamma_4(\epsilon)n^4$ different ways, it has at least $\gamma_4\theta_4^5 n^9$ shattered ring decompositions that live inside the set A_3. Since the trivial maximum number of these shattered ring decompositions is n^9, by Lemma 2.31 (because the number of internal vertices is 9), the number of copies of K_1 in the van Kampen complex K_{A_3} is within a constant of its trivial maximum, as we wanted.

The next step is to select another $2r$-gon by choosing another internal vertex, this time of K_1. We can do this by picking all the faces of K_1 that contain some given internal vertex that is incident to at least one unfixed edge. For instance, we might take the leftmost internal black vertex in Figure 2.11.

This gives us a 6-gon F, since this vertex is contained in six faces of K_1. Let K_2 be the partially fixed surface obtained by replacing F with a shattered ring decomposition and declaring all its internal edges to be unfixed. It is already challenging to draw K_2 in detail, and we shall see shortly that it is not impor-

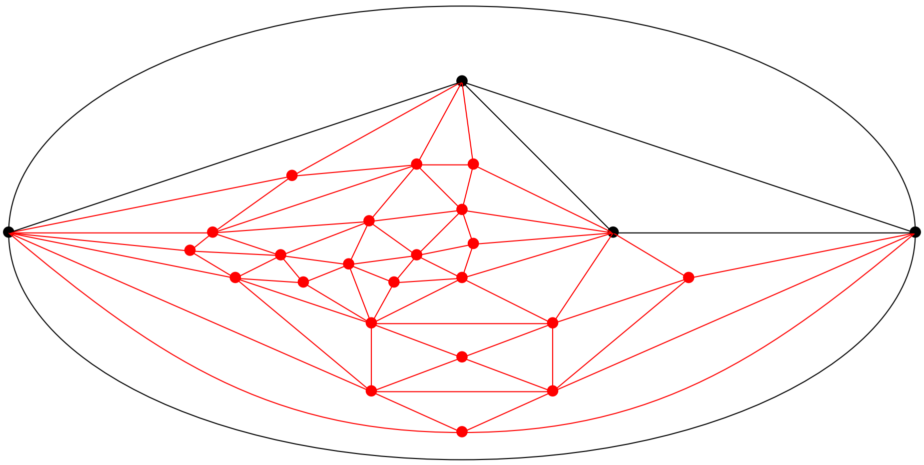


Figure 2.12: The partially fixed surface K_2 obtained after the second replacement. The fixed part is shown in black, and the unfixed part in red. All labels and directions have been omitted for simplicity.

tant to track the precise structure of the surfaces that we obtain at each step. Nevertheless, we include an illustration of K_2 in Figure 2.12 to help clarify the process.

Since any given 6-PF in A_3 is θ_3-popularly ring decomposable in A_2 in $\gamma_3 n^6$ different ways, it has $\gamma_3\theta_3^7 n^{13}$ different shattered ring decompositions. Thus, we may obtain a copy of K_2 in K_{A_2} by taking any one of the $\gamma_4\theta_4^5 n^9$ copies of K_1 and then replacing the image of the 6-gon F in that copy by any one of its $\gamma_3\theta_3^7 n^{13}$ shattered ring decompositions. We now claim that this gives us

$$(\gamma_4\theta_4^5 n^9)(\gamma_3\theta_3^7 n^{13}) = \gamma_3\gamma_4\theta_3^7\theta_4^5 n^{22}$$

copies of K_2 in K_{A_2}, but to verify this we must ensure that each copy we have just described is counted at most once.

Suppose that K, K' are copies of K_1 in $\mathcal{K}_1$ that, following replacements of their respective copies of F, both give the same copy of K_2. Then K and K' must agree on all but the internal edges of F. However, we chose F in such a way that one of the internal edges of F is fixed and thus shared between K and K'. But since all the edges of a $2r$-gon are determined once the boundary edges and a single internal edge are chosen (by the linearity of the underlying hypergraph), we see that $K = K'$.

Therefore we do not overcount, and the number of copies of K_2 in K_{A_2} is indeed $\gamma_3\gamma_4\theta_3^7\theta_4^5 n^{22}$. Again, it is easy to see that this is within a constant of the trivial maximum, since a shattered ring decomposition of a 6-PF has thirteen internal vertices, so the number of internal red vertices after the second unfixing is 22 (as the sceptical reader can verify from Figure 2.12).

The remaining two steps are similar. At the next step, we can replace the 8-PF around the rightmost, internal black vertex in Figure 2.12 by its shattered ring decomposition, with all the internal edges unfixed, to create a partially fixed surface K_3.

By Lemma 2.29, the number of shattered ring decompositions in A_1 is at least $\gamma_2\theta_2^9 n^{17}$, so that the number of copies of K_3 in K_{A_1} is at least $\gamma_2\theta_2^9 n^{17}$ times the number of copies of K_2 in K_{A_2}. But we will also have added $8+8+1=17$ new internal red vertices, so the trivial maximum increases by a factor of n^{17}. Therefore, the number of copies of K_3 is within a factor $\gamma_2\gamma_3\gamma_4\theta_2^9\theta_3^7\theta_4^5$ of the maximum possible.

In K_3 there is one remaining internal vertex that is incident to fixed edges. This vertex is the internal vertex of an 8-PF in K_3, so we may finish by replacing this 8-PF with a shattered ring decomposition to obtain a partially fixed surface K_4, for which only the two boundary edges are fixed. As before, Lemma 2.29 gives us at least $\gamma_1\theta_1^9 n^{17}$ shattered ring decompositions in K_{A_0}, and therefore Lemma 2.31 tells us that the number of copies of K_4 is within the constant factor $\gamma_1\gamma_2\gamma_3\gamma_4\theta_1^9\theta_2^9\theta_3^7\theta_4^5$ of the trivial maximum.

Drawings of the full structure of K_3 and K_4 would be too complicated to be illuminating, but we include Figure 2.13, which gives a global view of the replacement sequence we have performed. In this figure we show K_1, K_2, K_3 and K_4 but instead of drawing all the unfixed edges, we simply indicate where they are with red hatching.

Recall that the family $\mathcal{K}_4$ was obtained by starting with a given flappy cuboctahedron, which yielded a van Kampen diagram with boundary labelled d and d_j. By performing this sequence of popular replacements for each choice of $j \in \{1, \ldots, M\}$ we obtain M different collections of copies of the same van Kampen diagram K_4. Each of these collections has a fixed boundary, but one of the two fixed boundary edges differs from collection to collection. By taking the union over all these collections, we obtain a final collection $\mathcal{K}$ of copies of K_4 in which only the label on one of the two boundary edges is fixed.

Now we need an upper bound for the maximum number of copies of the partially fixed van Kampen diagram K_4', which is the same as K_4 except that only one of the two boundary edges is fixed. We cannot immediately apply Lemma 2.31 since the entire boundary is not fixed. But we can modify K_4 by attaching one new triangular face onto the unfixed boundary edge and fixing the other two edges of this face. We thus obtain a new partially fixed van

Kampen diagram K_4'' with a boundary consisting of three fixed edges, and every internal edge is unfixed. The maximum number of copies of K_4'' is at most the maximum number of copies of K_4', since adding extra fixed edges cannot increase the number. We can now apply Lemma 2.31 to K_4'', which has the same number of internal vertices as K_4. Therefore the maximum number of copies of K_4'' is the same as that of K_4, and hence the maximum number of copies of K_4' is at most that of K_4.

But the size of the collection $\mathcal{K}$ is $M|\mathcal{K}_4|$, and $|\mathcal{K}_4|$ is within a constant factor $\gamma_1\gamma_2\gamma_3\gamma_4\theta_1^9\theta_2^9\theta_3^7\theta_4^5$ of the maximum possible. Therefore if

$$M\gamma_1\gamma_2\gamma_3\gamma_4\theta_1^9\theta_2^9\theta_3^7\theta_4^5$$

$$\geq M\epsilon^{2^{450}} > 1$$

then we have our contradiction. Therefore we may take $B = A_4$, which has density at least $\alpha_4 \geq \epsilon^{2^{400}}$. $\square$

Our 'removal lemma' for flappy cuboctahedra (Theorem 2.6) follows from this lemma. We restate it here for convenience.

Theorem 2.6. *Let A be a tripartite linear 3-uniform hypergraph with vertex sets X, Y and Z of size n and suppose that A contains at least ϵn^5 cuboctahedra. Then there is a subhypergraph B of A with at least αn^2 faces, where $\alpha = \alpha(\epsilon) = \epsilon^{2^{453}}$, that contains no flappy cuboctahedra.*

Proof. We apply Lemma 2.32. This gives us a subgraph A' of A of density at least $\epsilon^{2^{400}}$ such that the associated graph $G(A')$ has maximum degree at most $\epsilon^{-2^{450}}$.

We now pick a maximal independent set I_X of vertices from the vertex class X in the graph $G(A')$ as follows. We first pick the vertex $v \in X$ that has highest degree in A' and add it to I_X. Then we discard all vertices in the neighbourhood of v in $G(A')$ and repeat, picking at each stage the remaining vertex of highest degree in A'. Since the maximum degree of $G(A')$ is at most $\epsilon^{-2^{450}}$, we end up picking at least $\epsilon^{2^{450}}n$ vertices from $G(A')$ corresponding to vertices belonging to at least a fraction $\epsilon^{2^{450}}$ of the edges of A'. Since A' has at least ϵn^5 cuboctahedra, A' must have edge density at least ϵ. Let A_1 be the subgraph of A' induced by I_X, Y an Z. Then A_1 has density at least $\epsilon^{2^{451}}$, and inside A_1 there is no flappy cuboctahedron with its degree 1 vertices belonging to X.

Since $G(A_1)$ also has maximum degree at most $\epsilon^{-2^{450}}$, we may similarly choose an independent set I_Y in the graph $G(A_1)$ of at least $\epsilon^{2^{450}}n$ vertices from

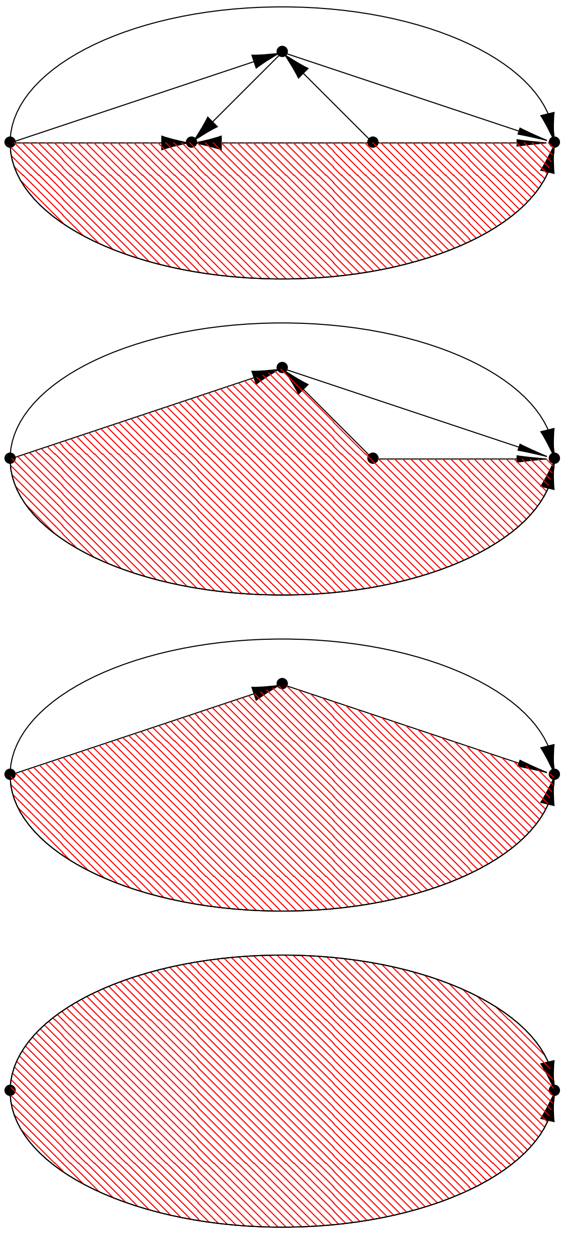


Figure 2.13: The sequence of four popular replacements from the proof of Lemma 2.32. Starting with a fixed surface corresponding to a flappy cuboctahedron, we progressively unfix all but the two boundary edges. Our unfixing process modifies the triangulation, and we represent the modified part with the red hatching (for example, the top figure represents K_1, shown in full detail in Figure 2.11). All edges in the triangulation represented by the red hatching are unfixed.

Y, accounting for at least a fraction $\epsilon^{2^{450}}$ of the edges of A_1. This gives us a set A_2 of density at least $\epsilon^{2^{452}}$ with no flappy cuboctahedron with its degree 1 vertices belonging to either X or Y.

Finally, we choose an independent set I_Z in the graph $G(A_2)$ of at least $\epsilon^{2^{450}}n$ vertices from Z, accounting for the greatest fraction of edges of A_2. This gives us a subgraph $A_3 = B$ of density at least $\epsilon^{2^{453}}$ with no flappy cuboctahedra. $\square$

2.5.4 The general case

Almost all of the complexity of the general case is contained in the detailed account given for the flappy cuboctahedron in the previous section. What remains is to describe how the replacement steps work in general, so that we can see that the argument for the cuboctahedron generalizes straightforwardly to arbitrary flappy structures.

The outline of the approach is as above. Given a surface K with boundary of length 2 and a tripartite linear hypergraph A, we shall define the auxiliary graph $G(A, K)$ on the vertex set of A by joining vertices d and d' by an edge if there is a copy of K in the van Kampen complex K_A with its two boundary edges labelled d and d'.

The main lemma will show that we may pass to a dense subgraph B of A such that the auxiliary graph $G(B, K)$ has bounded degree for each K of bounded size. If this is the case, then the elimination of flappy structures is straightforward – as in the proof of Theorem 2.6, we will simply pass down to independent sets in the graphs $G(B, K)$ in such a way that we avoid discarding too much of B.

The proof of the main lemma is similar to that of Lemma 2.32. Given M fixed copies of the surface K with boundary edges labelled d and d_j (for $j = 1, \ldots, M$), we shall unfix the edges by using popular decompositions of constituent $2r$-gons that surround internal vertices. At each stage we have, for each j, a collection of almost maximal size of copies of a partially fixed surface with boundary edges labelled d and d_j. We aim to show that once all edges incident to internal vertices are unfixed, we will have more than the trivial maximum number of copies of a certain partially fixed surface in the van Kampen complex of B unless M is bounded above by some constant that is independent of n (which will have a power dependence on ϵ, with the exponent depending on the number of faces of K).

Lemma 2.33. *Let A be a tripartite linear 3-uniform hypergraph with vertex sets X, Y, Z of size n with at least ϵn^5 cuboctahedra. Let $b \geq 100$. Then we can pass* *to a subgraph B of density at least $\epsilon^{b^{20b}}$ such that for each surface K with at most b faces and a boundary of length 2, the maximum degree in the graph $G(B,K)$ is at most $\epsilon^{-b^{20b}}$.*

Proof. We begin the proof, as we began the proof of Lemma 2.32, by applying Theorem 2.26, which we do with $k = 2b$. We obtain a sequence $A = A_0 \supset A_1 \supset \ldots$ with the property that A_i is $\alpha_i(\epsilon, 2b)$ dense and for each $r = 2, \ldots, k$ we have that every $2r$-PF in A_i is $\theta_i(\epsilon, 2b)$-popularly decomposable in A_{i-1} in at least $\gamma_i(\epsilon, 2b)n^{2r}$ different ways, where each of α_i, γ_i and θ_i are at least $\epsilon^{(2b)^{15i}} \geq \epsilon^{b^{20i}}$. Our set B will be A_b which has density at least $\epsilon^{b^{20b}}$. Note that the number of internal vertices of any surface with at most b faces is at most $3b/4 < b$, since each internal vertex is contained in at least four faces and each face contains at most three internal vertices.

Now let K be a surface with at most b faces and with all its edges fixed. Our goal is to unfix all edges except the boundary edges. As before, our unfixing steps involve picking vertices from the diagram, removing all of their incident faces and re-triangulating the resulting $2r$-gonal hole using the shattered ring decomposition of the $2r$-gon, taking all internal edges of this shattered ring decomposition to be unfixed. Starting with $K = K_0$, this process will lead us to construct a sequence $K = K_0, K_1, K_2, \ldots$ of partially fixed surfaces and associated collections $\mathcal{K}_i$ of copies of these surfaces, where the copies in the family $\mathcal{K}_i$ live in the set A_{s-i}.

In the previous section, we performed the replacements one by one and ensured at each stage that the size of $\mathcal{K}_i$ is within a constant of the maximum possible. For the general case, it will be simplest to perform the latter check at the end, once all replacements have been made and the we have reached a partially fixed surface K_s in which all edges incident to internal vertices are unfixed.

At each stage, we pick any vertex v inside K_i (not on the boundary) such that v is incident to fixed edges. We then consider the faces containing v – there are $2r_i$ of them giving a $2r_i$-PF (the number must be even because the surfaces are built from kaleidoscopic triangulations). We use popular decomposability to replace this $2r_i$-PF with a shattered ring decomposition with unfixed internal edges, giving us K_{i+1}. As before, the collection of copies $\mathcal{K}_{i+1}$ is obtained from $\mathcal{K}_i$ by choosing each possible replacement for each member of $\mathcal{K}_i$. As in the cuboctahedron case, we will have that the size of $\mathcal{K}_{i+1}$ will be equal to at least the size of $\mathcal{K}_i$ times the minimum number of different ring decompositions of the $2r_i$-PF in the set A_{s-i-1}. We do not overcount, since if two copies of K_i agree on all edges apart from those incident to v then, since v is also incident to a fixed edge, they must agree everywhere.

At each stage we reduce the number of internal vertices incident to fixed edges by exactly one, so the number of unfixing steps that we need to perform is equal to the number of internal vertices of the surface K, which is at most $3b/4$. Moreover, the maximum degree of a vertex in K is bounded above by b and this increases by at most two with each popular replacement. Thus, the maximum value of r for which we ever perform a popular replacement of a $2r$-PF is bounded above by $(b + 2(3b/4))/2 \leq 2b = k$.

We now consider the surface K_s that we get at the end of this process. Each time we do a popular replacement of a $4r_i$-PF, we increase the size of the family by a factor $\gamma_{k+1-i}\theta_{k+1-i}^{2r_i-1}n^{4r_i+1}$, by Lemma 2.29. So at the end of the process, the size of the collection $\mathcal{K}_s$ is at least

$$\gamma_b^b \theta_b^{4b^2} \prod_{i=1}^{s} n^{4r_i+1} \geq \epsilon^{b^{20b}} \prod_{i=1}^{s} n^{4r_i+1}.$$

The number of internal vertices of $\mathcal{K}_s$ is $\sum_{i=1}^{s}(4r_i + 1)$, since at each step of the unfixing process we replace one internal vertex by the $4r_i + 1$ internal vertices of a shattered ring decomposition. So, by Lemma 2.31, the maximum possible size of a collection of copies of K_s that agree on the boundary edges is $\prod_{i=1}^{s} n^{4r_i+1}$.

Therefore $|\mathcal{K}_s|$ is within a constant factor of the maximum possible. Indeed the constant factor η is bounded by

$$\eta \geq \epsilon^{b^{20b}}.$$

As before, we may repeat the same unfixing process (in the same order) for each different choice of label d_i $(i = 1, \ldots, M)$. Each different choice gives us a collection of surfaces with fixed boundary labels. The union of these collections is $\mathcal{K}$, a collection of copies of the partially fixed surface K' obtained by unfixing the appropriate boundary edge of K_s. By the same trick as in the previous section, we can apply Lemma 2.31 to deduce that the maximum possible number of copies of K' is in fact the same as the maximum possible number of copies of K_s, and therefore we obtain a contradiction if $M\eta > 1$. Therefore $M \leq \epsilon^{-b^{20b}}$, which proves the lemma. □

We are finally ready to prove Theorem 2.9, which we restate here for convenience.

Theorem 2.9. *Let A be a tripartite linear 3-uniform hypergraph with vertex sets of size n, and suppose that A contains at least ϵn^5 cuboctahedra. Then for any positive integer b there is a subhypergraph of A with at least $\epsilon^{b^{25b}} n^2$ faces that contains no flappy spherical hypergraphs with b faces or fewer.*

Proof. We apply Lemma 2.33 to obtain a subgraph A' of A such that the graph $G(A', H)$ has maximum degree at most $\epsilon^{-b^{20b}}$ for any flappy spherical hypergraph H with fewer than b faces. The goal is now to pass to subsets V_i of each vertex class with the property that $G_i(A_V, H)$ contains no edges for any choice of a flappy, spherical hypergraph H with at most b faces, where A_V is the subgraph of A induced by $V_1 \times V_2 \times V_3$.

In order to do this, we introduce the graph $G(A', b)$ which is the union of all graphs $G_i(A', H)$ where H is a flappy, spherical hypergraph with at most b faces. Since a flappy, spherical hypergraph has $3b/2+1$ vertices, the number of different flappy, spherical hypergraphs with at most b faces is at most $(3b/2+1)^{b+1}$, so $G(A', b)$ has maximum degree at most $(3b/2+1)^{b+1}\epsilon^{-b^{20b}}$.

Now, as in the proof of Theorem 2.6, we select our subsets V_i by passing to independent sets in the $G(A', b)$ in such a way that the number of faces in the induced subgraph A_V is maximised. Doing this gives us a subgraph B which is guaranteed to have at least

$$\left((3b/2+1)^{-(b+1)}\epsilon^{b^{20b}}\right)^3 n^2 \geq \epsilon^{b^{25b}} n^2$$

faces, and which contains no flappy spherical hypergraphs with fewer than b faces. □

Remark 2.34. Of course, Theorem 2.9 implies a version of Theorem 2.6, although the bound is somewhat worse because Theorem 2.9 uses crude estimates for the number of replacements required (whereas in the proof of Theorem 2.6 we determine an exact sequence of four replacements for the flappy cuboctahedron, and determine each r_i required).

2.6 From Theorem 2.9 to a metric group

In this short section we show how to deduce Theorem 2.4, our main theorem, from Theorem 2.9. This turns out to be quite easy. For convenience we restate the theorem here, and again we give explicit bounds. Note that in our metric group we allow infinite distances.

Theorem 2.4. *Let X, Y, Z be sets of size n, let $E \subset X \times Y$, and let $\lambda : E \to Z$ be a partial Latin square with at least ϵn^5 cuboctahedra. Then for every positive integer b there exist a subset $A \subset E$ of density at least $\epsilon^{b^{25b}}$, a metric group G, and maps $\phi : X \to G$, $\psi : Y \to G$ and $\omega : Z \to G$, such that the images $\phi(X), \psi(Y)$ and $\omega(Z)$ are 1-separated, and $d(\phi(x)\psi(y), \omega(z)) \leq b^{-1}$ for every $(x, y, z) \in X \times Y \times Z$ such that $(x, y) \in A$ and $\lambda(x, y) = z$.*

The group G has what may at first seem a slightly surprising definition: it is simply the free group generated by $X \cup Y \cup Z$. (If necessary, we make copies in order to ensure that the sets X, Y and Z are disjoint.) But the point is that when we place a metric on G, we are giving it a great deal of structure – it will an approximate version of what we do when we impose relations.

The metric arises naturally from the following question: if we are given a partial Latin square (X, Y, Z, A, λ), then how can we tell whether it is isomorphic (in an obvious sense) to part of the multiplication table of a group? One quickly observes that a universal construction yields a group H such that *if* the partial Latin square is isomorphic to part of the multiplication table of some group, then it must be isomorphic to part of the multiplication table of H. Indeed, H is the group with the following presentation. The generators are the elements of $X \cup Y \cup Z$ and the relations are all those of the form $xy = z$ such that $(x, y) \in A$ and $\lambda(x, y) = z$.

If we now define ϕ, ψ and ω to be the obvious inclusion maps, we have the property that $\phi(x)\psi(y) = \omega(z)$ whenever $(x, y) \in A$ and $\lambda(x, y) = z$. However, this is not enough, because all it gives us is a 'homomorphism' rather than an 'isomorphism'. The problem is that we may be able to use the relations to deduce that two generators are equal.

If that is the case, then corresponding to the proof of equality, say between generators x_1 and x_2, there will be a van Kampen diagram with boundary word $x_1 x_2^{-1}$. This will give us a triangulation of the disc, and because every relation is of the form $xy = z$ for some $x \in X, y \in Y$ and $z \in Z$, there is a proper 3-colouring of the edges of the triangulation, which implies that it is kaleidoscopic. Therefore, the van Kampen diagram gives rise to a flappy spherical hypergraph.

From this it follows that if we have the conclusion of Theorem 2.9, then there is no van Kampen diagram of area less than b with boundary word uv^{-1} for two unequal generators u, v. Therefore, defining a distance on G by taking $d(w_1, w_2)$

to be the smallest area of a van Kampen diagram (with the given relations) with boundary word $w_1w_2^{-1}$, we find that the generators are b-separated. Also, it is trivial that if $(x,y) \in A$ and $\lambda(x,y) = z$, then $d(xy,z) \leq 1$, since we have the relation $xy = z$. Therefore, if we rescale the distance by a factor of b^{-1}, and let ϕ, ψ and ω be the maps that take the elements of X, Y and Z to the corresponding generators of G, then we obtain the conclusion of Theorem 2.4.

It is easy to see that the correspondence we have just described goes the other way as well. Suppose that X, Y and Z are subsets of a metric group G, that $A \subset X \times Y$, and that $\circ : A \to Z$ is a binary operation with the property that $d(x \circ y, xy) \leq \delta$ for every $(x,y) \in A$. If we take the relations from the multiplication table as above, and if w is the boundary word of a van Kampen diagram of area at most k, then in k steps we can contract the diagram down to a point. At each stage of the contraction, we have a new boundary word, and the corresponding element of G is at distance at most δ from the element corresponding to the previous boundary word. Therefore, the element corresonding to w has distance at most $k\delta$ from the identity. If $k\delta < 1$, it follows that the boundary word is not of the form $x_1x_2^{-1}$, $y_1y_2^{-1}$ or $z_1z_2^{-1}$, and therefore there are no flappy spherical hypergraphs of size less than δ^{-1}.

Because of this correspondence, we see also that Green's result [40] implies that Theorem 2.9 cannot be improved to a result where one removes all flappy structures (as opposed to all flappy structures up to some given size). Indeed, let X be a maximal ϵ-separated subset of SO(3) and define a partial binary operation $\circ$ on X by setting $x \circ y = z$ if $d(xy,z) \leq \epsilon$. Then let H be the hypergraph corresponding to the multiplication table of $\circ$. If H had a dense subhypergraph with no flappy spherical hypergraphs at all, then it would correspond to a dense subset of X^2 that was isomorphic to a subset of a group, because we could take $b = \infty$. But this would contradict what Green proved.

2.7 Concluding remarks

It is important to stress that although algebraically G is just a free group, the metric gives it a much more interesting structure. Indeed, one can think of this metric as an approximate group presentation: instead of declaring that certain words are equal to the identity, we declare that they are *close* to the identity, and then we take the distance to be the largest one that is compatible with these 'approximate relations'. (Note that this should be read as 'approximate group-presentation' and not 'approximate-group presentation'.)

Theorem 2.4 gives us in particular a metric group G and three 1-separated subsets $\phi(X), \psi(Y), \omega(Z)$ of G of comparable size with the property that for a constant proportion of pairs $(x, y) \in \phi(X) \times \psi(Y)$ there exists $z \in \omega(Z)$ such that $d(xy, z) \leq \delta$, where $\delta = b^{-1}$. In particular, we can conclude that there is an approximate group H of size not much larger than $|\phi(X)|$ and translates xH and Hy of H such that a constant proportion of the points of $\phi(X)$ belong to xH and a constant proportion of the points of $\psi(Y)$ belong to Hy. In the appendix we show that a suitable 'metric entropy version' of this result holds, which allows us to replace equality by approximate equality and obtain an appropriate conclusion, where the notion of an approximate group is replaced by that of an approximate group that is also approximate in a metric sense. We call these structures 'rough approximate groups'. (To the best of our knowledge, this concept was first formulated by Tao [82], and a slight adaptation of it was introduced and studied by Hrushovski [48], who called it a metrically approximate subgroup.)

It would be very interesting to go further and describe in a more concrete way the structure of rough approximate groups, ideally obtaining an analogue of the results of Breuillard, Green and Tao on approximate groups [14]. We have not attempted to formulate a conjecture along these lines, but examples such as taking a maximal δ-separated subset of a small ball about the identity in SO(3), where the size of the ball tends to zero with δ but much more slowly than δ, suggest that Lie groups of bounded rank are likely to play a role, and also that the part played by nilpotency may be significantly different.

It is natural to ask whether there is an analogue of the results of this chapter for Abelian groups. In the following chapter we address this question, identifying a structure that plays the role that the cuboctahedron plays for general groups, in the sense that if the number of copies of that structure in a partial Latin square is within a constant of maximal, then the partial Latin square has Abelian-group-like behaviour. The proof turns out to be quite a lot harder, because it is necessary to consider surfaces of higher genus, and that leads to significant complications.

2.A Rough approximate groups

Let G be a group. A subset H of G is a *k-approximate subgroup* if it contains the identity, it is closed under taking inverses, and there exists a set K of size at most k such that $HH \subset KH$ – that is, if the product set HH can be covered by a bounded number of (left) translates of H. If G is a metric group, we shall say that a subset H is a (k,δ)-*rough approximate subgroup* if there is a set K of size at most k such that $HH \subset (KH)_\delta$, where for any subset U we write U_δ denotes the δ-expansion $\{x : d(x,U) \leq \delta\}$ of U. Thus, H is a rough approximate subgroup if every point in HH can be approximated by a point in one of a bounded number of translates of H. By a rough approximate group, we mean simply a rough approximate subgroup of some metric group. (As with approximate groups themselves, it is possible to define rough approximate groups more intrinsically, but since ours arise naturally as subsets of an ambient group, we shall not do this.)

Theorem 2.4 yields for us three 1-separated subsets X, Y, Z of a metric group G (here X, Y and Z refer to the sets $\phi(X)$, $\psi(Y)$ and $\omega(Z)$, but we have dropped the maps for convenience). These sets are all of roughly the same size, and there is a small positive number δ such that $d(xy, Z) \leq \delta$ for a positive proportion of pairs $(x, y) \in X \times Y$. In this appendix we shall deduce that there is a rough approximate subgroup H of G such that X has substantial overlap with a left translate of H, Y has substantial overlap with a right translate, and Z has substantial overlap with a two-sided translate. The (slightly stronger) precise statement is Theorem 2.48 below. The arguments are mostly contained in either [83] or [82], and those that are not are fairly straightforward modifications or extensions of those arguments. It is for that reason, and because the result is something of an optional extra to our main result, that we present it in an appendix rather than in the main body of the chapter.

2.A.1 Metric entropy definitions and some basic observations

Given a subset X of a metric space, and another subset Δ, we say that Δ is an ϵ-*net* of X if for every $x \in X$ there exists $y \in \Delta$ such that $d(x, y) < \epsilon$. An ϵ-*separated subset* of X is a subset Γ such that $d(x, x') \geq \epsilon$ for every pair of distinct elements $x, x' \in \Gamma$. Write $\nu_\epsilon(X)$ for the smallest size of an ϵ-net of X, and $\sigma_\epsilon(X)$ for the largest size of an ϵ-separated subset. We begin with three very basic lemmas.

Lemma 2.35. *Let X be a subset of a metric space and let $\epsilon > 0$. Then $\nu_\epsilon(X) \leq \sigma_\epsilon(X) \leq \nu_{\epsilon/2}(X)$.*

Proof. Let Γ be an ϵ-separated set of maximal size. Then in particular it is maximal. It follows that it is an ϵ-net. This proves the first inequality.

Now let Δ be an $(\epsilon/2)$-net. Then the balls of radius $\epsilon/2$ about the points of

Δ cover X, and no ϵ-separated set can contain more than one element in any of these balls. This proves the second inequality. $\square$

Lemma 2.36. *Let X and Y be subsets of metric spaces and let d be the metric on $X \times Y$ defined by $d((x, y), (x', y')) = d(x, x') \vee d(y, y')$. Then $\nu_\epsilon(X \times Y) \leq \sigma_{\epsilon/2}(X)\sigma_{\epsilon/2}(Y)$.*

Proof. By Lemma 2.35, we have that

$$\nu_\epsilon(X \times Y) \leq \nu_{\epsilon/2}(X)\nu_{\epsilon/2}(Y) \leq \sigma_{\epsilon/2}(X)\sigma_{\epsilon/2}(Y).$$

$\square$

Lemma 2.37. *Let X be a subset of a metric group and let $\epsilon > 0$. Then $\nu_\epsilon(X) = \nu_\epsilon(X^{-1})$ and $\sigma_\epsilon(X) = \sigma_\epsilon(X^{-1})$.*

Proof. This is an immediate consequence of the fact that

$$d(x, y) = d(y, x) = d(e, y^{-1}x) = d(x^{-1}, y^{-1})$$

for any two elements x, y of a metric group. $\square$

We shall write $\overline{\nu}_\epsilon(X)$ for the size of the smallest non-strict ϵ-net of X – that is, of the smallest set Δ such that for every $x \in X$ there exists $y \in \Delta$ with $d(x, y) \leq \epsilon$.

Lemma 2.38. *Let X, Y, Z be 1-separated subsets of a metric group G, let $\delta < \frac{1}{100}$, let $\epsilon < 1/6$, and suppose that $|Z| \leq \delta^{-1}|X|^{1/2}|Y|^{1/2}$ and that $d(xy, Z) \leq \epsilon$ for at least $\delta|X||Y|$ pairs $(x, y) \in X \times Y$. Then there are subsets $X' \subset X$ and $Y' \subset Y$ with $|X'| \geq \delta^7|X|$ and $|Y'| \geq \delta^7|Y|$ such that $\overline{\nu}_{6\epsilon}(\tilde{X'}\tilde{Y'}) \leq \delta^{-16}|X|^{1/2}|Y|^{1/2}$ and such that $d(xy, Z) \leq \epsilon$ for at least $\delta|X'||Y'|/4$ pairs $(x, y) \in X' \times Y'$.*

Proof. Form a bipartite graph G with vertex sets X, Y by joining x to y if and only if $d(xy, z) \leq \epsilon$. Then by hypothesis G has density δ.

We shall apply Lemma 2.17, but in order to do so we must first balance the sizes of the vertex sets. Suppose without loss of generality that $|X| \leq |Y|$. From the above discussion, we recall that $|Y| \leq \delta^{-4}|X|$. We now discard vertices of minimal degree from Y one by one, until we arrive at a subset $Y_1 \subset Y$ with $|Y_1| = |X|$. The edge density of the graph $G|_{X \times Y_1}$ is still at least δ.

Applying Lemma 2.17 with $k = 1$, we can find $X' \subset X$ and $Y' \subset Y_1$ with $|X'| \geq \delta^2|X|/16 \geq \delta^7|X|$ and $|Y'| \geq \delta^2|Y_1|/16 \geq \delta^6|Y|/16 \geq \delta^7|Y|$ such that between any $x \in X'$ and $y \in Y'$ there are at least $\delta^9|X||Y|$ paths of length 3 (with the two vertices in between not required to live in X' and Y') and such that the graph $G|_{X' \times Y'}$ has density at least $\delta/4$.

For each $x \in X'$ and $y \in Y'$, let $T(x, y)$ be the set of triples $(z_1, z_2, z_3) \in Z^3$ such that there exist $x_1 \in X$ and $y_1 \in Y$ with $d(xy_1, z_1), d(x_1y_1, z_2)$ and $d(x_1y, z_3)$ all at most ϵ. Since X, Y and Z are all 1-separated, there is a bijection between triples in $T(x, y)$ and paths of length 3 from x to y in the graph, so each set $T(x, y)$ has size at least $\delta^9|X||Y_1| \geq \delta^{13}|X||Y|$.

Suppose now that (z_1, z_2, z_3) belongs to $T(x, y)$ and x_1, y_1 are as above. Then from the three approximate relations and the fact that

$$xy = xy_1(x_1y_1)^{-1}x_1y,$$

it follows that

$$d(xy, z_1z_2^{-1}z_3) \leq 3\epsilon.$$

Now let $\Gamma = \{(x_1y_1), \dots, (x_my_m)\}$ be a 6ϵ-separated subset of $X'Y'$. Then the balls of radius 3ϵ about the x_iy_i are disjoint, from which it follows that the sets $T(x_i, y_i)$ are disjoint. But each one has size at least $\delta^{13}|X||Y|$ and their union has size at most $|Z|^3$, so $m \leq \delta^{-13}|Z|^3|X|^{-1}|Y|^{-1} \leq \delta^{-16}|X'|^{1/2}|Y'|^{1/2}$. This bound holds for all 6ϵ-separated subsets, so the result now follows from Lemma 2.35. $\square$

We remark that since X and Y are 1-separated sets, we could if we wanted replace the cardinalities $|X'|$ and $|Y'|$ in the statement above by the quantities $\sigma_1(X')$ and $\sigma_1(Y')$.

One of the main results of [83] is that if X, Y are finite subsets of a group and $|XY| \leq C|X|^{1/2}|Y|^{1/2}$, then there exists an approximate group H and sets K, L of bounded size such that $X \subset KH$ and $Y \subset HL$. (One can of course take K and L to be the same by taking their union.) In the next subsection, we shall prove an analogous statement for our metric-entropy context.

2.A.2 Products with small metric entropy come from rough approximate groups

The main theorem we prove in this subsection is the following metric-entropy variant of Theorem 4.6 of [83].

Theorem 2.39. *Let G be a metric group, let $\beta \geq 2048\epsilon$, and let $X, Y \subset$*

G be subsets such that $\nu_\epsilon(XY) \leq C\sigma_\beta(X)^{1/2}\sigma_\beta(Y)^{1/2}$. Then there exists a $(16C^{16}, 256\epsilon)$-rough approximate group $H \subset G$ and sets K, L of sizes at most $256C^{32}$ and $2048C^{48}$, respectively, such that KH is a 584ϵ-net of X, HL is a 2304ϵ-net of Y, and $\nu_{128\epsilon}(H) \leq 8C^{15}\sigma_\beta(X)^{1/2}\sigma_\beta(Y)^{1/2}$.

We begin with an analogue of the Ruzsa triangle inequality (which can also be found in [82]).

Lemma 2.40. *Let G be a metric group and let U, V, W be subsets of G. Then $\nu_\epsilon(U)\nu_\epsilon(VW^{-1}) \leq \sigma_{\epsilon/4}(UV^{-1})\sigma_{\epsilon/4}(UW^{-1})$.*

Proof. Let Γ_1 be an ϵ-separated subset of U and let Γ_2 be an ϵ-separated subset of VW^{-1}. Define $\phi : \Gamma_1 \times \Gamma_2 \to UV^{-1} \times UW^{-1}$ by choosing for each $x \in \Gamma_2$ a pair of elements $(v(x), w(x)) \in V \times W$ such that $v(x)w(x)^{-1} = x$, and then for each $(u, x) \in \Gamma_1 \times \Gamma_2$ defining $\phi(u, x)$ to be $(uv(x)^{-1}, uw(x)^{-1})$.

Suppose now that (u_1, x_1) and (u_2, x_2) are elements of $\Gamma_1 \times \Gamma_2$ such that $d(\phi(u_1, x_1), \phi(u_2, x_2)) < \delta$, where for our product metric we take the maximum of the metrics on UV^{-1} and UW^{-1}. Then $d(u_1v(x_1)^{-1}, u_2v(x_2)^{-1}) < \delta$ and $d(u_1w(x_1)^{-1}, u_2w(x_2)^{-1}) < \delta$. Since G is a metric group, it follows that

$$\begin{aligned} d(x_1, x_2) &= d(v(x_1)w(x_1)^{-1}, v(x_2)w(x_2)^{-1}) \\ &= d(v(x_1)u_1^{-1}u_1w(x_1)^{-1}, v(x_2)u_2^{-1}u_2w(x_2)^{-1}) \\ &< \delta + \delta = 2\delta. \end{aligned}$$

Therefore, if $\delta \leq \epsilon/2$ we can deduce that $x_1 = x_2$, since they are both elements of Γ_2. But then $d(u_1, u_2) = d(u_1v(x_1)^{-1}, u_2v(x_1)^{-1}) < \delta$, which implies that $u_1 = u_2$ as well.

Since Γ_1 and Γ_2 were arbitrary ϵ-separated subsets, it follows that

$$\sigma_\epsilon(U)\sigma_\epsilon(VW^{-1}) \leq \sigma_{\epsilon/2}(UV^{-1} \times UW^{-1}),$$

and hence by Lemmas 2.35 and 2.36, that

$$\nu_\epsilon(U)\nu_\epsilon(VW^{-1}) \leq \sigma_{\epsilon/4}(UV^{-1})\sigma_{\epsilon/4}(UW^{-1}).$$

∎

Corollary 2.41. *Let $\epsilon, \delta > 0$ and let X, Y be a subsets of a metric group such that $\nu_\epsilon(XY) \leq C\sigma_\delta(X)^{1/2}\sigma_{16\epsilon}(Y)^{1/2}$. Then $\nu_{8\epsilon}(XX^{-1}) \leq C^2\sigma_\delta(X)$.*

Proof. By Lemma 2.40, Lemma 2.37 and our hypothesis, we have that

$$\nu_{8\epsilon}(Y^{-1})\nu_{8\epsilon}(XX^{-1}) \leq \sigma_{2\epsilon}(Y^{-1}X^{-1})^2 = \sigma_{2\epsilon}(XY)^2 \leq \nu_\epsilon(XY)^2$$

$$\leq C^2\sigma_\delta(X)\sigma_{16\epsilon}(Y).$$

By Lemmas 2.37 and 2.35, $\nu_{8\epsilon}(Y^{-1}) = \nu_{8\epsilon}(Y) \geq \sigma_{16\epsilon}(Y)$, so the result follows. □

Our next lemma is a version of the Ruzsa covering lemma.

Lemma 2.42. *Let $\epsilon > 0$ and let A, B be subsets of a metric group such that $\nu_\epsilon(AB) \leq C\sigma_{2\epsilon}(B)$. Then there exists a set K of size at most C such that KBB^{-1} is a 2ϵ-net of A.*

Proof. Let $K \subset A$ be maximal such that for any two distinct elements $x, x' \in K$ the distance between the sets xB and $x'B$ is at least 2ϵ. Then if $y \in A$ there must be some $x \in K$ such that $d(xB, yB) < 2\epsilon$, by maximality, from which it follows that $d(y, xBB^{-1}) < 2\epsilon$. Therefore, KBB^{-1} is a 2ϵ-net of A.

Now let Γ be a 2ϵ-separated subset of B. Then $K\Gamma$ is a 2ϵ-separated subset of KB, which is contained in AB. It follows that $K\sigma_{2\epsilon}(B) \leq \sigma_{2\epsilon}(AB)$, which by Lemma 2.35 is at most $\nu_\epsilon(AB)$. By hypothesis this is at most $C\sigma_{2\epsilon}(B)$ and the result follows. □

Next we need a notion of 'popular differences' that will be suitable for this metric-entropy context.

Definition 2.43. *Let A be a subset of a metric group. We say that an element $d \in A^2$ is* (ϵ, δ, m)-popular *if there are m pairs $(x_i, y_i) \in A^2$ such that the sets $\{x_1, \ldots, x_m\}$ and $\{y_1, \ldots, y_m\}$ are δ-separated and $d(y_i^{-1}x_i, d) < \epsilon$ for every i,*

Lemma 2.44. *Let $\delta \geq 2\epsilon$, let A be a subset of a metric group such that $\nu_\epsilon(AA^{-1}) \leq C\sigma_\delta(A)$ and let S be the set of $(2\epsilon, \delta, \sigma_\delta(A)/2C)$-popular elements of $A^{-1}A$. Then $\sigma_\delta(S) \geq \sigma_\delta(A)/2C$.*

Proof. Let Γ be a δ-separated subset of A of size $\sigma_\delta(A)$. Choose a partition of AA^{-1} into $\nu_\epsilon(AA^{-1})$ sets, each contained in an open ball of radius ϵ, and write $z \sim w$ if z and w belong to the same cell of the partition.

If we choose a random cell from the partition, then the expected number of pairs $(x_1, x_2) \in \Gamma^2$ with $x_1x_2^{-1}$ in that cell is at least $\sigma_\delta(A)^2/\nu_\epsilon(AA^{-1})$. It follows that there are at least $\sigma_\delta(A)^4/\nu_\epsilon(AA^{-1}) \geq \sigma_\delta(A)^3/C$ quadruples

$(x_1, x_2, x_3, x_4) \in \Gamma^4$ such that $x_1 x_2^{-1} \sim x_3 x_4^{-1}$, and hence, since the cells are contained in balls of radius ϵ, such that $d(x_3^{-1}x_1, x_4^{-1}x_2) < 2\epsilon$. It follows that for a randomly chosen $(x_1, x_3) \in \Gamma^2$ the expected number of pairs $(x_2, x_4) \in \Gamma^2$ such that $d(x_3^{-1}x_1, x_4^{-1}x_2) < 2\epsilon$ is at least $\sigma_\delta(A)/C$. Since $\delta \geq 2\epsilon$, it is not possible to find $x, y, z \in \Gamma$ such that $x^{-1}y = x^{-1}z$ or such that $x^{-1}z = y^{-1}z$. It follows that the maximum number of pairs (x_2, x_4) with $d(x_3^{-1}x_1, x_4^{-1}x_2) < 2\epsilon$ is at most $\sigma_\delta(A)$. Therefore, there are at least $\sigma_\delta(A)^2/2C$ pairs $(x_1, x_3) \in \Gamma$ such that $x_3^{-1}x_1$ is $(2\epsilon, \delta, \sigma_\delta(A)/2C)$-popular.

By averaging we can find some x_i for which there are at least $\sigma_\delta(A)/2C$ popular pairs (x_i, x_j). If (x_i, x_j) and (x_i, x_k) are two distinct such pairs, then $d(x_j^{-1}x_i, x_k^{-1}x_i) \geq \delta$. It follows that there is a δ-separated subset of S of size at least $\sigma_\delta(A)/2C$, as claimed. □

Lemma 2.45. *Let $\delta \geq 4\epsilon$, let A be a subset of a metric group such that $\nu_\epsilon(AA^{-1}) \leq C\sigma_\delta(A)$ and let S be the set of $(2\epsilon, \delta, \sigma_\delta(A)/2C)$-popular elements of $A^{-1}A$. Then $\nu_{16\epsilon}(AS^3A^{-1}) \leq 8C^7\sigma_\delta(A)$.*

Proof. Let x_0, x_7 be elements of A and let $d_1, d_2, d_3 \in S$. Since each d_i is popular, we can approximate $x_0 d_1 d_2 d_3 x_7^{-1}$ as $x_0 x_1^{-1} x_2 x_3^{-1} x_4 x_5^{-1} x_6 x_7^{-1}$ in several ways. More precisely, for each $i = 1, 3, 5$ we have at least $\sigma_\delta(A)/2C$ independent choices for the pair (x_i, x_{i+1}), and the individual coordinates of these choices form δ-separated sets.

Each such product gives us an element $(x_0x_1^{-1}, x_2x_3^{-1}, x_4x_5^{-1}, x_6x_7^{-1})$ of the set $(AA^{-1})^4$. If $(x_0x_1^{-1}, x_2x_3^{-1}, x_4x_5^{-1}, x_6x_7^{-1})$ and $(x_0x_1'^{-1}, x_2'x_3'^{-1}, x_4'x_5'^{-1}, x_6'x_7^{-1})$ are two different such quadruples, then if their first i coordinates agree and the $(i+1)$st coordinate is different, then $x_j = x_j'$ for $0 \leq j < 2i$, and hence for $j = 2i$ as well, so we find that the two $(i+1)$st coordinates are $x_{2i}x_{2i+1}^{-1}$ and $x_{2i}x_{2i+1}'^{-1}$, which are separated by at least $\delta \geq 4\epsilon$.

We also have that if two elements of AS^3A^{-1} are separated by at least 16ϵ and for each one we choose a quadruple as above, then at least one coordinate of the two quadruples will be separated by at least 4ϵ, since the products of the two quadruples give the two elements.

It follows that

$$\sigma_{16\epsilon}(AS^3A^{-1})(\sigma_\delta(A)/2C)^3 \leq \sigma_{4\epsilon}((AA^{-1})^4) \leq \nu_\epsilon(AA^{-1})^4.$$

Since $\nu_\epsilon(AA^{-1}) \leq C\sigma_\delta(A)$, this implies the result. □

Lemma 2.46. *Let $\delta \geq 2\epsilon$ and let S be a subset of a metric group such that $S = S^{-1}$ and $\nu_\epsilon(S^3) \leq C\sigma_\delta(S)$. Then S^2 is a $(C, 2\epsilon)$-rough approximate group.*

Proof. By Lemma 2.42 with $A = S^2$ and $B = S$ there is a set K of size at most C such that KS^2 is a 2ϵ-net of S^2. □

Lemma 2.47. *Let $\delta \geq 2\epsilon_1$, let A be a subset of a metric group, let H be a (C_2, ϵ_2)-rough approximate group, and suppose that $\nu_{\epsilon_1}(AH) \leq C_1\sigma_\delta(H)$. Then there is a set K of size at most C_1C_2 such that KH is a $(2\epsilon_1 + \epsilon_2)$-net of A.*

Proof. By Lemma 2.42 there is a set K_1 of size at most C_1 such that K_1H^2 is a $2\epsilon_1$-net of A. By the definition of an approximate group there is also a set K_2 of size at most C_2 such that K_2H is an ϵ_2-net of H. But then K_1K_2H is a $(2\epsilon_1 + \epsilon_2)$-net of A. □

Proof of Theorem 2.39.

If X, Y are subsets of a metric group and $\nu_\epsilon(XY) \leq C\sigma_\beta(X)^{1/2}\sigma_\beta(Y)^{1/2}$, then by Corollary 2.41 we have the inequality $\nu_{8\epsilon}(XX^{-1}) \leq C^2\sigma_\beta(X)$. By Lemmas 2.44 and 2.45 we obtain a set S with $S = S^{-1}$ and $\sigma_\beta(S) \geq \sigma_\beta(X)/2C^2$ such that $\nu_{128\epsilon}(XS^3X^{-1}) \leq 8C^{14}\sigma_\beta(X)$.

It follows that $\nu_{128\epsilon}(S^3) \leq 16C^{16}\sigma_\beta(S)$. Therefore, by Lemma 2.46, S^2 is a $(16C^{16}, 256\epsilon)$-rough approximate group.

We also have that $\nu_{128\epsilon}(XS^2) \leq 16C^{16}\sigma_\beta(S^2)$. Therefore, by Lemma 2.47 there is a set K of size at most $256C^{32}$ such that KS^2 is a 512ϵ-net of X.

By Lemma 2.40,

$$\begin{aligned}
\nu_{1024\epsilon}(X)\nu_{1024\epsilon}(S^2Y) &\leq \sigma_{256\epsilon}(XS^2)\sigma_{256\epsilon}(XY) \\
&\leq \nu_{128\epsilon}(XS^2)\nu_{128\epsilon}(XY) \\
&\leq 16C^{16}\sigma_\beta(S^2).C\sigma_\beta(X)^{1/2}\sigma_\beta(Y)^{1/2}.
\end{aligned}$$

But

$$\sigma_\beta(X) \leq \sigma_\beta(XY) \leq \nu_{\beta/2}(XY) \leq C\sigma_\beta(X)^{1/2}\sigma_\beta(Y)^{1/2},$$

so $\sigma_\beta(X) \leq C^2\sigma_\beta(Y)$ and therefore $\sigma_\beta(X)^{1/2}\sigma_\beta(Y)^{1/2} \leq C\sigma_\beta(Y)$. Also, since $\beta \geq 2048\epsilon$,

$$\sigma_\beta(S^2) \leq \nu_{128\epsilon}(XS^3X) \leq 8C^{14}\sigma_\beta(X) \leq 8C^{14}\nu_{1024\epsilon}(X).$$

It follows that $\nu_{1024\epsilon}(Y^{-1}S^2) = \nu_{1024\epsilon}(S^2Y) \leq 128C^{32}\sigma_\beta(Y)$.

Therefore, by Lemma 2.47 again it follows that there is a set L of size at most $2048C^{48}$ such that LS^2 is a 2304ϵ-net of Y^{-1}, which implies that S^2L^{-1} is a 2304ϵ-net of Y. $\square$

We conclude this appendix by combining Lemma 2.38 and Theorem 2.39. We shall present the result (mostly) without explicit constants, but it is not hard to obtain them.

Theorem 2.48. *Let X, Y, Z be 1-separated subsets of a metric group G, let $0 < \delta < 1/100$, let $\epsilon > 0$ be sufficiently small, and suppose that $|Z| \leq \delta^{-1}|X|^{1/2}|Y|^{1/2}$ and that $d(xy, Z) \leq \epsilon$ for at least $\delta|X||Y|$ pairs $(x, y) \in X \times Y$. Then there exist subsets $X'' \subset X$, $Y'' \subset Y$ and $Z'' \subset Z$ with $|X''| = \delta^{O(1)}|X|$, $|Y''| = \delta^{O(1)}|Y|$ and $|Z''| = \delta^{O(1)}|Z|$, a $(\delta^{-O(1)}, O(\epsilon))$-rough approximate group $H \subset G$, and elements u, v, w of G such that $\nu_{O(\epsilon)}(H) = \delta^{-O(1)}|X|^{1/2}|Y|^{1/2}$, $X'' \subset (uH)_{O(\epsilon)}$, $Y'' \subset (Hv)_{O(\epsilon)}$, $Z'' \subset (X''Y'')_\epsilon \cap (uwHv)_{O(\epsilon)}$ and $d(xy, Z'') \leq \epsilon$ for $\delta^{O(1)}|X''||Y''|$ pairs $(x, y) \in X'' \times Y''$.*

Proof. Lemma 2.38 gives us $X' \subset X$ and $Y' \subset Y$ with $|X'| \geq \delta^7|X|$ and $|Y'| \geq \delta^7|Y|$ such that $\nu_{O(\epsilon)}(X'Y') = \delta^{-O(1)}|X|^{1/2}|Y|^{1/2}$ and such that $d(xy, Z) \leq \epsilon$ for $\delta^{O(1)}|X'||Y'|$ pairs $(x, y) \in X' \times Y'$. Applying Theorem 2.39 (with $\beta = 1$), we obtain a $(\delta^{-O(1)}, O(\epsilon))$-rough approximate group $H \subset G$ and sets K, L of sizes $\delta^{-O(1)}$ such that $X' \subset (KH)_{O(\epsilon)}$ and $Y' \subset (HL)_{O(\epsilon)}$.

We will pick $u \in K$ and $v \in L$ at random, and let $X'' = X' \cap (uH)_{O(\epsilon)}$ and $Y'' = Y' \cap (Hv)_{O(\epsilon)}$. By averaging there are choices $u \in K$ and $v \in L$ such that $|X''| = \delta^{O(1)}|X|$, $|Y''| = \delta^{O(1)}|Y|$ and $d(xy, Z) \leq \epsilon$ for $\delta^{O(1)}|X''||Y''|$ pairs $(x, y) \in X'' \times Y''$.

Observe that since $X'' \subset (uH)_{O(\epsilon)}$ and $Y'' \subset (Hv)_{O(\epsilon)}$, we have that $X''Y'' \subset (uHHv)_{O(\epsilon)}$. Since H is a $(\delta^{-O(1)}, O(\epsilon))$-rough approximate subgroup of G, this means that there exists a set $M \subset G$ of size $\delta^{-O(1)}$ such that $X''Y'' \subset (uMHv)_{O(\epsilon)}$.

Since $X''Y'' \subset (uMHv)_{O(\epsilon)}$, we have that $(X''Y'')_\epsilon \subset (uMHv)_{O(\epsilon)}$. Let

$$Z' = Z \cap (X''Y'')_\epsilon \subset (uMHv)_{O(\epsilon)}$$

and observe that $d(xy, Z') \leq \epsilon$ for $\delta^{O(1)}|X''||Y''|$ pairs $(x, y) \in X'' \times Y''$.

Now we choose $w \in M$ uniformly at random, and let $Z'' = Z' \cap (uwHv)_{O(\epsilon)}$. Since $|M| = \delta^{-O(1)}$, we have in expectation that $d(xy, Z'') \leq \epsilon$ for $\delta^{O(1)}|X''||Y''|$ pairs $(x, y) \in X'' \times Y''$. Suppose without loss of generality that $|X| \geq |Y|$. If

$d(xy, Z'') \leq \epsilon$ for at least $\delta^{O(1)}|X''||Y''|$ pairs $(x, y) \in X'' \times Y''$, then there exists a choice of $y \in Y''$ such that $d(xy, Z'') \leq \epsilon$ for $\delta^{O(1)}|X''|$ choices of $x \in X''$. Since X'' is 1-separated, this implies that $|Z''| = \delta^{O(1)}|X''| = \delta^{O(1)}|Z|$. Therefore there is some choice of $w \in M$ satisfying our requirements. $\square$

Chapter 3

Latin squares and multiplication tables of abelian groups

This chapter is based on recent joint work with W. T. Gowers [37].

3.1 Introduction

The work in this chapter follows on from that in Chapter 2, but we have written it to be as self-contained as possible.

In Chapter 2, we proved that a statistical condition – the presence of many cuboctahedra in the multiplication table of a partial binary operation – gives rise to a structural property that says that a large part of the partial binary operation has an approximate metric group structure. In this chapter we find a similar condition that will guarantee that a large part has an approximate metric abelian group structure – a notion that we will explain shortly.

3.1.1 Results of Chapter 2 and rough approximate groups

We shall begin with a brief discussion of the results from Chapter 2 in order to set the scene. We first recall the following definition and theorem which motivated the work in Chapter 2.

Definition 3.1 (Quadrangle condition, [13]). *We say that a Latin square A*

satisfies the quadrangle condition *if the label d in a configuration*

$$\begin{matrix} a & b \\ c & d \end{matrix}$$

is uniquely determined by a, b, c. Equivalently, A satisfies the quadrangle condition if A contains n^5 copies of the following configuration, which we call a cuboctahedron:

$$\begin{matrix} a & b & & \\ c & d & & \\ & & a & b \\ & & c & d \end{matrix}$$

Theorem 3.2 (Brandt, [13])**.** *Suppose that A is a Latin square satisfying the quadrangle condition. Then A is the multiplication table of a group G of order n.*

In Chapter 2, we tackled a '1%' version of Brandt's theorem. For this question, we assume that the Latin square A satisfies only a partial associativity condition, namely that it contains ϵn^5 cuboctahedra for some positive ϵ. We then investigated whether some dense subset of A must be somehow group-like.

We observed that a partial Latin square A embeds into a group multiplication table if and only if the corresponding hypergraph does not contain any *flappy, spherical hypergraphs*. In order to make this statement precise, we make the following definition.

Definition 3.3. *Given a partial Latin square A we let G_A be the group with $3n$ generators given by the set of rows, columns and labels of A with relations $xyz^{-1} = e$ for each triple $(x, y, z) \in A$. We also define (as in Chapter 2) the* van Kampen complex of A*, written $\mathcal{K}_A$, which is the directed simplicial complex containing $3n$ edges (1-simplices) corresponding to the generators of G_A and a 2-simplex bounded by edges x, y, z for each relation $xyz^{-1} = e$ in G_A.*

A *flappy, spherical hypergraph S* corresponds to a van Kampen diagram V_S with boundary of length 2. A specific copy of S in A corresponds to a specific copy of V_S in $\mathcal{K}_A$ (meaning a specific assignment of edges of V_S to 1-simplices of $\mathcal{K}_A$ in such a way that each face of V_S is mapped to a 2-simplex of $\mathcal{K}_A$). Moreover, the van Kampen diagram V_S can be drawn in the plane (i.e. it is homeomorphic to a disk). The word *spherical* is used because if the boundary edges were glued together then the resulting surface would be homeomorphic to a sphere.

While we could not ensure that the partial associativity assumption allows us to pass to a dense subset of A with no such van Kampen diagrams (indeed we provide strong evidence to suggest this is too much to hope for), we showed that for each K we can find a dense subset B_K with no van Kampen diagram with boundary of length 2 and fewer than K faces (see Theorem 2.9).

We showed that this property provides an approximate metric group structure in the following sense. First, we take G to be the free group on $3n$ generators given by the set of rows, columns and labels of B_K. The metric on this group is the *van Kampen metric*, where the distance between two words $w_1, w_2 \in G$ is defined to be the area of the smallest van Kampen diagram whose boundary is $w_1 w_2^{-1}$

The lack of small van Kampen diagrams in B_K proves that the set of generators is well-separated, and we can use this to obtain an approximate embedding of B_K into the multiplication table of G (see Theorem 2.4 for more details). In Appendix 2.A we combined this result with metric-entropy analogues of various results in sum-set theory to obtain a rough approximate group into whose multiplication table a large fraction of B_K can embed.

We were thus able to identify a configuration of bounded size (the cuboctahedron) such that if this configuration appears with a positive proportion of the maximum frequency then there must be an underlying approximate metric group structure. In this chapter we would instead like to identify a bounded-size configuration that would yield an approximate metric *abelian* group structure. We give more details on what is meant by this in the next section.

3.1.2 Approximate metric abelian group structure

We approach the abelian question in a manner similar to Chapter 2. There, we began by classifying the obstructions to a partial Latin square A embedding into a group multiplication table in terms of van Kampen diagrams, and then we worked on eliminating as many of these obstructions as possible (which turned out to be all those of bounded size).

Now, we would like to classify the larger collection of obstructions to embedding A into an abelian group multiplication table. Such an obstruction corresponds to a van Kampen diagram in $\mathcal{K}_A$ with a boundary word that simplifies to the length 2 word xy^{-1} if we assume that all elements commute. For example,

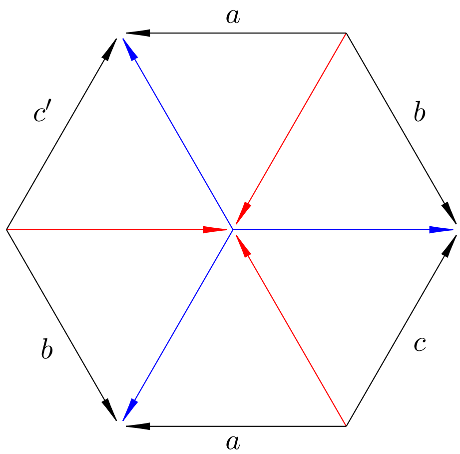


Figure 3.1: A van Kampen diagram with boundary word $a^{-1}bc^{-1}ab^{-1}c'$ which simplifies to $c^{-1}c'$ if the elements commute. This is therefore an example of a van Kampen diagram that is not permitted in a subset of an abelian group multiplication table.

the van Kampen diagram in Figure 3.1 corresponds to the configuration

$$\begin{array}{ccc} a & b & \\ & c & a \\ c' & & b \end{array}$$

which is a flappy version of a hypergraph we shall later call $\mathcal{T}$ (the flap vertices are the labels c and c').

We can identify boundary edges with the same label in such a van Kampen diagram. This defines a surface with a boundary of length 2, which we call a *van Kampen surface*. By this we mean a van Kampen diagram that is drawn on some oriented surface which need not be the plane. When we refer to a van Kampen surface V we usually mean the abstract structure as a simplicial complex[1] (combined with the information about which edges represent rows, columns or vertices) but we do not assign specific generators of G_A to the edges. When we are referring to a specific assignment of generators to edges of V, we will refer to a *copy of V in $\mathcal{K}_A$*.

The example in Figure 3.1 is a triangulated torus (note that we allow our triangulations to involve distinct triangles sharing all three vertices, such as the top and bottom triangles in Figure 3.1).

The obstructions, therefore, to a partial Latin square embedding into an

[1]Strictly speaking, van Kampen surfaces are not simplicial complexes because we allow distinct faces to intersect on more than a single edge. For instance, in Figure 3.1 the top and bottom faces intersect in an edge and the opposite vertex.

abelian group multiplication table are characterised by van Kampen surfaces of boundary length 2. This classification is strikingly similar to that in Chapter 2 – the only difference arises from the distinction between van Kampen diagrams and van Kampen surfaces, namely that the latter may be drawn on a higher-genus oriented surface.

The obvious hope, therefore, is that starting from the presence of many copies of a suitable configuration in a partial Latin square A, we might be able to pass to a large subset B_K in which there are no van Kampen surfaces with a boundary of length 2 and fewer than K faces (rather than just no such van Kampen diagrams).

The cuboctahedron is not a suitably powerful starting configuration, since the multiplication tables of non-abelian groups yield many copies of van Kampen surfaces with a boundary of length 2 and yet have the maximum number of cuboctahedra. However, before thinking further about which configurations may be more suitable for promoting abelian structure, we pause to consider what structure we might be left with if we were successful in our aim of eliminating small van Kampen surfaces with a boundary of length 2.

In Chapter 2, the elimination of small van Kampen diagrams with boundary of length 2 was used to provide a well-separated condition for the set of generators in the free group under the van Kampen metric described earlier. With the elimination of small van Kampen surfaces with boundary of length 2, we are able to prove a well-separated condition with respect to the following, smaller metric.

Let G be the free group with generators given by the set of rows, columns and labels of a partial Latin square B_K such that $\mathcal{K}_{B_K}$ contains no van Kampen surfaces with a boundary of length 2 and fewer than K faces. Let G^* be the free abelian group on the same set of generators. We define an *abelian van Kampen metric* d on words in G as follows. For words w_1 and w_2 in G we let the distance $d(w_1, w_2)$ be the area of the smallest van Kampen diagram with the relations xyz^{-1} for $(x, y, z) \in B_K$ and boundary word equal to $w_1 w_2^{-1}$ in G^*. We emphasise that the boundary word does not have to be precisely $w_1 w_2^{-1}$, but simply any word equal to $w_1 w_2^{-1}$ in G^*. If no such diagram exists, we set the distance $d(w_1, w_2) = \infty$. It is an easy exercise to prove that d is a metric on G (or rather a pseudometric, since distinct words in G which are equivalent in G^* are at distance 0), and the distance under d between words is at most distance under the (non-abelian) van Kampen metric.

Observe that if $w_1 = x$ and $w_2 = y$ are single generators, then the words w^*

in G^* that are equal to $w_1 w_2^{-1}$ are obtained by adding some number of inverse pairs to the word xy^{-1} and then reordering. Thus, van Kampen diagrams with boundary words w^* are van Kampen surfaces with a boundary of length 2. The fact that B_K contains no small van Kampen surfaces with a boundary of length 2 gives us that the generators of G are well-separated in the abelian van Kampen metric. Specifically, we obtain the following.

Proposition 3.4. *If A is an $n \times n$ partial Latin square with density α such that $\mathcal{K}_A$ contains no van Kampen surfaces of boundary length 2 with fewer than b faces, then there exists a metric abelian group G, and maps $\phi : X \to G$, $\psi : Y \to G$ and $\omega : Z \to G$, such that the images $\phi(X), \psi(Y)$ and $\omega(Z)$ are 1-separated, and $d(\phi(x)\psi(y), \omega(z)) \leq b^{-1}$ for every $(x, y, z) \in X \times Y \times Z$ such that $(x, y) \in A$ and $\lambda(x, y) = z$.*

From this point, we could continue as in Appendix 2.A – in fact the arguments are essentially unchanged, and we omit the full details. By defining a *rough approximate abelian group* to be a rough approximate subgroup of some metric abelian group, we can show that if A is an $n \times n$ partial Latin square with density α such that $\mathcal{K}_A$ contains no van Kampen surfaces of boundary length 2 with fewer than K faces, then we can find a dense subset of A which embeds into a rough approximate abelian group (for a precise statement, replace 'group' with 'abelian group' throughout the statement of Theorem 2.48).

3.1.3 Characterising rough approximate abelian groups

In this section, we will briefly explain why rough approximate abelian groups may be more 'group-like' than their non-abelian counterparts. We start by giving an example.

Consider the case in which G is the circle group S_1, represented (for example) as the interval $[0, 1]$ under addition modulo 1. We take the obvious choice for the metric d, namely

$$d(x, y) = \min\{|x - y|, 1 - |x - y|\}.$$

Now choose $\epsilon > 0$ and let Γ be any maximal ϵ-separated subset of S_1. Given another small constant $\theta > 0$, we may partially define a commutative operation $*$ on Γ by $x * y = z$ if $d(x + y, z) < \theta\epsilon$.

It is, however, possible to pass to a dense subset of the multiplication table of $*$ that embeds into the multiplication table of a group (in fact, that of S_1).

Proposition 3.5. *There exists a subset B of the multiplication table A of $(\Gamma, *)$ of size at least $|A|/32$ which embeds into the multiplication table of S_1.*

Proof. We write $(\Gamma)_\delta$ for the δ-expansion of the set Γ in the metric d, and $(\gamma)_\delta$ for the δ-neighbourhood of the point $\gamma \in S_1$.

Let $K = \lfloor (4\theta\epsilon)^{-1} \rfloor \geq (8\theta\epsilon)^{-1}$. We choose random cosets X and Y of the discrete subgroup

$$G_K = \{i/K : i = 0, \ldots, K-1\}$$

of S_1, and let Z be the coset $X + Y$.

The probability that X has non-empty intersection with $(x)_\delta$ for some given x is at least $2K\delta$.

Suppose that $x, y, z \in \Gamma$ such that $x * y = z$. Then we have that

$$(x)_{\theta\epsilon} + (y)_{\theta\epsilon} = (x + y)_{2\theta\epsilon}$$

so certainly we must have

$$(z)_{\theta\epsilon} \subset (x)_{\theta\epsilon} + (y)_{\theta\epsilon}.$$

Therefore the probability that X, Y and Z have non-empty intersections with $(x)_{\theta\epsilon}, (y)_{\theta\epsilon}$ and $(z)_{\theta\epsilon}$ respectively is at least $4K^2\theta^2\epsilon^2/2 \geq 1/32$. By linearity of expectation and Markov's inequality we find that there exists some choice of cosets X, Y, Z such that $X + Y = Z$ and X, Y and Z have non-empty intersections with $(x)_{\theta\epsilon}, (y)_{\theta\epsilon}$ and $(z)_{\theta\epsilon}$ for a proportion at least $1/32$ of triples $x * y = z$ in Γ.

Let Γ_X be the set of $x \in \Gamma$ such that $X \cap (x)_{\theta\epsilon}$ is non-empty, so that to each $x \in \Gamma_X$ we can assign the unique point $x' \in X \cap (x)_{\theta\epsilon}$. We can define Γ_Y and Γ_Z similarly, as well as the points y' and z'.

We note that if $x \in \Gamma_X$, $y \in \Gamma_Y$ and $Z \in \Gamma_Z$ are such that $x * y = z$, then $d(x + y, z) < \theta\epsilon$ and so

$$d(x' + y', z') < 4\theta\epsilon$$

. But both $x' + y'$ and z' belong to the coset G_K of S_1 which is at least $4\theta\epsilon$-separated. This implies that $x' + y' = z'$.

Letting B be the subset of the multiplication table of $(\Gamma, *)$ corresponding to $\Gamma_X \times \Gamma_Y \times \Gamma_Z$, we have shown that B embeds into the multiplication table of S_1, and furthermore that B contains at least a proportion $1/32$ of the entries of the multiplication table of $(\Gamma, *)$, as required. $\square$

It seems conceivable that a variant of Proposition 3.5 is true more generally for rough approximate abelian groups. If this were the case, we could hope to prove the surprisingly strong statement that any Latin square B_K such that $\mathcal{K}_{B_K}$ contains no van Kampen surface of boundary length 2 and fewer than K faces, in fact has a dense subset containing no van Kampen surfaces of boundary length 2 whatsoever.

Proving such a generalisation may be difficult, however, since a lot of complexity can be encoded into the metric d. Even classifying translation-invariant metrics on $\mathbb{Z}$ is surprisingly challenging. For example, one can take two rationally independent irrationals α, β and take the metric derived from the map that takes n to $(\alpha n, \beta n)$ modulo 1, which gives a dense subset of a two-dimensional torus.

3.1.4 Finding configurations to replace the cuboctahedron

Now let us return to considering which configuration might play the role of the cuboctahedron in the abelian context. As we have already discussed, the presence of many cuboctahedra is, by itself, insufficient to eliminate small van Kampen surfaces of boundary length 2, essentially because multiplication tables of non-abelian groups still contain the maximum number of cuboctahedra.

The starting point for our investigations is the following definition.

Definition 3.6 (Thomsen condition)**.** *We say that a Latin square A satisfies the* Thomsen condition *if in any copy of the configuration*

$$\begin{array}{ccc} a & b & \\ & c & a \\ c' & & b \end{array}$$

we always have $c = c'$. Equivalently, A contains n^4 copies of the hypergraph $\mathcal{T}$ which has grid representation

$$\begin{array}{ccc} a & b & \\ & c & a \\ c & & b \end{array}$$

Like the quadrangle condition, the Thomsen condition in a Latin square provides a group structure, but we may further guarantee that the underlying group is abelian.

Theorem 3.7. *A Latin square A satisfies the Thomsen condition if and only if A is the multiplication table of an abelian group G of order n.*

So we may ask for a 1% version of this result.

Question 3.8. *Suppose A is a Latin square containing at least ϵn^4 copies of $\mathcal{T}$. Then given any $K > 0$, can we find a dense subset $B \subset A$ such that G_B contains no van Kampen surfaces with boundary of length 2 and at most K faces?*

We believe that the answer to this question is yes, although we have not yet seen how to prove this. The configuration $\mathcal{T}$ does not break down into rectangles like the cuboctahedron, and this has prevented us from finding direct analogues of certain statements from Chapter 2.

We will now give a proposition which suggests an obvious alternative approach. We start with a definition.

Definition 3.9. *By a* row-column skew cuboctahedron *we mean the hypergraph represented by the configuration*

$$\begin{array}{cccc} a & b & & \\ c & d & & \\ & & a & c \\ & & b & d \end{array}$$

The reason for the name 'row-column skew cuboctahedron' will become clearer in Section 3.2.

Note that the maximum possible number of copies of the row-column skew cuboctahedron in a partial Latin square A is n^5, since a copy is uniquely determined by selecting two points labelled a (giving n^3 choices) and deciding on the labels b and c.

Proposition 3.10. *A Latin square A contains n^5 copies of the row-column skew cuboctahedron if and only if A contains n^4 copies of $\mathcal{T}$. Therefore A contains n^5 copies of the row-column skew cuboctahedron if and only if A is the multiplication table of an abelian group.*

Proof. Suppose that A contains n^4 copies of $\mathcal{T}$. Then for each pair of points labelled c, we must be able to extend to a copy

$$\begin{array}{ccc} a & b & \\ & c & a \\ c & & b \end{array}$$

of $\mathcal{T}$ in at least n ways. But each pair of different extensions gives the configuration

$$\begin{array}{cccc} a' & b' & & \\ a & b & & \\ & c & a & a' \\ c & & b & b' \end{array}$$

which contains a row-column skew cuboctahedron. So we obtain n^2 row-column skew cuboctahedra for each pair of points with the same label, and in a Latin square there are n^3 choices for such a pair.

Conversely, if A contains n^5 row-column skew cuboctahedra then taking a configuration

$$\begin{array}{ccc} a & b & \\ & c & a \\ c' & d & b \end{array}$$

we note that we have a rectangle

$$\begin{array}{cc} a & b \\ c' & d \end{array}$$

and a rectangle

$$\begin{array}{cc} a & c \\ b & d \end{array}$$

and together these must form a row-column skew cuboctahedron (since we have the maximum number of row-column skew cuboctahedra). Therefore $c = c'$. $\square$

Proposition 3.10 shows us that in a complete Latin square the Thomsen condition is equivalent to demanding n^5 row-column skew cuboctahedra. Since row-column skew cuboctahedra seem much closer to non-skew cuboctahedra in structure, we can perhaps hope to mirror the arguments from Chapter 2 more closely if we replace $\mathcal{T}$ with the row-column skew cuboctahedron.

Unfortunately, this approach does not work. Consider a partial Latin square constructed as follows. Take some highly non-abelian group G such as $PSL_2(q)$, of order about $n/2$. Then construct A by taking two copies of the multiplication table of G and placing them in the top left quadrant and bottom right quadrant respectively, so that they share no rows or columns (but share all their labels). Now take the transpose of the bottom right quadrant.

All rectangles in the resulting partial Latin square A belong to either the top left quadrant or the bottom quadrant. It is clear that for any rectangle

$$\begin{matrix} a & b \\ c & d \end{matrix}$$

in the top left quadrant of A there is a rectangle

$$\begin{matrix} a & c \\ b & d \end{matrix}$$

in the bottom right quadrant, so there are many (about $n^5/32$) row-column skew cuboctahedra. However, rectangles

$$\begin{matrix} a & b \\ c & d' \end{matrix}$$

with $d \neq d'$ are very likely to appear frequently in the bottom right quadrant, since if $d = d'$ this would correspond to a row-column skew cuboctahedron in the multiplication table of $PSL_2(q)$ and these are rare (since they witness commuting elements). So in order to pass to a subset with no flappy cuboctahedra (which are van Kampen diagrams with a boundary of length 2), we essentially have to pass either to the top left quadrant or to the top right quadrant. But to do so destroys almost all of our row-column skew cuboctahedra, and we have no hope of passing to a further dense subset that will embed into an abelian group multiplication table.

The problem is that in A our cuboctahedra are almost all fully contained in either the top left quadrant or the bottom right quadrant, whereas our row-column skew cuboctahedra almost all straddle the two quadrants. So our skew and non-skew cuboctahedra fail badly to 'mix' in some way.

This example shows that, in our quest for a bounded-size configuration yielding rough approximate abelian group structure, we must find a configuration that somehow forces our skew and non-skew cuboctahedra to mix in a way that prevents the above counterexample from applying.

We therefore introduce the hypergraph H_c which corresponds to the following configuration.

$$\begin{array}{ccccc}
c & d & & & \\
a & b & & & \\
 & & & c & d \\
 & & c & a & b \\
 & & d & b &
\end{array}$$

Observe that H_c consists essentially of a cuboctahedron and a skew cuboctahedron on the same set of labels which agree on the top left rectangle and the one point labelled a in the bottom right. Note also the presence of a copy of $\mathcal{T}$ in the bottom right. The maximum possible number of copies of H_c in a Latin square is n^5, since a whole copy of H_c is determined by the cuboctahedron consisting of two rectangles labelled (a, b, c, d).

We observe that H_c enjoys the same '100%' result as $\mathcal{T}$ and the skew cuboctahedron.

Proposition 3.11. *Let A be a Latin square. Then A contains n^5 copies of H_c if and only if it contains n^4 copies of $\mathcal{T}$ (and is thus the multiplication table of an abelian group).*

Proof. If A contains n^5 copies of H_c then A contains n^4 copies of the configuration

$$\begin{array}{ccc}
 & c & d \\
c & a & b \\
d & b &
\end{array}$$

and therefore n^4 copies of $\mathcal{T}$. In the other direction, we note that if A contains n^4 copies of the configuration

$$\begin{array}{ccc}
 & c & d \\
c & a & b \\
d & b &
\end{array}$$

then since A is a Latin square then A contains n^5 copies of the configuration

$$\begin{array}{ccccc}
c & d & & & \\
a' & b & & & \\
 & & & c & d \\
 & & c & a & b \\
 & & d & b &
\end{array}$$

and then $a = a'$ is guaranteed by the quadrangle condition (which follows from Thomsen's condition by Theorem 3.7). $\square$

We remark also that a dense subset of the multiplication table of an abelian group contains a constant proportion of the maximum possible number of copies of H_c. However, this claim is not obvious[2], and we do not need it for our result so we will not give a detailed proof.

Unlike the row-column skew cuboctahedron, the presence of many copies of H_c is sufficiently powerful to provide a partial abelian torsor.

Theorem 3.12. *Suppose A is a partial Latin square with at least ϵn^5 copies of H_c. Then given any $K > 0$, we can find a subset $B \subset A$ of density at least $\beta = \beta(\epsilon, K)$ such that G_B contains no van Kampen surfaces with boundary of length 2 and at most K faces.*

The precise dependence of $\beta(\epsilon, K)$ on the parameters will be determined later (see Theorem 3.48).

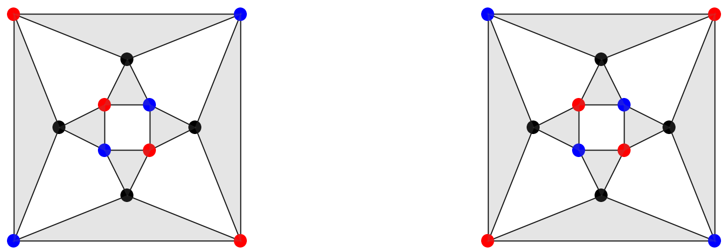
Although the overall approach to Theorem 3.12 will be along very similar lines to that of the non-abelian case in Chapter 2, there will be technical differences and additional challenges at almost every stage. The decompositions used will have to be thoroughly reworked, since the notions of point decompositions and ring decompositions used in Chapter 2 relied completely on the fact that a non-skew cuboctahedron is built from two rectangles – the same is not true for H_c. Although a popular replacement strategy will also be used in this work, it is not immediately clear what form our popular replacement lemma should take. The unfixing process is greatly complicated by the considerations required to deal with higher genus van Kampen surfaces, leading to topological difficulties that were almost entirely absent in Chapter 2. Nevertheless, we shall present the argument in such a way that the overlap with Chapter 2 is emphasised as much as possible.

The argument we have found for H_c is not symmetric, in the sense that it does not treat rows, columns and labels equally, and it seems not to suit a highly symmetric configuration such as $\mathcal{T}$. However, this difficulty does not feel like a fundamental one, and we cannot see any compelling reason to think that a more symmetric approach, answering Question 3.8, does not exist.

3.2 Preliminaries

In Chapter 2 we defined a k-PF in Definition 2.8. In this chapter, we modify the definition slightly by dropping the condition that the petal vertices must

[2]The claim would follow from the proof of Theorem 4.8 in the next chapter.



(a) A non-skew cuboctahedron. (b) A row-column skew cuboctahedron.

Figure 3.2: Two kinds of cuboctahedron.

all come from the same label class. This leaves us with the following, modified version of the definition.

Definition 3.13. *In hypergraph terms, a k-PF is a cycle of k faces in which one vertex is shared between adjacent faces. The k-PF thus contains k vertices of degree 2 and k vertices of degree 1. We call the vertices of degree 1 the* petal vertices *and the vertices of degree 2 the* inner vertices*. The inner vertices are arranged in a cycle, so can naturally be represented by a sequence up to cyclic permutations.*

Recall also that configurations in our Latin square can be thought of both as hypergraphs and as subsets of the van Kampen complex – see the discussion in Sections 2.1.2 and 2.4 of Chapter 2. When we refer to a k-PF we shall usually be referring to the hypergraph representation, although we may sometimes mean the representation as a van Kampen surface if this is clear from the context.

Recall also that a cuboctahedron can be split naturally into two rectangles, each of which correspond to a 4-PF. The pair of 4-PFs share their four petal vertices, which all belong to the label class. This representation of the cuboctahedron is shown in Figure 3.2a, where we have coloured the vertices according to their vertex class.

In the previous section we described a row-column skew cuboctahedron:

$$\begin{array}{cccc} a & b & & \\ c & d & & \\ & & a & c \\ & & b & d \end{array}$$

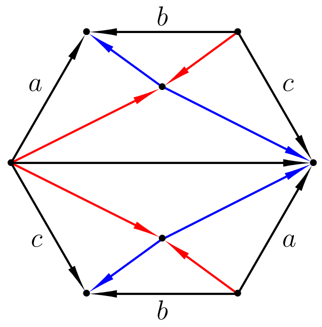


Figure 3.3: The van Kampen surface corresponding to a skew cuboctahedron. Note that the surface is homeomorphic to a torus, since the pairs of outer edges with the same label are identified.

This configuration also naturally splits into two rectangles

$$\begin{matrix} a & b \\ c & d \end{matrix}$$

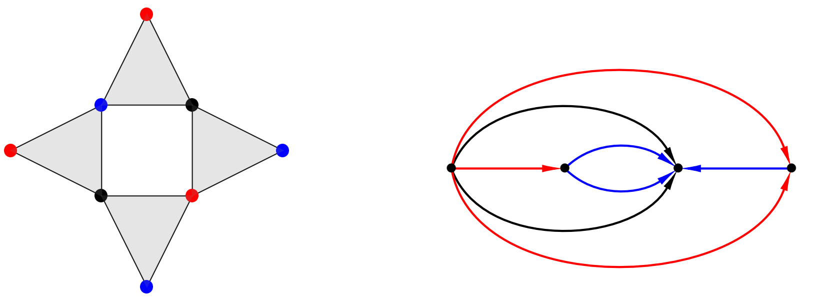
and

$$\begin{matrix} a & c \\ b & d \end{matrix}$$

which also correspond to 4-PFs which share their four petal vertices, which all belong to the label class. This gives the representation in Figure 3.2b. Note that, unlike in Figure 3.2a, the outer 4-PF is 'rotated' with respect to the inner one, in the sense that row vertices now pair up with column vertices and vice versa. We also note that a cuboctahedron and a skew cuboctahedron are isomorphic as hypergraphs, but not as tripartite hypergraphs, since the isomorphism does not preserve the tripartition. This explains why we call the skew cuboctahedron a cuboctahedron, and the rotation of the outer 4-PF is the reason for the word *skew*.

One very important difference between the cuboctahedron and the skew cuboctahedron can be seen when we consider the corresponding van Kampen surfaces. As we saw numerous times in Chapter 2, the van Kampen surface corresponding to a cuboctahedron is an octahedron, which is homeomorphic to a sphere. However, the van Kampen surface associated with the skew cuboctahedron is in fact homeomorphic to the torus, as can be seen in Figure 3.3.

One consequence of the rotation used in building the skew cuboctahedron is that we can find 4-PFs inside it which have petal vertices coming from more



(a) A row-column 4-PF. Red vertices correspond to row vertices, blue to column and black to label.

(b) The van Kampen diagram for a row-column 4-PF. Note that the boundary has two components, consisting of the outer red edges and the inner blue edges, so the diagram is homeomorphic to a cylinder.

Figure 3.4: Hypergraph and van Kampen representations of a row-column 4-PF.

than one class. For instance, the configuration

$$\begin{array}{ccc} a & & \\ c & & \\ & a & c \end{array}$$

is a 4-PF with hypergraph representation as shown in Figure 3.4a. Note that the van Kampen representation of this 4-PF is not as one may expect – it is in fact homeomorphic to a cylinder as shown in Figure 3.4b. In Chapter 2 such $2r$-PFs were never considered – as mentioned earlier, the Definition 2.8 insisted that all petal vertices should come from the same class. Because of the significant differences that arise when petal vertices come from multiple classes, it will be important to distinguish these new PFs. We shall call PFs in which all petal vertices do come from the same class *non-skew*, while ones with petals of multiple types are called *skew*. Specifically, we make the following definition.

Definition 3.14. *We say that a k-PF (as in Definition 3.13) is* non-skew *if all its petal vertices come from the same vertex class. If petal vertices come from multiple classes we call the k-PF skew. We also say that a k-PF is* row-column semi-skew*, or simply* row-column*, if all petal vertices come from just the row and column classes (in other words there are no label petals). Similarly we may talk about* row-label *or* label-column *k-PFs.*

Observe that the 4-PF shown in Figure 3.4a is a row-column 4-PF, since the four petal vertices all come from the row and column classes. The inner vertices come from the label, row, label and column classes as we traverse the cycle clockwise from the top. We can thus associate to a row-column 4-PF a sequence, up to cyclic permutations, representing the vertex classes of its inner vertices – in this case the sequence is (L, R, L, C). The following definition generalises this example.

Definition 3.15. *To a k-PF F we may associate a length k sequence S_F (up to cyclic permutation) from $\{R, C, L\}$, corresponding to the sequence of inner vertices of F taken from some starting position (of course $R/C/L$ denotes a row/column/label vertex, respectively). In this sequence, no letter may appear in consecutive positions (where the first and last elements are seen as consecutive because the inner vertices of F lie on a cycle). We call S_F the* inner vertex profile *of F. We also refer to an inner vertex profile S of length k without reference to a k-PF F. In this case we mean a cyclic sequence from $\{R, C, L\}$ of length k in which no letter may appear in consecutive positions.*

Our next task is to provide a proof that dense partial Latin squares contain many row-column $2r$-PFs of each possible inner vertex profile.

Proposition 3.16. *Let A be a partial Latin square of density α and let S be an inner vertex profile of length $k = 2r$ in which every other entry is L (henceforth we refer to this property as 'label-alternating'). Then the number of different $2r$-PFs F in A with $S_F = S$ is at least $\alpha^{2r^3} n^{2r}$.*

If S contains only two different types of vertex (L and R only, or L and C only) then the proposition follows from the non-skew version, Lemma 2.11 in Chapter 2.

Otherwise, we require a (semi) skew version of Lemma 2.11 from Chapter 2. This provides a good example of how very easy steps in the non-skew arguments become rather more tricky in the skew/semi-skew context. Indeed the proof is sufficiently involved to require some set-up.

The proof strategy for Proposition 3.16 is as follows. Given an inner vertex profile S, we try and embed a $2r$-PF with inner vertex profile S in a non-degenerate way into a larger configuration that we can count with Cauchy–Schwarz. We will then show that the presence of many copies of the larger configuration guarantees the presence of many copies of the $2r$-PF.

For example, suppose that our sequence is $S = RLCLRL$. Then a 6-PF F

with $S_F = S$ has a grid representation given by

$$
\begin{array}{ccccc}
a & b & & & \\
 & & b & & \\
 & & c & & \\
 & & & c & a
\end{array}
$$

This can be seen as follows: traversing the sequence from the top left to the bottom right, you start at a and go first to a point in the same row, then a point with the same label, then a point with the same column, and so on, giving the sequence $RLCLRL$.

Now let C_S be the configuration with grid representation given by the following.

$$
\begin{array}{ccccccccc}
a & b & & & & & & & \\
 & c & a & & & & & & \\
 & & & a & b & & & & \\
 & & & & c & a & & & \\
 & & & & & & a & b & \\
 & & & & & & & c & a
\end{array}
$$

Observe that C_S contains a 6-PF with inner vertex profile S, as shown in red in the following diagram.

$$
\begin{array}{ccccccccc}
{\color{red}a} & {\color{red}b} & & & & & & & \\
 & c & a & & & & & & \\
 & & & a & {\color{red}b} & & & & \\
 & & & & {\color{red}c} & a & & & \\
 & & & & & & a & b & \\
 & & & & & & & {\color{red}c} & {\color{red}a}
\end{array}
$$

Consider the configuration consisting of two points sharing a row, represented as follows (where asterisks are used in place of labels whose value is irrelevant).

$$
\begin{array}{cc}
* & *
\end{array}
$$

The number of such configurations in a partial Latin square of density α is at least $\alpha^2 n^3$. If we let $f(a, x)$ count the number of such configurations in which the left point is labelled a and the rightmost column is column x then we have

$\sum_{a,x} f(a,x) \geq \alpha^2 n^3$ and so by Cauchy–Schwarz

$$\sum_{a,x} f(a,x)^2 \geq \alpha^4 n^4.$$

The last sum counts the number of configurations

$$\begin{matrix} a & * & \\ & * & a \end{matrix}$$

in A. We can then let $g(a,b,c)$ count the number of configurations

$$\begin{matrix} a & b & \\ & c & a \end{matrix}$$

and since $\sum_{a,b,c} g(a,b,c) \geq \alpha^4 n^4$ we get

$$\sum_{a,b,c} g(a,b,c)^3 \geq \alpha^{12} n^6$$

by Jensen's inequality. This sum counts the number of copies of C_S.

But note also that fixing all vertices in the red 6-PF in the diagram of C_S above fixes the rest of the diagram by linearity. In other words, each 6-PF with inner vertex profile S extends to at most one copy of C_S. Therefore the number of 6-PFs with inner vertex profile S in A is at least $\alpha^{12} n^6$ also.

More generally, we will not be so lucky and our $2r$-PFs will extend to more than a unique copy of our larger configuration. We will have to track this overcounting.

Before we begin, we deal with one separate case which will not fit into our general framework. The case is that of the row-column 4-PF: $S = RLCL$. We may swap coordinates and take $S = LRCR$ instead – this makes no difference to the problem, but makes this case easier to visualise. We note that the configuration corresponding to a 4-PF with inner vertex profile S is

$$\begin{matrix} a & * & \\ & * & a \end{matrix}$$

and we showed above that the number of such configurations in A is at least $\alpha^4 n^4$. So this case is done.

Since we may assume that S contains at least one instance of each of L, R and C, and that $S \neq LRLC$, we may now assume that $2r \geq 6$. We may further assume, without loss of generality (using the fact that we can cyclically permute S) that S contains the subsequence $LR \ldots C \ldots RL$.

We begin by defining a configuration U_S which takes the role of the configuration

$$\begin{array}{ccc} a & * & \\ & * & a \end{array}$$

from the 6-PF example above.

Definition 3.17. *Given a label-alternating sequence S, we define a configuration U_S as follows. We start with a labelled point. Then, for each non-label term in the sequence, we add a new unlabelled point, in the same row/column as the previous one according to whether the term is R or C. We make the last point labelled with the same label as the first point. E.g. if $S = RLCLCLRL$ then the configuration U_S is*

$$\begin{array}{ccc} a & * & \\ & * & \\ & * & a \end{array}$$

It is not immediate that we can find many copies of U_S inside A, but this becomes clearer when we define a further configuration V_S.

Definition 3.18. *Given a label-alternating sequence S, we define a configuration V_S as follows. We take two complete $r \times r$ subgrids that share their label on one pair of corresponding points, and share the column of a different pair of corresponding points (not in the same column as the first). For instance when $r = 3$ we have V_S as follows:*

$$\begin{array}{ccccc} a & * & * & & \\ * & * & * & & \\ * & * & * & & \\ & & * & * & * \\ & & * & * & * \\ & & * & * & a \end{array}$$

Note that we may always embed U_S inside V_S, since we are assuming that S contains the subsequence $LR \ldots C \ldots RL$.

We are finally ready to define the configuration C_S in general.

Definition 3.19. *Given a label-alternating sequence S, we define a configuration C_S as follows. We consider the configuration V_S and choose a specific copy of*

U_S within. We then take r row/column disjoint copies of V_S in which the labels on the points representing our copies U_S agree between the copies of V_S. E.g. when $S = RLCLRL$ we can take C_S to be the configuration made from 3 copies of the following configuration

$$\begin{array}{ccccc} a & * & b & & \\ * & * & * & & \\ * & * & * & & \\ & & * & * & * \\ & & * & * & * \\ & & c & * & a \end{array}$$

Lemma 3.20. *Let S be a label-alternating sequence of length $2r$ containing the subsequence $LR \dots C \dots RL$. Then define a configuration C_S as in Definition 3.19. Then the number of copies of C_S in A is at least $\alpha^{2r^3} n^{3r^2-r}$.*

Proof. By an easy Cauchy–Schwarz argument, the number of copies of V_S is at least $\alpha^{2r^2} n^{4r-2}$. Letting $f(a_1, \dots, a_r)$ count the number of copies of V_S with labels $a_1, \dots, a_r$ on the points corresponding to our selected copy of U_S (for the construction of V_S), we see that $\sum_{a_i} f(a_1, \dots, a_r) \geq \alpha^{2r^2} n^{4r-2}$ and therefore

$$\sum_{a_i} f(a_1, \dots, a_r)^r \geq \alpha^{2r^3} n^{(4r-2)+(r-1)(3r-2)}$$

$$= \alpha^{2r^3} n^{3r^2-r}$$

and this final sum counts the number of copies of C_S in A. □

All that remains is to bound the maximum number of copies of C_S that extend a given $2r$-PF F with inner vertex profile S.

Lemma 3.21. *Let S be a label-alternating sequence of length $2r$ containing the subsequence $LR \dots C \dots RL$. Then define a configuration C_S as in Definition 3.19. Let F be a given $2r$-PF with inner vertex profile S. Then the number of copies of C_S that extend F is at most n^{3r^2-3r}.*

Proof. Let C be a generic copy of C_S containing F. Note that the number of points that belong to C but not F is $2r^3 - 2r$. The number of rows and columns that C occupies is $(4r-1)r$, while the number of label vertices is $2r^3 - r^2$. The number of vertices in F is $4r$. Therefore the number of vertices that belong to C but not to F is

$$2r^3 - r^2 + (4r-1)r - 4r$$

$$= 2r^3 + 3r^2 - 5r.$$

Now we will order the points of $C \setminus F$ such that every point p contains a vertex v_p not contained in any earlier point of $C \setminus F$. We may clearly leave all the points with degree 1 label vertices (represented by asterisks) until last, since their label vertices appear in no other point of $C \setminus F$. So we need only think about the points in the copies of U_S that appear in each copy of V_S in C (apart from those in F). Considering a copy U of U_S in some copy of V_S in C, we see that F contains two of these points – indeed the points contained in F are 'consecutive' in U, meaning they share a row or column (which belongs to F, so cannot be chosen as a v_p). Also, the label vertices of every point in U belongs to F.

We may consider the points of U as edges in the bipartite 2-graph between the row set and the column set. In this graph, U corresponds to a tree T_U consisting of the union of a path P with a star at each vertex and $r+1$ edges in total. The vertices belonging to F are three vertices along the path P. But it is now clear that we can order these edges such that each each edge contains a vertex not contained in any earlier edge or in F – we just 'grow' the tree starting from the points in F. Starting from the two edges and three vertices of T_U that belong to F, we add in one edge at a time. Each time we add in a new edge (corresponding to some point p) we also add a new vertex, and this vertex is v_p.

Since we can order the points in U in the required way, and each copy of V_S in C has a disjoint set of row/column coordinates, we can obtain our ordering of the set of points $\{p \in C \setminus F\}$ and the corresponding sequence of v_p.

It remains to observe that if we select specific rows, columns and labels for each vertex of C that does not belong to $\{v_p\}$ then we uniquely determine C by linearity of the underlying hypergraph. This follows by induction – consider some point p. This point has three vertices, one of which is v_p. Neither other vertex can be $v_{p'}$ for some p' later in the order than p, by definition, so by the induction hypothesis both other vertices of p have been determined. This uniquely determines v_p.

Therefore the maximum number of copies of C_S that extend a specific $2r$-PF F with $S_F = S$ is at most n^{v-f} where v is the number of vertices of $C \setminus F$ and f is the number of faces. From our earlier calculations, we see that

$$n^{v-f} = n^{2r^3+3r^2-5r-(2r^3-2r)} = n^{3r^2-3r}.$$

□

Proof of Proposition 3.16. As shown above, the result follows if $r = 2$. We may thus assume that $r \geq 3$ and moreover that S contains the subsequence $LR \dots C \dots RL$. We may now apply Lemma 3.20 which shows that the number of configurations C_S in A is at least $\alpha^{2r^3} n^{3r^2 - r}$. Then Lemma 3.21 shows that each row-column $2r$-PF with inner vertex profile S can extend to a copy of C_S in at most n^{3r^2-3r} ways. Therefore the number of distinct row-column $2r$-PFs with inner vertex profile S in A is at least

$$\alpha^{2r^3} n^{(3r^2-r)-(3r^2-3r)}$$

$$= \alpha^{2r^3} n^{2r}$$

as required. □

Remark 3.22. The proof strategy for Proposition 3.16 can be applied to general k-PFs for any $k > 3$. However there are a number of different cases for different broad classes of inner vertex profile S, so for the sake of simplicity we do not provide the full result.

3.3 Overview

We begin by providing a broad outline of the program that we follow to prove Theorem 3.12. The outline is divided into 5 steps, each of which mirror corresponding steps in Chapter 2.

Before describing these steps, we must recall a key definition from Chapter 2, modified slightly for the skew context.

Definition 3.23. *Let $\mathcal{F}$ be a collection of $2r$-PFs. We say that the set $\mathcal{F}$ of $2r$-PFs satisfies a C-*well-defined condition *(or simply that $\mathcal{F}$ is C-well-defined) if the largest subset $\mathcal{F}' \subset \mathcal{F}$ that all share some specific $2r-1$ petal vertices but differ (pairwise) on the last petal vertex has size at most C.*

We can think of the C-well-defined condition as saying that fixing $2r-1$ petal vertices of a $2r$-PF in $\mathcal{F}$ leaves at most C different choices for the last petal vertex. Indeed, we can even think of defining a multivalued function ϕ taking a $2r-1$ tuple of vertices $(v_1, \dots, v_{2r-1})$ to the set of vertices v_{2r} such that there exists a $2r$-PF in $\mathcal{F}$ with petal vertices $(v_1, \dots, v_{2r})$. Then we are saying that $\mathcal{F}$ is C-well-defined if ϕ is at most C-valued.

A number of different kinds of decomposition are defined in Chapter 2, including the point decomposition (Definition 2.13), the ring decomposition (Definition 2.15) and the shattered ring decomposition (Definition 2.27). All of these decompositions will return in this work, but will have to be heavily modified for the semi-skew context. We shall present the formal definitions as we need them in the technical arguments, but it will help to recall the broad ideas used in the non-skew version for the purpose of understanding the following overview.

1. By defining an appropriate notion of a point decomposition in the skew context, we pass to a dense subset in which almost all row-column (semi-skew) $2r$-PFs satisfy a C-well-defined condition, where the 'almost all' can be made arbitrarily strong by increasing C.

2. We pass to a further dense subset in which every row-column $2r$-PF is decomposable in many ways into $2r$-PFs satisfying the C-well-defined condition. We then deduce a C-well-defined condition that holds for all row-column $2r$-PFs.

3. We then prove that if, in a dense partial Latin square, the set of all row-column $2r$-PFs has the C-well-defined condition, then we can pass to a dense subset in which every row-column $2r$-PF is popularly decomposable (in the original partial Latin square) in many ways.

4. We then put together a popular replacement lemma by combining the above step with the non-abelian work. We arrive at a sequence of dense subsets in which each one has the property that every row-column $2r$-PF and every non-skew $2r$-PF (for bounded r) is popularly replaceable in the previous subset.

5. Finally, we prove that this is sufficient to eliminate all flappy structures of bounded size.

An important remark is that we have fixed in this outline the row, column and label vertex sets and we are treating them differently. What we are achieving for row-column $2r$-PFs does not match what we achieve for row-label $2r$-PFs, for instance. This asymmetry is a source of extra challenges, particularly in step 5 when we have to apply our restricted popular replacement lemma. Unfortunately, there are some steps in our argument that do not seem to admit an obvious symmetric formulation, and so for the time being we must make do with the restricted arguments.

3.4 Step 1: A C-well-defined condition for almost all row-column $2r$-PFs

Given an $n \times n$ partial Latin square A and some positive real number ϵ, we can define an auxiliary graph $G_A(\epsilon)$ with vertex set given by the n^2 cells of the $n \times n$ grid and a directed edge between two points if they complete to at least ϵn different copies of H_c in which the chosen cells (with the corresponding edge in $G_A(\epsilon)$) contain the points shown in bold in the following diagram.

$$\begin{array}{ccccc}
c & d & & & \\
a & \mathbf{b} & & & \\
 & & \nwarrow & c & d \\
 & & c & \mathbf{a} & b \\
 & & d & b &
\end{array}$$

Definition 3.24. *We say that a row-column $2r$-PF F has an (ϵ, δ)* point decomposition *if the common neighbourhood in $G_A(\epsilon)$ of the points in F has size at least δn^2.*

Given a partial Latin square A with many copies of H_c, we apply a dependent random selection argument to pass to a dense subset of A in which almost every row-column $2r$-PF has an (ϵ, δ) point decomposition.

Lemma 3.25. *Let $\epsilon, \theta > 0$ and let $k > 1$ be a fixed integer. Given an $n \times n$ partial Latin square A containing at least ϵn^5 copies of H_c, we can find a subset $B_1 \subset A$ of density at least $\beta_1 = \epsilon/8$ such that a proportion at least $1 - \theta$ of row-column $2r$-PFs (for $1 < r \leq k$) in B_1 have an $(\epsilon/2, \delta)$ point decomposition in B_1, where $\delta = \delta(\epsilon, \theta) = \theta \epsilon^{\mathcal{O}(k^3)}$.*

Proof. We apply a dependent random selection to the digraph $G = G_A(\epsilon/2)$. Given a vertex v in this graph, the in-neighbourhood of v consists of all vertices w with an edge vw directed from w to v. The number of copies of H_c in A is at least ϵn^5, and an edge in G corresponds to a set of at least $\epsilon n/2$ copies of H_c. Letting X be the number of edges in G and Y be the number of non-edges, we have that

$$Xn + Y\epsilon n/2 \geq \epsilon n^5$$

$$\Rightarrow Xn + \epsilon n^5/2 \geq \epsilon n^5$$

and so G has average out-degree at least $\epsilon n^2/4$.

We choose a vertex v in G uniformly at random, and we let $N(v)$ be the in-neighbourhood of v.

Let $C = x_1y_1 \dots x_ry_r$ be a given row-column $2r$-PF in A (i.e. the x_i and y_i are points of A, equivalently vertices of G). Let $N(C)$ be the set of common in-neighbours of $x_1, \dots, y_r$ in G. We shall say that C is *bad* if $|N(C)| < \delta n^2$ (for some δ to be chosen later). If C is bad, we have that

$$\mathbb{P}(C \subset N(v)) = \frac{|N(C)|}{n^2} < \delta.$$

Let $Z_r(P)$ count the number of bad $2r$-PFs with inner vertex profile P in $N(v)$. We have $\mathbb{E}Z_r(P) < \delta n^{2r}$.

Now let $Z_r = \sum_{P_r} Z_r(P)n^{-2r}$, where the inner sum is over all inner vertex profiles P_r of size $2r$, and let $Z = \sum_{r=2}^{k} Z_r$. The number of such P_r is clearly at most 3^r, so we have

$$\mathbb{E}Z < \sum_{r=2}^{k} 3^r\delta \leq 3^{k+1}\delta.$$

Moreover, the average degree of G tells us that

$$\mathbb{E}(|N(v)|) \geq \epsilon n^2/4.$$

In particular, we have

$$\mathbb{E}\left(|N(v)|n^{-2} - \frac{\epsilon}{8} - \frac{\epsilon Z}{8.3^{k+1}\delta}\right) \geq 0$$

so there is a choice of vertex v such that $|N(v)| \geq \epsilon n^2/8$ and $|N(v)|n^{-2} \geq \frac{\epsilon Z}{8.3^{k+1}\delta}$. The first inequality gives us that the total count, $X_r(P)$, of row-column $2r$-PFs in $N(v)$ with inner vertex profile P is at least $(\epsilon/8)^{2r^3}n^{2r}$ (by Proposition 3.16), while the second inequality implies that $Z \leq 8.3^{k+1}\delta/\epsilon$. So

$$Z_r(P)n^{-2r} \leq \frac{8.3^{k+1}\delta}{\epsilon} \leq \frac{8.3^{k+1}\delta}{\epsilon(\epsilon/8)^{2k^3}}n^{-2r}X_r(P)$$

and so $Z_r(P) \leq \theta X_r(P)$ when $\delta = \delta(\epsilon, \theta)$ is chosen as $\epsilon(\epsilon/8)^{2k^3}\theta/8.3^{k+1} = \theta\epsilon^{\mathcal{O}(k^3)}$.

Setting $B_1 = N(v)$ for this choice of v, we have that $\beta_1 = \beta_1(\epsilon) \geq \epsilon/8$ and a proportion at least $1 - \theta$ of row-column $2r$-PFs (for $1 < r \leq k$) in B_1 have an $(\epsilon/2, \delta)$ point decomposition. $\square$

Next, we prove a C-well-defined condition for row-column $2r$-PFs that have an (ϵ, δ) point decomposition.

Lemma 3.26. *Let B be a partial Latin square, and let $\mathcal{F}_P$ be a collection of row-column $2r$-PFs with inner vertex profile P that all have an (ϵ, δ) point decomposition. Then the set $\mathcal{F}_P$ is $C(\epsilon, \delta) = (\delta\epsilon^{2r})^{-1}$-well-defined.*

Proof. Before giving a general proof, we will first step through an example case. We shall consider the inner vertex profile $P = (C, L, C, L, R, L)$. After seeing the proof for this choice of P, the general approach will become much clearer.

A 6-PF F with inner vertex profile P is the following configuration:

$$\begin{array}{cccc}
a & & & \\
b & & & \\
 & b & & \\
 & c & & \\
 & & c & a
\end{array}$$

Let us say that F is *good* if $F \in \mathcal{F}_P$.

We want that, given five petal vertices of a 6-PF of this kind, there are at most C different possibilities for the last petal vertex. We recall that in the grid representation, the labels, columns and rows correspond to vertices of the hypergraph, while the labelled points correspond to faces of the hypergraph. A vertex has degree 1 if there is only a single point in the corresponding row/column/label set. In the semi-skew 6-PF shown above, the petal vertices are the top four rows and the rightmost two columns (i.e. the vertices of degree 1).

Suppose that there are K choices for the 6th petal. In other words, we assume the existence of a subset of $\mathcal{F}'$ of $\mathcal{F}_P$ of size K in which all members share the first 5 petal vertices but differ on the 6th.

We represent $\mathcal{F}'$ with the following diagram.

$$K \times \left[\begin{array}{c|cccc}
 & * & * & c_3 & * \\
\hline
r_1 & *_1 & & & \\
r_2 & *_2 & & & \\
r_3 & & *_2 & & \\
r_4 & & *_3 & & \\
* & & & *_3 & *_1
\end{array}\right]$$

This diagram shows the grid representation of F in the bottom right, with the row and column headings giving the row and column vertices. If a vertex varies over different copies of F in $\mathcal{F}'$ then we represent it with an asterisk, while if the vertex is shared by all copies of F in $\mathcal{F}'$ then we represent it with a subscripted Roman letter. For the label vertices (which may vary over different copies of F in $\mathcal{F}'$) we use asterisks with subscripts to indicate which points share labels.

This diagram shows the row and column vertices belonging to the $F \in \mathcal{F}'$ as the leftmost column and top row. If the vertex is not a petal, we represent it with an asterisk (since we have no information about these rows/columns), but r_1, r_2, r_3, r_4 and c_3 represent the fixed petal vertices shared by all members of $\mathcal{F}'$. The entry in the fourth column varies over the different members of $\mathcal{F}'$. We write '$K\times$' to indicate that the diagram represents a set of K different 6-PFs, one for each member of $\mathcal{F}'$.

Since $\mathcal{F}' \subset \mathcal{F}_P$, we have that each member of $\mathcal{F}'$ is good, meaning that for each member of $\mathcal{F}'$, the common neighbourhood of the corresponding points in $G_A(\epsilon)$ has size at least δn^2.

Recall that an edge in $G_A(\epsilon)$ represents the fact that the corresponding points complete to ϵn different copies of H_c. In particular, we get an edge between a point a and a point b if and only if a and b form opposite corners of an ϵ-popular column-non-skew 4-PF and an ϵ-popular row-column 4-PF.

The following diagram represents the set of configurations obtained by taking a 6-PF from $\mathcal{F}'$ and including some points from the popular 4-PFs obtained from the edges of $G_A(\epsilon)$. The points corresponding to the choices of 6-PF from $\mathcal{F}'$ are shown in red.

$$K\delta n^2 \times \left[\begin{array}{c|ccccccccc}
 & * & * & c_3 & * & * & * & * & * & * \\
\hline
r_1 & \color{red}{*_1} & & & & & & & & \\
r_2 & \color{red}{*_2} & & & & & & & & \\
* & *_4 & & & & & & & & \\
r_4 & & \color{red}{*_2} & & & & & & & \\
r_5 & & \color{red}{*_3} & & & & & & & \\
* & & *_4 & & & & & & & \\
* & & & \color{red}{*_3} & \color{red}{*_1} & *_4 & & & & \\
* & & & & & & *_1 & *_2 & *_3 & *_4
\end{array}\right]$$

Now we can apply the pigeonhole principle. By multiplying the number of configurations represented by the diagram by a factor $1/n$, we can 'fix' a row, column or label in the diagram. In particular, we can obtain the following collection.

$$K\delta \times \left[\begin{array}{c|ccccccccc}
 & * & * & c_3 & * & * & * & * & * & c_9 \\
\hline
r_1 & \color{red}{*_1} & & & & & & & & \\
r_2 & \color{red}{*_2} & & & & & & & & \\
r_3 & *_4 & & & & & & & & \\
r_4 & & \color{red}{*_2} & & & & & & & \\
r_5 & & \color{red}{*_3} & & & & & & & \\
* & & *_4 & & & & & & & \\
* & & & \color{red}{*_3} & \color{red}{*_1} & *_4 & & & & \\
* & & & & & & *_1 & *_2 & *_3 & *_4
\end{array}\right]$$

Next, we continue to pigeonhole but we use the popularity of certain 4-PFs in order to gain a factor ϵ (as opposed to a factor $1/n$) for each new fixed vertex. For example, we note that the row-column 4-PF with petal vertices corresponding to the 1st and 3rd rows and 6th and 9th columns is ϵ-popular. Since the 1st and 3rd rows and the 9th column are all fixed, we may fix the 6th column at the cost of a factor ϵ.

Similarly, we can fix the 7th column at the cost of a factor ϵ (using the popularity of the row-column 4-PF with petal vertices corresponding to the 2nd and 3rd rows and 7th and 9th columns).

We thus obtain the following diagram.

$$K\delta\epsilon^2 \times \left[\begin{array}{c|ccccccccc}
 & * & * & c_3 & * & * & c_6 & c_7 & * & c_9 \\ \hline
r_1 & *_1 & & & & & & & & \\
r_2 & *_2 & & & & & & & & \\
r_3 & *_4 & & & & & & & & \\
r_4 & & *_2 & & & & & & & \\
r_5 & & *_3 & & & & & & & \\
* & & *_4 & & & & & & & \\
* & & & *_3 & *_1 & *_4 & & & & \\
* & & & & & & *_1 & *_2 & *_3 & *_4
\end{array}\right]$$

Next, we can fix the 5th row at the cost of a factor ϵ using the popularity of the semi-skew 4-PF with petal vertices corresponding to the 3rd and 5th rows and 7th and 9th columns. This the allows us to fix the 8th column at the cost of a factor ϵ. Then we can use the popularity of the column-non-skew 4-PF with petal vertices corresponding to the 3rd, 5th, 8th and 9th columns to fix the 5th column, allowing us to fix the 4th column by using the 4-PF with petal vertices corresponding to the 4th, 5th, 6th and 9th columns.

This leaves us with the following diagram.

$$K\delta\epsilon^6 \times \left[\begin{array}{c|ccccccccc}
 & * & * & c_3 & c_4 & c_5 & c_6 & c_7 & c_8 & c_9 \\ \hline
r_1 & *_1 & & & & & & & & \\
r_2 & *_2 & & & & & & & & \\
r_3 & *_4 & & & & & & & & \\
r_4 & & *_2 & & & & & & & \\
r_5 & & *_3 & & & & & & & \\
r_6 & & *_4 & & & & & & & \\
* & & & *_3 & *_1 & *_4 & & & & \\
* & & & & & & *_1 & *_2 & *_3 & *_4
\end{array}\right]$$

However, recall that the red points in the above diagram correspond to a specific choice of 6-PF from $\mathcal{F}'$, and each member of $\mathcal{F}'$ differs in its 6th petal vertex (corresponding to the fourth column above). Therefore each of the configurations represented by the diagram above come from the same choice of 6-PF in $\mathcal{F}'$, which allows us to fix $*_1, *_2$ and $*_3$. But this immediately fixes all other vertices in the diagram, meaning that the diagram can represent at most 1 configuration. Therefore $K\delta\epsilon^6 \leq 1$ and $K \leq (\delta\epsilon^6)^{-1}$.

The general argument works in exactly the same way. We begin with a collection of $K\delta n^2$ configurations obtained from a collection of K different $2r$-PFs agreeing on $2r-1$ vertices combined with the popular 4-PFs obtained from the fact that each $2r$-PF is good. We then simply use the pigeonhole principle combined with the popularity of 4-PFs given by the edges in $G_A(\epsilon)$ to pass to sub-collections which agree on more and more vertices. Eventually, we obtain $K\delta\epsilon^{2r}$ configurations agreeing on all their vertices, and therefore $K \leq (\delta\epsilon^{2r})^{-1}$. $\square$

By combining Lemmas 3.25 and 3.26 we may pass to a dense subset B_1 of our initial Latin square A in which there is a set $\mathcal{F}$ containing almost all row-column $2r$-PFs (for $1 < r \leq k$) in B_1 satisfying the C-well-defined condition. This will be done formally in Corollary 3.33 later.

3.5 Step 2: A C-well-defined condition for all row-column $2r$-PFs

Next we need to boost from an 'almost all' to an 'all'. This will require formulating a ring decomposition for row-column $2r$-PFs in which all the constituent PFs in the decomposition are also row-column semi-skew.

We start by recalling the definition of the link graph.

Definition 3.27. *Given a partial Latin square (i.e. a tripartite, linear 3-graph), we denote by $G^{(l)}$ the* link graph *of G; that is to say the 2-graph on the same vertex set as G obtained by replacing each 3-edge of G with the corresponding triangle of 2-edges.*

Definition 3.28. *Given a row-column $2r$-PF $F_1 = x_1x_2...x_{2r}$, a* ring decomposition *of F_1 consists of the following. We take any other row-column $2r$-PF $F_2 = y_1y_2...y_{2r}$ with the same inner vertex profile as F_1. Let $x'_1, x'_2, ..., x'_{2r}$ be the $2r$ inner vertices of F_1, which form a $2r$-cycle in the link graph described above, and similarly let $y'_1y'_2...y'_{2r}$ be the cycle formed by the inner vertices of F_2. Observe that x'_i and y'_i both lie in the same vertex class (row, column or label).*

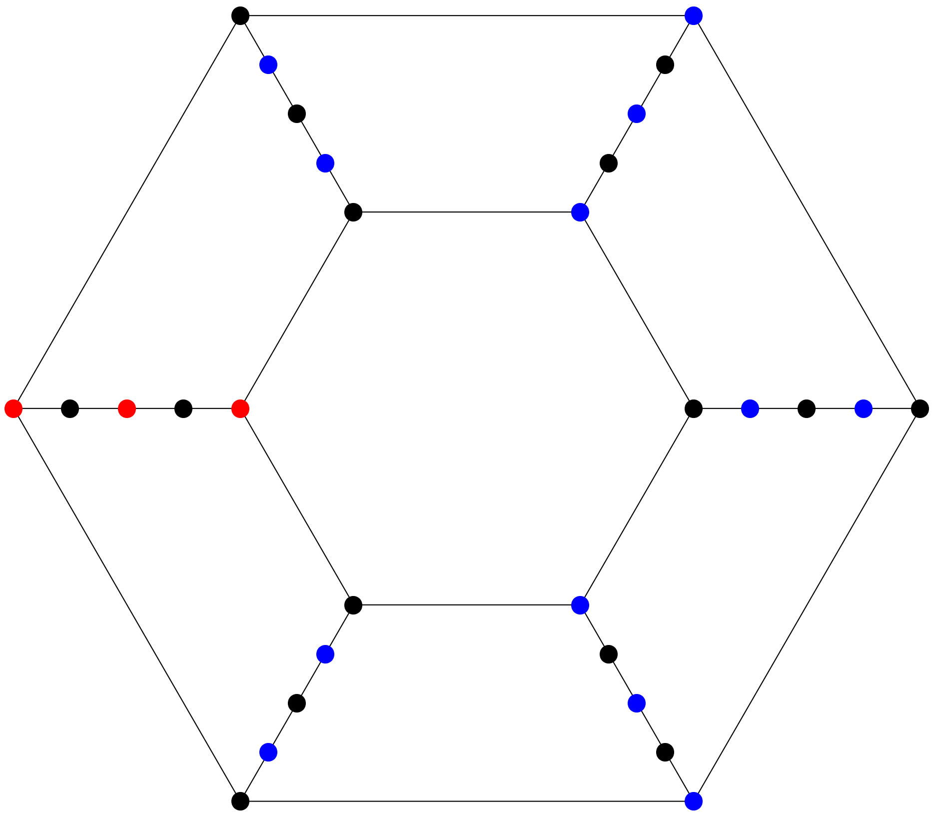


Figure 3.5: A ring decomposition of a 6-PF F_1 with inner vertex profile $P = (C, L, C, L, R, L)$, read clockwise from vertical. Column vertices are shown in blue, label vertices in black and row vertices in red. The 6-PF F_1 corresponds to the inner hexagon, while F_2 corresponds to the outer hexagon. The figure depicts the configuration in the link graph, so the presence of an edge in the above diagram represents the presence of a point containing those two vertices in B_2.

Then for each i we take a label-alternating path of length 4 between x'_i and y'_i (i.e. a path of length 4 in which alternate vertices come from the label class). Any choice of F_2, combined with any choice of the $2r$ label-alternating paths from x'_i to y'_i (for $i = 1, \dots, 2r$) gives a ring decomposition. To clarify with an example, suppose that F_1 is the 6-PF with inner vertex profile $P = (C, L, C, L, R, L)$ that we considered before. Then Figure 3.5 shows a ring decomposition of F_1 in the link graph $G_1^{(l)}$.

In order to construct ring decompositions, we pass to a dense subset of rows, columns and labels in such a way that any pair of row/column/label vertices have many paths of length 3 or 4 between them. We can allow these paths to exist in the larger set, provided that the dense subsets of the vertices induce a positive fraction of edges. This objective will be achieved using the standard dependent random selection idea – indeed it is just the original application as found in the BSG proof.

Lemma 3.29. *Let G be a partial Latin square, seen as a tripartite, linear 3-graph with vertex classes X, Y and Z of size n and edge density δ. Then we can pass to subsets $X' \subset X$, $Y' \subset Y$, $Z' \subset Z$ such that*

1. *each of X', Y', Z' has size at least $\delta^{\mathcal{O}(1)}n$*
2. *$G' = G|_{X' \cup Y' \cup Z'}$ has edge density at least $\Omega(\delta)$*
3. *for any pair of vertices in different vertex classes there are at least $\delta^{\mathcal{O}(1)}n^2$ paths of length three between those vertices that belong to the bipartite link graph between these vertex classes, and*
4. *for every pair of vertices in the same vertex class there are at least $\delta^{\mathcal{O}(1)}n^3$ paths of length four between those vertices in the bipartite link graph between that vertex class and either of the other two vertex classes.*

Proof. It is simplest, though not most direct, to simply apply Lemma 2.19 from Chapter 2 three times, once for each choice of two distinct vertex classes. If we pass to a dense subset of the edges in a bipartite link graph $G^{(l)}|_{U \times V}$, we crucially still have a dense subset of the edges of G. □

By making the "almost all" sufficiently good following our applications of Lemmas 3.25 and 3.26, we can ensure that we still have almost all row-column $2r$-PFs belonging to $\mathcal{F}$ (the set with the C-well-defined condition) after dropping down to our dense subset of rows, columns and labels.

Lemma 3.30. *Suppose that B_1 is a partial Latin square of density β_1. Then there exists a subset B_2 of B_1 of density $\beta_1^{\mathcal{O}(1)}$ in which every row-column $2r$-PF (for $1 < r \leq k$) has at least $\beta_1^{\mathcal{O}(k^3)}n^{8r}$ ring decompositions into row-column PFs (with at most k inner vertices) belonging to B_1.*

Proof. We pass to the set B_2 by applying Lemma 3.29. This is achieved as follows. We consider the hypergraph G_1 corresponding to B_1, and we write $G_1^{(l)}$ for the link graph of G_1. Since G_1 is a dense linear hypergraph, $G_1^{(l)}$ is a dense, tripartite 2-graph to which we can apply Lemma 3.29 and obtain a dense subset B_2 with link graph $G_2^{(l)}$. The edge density of $G_1^{(l)}$ is β_1 so the edge density of $G_2^{(l)}$ is $\Omega(\beta_1)$, but since the vertex set of $G_2^{(l)}$ has size $\beta_1^{\mathcal{O}(1)}$ we have that the density of B_2 as a subset of our original Latin square is $\beta_1^{\mathcal{O}(1)}$.

Now suppose that $F_1 = x_1x_2...x_{2r}$ is a row-column $2r$-PF in B_2. We form a ring decomposition by taking any other row-column $2r$-PF $F_2 = y_1y_2...y_{2r}$ in

B_2 with the same inner vertex profile as F_1. Let $x'_1, x'_2, ..., x'_{2r}$ be the $2r$ inner vertices of F_1, which form a $2r$-cycle in $G_2^{(l)}$, and similarly let $y'_1 y'_2 ... y'_{2r}$ be the cycle formed by the inner vertices of F_2. Observe that x'_i and y'_i both lie in the same vertex class (row, column or label) and so we may find $\beta_1^{\mathcal{O}(1)} n^3$ paths of length 4 between x'_i and y'_i whose alternate vertices come from the label class. Any choice of F_2, combined with any choice of the $2r$ alternating-label paths from x'_i to y'_i (for $i = 1, \ldots, 2r$) gives a ring decomposition.

We observe that the number of choices of F_2 in B_2 is at least $\beta_1^{\mathcal{O}(k^3)} n^{2r}$ by Proposition 3.16, while each path of length 4 contributes $\beta_1^{\mathcal{O}(1)} n^3$ choices. So in total we have $\beta_1^{\mathcal{O}(k^3)} n^{8r}$ different ring decompositions, and this is clearly within a constant factor of the maximum possible. $\square$

It is not sufficient simply to obtain many ring decompositions for each row-column $2r$-PF. We will need the constituent PFs in the decompositions to satisfy a certain additional property.

Definition 3.31. *Observe that a ring decomposition naturally gives rise to a collection of* $2r + 2$ *row-column PFs. As well as* F_1 *and* F_2*, we also have the* $2r$ *row-column 10-PFs that lie between* F_1 *and* F_2 *in Figure 3.5. Suppose that we have a property of row-column PFs called 'good'. Then we call a ring decomposition of* F_1 good *if all of the constituent* $2r+2$ *row-column PFs, except possibly* F_1 *itself, are good. Moreover, we say that* F_1 *has* many good ring decompositions *if* F_1 *has* $\mathcal{O}(n^{8r})$ *good ring decompositions.*

We now show that if almost all row-column $2r$-PFs in B_2 are good (for $1 < r \leq k$) then we can pass to a further dense subset $B_3 \subset B_2$ in which each row-column $2r$-PF F_1 has many good ring decompositions.

Lemma 3.32. *Let* B_1 *be a partial Latin square with density* β_1*, and let* k *be a positive integer. Then there exists* $\theta(\beta_1, k) = \beta_1^{\mathcal{O}(k^3)}$ *such that if at least a proportion* $1 - \theta$ *of row-column* $2r$*-PFs (for* $1 < r \leq k$*) in* B_1 *are good then there exists a subset* B_3 *of* $B_1 s$ *of density* $\beta_3 = \beta_1^{\mathcal{O}(1)}$ *in which every row-column* $2r$*-PF (for* $1 < r \leq k$*) has at least* $\beta_1^{\mathcal{O}(k^3)} n^{8r}$ *good ring decompositions into row-column PFs (with at most* k *inner vertices) in* B_1*.*

Proof. Let B_2 of density $\beta_2 = \beta_1^{\mathcal{O}(1)}$ be as given by applying Lemma 3.30.

The rest of this lemma mirrors Lemma 2.14 from Chapter 2, although we must modify it to work for our new definitions.

Recall that a ring decomposition of a given row-column $2r$-PF F_1 is composed of a second $2r$-PF F_2 which we call the *back face*, and $2r$ row-column 10-PFs which we call the *side faces*. We shall call F_1 *indecomposable* if it has fewer than $\mu(\beta_1, k)n^{8r}$ good ring decompositions in B_1 (for some sufficiently small μ to be chosen later). Moreover, we shall call an indecomposable PF *bad on the back face* if at least one third of its ring decompositions in B_1 have a bad (i.e. not good) cycle on the back face, and *bad on the side faces* if at least one third of its decompositions have a bad 10-PF as a side face.

Since every row-column $2r$-PF has at least $\beta_1^{\mathcal{O}(k^3)}n^{8r}$ ring decompositions in B_1, we have that if $\mu = \beta_1^{\mathcal{O}(k^3)}$ is chosen sufficiently small then a given row-column PF F_1 is indecomposable then it must be either bad on the back face or bad on the side faces. Furthermore, we have that if F_1 is bad on the back face then B_2 must contain at least $\beta_1^{\mathcal{O}(k^3)}n^{2r}$ bad row-column $2r$-PFs. But the number of bad row-column $2r$-PFs in B_2 is in fact at most θn^{2r} and so for sufficiently small $\theta = \beta_1^{\mathcal{O}(k^3)}$ we have a contradiction.

Therefore every indecomposable row-column PF F_1 in B_2 must be bad on the side faces. If F_1 is bad on the side faces, then since F_1 has at least $\beta_1^{\mathcal{O}(k^3)}n^{8r}$ ring decompositions then some point of F_1 must be contained in at least $\beta_1^{ck^3}n^8$ distinct bad row-column 10-PFs for some absolute constant $c > 0$. Since the number of bad row-column 10-PFs in B_2 is at most θn^{10}, this means that the number of totally disjoint $2r$-PFs (i.e. sharing no points) is at most $\theta n^2/\beta_1^{ck^3}$. This holds for each $1 < r \le k$. Therefore, by discarding one point from every indecomposable row-column $2r$-PF, for each $1 < r \le k$, we discard at most $\theta k n^2/\beta_1^{ck^3}$ points. But for θ sufficiently small, $\theta k/\beta_1^{\mathcal{O}(k^3)} < \beta_2/2$ so we discard at most half of the points from B_2 and are left with no indecomposable row-column $2r$-PFs (for $1 < r \le k$). In fact we may take $\theta = \beta_1^{\mathcal{O}(k^3)}$ □

Corollary 3.33. *Let A be a Latin square with at least ϵn^5 copies of H_c, and let k be a positive integer. Then there exists $B_3 \subset A$ of density $\beta_3 = \epsilon^{\mathcal{O}(1)}$ such that there is a $C = C(\epsilon, k) = \epsilon^{-\mathcal{O}(k^4)}$-well-defined condition satisfied by every row-column $2r$-PF in B_3 (for $1 < r \le k$).*

Proof. First we find B_1 by applying Lemmas 3.25 and 3.26 where we take $\theta = \epsilon^{ak^3}$ for some absolute constant a to be chosen later. As a result, we have a C-well-defined condition that holds for a proportion $1 - \theta$ of row-column $2r$-PFs (for $1 < r \le k$) in B_1, and $C = (\theta\epsilon^{\mathcal{O}(k^3)}\epsilon^{2r})^{-1} = \epsilon^{-\mathcal{O}(k^3)}$. Let the set of row-column $2r$-PFs for which we have the C-well-defined condition be called $\mathcal{F}$.

We will then apply Lemma 3.32, saying that a row-column PF is good if it belongs to $\mathcal{F}$. In order to apply Lemma 3.32, we must have $\theta < \epsilon^{a'k^3}$ for some absolute constant a', so we must simply take $a < a'$. The result is a subset B_3

of B_2 of density $\beta_3 = \epsilon^{\mathcal{O}(1)}$ in which every row-column $2r$-PF (for $1 < r \leq k$) has at least $\epsilon^{ck^3} n^{8r}$ good ring decompositions for some absolute constant $c > 0$.

Suppose now that we have a collection $\mathcal{F}'$ consisting of K different row-column $2r$-PFs in B_3 with the same inner vertex profile that agree on $2r - 1$ petal vertices but disagree on the last. Each $F \in \mathcal{F}'$ has at least $\epsilon^{ck^3} n^{8r}$ good ring decompositions. Therefore, in total we obtain at least $K\epsilon^{ck^3} n^{8r}$ good ring decompositions of row-column $2r$-PFs which agree on $2r - 1$ petal vertices, for some absolute constant $c > 0$. If $K\epsilon^{ck^3} \geq K'$ then we get at least $K'n^{8r}$ such good ring decompositions.

Therefore, by the pigeonhole principle, we may find a collection of at least K' of these decompositions that agree on a further $8r$ petal vertices of our choice. We may choose to fix the petal vertices on 3 of the edges from every path of length 4 connecting to the back face, and also the petal vertices on $2r - 1$ of the edges of the back face. We choose to fix the petal vertices on the edges of the back face that match up with the edges with fixed petals in the decomposed $2r$-PF. Finally, we fix the petal vertex on the fourth edge from one of the paths of length 4. Figure 3.6 shows these choices for the example 6-PF earlier – we colour an edge red if the corresponding petal vertex is fixed.

We now note that the back faces of each of these good ring decompositions share $2r - 1$ out of $2r$ petal vertices. Since these faces correspond to PFs belonging to $\mathcal{F}$, which satisfies a C-well-defined condition, there are at most C choices for the last. So by pigeonholing we may fix the petal vertex on this last edge at a cost of a factor $1/C$ in the size of our collection. Then we note that we may continue this process of 'propagating' fixed petal vertices through the side faces. At a cost of a factor $C^{-(2r-1)}$ in the size of our collection we may fix all remaining unfixed petal vertices in the paths of length 4 that separate the side faces, and with a final cost of a factor C^{-1} we may fix the last petal vertex. We are left with a collection of at least $K'C^{-(2r+1)}$ ring decompositions sharing all petal vertices.

But recall that the original collection originated by taking decompositions of specific $2r$-PFs that each differed on the $2r$th petal vertex. Therefore if we are left with decompositions that share all their petals, they must all be decompositions of the same member of $\mathcal{F}'$. This allows us to fix all the rest of the vertices in our configuration, and we find that $K'C^{-(2r+1)} \leq 1$.

Since $C = (\epsilon^{-\mathcal{O}(k^3)})$, $K' = \epsilon^{\mathcal{O}(k^3)} K$, and $r \leq k$, we have that $K \leq \epsilon^{-\mathcal{O}(k^4)}$ from which the result follows. $\square$

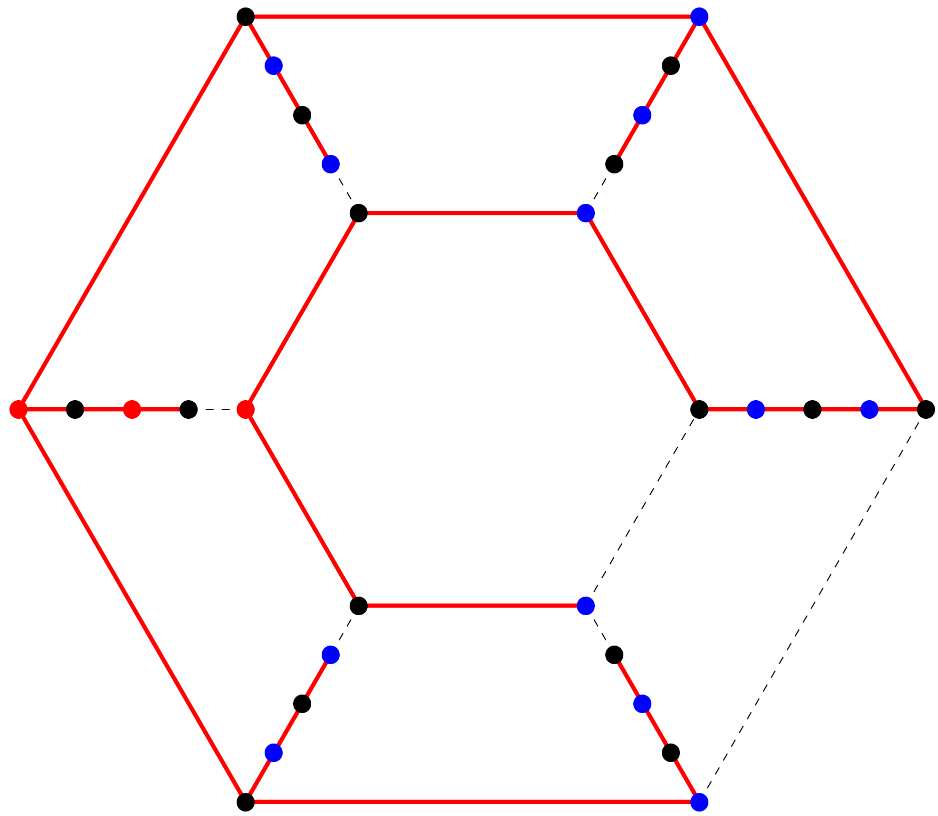


Figure 3.6: With this figure we illustrate a collection of good ring decompositions in B_1. Recall that edges in this figure represent points of the partial Latin square B_1 in the following way: given a point of B_1 we consider the corresponding hyperedge and discard the petal vertex to obtain a 2-edge in the link graph. We draw these edges in red if the petal vertex is shared between all good ring decompositions in the collection – all other vertices (including those pictured) may vary between decompositions across our collection. We show edges whose petal vertex is unfixed (may vary) as dashed black lines. As before, column vertices are shown in blue, label vertices in black and row vertices in red.

Since satisfying the C-well-defined condition is hereditary, we can now afford to be more cavalier when passing to dense subsets.

3.6 Step 3: Popular decompositions for row-column $2r$-PFs

The C-well-defined condition for row-column $2r$-PFs in B_3 immediately shows that almost all row-column $2r$-PFs in B_3 are η-popular, for some sufficiently small η. This uses the fact that the number of each type of row-column $2r$-PF is within a constant factor of the maximum in any dense subset of A.

We can then re-apply Lemma 3.32 to find a dense subset B_4 in which every row-column $2r$-PF (for bounded r) has many ring decompositions into η-popular row-column PFs. We then define a shattered ring decomposition by applying popularity to move all the popular constituent row-column PFs around, and deduce that in B_4 every row-column $2r$-PF (for bounded r) has many shattered $2r$-PFs.

We now formalise this discussion.

Definition 3.34. *Recall that a ring decomposition of F_1 consists of a back face F_2 and $2r$ side faces $F_3, \ldots, F_{2r+2}$. The corresponding* shattered ring decomposition *of F_1 is the collection of $2r+1$ row-column PFs $F'_2, F'_3, \ldots, F'_{2r+2}$ such that each F'_i has the same inner vertex profile as F_i and moreover shares all petal vertices with F_i.*

Figure 3.7 shows a shattered ring decomposition of a row-column 6PF as a hypergraph.

Note that the maximum number of shattered ring decompositions for a given row-column $2r$-PF is n^{10r+1}, since there are at most n^{8r} different possibilities for $F_2, F_3, \ldots, F_{2r+2}$, and then at most n choices for each F'_i.

Corollary 3.35. *Let k be a positive integer. Let B_3 be a partial Latin square of density β_3 in which there is a C-well-defined condition that holds for all row-column $2r$-PFs for $1 < r \leq k$. Then there exists a subset $B_4 \subset B_3$ of density at least $\beta_4^{\mathcal{O}(1)}$ in which every row-column $2r$-PF (for bounded r) has at least $\beta_3^{\mathcal{O}(k^4)} C^{-\mathcal{O}(k)} n^{10r+1}$ shattered ring decompositions belonging to B_3.*

Proof. By Proposition 3.16, B_3 contains at least $\beta_3^{\mathcal{O}(k^3)} n^{2r}$ different row-column $2r$-PFs (for each possible inner vertex profile).

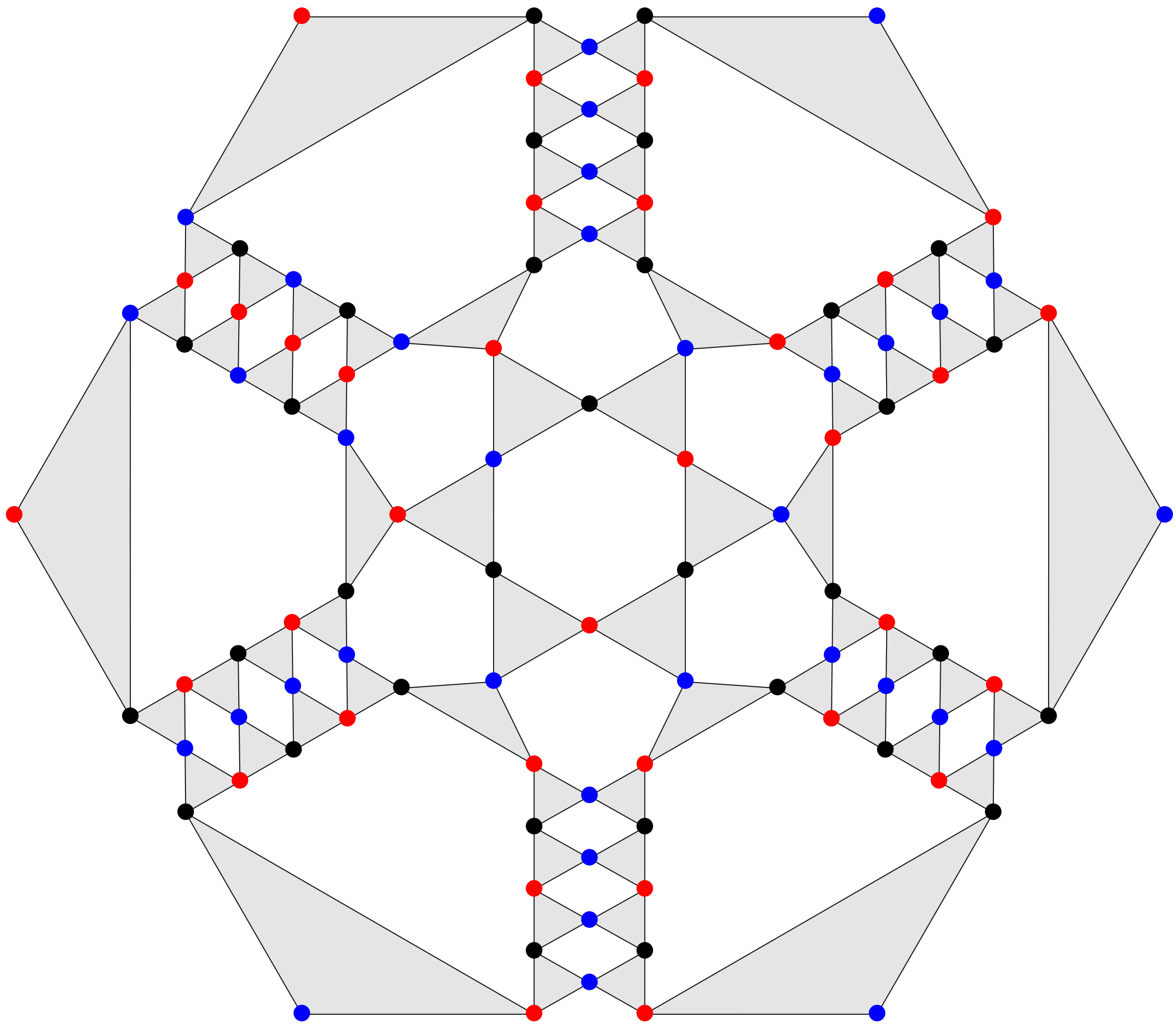


Figure 3.7: The hypergraph representing a shattered ring decomposition of a row-column 6-PF with inner vertex profile $S_F = LRLRLC$. Row vertices are given in red, column in blue and label in black, and the triangles represent faces of the hypergraph.

Fix some inner vertex profile P for a row-column $2r$-PF. Let $X(x_1, \ldots, x_{2r})$ denote the number of $2r$-PFs with inner vertex profile P. We have that

$$\sum_{x_1,\ldots,x_{2r}} X(x_1, \ldots, x_{2r}) \geq \beta_3^{\mathcal{O}(k^3)} n^{2r},$$

but also

$$\sum_{x_1,\ldots,x_{2r}} \mathbb{1}\Big(X(x_1, \ldots, x_{2r}) \neq 0\Big) \leq Cn^{2r-1}.$$

Therefore

$$\sum_{x_1,\ldots,x_{2r}} \mathbb{1}\Big(X(x_1, \ldots, x_{2r}) < \eta n\Big) X(x_1, \ldots, x_{2r}) \leq C\eta n^{2r}$$

and thus the proportion of row-column PFs with inner vertex profile P which are not η popular is at most $\alpha(C, \eta, \beta_3) = C\eta/\beta_3^{\mathcal{O}(k^3)}$ which can be made arbitrarily small by decreasing η.

Therefore we may apply Lemma 3.32 to the set B_3, now saying that a row-column PF is good if it is η-popular. We will choose $\eta = \beta_3^{\mathcal{O}(k^3)}/C$ sufficiently small.

Thus we pass to a subset B_4 of density $\beta_4 = \beta_3^{\mathcal{O}(1)}$ in which every row-column $2r$-PF (for $1 < r \leq k$) has $\beta_3^{\mathcal{O}(k^3)} n^{8r}$ ring decompositions into η-popular PFs F_2, F_3, $\ldots$, F_{2r+2} in B_3. Since each F_i is η-popular in B_3, we get at least ηn choices for each F_i' in the corresponding shattered ring decomposition. We therefore conclude that every row-column $2r$-PF (for $1 < r \leq k$) in B_4 has at least $\beta_3^{\mathcal{O}(k^3)} \eta^{\mathcal{O}(k)} n^{10r+1} = \beta_3^{\mathcal{O}(k^4)} C^{-\mathcal{O}(k)} n^{10r+1}$ shattered ring decompositions belonging to B_3. □

3.7 Step 4: The popular replacement lemma

Now we apply a core result from Chapter 2 to B_4. Note that these results are only applicable to B_4 if we can prove that B_4 contains many cuboctahedra. But row-column 4-PFs satisfy a C-well-defined condition in B_4, and this is sufficient to prove that B_4 contains many cuboctahedra.

To begin, we recall the relevant result from Chapter 2 (namely Theorem 2.26).

Theorem 3.36. *Starting with a 3-uniform, linear hypergraph A containing at least γn^5 cuboctohedra, we may find a sequence $A = A_0 \supset A_1 \supset \ldots$ with the property that A_i is α_i dense and for each $r = 2, \ldots, k$ we have that every non-skew $2r$-PF F in A_i has at least $\gamma_i n^{4r+1}$ different non-skew shattered ring de-*

compositions sharing petals with F. The parameters α_i and γ_i may all be chosen to be $\gamma^{k^{\Theta(i)}}$.

Remark 3.37. Recall that shattered ring decompositions for non-skew $2r$-PFs are, according to Definition 2.15 in Chapter 2, slightly different to those for semi-skew $2r$-PFs in this work. For a non-skew $2r$-PF, a ring decomposition of the $2r$-PF F_1 involves another $2r$-PF F_2 with edges in the link graph (rather than paths of length 4) connecting corresponding inner vertices of F_1 and F_2. This accounts for the power n^{4r+1} rather than n^{10r+1}. We call these decompositions *non-skew shattered ring decompositions* to emphasise the difference.

Corollary 3.38. *Let B_4 be a partial Latin square of density β_4 in which row-column $2r$-PFs satisfy a C-well-defined label completion condition. There exists a subset $B_5 \subset B_4$ of density $\beta_5 = \beta_5(\beta_4, C, k) = (\beta_4/C)^{k^{\Theta(1)}}$ in which for each $1 < r \leq k$, all non-skew $2r$-PFs have at least $\nu(\beta_4, C, k)n^{4r+1} = (\beta_4/C)^{k^{\Theta(1)}} n^{4r+1}$ non-skew shattered ring decompositions belonging to B_3.*

Proof. Let $X(x_1, \ldots, x_4)$ denote the number of row-non-skew 4-PFs (which are of course also row-column 4-PFs). We have that

$$\sum_{x_1,\ldots,x_4} X(x_1, \ldots, x_4) \geq \beta_4^4 n^4$$

but also

$$\sum_{x_1,\ldots,x_4} \mathbb{1}\Big(X(x_1, \ldots, x_4) \neq 0\Big) \leq Cn^3.$$

Therefore, by convexity,

$$\sum_{x_1,\ldots,x_4} X(x_1, \ldots, x_4)^2 \geq \beta_4^8 n^5/C.$$

But note that $\sum_{x_1,\ldots,x_4} X(x_1, \ldots, x_4)^2$ counts the number of pairs of row 4-PFs that agree on all four petal vertices, and this precisely counts cuboctahedra.

Therefore we may apply Theorem 3.36 to B_4 with $\gamma = \beta_4^8/C$, obtaining B_5. $\square$

Remark 3.39. It may seem strange that the density β_5 in Corollary 3.38 depends on C. After all, in Corollary 3.35 the density β_4 avoids a dependence on C, but Corollary 3.35 looks very similar to Corollary 3.38. The difference is that by applying Theorem 3.36 we end up with a certain amount of duplication of effort – the proof of Theorem 3.36 involves recovering a C'-well-defined condition for non-skew $2r$-PFs from the presence of γn^5 cuboctahedra (mimicking the work in Section 3.4, but in the non-skew environment), and in our application γ depends on the C in our C-well-defined condition for row-column $2r$-PFs. This is obviously an unnecessary extra step for row and column-non-skew $2r$-PFs, since these are row-column PFs and thus already satisfy our C-well-defined condition (and indeed have many shattered ring decompositions). However, the additional work is needed to deal with label-non-skew $2r$-PFs, so the worse dependency of β_5 is necessary without a serious modification to the approach.

The next lemma shows that we can simultaneously get all row-column $2r$-PFs and all non-skew $2r$-PFs to have many shattered ring decompositions.

Lemma 3.40. *Let k be a positive integer. Let B_3 be a partial Latin square of density β_3 in which there is a C-well-defined condition that is satisfied by all row-column $2r$-PFs for $1 < r \leq k$. Then there exists a partial Latin square $B_5 \subset B_3$ of density at least $\beta_5 = (\beta_3/C)^{k^{\mathcal{O}(1)}}$ and $\nu = (\beta_3/C)^{k^{\mathcal{O}(1)}}$ such that, for each $1 < r \leq k$, every row-column $2r$-PF in B_5 has at least νn^{10r+1} shattered ring decompositions and every non-skew $2r$-PF has at least νn^{4r+1} non-skew shattered ring decompositions.*

Proof. We simply apply Corollary 3.35, followed by Corollary 3.38. □

We can repeatedly apply Lemma 3.40 to prove the following.

Lemma 3.41. *Let A be a Latin square containing at least ϵn^5 copies of H_c. Then we can construct a nested sequence B_i ($5 \leq i \leq t$) of subsets of A with densities $\beta_i = \beta_i(\epsilon) = \epsilon^{k^{\mathcal{O}(i)}}$ in which, for each $1 < r \leq k$, every row-column $2r$-PF has at least $\nu_i n^{10r+1}$ shattered ring decompositions belonging to B_i and every non-skew $2r$-PF in B_{i+1} has at least $\nu_i n^{4r+1}$ non-skew shattered ring decompositions belonging to B_i, where $\nu_i = \nu_i(\epsilon, k) = \epsilon^{k^{\mathcal{O}(i)}}$.*

Proof. We now put everything together. By Corollary 3.33, we can pass to a subset B_3 of our initial Latin square A of density $\beta_3 = \epsilon^{\mathcal{O}(1)}$ in there is a $C = \epsilon^{-\mathcal{O}(k^4)}$-well-defined condition that is satisfied by all row-column $2r$-PFs (for $1 < r \leq k$). We can then repeatedly apply Lemma 3.40. At each stage the density changes from β_i to $\beta_{i+1} = (\beta_i/C)^{k^{\mathcal{O}(1)}}$, meaning that we obtain $\beta_i = \beta_i(\epsilon) = \epsilon^{k^{\mathcal{O}(i)}}$. The same is true for ν_i and the result follows. □

Lemma 3.41 is our popular replacement lemma. It remains to show that the resulting sequence of sets allows us to perform the replacements we need to guarantee the non-existence of any kind of flappy structures of bounded size.

3.8 Step 5: Unfixing a van Kampen surface

In this section, we apply our popular decomposition results in order to prove Theorem 3.12.

Given a van Kampen surface V we can associate with it a hypergraph S. We shall usually denote the van Kampen surface by V_S to specify the hypergraph S to which it corresponds.

Given a Latin square G_A we can form the group G_A as defined in Definition 3.3. Given a van Kampen surface V_S, we will be interested in counting copies of V_S in the van Kampen complex $\mathcal{K}_A$ of A – see Definition 3.3. Formally, a copy of V_S in $\mathcal{K}_A$ is a mapping of edges of V_S to 1-simplices of $\mathcal{K}_A$ in such a way that each face of V_S is mapped to a 2-simplex of $\mathcal{K}_A$. Note that if A is a partial Latin square and $B \subset A$, we have that $\mathcal{K}_B \subset \mathcal{K}_A$ and so copies of V_S in $\mathcal{K}_B$ belong to $\mathcal{K}_A$ also.

Given a collection of copies $\mathcal{C}$ of a van Kampen surface V_S in $\mathcal{K}_A$, we will say that an edge e of V_S is *unfixed* (in $\mathcal{C}$) if the assignment of a generator of G_A to e may vary across $\mathcal{C}$. For example, if $\mathcal{C}$ represents the set of all row-column 4-PFs in A (with van Kampen diagram shown in Figure 3.4b) then all edges are unfixed. If an edge e is such that every copy of V_S in $\mathcal{C}$ has the same assignment of a generator of G_A to the edge e (in other words, the collection $\mathcal{C}$ agrees on e) then we say that e is *fixed* (in $\mathcal{C}$).

We begin with a lemma which gives the maximum size of a collection of copies of some van Kampen surface with boundary of length 2 in which one of the boundary edges is fixed. This lemma corresponds to the generalisation of Lemma 2.31 in Chapter 2 to van Kampen surfaces.

Lemma 3.42. *Let V_S be a van Kampen surface with v vertices, e edges and f faces, and a boundary of length 2 (containing edges e_1 and e_2). Then the maximum size of a collection $\mathcal{C}$ of copies of V_S in $\mathcal{K}_A$ that agree on one of the two boundary edges (e_1, say) is at most n^{e-f-1}.*

Proof. Note that picking two edges of some face of V_S uniquely determines the third. We shall pick a collection E' of f edges of V_S as follows. We order the f faces of V_S in a sequence $F_1, F_2, \ldots, F_f$ in such a way that F_i contains some edge e'_i which is neither the edge e_1 on the boundary of V_S or an edge belonging to any F_j with $j < i$, then we take $E' = \{e'_1, \ldots, e'_f\}$. We must show that such an ordering of the faces is possible. Consider a spanning tree in the graph on faces where we connect two faces with an edge if the faces share a geometrical edge (in V_S). Then iteratively take leaves from this spanning tree until only a single face remains. When F_i with $i < f$ is chosen for our sequence, it is a leaf on our spanning tree (restricted to the remaining faces), and so it is adjacent to some face F which has not yet been chosen. We may take the edge of V_S contained in both F_i and F to be e'_i. We can choose any face to be the last remaining face, so in particular we can choose it to be the face adjacent to the boundary edge e_2, so we may take $e'_f = e_2$.

Now we observe that a collection of copies of V_S that agree on all but the edges in E' in fact agree on all edges $e'_i \in E'$ as well. This follows by induction: e'_i belongs to face F_i and no edge of F_i is some other e'_j for $j > i$ by how we chose our sequence. Therefore the copies of V_S in our collection agree on two edges of F_i and thus agree on e_i also.

In any collection $\mathcal{C}$ of copies of V_S that agree on the boundary edge e_1, by the pigeonhole principle we can find a subcollection $\mathcal{C}'$ of size $n^{e-1-f}|\mathcal{C}|$ that agree on all but the edges in E'. By the above, $|\mathcal{C}'| \leq 1$ so $|\mathcal{C}| \leq n^{e-f-1}$ as required. $\square$

A key remark is that it will turn out that the maximum possible number of copies of V_S in Lemma 3.42 is unchanged even if we insist that all copies also agree on the other boundary edge. We do not need to prove this at this stage, but it is nevertheless a fact which underpins our strategy.

The overall idea for this section is the same as in the corresponding section of Chapter 2, namely Section 2.5. Let S be some flappy hypergraph for which the 'flap vertices' are from vertex class $X \in \{R, C, L\}$. Given a partial Latin square B, we form an auxiliary graph $H_B(S)$ on X in which ab is an edge if a and b form the flap vertices of some copy of S. In other words, the edge ab witnesses the existence of a copy of the van Kampen surface V_S with boundary word ab^{-1} belonging to the group G_B.

Given a Latin square A with many copies of H_c, we can apply our popular replacement lemma, Lemma 3.41 to obtain a sequence of sets $(B_i)_{i=0}^{T}$ in which all label-non-skew $2r$-PFs and row-column $2r$-PFs (for bounded r) have many shattered ring decompositions in B_{i-1}. Let $B = B_T$.

If we can prove that $H_B(S)$ has bounded degree, we will be able to find a large independent set in X, which will provide us with a dense subset of B in which there are no flappy structures S.

If $H_B(S)$ does not have bounded degree, then we can find a vertex v of large degree. This corresponds to a collection $\mathcal{C}_0$ of copies of V_S in $\mathcal{K}_B$ which all agree on one of the flap edges (but may differ on the other edges).

Roughly speaking, replacements with shattered ring decompositions allow us to do the following. Given a single copy $c_z \in \mathcal{C}_0$ (with $\mathcal{C}_0$ given as above) of V_S we can select some row-column $2r$-PF (or a non-skew $2r$-PF) F, and then we can replace F with a different structure F' in many ways. This gives us a larger collection $\mathcal{C}_1(c_z)$ of copies of a new van Kampen surface V_{S_1} which agree on all but the edges corresponding to those of F'. Inductively, we will eventually end up with a collection $\mathcal{C}_t(c_z)$ of copies of V_{S_t} which agree only on both of the boundary edges. In other words, all edges of V_{S_t} are unfixed in $\mathcal{C}_t(c_z)$ apart from the two boundary edges.

Strictly speaking, the collections $\mathcal{C}_i(c_z)$ exist in the van Kampen complexes of the sets $B_{T-i} \subset A$ rather than in the van Kampen complex of B itself. This is because the shattered ring decompositions for a PF in B_i belong to B_{i-1}. However, provided that T is large enough compared to the size of V_{S_0}, we will reach the collection $\mathcal{C}_t(c_z)$ for $t < T$ and so all collections belong to $\mathcal{K}_{B_0}$. In Section 3.8.3 we will recall this technicality and ensure that parameters are chosen correctly, but until then we will not track exactly which of the $\mathcal{K}_{B_i}$ our collections $\mathcal{C}_j(c_z)$ actually belong to at each step j – our first priority will be to ensure that we can reach some collection $\mathcal{C}_t(c_z)$ as described above using some bounded-length sequence of popular replacements.

Intuitively, we can think of this as an 'unfixing process'. We start with a single copy $c_z \in \mathcal{C}_0$ of V_{S_0}, and in this singleton collection all edges are fixed. When we replace F with F' in step i, we replace some subsurface V of the van Kampen surface $V_{S_{i-1}}$ with a new van Kampen surface V' to obtain V_{S_i}. Moreover, we obtain a collection $\mathcal{C}_i(c_z)$ of copies of V_{S_i} in which all the internal edges of V' are unfixed. We refer to this step as *performing a popular replacement on F (or V)*. Our goal is to use these popular replacements to arrive at a collection $\mathcal{C}_t(c_z)$ in which only the boundary edges are fixed and where $|\mathcal{C}_t(c_z)|$ is within a constant of the maximum possible.

Once we have built the collections $\mathcal{C}_t(c_z)$ for each $c_z \in \mathcal{C}_0$, we can take

$$\mathcal{C} = \bigcup_{c_z \in \mathcal{C}_0} \mathcal{C}_t(c_z)$$

which is a collection of copies of V_{S_t} in which only one of the two boundary edges (that corresponding to the vertex v of $H_B(S)$) is fixed. And if the degree of v is sufficiently large, the size of this collection will violate the trivial upper bound from Lemma 3.42, essentially because of our key remark above.

Since we will be wanting to lower bound the size of the collections $\mathcal{C}_i(c_z)$, it is important to ensure that we do not 'overcount' by arriving at the same copy of V_{S_i} using different sequences of replacements from V_{S_0}. However, this is easily verified by induction, provided that whenever we choose some F in S_i to replace, we ensure that one of the inner vertices of F (corresponding to some inner edge of the band in van Kampen terms) is still fixed (i.e. constant across $\mathcal{C}_i(c_z)$). This is because if two copies of $V_{S_{i+1}}$ agree everywhere but come from copies C_1 and C_2 of V_{S_i}, then C_1 and C_2 must agree on all edges that do not correspond to inner vertices of F as well as one of the vertices corresponding to the fixed inner vertex of F. But this is sufficient to uniquely determine a copy of V_{S_i}, so $C_1 = C_2$.

In Chapter 2, this unfixing process (done in Section 2.5) is achieved entirely using non-skew $2r$-PF replacements. When unfixing a non-skew $2r$-PF, we may replace it with a shattered ring decomposition. This involves picking some vertex of S_i and considering the polygon formed by the triangular faces incident on that vertex. The internal edges of this polygon are then replaced with a shattered ring decomposition, as shown in Figure 3.8.

Unfortunately, unfixing non-skew $2r$-PFs alone will not work in the case of a general van Kampen surface. The difference arises from the genus of the surface – when the genus is non-zero, it is insufficient to simply unfix edges of the van Kampen diagram by unfixing around vertices.

In the general case we must also unfix row-column $2r$-PFs. It is harder to draw a van Kampen picture of a shattered ring decomposition of a row-column $2r$-PF, but we can show hypergraph pictures (see Figures 3.7 and 3.18).

There are several features of the general case that present significant complications that do not arise in the genus-0 case. The biggest complication arises from the fact that we are restricted to row-column $2r$-PF replacements rather than having the flexibility to replace general skew $2r$-PFs. This is an asymmetrical constraint that will cause difficulties throughout. Another challenge arises because, while shattered decompositions of non-skew $2r$-PFs are possible to visualise (allowing the popular replacement argument to be followed with figures), decompositions of semi-skew $2r$-PFs are much more difficult to visualise and it will not be feasible to draw complete pictures of the intermediate van Kampen surfaces.

Nevertheless, we begin by giving a detailed example, and then we shall provide the general approach. The example we choose is that of the label-flappy version of the hypergraph $\mathcal{T}$ which we call $\mathcal{T}'$, whose van Kampen representation was depicted earlier in Figure 3.1.

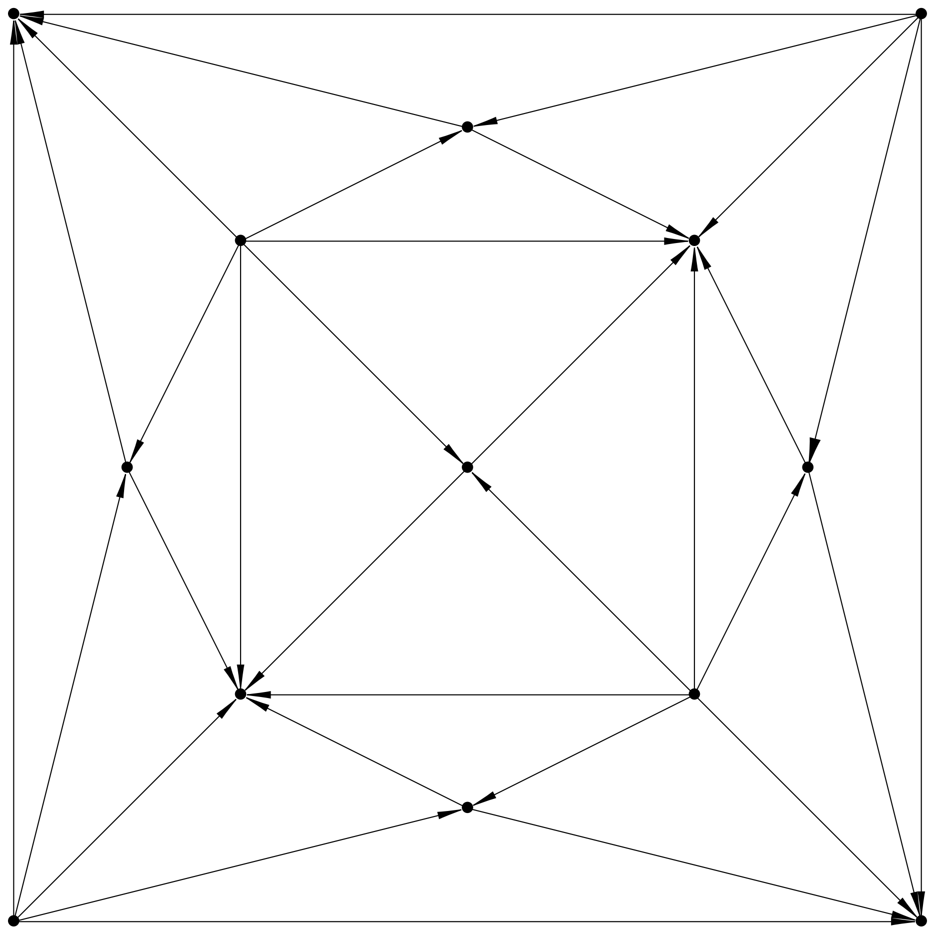


Figure 3.8: A van Kampen diagram representing a shattered ring decomposition of a non-skew 4-PF. We have omitted the labels on the edges for simplicity. More generally, the shattered ring decomposition for a non-skew $2r$-PF involves two $2r$-gons with corresponding vertices joined with edges and every face triangulated.

3.8.1 Unfixing a flappy version of the hypergraph $\mathcal{T}$

In this section, we shall give several examples involving $\mathcal{T}'$. First we shall discuss the approach that we might take if we had a much less restrictive popular replacement lemma. This will help to isolate some of the difficulties.

Let us begin by recalling some facts about the van Kampen surface $V_{\mathcal{T}'}$ representing $\mathcal{T}'$. First of all, the number of vertices in $V_{\mathcal{T}'}$ is $v = 3$. As drawn in Figure 3.1, there is a central vertex in the middle of the hexagon and only two vertices on the boundary of the hexagon. The number of edges is $e = 10$, and the number of faces is $f = 6$. Therefore the maximum possible size of a collection of copies of $V_{\mathcal{T}'}$ that agree on one of the boundary edges is $n^{e-f-1} = n^3$ by Lemma 3.42.

An illustrative thought experiment

As described above, our task is to start with a single copy c_z of V_S belonging to $\mathcal{C}_0$, and to apply popular replacements in order to produce a large collection $\mathcal{C}_t(c_z)$ in which all but the two boundary edges are unfixed and $|\mathcal{C}_t(c_z)|$ is within a constant factor of the maximum possible. Our popular replacement lemma allows us to replace row-column $2r$-PFs and label-non-skew $2r$-PFs, and upon replacing F with F' in stage i, all of the internal edges of the part of the van Kampen surface V_{S_i} corresponding to F' become unfixed.

It will be helpful to begin with a simplified version of our task. In this simplified setting, we shall imagine that we have at our disposal a better version of our popular replacement lemma. In this better version, we suppose that all $2r$-PFs are ϵ-popular, meaning that any $2r$-PF can be 'replaced' with ϵn different options that differ only on the inner vertices (i.e. the inner edges of the band in van Kampen terms).

In fact, such a popular replacement lemma is too much to ask for – for instance, one can construct examples of partial Latin squares in which no dense subset has all 4-PFs popular. Nevertheless, it is instructive to see how the unfixing process would be carried out if such a popular replacement lemma were achievable.

Given a copy of $\mathcal{T}'$, Figure 3.9 shows a sequence of three popular replacements that can be used to unfix all but the two boundary edges of the corresponding van Kampen surface $V_{\mathcal{T}'}$. Since each popular replacement unfixes a collection of edges containing at least one fixed edge, we do not need to worry about over-counting – the size of collection $\mathcal{C}_i(c_z)$ is $(\epsilon n)^i$. Therefore we obtain $\epsilon^3 n^3$ copies of

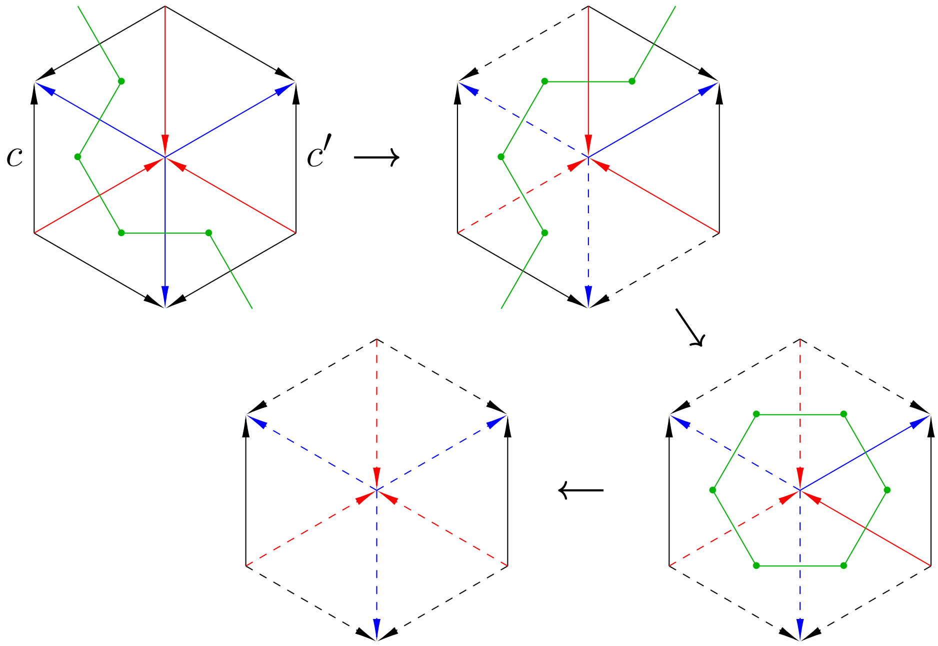


Figure 3.9: A sequence of three popular replacements for $\mathcal{T}'$. Recall that opposite edges on the boundary of the hexagon are identified, apart from the edges labelled c and c' which represent the two boundary edges. The dashed edges are unfixed, and the green lines represent the $2r$-PFs being replaced at each stage.

$\mathcal{T}'$ that agree on the two boundary edges. Since the maximum possible number of copies of $V_{\mathcal{T}'}$ from each c_z that agree on just one of the two boundary edges is n^3, we achieve our popular replacement objective.

In full, suppose that the graph $H_B(\mathcal{T}')$ with edges corresponding to copies of $V_{\mathcal{T}'}$ has maximum degree K. Then we can find K copies of $V_{\mathcal{T}'}$ that agree on one of the two boundary edges and disagree on the other. By applying the above popular replacement argument, each of these copies give rise to a collection of $\epsilon^3 n^3$ copies of $V_{\mathcal{T}'}$ with the same boundary edges. Therefore, taking the union of these collections gives a collection of $K\epsilon^3 n^3$ copies of $V_{\mathcal{T}'}$ which agree on one of the two boundary edges. But the maximum possible number of copies of $V_{\mathcal{T}'}$ that agree on just one of the two boundary edges is also n^3, so we have that $K < \epsilon^{-3}$, which gives us that $H_B(\mathcal{T}')$ has bounded degree as desired.

Note that it is possible to choose a bad sequence of popular replacements. For instance, we might choose the two popular replacements shown in Figure 3.10, which would leave us with a collection too small to give an effective bound on K. Part of the challenge in providing a general argument will arise from making sure that we can pick a sequence of replacements of the appropriate length.

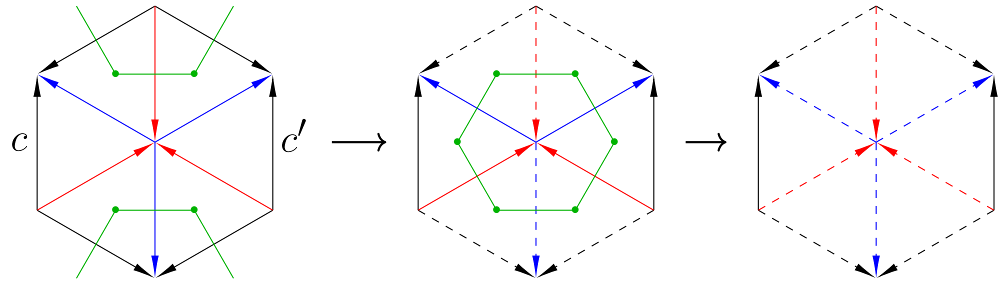


Figure 3.10: A sequence of two popular replacements for $\mathcal{T}'$. This sequence is not as long as that shown in Figure 3.9 and must therefore be avoided in our popular replacement argument.

What happens in reality

In reality, our popular replacement lemma is not nearly as convenient as we might hope. Firstly, we are restricted to unfixing row-column $2r$-PFs or non-skew $2r$-PFs rather than general $2r$-PFs. Secondly, when we replace a certain $2r$-PF, we do not replace it with the same structure. Instead, we replace with a shattered ring decomposition, and this will lead to a rapid explosion of complexity in the pictures that we need to visualise.

We start with a collection $\mathcal{C}_0(c_z)$ containing a single copy of V_{S_0} where $S_0 = \mathcal{T}'$. We begin the popular replacement process by unfixing around the central vertex of the hexagon. This will give us some new collection $\mathcal{C}_1(c_z)$ of copies of a new van Kampen surface V_{S_1} in which all but four edges are unfixed. We show V_{S_1} in Figure 3.11.

The size of the collection $\mathcal{C}_1(c_z)$ is $\Omega(n^{13})$ since we are replacing a non-skew 6-PF.

Next, we will choose a row-column $2r$-PF in V_{S_1} to unfix. We will choose the 10-PF shown in green in Figure 3.12.

Replacing this row-column $2r$-PF with its shattered ring decompositions gives a collection $\mathcal{C}_2(c_z)$ of copies of a new van Kampen surface V_{S_2} with just three edges fixed, and the elements of $\mathcal{C}_2(c_z)$ agree on the fixed edges. Unfortunately, it now becomes impractical to draw V_{S_2} in full detail due to the complexity of the shattered ring decomposition of the row-column 10-PF (V_{S_2} contains 130 faces!). We can, however, draw V_{S_2} as in Figure 3.13 by representing the unfixed component using the shaded red area.

We can also count the size of the collection $\mathcal{C}_2(c_z)$. Since the number of shattered ring decompositions of our row-column 10-PF is $\Omega(n^{51})$, the total size of $\mathcal{C}_2(c_z)$ is $\Omega(n^{64})$ (since we avoid over-counting thanks to ensuring that the 10-PF chosen in Figure 3.12 contained a fixed internal edge. Note also that

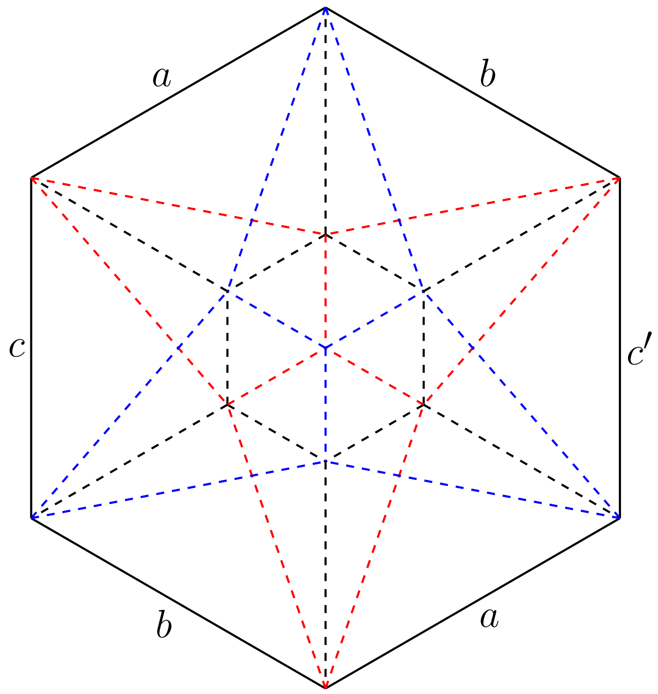


Figure 3.11: The van Kampen surface obtained from $V_{\mathcal{T}'}$ by replacing the label-non-skew 6-PF surrounding the central vertex. We have omitted edge directions for clarity.

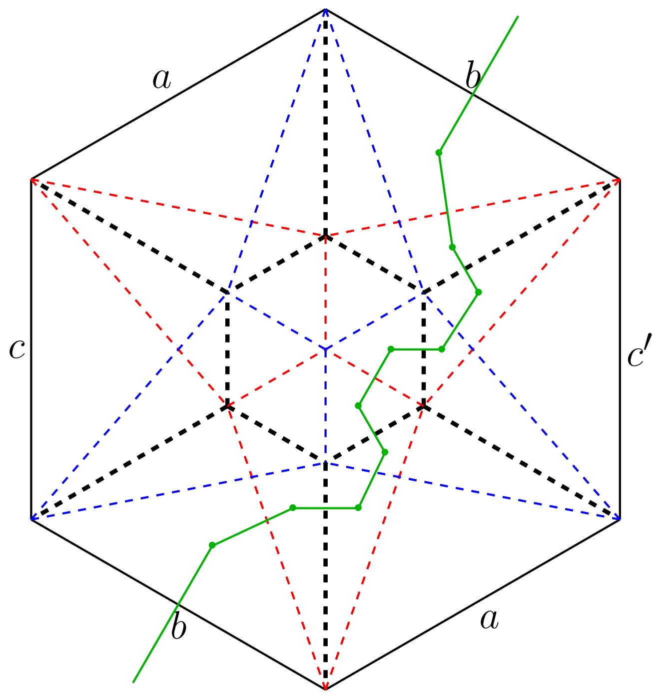


Figure 3.12: The van Kampen surface obtained from $V_{\mathcal{T}'}$ by replacing the label-non-skew 6-PF surrounding the central vertex. A green line has been added to show a row-column 10-PF which can be replaced.

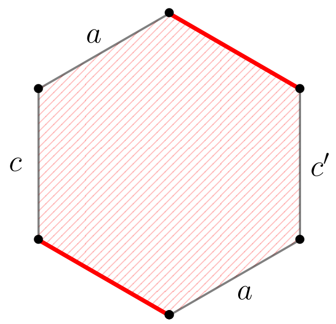


Figure 3.13: An abstract representation of the van Kampen surface obtained by replacing the 10-PF shown in green in Figure 3.12. The unfixed component is shaded in red, but the precise structure is omitted. The boundary edges are c and c' while the edge a is the final remaining fixed non-boundary edge.

V_{S_2} has 130 faces, each containing three edges. All but the boundary edges are contained in two faces, so the number of edges in total is $390/2 + 1 = 196$. Of these, the two boundary edges and one additional edge are fixed.

We shall now use one final row-column $2r$-PF replacement to unfix the last non-boundary edge. Without drawing V_{S_2} we cannot exhibit a row-column $2r$-PF which passes through the last fixed non-boundary edge, and it is not clear that one exists.

Suppose that one does exist, so we can find some row-column $2r$-PF which passes through the last fixed non-boundary edge of V_{S_2}. Then after replacing we will obtain a collection $\mathcal{C}_3(c_z)$ of copies of some van Kampen surface V_{S_3} which agree only on the two boundary edges. The size of the collection $\mathcal{C}_3(c_z)$ will be $\Omega(n^{64+10r+1}) = \Omega(n^{65+10r})$. Moreover, the number of faces of V_{S_3} will be $130 - 2r + 22r = 130 + 20r$ and the number of edges will be $3(130 + 20r)/2 + 1 = 196 + 30r$. Therefore the maximum possible size of a collection of copies of V_{S_3} agreeing on the two boundary edges is $n^{196+30r-(130+20r)-2+1} = n^{65+10r}$. So $|\mathcal{C}_3(c_z)|$ is within a constant of the maximum possible.

Importantly, the maximum possible size of a collection of copies of V_{S_3} which agree only on one of the two boundary edges is still n^{65+10r} by Lemma 3.42, which is the same as $|\mathcal{C}_3(c_z)|$ up to a constant factor. This allows us to finish the popular replacement argument as before.

So all that remains unproved is whether there does indeed exist some row-column $2r$-PF containing the final fixed non-boundary edge in V_{S_2} as shown in Figure 3.13. Rather than providing an argument for this specific case, we shall now move on and give a general approach.

3.8.2 The general case

The example with $\mathcal{T}'$ above illustrates the main difficulties. We must ensure that there is always a suitable choice of row-column $2r$-PF to replace at each stage, and we must also correctly count the sizes of our collections and compare them, at the end, with the maximum possible size.

It will help to introduce one final alternative representation for our flappy structures, distinct from the hypergraph and the van Kampen representations.

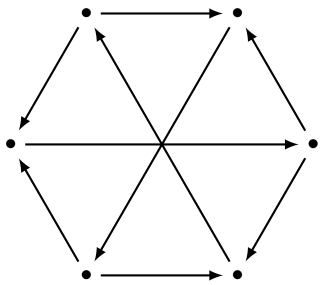
Definition 3.43. *Given a hypergraph S with van Kampen surface V_S we let G_S be the graph constructed as follows. The vertex set of G_S consists of the faces of V_S, but for clarity we write v_f for the vertex that corresponds to the face f. There is an edge $v_f v_g$ if f, g are adjacent in V_S.*

Recall that the edges of the van Kampen diagram V_S are oriented and each face contains a row, column and label edge. We can thus 2-colour the faces of V_S by colouring the face f red if the direction of the row edge is clockwise around f (from a perspective exterior to the surface) and blue otherwise. This 2-colouring of the faces is proper, in the sense that adjacent faces must receive different colours. Using this colouring, we can put directions on the graph G_S as follows.

Definition 3.44. *Let D_S be a directed graph constructed by orienting the edges of G_S as follows. Consider the above 2-colouring of the faces of V_S. This 2-colouring corresponds to a red/blue colouring of the vertices of G_S. We orient the edge $v_f v_g$ of G_S from the blue vertex to the red vertex if the edge of V_S shared by both f and g is a label edge, and from red to blue otherwise.*

The orientations in D_S ensure that the faces of a row-non-skew or a column-non-skew $2r$-PF in V_S correspond to a directed cycle in D_S, while the faces of a label-non-skew $2r$-PF correspond to a cycle containing no directed path of length 2. We have oriented the edges according to the direction that one must leave a face of V_S if one is tracing the faces of a row-column $2r$-PF. To give an example, the hypergraph $\mathcal{T}$ has $D_{\mathcal{T}}$ as given in Figure 3.14a and the hypergraph $\mathcal{T}'$ has $D_{\mathcal{T}'}$ as given in Figure 3.14b.

Observe that the edges of D_S are in one to one correspondence with the non-boundary edges of V_S. We can visualise our unfixing process by tracking the graphs D_{S_i} and calling an edge of D_{S_i} unfixed if the corresponding edge of V_{S_i} is unfixed (in the collection $\mathcal{C}_i(c_z)$).



(a) $D_{\mathcal{T}}$ (b) $D_{\mathcal{T}'}$

Figure 3.14: Examples of the directed graph D_S.

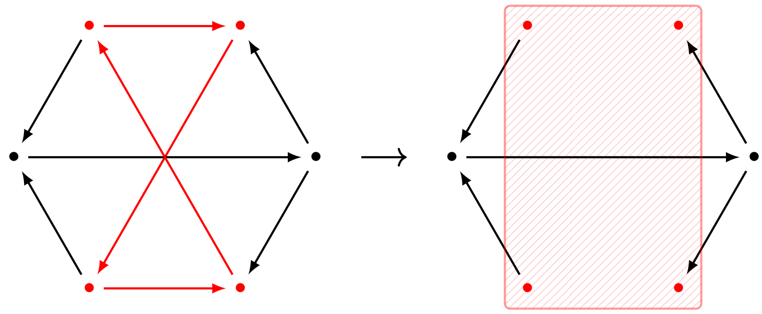
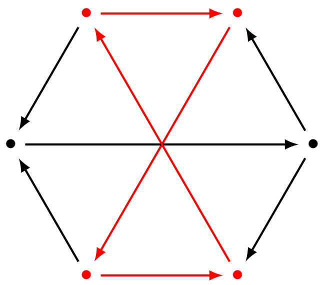


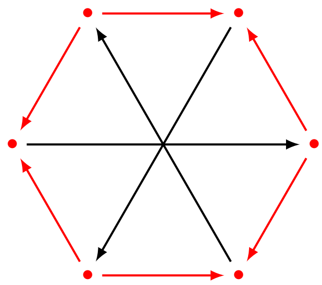
Figure 3.15: On the left we have a copy of $D_{\mathcal{T}}$, with a directed cycle (corresponding to a row-column 4-PF in $\mathcal{T}$) ready to be replaced. On the right, we show the directed graph after replacement, with the unfixed subgraph shown as a red hatched rectangle. This is the best we can do to visualise the directed graph after replacement, since the unfixed component is in reality very complicated. Indeed the red hatched box would contain 40 vertices if the picture were drawn in full. The claim in Lemma 3.46 is that the subgraph represented by the red hatching has the path completion property.

We start with D_{S_0}, all of whose edges are fixed (so the unfixed subgraph is empty). When we perform a popular replacement of a $2r$-PF F in V_S, we remove the cycle C of edges of D_S corresponding to F and we replace C with a new graph C' representing the new part of the van Kampen surface introduced by the replacement, and we add C' to the unfixed subgraph. This gives us a new van Kampen surface $V_{S'}$ with corresponding directed graph $D_{S'}$. Since the edges in D_S from outside C into C still exist in $D_{S'}$ we may view C' as extending (rather than replacing) the vertex set of C, so that D_S is a subgraph of $D_{S'}$. A step of this process is shown in Figure 3.15, and later in Figure 3.19.

Our task is to use popular replacements to unfix all edges, resulting in a directed graph D_{S_t}, corresponding to our van Kampen surface V_{S_t}, in which all edges are unfixed. As before, in order to avoid over-counting issues when we track the size of the corresponding collections $\mathcal{C}_i(c_z)$ we must make sure that each step of the unfixing process we replace some cycle C in D_{S_i} that contains at least one fixed edge.



(a) The red subgraph shown here has the path completion property.



(b) The red subgraph shown here does not have the path completion property, since if we take $e_1 = e_2$ to be any choice of one of the three fixed edges we cannot find the required directed path.

Figure 3.16: The path completion property.

As we saw in the example with $\mathcal{T}'$ above, one of the difficulties that we shall face involves ensuring that at each step, after we have performed several popular replacements, we can still find directed cycles containing some fixed edges. Ideally, we would like to achieve this without being forced to keep track of the precise structure of the van Kampen surface that we are considering.

The following definition and lemma will help us find directed cycles.

Definition 3.45. *Given some subgraph D' of D_S, we say that D' has the* path completion property *(in D_S) if the following holds. For any edges $e_1 = xy$ (directed from x to y) and $e_2 = zw$ (directed from z to w) of D_S not belonging to D' but such that $y \in D'$ and $z \in D'$, we have that there exists a directed path from y to z contained entirely in D'. We do not insist that e_1 and e_2 are distinct edges.*

Figure 3.16 gives an example of a subgraph without the path completion property and an example with the property.

The following lemma shows that when we apply a popular replacement, we obtain the path completion property in the new part of the directed graph. The precise statement is quite technical, but Figure 3.15 provides further explanation.

Lemma 3.46. *Let S be some hypergraph corresponding to a van Kampen surface V_S. Suppose that we take a $2r$-cycle C from D_S corresponding to either a row-column $2r$-PF (if C is directed) or a label-non-skew $2r$-PF in S. Let S' be the hypergraph obtained by performing popular replacement on the $2r$-PF corresponding to C, and let D' be the subgraph of $D_{S'}$ corresponding to the new subgraph replacing C. Then D' has the path completion property in $D_{S'}$.*

Proof. Proving this lemma will require understanding the structure of the shat-

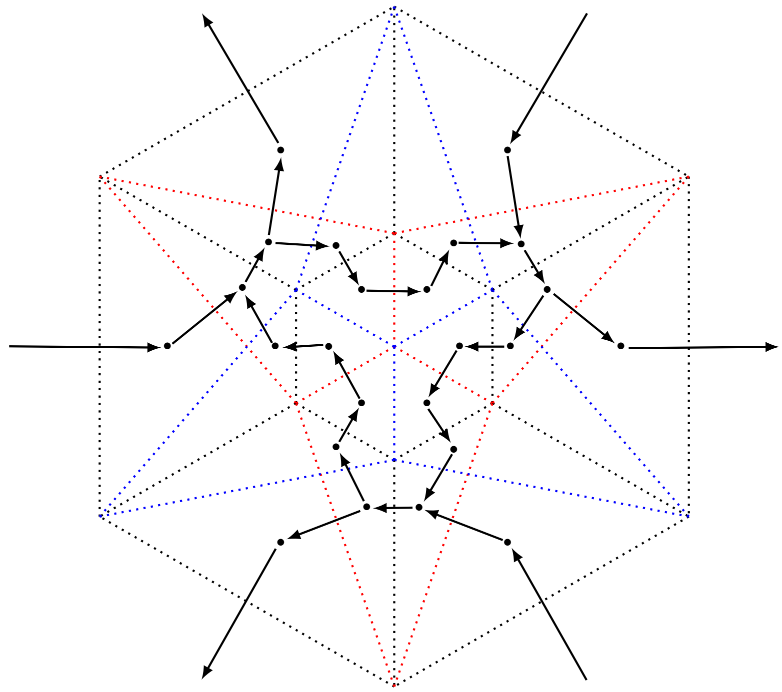


Figure 3.17: Path completion in the shattered ring decomposition for the label-non-skew $2r$-PF. Only the edges of D_S providing the required directed paths have been shown.

tered ring decompositions. Let us begin with the case of a non-skew label $2r$-PF F. Drawing only the label edges of the van Kampen surface V representing the shattered ring decomposition of F we get two concentric $2r$-gons with their corresponding vertices joined by edges. Let us number the edges of the outer $2r$-gon $e_1, \ldots, e_{2r}$ and suppose that the edges of the inner $2r$-gon are labelled $e'_1, \ldots, e'_{2r}$ with e_i corresponding to e'_i. Let the edge joining the vertex between e_i and e_{i+1} with the vertex between e'_i and e'_{i+1} be called e''_i. Suppose that the edge of D_S corresponding to e_1 is oriented into D'. The edges of D_S with precisely one vertex in D' and edges oriented out of D' are precisely those corresponding to e_{2i}. We will be done (by symmetry) if we can show that the edges corresponding to e_1 and e_{2i} can be joined by a directed path inside D'.

This can be achieved as follows. We enter through e_1 and pass through two faces inside the 4-gon bounded by e_1, e''_1, e'_1 and e''_{2r}, leaving through e''_1. We then pass through 2 faces of the 4-gon bounded by e_2, e''_1, e'_2 and e''_2 and we can choose to exit either through e_2 or e'_2. If we exit through e'_2 then, passing through 2 faces of the internal $2r$-gon bounded by the e'_i, we exit at e'_3 into the 4-gon bounded by e_3, e''_3, e'_3 and e''_2. Then we pass through two faces inside this 4-gon, leaving through e''_3. We may then leave through e_4 or e''_4. Continuing in the same way we may exit the outer $2r$-gon at any e_{2i}. An example is shown in Figure 3.17.

The case where C corresponds to a row-column $2r$-PF is more complex. The van Kampen surfaces are too complicated to draw. Instead, we will think about the hypergraph H_C representing the shattered ring decomposition. Note that edges of D_S correspond to vertices of the hypergraph H_C, and the edges with precisely one endpoint in D' correspond to the degree 1 vertices of H_C.

Recall that H_C consists of an inner row-column $2r$-PF surrounded by 10-PFs that share 4 petals with each adjacent 10-PF. The degree 1 vertices of H_C are to be found on each 10-PF – one per 10-PF.

Let p be a degree 1 vertex and let p' be the vertex p' that is both a petal of the 10-PF with petal vertex p and of the inner row-column $2r$-PF. Let e_p be the directed edge corresponding to vertex p and $e_{p'}$ the directed edge corresponding to p'. Then there is either a directed path joining e_p to $e_{p'}$ (if e_p is directed into D') or a directed path from $e_{p'}$ to e_p. This follows because the inner vertices of each 10-PF alternate coming from the label class, so if (say) the orientation of e_p is into D', then there is a path that alternates label vertices from p to p' and this path corresponds to a directed path in D'. Moreover, we note that a row-column $2r$-PF corresponds to a directed cycle in D', so the subgraph H' of the graph D_S corresponding to the inner row-column $2r$-PF has the path completion property in D_S. Therefore D' has the path completion property in D_S – we simply follow a directed path from p to p', use H' to move to the correct 10-PF and then exit analogously. An example explaining this is given in Figure 3.18.

□

The following corollary shows that as we extend our unfixed subgraph by performing popular replacements, the (weakly) connected components of the unfixed subgraph satisfy the path completion property. Again, the statement is very technical but the Figure 3.19 provides an example.

Corollary 3.47. *Let S be a hypergraph with corresponding van Kampen surface V_S. Let D be some subgraph of D_S such that every connected component of D has the path completion property. Now let C be a directed cycle in D_S. Suppose that we perform a popular replacement on the $2r$-PF corresponding to C, leading to a new van Kampen surface $V_{S'}$. In $D_{S'}$ we have replaced C with some new graph C'. Let D' be the subgraph of $D_{S'}$ consisting of the edges of D not contained in C combined with all of C', i.e. $D' = (D \setminus C) \cup C'$. Then every connected component of D' has the path completion property.*

Proof. Take edges $e_1 = xy$ (directed from x to y) and $e_2 = zw$ (directed from z to w) in $D_{S'}$ such that $y, z \in D'$ and moreover both belong to the same component

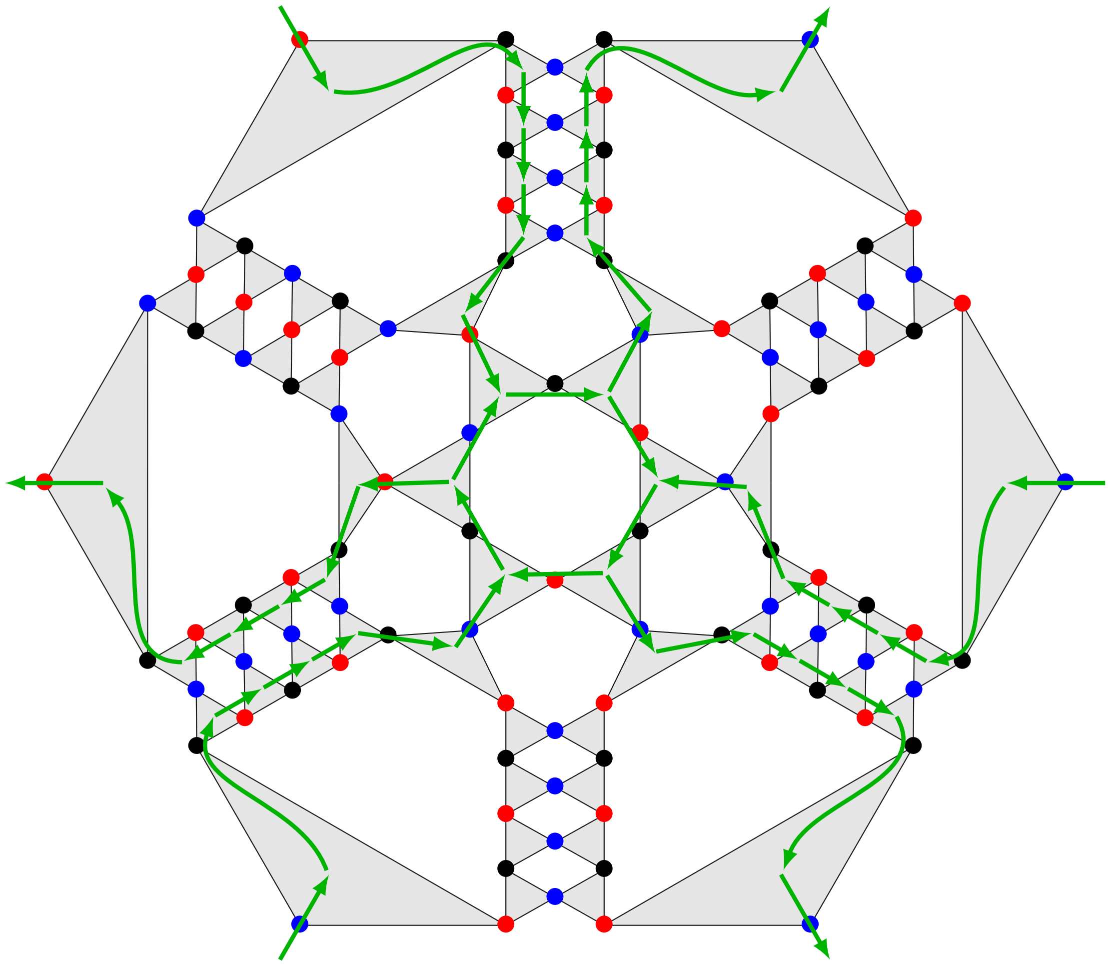


Figure 3.18: We show the path completion property in the shattered ring decomposition of a row-column 6-PF. We have given the decomposition in hypergraph form (since the van Kampen surface is very difficult indeed to draw). Since vertices of the hypergraph correspond to edges of the van Kampen surface, they correspond also to edges of D_S. The vertices of D_S correspond to faces of the hypergraph. In the figure, edges of D_S are shown in green, while the red, blue and black vertices correspond to row, column and label vertices (respectively) of the shattered ring decomposition (as a hypergraph).

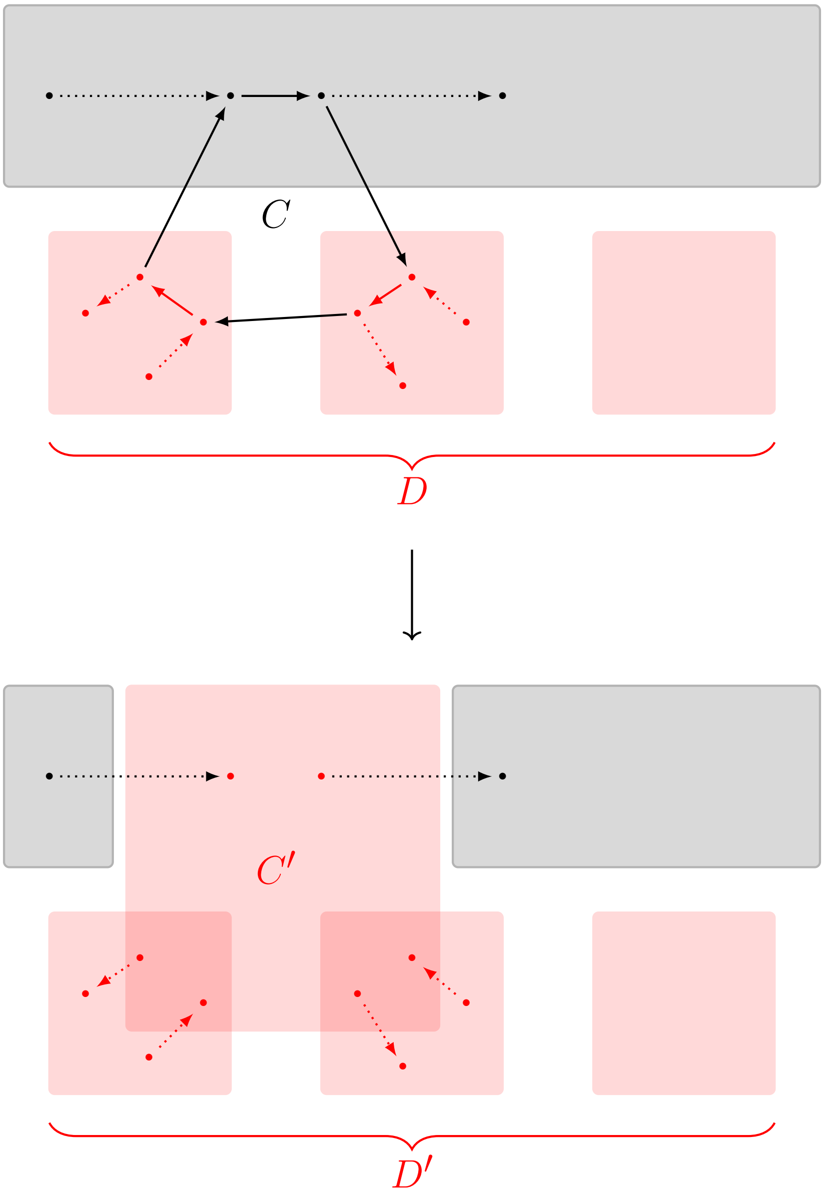


Figure 3.19: What happens to the directed graph D_S after a popular replacement step. Here we are replacing the directed cycle C shown in D_S at the top of the figure. The unfixed subgraph D is shown in red, divided into several connected components. The dashed lines represent edges of D_S which join to C. In the bottom of the figure we have drawn $D_{S'}$, the directed graph after popular replacement. The subgraph C has been replaced with C' and unfixed, but the edges into C remain in C'. The new unfixed component is D'.

of D'.

Since C' contains the vertices of C, we may view the vertices of D_S as a subset of the vertices of $D_{S'}$.

Note that both y and z are vertices both in $D_{S'}$ and in D_S since they belong to edges that are not contained in D'.

If both y and z belong to C' then the result follows from Lemma 3.46.

Now suppose that y belongs to C' and z does not. Since z does not belong to C', the vertex z must also exist in the graph D_S and belong to some component D_1 of D. If y also belongs to D_1, then by the path completion property in D_1 we have some directed path P from y to z in D_1. This path may use edges from C, so might not exist in D'. Let uv be the directed edge at which the path P leaves the cycle C for the last time. Then the path completion property in C' tells us that we can find a directed path from y to u, and P provides us with a directed path from u to z, so we are done.

So we may assume that y does not belong to D_1. Since y, z are connected in D', we must have that cycle C intersects D_1 in D_S. Since C is a directed cycle and not entirely contained in D_1 (because $y \notin D_1$) we have some directed edge uv in C such that $u \notin D_1$ and $v \in D_1$. Then there is a directed path P from v to z by the path completion property in D_1. Note that P may not entirely exist in $D_{S'}$, since the edges of C get removed. However, since z does not belong to C', part of the path P exists in $D_{S'}$ – indeed, in $D_{S'}$ we can trace a path backwards from z to a directed edge $u'v'$ such that $u' \in C'$ and $v' \notin C'$. But by the path completion property in C' we can find a directed path from y to u' in C', which then completes to a directed path from y to z in $D_{S'}$.

Next suppose that z belongs to C' and y does not. In this case we are done by the previous case – by reversing the directions on all edges and applying the previous case we can find a directed path from z to y, and then we can reverse directions back to give our path from y to z. This is possible because reversing directions on all edges preserves the path completion property in subgraphs.

So we may assume that neither y nor z belong to C'. Therefore they both belong to D in D_S. Suppose first that they both belong to the same component D_1 of D. Then by the path completion property in D_1 we can find a directed path P from y to z in D_1. If P does not meet the cycle C then P exists also in $D_{S'}$ and we are done. If P does meet the cycle C, it must enter the cycle C for the first time through a directed edge uv and leave for the last time through a directed edge $u'v'$. By the path completion property in C' we can find a directed path from v to u inside C' and use this in combination with the start and end of P to obtain a directed path from y to z.

Lastly we consider the case where neither y nor z belong to C' and they belong to different components D_1 (containing y) and D_2 (containing z) of D. Since y and z are in the same component of D', the cycle C must connect D_1 and D_2. Therefore we can find a directed edge uv of C leaving D_1 for the first time and a directed edge $u'v'$ of C entering D_2 for the last time. By the path completion properties in D_1, C' and D_2, we get directed paths from y to u, from v to u' and from v' to z in D', which can be put together to give our directed path from y to z in D'. □

We are now ready to describe our popular replacement strategy in full detail. Let $S = S_0$ be some hypergraph with corresponding van Kampen surface V_{S_0} with a boundary of length 2. We may assume that there is no proper subset of the faces of V_{S_0} that has boundary of length 2. We would like to show that, using a sequence of popular replacements, we can unfix all but the two boundary edges in some van Kampen surface V_{S_j} (at some stage j), and moreover that the number of copies of V_{S_j} in our collection following our sequence of replacements is within a constant of the maximum possible.

Case 1: Boundary edges of V_{S_0} are not label edges

Consider the graph D_{S_0}. Note that this graph has the property that all vertices have out-degree at least 1 – if a vertex v corresponds to a face of V_{S_0} that is adjacent to three faces, then v either has out-degree 2 and in-degree 1 or out-degree 1 and in-degree 2, while if v corresponds to a face of V_{S_0} that is only adjacent to two faces (i.e. the face lies on the boundary) then since the boundary is not a label edge, we must have that v has both in- and out-degree 1.

In this case, we shall perform the whole popular replacement process with row-column $2r$-PF replacements – there will be no need to use label-non-skew $2r$-PFs. As we proceed with replacements, we shall generate sequences $\mathcal{C}_i(c_z)$ (for each $c_z \in \mathcal{C}_0$) of sets of copies of van Kampen surfaces V_{S_i}. We will also track the sequence of directed graphs D_{S_i}. Moreover, we shall also consider a subgraph U_{S_i} of D_{S_i} that tracks the 'unfixed' edges of V_{S_i}. In particular, if an edge of V_{S_i} can vary between different copies in $\mathcal{C}_i(c_z)$ then the corresponding edge of D_{S_i} belongs (along with both its vertices) to U_{S_i}. The process begins with U_{S_0} empty.

We begin by picking our first row-column $2r$-PF to replace. We do this by picking any directed cycle in D_{S_0} – since all vertices have out-degree at least 1 it is trivial to find such a cycle C_0. Suppose that C_0 has length $2r_1$. We perform our popular replacement on C_0, giving us S_1 and D_{S_1}. The subgraph U_{S_1} consists of the subgraph of D_{S_1} corresponding to the shattered ring decomposition used to replace C_0. Observe that U_{S_1} has the path completion property by Lemma 3.46.

Next, we pick some cycle C_1 in D_{S_1} that does not lie entirely in U_{S_1}. This is straightforward – we start at some unfixed edge and follow a directed path. Since the unfixed component U_{S_1} satisfies the path completion property we can think of U_{S_1} as a single 'blob' – any path that enters U_{S_1} can be continued to leave U_{S_1} through any edge directed out from U_{S_1}. Moreover, there is at least one edge directed out from U_{S_1} (since the same is true of C_0). So we can imagine collapsing U_{S_1} to a single vertex, and searching for a directed cycle in the resulting graph D_{S_1}/U. Any directed cycle in D_{S_1}/U corresponds to some directed cycle C_1 in D_{S_1} that is not entirely contained in U_{S_1} – moreover, C_1 only enters U_{S_1} a maximum of once (the cycle does not enter and leave U_{S_1} multiple times).

We can now replace C_1 with its shattered ring decompositions. We claim, by induction, that we can continue this process until $D_{S_t} = U_{S_t}$ for some t. In order to perform step $j + 1$, we must show the following properties hold after step j:

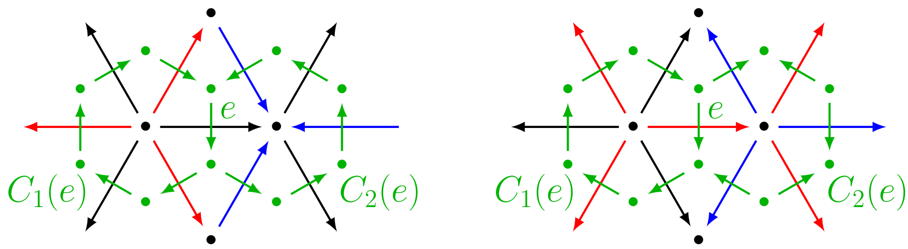
(1) Each (weakly) connected component of U_{S_j} satisfies the path completion property.

(2) For (weakly) connected component U of U_{S_j} there is at least one directed edge $e = \overrightarrow{uv}$ such that $u \in U$ but $e \notin U$.

If these properties do hold then we can pick a directed cycle C_j in D_{S_j} as follows. We form the multi-graph D_U by collapsing each connected component of U_{S_j} in D_{S_j} to a single vertex and eliminating all unfixed edges. Note that D_U will contain loops if there are edges that are fixed (and therefore do not belong to U_{S_j}) but have both endpoints in some component of U_{S_j}. By condition (2) we will still have that any vertex in D_U has out-degree at least 1, so we can find a cycle in D_U (which may simply be a loop). Condition (1) allows us to translate this cycle back into a cycle in D_{S_j}, and moreover this cycle both contains an unfixed edge and enters each connected component of U_{S_j} at most once.

The first item follows immediately for all j from Corollary 3.47.

The second item is less immediate. Suppose that at some stage there is a connected component U of U_{S_j} such that all directed edges $e = \overrightarrow{uv}$ with $u \in U$ belong to U.



(a) When e is a label edge both $C_1(e)$ and $C_2(e)$ are directed.

(b) When e is a non-label edge, $C_1(e)$ is directed but $C_2(e)$ is not.

Figure 3.20: The cycles $C_1(e)$ and $C_2(e)$ in D_S, shown when e is a label edge in Figure 3.20a and when e is a non-label edge in Figure 3.20b. In general, these cycles need not be the same size. We use red for row edges, blue for column edges and black for label edges in the underlying van Kampen surface V_S.

By assumption, U cannot contain both vertices at the end of some edge e not belonging to U. So the vertex set of U must be a proper subset of the vertices of D_{S_j} unless U contains all edges of D_{S_j} (in which case we have unfixed everything in D_{S_j}, and we have finished with the popular replacements). Therefore the edges with one end in U correspond to the boundary of some proper subset of the faces of V_{S_j}, and moreover, since the edges in this boundary are unfixed, this boundary exists as the boundary of a proper subset of faces of V_{S_0} also. But such a boundary must have length at least 3, since no proper subsurface of V_{S_0} has boundary of length less than 3.

Let $\overline{S_j}$ be the hypergraph obtained by 'closing the flap' of S_j (by identifying the petal vertices on the flap edges). We note that $D_{\overline{S_j}}$ is precisely D_{S_j} with one additional directed edge corresponding to the (now closed) flaps.

Since $V_{\overline{S_j}}$ is a closed surface, the edges of $V_{\overline{S_j}}$ are in one to one correspondence with directed edges in $D_{\overline{S_j}}$. We can view U as a subgraph of D_{S_j} which is a subgraph of $D_{\overline{S_j}}$, so U is also a subgraph of $D_{\overline{S_j}}$. In $D_{\overline{S_j}}$, we have that the number of directed edges with one end inside and one end outside of U must be at least 3.

Observe that in $D_{\overline{S_j}}$ we can associate to each edge e two cycles $C_1(e)$ and $C_2(e)$. These cycles are those obtained by considering the non-skew $2r$-PFs around the vertices at the end points of the edge corresponding to e in $V_{\overline{S_j}}$ (see Figure 3.20).

Both of the cycles $C_1(e)$ and $C_2(e)$ contain the edge e. If e corresponds to a label edge of $V_{\overline{S_j}}$, then both $C_1(e)$ and $C_2(e)$ are directed (because the corresponding non-skew $2r$-PFs both contain label vertices as inner vertices), whereas if e corresponds to a row or column edge then one of the cycles is directed but the other is not. Without loss of generality, we will say that $C_1(e)$ is always directed.

Consider any edge e_1 directed into U with exactly one endpoint in U. Then consider $C_1(e_1)$. This is a directed cycle, and since exactly one endpoint of e_1 lies in U, we have that the cycle exits U elsewhere. The edge e' from which it exits U is directed out from U. Therefore we are done unless the edge e' happens to be the edge present in $D_{\overline{S_j}}$ but not in D_{S_j}. If this is the case, take a second edge e_2 directed into U with exactly one endpoint in U; if no such edge exists then, since there are at least three edges with exactly one endpoint in U, we must have at least two directed out of U and so we must have an edge directed out from U even after discarding the edge in $D_{\overline{S_j}} \setminus D_{S_j}$. The edge e_2 also belongs to a directed cycle $C_1(e_2)$, which must also leave U at some other edge e''. As before, we are done unless $e'' = e'$, the edge in $D_{\overline{S_j}} \setminus D_{S_j}$. But in this case, since e' does not correspond to a label edge (since the boundary of V_{S_j} does not contain label edges) and both $C_1(e_1)$ and $C_1(e_2)$ contain e' we have that $C_1(e') = C_1(e_1) = C_1(e_2)$ and so there must be a further edge, distinct from e', directed out of U with exactly one vertex in U (since both e_1 and e_2 must be balanced with edges going the other direction).

So we have established both items (1) and (2) from above, and so by induction we may continue to choose such cycles until we reach stage t at which $U_{S_t} = D_{S_t}$. At this point we have unfixed all non-boundary edges of V_{S_t}.

It remains to confirm that the size of our collection $\mathcal{C}_t(c_z)$ of copies of V_{S_t} that agree on the two boundary edges is within a constant factor of the maximum possible.

In order to achieve this, we note that at step j we replaced a row-column $2r_j$-PF with a shattered ring decomposition. This involved removing $2r_j$ faces and $2r_j$ edges from V_{S_j}, before adding $22r_j$ faces and $(20r_j \times 3 + 2r_j \times 2)/2 = 32r_j$ edges. So in total the number of faces increases by $20r_j$ and the number of edges by $30r_j$.

So if V_{S_0} has e_0 edges and f_0 faces then V_{S_t} has $e_0 + 30r$ edges and $f_0 + 20r$ faces, where $r = \sum_{i=1}^{t} r_i$. Thus the total number of copies of V_{S_t} with one boundary edge fixed is $e_0 + 30r - (f_0 + 20r) - 1 = e_0 - f_0 + 10r - 1$ by Lemma 3.42.

The total size of the collection $\mathcal{C}_{i+1}(c_z)$ is larger than the size of $\mathcal{C}_i(c_z)$ by a factor $c(\epsilon, k, i)n^{10r_i+1} = \Omega(n^{10r_i+1})$. Therefore, the size of $\mathcal{C}_t(c_z)$ is $\Omega(n^{10r+t})$. So we need to check that $t = e_0 - f_0 - 1$.

Let e_j^F count the number of fixed edges of D_{S_j}, i.e. those edges that do not belong to U_{S_j}, and let f_j^F count the number of vertices of D_{S_j} that do not belong

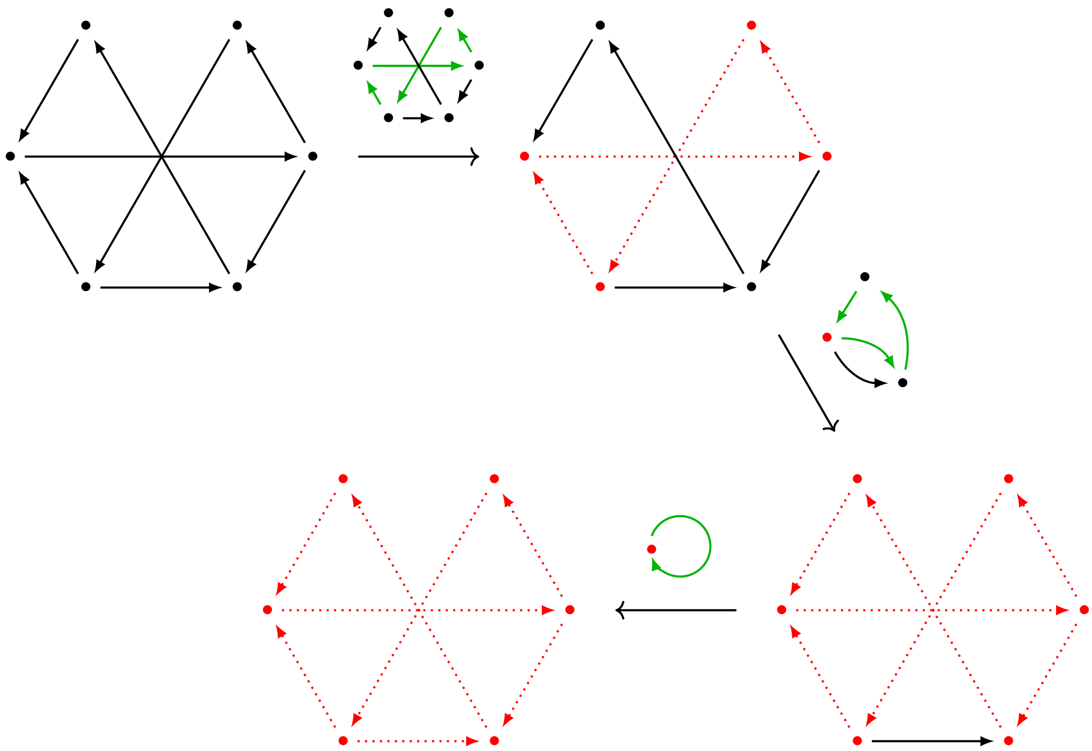


Figure 3.21: A sequence of popular replacements unfixing all edges for a row-flappy version of the hypergraph $\mathcal{T}$. The graphs D_{S_j} are shown, with the unfixed subgraph U_{S_j} shown in red. Note that the dotted red edges are not really present – they represent edges that are replaced in the popular replacement steps. Between the D_{S_j} we show the multi-graph D_U in which the unfixed components are collapsed to single vertices, and we show the cycle C_j selected for replacement in step j in green. All replacement steps unfix directed cycles, corresponding to row-column $2r$-PFs.

to U_{S_j}. Let u_j count the number of connected components of U_{S_j}. We consider the quantity $Q_j = e_j^F - f_j^F - 1 - u_j$.

Note that $Q_0 = e_0 - f_0 - 1$. We claim that $Q_j = Q_0 - j$ and that $Q_t = 0$ so that $t = e_0 - f_0 - 1$ as desired. The proof is by induction on j. If C_j contains vertices from s different components of U_{S_j} (say) then $u_{j+1} = u_j - s + 1$ and also $e_{j+1}^K - f_{j+1}^F = e_j^F - f_j^F - s$ (C_j has exactly one more vertex than edge contained in each component of U_{S_j}). So we always have $Q_{j+1} = Q_j - 1$ and the proof in this case is complete.

In Figure 3.21 we provide an example sequence of replacements in the case where S_0 is a row-flappy version of the hypergraph $\mathcal{T}$.

Case 2: Boundary edges of V_{S_0} are label edges

The argument will be similar to that of Case 1. However, we note that the same approach cannot work unchanged. In Case 1, we restricted ourselves to replacing only directed cycles in D_{S_j}, but that cannot work in Case 2 because there is a vertex of out-degree 0 in D_{S_0}. This happens because one of the two faces adjacent to the boundary in the van Kampen surface has both its row and column edge directed inwards in D_{S_j} (but the label edge is on the boundary so does not correspond to an edge in D_{S_j}).

In order to deal with this, we begin by using label-non-skew $2r$-PF replacements in order to unfix around every vertex in V_{S_0} that is incident to only row and column edges. Throughout this part of the unfixing process, each unfixed graph U_{S_j} consists of j disconnected components, each with the path completion property. By the time this part of the process is completed, at stage s say, all vertices of D_{S_s} belong to U_{S_s} and all edges of D_{S_s} that do not belong to U_{S_s} correspond to label edges in V_{S_s}.

We will now begin the row-column $2r$-PF replacement stage, just as in the previous case. As before, we must show the following properties after each step $j \geq s$:

(1) Each (weakly) connected component of U_{S_j} satisfies the path completion property.

(2) For (weakly) connected component U of U_{S_j} there is at least one directed edge $e = \overrightarrow{uv}$ such that $u \in U$ but $e \notin U$.

As before, item (1) is easily verified by Corollary 3.47 and Lemma 3.46 for step $j = s$ since U_{S_j} is a disjoint union of the subgraphs given by each popular replacement of a non-skew $2r$-PF. So item (2) presents most of the difficulties.

Suppose that at some stage j there is a connected component U of U_{S_j} such that all directed edges with one end in U are directed into U. Since all directed edges with exactly one vertex in U do not lie in U_{S_j}, they must all be edges corresponding to label edges of V_{S_j} that also exist in V_{S_0}. In particular, U corresponds to some union of faces of V_{S_j} with a label boundary. The boundary must have length greater than 2 (because no proper subset of faces of V_{S_0} has a boundary of length 2).

As before, we consider $\overline{S_j}$ in which the flaps of S_j have been identified. We can view U as a subgraph of $D_{\overline{S_j}}$ by observing that $D_{\overline{S_j}}$ is obtained from D_{S_j} by adding a single edge. So there are at least three directed edges in $D_{\overline{S_j}}$ with exactly one vertex in U. Each of these edges corresponds to a label edge of $V_{\overline{S_j}}$. At most one of these edges does not exist in D_{S_j}, so for our contradiction we simply need to show that there are at least two directed edges in $D_{\overline{S_j}}$ with exactly one vertex in U directed outwards from U.

Now recall the definition of the cycles $C_1(e)$ and $C_2(e)$ from the previous section. If e is an edge corresponding to a label edge of $V_{\overline{S_j}}$, both of these cycles are directed. We will say that $C_1(e)$ is the cycle corresponding to alternating row and label edges, while $C_2(e)$ is the cycle corresponding to alternating column and label edges. If there are fewer than two directed edges in $D_{\overline{S_j}}$ with exactly one vertex in U and directed into U then we are done. So we may take e_1 and e_2 directed into U. Consider $C_1(e_1)$ and $C_1(e_2)$. These are both directed cycles that have proper intersection with U. Therefore they each leave U through a directed edge, and we are done unless this edge equals in both cases e', the edge in $D_{\overline{S_j}} \setminus D_{S_j}$. But if both $C_1(e_1)$ and $C_1(e_2)$ leave U through the edge e' then $C_1(e_1) = C_1(e_2) = C_1(e')$ and therefore there must be another directed edge, distinct from e', leaving U (in order to balance both e_1 and e_2).

Therefore we have established items (1) and (2) for all stages $j \geq s$. This means that we can choose row-column $2r$-PFs to unfix exactly as in Case 1. We proceed like this until we reach some stage t at which $U_{S_t} = D_{S_t}$.

It remains to confirm once again that the size of $\mathcal{C}_t$ is within a constant of the maximum possible.

We can define e_j^F, f_j^F, u_j and Q_j as before, and as before we find that $Q_0 = e_0 - f_0 - 1$, $Q_j = Q_0 - j$ and $Q_t = 0$. Note also that in the first s steps of the replacement process we replace label-non-skew $2r_j$-PFs. When we do this, we remove $2r_j$ faces and $2r_j$ edges from V_{S_j} before adding in $10r_j$ faces and $(10r_j \times 3 + 2r_j \times 2)/2 = 14r_j$ edges. In total we add $8r_j$ faces and $12r_j$ edges. In later stages ($j > s$), we add $20r_j$ faces and $30r_j$ edges as before.

So by Lemma 3.42 the maximum possible size of a collection of copies of V_{S_t} that agree on one of the two boundary edges is

$$n^{e_0 - f_0 - 1 + 4\sum_{i=1}^{s} r_i + 10\sum_{i=s+1}^{t} r_i}.$$

But we also have that in stages $j \leq s$ we replace a non-skew $2r_j$-PF which increases the size of our collection by a factor $\Omega(n^{4r_j+1})$ and in stages $j > s$ we replace a row-column $2r_j$-PF which increases the size of our collection by a

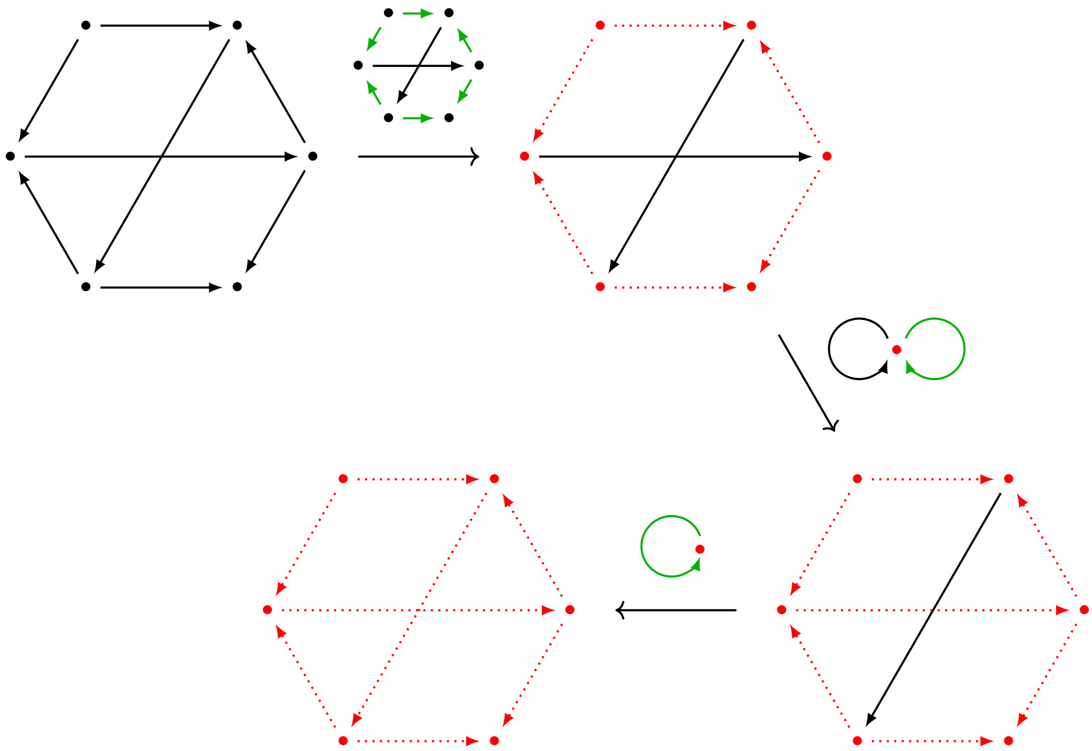


Figure 3.22: A sequence of popular replacements unfixing all edges in $D_{\mathcal{T}'}$. The graphs D_{S_j} are shown, with the unfixed subgraph U_{S_j} shown in red. Note that the dotted red edges are not really present – they represent edges that are replaced in the popular replacement steps. Between the D_{S_j} we show the multi-graph D_U in which the unfixed components are collapsed to single vertices, and we show the cycle C_j selected for replacement in step j in green. The first replacement unfixes a cycle which corresponds to a label-non-skew 6-PF – hence why the cycle is not directed.

factor $\Omega(n^{10r_j+1})$. So

$$|\mathcal{C}_t| = \Omega(n^{t+4\sum_{i=1}^{s} r_i+10\sum_{i=s+1}^{t} r_i})$$

$$= \Omega(n^{e_0-f_0-1+4\sum_{i=1}^{s} r_i+10\sum_{i=s+1}^{t} r_i})$$

as desired.

In Figure 3.22 we provide an example sequence of replacements in the case of $S_0 = \mathcal{T}'$ to illustrate all steps of the replacement argument in action.

3.8.3 Final details

In this section we shall show exactly how to use the above argument, coupled with our popular replacement lemma (Lemma 3.41) to complete a full proof of Theorem 3.12.

We begin by restating Theorem 3.12 in a somewhat more precise manner.

Theorem 3.48. *There exists a constant $C > 0$ such that for any partial Latin square A with at least ϵn^5 copies of H_c and any $K > 0$, we can find a subset $B \subset A$ of density $\beta \geq \epsilon^{2^{CK^2}}$ such that G_B contains no van Kampen surfaces with boundary of length 2 and at most K faces.*

Proof. We apply Lemma 3.41. Note that the number of replacements that we will need to apply (in both case 1 and case 2 above) is precisely $e-f-1$ where e and f are the number of edges and faces of the van Kampen surface that we wish to target. In the case of a van Kampen surface S_0 of boundary length 2, we have that $e = 3f/2+1$ so $e-f-1 = f/2$. So we may take $t = K/2+5 = \mathcal{O}(K)$ and all popular replacement arguments will terminate within $m \leq t$ steps. Furthermore, the size of the largest $2r_i$-PF that we need to replace is at most the number of faces of S_i. Since at each stage the number of faces in S_i increases (crudely) by no more than a factor 11 (since we could in principle have $2r_i$ equalling the number of faces in S_i, and then the number of faces in S_{i+1} is at most $22r_i$), we have that S_m has at most $11^m K$ faces, which is at most $2^{\mathcal{O}(K)}$. So we should take $k = 2^{\mathcal{O}(K)}$.

Thus Lemma 3.41 allows us to find a nested sequence B_i ($5 \leq i \leq t$) of subsets of A with a popular replacement condition. The density of B_t is $\epsilon^{2^{\mathcal{O}(K^2)}}$. The factor involved in the number of shattered ring decompositions is at least $\nu = \epsilon^{2^{\mathcal{O}(K^2)}}$.

Recall that given any van Kampen surface S_0 of boundary length 2, we can define an auxiliary graph $H_{B_t}(S_0)$ on the vertex set of B_t (i.e. the set of rows, columns and labels of B_t) where xy is an edge if there is a copy of V_{S_0} in $\mathcal{K}_{B_t}$ where the flaps correspond to vertex x and vertex y.

By applying the popular replacement arguments detailed in the previous sections, we find that given any van Kampen surface S_0 of boundary length 2, we can consider the graph $H_{B_t}(S_0)$ and we find that it has bounded maximum degree. Specifically, the degree is bounded by $\nu^{-t} = \epsilon^{-2^{\mathcal{O}(K^2)}}$.

Let H_{B_t} be the union of all the graphs $H_{B_t}(S_0)$ for S_0 a van Kampen surface with boundary of length 2 and at most K faces. We see that the number of such S_0 is at most $K^{\mathcal{O}(K)}$ (this is because such an S_0 corresponds to a 3-uniform hypergraph with K 3-edges on at most $3K$ vertices, and the number of these is straightforward to bound). Therefore the maximum degree of H_{B_t} is at most $K^{\mathcal{O}(K)}\epsilon^{-2^{\mathcal{O}(K^2)}} = \epsilon^{-2^{\mathcal{O}(K^2)}}$.

We now want to take a maximal independent set in H_{B_t}. All edges of H_{B_t}

live inside a single vertex class (either rows, columns or labels) so we simply have to pass to independent sets in each vertex class separately. However, we must do it in such a way that we keep a dense subset of points of B_t. So we begin by passing to an independent set in the label vertices (say) by keeping the vertices which are contained in the largest number of 3-edges of B_t. This gives us B'_t of density at least $\epsilon^{2^{\mathcal{O}(K^2)}}$. We can then pass to a independent set of the row vertices while keeping the maximum possible number of edges, which reduces the density by a further factor $\epsilon^{2^{\mathcal{O}(K^2)}}$ and giving us B''_t. Finally we produce B'''_t by passing to an independent set in the column vertices in the same way.

We are left with a partial Latin square B of density $\epsilon^{2^{\mathcal{O}(K^2)}}$ with the property that G_B has no van Kampen surfaces of boundary length 2. $\square$

3.9 Concluding remarks

By applying Theorem 3.48 and Proposition 3.4, we have shown that any partial Latin square with many copies of H_c contains a dense subset with an approximate metric abelian group structure. In particular, following the arguments of Appendix 2.A, we can find a dense subset which approximately embeds into the multiplication table of a rough approximate abelian group.

Two obvious directions remain for future work. The first is Question 3.8, which asks whether the Thomsen condition is sufficient to achieve the same result – we restate the question here.

Question 3.49. *Let A be a partial Latin square containing at least ϵn^4 copies of the hypergraph $\mathcal{T}$. Must A contain a dense subset that embeds into the multiplication table of a rough approximate abelian group?*

The next question concerns whether the rough approximate group is the end of the story. Since we are unable to provide examples of rough approximate abelian groups that do not contain large subsets isomorphic to an abelian group, we wonder whether it may be possible to strengthen our results. In particular, we would like to have an answer to the following question.

Question 3.50. *Can we remove the word 'rough' from the statement of our main result? In particular, does any partial Latin square with many copies of H_c (or, more ambitiously, $\mathcal{T}$) contain a dense subset that embeds into the multiplication table of an approximate abelian group?*

Chapter 4

On the Brown–Erdős–Sós conjecture in finite groups

This chapter is based on the following preprint [57].

The main result of this chapter was discovered simultaneously and independently by Nenadov, Sudakov and Tyomkyn [64], and a slightly weaker result avoiding the arithmetic machinery was obtained independently by Wong [89].

4.1 Introduction

A central open problem in extremal combinatorics is the Brown–Erdős–Sós conjecture [15], which states that for any fixed positive integer t, any sufficiently large, dense, 3-uniform hypergraph H contains a collection of t edges spanning at most $t + 3$ vertices. This conjecture can be generalised to higher uniformity, but we shall focus on the 3-uniform case in this chapter.

Since its formulation in 1973 there has been a great deal of work on this problem. Ruzsa and Szemerédi [69] resolved the first non-trivial case ($t = 3$), but the conjecture remains open for all $t > 3$. The most powerful result on this problem to date is due to Sárközy and Selkow [74], who showed that any 3-uniform hypergraph in which every set of t edges spans strictly more than $t + 2 + \lfloor \log_2 t \rfloor$ vertices has at most $o(n^2)$ edges.

In work of Solymosi [75], it was observed that it is sufficient to consider the case where H is additionally assumed to be linear. It is also clear that we may assume that H is tripartite, since given a dense 3-graph H we may obtain a dense, tripartite 3-graph H' by taking three copies of the vertex set of H and placing edges between these partitions corresponding to the edges of H.

Given a dense, linear, tripartite, 3-uniform hypergraph H on $n+n+n$ vertices we can associate a partially labelled $n \times n$ grid by labelling position (a,b) with label c if $(a,b,c) \in E(H)$. Thus the Brown–Erdős–Sós conjecture can be formulated in terms of a quasigroup – this is noted in [75] and [76], for example.

Conjecture 4.1 (Brown–Erdős–Sós)**.** *Fix $t \in \mathbb{Z}^+$ and $\epsilon > 0$. Then there exists $N = N(t,\epsilon)$ such that for any quasigroup G of order $n > N$ and any subset A of the multiplication table of G of density ϵ, we can find a set of t triples in A spanning at most $t+3$ vertices (i.e. rows, columns or labels).*

It is therefore natural to ask the same question when G is in fact a group, as the additional structure perhaps forces higher local density and makes the problem more tractable.

Conjecture 4.2 (Brown–Erdős–Sós for groups)**.** *Fix $t \in \mathbb{Z}^+$ and $\epsilon > 0$. Then for any sufficiently large group G and any subset A of the multiplication table of G of density ϵ, there exists a set of t triples in A spanning at most $t+3$ vertices.*

In 2015, Solymosi [75] resolved this open case, showing that Conjecture 4.2 holds for $t = 4$.

Recently, Solymosi and Wong [76] showed that in fact much more is true, proving that the Brown–Erdős–Sós threshold can be surpassed in the groups setting. In particular, they prove that dense subsets of sufficiently large group multiplication tables contain sets of t triples in A spanning asymptotically only $3t/4$ vertices. Since their result concentrates on the case of large t, they do not match Conjecture 4.2 for small t, but they do prove that it holds for infinitely many t.

Given that the Brown–Erdős–Sós threshold can be surpassed in the groups setting, one may ask what the correct threshold should be in this case. Since A corresponds to a linear hypergraph, we cannot find sets of t triples in A spanning fewer that $\sqrt{t}$ vertices, but can we approach lower bound?

Question 4.3. *Let t be a fixed positive integer. What is the smallest number of vertices $F(t)$ that can be found in the span of t triples in a dense triple system coming from a group G?*

In this chapter we answer this question and resolve Conjecture 4.2. By using machinery from arithmetic combinatorics, including the multidimensional

Szemerédi theorem and a multidimensional variant of the density Hales–Jewett theorem, we prove that any dense subset of a sufficiently large group multiplication table contains a large subgrid belonging to one of two families: either the subgrid matches part of the addition table of a cyclic group, or the subgrid matches the entire addition table of $\mathbb{F}_p^m$ for some small prime p and large m. A precise statement appears in Theorem 4.8 following some notation.

This structural result is best possible, since A may simply belong to one of the above pair of families. This reduces Question 4.3 to a discrete optimisation problem. We tackle this optimisation problem in Section 4.4, allowing us to show that $F(t) = \mathcal{O}(\sqrt{t})$ and resolving Conjecture 4.2 for all t.

4.2 Notation and Statements

We write $\mathbb{Z}_n$ for the group of integers modulo n under addition and we write $[k]$ for the set $\{0, 1, \ldots, k-1\}$. We begin with some definitions.

Definition 4.4. *By the* multiplication table *of a group G we mean the collection of triples (a, b, ab) for $a, b \in G$. The* vertex set *will be given by three disjoint copies of G called the* row vertices, column vertices *and* label vertices. *We shall refer to triples as the edges or* faces *of the corresponding tripartite 3-uniform hypergraph. Typically we will represent this as a labelled grid, with entry (a, b) given label ab. In the case that G is an abelian group, we will call the multiplication table an* addition table.

Definition 4.5. *By a* subgrid *of a labelled grid, we mean the labelled grid contained in the intersection of some subset of the rows and columns.*

Definition 4.6. *When we say that a labelled grid is* isomorphic *to another labelled grid, we mean that we can biject the row set, column set and label set in such a way that the resulting map is a graph isomorphism between the corresponding 3-graphs.*

Using this notation we reformulate Question 4.3 in a precise way.

Question 4.7. *Let t be a fixed positive integer. Let $F(t)$ be minimal such that, given any dense subset A of a sufficiently large group multiplication table, we may find a set of t faces of A spanning at most $F(t)$ vertices. How does $F(t)$ grow with t? Is $F(t) \leq t + 3$ for all t?*

In order to answer this question, we prove the following structural result.

Theorem 4.8. *Fix $k \in \mathbb{Z}^+$ and $\epsilon > 0$. Then there exists $N = N(k, \epsilon)$ such that, for any group G of order $n > N$ and any subset A of the multiplication table of G of density at least ϵ, A contains either a subgrid isomorphic to the addition table of $[k]$ as a subset of $\mathbb{Z}_K$ for some $K \geq k$, or a subgrid isomorphic to the addition table of $\mathbb{Z}_p^k$ for some $p < k$ prime.*

Remark 4.9. This result is 'best possible' in terms of finding configurations with many edges spanned by few vertices, since if A is simply taken to be the addition table of $[n/2]$ as a subset of $\mathbb{Z}_n$, say, then any subgrid of A is isomorphic to part of a larger addition table and we cannot improve on the first case of the theorem. Similarly, if A is taken to be the addition table of $\mathbb{Z}_p^t$ for small p and large t then we cannot improve on the second case.

4.3 Proof of main result

We begin by introducing the arithmetic machinery that we use later. We begin with a multidimensional version of Szemerédi's theorem [31].

Theorem 4.10 (Multidimensional Szemerédi Theorem)**.** *Let $k, t \in \mathbb{Z}^+$ and let $\epsilon > 0$. Then there exists $N = N(\epsilon, k, t)$ such that for any $n > N$ and any $A \subset \mathbb{Z}_n^t$ of density at least ϵ, we can find $a_1, a_2, \ldots, a_t, d \in \mathbb{Z}_n$ such that*

$$(a_1 + i_1 d, a_2 + i_2 d, \ldots, a_t + i_t d) \in A$$

for each $i_j \in \{0, \ldots, k-1\}$. In other words, A contains the Cartesian product of t arithmetic progressions of length k with the same common difference.

We shall also need a multidimensional version of the density Hales–Jewett theorem [32]. We recall the definition of a combinatorial line.

Definition 4.11. *By a* combinatorial line *in $\mathbb{Z}_m^n$, we mean a set U of the form*

$$U = \{(x_1, \ldots, x_n) \,|\, x_i \text{ constant on } I,\ x_j = z_j \text{ for } j \notin I\}$$

for some indexing set $I \subset \{1, \ldots, n\}$ and some $z \in \mathbb{Z}_m^n$. By a combinatorial subspace of dimension k, *we mean a set U of the form*

$$U = \{(x_1, \ldots, x_n) \,|\, x_i \text{ constant on each } I_s,\ x_j = z_j \text{ for } j \notin \cup_s I_s\}$$

for some collection of k disjoint indexing sets $I_s \subset \{1, \ldots, n\}$, and some $z \in \mathbb{Z}_m^n$.

Theorem 4.12 (Density Hales–Jewett)**.** *Fix $m \in \mathbb{Z}^+$ and let $\epsilon > 0$. Then there exists $N = N(\epsilon, m)$ such that for any $n > N$ and any $A \subset \mathbb{Z}_m^n$ of density at least ϵ, we can find an entire combinatorial line inside A.*

The density Hales–Jewett theorem easily implies its own multidimensional variant – for a proof, see [22] for example.

Corollary 4.13 (Multidimensional density Hales–Jewett)**.** *Let m, k be fixed positive integers and let $\epsilon > 0$. There exists $N = N(\epsilon, m, k)$ such that for any $n > N$ and any $A \subset \mathbb{Z}_m^n$ of density at least ϵ, we can find an entire combinatorial subspace of dimension k inside A.*

We will need a further variant of density Hales–Jewett, which follows easily from Corollary 4.13 by applying the same idea used to extend from Theorem 4.12 to Theorem 4.13.

Corollary 4.14. *Let k, t be fixed positive integers, p a fixed prime, and let $\epsilon > 0$. There exists $N = N(\epsilon, p, k, t)$ such that for any $n > N$ and any $A \subset (\mathbb{Z}_p^n)^t$ of density at least ϵ, we can find a subspace Γ of dimension k and $a_1, \ldots, a_t \in \mathbb{Z}_p^n$ such that*

$$(a_1 + \Gamma) \times (a_2 + \Gamma) \times \cdots \times (a_t + \Gamma) \subset A.$$

Proof. We simply identify $(\mathbb{Z}_p^n)^t$ with $\mathbb{Z}_{p^t}^n$ in the obvious way. We can then apply Theorem 4.12 to find a combinatorial subspace of dimension k inside A, which gives us an affine subspace of dimension k. The result follows by translating back to $(\mathbb{Z}_p^n)^t$. $\square$

Lastly, we will need Pyber's theorem [66] which provides us with a large abelian subgroup of G.

Theorem 4.15 (Pyber's Theorem)**.** *There is a universal constant $c > 0$ such that any group G of order n contains an abelian subgroup of order at least $e^{c\sqrt{\log(n)}}$.*

We are now ready to prove our main result.

Proof of Theorem 4.8. We begin by applying Theorem 4.15, which states that G contains an abelian subgroup G' of order at least $\exp(c\sqrt{N})$ for some absolute constant $c > 0$. In particular, $N' = |G'|$ tends to infinity with N.

Note that the multiplication table of G can be partitioned into the Cartesian products of left cosets of G' with right cosets of G'. Since A has at least density

ϵ in the full multiplication table $G \times G$, we know that there exists $r, s \in G$ such that A has density at least ϵ in the Cartesian product $rG' \times G's$. The part of the multiplication table corresponding to this Cartesian product is isomorphic to the addition table of G'. Let $A' = A \cap (rG' \times G's)$ be the ϵ-dense subset of $rG' \times G's$ obtained from A.

Note that G' is a finite abelian group, and can therefore be written as a direct product of cyclic groups of prime power order.

Suppose that G' has a cyclic factor $\mathbb{Z}_M$. Then, as above, we can find a subset A'' which is ϵ-dense in a Cartesian product of two cosets of $\mathbb{Z}_M$ in G, and this Cartesian product is isomorphic to the addition table of $\mathbb{Z}_M$. Thus A'' corresponds to an ϵ-dense subset of the $M \times M$ addition table. By Theorem 4.10, if $M > M(k, \epsilon)$ is sufficiently large then we can find a Cartesian product of two arithmetic progressions $(a, a+d, \ldots, a+(k-1)d)$ and $(b, b+d, \ldots, b+(k-1)d)$ in A''. The labels in this subgrid belong to the set $\{a+b, a+b+d, \ldots, a+b+2d\}$. Indeed, this subgrid is isomorphic to the addition table $\{0, \ldots, k-1\} \times \{0, \ldots, k-1\} \subset \mathbb{Z}^2$ and so we are in the first case of the statement of the theorem.

So we are done if G' contains a cyclic factor $\mathbb{Z}_M$ with $M > M(k, \epsilon)$. Therefore we may assume that all factors of G' are cyclic groups with bounded (prime power) order. Since $|G'|$ tends to infinity with N, we see that for any positive integer m, if N is sufficiently large then we may find (by the pigeonhole principle) a cyclic factor $\mathbb{Z}_{p^a}$ which appears to the power m. In particular, G' contains $\mathbb{Z}_p^m$ as a subgroup.

Once again we note that this means that we may find $A'' \subset A$ which is ϵ-dense in the Cartesian product of two cosets of $\mathbb{Z}_p^m$ inside G, and this product is isomorphic to the multiplication table of $\mathbb{Z}_p^m$. If m is sufficiently large, then by Corollary 4.14 we can find the complete Cartesian product of $a + \mathbb{Z}_p^k$ and $b + \mathbb{Z}_p^k$ inside A''. This complete Cartesian product is isomorphic to the addition table of $\mathbb{Z}_p^k$. If $p \geq k$ then we can find the addition table of $\mathbb{Z}_p$ and we are in the first case of the theorem, and otherwise we have $p < k$ and are in the second case. $\square$

Theorem 4.8 reduces Question 4.7 to the problem of finding the minimal number $f(t)$ of vertices spanned by a set of t faces in the entire addition table of $[k] \subset \mathbb{Z}_K$ for $K \geq k$ large compared to t, and the minimal number $g_p(t)$ of vertices spanned by a set of t faces in the entire addition table of $\mathbb{Z}_p^k$ for some fixed prime p and k large. Indeed we have that $F(t) = \max_p(f(t), g_p(t))$. Finding this minimum is a discrete optimisation problem, but providing an exact, closed form answer for all t is tricky because of certain divisibility considerations.

4.4 Finding locally dense configurations

In the interests of keeping this chapter brief, we will not attempt to give the best possible bounds. We will instead show that $F(t) = \mathcal{O}(\sqrt{t})$, and, because of the connection with Conjecture 4.1, we shall show that $F(t) \leq t + 3$ for all t.

For the analysis of the discrete optimisation problem arising from Theorem 4.8, it simplifies the calculations to try and maximise the number of faces induced by a fixed number v of vertices rather than minimise the number of vertices spanned by a fixed number t of faces. Thus we let $f'(v)$ be the maximal number of faces induced by v vertices in the addition table of $[k] \subset \mathbb{Z}_K$ for $K \geq k$ large compared to v, and observe that if $f'(v) \geq t$ then $f(t) \leq v$. Similarly, we let $g'_p(v)$ be the maximal number of faces induced by v vertices in the addition table of $\mathbb{Z}_p^k$ for some prime p and k large, and observe that if $g'_p(v) \geq t$ then $g_p(t) \leq v$.

Proposition 4.16. *We have that $f'(v) \geq (1 + o(1))v^2/12$, and therefore $f(t) \leq (\sqrt{12} + o(1))\sqrt{t}$.*

Proof. We work in the addition table of $[k] \subset \mathbb{Z}_K$ for $K \geq k \geq v$. Given r rows and r columns, we can optimise the density of our configuration by including the s most numerous labels. The labels in the addition table are constant along falling diagonals. In the worst case, each falling diagonal corresponds to a different label, in which case the most numerous label occurs r times, the next two most numerous occur $r - 1$ times each, etc. Therefore, by including the s most numerous labels, we include a total of at least

$$r + (r-1) + (r-1) + (r-2) + (r-2) + \cdots + (r - \lceil (s-1)/2 \rceil)$$
$$= sr - s(s-1)/4 - \frac{1}{2}\lfloor s/2 \rfloor$$

different faces. The total number of vertices is $2r + s$ so we seek to maximise this expression with respect to the constraint that $2r + s \leq v$. Taking $r = \lfloor v/3 \rfloor$ and $s = \lceil v/3 \rceil$, and noting that $f'(v)$ is an increasing function of v, the proposition follows. $\square$

Proposition 4.17. *We have that $g'_p(v) \geq (1+o(1))v^2/49$ for all p, and therefore $g_p(t) \leq (7 + o(1))\sqrt{t}$.*

Proof. We work in the addition table T of $\mathbb{Z}_p^k$ for k large. If $p \geq v/3$ then the construction in the proof of Proposition 4.16 finds a configuration in the addition table of $\mathbb{Z}_p$ with $(1 + o(1))v^2/12$ faces and so we are done.

Otherwise, let l be minimal such that $3p^{l+1} > v$. Since $v \geq l+1$, T contains a subgrid isomorphic to the multiplication table of $\mathbb{Z}_p^{l+1}$. We can partition this multiplication table into the Cartesian products of the cosets of $\mathbb{Z}_p^l$. These Cartesian products are arranged in a grid and correspond to entries of the addition table $\mathbb{Z}_p \times \mathbb{Z}_p$.

We form our configuration by taking a union of these blocks. Let $v = \lambda p^l$, and so $\lambda \in [3, 3p)$. The number B of blocks that we can use is precisely the maximum number of faces induced by $\lfloor \lambda \rfloor$ vertices in the addition table of $\mathbb{Z}_p$. The number of vertices in the resulting configuration will be at most v, and the number of edges will $Bp^{2l} = Bv^2/\lambda^2$.

Since $p > \lambda/3$ we could use the construction idea from Proposition 4.16. Unfortunately, we cannot assume that λ is large (in which case we could take approximately $\lambda^2/12$ blocks and therefore approximately $v^2/12$ faces) and the worst cases for this construction will in fact be decided by the best options for small λ.

In order to minimise the calculation, we will instead simply take an $a \times a$ grid of these blocks, and we shall choose a maximal subject to our constraint on the number of vertices.

If we take the bottom $a \times a$ grid of these Cartesian products we obtain a configuration with ap^l rows, ap^l columns and at most $(2a-1)p^l$ labels. The configuration has a^2p^{2l} faces. Taking a maximal so that $4a - 1 \leq v/p^l = \lambda$, we obtain a configuration C with at most v vertices.

By the maximality of a we see that $a = \lfloor \lambda/4 + 1/4 \rfloor$ so in particular $a \geq \max(1, \lambda/4 - 3/4)$. The number of faces of the configuration C is a^2p^{2l} which is at least

$$\max\left(\frac{v^2}{\lambda^2}, \frac{(\lambda-3)^2}{16\lambda^2}v^2\right)$$

which takes its minimal value of $v^2/49$ when $\lambda = 7$. □

Remark 4.18. It is not hard to show that Proposition 4.16 is in fact best possible, and $1/12$ is the correct constant in the limit. On the other hand, Proposition 4.17 does not give the correct constant. As mentioned in the proof, combining the construction in Proposition 4.16 with a careful analysis of small λ allows improvements to be made quite easily. We can also make use of leftover vertices (when λ is not an integer, a union of blocks uses only $\lfloor \lambda \rfloor p^l < v$ vertices, leaving some left unused) to interpolate between the constructions for integer values of λ. Using these techniques, we were able to improve the constant from $1/25$ to $5/64$. However, the calculations are quite involved and not of much interest, so we have tried to find a compromise between giving the best bounds that we can and providing a streamlined result.

Propositions 4.16 and 4.17 provide the following corollary.

Corollary 4.19. *$F(t) = \mathcal{O}(\sqrt{t})$ (in fact, $F(t) \leq (7 + o(1))\sqrt{t}$).*

Therefore the Brown–Erdős–Sós threshold of $(t+3, t)$ is far below what can be found given the extra group structure. Nevertheless, we confirm that we do indeed prove the Brown–Erdős–Sós conjecture in the context of group multiplication tables, which essentially involves checking that sufficiently dense configurations exist for the small values of t, as well as for large t as verified by Corollary 4.19.

Proposition 4.20. *We have that $F(t) \leq t + 3$ for all t.*

Proof. Although much better bounds than $t + 3$ are possible for large t, it will be most convenient to simply find configurations t faces spanning at most $t + 3$ vertices in the addition table of $[k] \subset \mathbb{Z}_K$ for $K \geq k$ large, and also in the addition table of $\mathbb{Z}_p^k$ for k large. The result is trivial for $t < 3$.

For the first case, we note that taking the points in positions $(0, 0)$, $(0, 1)$, and $(1, 0)$ gives the configuration

$$\begin{matrix} b & \\ a & b \end{matrix}$$

which has 6 vertices spanning 3 faces. Next, we can include the point in position $(1, 1)$, which introduces one new vertex (a new label) and one new face. Then the point in position $(2, 0)$ introduces one new vertex (a new column) and one new face, and then the point in position $(2, 1)$ introduces one new vertex (a new label) and one new face. Continuing, we introduce the points in positions $(i, 0)$ and $(i, 1)$ for each i until we have t faces. At this point we have a configuration with t faces spanning $t + 3$ vertices.

In the second case, we can use the above argument to find an $(r + 3, r)$-configuration for r up to $2p - 1$ by taking the bottom two rows, minus the final face, of the multiplication table of some copy of $\mathbb{Z}_p$. When we add in the final point in position $(p - 1, 1)$ we re-use the label in position $(0, 0)$ so we get an $(r + 2, r)$ configuration. We can then start again in a new copy of $\mathbb{Z}_p$, including the corresponding points one by one in the same order as before. Our first point introduces two new vertices (a new row and new column) for just one more face, but since we are adding it to an $(r + 2, r)$-configuration we get back to an $(r + 3, r)$-configuration. Thereafter we add at most one new vertex with every new face. Once we finish the bottom two rows of the next copy of $\mathbb{Z}_p$ we can start again in another copy, and we can continue until we have t faces. At that point we will span most $t+3$ vertices. $\square$

Proposition 4.20 verifies Conjecture 4.2.

4.5 Concluding remarks

In this chapter, we have seen that the Brown–Erdős–Sós conjecture is true for hypergraphs with an underlying group structure, and in fact much better is possible. We give a bound of $\mathcal{O}(\sqrt{t})$ on the minimum size of a collection of vertices spanned by t edges, which is tight up to the implied constant. Theorem 4.8 provides an explanation for this local density by showing that large subgrids manifesting the group structure can be found in any dense subset of a group multiplication table.

In light of the work in the previous chapters, one might wonder to what extent the local density remains when the group structure is loosened. For instance, what is the smallest number of vertices spanned by t edges in a dense subset of the multiplication table of a rough approximate group? In these structures the arithmetic machinery that we use is less obviously applicable, and a result akin to Theorem 4.8 seems much more difficult to obtain.

Chapter 5

The length of an s-increasing sequence of r-tuples

This chapter is based on joint work with W. T. Gowers [35]. The paper is due to appear in Combinatorics, Probability and Computing.

5.1 Introduction

This chapter concerns a deceptively simple problem formulated recently by Po-Shen Loh [56]. As he put it in an interview [21], "I thought it had to be trivial, it's so easy to describe, surely it will fall from some simple argument like the pigeonhole principle, and I will be done. I wasn't done in one hour, actually I'm still not done, and in fact there have been quite a few people who tried it and they also are not done."

We too are not done, but we have made some partial progress. Along the way, like Loh, we have noticed interesting connections to other parts of combinatorics, which we shall describe later and which lend support to Loh's view that his problem is, despite its simplicity, a deep and interesting one. Two other recent papers about it are [86] and [88].

5.1.1 2-increasing sequences of triples

We start by defining a simple relation on triples of integers.

Definition 5.1. *Let $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$ be two triples of integers. Say that a is* 2-less than *b, or $a <_2 b$, if $a_i < b_i$ for at least two coordinates i.*

For example, $(3,3,9) <_2 (5,6,1) <_2 (7,7,7) <_2 (7,8,9)$, but $(1,2,3)$ is not 2-less than $(1,2,4)$.

We think of this relation as a sort of ordering, even though in fact it is not, since it is not transitive: for instance $(1,2,3) <_2 (2,3,1) <_2 (3,1,2) <_2 (1,2,3)$. (This is the Condorcet paradox, and indeed Loh notes connections between his problem and questions in voting theory.) With that in mind, we make a further definition.

Definition 5.2. *A sequence (a^i) of integer triples is* 2-increasing *if for all $i < j$ we have $a^i <_2 a^j$.*

Note that because of the lack of transitivity, this is strictly stronger than saying that $a^i <_2 a^{i+1}$ for each i.

We are now ready to state Loh's problem. Here and throughout this chapter we write $[n]$ for the set $\{1, 2, \ldots, n\}$.

Problem 5.3. *For each n, let $F(n)$ be the maximal length of a 2-increasing sequence of triples with each coordinate belonging to $[n]$. How does $F(n)$ grow with n?*

In [56], Loh demonstrates that the problem of determining F is equivalent to the following problem with a Ramsey-theoretic flavour: what is the longest 2-coloured directed path that we are guaranteed to find in any edge 3-colouring of the transitive tournament on n vertices? In this chapter we shall focus on the triples formulation, and we shall not return to the coloured tournament setting.

An instructive example of a 2-increasing sequence is the following sequence of length 8, which is of maximal length when $n = 4$:

$$\begin{gathered}(1,1,1)\\(1,2,2)\\(2,1,3)\\(2,2,4)\\(3,3,1)\\(3,4,2)\\(4,3,3)\\(4,4,4)\end{gathered}$$

The following proposition gives the easy bounds on F for general n.

Proposition 5.4. *For all n we have $F(n) \leq n^2$. Moreover, whenever n is a perfect square we have $F(n) \geq n^{3/2}$.*

Proof. The upper bound follows from the trivial remark that in a set of more than n^2 triples with coordinates from $[n]$ we must have two triples that are equal in their first two coordinates, by the pigeon-hole principle. But neither of these is 2-less than the other.

For the lower bound, we generalize the construction used in the example above. Say $n = m^2$ is a perfect square. We let the sequence of first coordinates be m consecutive copies of $1, \dots, m^2$. Then we let the sequence of second coordinates be m consecutive copies of $1, \dots, m$, followed by m copies of $m+1, \dots, 2m$, etc, finishing with m copies of $m^2-m+1, \dots, m^2$. Finally, we let the sequence of third coordinates be m consecutive 1s, followed by m consecutive 2s, etc, finishing with m consecutive m^2s. For example, with $n = 9$ we have the construction

$$\begin{array}{c}
1\,2\,3\,4\,5\,6\,7\,8\,9\,1\,2\,3\,4\,5\,6\,7\,8\,9\,1\,2\,3\,4\,5\,6\,7\,8\,9 \\
1\,2\,3\,1\,2\,3\,1\,2\,3\,4\,5\,6\,4\,5\,6\,4\,5\,6\,7\,8\,9\,7\,8\,9\,7\,8\,9 \\
1\,1\,1\,2\,2\,2\,3\,3\,3\,4\,4\,4\,5\,5\,5\,6\,6\,6\,7\,7\,7\,8\,8\,8\,9\,9\,9
\end{array}$$

where to save space we have written the triples as columns rather than rows.

It is easy to check that this gives a 2-increasing sequence, and it has length m^3, as required. $\square$

Faced with the above bounds, it is natural to think that the lower bound is probably closer to the truth, since the proof giving the upper bound is very weak. However, the main result of Loh's paper [56] may reduce one's confidence in this view. For use in the proof, and later in this chapter, we make the following definition.

Definition 5.5. *Two triples t_1 and t_2 are* 2-comparable *if one of them is 2-less than the other. A set of triples is* 2-comparable *if any two of them are 2-comparable.*

Proposition 5.6. $F(n) \leq n^2 / \exp(\Omega(\log^*(n)))$.

Proof. Let $T = (t_i)$ be a 2-increasing sequence of triples taking values in $[n]$, and let $t_i = (a_i, b_i, c_i)$. Now construct a tripartite graph with vertex sets $A = B = C = [n]$ by taking each triple t_i and thinking of it as a triangle with vertices $a_i \in A, b_i \in B$ and $c_i \in C$. That is, we put in the edges $a_i b_i, b_i c_i$ and $a_i c_i$.

Note that no two of these triangles can share an edge. For example, if the edges $a_i b_i$ and $a_j b_j$ are the same, then $a_i = a_j$ and $b_i = b_j$, which implies that neither of the triples (a_i, b_i, c_i) and (a_j, b_j, c_j) can be 2-less than the other. Furthermore, these are the only triangles in the graph, since if we have a triangle with all three of its edges coming from different triples, then we have three triples in our collection, of the form $(x, b, c), (a, y, c), (a, b, z)$, which must be 2-comparable. If $x < a$, then we can deduce from the 2-comparability that $b < y$, which in turn gives us that $c > z$, which then implies that $x > a$, a contradiction. Similarly, if $x > a$ we can deduce that $x < a$ and again obtain a contradiction.

It follows that no two triangles in the graph we have just constructed share an edge. But by the triangle removal lemma [69], any such graph has $o(n^2)$ edges, and using the best-known bounds, due to Fox [28], we obtain the result stated. □

After seeing this proof, one might now expect that the correct bound is of the form $n^{2-o(1)}$, with a lower bound provided by a suitable modification of Behrend's surprisingly dense set that contains no arithmetic progression of length 3 [9]. However, it does not take long to see that this does not work: in brief, the reason is that the 2-comparable and 2-increasing conditions impose far stronger constraints on the graph than the ones used in the above proof. (For more details, see Section 2.3 of Loh's paper.)

We end the description of the problem with a simple product argument that shows that if for any fixed k one could obtain any improvement at all over the lower bound of $k^{3/2}$, then we could deduce that asymptotically $F(n)$ beats $n^{3/2}$ in the exponent (meaning that there exists some $\alpha > 3/2$ such that $F(n) > n^\alpha$ for all sufficiently large n).

Lemma 5.7. *Suppose that for some n we have $F(n) = n^\alpha$. Then there are arbitrarily large m such that $F(m) \geq m^\alpha$.*

Proof. We define the product $\otimes$ of two sequences in an obvious way: given two 2-increasing sequences (a_i, b_i, c_i) and (d_j, e_j, f_j), we form the sequence

$$((a_i, d_j), (b_i, e_j), (c_i, f_j)),$$

where the indices (i, j) are arranged lexicographically. Also, take the lexicographical ordering on the pairs themselves. Then if $(i, j) < (k, l)$ we either have $i < k$, in which case

$$((a_i, d_j), (b_i, e_j), (c_i, f_j)) < ((a_k, d_l), (b_k, e_l), (c_k, f_l))$$

just because $(a_i, b_i, c_i) < (a_k, b_k, c_k)$, or we have $i = k$ and $j < l$, in which case we are done because of the second coordinates. Finally, we can just inject pairs (x, y) with $x, y \in [n]$ into $[n^2]$ with an injection that respects the lex ordering.

So if we have a sequence of tuples T with $|T| = n^\alpha$ then by taking $T \otimes \cdots \otimes T$ we can boost the construction to arbitrarily large m. $\square$

Observe also that since for every m there is an integer power of n that lies between m/n and m, we can also deduce from the assumption of the lemma that $F(m) \geq (m/n)^\alpha$ for every m. Therefore, for every $\beta < \alpha$ and all sufficiently large m, we have that $F(m) \geq m^\beta$.

In the light of this result, it is natural to try a computer search to see whether it throws up any small examples that give rise to an exponent greater than 3/2. We have tried this and failed to find any, which lends some support to the following conjecture, which is also suggested by remarks that Loh makes in his paper.

Conjecture 5.8. *$F(n) \leq n^{3/2}$ for all n.*

5.1.2 Weakening the main condition to 2-comparability

The proof of Loh's upper bound, Proposition 5.6, did not make full use of the property that the sequence of triples is 2-increasing: all that was needed was that it was 2-comparable (recall that this means that for any two triples in the sequence, one is 2-less than the other). It is therefore natural to consider the following weakening of Problem 5.3.

Problem 5.9. *For each n, let $G(n)$ be the maximal size of a 2-comparable set of triples with each coordinate belonging to $[n]$. How does $G(n)$ grow with n?*

From the remarks we have just made, and the fact that $G(n) \geq F(n)$ for every n, we have the following result.

Proposition 5.10. *For all n we have $G(n) \leq n^2/\exp(\Omega(\log^*(n)))$. Moreover, whenever $n = m^2$ we have $G(n) \geq n^{3/2}$.*

Also, essentially the same product argument shows that Lemma 5.7 is true for G just as it is for F.

It is also worth mentioning that $G(n)$ can be viewed as the largest clique inside the graph on $[n]^3$ where vertices (corresponding to triples) are adjacent if they are 2-comparable. This formulation is particularly useful for numerical experimentation, since there are fast algorithms that perform well at finding large cliques.

One can use exactly the same argument as presented in [56] to translate G into a Ramsey problem on edge-coloured tournaments, but the equivalent formulation involves a slightly unwieldy condition: what is the longest 2-coloured directed path that we are guaranteed to find in any edge 3-colouring of a tournament on n vertices that does not contain a 2-coloured directed cycle?

However, it is not necessary to motivate the study of G by returning to a Ramsey problem on tournaments. In fact, the problem of determining the growth of G has already been studied under the name of 'tripod packing'. This problem, exactly equivalent to determining $G(n)$, was first posed by Stein in 1967 [78]. It turns out that the lower bound $G(n) \geq n^{3/2+o(1)}$ is not best possible. This was discovered in the context of tripod packing, and there have since been a number of papers gradually improving the exponent from 3/2 [79, 80, 81, 87].

In our work, we give the best known lower bound on $G(n)$ (see Section 5.1.4), and provide some connections between this question and other problems in extremal combinatorics.

5.1.3 Generalizing to s-increasing sequences of r-tuples

It is natural to consider what happens if we generalize the problem in an obvious way from 2-increasing or 2-comparable sequences of triples to s-increasing or s-comparable sequences of r-tuples. It is important to highlight at this point that this generalisation does not preserve the connections to Ramsey theory. There is also a natural generalisation of the Ramsey theory questions that motivated Loh, but it does not coincide with the problem discussed in this chapter for general r and s. For a brief discussion of the generalisation of the Ramsey question, see [88] for example.

Definition 5.11. *An r-tuple $a = (a_1, \ldots, a_r)$ of integers is* s-less than *an r-tuple $b = (b_1, \ldots, b_r)$ if $a_i < b_i$ for at least s values of i. In that case we write $a <_s b$. An* s-increasing *sequence of r-tuples is a sequence $(a^1, \ldots, a^m)$ such that $a^i <_s a^j$ whenever $i < j$. Two r-tuples are* s-comparable *if one is s-less than the other, and an* s-comparable *set of r-tuples is a set $\{a^1, \ldots, a^m\}$ such that any two distinct elements of the set are s-comparable.*

It will be convenient to refer to an s-increasing sequence of r-tuples as an (r, s)-sequence and an s-comparable sequence of r-tuples as an $[r, s]$-sequence.

Let $F_{r,s}(n)$ be the greatest possible length of an (r, s)-sequence and let $G_{r,s}(n)$ be the greatest possible length of an $[r, s]$-sequence such that the r-tuples take values in $[n]$. The following proposition generalizes Proposition 5.4.

Proposition 5.12. *For all r, s and n we have $F_{r,s}(n) \leq G_{r,s}(n) \leq n^{r-s+1}$. Moreover, whenever n is a perfect sth power, we have $G_{r,s}(n) \geq F_{r,s}(n) \geq n^{r/s}$.*

Proof. As with Proposition 5.4, the upper bound follows instantly from the pigeonhole principle. Also, it is trivial that $F_{r,s}(n) \leq G_{r,s}(n)$ for every r, s and n.

The lower bound is obtained by generalizing the construction in Proposition 5.4 in a straightforward, but not quite trivial, way. We can describe it succinctly as follows. Just for this proof, we will use the notation $[q]$ to stand for the set $\{0, 1, \ldots, q-1\}$ instead of the set $\{1, 2, \ldots, q\}$.

Let $n = m^s$. Then write the integers in $[m^r]$ in base m. Given any subset A of $[r]$ of size s, and any integer $k \in [m^r]$, let $f_A(k) \in [m^s]$ be the number you get by restricting the base-m representation of k to the digits indexed by A. Now let $A_i = \{i, i+1, ..., i+s-1\}$ mod r for each $i \in [r]$, and define a sequence $T_0, T_1 \ldots, T_{m^r-1}$ of r-tuples by setting T_k to be $(f_{A_1}(k), ..., f_{A_r}(k))$ for each $k \in [m^r]$.

If $i < j$ then $f_{A_t}(i) < f_{A_t}(j)$ for any set A_t that contains the highest coordinate that is less in the base m representation of i than in the base m representation of j. There are s such sets A_t, and so this sequence of r-tuples is s-increasing. $\square$

Note that it was not important in the above construction that the sets A_i were intervals mod r: all we needed was a collection of r subsets of $[r]$, each of size s, such that every element of $[r]$ belonged to precisely s of the sets.

The result of Loh can also be easily generalized to improve the upper bound by a tiny fraction.

Proposition 5.13.

$$G_{r,s}(n) = o(n^{r-s+1})$$

Proof. First we note that, given an s-comparable sequence of r-tuples, by restricting to the first $r-s+2$ coordinates we obtain a 2-comparable sequence of $(r-s+2)$-tuples. Therefore it suffices to prove this result in the case $s = 2$.

The proof follows the same argument as Proposition 5.6, but the hypergraph removal lemma is required to replace the triangle removal lemma. As a consequence, the improvement obtained is much smaller.

Given a 2-comparable sequence of r-tuples (t_i), we construct an $(r-1)$-uniform, r-partite hypergraph G by taking, for each i, a copy of the hypergraph

H given by hyperedges corresponding to all subsets of size $(r-1)$ from the r-tuple t_i.

First of all we note that these copies of H must be edge-disjoint between the tuples t_i, because no two tuples can share $r-1$ coordinates. Also, any two distinct hyperedges from the same copy of H uniquely determine the t_i corresponding to that copy.

It remains to show that we cannot have a copy of H given by a collection of hyperedges each coming from a different copy of H. If we had such a configuration, we would require the tuples

$$(y_1, x_2, \ldots, x_r), (x_1, y_2, x_3, \ldots, x_r), \ldots, (x_1, \ldots, x_{r-1}, y_r)$$

to lie in our sequence, for some $x_1, \ldots, y_r$. But simply by restricting to the first three coordinates of the first three of these tuples, we get a contradiction by the same reasoning as in Proposition 5.6. □

It is now tempting to conjecture that the lower bound is sharp not just for 2-increasing sequences of triples, but more generally for s-increasing sequences of r-tuples. However, this turns out to be false. One way of seeing this is simply to note that the following example (discovered by a computer search, though it could probably have been found by hand) shows that $F_{4,2}(3) \geq 10 > 3^2$.

(1,1,1,1)
(1,1,2,2)
(1,2,1,3)
(2,1,3,1)
(2,2,2,2)
(3,3,1,1)
(1,3,2,3)
(3,1,3,2)
(2,2,3,3)
(3,3,3,3)

But there is also a more conceptual argument, which makes it completely obvious that $n^{r/s}$ is not the right bound for all pairs (r, s). If we fix n to be 2, say, then for two random r-tuples a and b, the expected number of coordinates for which $a_i < b_i$ is $r/4$, so by standard arguments the probability that a is not $r/8$-less than b is exponentially small in r. It follows easily that $F_{r,r/8}(2)$

is exponentially large in r, whereas if the $n^{r/s}$ bound were sharp, then $F_{r,r/8}(2)$ would be at most 2^8.

These counterexamples slightly weaken the case for believing that $F_{3,2}(n) \leq n^{3/2}$, and they suggest that giving an exact formula for $F_{r,s}(n)$ is unlikely to be possible for all triples (r, s, n). They also tell us that any proof that $F_{3,2}(n) \leq n^{3/2}$ will have to have some aspect that cannot be generalized to all pairs (r, s) – indeed, not even to the pair $(4, 2)$.

5.1.4 Our main results

Our main result is the following theorem, which is presented in the next section. It provides a non-trivial power-type improvement to the upper bound for Problem 5.3.

Theorem 5.14. *There exists $\epsilon > 0$ such that every 2-increasing sequence of triples taking values in $[n]$ has size at most $n^{2-\epsilon}$.*

This is the first improvement over Loh's $n^2/\exp(\Omega(\log^*(n)))$ bound. Our proof makes essential use of the assumption that the sequence in question is 2-increasing and not just 2-comparable, so it does not yield an improvement for Problem 5.9. Also, the explicit ϵ we obtain is very small indeed, though as we shall explain later, if we had unlimited computer power then it could probably be improved substantially, though not to the point where it matches the lower bound.

Our second main result concerns the problem for 2-comparable sets of triples. We have not been able to improve on Loh's upper bound in this case, but we give a construction that beats the $n^{3/2}$ lower bound, yielding the following result.

Theorem 5.15. *For arbitrarily large n there exist 2-comparable sets of triples of size at least $n^{1.546}$.*

Since our work, we have been made aware that the existence of constructions beating $n^{3/2}$ was already known thanks to the equivalent formulation as Stein's tripod packing problem as discussed in Section 5.1.2. Prior to our work, the best known construction had size $n^{1.534}$ [87]. There seems to be some chance that our construction is in fact best possible – we discuss this in more detail in Section 5.3.

We also obtain the following corollary, which translates back to a Ramsey theory setting.

Corollary 5.16. *For arbitrarily large N there exist 3-coloured tournaments on N vertices with no 2-coloured directed paths of length greater than $N^{0.647}$.*

Proof. Given a 2-comparable set of triples T with coordinates belonging to $[n]$, we construct a 3-coloured directed graph $G(T)$ with one vertex $v(t)$ for each triple $t \in T$. We let the edge $v(t_1)v(t_2)$ be directed to respect the $<_2$ ordering between t_1 and t_2, and we colour the edge red if the second and third coordinates increase, blue if the first and third increase and green if the second and third increase. If all coordinates increase we may choose a colour randomly. It is clear that this is a well-defined 3-coloured tournament on $|T| \leq G(n)$ vertices, and that the longest 2-coloured directed path has length n. Setting $N = G(n)$ for some n such that $G(n) \geq n^{1.546}$, the result follows. $\square$

We shall describe the construction that proves Theorem 5.15 in Section 5.3, before moving on to discuss a few interesting variants of the problem and connections to widely studied Turán-type problems.

These two results suggest that the problems for 2-increasing sequences and 2-comparable sets of triples are fundamentally different, despite what the bounds for small examples suggest, though of course they do not actually prove that the exponents for the functions $F_{3,2}$ and $G_{3,2}$ are distinct.

In the final section we discuss the generalized problem for $[r, s]$-sequences. Our focus will switch from fixing r and s to fixing n and the ratio r/s. This problem has some similarities with well-known results about unit vectors with upper bounds on their inner products, where the form of the bound depends strongly on whether the upper bound is positive, negative, or zero. We prove the following theorem, which shows a similar change in behaviour, for similar reasons, though our proofs are somewhat different, and the differences appear to be necessary.

Theorem 5.17. *Let $n \in \mathbb{N}$ and $\beta \in (0, 1)$ be fixed. Then*

(i) if $\beta < (1 - 1/n)/2$, then $G_{r,\beta r}(n)$ grows exponentially in r,

(ii) if $\beta = (1 - 1/n)/2$, then $G_{r,\beta r}(n)$ grows at least linearly in r, and

(iii) if $\beta > (1 - 1/n)/2$, then $G_{r,\beta r}(n)$ is bounded independently of r.

The significance of the number $(1 - 1/n)/2$ is that if a and b are random r-tuples taking values in n, then the expected proportion of coordinates i for which $a_i < b_i$ is $r(1 - 1/n)/2$. (This quickly implies (i), as we have already observed in the case $n = 2$ and $\beta = 1/8$.)

5.2 An upper bound for $(3, 2)$-sequences

In this section we shall prove our upper bound for $F(n)$. It may be of interest that this approach was discovered only after a significant amount of time considering a different, but more "obvious" approach. The idea was to decompose sequences into smaller subsequences and use a combination of induction and Cauchy-Schwarz to prove the conjectured bound. Despite this method initially seeming promising, we did not manage to make it work. In Section 5.6 we shall give a brief discussion of the obstacles that we encountered.

For the purposes of obtaining a convenient inductive hypothesis later, it will be useful to generalize Problem 5.3 so that instead of taking the triples from $[n]^3$, we shall take them from a grid $[r] \times [s] \times [t]$, where the sides may have unequal lengths. The maximal length of a 2-increasing sequence now depends on the three parameters r, s and t, and the trivial upper bound is $\min\{rs, rt, st\}$. Note that if we could ever find an example of a 2-increasing sequence of length greater than $(rst)^{1/2}$, then taking the product (in the sense described earlier) of this example and two further copies with the roles of the coordinates cycled round would give a 2-increasing sequence of length greater than $(rst)^{3/2}$ taking values in $[rst]$.

We now state our main result in a slightly generalized form.

Theorem 5.18. *There exists $\theta < 2/3$ such that any 2-increasing sequence of triples from $[r] \times [s] \times [t]$ has size at most $(rst)^{\theta}$.*

Note that if $r = s = t = n$, then the bound we obtain is $n^{3\theta}$. Thus, any improvement on $2/3$ for the exponent θ translates directly into an improvement on the exponent 2 for the problem as it was stated before. Unfortunately the improvement over $\frac{2}{3}$ that we obtain is tiny. The main reason for this is that we need as a base case for an inductive argument an n for which the trivial bound is beaten by a reasonable-sized constant. Finding such an n by brute force is not computationally feasible, so we are forced instead to use Loh's upper bound (Proposition 5.6). But then the n in question is huge, so the exponent in the base case is only very slightly less than 2. So in a certain sense, the weakness in our argument is not a fundamental one, since a finite-time computation would, if a much better bound exists at all, allow us to improve our result significantly. Unfortunately, the finite time needed is huge.

However, our argument gives a strictly worse exponent than the one assumed for the base case, so even with unlimited computational power at our disposal we would not obtain a bound that was all that close to the conjectured $(rst)^{1/2}$.

The reasons for this will become clearer later, and we shall discuss this point further at the end of the section.

5.2.1 Proof of Theorem 5.18

The proof will be by induction. Since the argument cannot hope to produce anything other than a θ very close to $\frac{2}{3}$, we shall not put much effort into optimizing the details and shall aim instead for simplicity and clarity.

A weakened 2-increasing condition

We begin with a brief discussion of a weakened form of the problem, including a small digression.

Definition 5.19. *Given r-tuples $x = (x_1, x_2, \ldots, x_r)$ and $y = (y_1, y_2, \ldots, y_r)$, say that x is* weakly s-less *than y (written $x \leq_s y$) if at least s of the relations $x_i \leq y_i$ hold.*

Definition 5.20. *Given a sequence $T = (t_i)$ of r-tuples, we say that T is* weakly s-increasing *if for all $i < j$ $t_i \leq_s t_j$, and no r-tuple is repeated.*

With a slight abuse of notation, we shall sometimes also say that a *set* T of r-tuples is weakly s-increasing if it can be sorted into a weakly s-increasing sequence.

An s-increasing sequence of r-tuples is trivially also weakly s-increasing.

We shall require the following lemma in our improvement of the upper bound for 2-increasing sequences.

Lemma 5.21. *Let T be a weakly 2-increasing sequence of triples from $[n]$. Then $|T| \leq 6n^2$.*

Proof. We start by forming a collection T_1 where for each fixed (x, y) we throw out from T all triples (x, y, z) such that z is maximal or minimal. We then form a collection T_2 where for each fixed (x, z) we throw out from T_1 all triples (x, y, z) such that y is maximal or minimal. Finally we form a collection T_3 where for each fixed (y, z) we throw out from T_2 all triples (x, y, z) such that x is maximal or minimal.

Clearly we have thrown out a maximum of $6n^2$ triples. Now suppose that T_3 is non-empty, so it contains a triple (a, b, c). Then T must contain a triple (a_0, b, c_1) with $a_0 < a$ and $c_1 > c$, since T_2 contains a triple (a_0, b, c), which also belongs to T_1, and that implies that T contains a triple (a_0, b, c_1). By similar arguments T must contain triples (a_1, b_0, c) and (a, b_1, c_0), with $a_0 < a < a_1$, $b_0 < b < b_1$ and $c_0 < c < c_1$. These triples form a pure directed 3-cycle. □

We shall use Lemma 5.21 in a central way in the inductive step of our main argument.

It is natural to ask about weakly increasing sequences in more generality, and they turn out to be quite interesting: in particular, they are related to the famous Füredi-Hajnal conjecture [30].

By defining a sequence $T = (t_i)$ of r-tuples from $[n]$ by letting the first $(s-1)$-coordinates be equal to 1 for all the tuples, and setting the last $r-s+1$ coordinates to range through all the n^{r-s+1} possibilities in lexicographic order, we find that the maximum length of a weakly s-increasing sequence of r-tuples is at least n^{r-s+1}. But is it possible to do significantly better?

This question is partially settled by the following theorem, which follows from the result of Marcus and Tardos [61], which settled the Füredi-Hajnal conjecture. More specifically, we shall use a multidimensional extension of it by Marcus and Klazar [60]. We shall not require the full generality of this result in our proof of Theorem 5.18, but we want to highlight the connection with permutation patterns.

Theorem 5.22. *The maximum length of a weakly* 2*-increasing sequence of* r*-tuples from* $[n]$ *is* $\Omega(n^{r-1})$.

Proof. Let $T = (t_i)$ be a sequence of distinct r-tuples from $[n]$. Define a directed graph G that has one vertex for each t_i and an edge (t_i, t_j) directed from t_i to t_j $t_i \leq_2 t_j$. Note that edges may be directed in both directions.

Then T is weakly 2-increasing if and only if G admits a topological sorting (ie the vertices of G can be ordered that if vertex u is below vertex v, then the edge is directed from u to v). It is well known that a directed graph can be topologically sorted if and only if it contains no "pure directed cycle", by which we mean a directed cycle of edges that each have only one direction.

An example of a collection of tuples leading to a pure directed cycle in G is the r cyclic permutations of the tuple $(1, 2, \ldots, r)$. Defining

$$t_1 = (1, 2, \ldots, r), t_2 = (2, 3, \ldots, r, 1), \ldots, t_r = (r, \ldots, r-1)$$

we see that $t_i \leq_2 t_{i+1}$ for each $1 \leq i \leq r-1$ and $t_r \leq_2 t_1$. Moreover, none of these relationships hold the other way around.

Notice that we may also take the above construction with the numbers $1, \ldots, r$ replaced with $x_1, \ldots, x_r$ provided that $x_1 < x_2 < \cdots < x_r$. Call this collection of tuples $C(x_1, \ldots, x_r)$.

Now if we view the collection of r-tuples as an r-dimensional binary matrix with entry 1 in position t_i for each t_i in T, the collection $C(x_1, \ldots, x_r)$ corresponds to an $r \times r \times \cdots \times r$ permutation sub-matrix.

By the result of Marcus and Klazar [60], generalizing the Marcus-Tardos theorem to d dimensions, we find that it is possible to embed any d-dimensional $k \times \cdots \times k$ permutation matrix inside an $n \times \cdots \times n$ matrix M, after possibly changing some 1s to 0s in M, provided that M has mass at least $c(k,d)n^{d-1}$, where $c(k,d)$ is a constant depending on k and d only. Our theorem follows immediately. $\square$

Of course a version of Lemma 5.21 follows from Theorem 5.22 and hence from the Marcus-Klazar result, but with a much larger constant replacing the 6 in $|T| \leq 6n^2$.

We shall now move on to providing the base cases that we need for the induction.

The base case

First, we let N be the minimal positive integer such that $(2(N+1)^3/N^3)^{2/3} \leq 5/3$. We write N for this constant, which will appear throughout the inductive step, for the sake of conciseness.

For our base case, we need to find a positive integer k and a real number $\theta < 2/3$ such that if $\min\{r, s, t\} \leq Nk$, then every 2-increasing subset of $[r] \times [s] \times [t]$ has size at most $(rst)^\theta$. We obtain this by combining Loh's result (Proposition 5.6 above) with some simple observations.

First, we choose an integer k with the property that any 2-increasing sequence of triples in $[k]^3$ has length at most δk^2, where $20\delta^{1/10} = k^{-\epsilon}$ and ϵ is some positive constant. The existence of such a k and ϵ follows from Proposition 5.6.

Having chosen k and ϵ, let $\theta_1 = (2-\epsilon)/3$. It will turn out that we need to take $\theta \geq \theta_1$ for our inductive hypothesis to work.

Once we have chosen our k, we need every 2-increasing sequence of triples from $[r] \times [s] \times [t]$ with $\min\{r, s, t\} \leq Nk$ to have length at most $(rst)^\theta$. This places further strong constraints on how small we are able to take θ.

Without loss of generality, $r \leq s \leq t$. Then in order to ensure that the condition is satisfied, we first note that whenever r, s and t are not all equal the trivial bound rs is equal to $(rst)^\tau$ for some $\tau(r,s,t) = \log(rs)/\log(rst) < 2/3$. The expression on the left-hand side decreases as t increases and increases as s increases, so it is maximized, for fixed r, when $s = t = r + 1$ (using our assumption that $r \leq s \leq t$ and that $r \neq t$). Now allowing r to vary between 1 and Nk we find that $\tau(r,s,t)$ is maximized when $r = Nk, s = t = Nk + 1$, when it takes the value $\log(Nk(Nk+1))/\log(Nk(Nk+1)^2)$. Let us call this maximum θ_2. We will need θ to be at least θ_2.

It remains to deal with the cases in which $r = s = t \leq k$. For this we need a simple lemma.

Lemma 5.23. *A 2-comparable set T of triples in $[r]^3$ has size at most $t(r)$, where $t(r) = 3r^2/4$ if r is even and $t(r) = 3r^2/4 + r/2 + 3/4$ if r is odd.*

Proof. Let A be the set of all $(x,y) \in [r]^2$ such that $(x,y,z) \in T$ for some z. If such a z exists, it is unique, by the 2-comparability condition, so let us call it $f(x,y)$.

Suppose that $(x_1,y_1),(x_2,y_2) \in A$ and $\max\{x_1,y_1\} = \max\{x_2,y_2\}$. Then $f(x_1,y_1)$ and $f(x_2,y_2)$ are distinct, since either $x_1 = x_2$, $y_1 = y_2$, or x_1 and x_2 are not ordered in the same way as y_1 and y_2. Here again we are using 2-comparability.

For $i = 1,2,\ldots,r$, let $A_i = \{(x,y) \in A : \max\{x,y\} = i\}$. Then trivially $|A_i| \leq 2i-1$, and the argument just given shows also that $|A_i| \leq r$. It follows that $|A| \leq \sum_{i=1}^{\lfloor r/2\rfloor}(2i-1)+r\lceil r/2\rceil$. If r is even, this equals $(r/2)^2+r^2/2 = 3r^2/4$. If r is odd, then it is $((r-1)/2)^2+(r+1)^2/2$, which equals the bound stated. □

Actually all we really need is that T has size strictly less than r^2 when $r > 1$: the above result improves our eventual bound, but not in an interesting way.

For each $r > 1$, define $\tau(r)$ so that $r^{3\tau(r)} = t(r)$: that is,

$$\tau(r) = \log(t(r))/3\log r.$$

Let $\theta_3 = \max\{\tau(r) : r \leq Nk\}$. We shall also need the inequality $\theta \geq \theta_3$ for our proof to work.

We now fix $\theta = \max(\theta_1,\theta_2,\theta_3)$ and proceed with the inductive step of the argument.

The inductive step

Let T be a 2-increasing subset of $[r] \times [s] \times [t]$. We form a quotient set $T' \subset [k]^3$ by dividing each dimension into k intervals as equally as possible. That is, if our divisions into intervals are $[r] = R_1 \cup \cdots \cup R_k$, $[s] = S_1 \cup \cdots \cup S_k$ and $[t] = T_1 \cup \cdots \cup T_k$, then $T' = \{(h, i, j) : T \cap (R_h \times S_i \times T_j) \neq \emptyset\}$. We will assume that $\min\{r, s, t\} > Nk$, since otherwise we have one of our base cases and therefore the required estimate $|T| \leq (rst)^\theta$.

This quotient operation does not preserve 2-comparability, but, crucially, we see that the quotient set T' is weakly 2-increasing. This follows simply from the fact that if we have a pure directed cycle of quotient triples then by taking a representative triple $t \in T$ from each quotient triple we get a directed cycle in T. It is for this reason that it is so useful to us that the weakly increasing property alone has strong consequences.

At this point, we could naively bound the number of triples in T by applying our inductive hypothesis to bound the number of triples contained in each quotient triple, and multiplying by our upper bound on the size $|T'|$ of the quotient set. This gives us a bound of

$$(6k^2)\left(\frac{(N+1)^3 rst}{N^3 k^3}\right)^\theta .$$

Unfortunately this is larger than $(rst)^\theta$ when $\theta < 2/3$, and so the bound is not quite powerful enough to complete the induction.

However, we can improve on it by grouping the quotient triples into collections for which we may obtain an improved estimate using our inductive hypothesis. For this, we use the following definition and lemma.

Definition 5.24. *Let H be a collection of integer triples entirely contained in one of the planes $(x, *, *)$, $(*, y, *)$ or $(*, *, z)$. Suppose that when we project H onto the two free coordinates (obtaining a collection H_p of integer pairs) we have no two elements of H_p that are 2-comparable as pairs, in the obvious sense. Then we say that H is a* collapsible *collection of triples.*

It turns out that we can apply our inductive hypothesis to bound more efficiently the number of triples from T in a collection of quotient labels when the collection is collapsible.

Lemma 5.25. *Let H be a collapsible collection of triples in the quotient set. Then the total number of triples from T contained in the quotient triples of H*

is at most

$$\left(\frac{2(N+1)^3|H|rst}{N^3k^3}\right)^\theta.$$

Proof. Let us assume (without loss of generality) that the triples in H agree in their third coordinate, and let this coordinate be z. So the triples can be written in a sequence as $(u_1, v_1, z), \dots, (u_m, v_m, z)$ with $u_1 \geq \dots \geq u_m$ and $v_1 \leq \dots \leq v_m$.

Let us partition H into two sets U and V, where U is the set of (u_i, v_i, z) such that $u_i > u_{i+1}$ and $V = H \setminus U$. Then if $i < j$ and $(u_i, v_i, z), (u_j, v_j, z) \in U$, we have that $u_i > u_j$. Also, if $i < j$ and $(u_i, v_i, z), (u_j, v_j, z) \in V$, then $v_i < v_j$, since if v_i were to equal v_j then $v_i = v_{i+1}$, which implies that $u_i > u_{i+1}$ and therefore that $(u_i, v_i, z) \in U$. Thus, we have partitioned H into two sets, in one of which the u_i strictly decrease, and in the other of which the v_i strictly increase.

Now let us partition U further into sets U_i, according to the value of the second coordinate. The main fact that enables us to get a good bound is that if $i \neq j$ and q is the quotient map, then no point in $q^{-1}(U_i)$ can share a third coordinate with a point in $q^{-1}(U_j)$. That is because if $i < j$, then points in $q^{-1}(U_i)$ have a higher first coordinate and a lower second coordinate than points in $q^{-1}(U_j)$.

Let us suppose then that $|U_i| = a_i$ and that c_i different third coordinates occur in $q^{-1}(U_i)$. Then $\sum a_i = |U|$ and $\sum c_i \leq \lceil t/k \rceil \leq (N+1)t/Nk$. Also, by our inductive hypothesis, the number of points in $q^{-1}(U_i)$ is at most $((N+1)^2 a_i rsc_i/N^2k^2)^\theta$, since they live in a Cartesian product of three sets that have sizes at most $(N+1)a_i r/Nk$, $(N+1)s/Nk$, and c_i. Summing, over i, we find that

$$|T \cap q^{-1}(U)| \leq \sum_i \left(\frac{(N+1)^2 a_i c_i rs}{N^2k^2}\right)^\theta.$$

Similarly, we can partition V into sets V_i with $|V_i| = b_i$ and at most c_i' different third coordinates occurring in $q^{-1}(V_i)$, then $\sum b_i = |V|$ and $\sum c_i' \leq (N+1)t/Nk$, and we have the bound

$$|T \cap q^{-1}(V)| \leq \sum_i \left(\frac{(N+1)^2 b_i c_i' rs}{N^2k^2}\right)^\theta.$$

Now

$$\sum (x_j y_j)^\theta \leq \left(\sum x_j\right)^\theta \left(\sum y_j^{\theta/(1-\theta)}\right)^{1-\theta} \leq \left(\sum x_j\right)^\theta \left(\sum y_j\right)^\theta$$

by Hölder's inequality, the monotonicity of l_p norms, and the fact that $\theta \geq 1-\theta$. Applying this to the sum of the above two expressions and using our bounds for $\sum a_j$ and $\sum b_h$, $\sum c_j$ and $\sum c'_h$, we deduce that

$$|T \cap q^{-1}(H)| \leq \left(\frac{2(N+1)^3|H|rst}{N^3k^3}\right)^{\theta}.$$

□

Now the key idea is to partition the quotient set into two parts, the first of which is a union of large collapsible collections and the second of which is a genuine 2-increasing sequence. The contribution to the size of T from the first part will be controlled by using the collapsibility, while the second part will be controlled by the bound on the length of a 2-increasing sequence in $[k]^3$ obtained in the base case.

This splitting is achieved using the following lemma.

Lemma 5.26. *Suppose that S is a collection of triples containing no collapsible collection of size C. Then S contains a 2-comparable subset of size at least $C^{-3}|S|$.*

Proof. For the plane $P_x = (x, *, *)$, let $S_x = S \cap P_x$. Clearly the triples in the set S_x are partially ordered by $<_2$, and the antichains in this set are precisely the collapsible collections.

Since S has no collapsible collection of size larger than C, we have that S_x has no antichain of length greater than C and therefore (by Mirsky's Theorem) it must have a chain S'_x of length at least $C^{-1}|S_x|$.

Let S_1 be the subset $\cup_x S'_x$. We see that $|S_1| \geq C^{-1}|S|$.

Now we do the same with the y-coordinate, obtaining a subset S_2, and then again with the z-coordinate, obtaining a subset S_3. We have that $|S_3| \geq C^{-3}|S|$, and for any subset of S_3 obtained by fixing a coordinate the elements of this subset are totally ordered by $<_2$.

This means that S_3 is 2-comparable, since for two triples to fail to be 2-comparable they must share a coordinate and thus must both lie in one of the planes that we have treated above. Since restricting S_3 to this plane gives a subset totally ordered by $<_2$, the triples must be 2-comparable. □

Let C be a fixed constant, which we shall specify later. We may repeatedly extract collapsible collections of size C from the quotient set T' until we are left with a set S at which point the extraction fails. When that happens, Lemma 5.26 implies that S must have a 2-comparable subset S' of size $C^{-3}|S|$.

However, since $S' \subset T'$ and T' is weakly 2-increasing, S' is also weakly 2-increasing, which implies that it corresponds to a 2-increasing sequence (since for 2-comparable sets the weakly 2-increasing property implies transitivity of the relation $<_2$). Since T' contains no 2-increasing sequence of length δk^2 by our base case, we have that $C^{-3}|S| \leq \delta k^2$.

Now we may use Lemma 5.25 to bound the number of triples in T. We have split the quotient set T' into a set S of size at most $C^3 \delta k^2$, and the rest of T' which partitions into collapsible collections of size C. We therefore find that

$$\begin{aligned} |T| &\leq \frac{|T'| - |S|}{C} \left(\frac{2(N+1)^3 Crst}{N^3 k^3} \right)^\theta + |S| \left(\frac{2(N+1)^3 rst}{N^3 k^3} \right)^\theta \\ &\leq \left(\frac{6k^2}{C} \left(\frac{2(N+1)^3 C}{N^3 k^3} \right)^\theta + C^3 \delta k^2 \left(\frac{2(N+1)^3}{N^3 k^3} \right)^\theta \right) (rst)^\theta. \end{aligned}$$

Taking C to be such that $C^3 \delta = 6C^{2/3}/C = A$ we get $C = 6^{3/10} \delta^{-3/10}$ and $A = 6^{9/10} \delta^{1/10}$. Therefore

$$|T| \leq \left(2.6^{9/10} \delta^{1/10} k^2 \left(\frac{2(N+1)^3}{N^3 k^3} \right)^\theta \right) (rst)^\theta$$

which, by our choice of N, is

$$\leq \left(20 \delta^{1/10} k^{2-3\theta} \right) (rst)^\theta.$$

But our choice of k from the base case gives us that

$$20 \delta^{1/10} k^{2-3\theta} \leq k^{-\epsilon} k^{2-3\theta} = k^{2-3\theta-\epsilon}$$

and

$$2 - 3\theta - \epsilon < 0$$

by our choice of θ so the induction follows and the proof of Theorem 5.18 is complete.

5.2.2 Remarks

Size of θ

Here we shall give a very brief examination of the size of the θ that emerges from the argument. It is not worth being too careful here, as we have made little effort to tighten up the argument and because the use of Proposition 5.6 means that the difference $\frac{2}{3} - \theta$ is unavoidably extremely small.

First of all, it is important to get an explicit version of Proposition 5.6 that gives us a constant to replace the Ω notation. For this we can use the best known bound for the triangle removal lemma, due to Fox [28], and we obtain a quantitative version of Proposition 5.6, namely that

$$F(n) \leq n^2 / \exp(\log^*(n)/405).$$

It is also easy to check that in the base case $\theta_2 \geq \theta_3$ so θ_3 is of no concern.

In order to get $\epsilon > 0$ in the expression $20\delta^{1/10} \leq k^{-\epsilon}$, using

$$\delta = \exp(-\log^*(k)/405)$$

as is allowed by the above, we need

$$\exp(\log^*(k)/405) > 20^{10}$$

and so we will need $k \geq T(405\log(20^{10})$, where T is the tower function. Note that $405\log(20^{10}) < 12133$. We have that

$$\theta_2 = \frac{\log(Nk) + \log(Nk+1)}{\log(Nk) + 2\log(NK+1)}$$

$$= 2/3 - \frac{1}{9Nk\log(Nk)} + \mathcal{O}(1/(Nk)^2)$$

and since k is huge this gives us $\theta_2 \approx 2/3 - \frac{1}{9Nk\log(Nk)}$. Certainly, if we take $k = T(12133)$, say, then we have $\theta_2 < 2/3 - \frac{1}{T(12133)}$.

All that remains is θ_1, which is given by $(2-\epsilon)/3$ where $k^{-\epsilon} > 20\delta^{1/10} = 20\exp(-\log^* k/4050)$. If we take $k = T(12133)$ then we have

$$\epsilon = \log(\exp(12133/4050)/20)/\log(T(12133)) > 1/T(12133)$$

and so certainly $\theta_1 < 2/3 - \frac{1}{T(12133)}$ also.

Putting this together, we are able to choose $\theta = 2/3 - \frac{1}{T(12133)}$. With more effort to optimize the proof, the $T(12133)$ might be able to be brought down somewhat but a significant change to the base case is required to avoid the tower function.

Limitations and Scope for Improvements

The key to the argument that we have just given is that we may use the weakly 2-increasing property on its own to bring the size of H down from order k^3 to order k^2. Once we have realized this fact, it is fairly clear that we should be able to make a power-type improvement over the trivial n^2 bound on T by partitioning the quotient structure, which can be controlled by using the weak increasing property, into collections for which we can apply the inductive hypothesis efficiently.

However, there is a fundamental slackness in the argument as described above, since even if we could take $\delta = k^{-1/2}$ in the base case (the best we could hope for), we would end up with $\epsilon \approx 1/20$ and a rather tiny improvement to the upper bound. Therefore, even if we had enough computational power to verify for any finite k the conjectured bound of $(rst)^{1/2}$ for the maximal length of 2-increasing sequences from $[r] \times [s] \times [t]$ with $r \leq k$, so that we would could get ϵ as close as we like to $1/2$ in the base case, we would only be able to obtain a bound for θ that was arbitrarily close to $2/3 - 1/20$ rather than to the $1/2$ that we would expect.

One way that we could hope to improve this is to gain a better understanding of the structure of weakly 2-increasing sets. In the current argument we observe that the quotient structure is weakly 2-increasing, which limits the number of labels to $\mathcal{O}(k^2)$, but then we fall back on rather primitive methods to decompose it into collapsible subsets. Indeed, collapsible subsets are not the only ones for which we can obtain a more efficient application of the inductive hypothesis. If we were always able to decompose weakly 2-increasing sets of triples into a wider class of subsets that allow for efficient induction we could hope to improve the argument substantially. It seems very likely, therefore, that one can do better than this, especially since the structure of weakly 2-increasing sets with almost the maximum size seems to be quite restricted.

Generalizing from $(3,2)$ to (r,s)

We shall discuss the general (r,s) problems in more detail later in the chapter, but it is natural to wonder at this point whether our arguments above can be generalized easily to larger cases. Unfortunately there seems to be a genuine difficulty, which we shall briefly sketch here.

First observe that if we are simply aiming for an $n^{r-s+1-\epsilon}$ bound in the (r,s) problem, it suffices to prove an $n^{r-s+1-\epsilon}$ bound in the $(r-s+2,2)$ problem. So we may concentrate generally on the $(r,2)$ case.

Our above argument contains three main lemmas which need to be generalized. The first is Proposition 5.6, which gives a $o(n^2)$ bound for the $(3,2)$-problem. The second is Lemma 5.21 which gives a bound on the size of a weakly 2-increasing sequence. Finally, we need an analogue of Lemma 5.25 which provides an improved bound for *collapsible* collections of quotient triples.

The first of these is simply Proposition 5.13.

For the second we just use Theorem 5.22 instead of Lemma 5.21.

Therefore our only real obstacle is the analogue Lemma 5.25, for which we need a definition of a collapsible subset. Let us take, for instance, the $(4,2)$ case. In order to make the rest of the argument run unhindered, we need to define collapsible collections to be collections with two coordinates fixed and the other two coordinates pairwise not 2-comparable. But we have not been able to prove a version of Lemma 5.25 that works for this definition.

It therefore seems that the above argument cannot be straightforwardly generalized to the (r,s) cases, and that a new idea is needed.

5.3 A lower bound for $[3,2]$-sequences

In this section we shall describe a construction that beats the $n^{3/2}$ lower bound. We will then discuss the upper bound, for which any improvement over the result of Loh has proved elusive.

5.3.1 A reformulation using labels in grids

In this section we will be presenting various examples of 2-comparable sets of triples. If they are presented just as lists, then it is somewhat tedious to check that they are 2-comparable. However, there is a simple reformulation that is much more convenient for the purposes of looking at and understanding small examples of $[3,2]$-sequences, and also $(3,2)$-sequences. We briefly describe it here.

Given a 2-increasing sequence T of triples from $[r] \times [s] \times [t]$, we define the *grid representation* of T by considering each triple as a labelled point in the grid $[r] \times [s]$. That is, we think of the triple (a, b, c) as the point (a, b) labelled with c. Thus the whole sequence T corresponds to a labelling of some of the points of an $r \times s$ grid with labels from $[t]$.

As an example, the grid representation of the set

$$T = (1,1,1), (1,2,2), (2,1,3), (2,2,4), (3,3,1), (3,4,2), (4,3,3), (4,4,4)$$

is

		2	4
		1	3
2	4		
1	3		

Of course there is no particular reason to consider the third coordinate to be the label coordinate, and it is sometimes instructive to look at the same example in three different ways.

Now let us think about the restrictions imposed on labelled subsets of the grid if they are grid formulations of 2-increasing sequences of triples.

We begin by considering what follows from the 2-comparability condition. Note that if two triples do not share a coordinate, then they are automatically 2-comparable, so the condition is equivalent to saying that if a and b are two triples that share one coordinate, then either a is less than b in both the other coordinates, or a is greater than b in both the other coordinates. It follows from this that in the grid representation, if two points are in the same row, then the point to the right has a higher label than the point to the left, and if two points are in the same column, then the higher point has a higher label than the lower point. To put this more concisely, labels strictly increase as you go along a row or up a column. If it is the label coordinate that is fixed, then the condition states that the points with a given label must form a sequence that moves up and to the right, or in other words a 2-increasing sequence of pairs. That is, if (x_1, x_2) and (y_1, y_2) have the same label, then either $x_1 < y_1$ and $x_2 < y_2$ or $y_1 < x_1$ and $y_2 < x_2$. Equivalently (given that the same label cannot occur twice in a row or column), if $x_1 < y_1$ but $x_2 > y_2$, then (x_1, x_2) cannot have the same label as (y_1, y_2).

The additional constraint in the 2-increasing case is that the relation $<_2$ is

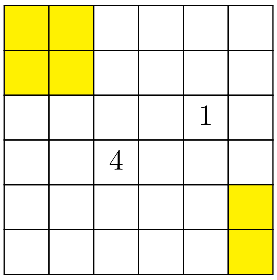


Figure 5.1: An example of the two regions described previously, highlighted in yellow. If the label 2 or 3 is placed within one of these regions, we get an intransitivity.

transitive when it is restricted to T. In the grid representation, a collection of triples that violates transitivity corresponds to having cell (a, b) filled with label c and cell (a', b') filled with label c', where $a' > a, b' > b$ and $c' < c$, and having a third cell (a'', b'') with label c'' and $c' < c'' < c$ where either $a'' > a'$ and $b'' < b$ or $a'' < a$ and $b'' > b$.

This configuration is much easier to express pictorially. Given two cells, with the one with smaller label c above and to the right of the other with larger label c', we define two regions of the grid. The first is the region above and to the left of both cells, and the second is the region below and to the right of both cells. To get a configuration that violates transitivity we simply place a label between c and c' in one of these regions. For an example, see Figure 5.1.

Thus, grid representations of 2-increasing sequences are characterized by the three properties below, and grid representations of 2-comparable sequences are characterized by the first two properties.

1. It increases along rows and up columns.
2. The set of points with any given label forms a 2-increasing sequence.
3. It must not contain a transitivity-breaking configuration of the kind just described.

5.3.2 A continuous generalization

Here we will give a natural continuous generalization of the $[3, 2]$ problem (and also the $(3, 2)$ problem), which extends the grid formulation discussed in the previous section. We use the word "cuboid" to mean an axis-parallel cuboid.

Definition 5.27. *Let I and J be two real intervals. Say that $I < J$ if $x < y$ for every $x \in I$ and $y \in J$. If I_1, I_2, I_3 and J_1, J_2, J_3 are real intervals, then*

$I_1 \times I_2 \times I_3 <_2 J_1 \times J_2 \times J_3$ if $I_h < J_h$ for at least two values of h. If C and C' are two cuboids, then they are 2-comparable *if $C <_2 C'$ or $C' <_2 C$. A sequence of cuboids $C_i \subset \mathbb{R}^3$ is* 2-increasing *if $C_i <_2 C_j$ whenever $i < j$. It is* 2-comparable *if any two distinct C_i are 2-comparable.*

Given a set of triples in $[r] \times [s] \times [t]$, we can convert it into as a collection of open unit cubes in the cuboid $[0, r] \times [0, s] \times [0, t]$ (where the triple (a, b, c) corresponds to the unit cube with corner (a, b, c) furthest from the origin). The resulting collection of cubes is 2-increasing/2-comparable if and only if the set or triples is 2-increasing/2-comparable.

This leads to the following generalization of the discrete question.

Problem 5.28. *Let $B = \{B_i\}$ be a set of disjoint open cuboids lying in $[0, 1]^3$. Define $\|B\|_\alpha$ by the formula*

$$\|B\|_\alpha = \big(\sum_i |B_i|^\alpha\big)^{1/\alpha}.$$

Let θ be the supremum over all α such that there exists a finite, 2-comparable collection B of at least two cuboids with

$$\|B\|_\alpha \geq 1.$$

What is the value of θ?

Observe that if $|B| = 1$ then we can take B to consist of the whole unit cube and then $\|B\|_\alpha = 1$ for all α, so we exclude this case. If $|B| > 1$ then for $\alpha > 1$ we have $\|B\|_\alpha < \|B\|_1 < 1$ since we cannot hope for the B_i to cover the whole of the encompassing cube. This tells us that θ exists and is at most 1.

Taking

$$B_1 = (0, 1/2) \times (0, 1/2) \times (0, 1)$$

and

$$B_2 = (1/2, 1) \times (1/2, 1) \times (0, 1)$$

and setting $B = \{B_1, B_2\}$ we have that $\|B\|_{1/2} = 1$, so $\theta \geq 1/2$.

We now show that this continuous generalization is, in a suitable sense, equivalent to the discrete problem.

Lemma 5.29. *Let θ be such that there exists a finite, 2-comparable/2-increasing collection B of at least two cuboids in $[0, 1]^3$ with*

$$\|B\|_\theta = 1.$$

Then for any $\epsilon > 0$ there exist n and a finite, 2-comparable/2-increasing collection T of integer tuples, each lying in $[n]^3$, with

$$|T| \geq n^{3\theta-\epsilon}.$$

The converse also holds, in the sense that given a collection T with $|T| = n^{3\theta}$ we get a collection B with $\|B\|_\theta = 1$.

Proof. The converse is easy, since, as already remarked, we can view the collection T as a collection of unit cubes inside $[0, n]^3$, which we can then scale down by a factor of n. This gives a collection B of at least two $1/n \times 1/n \times 1/n$ cuboids, and

$$\sum_{B_i} \left(\frac{1}{n^3}\right)^\theta = \frac{|T|}{n^{3\theta}} = 1.$$

The other implication is a little more subtle. What we would like to do is to take the collection B living inside $[0, 1]^3$ and discretize it. To begin with, we would take a fine grid and take all the points in it that live inside $\bigcup B$. Although this does not give us a 2-comparable/2-increasing set, it gives us a set that splits up nicely into a disjoint union of subgrids. We could then hope to take 2-comparable/2-increasing subsets of these subgrids that are as large as possible and put them together. However, this approach runs into difficulties, because the subgrids could be of very different sizes and shapes, which makes it unclear that we can fit long 2-increasing/2-comparable subsets inside all of them simultaneously. (Recall, for instance, the trivial upper bound of $\min\{rs, rt, st\}$, which, if r, s, t are sufficiently unbalanced, will be less than $(rst)^{1/2}$.)

So first we shall "treat" the collection B so that the cuboids are all of comparable dimensions. This is done as follows.

For any collection of cuboids B we define a sequence $B^1, B^2, \dots$ by setting $B^1 = B$ and defining B^k by replacing each $B_i \in B^{k-1}$ by a suitably scaled copy of B. Note that we have $|B^k| = |B|^k$ and also

$$\sum_{B_i \in B^k} |B_i|^\theta = \sum_{B_i \in B^{k-1}} |B_i|^\theta \big(\sum_{B_j \in B} |B_j|^\theta \big) = \sum_{B_i \in B^{k-1}} |B_i|^\theta.$$

Therefore, by induction we have

$$\sum_{B_i \in B^k} |B_i|^\theta = 1$$

for all k.

Now suppose we have a collection B such that $\|B\|_\theta = 1$. First we choose a positive integer m and perturb the cuboids in B so that their sidelengths are all multiples of m^{-1}. For any $\epsilon > 0$ we can choose m and the perturbation in such a way that the peturbed collection B' has $\|B'\|_{\theta-\epsilon} \geq 1$.

Let us fix our $\epsilon > 0$ and our choice of m. Then let $p_1, \dots, p_r$ be the primes less than or equal to m and define the *sidelength vector* of a cuboid in B' to be the vector $(a_1, \dots, a_r, b_1, \dots, b_r, c_1, \dots, c_r)$, where the sidelengths of the cuboid are $m^{-1}p_1^{a_1} \dots p_r^{a_r}$, $m^{-1}p_1^{b_1} \dots p_r^{b_r}$ and $m^{-1}p_1^{c_1} \dots p_r^{c_r}$.

We extend this definition of a sidelength vector to the cuboids in $(B')^k$ by assigning to a cuboid in $(B')^k$ the vector $(a_1, \dots, a_r, b_1, \dots, b_r, c_1, \dots, c_r)$ where the dimensions of the cuboid are $m^{-k}p_1^{a_1} \dots p_r^{a_r}$, $m^{-k}p_1^{b_1} \dots p_r^{b_r}$ and $m^{-k}p_1^{c_1} \dots p_r^{c_r}$. Having done this, we see that the sidelength vectors of cuboids in B_p^k are just sums of k of the sidelength vectors of cuboids in B'.

The total number of sidelength vectors for $(B')^k$ is the size of the k-fold iterated sumset of the set of sidelength vectors for B', and these all live in the box $[km]^{3r}$ so their number grows polynomially with k. Fix k large, and let v be the sidelength vector such that the sum of all $|B_i|^{\theta-\epsilon}$ such that $B_i \in (B')^k$ has sidelength vector v is maximized.

Let $C = \{C_i\}$ be the subcollection of $(B')^k$ consisting of the cuboids with sidelength vector v. Then

$$\sum_i |C_i|^{\theta-\epsilon} \geq (km)^{-3r}.$$

All the C_i have the same sidelengths: let these be d_1, d_2 and d_3. Now subdivide each of the three sides of the unit cube into intervals of equal lengths a, b and c, with $d_1/4 \leq a \leq d_1/2$, $d_2/4 \leq b \leq d_2/2$ and $d_3/4 \leq c \leq d_3/2$. Let D be a collection of cuboids obtained by selecting, for each C_i, precisely one of the $a \times b \times c$ cuboids that is entirely contained within C_i: by our choice of a, b, c such a cuboid must exist.

Since $\sum_i |C_i|^{\theta-\epsilon} \geq (km)^{-3r}$, we have that

$$|C|(d_1 d_2 d_3)^{\theta-\epsilon} \geq (km)^{-3r},$$

from which it follows that

$$|D|(abc)^{\theta-\epsilon} \geq (km)^{-3r}/64.$$

We may now obtain a discrete sequence of tuples by scaling up the collection of cuboids D by a factor of $A = 1/a$, $B = 1/b$ and $C = 1/c$ in the three dimensions (note that $A, B, C \in \mathbb{Z}$) so that the cuboids become unit cubes. Let T be the collection of integer tuples that we get by taking the furthest point in (the closure of) each cube from the origin. Then T is a collection of integer tuples lying in $[A] \times [B] \times [C]$, and it is 2-increasing/2-comparable if B is.

We now observe that ABC is exponentially large in k. This follows provided that we can show that at least one of a, b or c is exponentially small. But to any sidelength vector v we may associate a sequence S of k cuboids from B' such that the sidelengths defined by v are the products of the corresponding sidelengths from S. Since B' consists of more than one cuboid, the $1 \times 1 \times 1$ cuboid is not present, and consequently at least one third of the sidelengths in S are not length 1. Letting $h < 1$ be the largest non-unit sidelength of any cuboid in B', we deduce that at least one of a, b or c is at most $h^{k/3}$.

Since ABC is exponentially large in k, by taking k sufficiently large we can ensure that

$$|T| \geq (ABC)^{\theta - 2\epsilon}.$$

All that is now required is to build a collection of tuples from T that live inside a set $[n]^3$ rather than $[A] \times [B] \times [C]$. We saw how to do this at the beginning of Section 5.2. We let ϕ be the map that cycles the coordinates of each tuple round by one place, so $\phi(\{(a, b, c)\}) = \{(c, a, b)\}$. Define $T_1 = T$, $T_2 = \phi(T)$ and $T_3 = \phi^2(T)$. Then, using the definition of a product of two sequences given in the proof of Lemma 5.7, we can take the sequence $S = T_1 \otimes T_2 \otimes T_3$. S is a set of integer tuples each lying in $[n]^3$ where $n = ABC$, and $|S| = n^{3\theta - 6\epsilon}$, and it is 2-increasing/2-comparable if T is. $\square$

This lemma allows us to consider continuous constructions in our search for long 2-comparable sequences. It turns out, as we shall demonstrate in Section 5.3.3, that this is quite useful.

There is a clear resemblance between the definition of θ above and the definition of Hausdorff dimension. It seems almost certain that the correct exponent in the discrete problems is equal to the maximal Hausdorff dimension of a subset of $[0, 1]^3$ that is 2-increasing/2-comparable, but we have not attempted to prove this.

One reason the continuous problem helps is that it allows us to use variational arguments. The next lemma illustrates this. Although it is not strictly necessary for our purposes (we shall make use of it, but will then prove a stronger result without using it), it may be important in future developments. That is because, as we shall see in Section 5.6, to prove an upper bound of $n^{3/2}$ for the 2-increasing problem, it appears to be necessary to use extremality, and this lemma is almost the only way we have found of doing that.

Lemma 5.30. *Let $B = \{B_i\}$ be a finite collection of disjoint open cuboids lying in $[0,1]^3$ and let $\alpha > 0$. For each i, let $B_i = X_i \times Y_i \times Z_i$ and let $x_i = |X_i|, y_i = |Y_i|$ and $z_i = |Z_i|$. Given any $t \in [0,1]$, define $f(t)$ to be $\sum_{i:t\in X_i} x_i^{\alpha-1}(y_i z_i)^\alpha$. Then either f is constant for almost every t or there is a continuous piecewise linear bijection $\phi : [0,1] \to [0,1]$ such that if we set $C_i = \{(\phi(x), y, z) : (x, y, z) \in B_i\}$ for each i and $C = \{C_i\}$, then $\|C\|_\alpha > \|B\|_\alpha$.*

Proof. If f is not constant almost everywhere, then we can find t and u such that neither t nor u is the end point of any of the intervals X_i, and $f(t) \neq f(u)$.

Now choose small intervals I and J about t and u that do not contain the end points of any of the X_i and choose a piecewise linear bijection $\phi : [0,1] \to [0,1]$ that has gradient 1 outside $I \cup J$, increases the length of I by δ, and decreases the length of J by δ. Then $|\phi(X_i)| = |X_i|$ for every i such that X_i contains both t and u or neither t nor u. If it contains just t then $|\phi(X_i)| = |X_i| + \delta$ and if it contains just u then $|\phi(X_i)| = |X_i| - \delta$.

Now let us think about how the sum $\sum_i (x_i y_i z_i)^\alpha$ changes when we expand and contract the intervals X_i in this way. The effect of increasing x_i by δ is to increase the sum by $\alpha x_i^{\alpha-1} y_i z_i \delta + o(\delta)$ and the effect of decreasing it by δ is to decrease the sum by that amount. Therefore,

$$\|C\|_\alpha^\alpha - \|B\|_\alpha^\alpha = \alpha\delta(f(t) - f(u)) + o(\delta).$$

Since $f(t) \neq f(u)$, we can choose δ (possibly negative) such that the right-hand side is positive, and the result is proved. □

Note that the map $(x, y, z) \mapsto (\phi(x), y, z)$ preserves all the order relations we are interested in, so if B is 2-increasing or 2-comparable then so is C. So the lemma implies that if we have an extremal example in the continuous case, then all its cross sections (apart from those that intersect the boundaries of the cuboids) are of the same "size", as measured by the function f. Note too that if all the cuboids have the same size and shape, then the lemma implies that all cross sections that do not include a face of one of the cuboids intersect the same number of cuboids.

We believe that this property is also present in the discrete, 2-increasing setting, but we have not been able to prove this. Specifically, if we say that a 2-increasing, discrete sequence of triples is extremal if it is of length n^α where α is the maximal exponent (ie $F(n) = n^{\alpha+o(1)}$), then we conjecture that the following holds. There exists a function $C : \mathbb{N}^3 \mapsto \mathbb{N}$ such that if T is an extremal 2-increasing sequence of triples from $[r] \times [s] \times [t]$ then the number of triples in the plane $(*, *, z)$ is equal to $C(r, s, t)$, the number of triples in the plane $(*, y, *)$ is equal to $C(s, t, r)$ and the number of triples in the plane $(x, *, *)$ is equal to $C(t, r, s)$.

5.3.3 Long 2-comparable sequences

We begin with a very short but somewhat abstract argument that there are 2-comparable collections of triples that have length greater than $n^{3/2}$. The argument starts with the following example, given in its grid representation.

	3			4
3			4	
			1	2
	1	4		
1		2		

This lives in the set $[5] \times [5] \times [4]$ and contains ten triples. Since $10 = (5 \times 5 \times 4)^{1/2}$, this is not yet a suitable example. However, even the tiniest improvement would turn it into an example of what we want, since the number of triples is equal to the bound we are trying to improve.

This is where looking at the continuous problem helps. We cannot make a "tiny" improvement to a discrete example, but if we think of this set as a continuous example made out of unit cubes, then Lemma 5.30 implies that we *can* improve it, since the number of cubes in each layer is not constant; labels 1 and 4 appear three times each, while labels 2 and 3 appear only twice each. Then Lemma 5.29 allows us to convert our improved example back into a (much larger) discrete example that exhibits a similar improvement.

Rather than pursuing the above argument in detail, we shall use similar ideas to obtain better bounds and smaller examples. This time our starting point is the following length-five 2-comparable collection of tuples from $[3]^3$:

$$(1, 1, 1), (1, 2, 3), (2, 3, 1), (3, 1, 2), (3, 3, 3).$$

In the grid formulation, this is given by

	1	3
3		
1		2

Interestingly, this example is *not* on the boundary, since $5 < 3^{3/2} = 5.196...$. However, these two numbers are sufficiently close that by optimizing the corresponding continuous example one can still beat the power $3/2$, and that gives the best bound we currently know.

Let us therefore convert the example to the continuous variant by viewing it as a union of five $\frac{1}{3} \times \frac{1}{3} \times \frac{1}{3}$ cuboids living inside $[0,1]^3$. We now perform a distortion so that the cuboids have different sizes. Specifically, we shall simultaneously stretch and shrink the cubes by choosing some $x \in (0, 1/2)$ and dividing each copy of $[0,1]$ into the three intervals $(0,x), (x, 1-x)$ and $(1-x, 1)$. (Symmetry considerations show easily that we are not losing any important flexibility by doing this.) We shall then optimize x.

For the resulting collection of cuboids B we have

$$\|B\|_{1/2} = 2(x^3)^{1/2} + 3(x^2(1-2x))^{1/2}$$

which is optimized at $x = (7+\sqrt{5})/22 = 0.419\ldots$ giving

$$\|B\|_{1/2} = \sqrt{\frac{13}{22} + \frac{5\sqrt{5}}{22}} = 1.048\cdots > 1.$$

This shows already that $\theta > 1/2$, but we can work a little more and obtain a concrete lower bound on θ.

Note first that

$$\|B\|_\alpha = 2x^{3\alpha} + 3(x^{2\alpha}(1-2x)^\alpha,$$

so we want to find α as large as possible such that

$$\sup_{x \in (0,1/2)} (2x^{3\alpha} + 3x^{2\alpha}(1-2x)^\alpha) \geq 1.$$

The best we can do here is a numerical calculation, which reveals that the optimal α lies between 0.5154 and 0.5155. Therefore θ, the best possible exponent, is greater than 0.5154.

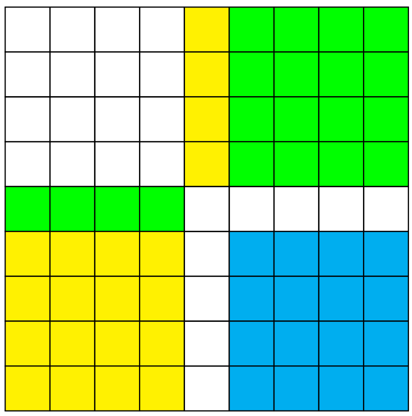
Applying Lemma 5.29 we instantly deduce Theorem 5.15.

It may be of interest to see some small examples of sequences breaking the

$n^{3/2}$ bound, since it is not immediately obvious how to extract simple ones from Lemma 5.29. We shall now give two, and explain a little how they were constructed.

The process for constructing explicit counterexamples with small n essentially follows the proof of the full upper bound, but we avoid the complexity of Lemma 5.29 by discretizing the continuous example above in a simple way. We simply subdivide all three dimensions equally and place discrete sequences inside each of the continuous cuboids in the resulting grid. In general this may not work, since we may not be able to fit long sequences inside the cuboids if their shapes are too different. However, a judicious choice of the parameter x in the continuous construction outlined above allows us to keep the cuboid dimensions in a good range.

For example, we can take the above construction but modify it by taking the sub-optimal $x = 4/9$. This value of x is chosen because our calculations above showed that we wanted $x > 1/3$, and if we take x to be a rational with small denominator we can subdivide coarsely and obtain a discrete sequence that lives inside a small grid. So we subdivide each dimension into 9 sections and scale up by a factor of 9 so that our subdivisions are into unit intervals, and we end up with the following:



where the yellow blocks correspond to cuboids with third dimension $(0, 4)$, the blue block corresponds to a cuboid with third dimension $(4, 5)$, and the green blocks correspond to cuboids with third dimension $(5, 9)$.

In order to convert this into a discrete sequence, we simply need to fill the cuboids with large 2-comparable collections of tuples. For instance, the bottom left cuboid is $(0, 4) \times (0, 4) \times (0, 4)$, so we want to treat it as a 4×4 grid with labels from the set $[4]$. We can fit $4^{3/2} = 8$ labels inside here. Similarly we can fit eight labels in the top right green block, and four labels in each of the other

				4			8	9
				3			6	7
				2	8	9		
				1	6	7		
6	7	8	9					
		3	4					5
		1	2				5	
3	4					5		
1	2				5			

(a) A length 28 2-comparable sequence of tuples in $[9]^3$, given in grid formulation. Observe that $28 = 9^{1.516\cdots} > 9^{3/2}$.

		1	3	4
3	4			
	1			2
1			2	

(b) Another example of a 2-comparable sequence given in grid formulation. Note that the length of the sequence is 9, while $rst = 80 < 9^2$.

Figure 5.2: Some long $[3, 2]$ sequences.

three blocks. This gives us a 2-comparable sequence of triples in $[9]^3$ of length 28, shown in Figure 5.2 alongside a smaller example of a 2-comparable sequence of tuples that beats the $(rst)^{1/2}$ bound.

An important observation is that all the sequences discussed in this section are disastrously far from being transitive. For example, in the coloured grid above we see that any choice of three labels from the leftmost green block, the rightmost yellow block and the blue block form an intransitive loop. As a result these constructions pose no problems for Conjecture 5.8.

It is also worth remarking that the best constructions above all began from the same starting point; namely the sequence of length 5 presented at the start of the section. We could hope that there are other short sequences to start from which could yield even better constructions. However, all the sequences that we have tried have yielded significantly worse bounds than the one above.

This, combined with an extensive search for counterexamples [1], leads us to think that the optimal α that we approximated earlier has a chance of being the correct exponent for the $[3, 2]$ problem. (The problem can be phrased as one about the largest clique in a certain explicit graph, to which one can apply state-of-the-art clique-finding algorithms. Although the failure to find any better construction does not prove that they do not exist, experience suggests that for explicit instances of the clique problem such algorithms tend to perform very well.)

[1] We thank Adam Wagner for performing this search.

5.4 Related conjectures

In this section we shall discuss several conjectures, some of them closely related to well-known questions, that would imply power bounds for the $[3, 2]$ or $(3, 2)$ problems.

5.4.1 Weakening the 2-comparability condition

We have not yet been able to improve on Loh's upper bound in the $[3, 2]$ case, and a power type improvement here is highly desirable. In this subsection we shall consider how far we can weaken the 2-comparability condition and still have some hope of a non-trivial power-type upper bound, since this may help in the search for a proof.

Recall that the grid representation of a $[3, 2]$ subset of $[n]^3$ is a subset $G \subset [n] \times [n]$ with its points given labels from $[n]$ in such a way that the following two conditions are satisfied.

Condition 1. The labels increase along rows and up columns.

Condition 2. Each label occupies a 2-increasing set of points from the grid.

Can we weaken these conditions without obviously allowing G to have size $n^{2-o(1)}$?

One weakening that goes too far is simply to omit Condition 2. In this case we can label (a, b) with the label $a+b-n/2$ provided $n/2 < a+b \leq 3n/2$, which allows us to place about $3n^2/4$ labels.

If Conditions 1 and 2 hold, and a labelled point P is in the same row as a labelled point Q and the same column as a labelled point Q', then Q and Q' must have different labels. That is because otherwise if Q is to the right of P and Q' is above P, then Condition 2 is violated, if Q is to the left of P and Q' is above P, then Condition 1 is violated, and the other two cases are similar. Let us give a name to this consequence.

Condition 3. Given any point $x \in G$, no point in the same row as x (excluding x itself) can share a label with a point in the same column as x.

Another weakening we might consider is to replace Condition 1 by Condition 3. However, if we associate matchings with the labels in an obvious way, Condition 3 is saying that these matchings are induced, so we can use the standard Behrend example to show that there are labelled sets of size $n^{2-o(1)}$ that satisfy Conditions 2 and 3. Indeed, take a set $A \subset [n]$ of size $n^{1-o(1)}$ that contains no arithmetic progression of length 3, and label the cells (x, y) on the line

$x + y = a \in A$ with the label $z = x - y$ provided that $x - y > 0$. In this way we label $n^{2-o(1)}$ cells, and it is easy to check that the labelling satisfies the two conditions.

However, a small strengthening of Condition 3 rules out Behrend-type constructions and leaves the possibility of a power bound wide open. We first give a definition.

Definition 5.31. *We denote by $S(c)$ the collection of cells from G with label c. We call such sets* label sets*. We also write $P(c)$ for the set of cells in $[n]^2$ that share both a row and a column with a cell from $S(c)$. We call $P(c)$ the* completion *of $S(c)$.*

The reason for the word "completion" is that $S(c)$ can be thought of as a matching, and $P(c)$ can be thought of as the smallest complete bipartite graph that contains it.

Condition 3 is equivalent to the statement that for all labels c, the cells in the set $P(c) \setminus S(c)$ are all empty.

For our new variant, we replace Condition 3 with the following stronger condition, which we call Condition 4. It states that Condition 3 holds and additionally that there is no cell x in G such that there exist labels c and d with c appearing to the left of x in the same row and d appearing to the right, and d appearing below x in the same column and c appearing above.

This condition rules out the following configuration appearing in a subgrid of G, where asterisks denote cells which may be labelled or empty:

	c	$*$
c		d
$*$	d	

A more appealing way to state the condition is as follows. We first extend Definition 5.31.

Definition 5.32. *Define the* upper completion *of a label set $S(c)$ to be the set $P_1(c)$ of cells from G that have points labelled c both directly above and directly to the left, and the* lower completion *$P_2(c)$ to be the set of cells from G that have points labelled c both directly below them and directly to the right.*

We have $P(c) = P_1(c) \cup P_2(c) \cup S(c)$, as illustrated in Figure 5.3.

This gives us the following way of stating the condition.

Condition 4. Given any two labels c and d, the sets $P_1(c)$ and $P_2(d)$ are disjoint.

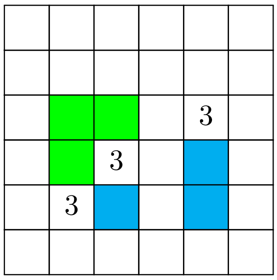


Figure 5.3: The upper and lower completions of a label set as given in Definition 5.32. $P_1(3)$ is highlighted in green and $P_2(3)$ in blue.

Problem 5.33. *Let $G \subset [n] \times [n]$ be labelled with points from $[n]$. Suppose that the labelling satisfies Conditions 2 and 4. How many labelled cells can G contain?*

Since Condition 4 is strictly weaker than Condition 1, this is a weakening of the $[3, 2]$ problem, so we cannot hope for a power bound as strong as $n^{3/2}$. However, the construction based on Behrend's AP3-free set does not come close to satisfying the two conditions, and a non-trivial power bound seems quite plausible.

Conjecture 5.34. *There exists $\epsilon > 0$ such that any labelling satisfying the conditions of Problem 5.33 has at most $n^{2-\epsilon}$ labels.*

A somewhat different weakening of the $[3, 2]$ problem can be obtained from the following observation. Given a subset $G \subset [n]^2$, we can regard it as a bipartite graph with copies of $[n]$ as its vertex sets. If we now assign labels to G, we can think of it as a labelled bipartite graph.

Proposition 5.35. *If the labelling of G corresponds to a 2-comparable set of triples, then the labelled bipartite graph just described contains no cycle with a sequence of labels that repeats itself twice. That is, there is no cycle of length $2k$ such that as you go along the edges, the sequence of labels is of the form $c_1c_2 \dots c_kc_1c_2 \dots c_k$.*

Proof. Suppose that a repeating cycle of this kind exists. In this bipartite-graphs formulation, Condition 2 says that no two edges with the same label can share a vertex or cross each other (if we imagine that the vertices are arranged in increasing order in two parallel rows). For each edge in the cycle, call it a *left edge* if it occurs to the left of its opposite counterpart (more formally, the vertices connected by the edge e_i are smaller than the vertices connected by the edge e_{i+k}, where addition is mod $2k$), and otherwise a *right edge.*

There must be some i such that e_i is a right edge and e_{i+1} is a left edge. Without loss of generality e_i is the edge xy and let e_{i+1} be the edge $x'y$. Then if k is even the edges e_{i+k} and e_{i+k+1} take the form zw and $z'w$, where w is both smaller than y and greater than y, a trivial contradiction. If k is odd, then they take the form zw and zw'. This time our assumptions give us that $x > z$, $x' < z$, $y > w$ and $y < w'$. The first two inequalities imply that $c_i > c_{i+1}$, by Condition 1, and the third and fourth imply that $c_i < c_{i+1}$, again giving a contradiction. □

Call a cycle of the kind discussed in the proposition above a *repeating cycle.*

Problem 5.36. *Let G be a bipartite graph with two vertex sets of size n and suppose that its edges can be labelled with n labels in such a way that there are no repeating cycles. How many edges can G have? In particular, is there an upper bound of $O(n^{2-\epsilon})$ for some positive ϵ?*

Note that the problem above does not say anything about orderings on the vertex sets or the set of labels, so it is a weakening to a more "purely combinatorial" problem. As the proposition shows, a positive answer to the last question would give a non-trivial power bound for the $[3, 2]$ problem.

5.4.2 Connections to extremal problems for hypergraphs

In this section we give one last perspective on the $[3, 2]$ problem, and give a connection to widely studied problems about hypergraphs, as well as to a well-known problem of Ruzsa [68].

Let G be a tripartite, 3-uniform, linear hypergraph with vertex sets $X = Y = Z = [n]$.

Definition 5.37. *We say that G is (u, v)-free if there is no collection of v edges spanned by at most u vertices.*

Problem 5.38. *What is the maximal size of G if it is (u, v)-free? In particular, when can we beat the trivial bound of $\Omega(n^2)$?*

This problem (and its generalization to r-uniform hypergraphs) has been studied by a large number of people, beginning with Brown, Erdős and Sós [15] who proved that $(u, u-2)$-free hypergraphs could contain $\Omega(n^2)$ edges. The well-known "$(6, 3)$ theorem" of Ruzsa and Szemerédi [69] was the next breakthrough, proving that $(6, 3)$-free hypergraphs can contain at most $o(n^2)$ edges, and the

Behrend construction [9] showed that this is almost tight in the sense that there are $(6, 3)$-free hypergraphs containing $n^{2-o(1)}$ edges.

The following conjecture of Brown, Erdős and Sós has been open since 1971.

Conjecture 5.39. *If G is $(u, u-3)$-free then it contains $o(n^2)$ edges.*

The next result shows how these questions are related to our problem.

Proposition 5.40. *Given a collection of triples T, regard it as a tripartite 3-uniform hypergraph $G(T)$ in the obvious way. If T forms a 2-increasing sequence then $G(T)$ is $(9, 5)$-free, and if T is a 2-comparable set then $G(T)$ is $(10, 6)$-free.*

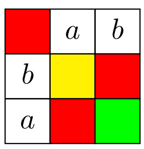
Proof. Define $F(r, s, t)$ (respectively $G(r, s, t)$) to be the maximum length of a 2-increasing (respectively 2-comparable) sequence of triples in $[r] \times [s] \times [t]$. In order to prove the proposition, we need to show that $F(r, s, t) \leq 4$ whenever $r + s + t \leq 9$ and $G(r, s, t) \leq 5$ whenever $r + s + t \leq 10$.

To show that $F(2, 3, 4) \leq 4$ (and even that $G(2, 3, 4) \leq 4$), note that if two triples (a, b, c) and (a', b', c) share a third coordinate c, then there can be no triples beginning (a, b') or (a', b), which implies that there are at most four triples (since no two triples can share two coordinates). But if all the triples have distinct third coordinates then again there are at most four triples.

Furthermore, $F(1, 4, 4)$ is bounded by $1 \times 4 = 4$ trivially, and similarly for $F(1, 3, 5)$ and $F(2, 2, 5)$ (and again the same bounds hold for G). So to prove the first statement it remains to bound $F(3, 3, 3)$.

This is a little more difficult. If any coordinate takes the same value three times, then there can be at most three triples, since the other two coordinates must be 11, 22 and 33, which between them rule out all other possibilities for those two coordinates. If in some coordinate at most one value occurs twice, then trivially there are at most four triples.

So we may assume that in each coordinate two values occur twice. In the grid representation, we are labelling points in $[3]^2$ with labels from $[3]$, and we may assume that two labels appear twice. Since the Cartesian products associated with these two label sets are 2×2 subgrids of the 3×3 grid, they must intersect in a cell. So up to symmetry we have the following configuration:



where the cells in red cannot be filled as they are part of a Cartesian product associated with a label set. The cell in yellow cannot be filled since $b > a$ and labels must increase up columns and along rows. Finally, we see that the cell in green cannot be filled without violating transitivity.

In order to bound $G(r, s, t)$ by 5 for $r + s + t \leq 10$ we need to consider the (r, s, t) combinations $(2, 3, 5), (2, 4, 4)$ and $(3, 3, 4)$, since if $1 = r \leq s \leq t$ then the trivial bound of rs suffices. The first case is easy, since if we label all six points of the grid $[2] \times [3]$, then all the labels have to be distinct, which they cannot be if we have only five lables. For the second case, we consider labelling points in $[4] \times [4]$ with labels from $[2]$; if a label is used four times then we cannot fill in any more points, and if each label is used three times then we get two associated 3×3 Cartesian products, which must intersect in a 2×2 subgrid. But any 2×2 subgrid of the Cartesian product associated with one of the label sets actually contains a labelled point, which contradicts Condition 3 of Subsection 5.4.1.

So it remains to check $G(3, 3, 4)$. Here it is easiest to imagine labelling $[3] \times [3]$ from $[4]$. If at most one label is used twice we are done, and if any label is used three times we are done. So once again we may assume that two labels are used twice, and we once again arrive at the configuration

	a	b
b		
a		

where the red and yellow cells are unlabelled for the same reason as before. So there can be at most one more label, and $G(3, 3, 4) = 5$ as desired. □

Of course we are interested in a power bound, but because both the $(9, 5)$ and $(10, 6)$ cases of Problem 5.38 are imposing stronger conditions than those in Conjecture 5.39, it is reasonable to hope that such a bound could hold. Indeed, if Conjecture 5.39 is true then it would seem highly likely that a stronger bound should be possible in the $(u, u - 4)$ cases.

Conjecture 5.41. *For every u there exists $\epsilon > 0$ such that a $(u, u - 4)$-free hypergraph with n vertices has at most $O(n^{2-\epsilon})$ edges.*

As is also the case with the weakenings discussed at the end of the previous subsection, Proposition 5.40 loses some of the strength of the 2-increasing and 2-comparable conditions, so it is quite possible that Conjecture 5.41 is false, but that a non-trivial power bound still holds for the $[3, 2]$ problem.

The strongest known result in the direction of Conjecture 5.39 is the following theorem of Sárközy and Selkow [74], again stated only in the 3-uniform setting.

Theorem 5.42. *If G is $(v+2+\lfloor \log_2 v \rfloor, v)$-free then G contains $o(n^2)$ edges. In particular, if G is $(8,4)$-free, $(9,5)$-free or $(10,6)$-free then we have an upper bound of $o(n^2)$.*

This of course directly implies a bound of $o(n^2)$ for both the $(3,2)$ and $[3,2]$ problems, but the proof of Theorem 5.42 uses the regularity lemma and consequently does not improve on the $n^2/\exp(\Omega(\log^* n))$ bound from Loh. However, the result above is unlikely to be best possible. Indeed it is easy to prove a bound of $n^{3/2}$ for the $(8,4)$ case – this follows from Theorem 1.2 of Loh [56]. From this a power bound for the $(10,5)$ case follows easily, but $(9,5)$ and $(10,6)$ are still out of reach.

An additional reason to try to improve the $(9,5)$ bound to one of power-type is that it would answer the following question of Ruzsa [68].

Problem 5.43. *Let $A \subset [n]$ be a set containing no non-trivial solutions to the equation $2x+2y=z+3w$ (meaning all solutions have $x=y=z=w$). How large can A be as a subset of n? In particular, can it have size $n^{1-o(1)}$?*

The equation $2x+2y=z+3w$ is the simplest example of one for which simple Cauchy-Schwarz arguments do not work, but neither does the Behrend construction. So the best known upper bound is obtained by arguments similar to the proofs of Roth's theorem, but the best known lower bounds are of the form n^α with $\alpha < 1$. It would be of great interest to know which bounds are nearer to the truth.

Proposition 5.44. *A power bound for the $(9,5)$-free version of Problem 5.38 implies a power bound for Problem 5.43.*

Proof. Suppose that we have a set $A \subset [n]$ with no solutions to the equation $2x+2y=z+3w$. Then define a 3-uniform, linear, tripartite hypergraph G with vertex sets $X=Y=Z=[n]$ and all the faces (x,y,z) with $x-y=z$ and $x+y \in A$. By translating A if necessary (modulo n) we can ensure that G has $\mathcal{O}(n|A|)$ faces.

It is easiest to visualise the graph G as a labelled bipartite graph G' on $X \times Y$, where the label on the edge (x,y) is z (for each face (x,y,z) of G). Since G is linear, each edge in G' has precisely one label. For a string S of letters, we say that the graph is *S-free* if there is no path through G' where the edges have labels following the pattern of S. We call such a path an *S-path.* For instance, the graph is *aa*-free if no two incident edges have the same label, which follows from the fact G is linear.

Since A is AP3-free, it follows that G' is *aba*-free. We will now show that G' is *abcab*-free, and we shall further show that any configuration of five faces supported on nine vertices gives rise to either an *aa*-path, and *aba*-path or an *abcab*-path.

First we show that G' is *abcab*-free. An *abcab*-path has five edges, which we denote $(f_1, f_2, f_3, f_4, f_5)$. Each f_i is an edge (x_i, y_i) with $x_i + y_i = a_i \in A$. Without loss of generality we have $x_1 = x_2$, $y_2 = y_3$, $x_3 = x_4$ and $y_4 = y_5$. The constraints on the labelling then translate into arithmetical constraints on the a_i and we find that

$$a_4 = a_1 + 2(a_3 - a_2)$$

and

$$a_5 = a_2 + 2(a_4 - a_3).$$

But then

$$2(a_5 + a_3) = 2a_2 + 4a_4 - 2a_3 = 3a_4 + 3a_1 \ ,$$

which cannot happen for $a_i \in A$.

We now claim that any configuration of five faces of G supported on nine vertices gives rise to either an *aa*-path, an *aba*-path or an *abcab*-path in G'. This is little more than a brute-force check – we need to confirm that a bipartite graph H on $U \times V$ with five edges, each labelled from W, has one of the required paths if $|U| + |V| + |W| \leq 9$.

It obviously suffices to check the case $|U| + |V| + |W| = 9$. If $|W| = 1$ then H must be a matching to avoid *aa*-paths, and so if H has five edges then $|U| + |V| \geq 10$. If $|W| \geq 5$ then $|U| + |V| \leq 4$ and so H cannot have more than four edges. So we may assume $2 \leq |W| \leq 4$.

If $|W| = 2$, then to avoid *aba*-paths and *aa*-paths H must be a union of components of size at most 2. The only non-trivial case (without loss of generality) is $|U| = 3$ and $|V| = 4$, and if H has five edges we find that two vertices of U must have degree 2 with disjoint neighbourhoods in V, meaning that the final fifth edge cannot exist without connecting the components.

If $|W| = 4$ we must have $|U| = 2$ and $|V| = 3$ without loss of generality. All but one edge is present. Therefore there is a vertex from U, say u_1, with all three vertices from V as neighbours. The other vertex from U, say u_2 has only two neighbours from V, say v_1 and v_2. The edges from v_1 and v_2 to u_2 must both be labelled with the label that u_1 does not see in order to avoid aa-paths through u_1. But this creates an aa-path through u_2.

The remaining cases are $|U| = |V| = |W| = 3$ and $|U| = 2, |V| = 4, |W| = 3$. The latter case is dealt with by observing that there must be a vertex u_1 that sees all three labels, and the neighbourhood of the other vertex $u_2 \in U$ intersects the neighbourhood of u_1 in at least one vertex, say v. But since u_2 sees two labels we get an aba-path with middle edge u_1v.

Now let us consider the case $|U| = |V| = |W| = 3$. If any vertex has degree 3 we will get an aa-path or an aba-path, so we may suppose that $U = \{u_1, u_2, u_3\}$ and that u_1 and u_2 have degree 2 and u_3 has degree 1. Clearly u_1 and u_2 must have a common neighbour v_1. Without loss of generality let u_1v_2 and u_2v_3 be edges of H. There is only one labelling of these four edges that avoids aa-paths and aba-paths, so we may assume without loss of generality that u_1v_2 and u_2v_3 have label w_1, u_1, v_1 has label w_2 and u_2v_1 has label w_3. Now there is one edge from u_3. It cannot go to v_1 without creating a vertex of degree 3, so without loss of generality we have the edge u_3v_2. This must be labelled w_3 to prevent aa-paths and aba-paths, but this leaves us with an $abcab$-path.

Putting this together, since G' is aa, aba and $abcab$-free we find that we have been able to use A to build a linear tripartite hypergraph G which is $(9, 5)$-free and has size $\mathcal{O}(n|A|)$. The result follows. $\square$

The above proposition can be regarded either as a strong motivation for trying to find a power-type improvement to the trivial bound for the $(9, 5)$ problem or as a lower bound on the difficulty of doing so.

5.5 The Generalized $[r, s]$ Problem

In this section we shall discuss the generalization to s-increasing or s-comparable sets of r-tuples, which we mentioned in the introduction.

In the introduction we gave a construction of an $[r, s]$-sequence of size $n^{r/s}$ and commented that by an easy probabilistic argument it is not generally sharp. Here is that argument in more detail.

Lemma 5.45. *Let n be fixed and let the ratio $\beta = s/r$ be fixed with $\beta < (1 - 1/n)/2$. Then $F_{r,s}(n)$ (and hence $G_{r,s}(n)$ also) grows exponentially with r.*

Proof. We use a simple first-moment argument, with modification. Let us choose a sequence of size S of r-tuples by selecting the digits of each tuple from $[n]$ uniformly and independently at random.

Then if we take two distinct tuples x_i and x_j (with $i < j$) from this collection, the number of coordinates in which the first is larger than the second is binomially distributed as the sum of r independent $\mathrm{Bin}((1-1/n)/2)$ distributions. The probability that x_i is not s-less than x_j is at most the probability this binomial distribution takes a value less than $s = \beta r$. This event is a binomial tail probability and consequently is exponentially small in r.

Therefore by taking S to be exponentially large, and by removing any x_i that forms part of a pair that do not have the appropriate s-increasing relation, we find an exponentially large (r, s)-sequence. $\square$

As discussed in the introduction, the trivial upper bound for the size of an s-comparable set of r-tuples from $[n]$ is n^{r-s+1}, and there is a natural construction of size $n^{r/s}$. In the earlier sections we concentrated on the problem of fixing r and s (as 3 and 2 respectively) and aimed to improve one or other of these bounds. This problem seems to be difficult, even for larger fixed values of r and s where the bounds can be very far apart indeed. We shall therefore concentrate on the regime where n is fixed and r and s vary but have a fixed ratio. We shall discuss the comparable version rather than the increasing version, but only since we have not found an interesting difference between the two problems in this regime.

Specifically, we will study the following problem.

Problem 5.46. *Let n be a fixed positive integer, and $0 < \beta < 1$ a fixed real number. Let $H_{n,\beta}(r)$ be the maximal size of an s-comparable collection of r-tuples, where $s = \beta r$. For fixed n and β, how does $H_{n,\beta}(r)$ grow with r?*

Lemma 5.45 tells us that if $\beta < (1-1/n)/2$, then $H_{n,\beta}(r)$ grows exponentially with r. We complement this lemma with the following result.

Proposition 5.47. *If $\beta > (1-1/n)/2$, then $H_{n,\beta}(r)$ is bounded.*

Before proving Proposition 5.47, we note a parallel with a problem concerning real vectors.

Problem 5.48. *Let $\eta \in [-1, 1]$ be a fixed real. Then how large may a collection $V = \{v_i\}$ of d-dimensional real unit vectors be if V has the property that for all $i \neq j$ we have $\langle v_i, v_j \rangle \leq \eta$?*

This problem is well understood [12]. In particular, it is known that when η is positive then the maximum size of V grows exponentially in d, while when η is negative the maximal size of V is bounded independently of d (by about $-\eta^{-1}$). When $\eta = 0$ then $|V| \leq 2d$, given by choosing an orthonormal basis $\{e_i\}$ and taking $V = \{e_i\} \cup \{-e_i\}$.

The parallels with Problem 5.46 are quite clear. In both problems we have a collection of objects constrained by some condition of pairs from the collection, and we have a parameter with a threshold value on one side of which the size of the collection may be exponentially large and on the other side of which the size of the collection is bounded. Moreover, the threshold is the expected value of the parameter when the two objects are chosen at random. It is tempting to conclude that Problem 5.46 can be tackled by cleverly identifying tuples of integers with vectors in a way that translates Problem 5.46 into Problem 5.48, but the authors were unable to find such a transformation.

Furthermore, there are some reasons to think that a transformation of this kind does not exist. In the unit-vectors problem, if we want to deduce that the size of V is bounded when η is negative, it is enough to assume not that *every* inner product is at most η, but merely that the *average* inner product is at most η. However, a similar weakening of the hypotheses for our problem in case (iii) is no longer sufficient for boundedness. Take, for example, the case $n = 3$, and for an arbitrary m take a collection of $3m$ r-tuples, where m of them are equal to $(1, 1, \ldots, 1)$, m of them are equal to $(2, 2, \ldots, 2)$ and m of them are equal to $(3, 3, \ldots, 3)$. Then if you choose two triples randomly from this collection, the average number of places where they differ is $2r/3$, which is significantly more than $(1-1/n)r/2$. It is easy to modify this example, if one wishes to, to make all the r-tuples distinct with a large value of m at only a small cost to the average.

It therefore appears that we are forced to use a more complicated argument in the proof of Proposition 5.47. We shall apply a dependent random selection argument to pass from a collection of r-tuples to a large subcollection that resembles one whose members have had their coordinates selected independently at random from some distribution that depends on the coordinate. In an example such as the above, this dependent random selection would tend to pick out a subset that consisted of multiple copies of the same sequence, which would then lead to a contradiction. In the general case, the calculation is more delicate but we obtain a similar contradiction if the number of r-tuples we start with is large enough.

We will begin by quoting three results from a preprint of the first author [34].

The first encompasses the dependent random choice aspect of the argument:

Theorem 5.49. *Let G be a bipartite graph with vertex sets X and Y of sizes m and n respectively and let δ, η, ϵ and γ be positive constants less than 1. Let $\delta_1(x, x')$ be the density of the shared neighbourhood of x and x'. Suppose that there are at least ϵm^2 pairs $(x, x') \in X^2$ such that $\delta_1(x, x') \geq \delta(1+\eta)^{1/2}$. Then there is a constant $\alpha \geq \delta$ and a subset $B \subset X$ of density at least $(\epsilon\gamma)^{8\eta^{-2}\log(\delta^{-1})^2}$ with the following two properties.*

1. *$\delta_1(x, x') \geq \alpha$ for all but at most $\gamma|B|^2$ pairs $(x, x') \in B^2$.*
2. *$\delta_1(x, x') \leq \alpha(1+\eta)$ for all but at most $\epsilon|B|^2$ pairs $(x, x') \in B^2$.*

The second is a straightforward lemma that translates between different formulations of quasirandomness. We import also the definition of the *box norm*, which is as follows:

$$\|f\|_\square^4 = \mathbb{E}_{x,x',y,y'} f(x,y)f(x,y')f(x',y)f(x',y').$$

Note that in the preprint [34] the box norm is referred to as the U_2 norm, written $\|.\|_{U_2}$. Recall that in a dense graph f we have $\|f\|_\square = \mathcal{O}(1)$.

Lemma 5.50. *Let X and Y be finite sets and let $f : X \times Y \to \{0,1\}$ be a bipartite graph. Let $\delta_1(x, x')$ be the density of the shared neighbourhood of $x, x' \in X$ and $\delta_2(y, y')$ be the density of the shared neighbourhood of $y, y' \in Y$. Let $\delta_2(y)$ be the density of the neighbourhood of $y \in Y$. Then the following are equivalent.*

(i) $\mathbb{E}_{x,x'}\big|\delta_1(x,x') - \|\delta_2\|_2^2\big|^2 \leq c_1\|f\|_\square^4$.

(ii) $\|f - 1 \otimes \delta_2\|_\square \leq c_2\|f\|_\square^4$.

The third is obtained from Lemma 5.50, and will be used to translate the quasirandomness into an applicable condition.

Lemma 5.51. *Let X, X' and Y be finite sets and let $f : X \times Y \to \{0,1\}$ and $f' : X' \times Y \to \{0,1\}$ be bipartite graphs. Let δ_1 be the density of the shared neighbourhood of its argument(s) as a subset of Y, let $\delta_2(y)$ be the density of the neighbourhood of $y \in Y$ as a subset of X and let $\delta_2'(y)$ be the density of the neighbourhood of $y \in Y$ as a subset of X'. Let $0 < c \leq 2^{-24}$, and suppose that*

$$\mathbb{E}_{x_1,x_2}\big|\delta_1(x_1,x_2) - \|\delta_2\|_2^2\big|^2 \leq c\|f\|_\square^4$$

and that

$$\mathbb{E}_{x_1',x_2'}\big|\delta_1(x_1',x_2') - \|\delta_2'\|_2^2\big|^2 \leq c\|f'\|_\square^4.$$

Then

$$\mathbb{E}_{x,x'}\big|\delta_1(x,x') - \langle\delta_1,\delta_2\rangle\big|^2 \leq 16c^{1/16}\|f\|_\square^2\|f'\|_\square^2.$$

Proof of Proposition 5.47. We prove the result by induction on n, with $n = 1$ being trivial.

Suppose we have a βr-comparable subset $S \subset [n]^r$. We will first pass to a subset T of S such that each tuple in T has entry i in at least $r/4n^2$ different positions. Indeed, suppose that there exists a subcollection $S' \subset S$ and an entry i such that every tuple from S' has entry i in fewer than $r/4n^2$ positions. Then, by replacing the entries i with $i+1$ (or $i-1$ if $i = n$) and relabelling so that the entries come from $[n-1]$, we get a collection of r-tuples with entries from $n-1$ and the r-tuples are pairwise more than

$$\frac{1-1/n}{2}r - 2r/4n^2 > \frac{1-\frac{1}{n-1}}{2}r$$

comparable. By our induction hypothesis the size of S' is therefore bounded independently of r. Therefore, for $|S|$ sufficiently large (independent of r) we can find the required subcollection T of size proportional to $|S|$ where the constant of proportionality is dependent on n but not on r.

We now consider the following set-up. We form a bipartite graphs $G_1 = X \times Y$ where X has one vertex for each tuple in T and $Y = [r]$, and the edge (x,k) is present in G_1 if the tuple corresponding to x has the entry 1 in the kth position. Write $\delta_1(x,x')$ for the density of the shared neighbourhood of x and x', and $\delta_2(k)$ for the density of the neighbourhood of $k \in Y$.

We will first apply Theorem 5.49 to get a certain quasirandomness property for the graph G_1.

Since the tuples in T have every possible entry at least $r/4n^2$ times, we see that the degree of each vertex $x \in X$ is at least $r/4n^2$. Consequently the average degree in Y is at least $|T|/4n^2$ and so the number of pairs x, x' from X that share a common neighbour is at least $|T|^2r/16n^4$. We thus find that at least $|T|^2/16n^4$ pairs $x, x' \in X$ have shared neighbourhood of size at least $r/16n^4$.

This allows us to apply Theorem 5.49 with $\epsilon \leq 1/16n^4$, $\delta(1+\eta)^{1/2} \leq 1/16n^4$ and $\gamma = \epsilon$ small. We thus find a constant $\alpha_1 \geq \delta$ and a subset B_1 of X of size proportional to $|X|$ (ie $|B_1| = \lambda(\epsilon,\gamma,\eta)|X|$ and λ is independent of r) such that

$$\alpha_1 \leq \delta_1(x,x') \leq (1+\eta)\alpha_1$$

for all but at most $2\epsilon|B_1|^2$ pairs $(x,x') \in B_1^2$.

Now we can define G_2 to be the bipartite graph $G_2 = X \times Y$ where X has one vertex for each tuple in B and $Y = [r]$, and the edge (x, k) is present in G_2 if the tuple corresponding to x has the entry 2 in the kth position. We can repeat the above argument to find a proportionally sized $B_2 \subset B_1$ such that

$$\alpha_2 \leq \delta_1(x, x') \leq (1+\eta)\alpha_2$$

for all but at most $2\epsilon'|B_1|^2$ pairs $(x, x') \in B_1^2$. By taking ϵ much smaller than ϵ' we can ensure that in G_1 we also have

$$\alpha_1 \leq \delta_1(x, x') \leq (1+\eta)\alpha_1$$

for all but at most $2\epsilon'|B_2|^2$ pairs $(x, x') \in B_2^2$.

Continuing this for all of the n graphs G_i defined to continue the obvious pattern of G_1 and G_2 above, we can eventually find (for any $\eta, \mu > 0$) a subset B of the set of tuples of size $\lambda(\mu, \eta)|S|$ and constants $\alpha_i > 0$ such that in the graph G_i

$$\alpha_i \leq \delta_1(x, x') \leq (1+\eta)\alpha_i$$

for all but at most $\mu|B|^2$ pairs $(x, x') \in B^2$.

This tells us that simultaneously all the graphs H_i, defined by taking the induced subgraph of G_i on $B \times [r]$, are in a certain unbalanced sense quasirandom. More precisely, since our graphs have the property that almost all pairs of vertices $x, x' \in B$ have shared neighbourhoods of approximately the same size, the LHS of condition (i) from Lemma 5.50 is small. Therefore Lemma 5.50 tells us that the H_i are quasirandom permutations of the rank 1 matrix $1 \otimes \delta_2$. These can be thought of as behaving like random bipartite graphs with a given degree sequence.

We will now apply Lemma 5.51, which translates the quasirandomness into an applicable condition. For distinct i and j we view the graphs H_i and H_j taking $H_i = X \times Y$ with vertices in X corresponding to the tuples in B and $Y = [r]$ and $H_j = X' \times Y$ with vertices in X' also corresponding to the tuples in B. Let $\delta_{i,j}(x, x')$ be the density of the shared neighbourhood of $x \in X$ and $x' \in X'$. By applying Lemma 5.51 we find that for any distinct i, j we have that for almost all pairs $(x, x') \in X \times X'$ the shared neighbourhood $\delta_{i,j}(x, x')$ has density approximately $\mathbb{E}_{x,x'}\delta_i(x)\delta_j x'$, where $\delta_i(x)$ is defined to be the density of the neighbourhood of vertex x in H_i (which is also the density of the number of positions in which the tuple corresponding to x has entry i). Specifically, we have

$$\mathbb{E}_{x,x'}\left|\delta_{i,j}(x,x') - \mathbb{E}_{x''}\delta_i(x'')\delta_j(x'')\right| < \theta$$

where θ can be made arbitrarily small provided B is sufficiently large.

Observe that for any pair (x, x') of tuples from B we must either have $x <_{\beta r} x'$ or $x' <_{\beta r} x$ since B is βr-comparable. Therefore we must have

$$\mathbb{E}_{x \neq x'}\left(\sum_{i<j} \delta_{i,j}(x,x')\right) \geq \beta$$

and therefore

$$\sum_{i<j} \mathbb{E}_x \delta_i(x)\delta_j(x) \geq \beta - \theta n^2$$

which gives

$$\mathbb{E}_x\left(\sum_{i<j} \delta_i(x)\delta_j(x)\right) \geq \beta - \theta n^2.$$

Now we note that we also must have

$$\sum_i \delta_i(x) = 1$$

for all tuples $x \in B$. It is an easy exercise to show that

$$\sum_{i<j} \delta_i(x)\delta_j(x)$$

is maximized subject to the constraint

$$\sum_i \delta_i(x) = 1$$

when $\delta_i(x) = 1/n$ for all i. Therefore

$$\mathbb{E}_x\left(\sum_{i<j} \delta_i(x)\delta_j(x)\right) \leq \frac{n(n-1)}{2n^2}$$

which implies that

$$\beta - \theta n^2 \leq \frac{n(n-1)}{2n^2} = \frac{1-\frac{1}{n}}{2}.$$

But if $\beta > (1 - 1/n)/2$ then by making θ sufficiently small we have a contradiction. So it must be that we cannot make θ arbitrarily small, and so $|S|$ is bounded independently of r. $\square$

We have not attempted to obtain an explicit bound on the dependence of $H_{n,\beta}(r)$ on $(1-1/n)/2-\beta$. If we were concerned to find as good a bound as possible, then instead of using Theorem 5.49 iteratively it would be more efficient to prove directly a version of the theorem that works for n bipartite graphs simultaneously.

The final case left to consider is the threshold $\beta = (1-1/n)/2$. Given the parallel with the vector problem, we expect the size of the collection in this threshold case to be unbounded but sub-exponential – perhaps even only linear in s.

We have not managed to prove this, but in the other direction it is not too difficult to find a linear-sized construction, at least when n is a prime power.

Proposition 5.52. *Let q be a fixed prime power and let $\beta = (1-1/q)/2$. Let $r = q^k$ and $s = \beta r$. Then there exists an s-increasing collection of r-tuples from $[q]$ of size $q^{k+1} = qr$.*

Proof. Let $\mathbb{F}_q$ be the field with q elements and consider the set of all affine functions from $\mathbb{F}_q^k$ to $\mathbb{F}_q$, that is to say functions f of the form $f : x \mapsto ax + b$ for $a \in \mathbb{F}_q^k$ and $b \in \mathbb{F}_q$, viewed in the obvious way as a collection of sequences of length q^k with entries from $[q]$.

Note that two distinct affine functions agree on a subspace of codimension 1 or disagree everywhere, so any distinct pair of sequences from this collection agree in at most q^{k-1} places. Additionally, we see that if two sequences of length q^k from $[q]$ agree in at most q^{k-1} places then they are certainly $(q^k - q^{k-1})/2$-comparable, and the proposition follows by taking our collection to be the set of affine functions considered above. $\square$

The remaining goal, therefore, is to establish a sub-exponential bound when $\beta = (1-1/n)/2$. This would establish the desired trichotomy and mirror the behaviour of Problem 5.48. This appears to be more difficult than the corresponding vector question, and is another possible direction for future work.

5.6 Conclusions and future directions

This chapter has raised more questions than it has answered, but we hope that we have given convincing motivation for a rich collection of related and surprisingly challenging problems.

				4	5	
					1	2
	5					
5		7				
		2	6			
		1		3		
3						
1	2		4			

		5				
			4			
5	6				8	
			1	8		
2				3		
	1	4				7
1		3			7	

				3			
			4				
				1	2		6
4							5
	1	5					
2			3			7	
1		3				6	

Figure 5.4: A counterexample to the decomposition conjecture. It is a 2-increasing sequence of 15 triples from $[7] \times [7] \times [8]$. We give it in grid representation for each of the three possible choices of label coordinate.

The development that we would most like to see is an improvement to the power bound that we achieve in Theorem 5.18, ideally to a bound of $n^{3/2}$. If this lower bound is indeed sharp, one would expect it to be provable by a clean inductive argument, but we have had trouble making this work.

To give an idea of the difficulty, we will describe one possible approach along these lines. Say that a 2-increasing collection of triples T in $[r] \times [s] \times [t]$ has a *decomposition* if we can pick a coordinate, say t without loss of generality, such that the following holds. We can find a partition of $[r] \times [s]$ into sets $R_i \times S_i$ where $R_i \subset [r]$ and $S_i \subset [s]$ for all i such that there exists a partition $[t]$ into sets T_i in such a way that all the triples of T lie in the sets $R_i \times S_i \times T_i$.

Suppose that all 2-increasing collections of triples have a decomposition. Then we could use induction to bound the total number of triples by

$$\sum_i (|R_i||S_i||T_i|)^{1/2},$$

which by Cauchy-Schwarz is at most $(rst)^{1/2}$, and the problem would be solved.

It is very tempting to conjecture that a decomposition always exists, since no counterexample is easily found by hand and it would also provide a natural proof of the conjectured power bound for 2-increasing sequences. However, we eventually came across a counterexample, which is given in its three grid representations in Figure 5.4.

It seems to be hard to find such counterexamples. The example above was found with the help of a computer search. Very briefly, the algorithm we used works as follows. It builds up a 2-increasing sequence triple by triple, and at each step it randomly chooses a minimal triple (in the usual partial order on $[n]^3$) that is 2-greater than all the triples chosen so far, halting when it runs out of possibilities. Then it checks for decomposability.

The check can be done in polynomial time quite easily. Given a grid representation, we can decompose it in the desired way if we can find a non-trivial partition of the grid into Cartesian products such that no label appears in more than one of the cells of the partition. If two labels occur in some row and also in some column, then the Cartesian product that contains one is forced to contain the other. So the algorithm replaces these two labels by a single label and iterates. If it ends up with just one label, that proves that the example is not decomposable using the label coordinate (and the converse holds too). It then carries out the test for each choice of label coordinate.

Our experiments with this program seem to indicate that indecomposable examples are quite rare, but this may simply be because we have not yet found the right model for random 2-increasing sequences. We have experimented with adding conditions such as choosing at each stage a minimal triple that satisfies an additional condition. When the program chooses a purely random minimal triple, we stumbled on an example with $n = 20$. (More precisely, we stumbled on an example that was almost indecomposable, and could be made indecomposable by removing a few triples.) This happened only once, and seems to have been quite lucky. Adding the additional condition that the sum of squares of the coordinates is minimized led to the example above – in this case we set $n = 8$ and removed one triple from the randomly generated sequence.

For large n, these random models seem to create 2-increasing sequences of size about $2n$, apart from one model that looks as though it is growing at a rate more like $n^{4/3}$. That model is to take a minimal triple at each stage but to maximize its smallest coordinate (and to make the choice randomly in the case of ties).

For all the examples we know of 2-increasing sequences that attain the bound $(rst)^{1/2}$ it *is* possible to find a decomposition of the kind that could be used for an inductive proof. That leads to the following more precise conjecture.

Conjecture 5.53. *Let T be a 2-increasing sequence in $[r] \times [s] \times [t]$. Then $|T| \leq (rst)^{1/2}$, and equality holds only if there is a decomposition of the kind discussed above for some choice of label coordinate.*

The non-decomposable example given above not extremal, since 15 is quite a bit smaller than $(7 \times 7 \times 8)^{1/2}$, so it does not disprove this conjecture. However, it shows that in order to prove the existence of a decomposition, it is necessary to use the extremality somehow, and it is not obvious how to do that. (This is why we felt that Lemma 5.30 could turn out to be important.)

It is possible to go one step further than Conjecture 5.53 in the hope of classifying all 2-increasing sequences of length $(rst)^{1/2}$. For this purpose we tentatively formulate the following conjecture, which has survived some small-scale searches for counterexamples.

Conjecture 5.54. *Let T be a 2-increasing sequence in $[r] \times [s] \times [t]$. Then $|T| \leq (rst)^{1/2}$, and equality holds if and only if it can be built up as follows:*

1. *Choose a coordinate, say the third without loss of generality, and partition $[r] \times [s]$ into sets $R_i \times S_i$ where all of the R_i and S_i are intervals. Using the obvious ordering on disjoint intervals (and calling intervals incomparable if they intersect), ensure that the rectangles $R_i \times S_i$ are ordered in a 1-increasing fashion.*

2. *Partition $[t]$ into an increasing set of disjoint intervals T_i such that $|R_i||S_i|$ is proportional to T_i (so that we get equality when we apply the Cauchy-Schwarz inequality).*

3. *Put extremal examples into the sets $R_i \times S_i \times T_i$.*

4. *If it is possible to permute two rows, columns or labels while preserving the 2-increasing property (with a different order) then feel free to do so.*

Note that the fourth operation above is necessary for the conjecture to be true. An example that shows why is the sequence given by grid representation

$$\begin{array}{|c|c|c|}
\hline
 & & 2 \\ \hline
 & 2 & \\ \hline
 & & 1 \\ \hline
2 & & \\ \hline
 & 1 & \\ \hline
1 & & \\ \hline
\end{array}$$

which will not decompose using just the first three operations.

It would also be extremely interesting to obtain a non-trivial power-type upper bound for the 2-comparable problem, especially now we know that $n^{3/2}$ is not the right lower bound. Another reason for being interested in this problem is that, as we have shown, it is closely related to some central and quite long-standing problems in extremal hypergraph theory and additive combinatorics.

Finally, there are many interesting generalizations of Loh's original problem, from the minimalist variants and extremal hypergraph problems described in

Section 5.4 to the generalized $[r, s]$ problems studied in the previous section. Many of the resulting questions are not yet answered, and some of them look as though they may be approachable. Perhaps the most annoying question to which we do not know the answer is the following.

Question 5.55. *Is there a single pair of integers $1 \leq s < r$ for which the trivial upper bound of n^{r-s+1} for the size of the largest s-comparable subset of $[n]^r$ can be improved by a non-trivial power of n?*

Chapter 6

Partition problems in high dimensional boxes

This chapter is based on joint work with M. Bucic, B. Lidický and A. Z. Wagner [16], to appear in the Journal of Combinatorial Theory (Series A).

6.1 Introduction

The following lovely problem, due to Kearnes and Kiss [50, Problem 5.5], was presented at the open problem session at the August 1999 meeting at MIT that was held to celebrate Daniel Kleitman's 65th birthday [70]. Let a set of the form

$$A = A_1 \times A_2 \times \ldots \times A_d,$$

where $A_1, A_2, \ldots, A_d$ are finite sets with $|A_i| \geq 2$ be called a *d-dimensional discrete box*. A set of the form $B = B_1 \times B_2 \times \ldots \times B_d$, where $B_i \subseteq A_i$ for all $i \in [d]$, is a *sub-box* of A. Such a sub-box B is said to be *proper* if $B_i \neq A_i$ for every i. The question of Kearnes and Kiss is as follows: can the box $A = A_1 \times A_2 \times \ldots \times A_d$ be partitioned into fewer than 2^d proper sub-boxes?

Within a day, Alon, Bohman, Holzman and Kleitman solved [3] this problem. Their eventual distillation of the proof, which we present in Section 6.2, is a "proof from the book".

Theorem 6.1 (Alon, Bohman, Holzman, Kleitman)**.** *Let A be a d-dimensional discrete box, and let $\{B^1, B^2, \ldots, B^m\}$ be a partition of A into proper sub-boxes. Then $m \geq 2^d$.*

The following interesting question was recently posed by Leader, Milićević

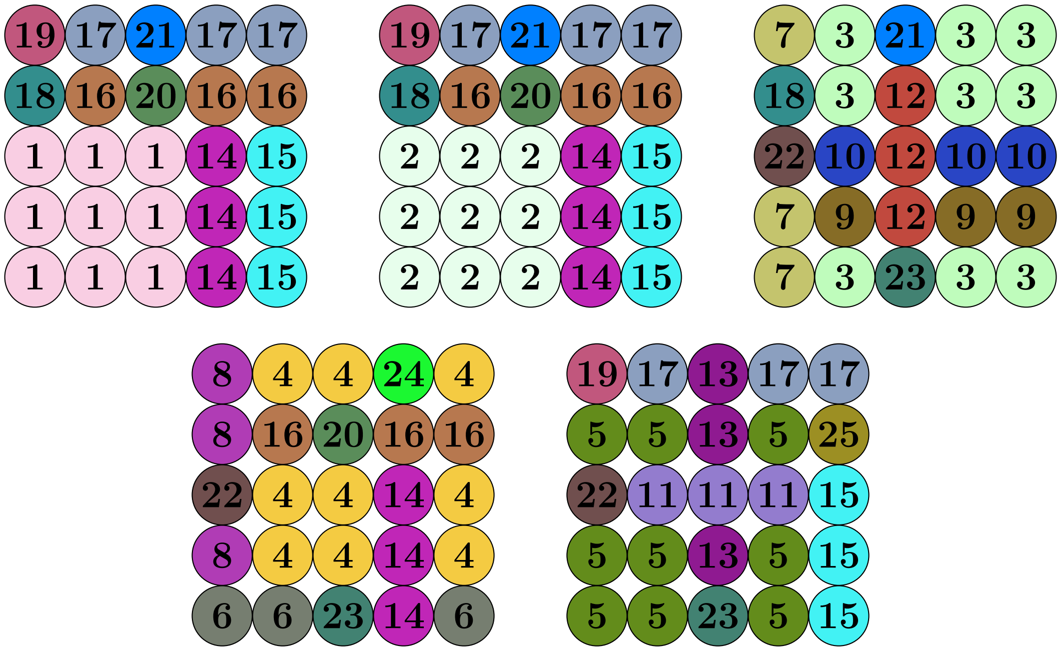


Figure 6.1: 25 odd boxes partitioning $[5]^3$.

and Tan [54]. Say that the d-dimensional box $A = A_1 \times A_2 \times \ldots \times A_d$ is *odd* if each $|A_i|$ is odd (and finite). It is easy to see that given a d-dimensional odd box A, there exists a partition of A into 3^d odd proper sub-boxes, by partitioning each side into three odd parts and taking all possible products.

Question 6.2 (Leader, Milićević, Tan [54])**.** *Let A be a d-dimensional odd box, and let $\{B^1, B^2, \ldots, B^m\}$ be a partition of A into odd proper sub-boxes. Does it follow that then $m \geq 3^d$?.*

Our first result is that the answer to this question is 'no', in general.

Theorem 6.3. *Let $d \in \mathbb{Z}^+$ be divisible by 3. Then there exists a partition of $[5]^d$ into $25^{d/3} \leq 2.93^d$ odd proper sub-boxes.*

The proof is based on an example which shows that it is possible to partition $[5]^3$ into 25 odd proper sub-boxes, see Figure 6.1. The example presented here is based on examples found by computer search, but has been tidied up by hand. The solution is not unique.

The situation changes, however, if we require the odd boxes in our partition to be products of intervals. Here we identify the sets indexing our starting box with $[n]$. We say that the box $B = B_1 \times B_2 \times \ldots \times B_d$ is a *brick* if for each $i \in \{1, 2, \ldots, d\}$ there exist integers $i_0 \leq i_1$, such that $B_i = \{i_0, i_0 + 1, \ldots, i_1\}$. For example:

- The set $B = \{2,3,4\} \times \{4\} \times \{1,6,7\}$ is an odd proper sub-box of $[7]^3$ but it is not a brick, as $\{1,6,7\}$ does not have the required form.
- The set $B = \{2,3,4\} \times \{3,4\}$ is a proper brick contained in $[5]^2$. However, it is not odd, as $|\{3,4\}| = 2$.

Our next result shows that the answer to Question 6.2 is 'yes' under the additional assumption that the sub-boxes are in fact proper, odd bricks.

Theorem 6.4. *Let $n \geq 2$ be odd, and $d \geq 1$ be an arbitrary integer. Let $\{B^1, B^2, \ldots, B^m\}$ be a partition of $[n]^d$ into proper, odd bricks. Then $m \geq 3^d$.*

There are a number of natural generalisations of Question 6.2, and we shall consider several of them in this chapter.

We start by weakening of the parity constraint. A key property enforced by a partition into odd, proper boxes is that any axis-parallel line through $[n]^d$ intersects at least 3 distinct sub-boxes, with the result that the most obvious construction involves dividing each dimension into 3 parts and taking the resulting 3^d sub-boxes. We therefore pose the following natural question, which we refer to as the *k-piercing* problem.

Question 6.5 (k-piercing)**.** *Let $n \geq k$ and $d \geq 1$, and let $\{B^1, B^2, \ldots, B^m\}$ be a partition of $[n]^d$ into proper boxes with the property that every axis-parallel line intersects at least k distinct B_i (we call this the* k-piercing property*). How small can m be?*

This question can obviously be phrased in a continuous setting, eliminating n by replacing $[n]$ with the interval $[0,1]$. For simplicity we shall not do this, but instead we will generally present bounds on m as a function of k and d only by considering sufficiently large n (for most of our results it is sufficient to take $n > 3k$). Note that somewhat similar problems, in some sense dual to piercing partitions, were considered in two earlier papers [53, 1].

The 2-piercing problem is precisely equivalent to the original problem of Kearnes and Kiss, and so the bound $m \geq 2^d$ holds. However, Theorem 6.3 tells us that $m < 3^d$ when $k = 3$. In fact the easy observation that 3^d cannot be a lower bound follows from a simple 2-dimensional construction shown in Figure 6.2.

Our later results will concentrate on the k-piercing problem. We show, perhaps surprisingly, that m is bounded by $c^d k$ for some c which is independent of k.

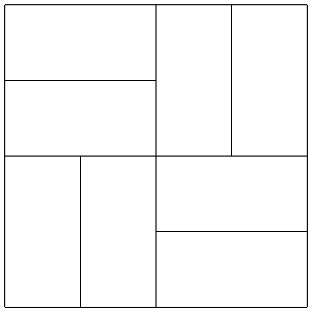


Figure 6.2: 8 bricks in two dimensions satisfying the 3-piercing property.

Theorem 6.6. *Let $k \geq 2$ and $d \geq 1$ be integers. For sufficiently large n there exists a partition $\{B^1, \ldots, B^m\}$ of $[n]^d$ into proper boxes having the k-piercing property with $m \leq 15^{d/2}k$.*

Recall that the answer to Question 6.2 changes fundamentally when boxes are replaced with bricks, with the trivial construction becoming best possible. In light of this, we also consider the special case of Question 6.5 when all the boxes are assumed to be bricks. We obtain a similar result, even under this additional restriction.

Theorem 6.7. *Let $k \geq 2$ and $d \geq 1$ be integers. For sufficiently large n there exists a partition $\{B^1, \ldots, B^m\}$ of $[n]^d$ into proper bricks having the k-piercing property with $m \leq 3.92^d k$.*

Both proofs will involve constructing an intermediate partition from a low-dimensional example and then solving a smaller instance of the same problem within each part. It seems almost certain that better examples exist, and in fact it is not out of the question that $m = (2 + o(1))^d$ for every fixed k, in both regimes.

For the lower bounds, there is a simple inclusion-exclusion argument which shows $m \geq d2^{d-1}(k-2) + 2^d$, but this only applies for bricks. With boxes, lower bounds are difficult to obtain, as neither the argument mentioned above nor the one used to prove Theorem 6.1 seem to extend to this problem. In fact, we fail to obtain any lower bound of the form $(1 + \varepsilon)^d k$ for any $\varepsilon > 0$. Such a bound almost certainly holds, and this presents a very interesting open problem.

In this setting, even the 2-dimensional case is not easy to resolve. The upper bound of $m \leq 4k - 4$ follows from Figure 6.3a and is easily seen to be tight in the case of bricks. With the aim of showing that this is best possible even for boxes, we introduce a graph theory question of an extremal flavour and solve it asymptotically.

Proposition 6.8. *Let $n \geq 2k-2$ be integers, and let $\{B^1, \ldots, B^m\}$ be a partition of $[n]^2$ into proper sub-boxes satisfying the k-piercing property with m minimal. Then $m = (4 + o_k(1))k$.*

This chapter is organized as follows. In Section 6.2 we give some set-up and preliminary observations. In Section 6.3 we prove Theorem 6.3 and Theorem 6.4. In Section 6.4 we consider the k-piercing problem and present our results, including Theorem 6.6, Theorem 6.7 and Proposition 6.8. A selection of open questions are given in Section 6.5.

Before beginning with the set-up for our investigations, we draw attention to other variants of the problem which have been considered in the literature, including geometrical results as in [41] concerning the minimal partitions obtained in Theorem 6.1, and extensions of these ideas in the context of cube tiling [51].

6.2 Set-up and previous results

We begin this section by recounting the proof of Alon, Bohman, Holzman and Kleitman of Theorem 6.1, as presented in [70].

Proof of Theorem 6.1. Let $A = [n]^d$ be a d-dimensional discrete box and let $\{B^1, B^2, \ldots, B^m\}$ be a partition of A into proper sub-boxes. Let O be a an odd-sized sub-box of A (meaning that each component of O has odd size, or equivalently that O has odd volume) picked uniformly at random.

Observe that, for a fixed i, the probability that the intersection $O \cap B^i$ has odd size is equal to 2^{-d}, since the probability that the intersection of a random odd subset of $[n]$ and a fixed proper subset of $[n]$ has odd size is precisely $1/2$. It follows that the expected number of B^i whose intersection with O has odd size is $m2^{-d}$.

But since $\{B^1, B^2, \ldots, B^m\}$ partitions A and O has odd size, at least one box B^i must intersect O in an odd number of places. Therefore $m2^{-d} \geq 1$ and the result follows. $\square$

Let $f_{\text{odd}}(n, d)$ denote the number of odd proper sub-boxes required to partition the box $[n]^d$. It follows easily from Theorem 6.1 that whenever $n \geq 2$ is even we have $f_{\text{odd}}(n, d) = 2^d$, and so we shall always assume that the first argument of f_{odd} is odd. Using this notation, Theorem 6.3 states simply that if d is divisible by 3, then $f_{\text{odd}}(5, d) \leq 25^{d/3}$.

Observe first that if $m \geq n$ are odd integers, and $\mathcal{B}$ is a partition of $[n]^d$ into odd proper sub-boxes, then one can obtain a partition of $[m]^d$ into $|\mathcal{B}|$ odd proper sub-boxes by identifying the element $\{n\}$ with the interval $\{n, n+1, \ldots, m\}$. Hence, if $2 < n \leq m$ are odd integers and $d \geq 1$, then

$$f_{\text{odd}}(n, d) \geq f_{\text{odd}}(m, d). \tag{6.1}$$

Note that if $\mathcal{B}_1$ and $\mathcal{B}_2$ are partitions of $[n]^{d_1}$ and $[n]^{d_2}$ respectively into odd boxes, then $\mathcal{B}_1 \times \mathcal{B}_2$ is a partition of $[n]^{d_1+d_2}$ into $|\mathcal{B}_1| \cdot |\mathcal{B}_2|$ odd boxes. Hence the function f_{odd} satisfies

$$f_{\text{odd}}(n, d_1 + d_2) \leq f_{\text{odd}}(n, d_1) \cdot f_{\text{odd}}(n, d_2) \tag{6.2}$$

for all $n \geq 2$ and $d_1, d_2 \geq 1$. Since by Theorem 6.1 we have that $f_{\text{odd}}(n, d) \geq 2^d$ for all n, d, Fekete's lemma [27] can be applied. It follows that for every $n \geq 2$, there exists a non-negative constant α_n depending only on n, such that $f_{\text{odd}}(n, d) = (\alpha_n + o_d(1))^d$, where $o_d(1) \to 0$ as $d \to \infty$.

By inequality (6.1) the sequence $(\alpha_n)_{n \in \mathbb{N}}$ is monotone decreasing. An interesting open question is to decide whether the limit of the subsequence $(\alpha_{2k+1})_{k \in \mathbb{N}}$ is equal to 2 – see Section 6.5 for more details.

The above considerations will apply to the k-piercing problem as well. Let us denote by $p_{\text{box}}(n, d, k)$ the answer to Question 6.5 and by $p_{\text{brick}}(n, d, k)$ the answer to the same question, but restricted to bricks. Note also that $p_{\text{box}}(n, d, k) \leq p_{\text{brick}}(n, d, k) \leq k^d$, which follows since both $p_{\text{box}}(n, d, k)$ and $p_{\text{brick}}(n, d, k)$ satisfy inequality (6.2). Let $p_{\text{box}}(d, k) = \lim_{n \to \infty} p_{\text{box}}(n, d, k)$ and $p_{\text{brick}}(d, k) = \lim_{n \to \infty} p_{\text{brick}}(n, d, k)$, which exist since both $p_{\text{box}}(n, d, k)$ and $p_{\text{brick}}(n, d, k)$ are decreasing in n. Furthermore, since they are bounded by k^d we know that for sufficiently large n both functions become constant, equal to $p_{\text{box}}(d, k)$ and $p_{\text{brick}}(d, k)$, respectively. With this in mind, from this point onward in this chapter we are always going to tacitly assume in piercing problems that n is large enough compared to d and k.

Notice that this implies that both $p_{\text{box}}(d, k)$ and $p_{\text{brick}}(d, k)$ satisfy (6.2) so Fekete's lemma implies that there exist β_k and γ_k such that $p_{\text{box}}(d, k) = (\beta_k + o_d(1))^d$ and $p_{\text{brick}}(d, k) = (\gamma_k + o_d(1))^d$. We will show in Theorem 6.6 that $\beta_k \leq 15^{1/2} \approx 3.87$ for all k, and in Theorem 6.7 that $\gamma_k \leq 3.92$.

The case of $k = 2$ is resolved completely by Theorem 6.1 as there is a trivial partition into 2^d bricks, by splitting the original box into two parts along each dimension, implying $p_{\text{brick}}(d, 2) \leq 2^d$. On the other hand, a partition being 2-

piercing is equivalent to it consisting only of proper boxes, so Theorem 6.1 implies that $2^d \le p_{\text{box}}(d,2)$. In particular, this implies a very surprising result that for $k=2$ the answer is the same for boxes and bricks: $p_{\text{box}}(d,2) = p_{\text{brick}}(d,2) = 2^d$.

6.3 Partitioning into odd boxes

We start with proving the upper bound given in Theorem 6.3.

Proof of Theorem 6.3. By inequality (6.2), it suffices to show that $f_{\text{odd}}(5,3) \le 25$. That is, we seek a partition of $[5]^3$ into 25 proper odd boxes. This partition can be seen in Figure 6.1. The list of the coordinates of the 25 boxes can be found in Appendix 6.A.

This solution was found by phrasing the problem as an integer program, with one (Boolean) variable for every possible odd sub-box, and one constraint per coordinate saying that the sum of variables that correspond to boxes which contain this point is one. We then used Gurobi [43], a commercially available solver, to find the counterexample. □

We now turn to lower bounds, starting with the easy observation that for each fixed n we have $\alpha_n > 2$.

Proposition 6.9. *Let $n > 2$ be odd, and $d \ge 1$. Then*

$$f_{\text{odd}}(n,d) \ge \left(2 + \frac{1}{2^{n-2}-1}\right)^d .$$

Proof. The assertion follows from a straightforward modification of the proof of Alon, Bohman, Holzman and Kleitman of Theorem 6.1. We simply take the sets R_i to be uniformly chosen at random amongst *proper*, odd-sized subsets of $[n]$. That is, R_i is a uniformly random element of the set $\{S \subset A : S \neq A \text{ and } |S| \text{ is odd}\}$. Define X_j, X and R as in the proof of Theorem 6.1 and note that

$$\mathbb{E}(X_j) = \mathbb{P}\left(|B^j \cap R| \text{ is odd}\right) = \left(\frac{2^{n-2}-1}{2^{n-1}-1}\right)^d .$$

As before we have that $X \ge 1$ with probability 1, hence $\mathbb{E}(X) = m\mathbb{E}(X_j) \ge 1$. After rearranging, this gives the required result. □

Note that Proposition 6.9 simply says that $\alpha_n \ge 2 + \frac{1}{2^{n-2}-1}$ for all odd n, but this sequence of lower bounds on the α_n converges to 2.

We will now consider the case where the members of our partition are proper, odd bricks. The idea behind the proof of Theorem 6.4 is to remove the 'top' and 'bottom' layers of a partition and prove that the number of remaining bricks has to be large, since their projection onto the first $d-1$ layers forms a partition of a $(d-1)$-dimensional odd box. While this does not directly work as stated, this proof method can be fixed by considering a stronger induction hypothesis.

Proof of Proposition 6.4. We will in fact prove the stronger claim that if $\mathcal{B} = \{B^1, B^2, \ldots, B^m\}$ is a set of odd proper bricks that cover every element of $[n]^d$ an odd number of times, then $m \geq 3^d$. We proceed by induction on d.

Recall that $n \geq 2$ is odd, $d \geq 1$ is an arbitrary (fixed) integer, and suppose $\mathcal{B}$ is as above. For any brick $B \in \mathcal{B}$ let $B = B_1 \times \cdots \times B_d$, where B_i are odd length intervals. Let $\mathcal{C}, \mathcal{D} \subset \mathcal{B}$ be defined as

$$\mathcal{C} = \left\{ B^i : B^i \cap \left(\underbrace{[n] \times [n] \times \ldots \times [n]}_{d-1} \times \{1\} \right) \neq \emptyset \right\},$$

and

$$\mathcal{D} = \left\{ B^i : B^i \cap \left(\underbrace{[n] \times [n] \times \ldots \times [n]}_{d-1} \times \{n\} \right) \neq \emptyset \right\}.$$

Note that $\mathcal{C} \cap \mathcal{D} = \emptyset$, as all the B^i are proper bricks. Moreover, as elements of $\mathcal{C}$ cover every point of $[n]^{d-1} \times \{1\}$ an odd number of times, by induction we have $|\mathcal{C}| \geq 3^{d-1}$, and similarly $|\mathcal{D}| \geq 3^{d-1}$. It remains to show that $|\mathcal{B} \setminus (\mathcal{C} \cup \mathcal{D})| \geq 3^{d-1}$.

For every point $(i_1, i_2, \ldots, i_d) \in [n]^d$ and any family of bricks $\mathcal{E}$, denote the number of bricks in $\mathcal{E}$ that contain $\{i_1\} \times \{i_2\} \times \ldots \times \{i_d\}$ by $x_{i_1, i_2, \ldots, i_d}(\mathcal{E})$, and note that by assumption $x_{i_1, i_2, \ldots, i_d}(\mathcal{B})$ is odd for all choices of the i_j.

For all $(i_1, i_2, \ldots, i_{d-1}) \in [n]^{d-1}$ define the quantity

$$y_{i_1, i_2, \ldots, i_{d-1}} = \sum_{j=1}^{n} x_{i_1, i_2, \ldots, i_{d-1}, j}(\mathcal{B} \setminus (\mathcal{C} \cup \mathcal{D})),$$

and note that $y_{i_1, i_2, \ldots, i_{d-1}}$ is odd for all choices of $i_1, \ldots, i_{d-1}$. Indeed, as $\mathcal{C} \cap \mathcal{D} = \emptyset$ we have that

$$y_{i_1, i_2, \ldots, i_{d-1}} = \sum_{j=1}^{n} x_{i_1, i_2, \ldots, i_{d-1}, j}(\mathcal{B}) - \sum_{j=1}^{n} x_{i_1, i_2, \ldots, i_{d-1}, j}(\mathcal{C}) - \sum_{j=1}^{n} x_{i_1, i_2, \ldots, i_{d-1}, j}(\mathcal{D})$$

$$
\begin{aligned}
&= \sum_{j=1}^{n} x_{i_1,i_2,\ldots,i_{d-1},j}(\mathcal{B}) - \sum_{C\in\mathcal{C}}\sum_{j=1}^{n} \mathbb{1}((i_1,i_2,\ldots,i_{d-1},j)\in C) \\
&\qquad - \sum_{D\in\mathcal{D}}\sum_{j=1}^{n} \mathbb{1}((i_1,i_2,\ldots,i_{d-1},j)\in D) \\
&= \sum_{j=1}^{n} x_{i_1,i_2,\ldots,i_{d-1},j}(\mathcal{B}) - \sum_{C\in\mathcal{C}(i_1,i_2,\ldots,i_{d-1})} |C_d| - \sum_{D\in\mathcal{D}(i_1,i_2,\ldots,i_{d-1})} |D_d|,
\end{aligned}
$$

where $\mathbb{1}(\cdot)$ denotes the indicator function of an event and $\mathcal{C}(i_1, i_2, \ldots, i_{d-1})$ denotes the set of those $C \in \mathcal{C}$ where $(i_1, i_2, \ldots, i_{d-1}) \in C_1 \times C_2 \times \ldots \times C_{d-1}$ (and similarly for $\mathcal{D}(i_1, i_2, \ldots, i_{d-1})$). Now as C_d, D_d are odd size intervals each term in the above sums is odd, so the total is odd.

Consider the projection of the bricks in $\mathcal{B} \setminus (\mathcal{C} \cup \mathcal{D})$ onto the first $d-1$ coordinates and note that it induces an odd cover of a $(d-1)$-dimensional odd cube, as follows. For any brick $B \in \mathcal{B}\setminus(\mathcal{C}\cup\mathcal{D})$ define $\pi(B) := B_1 \times B_2 \times \ldots \times B_{d-1}$ to be the projection of the box B onto the first $d-1$ coordinates. For all $(i_1, i_2, \ldots, i_{d-1}) \in [n]^{d-1}$ define the quantity

$$
z_{i_1,i_2,\ldots,i_{d-1}} = \sum_{B\in\mathcal{B}\setminus(\mathcal{C}\cup\mathcal{D})} \mathbb{1}\left((i_1,i_2,\ldots,i_{d-1})\in\pi(B)\right).
$$

Observe that

$$
z_{i_1,i_2,\ldots,i_{d-1}} \equiv y_{i_1,i_2,\ldots,i_{d-1}} \pmod 2
$$

for all choices of coordinates, and hence all of the $z_{i_1,i_2,\ldots,i_{d-1}}$ are odd. Since the set of bricks

$$
\{\pi(B) : B \in \mathcal{B} \setminus (\mathcal{C} \cup \mathcal{D})\}
$$

form a cover of $[n]^{d-1}$ in which each point is covered $z_{i_1,i_2,\ldots,i_{d-1}}$ times, it follows by induction that $|\mathcal{B} \setminus (\mathcal{C} \cup \mathcal{D})| \geq 3^{d-1}$ and the proof is complete. □

6.4 Piercing

In this section we will consider piercing problems related to Question 6.5. We start by giving some simple bounds, derived by generalising the arguments used for $k = 2$, which illustrate various difficulties that arise. In the following subsections we give some improvements to these bounds.

In the case of bricks, observe that a single brick of the partition that does not contain a corner vertex can be incident to only one edge of the original cube, as otherwise it would not be proper and thus fail the k-piercing property (even for

$k = 2$). Also, for each edge there needs to be at least k boxes which are incident to it. Combining these two observations we deduce that there needs to be at least $d2^{d-1}(k-2)$ different non-corner boxes, as there are $d2^{d-1}$ edges. Including the additional 2^d corner boxes this implies that there are at least $d2^{d-1}(k-2)+2^d$ different boxes. On the other hand, generalising the partition used for $k = 2$, splitting the original cube into k parts along each dimension gives a k-piercing partition into k^d bricks. So we have shown the following two easy bounds:

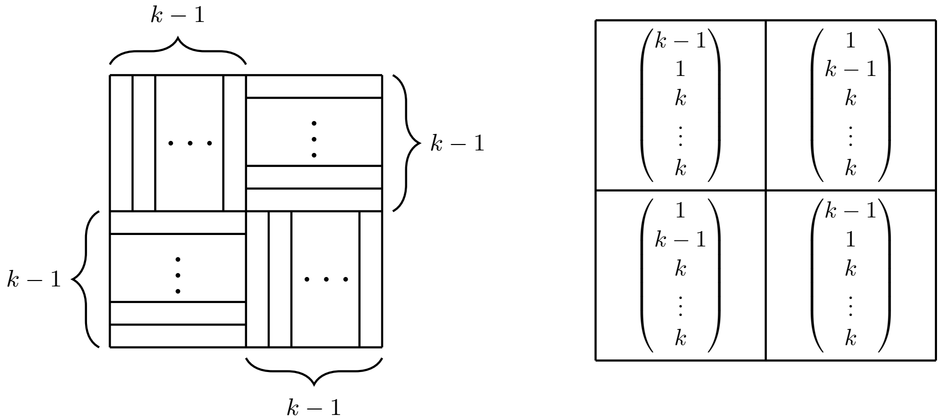
$$d2^{d-1}(k-2) + 2^d \leq p_{\text{brick}}(d, k) \leq k^d. \tag{6.3}$$

In the case of boxes, the lower bound no longer applies, as almost all the bricks counted as different above might become parts of a single box. The same kind of argument only gives $p_{\text{box}}(d, k) \geq d(k-1)+1$ by fixing a corner and counting all the boxes incident to an edge containing this corner, which need to be different. Furthermore, it is not clear how to exploit the k-piercing property in the argument used in Theorem 6.1 for $k > 2$. However, Theorem 6.1 is directly applicable in the case $k = 2$, which gives a lower bound of 2^d which also holds for all $k \geq 2$. From the other direction, it is also not clear how one could exploit the freedom afforded by using boxes instead of bricks when trying to find a partition, and in fact when $k = 2$ this turns out not to be possible. We can, however, reuse the bound for bricks to obtain the following simple bounds:

$$\max(d(k-1)+1, 2^d) \leq p_{\text{box}}(d, k) \leq k^d. \tag{6.4}$$

Note that the lower bound for p_{box} highlights a disconnection between our methods for dealing with the two most extreme regimes: firstly the case of k fixed and $d \to \infty$ in which the lower bound is 2^d, and secondly the case of d fixed and $k \to \infty$ in which the bound of $d(k-1)+1$ is relevant. We shall give our results in terms of both k and d so that they apply generally, and indeed the upper bounds we shall describe are the best we know across all regimes. Our lower bound efforts, however, are most relevant for the latter scenario (when d is small compared to k).

In the following subsections we will describe our various improvements to the above bounds. In the first subsection we will discuss upper bounds on $p_{\text{brick}}(d, k)$ and $p_{\text{box}}(d, k)$ and in the second subsection we discuss lower bounds.



(a) $p_{\text{brick}}(2, k) \leq 4(k-1)$ (b) $p_{\text{brick}}(d, k) \leq 4p_{\text{brick}}(d-1, k)$

Figure 6.3: (a) depicts a k-piercing configuration in two dimensions with $4(k-1)$ bricks. In (b), we use this idea to give a k-piercing construction with $k4^{d-1}$ boxes. In the first two dimensions we divide the cube into quadrants and then place optimal constructions in each quadrant satisfying the piercing conditions shown.

6.4.1 Upper bounds for the k-piercing problem

In this section we prove Theorems 6.6 and 6.7, giving a major improvement over the upper bound in (6.3) and (6.4). We begin by presenting a simple partition of $[n]^d$ into at most $4^{d-1}k$ bricks that satisfies the k-piercing property. This construction is so simple and natural that one might imagine that it could be best possible. This is not the case, however, and we will go on to present two different approaches for obtaining improvements in the base of the exponent, one of which is specific for boxes and gives a slightly better bound.

We define $f_d(a_1, \ldots, a_d)$ to be the minimum size of a partition of $[n]^d$ into boxes so that every line in dimension i hits at least a_i boxes, (we refer to this as the $(a_1, \ldots, a_d)$-*piercing condition*). In the first two dimensions, we split $[n]^d$ into 4 quadrants. In the top left and bottom right quadrants we place a construction satisfying the $(k-1, 1, k, \ldots, k)$-piercing condition. In the bottom left and top right quadrants we place a construction satisfying the $(1, k-1, k, \ldots, k)$-piercing condition. This is shown in Figure 6.3b. This gives a construction satisfying the k-piercing condition, and by observing that $f_d(1, k-1, \ldots, k) \leq f_{d-1}(k-1, k, \ldots, k) \leq f_{d-1}(k, k, \ldots, k)$ we arrive at the following bound which holds for $d \geq 2$:

$$f_d(k, \ldots, k) \leq 4f_{d-1}(k, \ldots, k).$$

Combining this with the fact that $f_1(k) = k$ we find that $f_d(k, \ldots, k) \leq 4^{d-1}k$.

In particular this shows that

$$p_{\text{box}}(d,k) \leq p_{\text{brick}}(d,k) \leq 4^{d-1}k. \tag{6.5}$$

So, in the notation introduced in Section 6.2, we have $\beta_k \leq \gamma_k \leq 4$.

One may wonder if these bounds are tight, and whether the construction described above is essentially best possible (at least in the case of bricks). We will now show that this is not the case, and give two different approaches for improving the base of the exponent further. In both of the following subsections we will reuse the general idea of splitting the cube along two dimensions. In the following subsection we work with bricks and prove Theorem 6.7 and in the subsequent subsection we exploit a simple observation which holds for boxes but not for bricks to get an even better bound.

Bricks

In some sense a more surprising part of the result (6.5) is the fact that for a fixed dimension d both $p_{\text{box}}(d,k)$ and $p_{\text{brick}}(d,k)$ are linear in k, but using the sub-multiplicative inequalities such as (6.2) can never give results linear in k. The idea of finding a small example and then using these inequalities as was done in the previous section for f_{odd} can only ever give something interesting when k is rather small. However, the idea behind the argument giving (6.5) is to use small examples in a different manner. The following observation gives a more general view of this idea.

Consider a partition of $[n]^d$ into bricks $A_1,\dots,A_m$ such that we can assign to each A_i a d-tuple $(a_{i,1}\dots,a_{i,d})$ of positive integers such that for any line in j-th dimension the sum of $a_{i,j}$, with i ranging over the bricks crossed by this line, is at least k. Whenever we have such a partition we obtain that $f_d(k,\dots,k) \leq \sum_{i=1}^m f_d(a_{i,1},\dots,a_{i,d})$ as we can solve the corresponding sub-problem within each brick of the partition. We will call such a partition *intermediate*.

The natural goal is to find small examples of intermediate partitions. For example, given a k-piercing example for small d, if we can group several bricks into sets A_i to obtain an intermediate partition then we obtain an upper bound on $f_d(k,\dots,k)$. For instance, in the proof of (6.5), we used the example shown in Figure 6.3a which gives a natural grouping into 4 bricks, yielding the intermediate example shown in Figure 6.3b.

The following lemma gives a way of obtaining, from an intermediate partition in d dimensions, a new intermediate partition in $d+1$ dimensions that does slightly better than the trivial approach of stacking two copies on top of one another.

Lemma 6.10. *Let $A_1, \dots, A_m$ be an intermediate partition of $[n]^d$. Let X and Y be corners of the cube such that, without loss of generality, the largest proper sub-brick containing X covers $A_1, \dots, A_s$, and let A_r be the brick containing corner Y. Then*

$$\begin{aligned} f_{d+1}(k, \dots, k) \leq \sum_{i=1}^{s} f_{d+1}(a_{i,1} \dots, a_{i,d}, 1) + \sum_{i=s+1}^{m} f_{d+1}(a_{i,1} \dots, a_{i,d}, k-1) + \\ \sum_{i=1, i \neq r}^{m} f_{d+1}(a_{i,1} \dots, a_{i,d}, 1) + f_{d+1}(a_{r,1} \dots, a_{r,d}, k-1). \end{aligned}$$

Proof. We split the cube in two parts along the $(d+1)$-st dimension. We use the given partition for both parts, but with the top part rotated in such a way that Y corresponds to X. We then rescale the top partition in such a way that A_r covers all of $A_1, \dots, A_s$ in the original partition (note that this may require a minor increase in the n we use). We add $k-1$ for the last dimension of A_r in the top part and all the bricks in the lower part except $A_1, \dots, A_s$, and we add 1 for the remaining bricks. This new partition is a new intermediate partition in $d+1$ dimensions; to see this, note that along first d dimensions all the lines satisfy the condition because we started with an intermediate partition. To see this is also true for any line ℓ parallel to the axis of the $(d+1)$-st dimension, we note that if ℓ intersects any of $A_1, \dots A_s$ in the lower part it will pass through A_r in the upper part, so it will intersect at least $1 + k - 1$ different boxes. Otherwise, if ℓ passes through some A_i, $i \geq s+1$ in the lower part, it will intersect some box in the upper part as well, again intersecting at least $k - 1 + 1$ boxes. The inequality now follows from the above observation. $\square$

We now apply this lemma to the 5-part intermediate partition derived from the one given in Figure 6.3 and given in Figure 6.4. We obtain the 3-dimensional intermediate partition shown in Figure 6.5. In particular, this implies that

$$\begin{aligned} f_d(k, \dots, k) \leq 2f_d(1, k-1, k-1, k, \dots, k) + 6f_d(1, 1, k-1, k, \dots, k) \\ + 2f_d(1, 1, k-2, k, \dots, k). \end{aligned}$$

Unfortunately, this bound still only implies that $f_d(k, \dots, k) \leq (4 + o_d(1))^d k$, but, modifying this partition slightly, we may consider Figure 6.6 and apply Lemma 6.10 once again. This does achieve an improvement in the base of the exponential term. In particular, we find:

$$f_d(k,\dots,k) \le 8f_d(1,1,k-1,k-1,k,\dots,k) + 5f_d(1,1,k-2,k-1,k,\dots,k) + \\ 8f_d(1,1,1,k-1,k,\dots,k) + 3f_d(1,1,1,k-2,k,\dots,k).$$

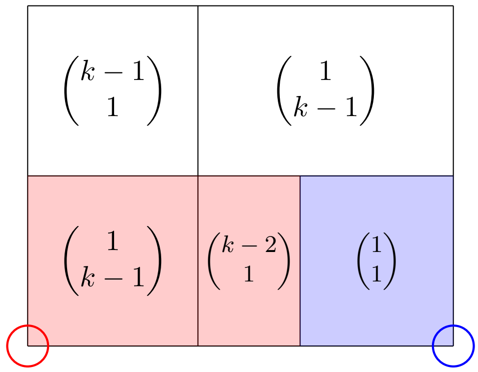


Figure 6.4: The intermediate partition in 2 dimensions, to which we apply Lemma 6.10. X is denoted by red circle, Y by a blue circle, the parts $A_1,\dots,A_s$ are shaded red and A_r is shaded blue.

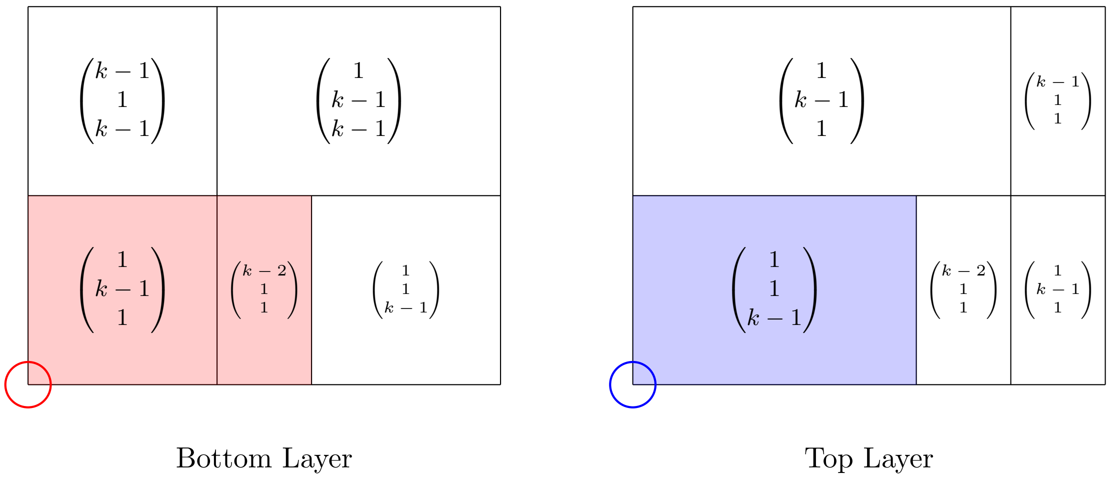


Figure 6.5: The intermediate partition in 3 dimensions, provided by the above lemma.

This already suffices to give an example with at most about $3.97^d k$ bricks. However, since the red bricks in Figure 6.6 have large piercing values in all but one dimension, it turns out that a further manual step can be made before applying Lemma 6.10. In particular, using the partition given in Figure 6.7 we obtain the following slight improvement:

$$f_d(k,\dots,k) \le 10f_d(1,1,k-1,k-1,k,\dots,k) + 3f_d(1,1,k-2,k-1,k,\dots,k) + \\ 6f_d(1,1,1,k-1,k,\dots,k) + 3f_d(1,1,1,k-2,k,\dots,k).$$

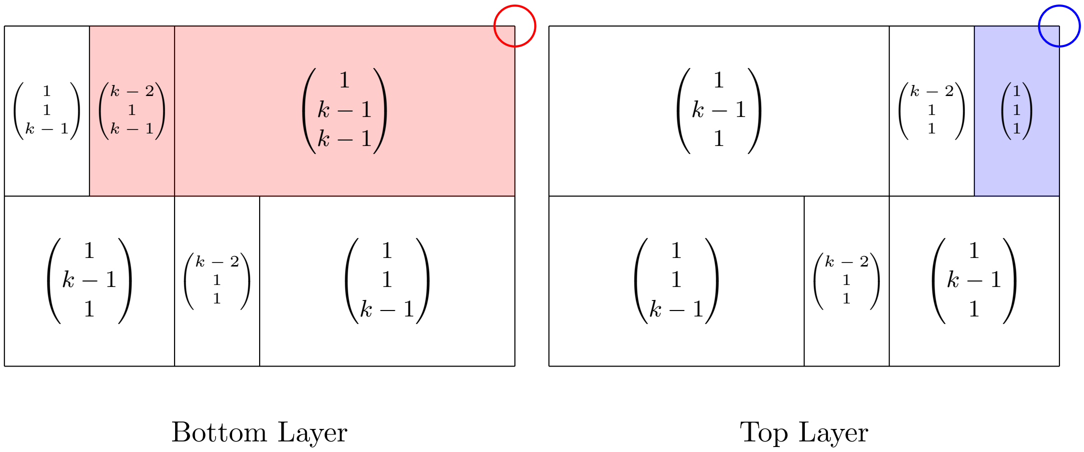


Figure 6.6: The intermediate partition in 3 dimensions, to which we apply Lemma 6.10. X is denoted by red circle, Y by a blue circle, the parts $A_1, \dots, A_s$ are shaded red and A_r is shaded blue.

This inequality implies

$$f_d(k, \dots, k) \leq 13 f_{d-2}(k, \dots, k) + 9 f_{d-3}(k, \dots, k),$$

which in turn implies $f_d(k, \dots, k) \leq x_0^d k$ where x_0 is the largest root of $x^3 - 13x - 9$, $x_0 \approx 3.91$. In particular, this shows that $\beta_k \leq \gamma_k \leq x_0$.

For small values of k the above inequality actually implies a somewhat stronger result, provided we take more care with the $k-1, k-2$ terms. E.g. for $k = 3$ we get:

$$f_d(3, \dots, 3) \leq 10 f_d(1, 1, 2, 2, 3, \dots, 3) + \\ 9 f_d(1, 1, 1, 2, 3, \dots, 3) + 3 f_d(1, 1, 1, 1, 3, \dots, 3)$$

and therefore

$$f_d(3, \dots, 3) \leq (10 \cdot 4 + 9 \cdot 2 + 3) f_{d-4}(3, \dots, 3) = 61 f_{d-4}(3, \dots, 3)$$

where we repeatedly used

$$f_d(2, a_1, \dots, a_{d-1}) \leq 2 f_d(1, a_1, \dots, a_{d-1}) = f_{d-1}(a_1, \dots, a_{d-1}),$$

which follows by taking two identical copies of the $(d-1)$-dimensional example. This inequality implies $\beta_3 \leq \gamma_3 \leq \sqrt[4]{61} \approx 2.79$.

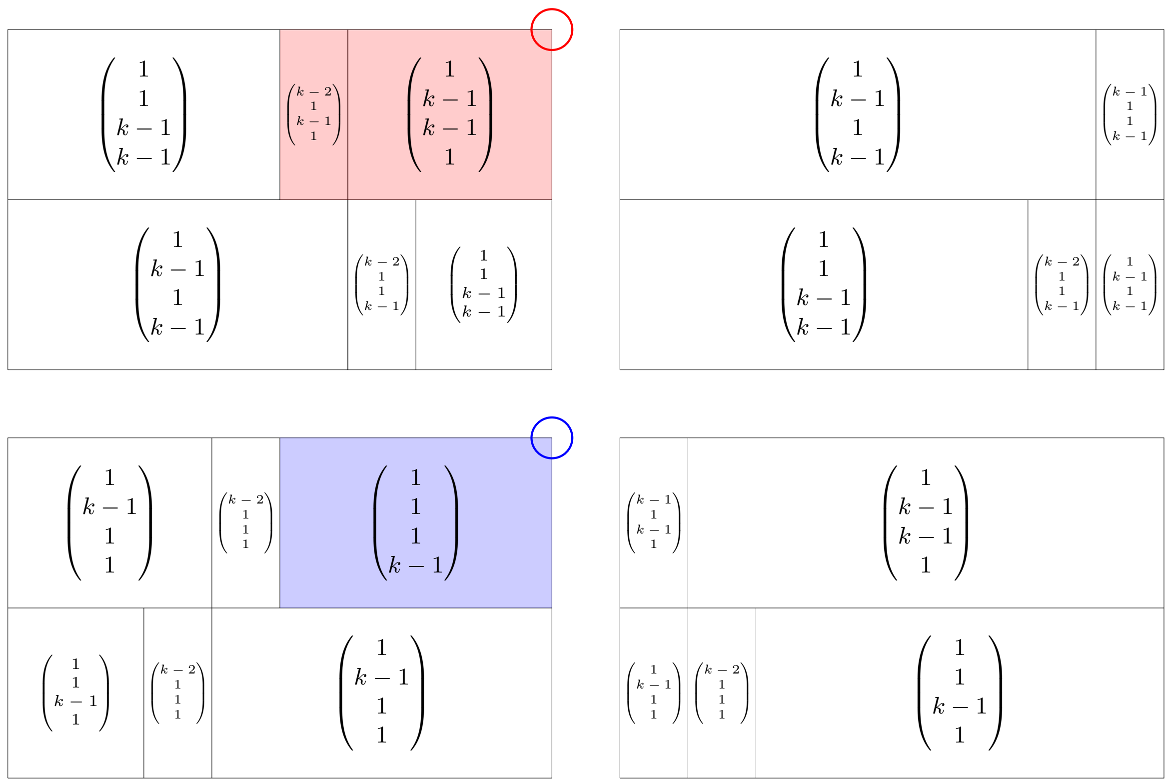


Figure 6.7: An intermediate partition in 4 dimensions. The third and fourth dimensions move between the rectangles horizontally and vertically respectively.

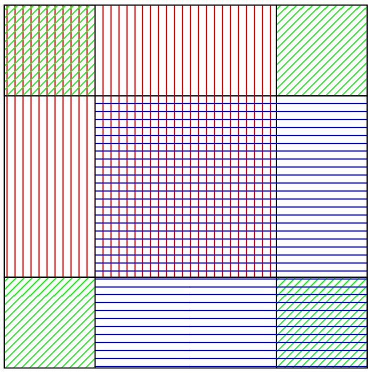


Figure 6.8: A square can be covered by 3 boxes, but not with 3 bricks.

Boxes

It is highly unclear how one could use the additional freedom afforded by using boxes instead of bricks. The only way that we have found exploits the fact that it is possible to cover a square using only three boxes, as shown in Figure 6.8. This allows us to obtain better examples using boxes than the ones using bricks described above.

We will reuse the intermediate partition given in Figure 6.4 to obtain a new intermediate 3-dimensional partition. This new intermediate partition will use three copies of the old one, stacked on top of each other such that in each layer the copy of A_r incident to vertex Y is stretched to make one of the three boxes used to cover a square in Figure 6.8 and divided into $k-2$ copies of itself along the third dimension. The full picture is as shown in Figure 6.9.

This implies that

$$f_d(k,\ldots,k) \leq 9f_d(1,1,k-1,k,\ldots,k) + 6f_d(1,1,k-2,k,\ldots,k).$$

Proof of Theorem 6.6. The above inequality directly gives that

$$f_d(k,\ldots,k) \leq 15f_{d-2}(k,\ldots,k),$$

showing that $\beta_k \leq \sqrt{15} \approx 3.87$. This proves the desired result. $\square$

6.4.2 Lower bounds

Initially, it was not clear to us whether the lower bound on p_{brick} given in (6.3) could in fact be tight. After all, it is tight for all k in two dimensions, as Figure 6.3a shows that $p_{\text{brick}}(2,k) \leq 4(k-1)$, which matches the lower bound.

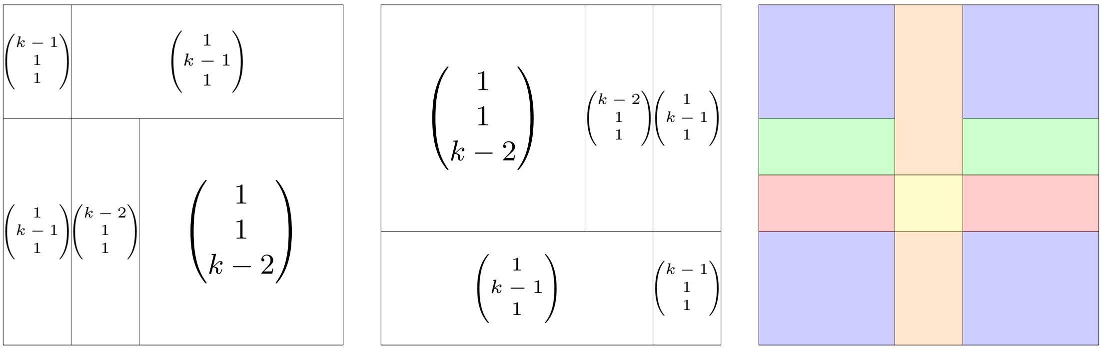


Figure 6.9: An intermediate partition based on the above observation. The blue box is labelled $\begin{pmatrix}1\\1\\k-2\end{pmatrix}$, red is labelled $\begin{pmatrix}k-2\\1\\1\end{pmatrix}$, orange is labelled $\begin{pmatrix}k-1\\1\\1\end{pmatrix}$, green is labelled $\begin{pmatrix}1\\k-1\\1\end{pmatrix}$ and yellow is labelled $\begin{pmatrix}1\\k-1\\1\end{pmatrix}$. All other boxes are in fact bricks.

In higher dimensions it matches the recursive lower bound obtained by the inclusion-exclusion principle through analysing the number of bricks touching faces of dimensions from 0 to $d-1$; for example, the proof of the lower bound of (6.3) used only faces of dimensions 0 (corners) and 1 (edges). It turns out, however, that $21 \leq p_{\text{brick}}(3,3)$, showing that the bound is not always tight. In fact, exploiting this fact and the aforementioned inclusion-exclusion inequality one can obtain a lower bound for $k=3$ which is better than (6.3) by a constant factor. We omit further details as both parts of the argument are quite cumbersome and result in only a very weak improvement.

The case of boxes seems much more difficult, and is non-trivial even in 2 dimensions.

In order to tackle the problem in 2 dimensions, we consider the following reduction. Given a partition of $[n]^2$ with the k-piercing property we construct an auxiliary graph with one vertex for each box. We place an edge between two boxes if and only if there is an axis parallel line intersecting both boxes. We colour the edges of this graph by colouring an edge e red if the line intersecting both boxes corresponding to e's endpoints is vertical line and blue if it is horizontal. Since the k-piercing constraint requires that any line intersects at least k boxes, we see that every vertex in our auxiliary graph is both contained inside a clique of at least k vertices with all edges coloured red and a clique of at least k vertices with all edges coloured blue. We therefore formulate the following question, which we find interesting in its own right.

Question 6.11. *Let $k \geq 1$ be an integer. What is the minimal N such that we can colour the edges of a graph on N vertices red and blue such that every vertex belongs to a monochromatic K_k of each colour?*

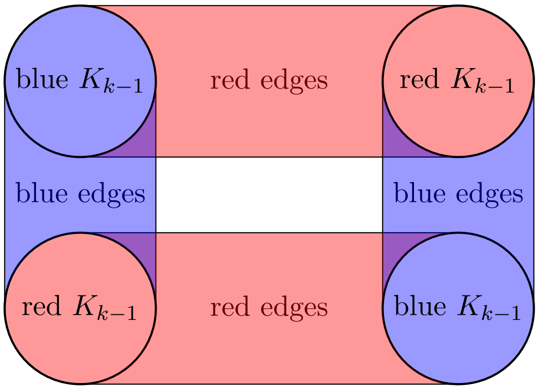


Figure 6.10: A graph in which every vertex is contained in a red K_k and a blue K_k.

Note that, by the above construction, the answer to the above question is a lower bound for $p_{\text{box}}(2, k)$. A construction arising from the example in Figure 6.3a which provides the bound $N \leq 4(k-1)$ can be seen in Figure 6.10.

At the time of writing, we conjectured that $N = 4(k-1)$ is best possible – however, we were only able to prove an asymptotic result. This conjecture, and therefore Question 6.11, has recently been resolved by Holzman [47].

Proposition 6.12. *In Question 6.11, we have $N \geq (4 + o_k(1))k$.*

Proof. Let R be the vertex set of a largest red clique and B the vertex set of a largest blue clique in the graph. Note that $|R \cap B| \leq 1$, as each edge can only have one colour. Define $A_0 = R \setminus B$ and $B_0 = B \setminus R$. Let $a_0 = |A_0| \geq k-1$ and $b_0 = |B_0| \geq k-1$.

In general, let R and B be the vertex sets of a largest red and blue clique on $G \setminus (A_0 \cup \ldots \cup A_{i-1} \cup B_0 \cup \ldots \cup B_{i-1})$, respectively. As before, $|R \cap B| \leq 1$ and we define $A_i = R \setminus B$ and $B_i = B \setminus R$. Let $a_i = |A_i|$ and $b_i = |B_i|$.

Given a vertex v in $A_0 \cup \ldots \cup A_{i-1}$ it belongs to a blue k-clique. This clique can have at most one vertex in each of $A_0, A_1, \ldots, A_{i-1}$, one of which is v itself. Similarly, by choice of B_i we know this clique can have at most $b_i + 1$ vertices outside of $A_0 \cup \ldots \cup A_{i-1} \cup B_0 \cup \ldots \cup B_{i-1}$. This implies that v has blue degree at least $k - 1 - (i-1) - (b_i + 1) = k - i - b_i - 1$ towards $B_0 \cup \ldots \cup B_{i-1}$. An analogous argument shows that any $w \in B_0 \cup \ldots \cup B_{i-1}$ has red degree at least $k - i - a_i - 1$ towards $A_0 \cup \ldots \cup A_{i-1}$.

In particular, letting $A = a_0 + \ldots + a_{i-1}$ and $B = b_0 + \ldots + b_{i-1}$, this implies that

$$AB \geq A(k - i - b_i - 1) + B(k - i - a_i - 1).$$

Now define c_{i-1} by $A + B = c_{i-1}(k-1)$. Since there are at least $c_{i-1}(k-1) + a_i + b_i$

vertices in G we get

$$AB + Ab_i + Ba_i \geq (k - i - 1)c_{i-1}(k-1).$$

For a fixed c_{i-1} the left-hand side is maximised for $A = c_{i-1}(k-1)/2 - (a_i - b_i)/2$ and $B = c_{i-1}(k-1)/2 + (a_i - b_i)/2$. This gives

$$c_{i-1}^2(k-1)^2/4 - (a_i - b_i)^2/4 + (a_i + b_i)c_{i-1}(k-1)/2 + (a_i - b_i)^2/2 \geq (k - i - 1)c_{i-1}(k-1)$$

$$\Rightarrow c_{i-1}^2(k-1)^2 + (a_i - b_i)^2 + 2(a_i + b_i)c_{i-1}(k-1) \geq 4(k - i - 1)c_{i-1}(k-1)$$

$$\Rightarrow c_{i-1}^2(k-1)^2 + (a_i + b_i)^2 + 2(a_i + b_i)c_{i-1}(k-1) \geq 4(k - i - 1)c_{i-1}(k-1)$$

$$\Rightarrow (c_{i-1}(k-1) + a_i + b_i)^2 \geq 4(k - i - 1)c_{i-1}(k-1).$$

Since $c_i(k-1) = c_{i-1}(k-1) + a_i + b_i$, we get

$$c_i \geq 2\sqrt{\frac{k-i-1}{k-1}c_{i-1}} \geq 2^{1+1/2+\ldots+1/2^i}\left(\frac{k-i-1}{k-1}\right)^{1/2+1/4+\ldots+1/2^i}$$

$$= 4 \times 2^{-1/2^i}(1 - i/(k-1))^{1-1/2^i}.$$

Choosing $i = \mathcal{O}(\log(k))$ gives the result. □

Proposition 6.8 follows immediately from this result, by the above reduction.

Note that Question 6.11 generalises naturally to $t > 2$ colours. The proof of Proposition 6.12 can be easily modified to give a lower bound of $(2t + o_k(1))k$ for this generalisation, and the construction on Figure 6.10 can also be modified to give an upper bound of $2t(k-1)$. While the lower bound for this question applies to the k-piercing question, giving a lower bound of $(2d + o_d(1))k$ in d dimensions which does beat the trivial bound of $d(k-1)$ from the start of the section, this bound is not particularly strong so we omit the full details. It seems that, in two dimensions, Question 6.11 captures the difficulty of the k-piercing problem, while the generalised version does not fully capture the difficulties of the higher dimensional piercing problem.

With this in mind we consider the following reduction. Given a k-piercing partition in d dimensions, consider the complete graph K_n with vertices being boxes. We colour an edge between two boxes in colour i if they are intersected by some $(d-1)$-dimensional plane orthogonal to the i-th dimensional axis. This gives a colouring in d colours, such that every edge gets at most $d-1$ colours.

Furthermore, every vertex is a part of a monochromatic K_t in each colour, where $t = p_{\text{box}}(d-1,k)$. We shall use this to give the following lower bound.

Theorem 6.13.

$$p_{\text{box}}(d,k) \geq e^{\sqrt{d/2}-1}(k-1).$$

Proof. We consider the complement of the colouring of the K_n described in the previous paragraph. In the complement each edge gets assigned only the colours it was not assigned in the above colouring. As each edge had at most $d-1$ colours, the new colouring assigns at least one colour to each edge. Furthermore, for every vertex v and every colour c, v belongs to a set of size t within which there is no edge of colour c.

We claim that this implies that for each colour there are at most $(n-t)^2$ edges of this colour. To see this, note that there needs to exist an independent set of size t in this colour and each of the remaining $n-t$ vertices can be incident to at most $n-t$ edges of this colour.

As our new colouring needed to cover all the possible edges at least once, this implies that

$$\begin{aligned} d &\geq \frac{n(n-1)}{2(n-t)^2} \\ \implies n-1 &\geq \left(1+\frac{1}{\sqrt{2d}-1}\right)(t-1) \\ \implies p_{\text{box}}(d,k)-1 &\geq \left(1+\frac{1}{\sqrt{2d}-1}\right)(p_{\text{box}}(d-1,k)-1). \end{aligned}$$

This gives

$$\begin{aligned} p_{\text{box}}(d,k) &\geq \prod_{i=2}^{d}\left(1+\frac{1}{\sqrt{2i}-1}\right)(k-1)+1 \\ &\geq e^{\sum_{i=2}^{d}\frac{1}{2\sqrt{2i}}}(k-1) \\ &\geq e^{\frac{1}{2\sqrt{2}}\sum_{i=2}^{d}\frac{1}{\sqrt{i}}}(k-1) \\ &\geq e^{\sqrt{d/2}-1}(k-1) \end{aligned}$$

as claimed.

□

6.5 Concluding remarks and open problems

There are a large number of very interesting questions that remain in this area, and we shall now list just a few.

It remains, of course, to determine the asymptotics of f_{odd}. The most important question seems to be the following.

Question 6.14. *Is* $f_{\text{odd}}(n, d) = (2 + o(1))^d$ *as* $n, d \to \infty$*?*

One may also consider the original question of Kearnes and Kiss with a relaxation of the condition that the boxes partition $[n]^d$. In their paper [54], Leader, Milićević and Tan ask how many proper boxes are required to form a double cover of $[n]^d$, and specifically whether at least 2^d are required. A natural construction involves taking three copies of a partition of $[n]^{d-1}$ and taking the products of these with the sets $\{1, 2\}$, $\{2, \dots, n\}$ and $\{1, 3, 4, \dots, n\}$ respectively, giving a double cover of size $(3/2)2^d$. We can show that this construction is not best possible (a simulated annealing approach found a double cover of size 11 in $[3]^3$, and Gurobi did even better by finding a construction of size 21 in $[3]^4$), but we have not been able to beat 2^d and the question remains open.

Regarding the k-piercing problem, there are several possible angles. Again, the most important question concerns improving the lower bound.

Question 6.15. *Does there exist an* $\varepsilon > 0$ *such that for a fixed* k *we have* $p_{\text{box}}(d, k) \geq (2 + \varepsilon)^d$*?*

The analogous question for p_{brick} would be a natural first step which we believe is interesting in its own right.

Along similar lines is the regime where d is fixed and k is allowed to grow. As discussed in Section 6.4, the bound for this problem is always linear in k, but finding the constant of linearity appears to be difficult.

Question 6.16. *Let* d *be fixed so that* $C_d = \limsup_{k\to\infty} p_{\text{box}}(d, k)/k$*. How does* C_d *grow with* d*? Must* C_d *be exponential in* d*?*

As noted in Section 6.4, we are only able to show that $e^{\sqrt{d/2}-1} \leq C_d \leq 15^{d/2}$. Proposition 6.8 shows that $C_2 = 4$, but finding C_3 is already beyond our methods. Answering this question would directly extend Theorem 6.1 and therefore likely requires some new ideas.

To finish, we shall describe one last problem which is of particular interest. We observe that in the k-piercing problem the requirement that the boxes B_i

partition $[n]^d$ can be dropped without trivialising the question, provided that we maintain the constraint that the B_i are disjoint.

Question 6.17. *Let $n \geq k$ and $d \geq 1$ be integers. Let $\{B^1, B^2, \dots, B^m\}$ be a collection of disjoint proper boxes in $[n]^d$ with k-piercing property (defined in the obvious way for non-partitions). What lower bounds can be shown for m? In particular, do we have $m \geq 2^d$?*

When $k = 2$ this generalises the original question of Kearnes and Kiss; however the proof of Theorem 6.1 relies on the B_i forming a partition and so the same approach cannot be used. Indeed, we know of no approach that gives a bound better than $(1 + o(1))^d$ for this question, although computer search finds no examples with $m < 2^d$.

6.A List of coordinates for the boxes in Figure 6.1

```
Box(1) = {1,2,3} x {1,2,3} x {1}
Box(2) = {1,2,3} x {1,2,3} x {2}
Box(3) = {2,4,5} x {1,4,5} x {3}
Box(4) = {2,3,5} x {2,3,5} x {4}
Box(5) = {1,2,4} x {1,2,4} x {5}
Box(6) = {1,2,5} x {1} x {4}
Box(7) = {1} x {1,2,5} x {3}
Box(8) = {1} x {2,4,5} x {4}
Box(9) = {2,4,5} x {2} x {3}
Box(10) = {2,4,5} x {3} x {3}
Box(11) = {2,3,4} x {3} x {5}
Box(12) = {3} x {2,3,4} x {3}
Box(13) = {3} x {2,4,5} x {5}
Box(14) = {4} x {1,2,3} x {1,2,4}
Box(15) = {5} x {1,2,3} x {1,2,5}
Box(16) = {2,4,5} x {4} x {1,2,4}
Box(17) = {2,4,5} x {5} x {1,2,5}
Box(18) = {1} x {4} x {1,2,3}
Box(19) = {1} x {5} x {1,2,5}
Box(20) = {3} x {4} x {1,2,4}
Box(21) = {3} x {5} x {1,2,3}
Box(22) = {1} x {3} x {3,4,5}
```

```
Box(23) = {3} x {1} x {3,4,5}
Box(24) = {4} x {5} x {4}
Box(25) = {5} x {4} x {5}
```

Chapter 7

The largest projective cube-free subsets of $\mathbb{Z}_{2^n}$

This chapter is based on joint work with A. Z. Wagner [59], to appear in the European Journal of Combinatorics.

7.1 Introduction

7.1.1 Theorems in $\mathbb{Z}_2^n$

We will consider four important results in the Boolean lattice, which we identify in the usual way with the elements of $\mathbb{Z}_2^n$.

We begin with Sperner's theorem from 1928, a cornerstone of extremal combinatorics. We recall that two distinct sets $A, B \subseteq [n]$ form a 2-*chain* if they are comparable, i.e. if $A \subset B$ or $B \subset A$. Similarly, k distinct sets form a k-*chain* if any two of them are comparable.

The *layers* of the Boolean lattice are the collections of sets that have the same size, so that the largest layer is $\binom{[n]}{\lfloor n/2 \rfloor}$ and has size $\binom{n}{\lfloor n/2 \rfloor}$.

Theorem 7.1 (Sperner, [77])**.** *If $\mathcal{F} \subset \mathbb{Z}_2^n$ does not contain a 2-chain then $|\mathcal{F}|$ is not larger than the largest layer. In particular, $|\mathcal{F}| \leq \binom{n}{\lfloor n/2 \rfloor}$.*

One of the many generalisations of Sperner's theorem is due to Erdős:

Theorem 7.2 (Erdős, [24])**.** *Let $n \geq k \geq 2$ be integers. If $\mathcal{F} \subset \mathbb{Z}_2^n$ does not contain a k-chain then $|\mathcal{F}|$ is not larger than the union of the $k-1$ largest layers.*

Kleitman generalised Sperner's theorem in a different direction. He considered families larger than $\binom{n}{\lfloor n/2 \rfloor}$ and asked which ones have the fewest 2-chains.

Let the layers of $\{0,1\}^n$ in decreasing order of size be denoted by $J_1, J_2, \dots, J_{n+1}$ so that $J_1 = \binom{[n]}{\lfloor n/2 \rfloor}$ and $\{J_n, J_{n+1}\} = \{\{\emptyset\}, \{[n]\}\}$. Say that a family $\mathcal{F} \subseteq \{0,1\}^n$ is *centred* if there exists an $i \in [n+1]$ such that for all j with $1 \leq j < i$ we have $J_j \subseteq \mathcal{F}$, and for all j with $i < j \leq n+1$ we have $J_j \cap \mathcal{F} = \emptyset$.

Theorem 7.3 (Kleitman, [52])**.** *Let $n \geq 2$ and M be integers. Amongst all families $\mathcal{F} \subseteq \mathbb{Z}_2^n$ of size $|\mathcal{F}| = M$, centred families minimise the number of 2-chains.*

Very recently Samotij, proving a conjecture of Kleitman, generalised Theorems 7.2 and 7.3.

Theorem 7.4 (Samotij, [71])**.** *Let $n \geq k \geq 2$ and M be integers. Amongst all families $\mathcal{F} \subseteq \mathbb{Z}_2^n$ of size $|\mathcal{F}| = M$, centred families minimise the number of k-chains.*

The main result of this chapter is that most of these results still hold if we replace $\mathbb{Z}_2^n$ by $\mathbb{Z}_{2^n}$. In order to state analogues of these theorems in $\mathbb{Z}_{2^n}$, we need to define analogues of the definition of *layers* and *chains*.

7.1.2 Theorems in $\mathbb{Z}_{2^n}$

Finding a natural partition of the elements of $\mathbb{Z}_{2^n}$ into $n+1$ layers is not too difficult. For all $1 \leq i \leq n$, let

$$L_i := \{x \in [2^n] : x \equiv 2^{i-1} \pmod{2^i}\}$$

be the i-th layer and define $L_{n+1} := \{0\}$. So L_1 consists of all odd numbers, L_2 consists of all numbers congruent to 2 mod 4, etc. In particular, we have $|L_{i-1}| = 2|L_i|$ for all $i \leq n$. Finding the right analogue of a chain in $\mathbb{Z}_{2^n}$ is much more challenging – indeed it was not obvious to us that a corresponding notion should exist. However, it turns out that the correct notion is that of the *projective cube.*

Following the notation of [8], given a multiset $S = \{a_1, \dots, a_d\}$ of size d, we define the *projective d-cube* generated by S to be

$$\Sigma^* S = \left\{ \sum_{i \in I} a_i : \emptyset \neq I \subseteq [d] \right\}.$$

Extremal properties of projective cubes have a vast literature, see e.g. [2, 5, 26, 42]. In particular, Rado [67], and later independently Sanders [73] and Folkman

[38, 39], showed that for any r and d, there exists a least number n so that for any partition of $[n]$ into r classes, some class contains a projective d-cube.

Throughout the rest of this chapter we work in the cyclic group $\mathbb{Z}_{2^n}$. Hence in the definition of $\Sigma^* S$ the summations are all modulo 2^n, and $\Sigma^* S$ is a subset of $\mathbb{Z}_{2^n}$. Following e.g. [65] we do not assume that the numbers $a_1, \ldots, a_d$ are distinct, but we will always view the d-cube $\Sigma^* S$ as a set, rather than a multiset. Hence $|\Sigma^* S| \leq 2^k - 1$, but $|\Sigma^* S|$ could be much smaller. We say that a set $A \subset \mathbb{Z}_{2^n}$ is *d-cube-free* if there does not exist a multiset S of size d with $\Sigma^* S \subseteq A$.

Examples

- If $S = \{a, b\}$, then $\Sigma^* S = \{a, b, a+b\}$ is a 2-cube (Schur triple). In the degenerate case where $a = b$ we have $\Sigma^* S = \{a, 2a\}$ which is also a 2-cube.
- The set $\{a, 2a, 3a, \ldots, da\}$ is a d-cube, as it is generated by $S = \{\underbrace{a, \ldots, a}_{d \text{ times}}\}$.
- If $n = 3$ then $\Sigma^*\{2, 5, 5\} = \{2, 4, 5, 7\}$, and hence the set $A = \{2, 3, 4, 5, 7\}$ is not 3-cube-free.
- The set $\{0\}$ is a d-cube for any d.

We are now ready to state our main results. We obtain the statements by replacing $\mathbb{Z}_2^n$ by $\mathbb{Z}_{2^n}$ and the expression "k-chain" by "2^{k-1}-dimensional projective cube" in Theorems 7.1–7.4. We begin with the resulting analogue of Sperner's theorem, which is an easy exercise.

Proposition 7.5 (Analogue of Sperner's theorem in $\mathbb{Z}_{2^n}$)**.** *If $\mathcal{F} \subset \mathbb{Z}_{2^n}$ does not contain a projective 2-cube, then $|\mathcal{F}|$ is not larger than the largest layer, i.e. L_1.*

Note that a projective 2-cube is simply a Schur triple, so Proposition 7.5 simply states that any sum-free set in $\mathbb{Z}_{2^n}$ has size at most 2^{n-1}. The analogue of Erdős' theorem (Theorem 7.2) is on largest sets without projective cubes:

Theorem 7.6 (Analogue of Erdős' theorem in $\mathbb{Z}_{2^n}$)**.** *Let $n \geq k \geq 2$ be integers. If $\mathcal{F} \subset \mathbb{Z}_{2^n}$ does not contain a projective 2^{k-1}-cube, then $|\mathcal{F}|$ is not larger than the union of the $k-1$ largest layers, i.e. $L_1 \cup L_2 \cup \ldots \cup L_{k-1}$.*

This theorem is sharp, since $L_1 \cup \ldots \cup L_{k-1}$ is 2^{k-1}-cube-free (indeed, amongst any collection of 2^{k-1} numbers there is a subset whose sum is divisible by 2^{k-1}). In order to state our version of Kleitman's theorem (Theorem 7.3) we need to define what a *centred* set is. Our definition of centred will be the same as in the Boolean lattice case: we say that $S \subset \mathbb{Z}_{2^n}$ is *centred* if there exists an $i \in [n+1]$

such that for all j with $1 \leq j < i$ we have $L_j \subseteq S$, and for all j with $i < j \leq n+1$ we have $L_j \cap S = \emptyset$. The analogue of Kleitman's theorem was raised as a question by Samotij and Sudakov [72] in the last line of their paper.

Conjecture 7.7 (Analogue of Kleitman's theorem in $\mathbb{Z}_{2^n}$)**.** *Let $n \geq 2$ and M be integers. Amongst all families $\mathcal{F} \subseteq \mathbb{Z}_{2^n}$ of size $|\mathcal{F}| = M$, centred families minimise the number of 2-cubes.*

For $M \leq 2^{n-1}$ Conjecture 7.7 is trivial, and our modest contribution is that the first non-trivial case of this conjecture is true. Theorem 7.8 states that the $M = 2^{n-1} + 1$ case of Conjecture 7.7 is true.

Theorem 7.8. *All sets of size $2^{n-1} + 1$ in $\mathbb{Z}_{2^n}$ contain at least $3 \cdot 2^{n-1}$ Schur triples.*

Following [72], we define the number of 2-cubes in a set $A \subseteq \mathbb{Z}_{2^n}$ by

$$\mathrm{ST}(A) = \left|\left\{(x, y, z) \in A^3 : x + y = z\right\}\right|,$$

so that if $x + y = z$ and $x \neq y$ then we consider (x, y, z) and (y, x, z) as different triples. Observe that a centred set of size $2^{n-1}+1$, e.g. the set $L_1 \cup \{2\}$, contains precisely $3 \cdot 2^{n-1}$ such 2-cubes, and hence Theorem 7.8 is sharp. The number of k-cubes in a set can be defined similarly, and indeed we conjecture that the analogue of Samotij's theorem (Theorem 7.4) also holds.

Conjecture 7.9 (Analogue of Samotij's theorem in $\mathbb{Z}_{2^n}$)**.** *Let $n \geq k \geq 2$ and M be integers. Amongst all families $\mathcal{F} \subseteq \mathbb{Z}_{2^n}$ of size $|\mathcal{F}| = M$, centred families minimise the number of 2^k-cubes.*

7.1.3 When d is not a power of two

While we have seen that 2^{k-1}-cubes in $\mathbb{Z}_{2^n}$ correspond to k-chains in $\mathbb{Z}_2^n$, the case of d-cubes where d is not a power of two does not seem to have an analogue in $\mathbb{Z}_2^n$. Hence it is not obvious what the extremal constructions should be, and indeed this case exhibits much more interesting behaviour. Table 7.1 illustrates our conjectured largest d-cube-free constructions, which we refer to as $\mathcal{C}_d$. We will always assume that n is sufficiently large for our constructions to make sense, in particular $n \geq d$ is always enough. Recall that Theorem 7.6 establishes that $\mathcal{C}_d$ is indeed best possible for $d = 2, 4, 8, \ldots$.

We give an explicit description of this construction $\mathcal{C}_d$ for all d in Section 7.3. While we cannot prove that these constructions are best possible (except when $d = 2^\ell$), we can show they are best amongst sets that are unions of layers.

d	$\mathcal{C}_d$, the largest conjectured d-cube-free subset of $\mathbb{Z}_{2^n}$	
2	L_1	✓
3	$L_1 \cup L_3$	
4	$L_1 \cup L_2$	✓
5	$L_1 \cup L_2 \cup L_4$	
6	$L_1 \cup L_2 \cup L_4 \cup L_6$	
7	$L_1 \cup L_2 \cup L_4 \cup L_5$	
8	$L_1 \cup L_2 \cup L_3$	✓
9	$L_1 \cup L_2 \cup L_3 \cup L_5$	
...	...	
26	$L_1 \cup L_2 \cup L_3 \cup L_4 \quad \cup \quad L_6 \cup L_7 \cup L_8 \quad \cup \quad L_{10} \cup L_{11}$	
...	...	

Table 7.1: The conjectured best constructions, with ticks next to those which we show are indeed best possible.

Theorem 7.10. *Let d, n be positive integers with $d \leq n$. Then $\mathcal{C}_d$ is the largest d-cube free subset of $\mathbb{Z}_{2^n}$ amongst all sets that can be written as a union of some layers.*

Our main tool in the proof of Theorem 7.10 is the following elementary lemma, which we believe is interesting in its own right.

Lemma 7.11. *Let $k \geq 1$ and $x \geq 0$ be integers. Given $2^k + x$ not necessarily distinct integers $a_1, a_2, \ldots, a_{2^k+x}$, at least one of the following two statements holds.*

1. *There exists a subset of these integers whose sum is divisible by 2^k but not by 2^{k+1}.*
2. *There exist $x+1$ disjoint non-empty sets $A_1, \ldots, A_{x+1} \subseteq \{1, 2, \ldots, 2^k + x\}$, such that for all $s \leq x + 1$, we have*

$$\sum_{i \in A_s} a_i \equiv 0 \pmod{2^{k+1}}.$$

The case of $x = 0$ in Lemma 7.11 follows from the standard statement that amongst m numbers there is a non-empty subset whose sum is divisible by m, but already the $x = 1$ case is far from trivial. Guaranteeing $x + 1$ subsets whose sum is divisible by 2^{k+1} in the second point of Lemma 7.11 is easy; the power of our lemma lies in the fact that we can take these sets to be disjoint from each other. Our proof of Lemma 7.11 relies on a series of compressions and a downward induction on x with base case $x = 2^k - 1$. Our proof of Lemma 7.11 is quite lengthy; it would be interesting to see a shorter proof.

Given Theorem 7.10, we would be surprised if these constructions were not best possible amongst all sets.

Conjecture 7.12. *Let d, n be positive integers with $d \leq n$. Then $\mathcal{C}_d$ is the largest d-cube free subset of $\mathbb{Z}_{2^n}$.*

This chapter is organised as follows. In Section 7.2 we focus on the $d = 2^\ell$ case and prove Theorem 7.6. The construction $\mathcal{C}_d$ is defined in Section 7.3, and there we also prove Lemma 7.11 and Theorem 7.10. Theorem 7.8, our partial result on the Samotij-Sudakov question, is proved in Section 7.4. Some further open questions and conjectures are given in Section 7.5.

7.2 When d is a power of two

Our main goal in this section is to prove Theorem 7.6. We will prove the following stronger statement, which immediately implies Theorem 7.6.

Theorem 7.13. *Let $\ell, n \in \mathbb{Z}^+$ be integers with $2^\ell \leq n$. If $A \subset \mathbb{Z}_{2^n}$ satisfies $|A| > \left(1 - \frac{1}{2^\ell}\right) 2^n$, then there exist integers $x, y \in [2^n]$ such that $\Sigma^*\{\underbrace{x, x, \ldots, x}_{2^\ell - 1}, y\} \subseteq A$.*

We will first need the following simple claim.

Claim 7.14. If $A \subset \mathbb{Z}_{2^n}$ has size $|A| > \left(1 - \frac{1}{2^\ell - 1}\right) 2^n$ then there exists an integer $x \in \mathbb{Z}_{2^n}$ such that $\{x, 2x, 3x, \ldots, \left(2^\ell - 1\right) x\} \subseteq A$.

Proof. Recall the definition of the layers $(L_i)_{i=1}^{n+1}$ from Section 7.1. For an integer $1 \leq a \leq n$, define the set $\mathcal{F}_a$ as

$$\mathcal{F}_a := \left\{ \left\{x, 2x, 3x, \ldots, \left(2^\ell - 1\right) x\right\} : x \in L_a \right\}.$$

Note that if $a \leq n - \ell + 1$ then all elements of $\mathcal{F}_a$ have size exactly $2^\ell - 1$. Indeed, if $i_1 x = i_2 x$ for some $1 \leq i_1 < i_2 \leq 2^\ell - 1$ then $2^n | (i_2 - i_1)x$. As $a \leq n - \ell + 1$, we have that x is not divisible by $2^{n-\ell+1}$, moreover 2^ℓ cannot divide $i_2 - i_1$.

The proof goes by contradiction, let A be a counterexample to the statement of Claim 7.14. Let $B = \{x, 2x, 3x, \ldots, \left(2^\ell - 1\right) x\}$ be an element of $\mathcal{F}_a$ and observe that $|B \cap L_a| = 2^{\ell-1}$, $|B \cap L_{a+1}| = 2^{\ell-2}$, etc, and $|B \cap L_{a+\ell-1}| = 1$. Note moreover that every element of $\bigcup_{i=a}^{a+\ell-1} L_i$ appears in precisely $2^{\ell-1}$ different elements of $\mathcal{F}_a$. As for every element $B \in \mathcal{F}_a$ there exists an element $x_B \in B$ with $x_B \notin A$, this implies that

$$\frac{|A \cap (L_a \cup L_{a+1} \cup \ldots \cup L_{a+\ell-1})|}{|L_a \cup L_{a+1} \cup \ldots \cup L_{a+\ell-1}|} \leq 1 - \frac{1}{2^\ell - 1}.$$

Now let b be an integer with $n - \ell + 2 \leq b \leq n + 1$ and observe that since $0 \notin A$ (as otherwise we could take $x = 0$) we have

$$\begin{aligned} \frac{|A \cap (L_b \cup L_{b+1} \cup \ldots \cup L_{n+1})|}{|L_b \cup L_{b+1} \cup \ldots \cup L_{n+1}|} &\leq 1 - \frac{1}{|L_{n-\ell+2} \cup \ldots \cup L_{n+1}|} \\ &= 1 - \frac{1}{2^{\ell-1}} \leq 1 - \frac{1}{2^\ell - 1}. \end{aligned}$$

Hence we can partition $\mathbb{Z}_{2^n}$ in at most $\lceil (n+1)/\ell \rceil$ parts such that the density of A in each part is at most $1 - \frac{1}{2^\ell - 1}$. This completes the proof of Claim 7.14. $\square$

Now we are ready to give the proof of Theorem 7.13.

Proof of Theorem 7.13. Let $A \subset \mathbb{Z}_{2^n}$ be a set of size $|A| > \left(1 - 2^{-\ell}\right) 2^n$. Let x be such that $\{x, 2x, 3x, \ldots, \left(2^\ell - 1\right) x\} \subseteq A$, as guaranteed by Claim 7.14. Note that as $|A| > \left(1 - 2^{-\ell}\right) 2^n$ we have

$$A' = A \cap (A - x) \cap (A - 2x) \cap \ldots \cap \left(A - \left(2^\ell - 1\right) x\right) \neq \emptyset.$$

Let y be an arbitrary element of A' and note that then we have $y, y + x, y + 2x, \ldots, y + \left(2^\ell - 1\right) x \in A$. Hence we have that

$$\begin{aligned} &\left\{x, 2x, 3x, \ldots, \left(2^\ell - 1\right) x, y, y + x, \ldots, y + \left(2^\ell - 1\right) x\right\} \\ &\qquad = \Sigma^*\{\underbrace{x, x, \ldots, x}_{2^\ell - 1}, y\} \subseteq A \end{aligned}$$

and A is not 2^ℓ-cube-free. This completes the proof of Theorem 7.13. $\square$

7.3 When d is not a power of two

Our goal in this section is to define the construction $\mathcal{C}_d$ for all integers d, n with n sufficiently large ($n \geq d$, say), then to prove Lemma 7.11 and use it to prove Theorem 7.10.

7.3.1 The construction $\mathcal{C}_d$

Our conjectured largest d-cube free subsets of $\mathbb{Z}_{2^n}$ always consist of the union of some of the first few layers, e.g. $\mathcal{C}_{10} = L_1 \cup L_2 \cup L_3 \cup L_5 \cup L_7$. Which layers we take does not depend on n, as long as the construction makes sense (e.g. L_7 does not exist if $n = 4$). Therefore, when defining $\mathcal{C}_d$ for all d we will always assume that there is enough space in $\mathbb{Z}_{2^n}$ for our construction to fit (i.e. no layers past L_n are included). It will always suffice to take $n \geq d$ in general. For positive integers a, b with $a \leq b$ we will use the notation $L_{[a,b]} := L_a \cup L_{a+1} \cup \ldots \cup L_b$.

We define $\mathcal{C}_d$ recursively as follows.

1. $\mathcal{C}_1 = \emptyset$.

2. If $d \geq 2$ then let ℓ be the largest integer such that $2^\ell \leq d$. Let

$$\mathcal{C}_d := L_{[1,\ell]} \ \cup \ \left\{2^{\ell+1} \cdot x \ : \ x \in \mathcal{C}_{d-2^\ell+1}\right\}.$$

In other words, $\mathcal{C}_d$ is the union of the first ℓ layers, skips $L_{\ell+1}$, and includes a copy of $\mathcal{C}_{d-2^\ell+1}$ in $L_{\ell+2} \cup L_{\ell+3} \cup \ldots$.

The same definition can be rephrased as follows. For any positive integer k, define $\alpha(k)$ to be the largest integer ℓ with $2^\ell \leq k$, and let $\beta(k) := k - \alpha(k) + 1$. Given $d \geq 2$, set $\ell_1 := \alpha(d) + 1$ and let $d_1 := \beta(d)$. Set $\ell_2 := \alpha(d_1) + 2$ and let $d_2 := \beta(d_1)$. Repeat until one of the d_i-s, say d_q, becomes equal to one. We will refer to the resulting sequence $(\ell_1, \ell_2, \ldots, \ell_q)$ as the *block vector* of $\mathcal{C}_d$. Then we construct $\mathcal{C}_d$ by including the first $\ell_1 - 1$ layers, *excluding* the next layer, including the next $\ell_2 - 1$ layers, excluding the layer after these, and so on. Hence, letting $M := \sum_{i=1}^q \ell_i$, we get that

$$\mathcal{C}_d = L_{[1,\ell_1-1]} \cup L_{[\ell_1+1,\ell_1+\ell_2-1]} \cup \ldots \cup L_{[M-\ell_q+1,M-1]}.$$

Example: Suppose we want to find $\mathcal{C}_{26}$.

- The largest power of two not greater than 26 is $2^4 = 16$. So we include the first four layers $L_1 \cup \ldots \cup L_4$ and do not include L_5. We replace 26 by $26 - 15 = 11$.

- The largest power of two not greater than 11 is $2^3 = 8$. Now we include the next three layers $L_6 \cup L_7 \cup L_8$ and skip L_9. We replace 11 by $11 - 7 = 4$.

- As $4 = 2^2$, we include the next two layers $L_{10} \cup L_{11}$. We replace 4 by $4 - 3 = 1$ and stop the algorithm since we hit 1.

So the block vector of $\mathcal{C}_{26}$ is $(5, 4, 3)$ and we have

$$\mathcal{C}_{26} = L_1 \cup L_2 \cup L_3 \cup L_4 \quad \cup \quad L_6 \cup L_7 \cup L_8 \quad \cup \quad L_{10} \cup L_{11} = L_{[1,4]} \cup L_{[6,8]} \cup L_{[10,11]}.$$

We will now use this example and Figure 7.1 to illustrate the intuition behind why this construction is d-cube free.

$$\mathcal{C}_{26} = L_1 \cup L_2 \cup L_3 \cup L_4 \;\cup\; L_6 \cup L_7 \cup L_8 \;\cup\; L_{10} \cup L_{11}$$

$$26 = \quad 15 \quad + \quad 7 \quad + \quad 3 \quad + \quad 1$$

Figure 7.1: $\mathcal{C}_{26}$ is the union of three blocks.

Suppose for the sake of a contradiction that $\mathcal{C}_{26}$ contains a 26-cube, say $\Sigma^*\{x_1, x_2, \ldots, x_{26}\} \subseteq \mathcal{C}_{26}$. Let us call $x_1, \ldots, x_{26}$ the *generators* of the cube. Each of these generators has to lie in either the first block $B_1 = L_1 \cup L_2 \cup L_3 \cup L_4$, the second block $B_2 = L_6 \cup L_7 \cup L_8$ or in the third block $B_3 = L_{10} \cup L_{11}$. Suppose that precisely 16 generators lie in B_1, 7 generators lie in B_2 and 3 lie in B_3. Consider the 16 generators lying in B_1. The numbers in B_1 are all not divisible by 16, but since we have 16 generators in B_1 we can find a subset sum, say S, that is divisible by 16. As $S \in \Sigma^*\{x_1, x_2, \ldots, x_{26}\} \subseteq \mathcal{C}_{26}$ we must have $S \in \mathcal{C}_{26}$, and as S is divisible by 16 we must have $S \in B_2 \cup B_3$. Assume $S \in B_2$. Now the 7 generators in B_2 together with S form 8 numbers, all divisible by 32, hence there is a subset sum S' that is divisible by $8 \cdot 32$ and thus must be in B_3. Now amongst the three generators in B_3 together with S' there is a subset sum divisible by $4 \cdot 2^9$ and is thus not in $\mathcal{C}_{26}$, which is a contradiction.

The difficulty with making the above intuition rigorous is the following observation. Suppose we are given that 17 instead of 16 of the generators lie in B_1. Then we can find two sets $S_1, S_2 \subset \{x_1, \ldots, x_{26}\} \cap B_1$ such that the sums of elements in S_1 and in S_2 are both divisible by 16. The issue is that if S_1 and S_2 are not disjoint (say they both contain x_1), then the number $\sum_{x \in S_1} x + \sum_{x \in S_2} x$ is not necessarily an element of $\Sigma^*\{x_1, x_2, \ldots, x_{26}\}$ and hence need not be contained in $\mathcal{C}_{26}$. Luckily Lemma 7.11 guarantees that we may take S_1 and S_2 to be disjoint and the proof is complete.

7.3.2 Proof of Theorem 7.10 assuming Lemma 7.11

The proof consists of two parts. First we use the ideas outlined in the previous section, together with Lemma 7.11, to show that $\mathcal{C}_d$ does not contain a d-cube. Then we use a simple construction to show that no other set that is a union of layers can be both d-cube free and larger than $\mathcal{C}_d$.

Claim 7.15. For any $d \geq 1$, $\mathcal{C}_d$ does not contain a d-cube.

Proof. The proof proceeds by induction on d, with $\mathcal{C}_1 = \emptyset$ not containing any 1-cube for any $n \geq d = 1$. Let $(\ell_1, \ell_2, \ldots, \ell_q)$ be the block vector of $\mathcal{C}_d$, so that $\mathcal{C}_d = L_{[1,\ell_1-1]} \cup L_{[\ell_1+1,\ell_1+\ell_2-1]} \cup \ldots \cup L_{[M-\ell_q+1,M-1]}$, where $M = \sum_{i=1}^{q} \ell_i$. Let the blocks of $\mathcal{C}_d$ be defined in the natural way as $B_1 = L_{[1,\ell_1-1]}$, $B_2 = L_{[\ell_1+1,\ell_1+\ell_2-1]}$, and in general for $1 \leq i \leq q$ we set $B_i = L_{[\ell_1+\ldots+\ell_{i-1}+1,\ell_1+\ldots+\ell_i-1]}$. Assume for the sake of a contradiction that $\Sigma^*\{x_1, \ldots, x_d\} \subset \mathcal{C}_d$. By rearranging we can find an integer d_1 such that $x_i \in B_1$ if and only if $i \leq d_1$. The proof splits into two cases according to how large d_1 is.

Suppose first that $d_1 \leq 2^{\ell_1-1} - 1$. Then $\{x_{d_1+1}, \ldots, x_d\} \subset \mathcal{C}_d \setminus B_1$ and hence

$$\Sigma^*\{x_{d_1+1}, \ldots, x_d\} \subset \mathcal{C}_d \setminus B_1.$$

Observe that $\{2^{\ell_1} \cdot x : x \in \mathcal{C}_{d-2^{\ell_1-1}+1}\} = \mathcal{C}_d \setminus B_1$, and hence

$$\Sigma^* \left\{ \frac{x_{d_1+1}}{2^{\ell_1}}, \frac{x_{d_1+2}}{2^{\ell_1}}, \ldots, \frac{x_d}{2^{\ell_1}} \right\} \subset \mathcal{C}_{d-2^{\ell_1-1}+1}.$$

This is a contradiction, as $\mathcal{C}_{d-2^{\ell_1-1}+1}$ does not contain a $\big(d - 2^{\ell_1-1} + 1\big)$-cube.

Hence we must have $d_1 \geq 2^{\ell_1-1}$. Applying Lemma 7.11 with $k = \ell_1 - 1$ we conclude that either there is a subset of $\{x_1, \ldots, x_{d_1}\}$ whose sum is divisible by 2^{ℓ_1-1} but not by 2^{ℓ_1}, or we can find $d_1 - 2^{\ell_1-1} + 1$ disjoint non-empty sets $A_1, \ldots, A_{d_1-2^{\ell_1-1}+1} \subseteq \{1, 2, \ldots, d_1\}$ such that for all $s \leq d_1 - 2^{\ell_1-1} + 1$ we have $\sum_{i \in A_s} a_i \equiv 0 \pmod{2^{\ell_1}}$. The first option is impossible, as $\mathcal{C}_d \cap L_{\ell_1} = \emptyset$, hence the second option must occur. For all j with $1 \leq j \leq d_1 - 2^{\ell_1-1} + 1$ let us set $s_j := \sum_{i \in A_j} a_i$. Then as the A_i were disjoint, we have that

$$\Sigma^*\{s_1, s_2, \ldots, s_{d_1-2^{\ell_1-1}+1},\ a_{d_1+1}, a_{d_2+2}, \ldots, a_d\} \subset \mathcal{C}_d \setminus B_1.$$

Hence $\mathcal{C}_d \setminus B_1$ contains a $\big(d - 2^{\ell_1-1} + 1\big)$-cube. As before, this implies that $\mathcal{C}_{d-2^{\ell_1-1}+1}$ contains a $\big(d - 2^{\ell_1-1} + 1\big)$-cube which is a contradiction. This completes the proof of Claim 7.15. $\square$

Claim 7.16. For any $d \geq 1$, if $S \subset \mathbb{Z}_{2^n}$ is a union of layers and $|S| > |\mathcal{C}_d|$, then S contains a d-cube.

Proof. As in the proof of Claim 7.15, let $(\ell_1, \ell_2, \ldots, \ell_q)$ be the block vector of $\mathcal{C}_d$, so that $\mathcal{C}_d = L_{[1,\ell_1-1]} \cup L_{[\ell_1+1,\ell_1+\ell_2-1]} \cup \ldots \cup L_{[M-\ell_q+1,M-1]}$, where $M = \sum_{i=1}^{q} \ell_i$. Let s be index of the first layer where $\mathcal{C}_d$ and S differ. Because $|L_{i-1}| = 2|L_i|$ for all $i \leq n$, we must have $L_s \subset S$ and $L_s \cap \mathcal{C}_d = \emptyset$. We will show that $L_{[1,s]} \cap S$ contains a d-cube.

For all $i \leq q$, let $M_k := \sum_{j=1}^{i} \ell_j$. Let k be the largest integer in the set $\{1, 2, \ldots, q\}$ that satisfies $M_k \leq s$. We split the proof into two cases, according to whether $s \leq M$ or $s > M$.

If $s \leq M$ then note that $s = M_k$ and we build a multiset T of size d, which will be the collection of the generators of the d-cube we find in S, as follows. First, include $2^{\ell_k} - 1$ copies of $2^{M_{k-1}}$ into T. Since $L_{[M_{k-1}+1,M_k]} \subset S$, every subset sum of these is in S. Next, for all $i \leq k-1$, include $2^{\ell_i - 1} - 1$ copies of $2^{M_{i-1}}$ into T. Given any subset of T, the largest power of two dividing its sum is determined by its smallest summands and their quantity, and hence it is easily verified that $\Sigma^* T \subset S$.

By construction

$$|T| = \sum_{i=1}^{k-1} \left(2^{\ell_i - 1} - 1\right) + 2^{\ell_k} - 1.$$

By the definition of the block vector of $\mathcal{C}_d$, we have that

$$\alpha\left(d - \sum_{i=1}^{k-1} \left(2^{\ell_i - 1} - 1\right)\right) = \ell_k - 1.$$

Hence $d - \sum_{i=1}^{k-1} \left(2^{\ell_i - 1} - 1\right) \leq 2^{\ell_k} - 1$ and so $|T| \geq d$, as required.

The case $s > M$ is easier; we construct T as follows. For all $i \leq q$, let T contain $2^{\ell_i - 1} - 1$ copies of $2^{M_{i-1}}$. Moreover, add to T one copy of the number 2^{s-1} (which is an element of L_s). Checking that $\Sigma^* T \subset S$ is similar to the above, and $|T| = d$ follows from the definition of the block vector of $\mathcal{C}_d$. This finishes the proof of Claim 7.16. □

Note that it is easy to modify the proof of Claim 7.16 to show that the constructions $\mathcal{C}_d$ are maximal, i.e. adding a single element to $\mathcal{C}_d$ makes it not d-cube-free. Now Theorem 7.10 follows from Claims 7.15 and 7.16. It only remains to prove Lemma 7.11, which is highly technical.

7.3.3 The proof of Lemma 7.11

Given a multiset $S = \{a_1, \ldots, a_d\}$ of size d, we define the *iterated sumset* of S as

$$S^* = \left\{ \sum_{i \in I} a_i : I \subseteq [k] \right\},$$

so that $S^* = (\Sigma^* S) \cup \{0\}$. The reader should be aware that we will be dealing with both sets and multisets in what follows. Anything that is not an iterated sumset is a multiset, also referred to as a collection. Iterated sumsets (of multisets) are just sets, as defined above. Given a residue t modulo 2^{k+1} we define $|t|$ to be the minimal absolute value of any integer in the residue class of t modulo 2^{k+1}. We will also refer to $|t|$ as the *absolute value* of t. Given an integer λ and a multiset $\mathcal{C}$ we define $\lambda \cdot \mathcal{C} = \{\lambda c : c \in \mathcal{C}\}$ (where the RHS is a multiset).

To begin the proof of Lemma 7.11, let $\mathcal{C}$ be a collection of $2^k + r$ non-zero residues modulo 2^{k+1} with the property that no sub-collection sums to 2^k modulo 2^{k+1}. Then we need to show $\mathcal{C}$ contains at least $r+1$ disjoint, non-empty subsets whose sums are 0 modulo 2^{k+1}.

The proof will involve two separate ideas. One, which we shall return to later, involves building up the iterated sumset $\mathcal{C}^*$ by introducing elements of $\mathcal{C}$ one by one, and analysing how it can grow. This idea was used by Alon and Freiman ([5], Lemma 4.2) in the following lemma.

Lemma 7.17 (Special case of Alon–Freiman Lemma)**.** *Any collection $\mathcal{C}$ of $2^{k+1} - 1$ non-zero residues modulo 2^{k+1} contains a non-empty sub-collection summing to 2^k modulo 2^{k+1}.*

Proof. Let $\mathcal{C} = \{c_1, \ldots, c_m\}$ where $m = 2^{k+1} - 1$ and let $\mathcal{C}_i = \{c_1, \ldots, c_i\}$. Note that $|\mathcal{C}_1^*| = 2$ and $\mathcal{C}_i^* \subseteq \mathcal{C}_{i+1}^*$ for all i. If $\mathcal{C}_{i+1}^* = \mathcal{C}_i^*$ for some i then $\mathcal{C}_i^*$ contains the cyclic subgroup of $\mathbb{Z}_{2^n}$ generated by c_{i+1}. Since 2^k is an element of every non-trivial subgroup of $\mathbb{Z}_{2^n}$, this completes the proof. $\square$

The second idea is a compression that involves replacing elements of $\mathcal{C}$ in a way that does not change the iterated sumset $\mathcal{C}^*$ or the number of sub-collections that sum to zero. These compressions fall into three categories, which we introduce with the following definition.

Definition 7.18. *Let $\mathcal{C}$ be a collection of non-zero residues modulo 2^{k+1} with the property that no sub-collection sums to 2^k modulo 2^{k+1}. Suppose that $\mathcal{C}$ contains at least $\lambda > 0$ copies of residues ± 1 and also a residue t with $1 < |t| \leq \lambda + 1$. A* type 1 compression *replaces t with $|t|$ copies of the residue 1 if $t \in [1, 2^k - 1]$ and* *with $|t|$ copies of -1 otherwise. Alternatively, suppose that $\mathcal{C}$ contains a residue $-t$ and two copies of the residue $2^k - t$. Then replacing the two copies of $2^k - t$ with two copies of $-t$ is called a* type 2 compression. *Finally, if we instead have that $\mathcal{C}$ contains at least 2^{k-1} copies of ± 1 and if we have elements u and v that lie in the range $[(3/2)2^{k-1}, 2^k - 1]$ then a* type 3 compression *replaces u and v with $u - 2^k$ and $v - 2^k$.*

Note that a type 1 compression which replaces the element t increases the number of elements in $\mathcal{C}$ by $|t| - 1$. Therefore, provided that we can show that the number of disjoint subsets summing to zero modulo 2^{k+1} does not increase by more than $|t| - 1$ we will be able to proceed by induction. Type 2 and type 3 compressions do not change the number of elements of $\mathcal{C}$ so we will not be able to immediately apply an induction hypothesis, but provided that we can show that these compressions do not add to the iterated sumset and do not increase the number of disjoint subsets summing to zero modulo 2^{k+1} then we will be able to apply them to modify $\mathcal{C}$ in potentially helpful ways. We now prove these properties.

Lemma 7.19. *Let $\mathcal{C}$ be a collection of non-zero residues modulo 2^{k+1} with the property that no sub-collection sums to 2^k modulo 2^{k+1}. Let $T_1(\mathcal{C})$ be the result of applying a type 1 compression to $\mathcal{C}$ (if possible), let $T_2(\mathcal{C})$ be the result of applying a type 2 compression to $\mathcal{C}$ (if possible) and let $T_3(\mathcal{C})$ be the result of applying a type 3 compression to $\mathcal{C}$ (if possible). Then $T_1(\mathcal{C})^* = \mathcal{C}^*$, $T_2(\mathcal{C})^* \subseteq \mathcal{C}^*$ and $T_3(\mathcal{C})^* \subseteq \mathcal{C}^*$. Moreover, if $T_2(\mathcal{C})$ or $T_3(C)$ contain m disjoint sub-collections summing to 0 modulo 2^{k+1} then so does $\mathcal{C}$. Lastly, if we write $T_1(C, t)$ for the result of applying a type 1 compression to $\mathcal{C}$ which replaces the element t then we have that if $T_1(\mathcal{C}, t)$ contains $m + |t| - 1$ disjoint sub-collections summing to 0 modulo 2^{k+1} then $\mathcal{C}$ contains at least m such sub-collections.*

Proof. We prove the lemma separately for $T_1(\mathcal{C})$, $T_2(\mathcal{C})$ and $T_3(\mathcal{C})$, starting with $T_1(\mathcal{C})$.

Without loss of generality we assume that $t \in [2, 2^k - 1]$ (otherwise we multiply by -1). Using the notation $x^{(\alpha)}$ to represent the multiset containing α copies of x, we begin by observing that, given $\alpha + \beta = t - 1$, we have

$$\{1^{(\alpha)}, (-1)^{(\beta)}, t\}^* = [-\beta, \alpha] \cup [t - \beta, t + \alpha]$$

and

$$\{1^{(\alpha+t)}, (-1)^{(\beta)}\}^* = [-\beta, t + \alpha].$$

However, since $\alpha + \beta = t - 1$ and so $\alpha = t - \beta - 1$ so

$$[-\beta, \alpha] \cup [t - \beta, t + \alpha] = [-\beta, t + \alpha].$$

It follows that $T_1(\mathcal{C})^* = \mathcal{C}^*$.

For T_2 we also assume without loss of generality that $t \in [1, 2^k - 1]$. We observe that

$$\{-t, 2^k - t, 2^k - t\}^* = \{-t, -2t, -3t, 2^k - t, 2^k - 2t\}$$

and

$$\{-t, -t, -t\}^* = \{-t, -2t, -3t\} \subseteq \{-t, 2^k - t, 2^k - t\}^*$$

whence $T_2(\mathcal{C})^* \subseteq \mathcal{C}^*$.

For T_3 we observe that, given $\alpha + \beta = 2^{k-1}$, we have

$$\{1^{(\alpha)}, (-1)^{(\beta)}, u - 2^k, v - 2^k\}^*$$

$$= [-\beta, \alpha] \cup [u - 2^k - \beta, u - 2^k + \alpha] \cup [v - 2^k - \beta, v - 2^k + \alpha] \cup [u + v - \beta, u + v + \alpha].$$

However, since u and v that lie in the range $[(3/2)2^{k-1}, 2^k - 1]$ we have that $u + v$ lies in the range $[-2^{k-1}, -2]$ and therefore

$$[u - 2^k - \beta, u - 2^k + \alpha] \cup [v - 2^k - \beta, v - 2^k + \alpha] \subseteq [-\beta, \alpha] \cup [u + v - \beta, u + v + \alpha]$$

and therefore

$$\{1^{(\alpha)}, (-1)^{(\beta)}, u - 2^k, v - 2^k\}^* \subseteq \{1^{(\alpha)}, (-1)^{(\beta)}, u + v\}^* \subseteq \{1^{(\alpha)}, (-1)^{(\beta)}, u, v\}^*.$$

This shows that $T_3(\mathcal{C})^* \subseteq \mathcal{C}^*$.

Now suppose that $S_1, \ldots, S_{m+t-1}$ are disjoint sub-collections of $T_1(\mathcal{C}, t)$ which all sum to 0 modulo 2^{k+1} (as above we assume without loss of generality that $t \in [2, 2^k - 1]$). Consider the t copies of the element 1 in $T_1(\mathcal{C}, t)$ that result from the type 1 compression replacing t. If these do not all appear in distinct S_i then by considering the S_j which do not contain any of these 1s we obtain at least m disjoint sub-collections of $\mathcal{C}$ summing to 0 modulo 2^{k+1}. If they do all appear in distinct S_i then combining the S_i in which they appear into a single big sub-collection S we may replace the t copies of 1 in S with a copy of t to obtain a sub-collection of $\mathcal{C}$. Combined with the rest of the S_j, we obtain m disjoint sub-collections of $\mathcal{C}$ summing to 0 modulo 2^{k+1}.

In the case of T_2 the situation is slightly different. Suppose that $S_1, \ldots, S_m$ are disjoint sub-collections of $T_2(\mathcal{C})$ which all sum to 0 modulo 2^{k+1}. The type 2 compression replaced two copies of $2^k - t$ with two copies of $-t$. Undoing these replacements in the S_i gives m disjoint sub-collections of $\mathcal{C}$ which all have sum either 0 or 2^k modulo 2^{k+1}. But since no sub-collection of $\mathcal{C}$ sums to 2^k modulo 2^{k+1} we get m disjoint sub-collections of $T_2(\mathcal{C})$ which all have sum 0.

Lastly, we consider the disjoint sums following a type 3 compression. Again, let $S_1, \ldots, S_m$ be disjoint sub-collections of $T_3(\mathcal{C})$ which all sum to 0 modulo 2^{k+1}. The type 3 compression replaced u and v with $u - 2^k$ and $v - 2^k$. As above, we see that undoing these replacements in the S_i gives m disjoint sub-collections of $\mathcal{C}$ which all have sum either 0 or 2^k modulo 2^{k+1}. But since no sub-collection of $\mathcal{C}$ sums to 2^k modulo 2^{k+1} we get m disjoint sub-collections of $T_3(\mathcal{C})$ which all have sum 0. $\square$

The proof of Lemma 7.11 will proceed by induction on both k and r. The induction on r will proceed downwards from $r = 2^k - 2$ – more details on this will follow. In light of Lemma 7.19 we will be able to use our induction hypothesis on r if we are ever able to apply a type 1 compression (to $\mathcal{C}$ itself or to any $\lambda \cdot \mathcal{C}$ where λ is odd). We will therefore care about the properties of 'maximally type 1 compressed' collections.

To this end, we return to the idea presented in Lemma 7.17. For our purposes the idea will need to be extended a little, requiring a more detailed analysis of how the iterated sumset of $\mathcal{C}$ can grow whilst avoiding the residue 2^k under the additional assumption that no $\lambda \cdot \mathcal{C}$ for λ odd can be type 1 compressed. This process is captured in the following lemma.

Lemma 7.20. *Let $k \geq 3$ and $r \geq 1$. Let $\mathcal{C}$ be a collection of $2^k + r$ non-zero residues modulo 2^{k+1} with the property that no sub-collection sums to 2^k modulo 2^{k+1}. Assume also that no $\lambda \cdot \mathcal{C}$ for λ odd can be type 1 compressed. Then either $\mathcal{C}$ contains $2^{k-1} + r$ even residues, or there is an odd residue t (modulo 2^{k+1}) such that $\mathcal{C}$ contains $2^{k-1} + r$ residues which are either $\pm t$ or $\pm(2^k - t)$.*

In order to prove this lemma, we shall need two technical lemmas that analyse the process of building $\mathcal{C}^*$ by taking into account new elements of $\mathcal{C}$ one by one.

Lemma 7.21. *Let $\mathcal{C}$ be a collection of $2^k + r$ non-zero residues modulo 2^{k+1} with the property that no sub-collection sums to 2^k modulo 2^{k+1}. Assume also that no $\lambda \cdot \mathcal{C}$ for λ odd can be type 1 compressed. For some fixed i let $\mathcal{C}_i \subset \mathcal{C}$ with $|\mathcal{C}_i| = i$. Now choose $\mathcal{C}_{i+1} = C_i \cup \{x_j\}$ for some $x_j \in \mathcal{C} \setminus \mathcal{C}_i$ so that $\mathcal{C}^*_{i+1} \setminus \mathcal{C}^*_i$ is* *maximal. We claim that* $|\mathcal{C}_{i+1}^* \setminus \mathcal{C}_i^*|$ *is greater than 2 unless one of the following cases holds:*

1. $|\mathcal{C}_i^*| \leq 5$ *or* $|\mathcal{C}_i^*| \geq 2^{k+1} - 5$.

2. *All elements of* $\mathcal{C} \setminus \mathcal{C}_i$ *are even.*

3. *All elements of* $\mathcal{C} \setminus \mathcal{C}_i$ *are either* $\pm u$ *or* $\pm(2^k - u)$ *modulo* 2^{k+1} *for some odd* u.

Proof. Suppose we reach a point where all remaining generators increase the sumset by at most 2. Let our pool of remaining generators be called $T = \mathcal{C} \setminus \mathcal{C}_i$, and our iterated sumset so far is $\mathcal{C}_i^*$. Assume that $|\mathcal{C}_i^*| \geq 6$ or we are in case (1).

If there are no odd elements in T then all remaining elements are even and we are in case (2). So assume that there is an odd generator in T. Without loss of generality (by multiplying everything by some odd residue λ) we may assume this generator is ± 1. Our goal is now to show that all remaining generators are ± 1 or $\pm(2^k - 1)$.

Since by assumption we know that including the ± 1 increases the iterated sumset size by at most 2, we have that $\mathcal{C}_i^*$ is a union of two intervals I_1 and I_2 (of course it could be a single interval, which is also a union of two intervals).

Now we consider the possibilities for other elements in T. Suppose $a \in T$. We wish to show $a = \pm 1$ or $\pm(2^k - 1)$.

We have that $|((I_1 + a) \cup (I_2 + a)) \setminus (I_1 \cup I_2)| \leq 2$. Clearly this allows a to be equal to ± 1. If a is ± 2 then we are done because we can do a type 1 compression and replace the ± 2 with two ± 1s. As usual, we may assume that $a \in [1, 2^k - 1]$ without loss of generality, by multiplying everything by -1.

Suppose $I_1 + a$ intersects I_1 but does not intersect I_2. Then since $|(I_1 + a) \setminus I_1| \leq 2$ we have $a \leq 2$ and so $a = 1$ (since $a = 2$ is forbidden), or $|I_1| \leq 2$. But the latter case also implies $a = 1$ since $I_1 + a$ intersects I_1. Similarly, if $I_2 + a$ intersects I_2 but does not intersect I_1 then we get that $a = 1$.

For a proper interval I we denote the $x \in I$ such that $x + 1 \notin I$ by $M(I)$ and the $x \in I$ such that $x - 1 \notin I$ by $m(I)$.

By the above, if $I_1 + a$ intersects I_1 we may assume that $I_1 + a$ also intersects I_2. So $I_1 + a$ contains the entire gap I_g between $M(I_1)$ and $m(I_2)$ which must therefore have size at most 2. So now let I be the whole interval consisting of I_1, I_2 and I_g. Note that $I + a$ contains at least $\min(a, 4)$ new elements that do not belong in I, since $|I| \leq 2^{k+1} - 6 + |I_g| \leq 2^{k+1} - 4$ by assumption. But

$$2 \geq |((I_1 + a) \cup (I_2 + a)) \setminus (I_1 \cup I_2)| \geq |(I + a) \setminus I| - |I_g| + |I_g|$$

since at most $|I_g|$ elements from $(I + a) \setminus I$ do not belong to $((I_1 + a) \cup (I_2 + a))$ and also $I_g \subseteq ((I_1 + a) \cup (I_2 + a))$. So $a \leq 2$ and therefore $a = 1$ (since $a = 2$ is forbidden).

Therefore we may assume that $I_1 + a$ is disjoint from I_1 and $I_2 + a$ is disjoint from I_2.

Observe that

$$|((I_1 + a) \cup (I_2 + a)) \setminus (I_1 \cup I_2)| = |(I_1 + a) \setminus I_2| + |(I_2 + a) \setminus I_1|$$

since I_1 and I_2 are disjoint.

We have that

$$|(I_1 + 2a) \setminus I_1| \leq |(I_2 + a) \setminus I_1| + |(I_1 + 2a) \setminus (I_2 + a)| = |(I_2 + a) \setminus I_1| + |(I_1 + a) \setminus I_2|$$

so

$$\begin{aligned} 2 &\geq |(I_1 + a) \setminus I_2| + |(I_2 + a) \setminus I_1| \\ &= |(I_1 + 2a) \setminus I_1|. \end{aligned}$$

Similarly $|(I_2 + 2a) \setminus I_2| \leq 2$. Since $|I_1| + |I_2| \geq 6$ we may assume without loss of generality that $|I_1| \geq 3$. But since $|(I_1 + 2a) \setminus I_1| \leq 2$, we deduce that $2a = \pm 1$ or $2a = \pm 2$, so a is ± 1 or $\pm(2^k - 1)$. $\square$

We need one final technical lemma that controls the size of $\mathcal{C}_3$.

Lemma 7.22. *Let $\mathcal{C}$ be a collection of $2^k + r$ non-zero residues modulo 2^{k+1} with the property that no sub-collection sums to 2^k modulo 2^{k+1}. Assume also that no $\lambda \cdot \mathcal{C}$ for λ odd can be type 1 compressed. Either $\mathcal{C}$ contains $2^{k-1} + r$ even residues, or there is an odd residue t (modulo 2^{k+1}) such that $\mathcal{C}$ contains $2^{k-1} + r$ residues which are either $\pm t$ or $\pm(2^k - t)$, or we can find 3 elements $\{x_1, x_2, x_3\}$ in $\mathcal{C}$ so that $|\{x_1, x_2, x_3\}^*| \geq 6$.*

Proof. Suppose we can find $x_1, x_2, x_3 \in \mathcal{C}$ odd and distinct. Then $\{x_1, x_2, x_3\}^*$ contains $\{x_1, x_2, x_3, x_1 + x_2, x_1 + x_3, x_2 + x_3\}$ which are all distinct and we are done.

Suppose there are at most 2^{k-1} even elements in $\mathcal{C}$. Then we may now assume that the odd residues, of which there are at least $2^{k-1} + r$, all lie in at most two residue classes, say u and v, and without loss of generality u appears at least twice. If $u \neq v$ then

$$\{0, u, v, 2u, u+v, 2u+v\} \subseteq \{u, u, v\}^*$$

and so $|\{u, u, v\}^*| \geq 6$ unless $u = 2^k \pm v$ or $u = \pm v$. In this case there is an odd residue t (modulo 2^{k+1}) so that u and v are both either $\pm t$ or $\pm(2^k - t)$ so we are done.

We may now assume that $\mathcal{C}$ contains more than 2^{k-1} even residues, so in particular it contains a repeated even residue u. Note also that if $\mathcal{C}$ does not contain $2^{k-1} + r$ even residues then it contains at least 2^{k-1} odd residues. Let v be an odd element of $\mathcal{C}$. Then $\{0, u, 2u, v, u+v, 2u+v\} \subseteq \{u, u, v\}^*$ and these are all distinct, so $|\{u, u, v\}^*| \geq 6$. $\square$

We are now ready to prove Lemma 7.20.

Proof of Lemma 7.20. By Lemma 7.22 we are either immediately done or we may find $\mathcal{C}_3$ of size 3 such that $|\mathcal{C}_3^*| \geq 6$. Now inductively define $\mathcal{C}_{i+1} = C_i \cup \{x\}$ for some choice of x in $\mathcal{C} \setminus \mathcal{C}_i$ so that $\mathcal{C}_{i+1}^* - \mathcal{C}_i^*$ is maximal. Let $j \geq 3$ be minimal so that $|\mathcal{C}_{j+1}^* \setminus \mathcal{C}_j| \leq 2$. Then $|\mathcal{C}_j^*| \geq 3(j-3) + 6 = 3j - 3$. Note that we always have that $|\mathcal{C}_{i+1}^* - \mathcal{C}_i^*| \geq 1$, since, by Lemma 7.17, the size of the iterated sumset increases by at least 1 whenever a new element is introduced. Since $|\mathcal{C}^*| \leq 2^{k+1} - 1$ we have

$$3j - 3 + 2^k + r - j \leq 2^{k+1} - 1$$

which implies that

$$j \leq 2^{k-1} - \left\lceil \frac{r}{2} \right\rceil + 1$$

and so

$$2^k + r - j \geq 2^{k-1} + \left\lceil \frac{3r}{2} \right\rceil - 1 \geq 2^{k-1} + r.$$

This is the number of remaining elements in $\mathcal{C}$. By Lemma 7.21 we may deduce that either:

1. $|\mathcal{C}_i^*| \leq 5$ or $|C_i^*| \geq 2^{k+1} - 5$.
2. All remaining elements of $\mathcal{C}$ are even.
3. All remaining elements of $\mathcal{C}$ are either $\pm u$ or $\pm(2^k - u)$ modulo 2^{k+1} for u odd.

In case (2) or (3) we are done since the number of remaining elements is at least $2^{k-1}+r$. Note also that $|\mathcal{C}_i^*| \geq |C_3^*| \geq 6$, while if $|C_i^*| \geq 2^{k+1}-5$ then we must have $2^{k-1}+r \leq 4$ since each remaining element increases the size of the iterated sumset by 1 and $|\mathcal{C}^*| \leq 2^{k+1}-1$, but $k \geq 3$ and $r \geq 1$ so this is impossible. $\square$

We are finally ready to prove Lemma 7.11.

Proof of Lemma 7.11. We prove the result by induction on k and r. The induction proceeds downwards on r starting at a base case $r = 2^k - 2$. The case $r = 0$ is also done separately. These base cases are covered in detail as follows.

1. $k \leq 2$: This requires checking that given $4+r$ residues modulo 8 that avoid a sum of 4 modulo 8 then we can find $r+1$ sums which are 0 modulo 8, since the $k = 1$ case has $0 \leq r \leq 2-2 = 0$ so is trivial. The above check for $k = 2$ can be done by hand.

2. $r = 0$: This follows immediately from the standard result that among n numbers there is a sum which is 0 modulo n. In our case we have 2^k numbers in $\mathcal{C}$ so there is a non-trivial sub-collection with sum 0 modulo 2^k. Since no sum is 2^k modulo 2^{k+1} the sum is in fact 0 modulo 2^{k+1}.

3. $r = 2^k - 2$: Observe that in this case we have $|\mathcal{C}| = 2^{k+1}-2$ and $|\mathcal{C}^*| = 2^{k+1}-1$. Therefore if we list the elements of $\mathcal{C} = \{x_1, \ldots, x_s\}$ and define $\mathcal{C}_i = \{x_1, \ldots, x_i\}$ then we must have that $|\mathcal{C}_i^*| = i+1$ for every i. In particular, $|\mathcal{C}_{i+1}^* \setminus C_i^*| = 1$ for each i. In particular, C_2^* has size 3. This holds for any ordering of the x_i, but if $u \neq \pm v$ then $|\{u,v\}^*| = 4$ so we find that all x_i are $\pm t$ for some residue t. Since $|\mathcal{C}| = 2^{k+1}-2$ we find that t cannot be even (otherwise all elements of $\mathcal{C}$ are even and so $|\mathcal{C}^*| \leq 2^{k-1}$ which is impossible) so by scaling we may assume all elements of $\mathcal{C}$ are ± 1. The only possibility that avoids a sum of 2^k is to have $2^k - 1$ copies of $+1$ and $2^k - 1$ copies of -1. In this case we get $2^k - 1 = r+1$ disjoint pairs which sum to 0 (by pairing up the $+1$s and -1s).

For the inductive case we may apply Lemma 7.20. We find that either $\mathcal{C}$ contains $2^{k-1}+r$ even residues, or there is an odd residue t (modulo 2^{k+1}) such that $\mathcal{C}$ contains $2^{k-1}+r$ residues which are either $\pm t$ or $\pm(2^k - t)$. In the first case we may simply divide everything by 2 – we are left with a collection $\mathcal{C}/2$ of $2^{k-1}+r$ residues modulo 2^k which avoid a sum of 2^{k-1} modulo 2^k. By induction on k this contains $r+1$ disjoint sums which are 0 modulo 2^k, and the corresponding disjoint sums in $\mathcal{C}$ are all 0 modulo 2^{k+1}.

So we are left in the case where there is an odd residue t (modulo 2^{k+1}) such that $\mathcal{C}$ contains $2^{k-1}+r$ residues which are either $\pm t$ or $\pm(2^k-t)$. Observe that having $+t$ and $+2^k-t$ is impossible as it gives a sum of 2^k, and similarly for $-t$ and $-(2^k-t)$.

Without loss of generality we assume that $\pm t$ occurs at least once. Then we can use type 2 compressions to obtain that all but at most one of the $2^{k-1}+r$ residues are $\pm t$. By scaling we can assume that $t=1$. We have thus reduced to the case where $\mathcal{C}$ contains at least $2^{k-1}+r-1 \geq 2^{k-1}$ (as if $r=0$ then done) copies of ± 1. Since we are done by induction on r if we are able to perform any type 1 compressions, we deduce that $\mathcal{C}$ does not contain any residues u with $|u| \in [2, 2^{k-1}]$.

Therefore all remaining generators have absolute value in the range $[2^{k-1}+1, 2^k-1]$. Note that if two members u, v of $\mathcal{C}$ lie in the range $[(3/2)2^{k-1}+1, 2^k-1]$ then we can do a type 3 compression to replace u and v with $u-2^k$ and $v-2^k$ respectively. Then since $u-2^k$ and $v-2^k$ have absolute value less than 2^{k-1} we can do a type 1 compression and we are done by induction unless $u-2^k = v-2^k = -1$, ie $u=v=2^k-1$. But after our type 2 compressions we had that all but at most one of the elements equal to ± 1 or $\pm(2^k-1)$ were in fact ± 1, so this cannot arise. Thus if we can find such u and v then we are done.

Similarly, if we find two members u, v of $\mathcal{C}$ lie in the range $[2^k+1, 2^k+2^{k-2}-1]$ then we can multiply by -1 to get two elements in the range $[(3/2)2^{k-1}+1, 2^k-1]$. We are then done as above.

If we find $u \in [(3/2)2^{k-1}+1, 2^k-1]$ and $v \in [2^k+1, 2^k+2^{k-2}-1]$ then we must have a sum congruent to 2^k modulo 2^{k+1}, since we have at least 2^{k-1} elements which are ± 1.

We may therefore assume that $\mathcal{C}$ contains at most one element with absolute value in the range $[(3/2)2^{k-1}+1, 2^k-1]$.

The last range to consider is the elements of $\mathcal{C}$ with absolute value in the range $[2^{k-1}+1, (3/2)2^{k-1}-1]$. Suppose we have elements u and v in $\mathcal{C}$ with absolute value in this range. Then order the elements of $\mathcal{C}$ as follows. Let $x_1=u$ and $x_2=v$. Then let $x_3, \ldots, x_{2^{k-1}+2} = \pm 1$. Then take the remaining elements in any order. Let $\mathcal{C}_i = \{x_1, \ldots, x_i\}$. Note that for $i \leq 2^{k-1}+2$, we have

$$\mathcal{C}_i^* = [u-\beta, u+\alpha] \cup [-\beta, \alpha] \cup [u+v-\beta, u+v+\alpha]$$

where there are α copies of $+1$ and β copies of -1 amongst $x_3, \ldots, x_{2^{k-1}+2}$. By the absolute values of u and v, we see that these 3 intervals are disjoint. So

$|\mathcal{C}^*_{2^{k-1}+2}| \geq 3(2^{k-1}+1)$.

Since adding each further element of $\mathcal{C}$ increases the size of the iterated sumset by at least 1, we get that

$$|\mathcal{C}^*| \geq 3(2^{k-1}+1) + 2^k + r - (2^{k-1}+2) \geq 2^{k+1}$$

which is impossible.

So we conclude that at most 1 element of $\mathcal{C}$ has absolute value in the range $[2^{k-1}+1, (3/2)2^{k-1}-1]$, and thus at most 2 elements of $\mathcal{C}$ have absolute value in the range $[2^{k-1}+1, 2^k-1]$. All other members of $\mathcal{C}$ must be equal to ± 1 or we can do type 1 compressions.

This means that we in fact have at least $2^k+r-2 \geq 2^k-1$ elements of $\mathcal{C}$ which are ± 1. But this means that we can do type 1 compression if *any* element of $\mathcal{C}$ is not equal to ± 1. Since at most 2^k-1 of these can be $+1$, we get at least $2^k+r-(2^k-1) = r+1$ disjoint pairs of $\{\pm 1\}$, giving us $r+1$ disjoint sums equalling 0. This finishes the proof of Lemma 7.11. $\square$

7.4 Families minimising the number of 2-cubes

Our goal in this section is to prove Theorem 7.8. Given n, let $S \subset \mathbb{Z}_{2^n}$ be a set of size $M = 2^{n-1}+1$. Recall that for $M \leq 2^{n-1}$ the centred family of size M is a subset of L_1 and hence contains no Schur triples, and so our $M = 2^{n-1}+1$ case is the first non-trivial case of Conjecture 7.7. Our goal is to prove that any family of size $2^{n-1}+1$ contains at least $3 \cdot 2^{n-1}$ Schur triples, which is the number of Schur triples in the centred set $T = L_1 \cup \{2\}$. We will use for all i the notation $S_i = S \cap L_i$ where L_i is the i-th layer as before, and similarly

$$S_{i+} = S \cap (L_{i+1} \cup L_{i+2} \cup \ldots \cup L_{n+1}) = S_{i+1} \cup S_{i+2} \cup \ldots \cup S_{n+1}.$$

Note that whenever the numbers x, y, z form a Schur triple, they cannot be in three different layers, nor all in the same layer.

Denote by $C(a,b,c)$ the set of Schur triples $(x,y,z) \in S^3$ with $x \in S_a$, $y \in S_b$ and $z \in S_c$. Similarly, let e.g.

$$C(a+,b,c) := \left\{(x,y,z) \in S^3 : \ x+y=z, \ x \in S_{a+}, y \in S_b, z \in S_c\right\}.$$

We will use the following elementary observation on the number of edges in an induced subgraph of a regular graph.

Claim 7.23. Let G be a directed graph, with bidirectional edges (i.e. $x \to y$ and $y \to x$) and loops allowed. Suppose every vertex has out-degree and in-degree equal to k, let $N := |V(G)|$ and let $R \subset V(G)$ be a set of size $|R| = m$. Then the number of edges in the induced subgraph $G[R]$ satisfies

$$E(G[R]) \geq \max\{m(k - N + m), k(2m - N), 0\}.$$

Proof. Since every vertex in R has k edges leaving it, and at most $N - m$ of these end in vertices not in R, it follows that at least $k - N + m$ point to vertices in R, and the first part follows. The middle inequality follows from the observation that there are km edges starting at vertices of R and there are $(N - m)k$ edges ending at vertices not in R. The third part holds since the number of edges cannot be negative. □

Claim 7.24. For any integer a with $1 \leq a \leq n$, we have

$$|C(a, a, a+)| \geq \max\left\{|S_a|\left(|S_{a+}| - |L_a| + |S_a|\right), |S_{a+}|(2|S_a| - |L_a|), 0\right\}.$$

Proof. Let $z \in S_{a+}$. Create a directed graph G_z on vertex set L_a by adding the edge $x \to y$ if $x + y = z$. Every vertex in this graph has in-degree and out-degree one (loops possible) and every edge corresponds to a Schur triple in $C(a, a, a+)$. Let

$$G = \bigcup_{z \in S_{a+}} G_z,$$

so that G is a directed graph with every vertex having in-degree and out-degree exactly $|S_{a+}|$, and every directed edge present at most once, and the bound follows from Claim 7.23. When $S = T$ we either have $|S_{a+}| = 0$ or $S_a = L_a$, with equality in both cases. □

Claim 7.25. For any integer a with $1 \leq a \leq n$, we have

$$|C(a+, a, a)| \geq \max\left\{|S_a|\left(|S_{a+}| - |L_a| + |S_a|\right), |S_{a+}|(2|S_a| - |L_a|), 0\right\}.$$

Proof. The proof is very similar to the proof of Claim 7.24. Fix an element $y \in S_{a+}$ and create a directed graph G_y on vertex set L_a by adding the edge $x \to z$ if $y = z - x$. Then in $G = \cup_{y \in S_{a+}} G_y$ every vertex has in-degree and out-degree $|S_{a+}|$, and the rest of the proof is exactly as in Claim 7.24. When $S = T$ we have equality as before. □

Proof of Theorem 7.8. Recall that $\mathrm{ST}(S)$ denotes the number of Schur triples in

S. By Claims 7.24 and 7.25 we have

$$\begin{aligned}\mathrm{ST}(S) &= \sum_{a=1}^{n} C(a+,a,a) + C(a,a+,a) + C(a,a,a+) \\ &\geq 3 \max \left\{ |S_a| \left(|S_{a+}| - |L_a| + |S_a| \right), |S_{a+}|(2|S_a| - |L_a|), 0 \right\}.\end{aligned}$$

It suffices to show that amongst sets of size M the function

$$f(S) = 3 \sum_{a=1}^{n} \max \left\{ |S_a| \left(|S_{a+}| - |L_a| + |S_a| \right), |S_{a+}|(2|S_a| - |L_a|), 0 \right\}$$

is never less than $3|L_1|$.

For every element $x \in \mathbb{Z}_{2^n}$, define $g(x)$ to be the integer satisfying $x \in L_{g(x)}$. Observe that if there exists an integer $b \geq 1$ such that $0 < |S_{b+}| \leq |L_b \setminus S_b|$ then we can replace all of S_{b+} by arbitrary elements of $L_b \setminus S_b$. This does not increase $f(S)$ and decreases $\sum_{x \in S} g(x)$. Setting $B := \max\{i : 0 < |S_i|\}$, the highest non-empty layer, we can assume that for all $b < B$ we have $|L_b \setminus S_b| < |S_{b+}|$. So we have the strict inequalities

$$\begin{aligned} |L_1| &< |S_1| + |S_2| + \ldots + |S_B| = |L_1| + 1 \\ |L_2| &< |S_2| + |S_3| + \ldots + |S_B| \\ |L_3| &< |S_3| + |S_4| + \ldots + |S_B| \\ &\ldots \\ |L_{B-1}| &< |S_{B-1}| + |S_B|. \end{aligned}$$

Hence we have

$$\begin{aligned} \frac{f(S)}{3} &\geq \left(\sum_{a=1}^{B-2} |S_a|(|S_a| + |S_{a+}| - |L_a|) \right) \ + \ |S_B|(2|S_{B-1}| - |L_{B-1}|) \\ &\geq \left(\sum_{a=1}^{B-2} |S_a| \right) \ + \ |S_B|(2|S_{B-1}| - |L_{B-1}|) \end{aligned}$$

Now note that subject to the constraints $0 \leq |S_{B-1}| \leq |L_{B-1}|$, $0 \leq |S_B| \leq |L_B|$ and $|S_{B-1}| + |S_B| \geq |L_{B-1}| + 1$, we have the inequality $|S_B|(2|S_{B-1}| - |L_{B-1}|) \geq$

$(|S_{B-1}| + |S_B| - |L_{B-1}|)|L_{B-1}|$. So we have

$$\begin{aligned}\frac{f(S)}{3} &\geq \left(\sum_{a=1}^{B-2} |S_a|\right) \;+\; (|S_{B-1}| + |S_B| - |L_{B-1}|)|L_{B-1}| \\ &\geq |L_1| + 1 - |S_{B-1}| - |S_B| + (|S_{B-1}| + |S_B| - |L_{B-1}|)|L_{B-1}| \\ &\geq |L_1| + 1 - (|L_{B-1}| + 1) + |L_{B-1}| = |L_1|,\end{aligned}$$

where in the last inequality we used that $|S_{B-1}| + |S_B| > |L_{B-1}|$. Hence $f(S) \geq 3|L_1|$ and this completes the proof of Theorem 7.8. □

7.5 Concluding remarks and open problems

The main conjectures raised in this chapter are the analogue of Samotij's theorem in $\mathbb{Z}_{2^n}$ (Conjecture 7.9) and that the constructions $\mathcal{C}_d$ are best possible (Conjecture 7.12). The first open case of Conjecture 7.12 is the $d = 3$ case, which we restate here.

Conjecture 7.26. *The largest 3-cube-free family in $\mathbb{Z}_{2^n}$ has size $(5/8) \cdot 2^n$.*

Recall that $L_1 \cup L_3$ is 3-cube-free, so this conjecture, if true, is sharp. Using Gurobi [43] we could check that for $n \leq 7$ the following stronger conjecture is also true.

Conjecture 7.27. *If $S \subset \mathbb{Z}_{2^n}$ is a set of size $|S| > (5/8) \cdot 2^n$ then there exist x, y such that $\Sigma^*\{x, x, x\} \subset A$ or $\Sigma^*\{x, 3x, y\} \subset A$.*

If true, Conjecture 7.27 might be easier to prove than Conjecture 7.26. Our hope is that an insightful proof of Conjecture 7.26 may quickly lead to a full proof of Conjecture 7.12.

Following [4, 25, 44], a natural complementary problem is to determine for all M, n (and k) the set $S \subset \mathbb{Z}_{2^n}$ of size $|S| = M$ with the largest number of Schur triples (or k-chains). In the Boolean lattice for a wide range of M the constructions with the largest number of comparable pairs are essentially towers of cubes. In $\mathbb{Z}_{2^n}$, say that a set $S \subset \mathbb{Z}_{2^n}$ is *anti-centred* if there exists an $i \in [n+1]$ such that for all j with $n \geq j > i$ we have $L_j \subseteq S$, and for all j with $i > j \geq 1$ we have $L_j \cap S = \emptyset$. It seems plausible that anti-centred families maximize the number of Schur triples and perhaps even k-cubes. Note that if $M = 2^{n-\ell}$ for some ℓ then an anti-centred family is the union of the $n - \ell + 1$ smallest layers of $\mathbb{Z}_{2^n}$ and hence contains M^2 Schur triples and M^k distinct k-cubes, both of which are optimal.

Another related problem, following [33, 63], is to determine the smallest maximal k-cube free set in $\mathbb{Z}_{2^n}$. That is, the smallest $S \subset \mathbb{Z}_{2^n}$ that is k-cube free, but the addition of any new element to S makes it not k-cube-free. Currently we do not have a conjecture for what the extremal families should be.

Chapter 8

Diffusion on graphs is eventually periodic

This chapter is based on a short note, joint with B. Narayanan [58]. The result has been published in the Journal of Combinatorics.

8.1 Introduction

In this chapter, we will be be concerned with 'chip-firing' games. Given a graph G with piles of chips at each vertex, in the traditional *chip-firing* game, one plays by repeatedly choosing a vertex that has at least as many chips as its degree, and then 'firing' this vertex by moving a chip from the vertex to each of its neighbours. This one-player game was introduced by Björner, Lovász and Shor [11], and the study of the dynamics of the chip-firing game and its variants has since grown rapidly, due both to its inherent appeal and the many connections to other areas of mathematics; see [19, 45, 46, 49] for some examples of recent developments, and the survey of Merino [62] for more background.

Here, we will primarily be interested in a variant of the traditional chip-firing game introduced by Duffy, Lidbetter, Messinger and Nowakowski [23] called *diffusion.* In diffusion on a finite graph G, each vertex of G is initially labelled with an integer interpreted as the number of chips at that vertex, and at each subsequent step, each vertex simultaneously fires one chip to each of its neighbours with fewer chips. In contrast to the parallel chip-firing game [10] where every vertex that has at least as many chips as its degree simultaneously fires a chip to each of its neighbours, note that the firing rule in diffusion may result in negative labels even when the initial labels are all positive integers. It is therefore not clear a priori if diffusion is bounded, and consequently, if it must exhibit periodic behaviour. Hence, it is natural to ask if diffusion, on any graph, and from any initial configuration, is always eventually periodic. Duffy, Lidbetter, Messinger and Nowakowski [23] raised this precise problem and conjectured, motivated by overwhelming numerical evidence, that diffusion is always eventually periodic with period either 1 or 2; our goal here is to prove this attractive conjecture.

A more formal description of diffusion, which is a cellular automaton on a finite graph, is as follows. Let G be an n-vertex graph on the vertex set $[n] = \{1, 2, \ldots, n\}$. At time $t = 0$, each vertex $v \in [n]$ is assigned an initial integer label $w_v(0)$. We then update these labels at discrete time steps according to the following rule: at time $t \geq 0$, for a vertex $v \in V(G)$, if $A_v(t)$ is the number of neighbours u of v with $w_u(t) > w_v(t)$, and $B_v(t)$ is the number of neighbours u of v with $w_u(t) < w_v(t)$, then we set

$$w_v(t+1) = w_v(t) + A_v(t) - B_v(t).$$

For each $t \geq 0$, let $w_G(t) \in \mathbb{Z}^n$ denote the vector $(w_1(t), w_2(t), \ldots, w_n(t))$. In this language, the diffusion process on G from the initial configuration $w_G(0) \in \mathbb{Z}^n$ is eventually periodic if the sequence $(w_G(t))_{t\geq 0}$ is eventually periodic. We shall establish the following, thereby settling the aforementioned conjecture due to Duffy, Lidbetter, Messinger and Nowakowski [23].

Theorem 8.1. *Diffusion on any finite graph, and from any initial configuration, is eventually periodic with period either 1 or 2; in other words, for any n-vertex graph G and any initial configuration $w_G(0) \in \mathbb{Z}^n$, the sequence $(w_G(t))_{t\geq 0}$ is eventually periodic with period either 1 or 2.*

This chapter is organised as follows. We prove Theorem 8.1 in Section 8.2, and in Section 8.3, we conclude with a discussion of some open problems.

8.2 Proof of the main result

Our proof of Theorem 8.1 hinges on the definition of an integer-valued potential function. We shall show that this potential is bounded below, and also that this potential is non-increasing with time; finally, we shall also show that once our potential function stops decreasing (and is consequently constant for all time), the diffusion process must then attain periodicity with period either 1 or 2. Of course, once we write down the appropriate potential, the rest of the argument is quite straightforward; finding the right definition is hence the crux of the matter.

Proof of Theorem 8.1. In diffusion on an n-vertex graph G from an initial configuration $w_G(0) \in \mathbb{Z}^n$, we define the potential $P(t)$ of the diffusion process at time t by

$$P(t) = \sum_{v=1}^{n} w_v(t) w_v(t+1).$$

Let us note two somewhat unexpected features of this potential. First, it is slightly surprising that our potential at a time t depends on the labels of the vertices at both times t and $t+1$. Second, and perhaps more surprisingly, this potential does not appear to take into direct account the structure of the underlying graph, in the sense that the potential merely involves a sum over the vertex set, and completely ignores the edge set!

We first observe that our potential function is bounded below.

Lemma 8.2. *For all $t \geq 0$, we have $P(t) \geq -n(n-1)^2/4$.*

Proof. This follows immediately from the observation that $|w_v(t+1) - w_v(t)| \leq n-1$ for each $v \in [n]$; therefore, for each $v \in [n]$, we have $w_v(t)w_v(t+1) \geq -(n-1)^2/4$, and the claim follows. □

To show that our potential function is non-increasing with time, we shall assign some labels to the edges of G at each time $t \geq 0$. Roughly speaking, at each time $t \geq 0$, we label each edge of G according to the directions in which chips are passed along that edge in the next two steps. More precisely, at a time $t \geq 0$, an edge uv of G with $1 \leq u < v \leq n$ gets assigned the label $(x_{uv}(t), y_{uv}(t))$ as follows: we set $x_{uv}(t) = \mathrm{sgn}(w_u(t) - w_v(t))$ and $y_{uv}(t) = \mathrm{sgn}(w_u(t+1) - w_v(t+1))$, where $\mathrm{sgn}(m)$ is equal to either -1, 0 or 1 respectively according to whether $m < 0$, $m = 0$ or $m > 0$. We now observe the following.

Lemma 8.3. *For all $t \geq 0$, we have $P(t+1) \leq P(t)$; furthermore, if any edge of G is labelled either $(1,1)$, $(-1,-1)$, $(0,1)$ or $(0,-1)$ at time t, then $P(t+1) < P(t)$.*

Proof. Observe that

$$P(t+1) - P(t) = \sum_{v=1}^{n} w_v(t+1)(w_v(t+2) - w_v(t)).$$

With the convention that $(x_{uv}(t), y_{uv}(t)) = (0,0)$ whenever uv is not an edge of

G, we have

$$w_v(t+2) = w_v(t) + \sum_{u \neq v} \operatorname{sgn}(v-u)(x_{uv}(t) + y_{uv}(t)).$$

Consequently, it follows that

$$\begin{aligned} P(t+1) - P(t) &= \sum_{v=1}^{n} w_v(t+1) \left(\sum_{u \neq v} \operatorname{sgn}(v-u)(x_{uv}(t) + y_{uv}(t)) \right) \\ &= \sum_{u<v} (x_{uv}(t) + y_{uv}(t))(w_v(t+1) - w_u(t+1)). \end{aligned}$$

Consider the contribution $(x_{uv}(t) + y_{uv}(t))(w_v(t+1) - w_u(t+1))$ from a pair of vertices $u, v \in [n]$ with $u < v$ to the above sum. Clearly, this contribution is zero if $x_{uv}(t)+y_{uv}(t) = 0$. Now, suppose that $x_{uv}(t)+y_{uv}(t) \neq 0$; of course, this is only possible when uv is in fact an edge of G. If $x_{uv}(t) + y_{uv}(t) > 0$, then $y_{uv}(t) \geq 0$ and this implies that $w_v(t+1) - w_u(t+1) \leq 0$, and if $x_{uv}(t) + y_{uv}(t) < 0$, then $y_{uv}(t) \leq 0$ and this implies that $w_v(t+1) - w_u(t+1) \geq 0$. Therefore, each term in the above sum is at most zero, and so $P(t+1) \leq P(t)$, proving the first claim.

Now, if any edge uv is labelled with one of the four labels $(1, 1)$, $(-1, -1)$, $(0, 1)$ or $(0, -1)$ at time t, then we see that the corresponding term $(x_{uv}(t) + y_{uv}(t))(w_v(t+1) - w_u(t+1))$ is negative. For example, if $x_{uv}(t) = 0$ and $y_{uv}(t) = 1$, then we have $x_{uv}(t) + y_{uv}(t) = 1$ and $w_v(t+1) - w_u(t+1) < 0$; the three other cases are similarly easy to handle, and this establishes the second claim. $\square$

We may now finish the proof as follows. By Lemma 8.2, we see that our potential $P(t)$ is bounded below for all $t \geq 0$, and by Lemma 8.3, we see that $P(t)$ is non-increasing with t. Since $P(t)$ is integer-valued, there exists some finite time T (depending on our graph G and the initial configuration $w_G(0)$) such that $P(t)$ is constant for all $t \geq T$. It further follows from Lemma 8.3 that at each time $t \geq T$, the label of each edge belongs to the set $\{(1, -1), (-1, 1), (0, 0), (1, 0), (-1, 0)\}$.

We claim that there exists a time $T' \geq T$ at which the label of each edge belongs to the set $\{(1, -1), (-1, 1), (0, 0)\}$. To see this, we first note that if an edge has labels (i, j) and (k, l) at times t and $t + 1$, then $j = k$. Furthermore, we also know that an edge cannot be labelled either $(1, 1)$, $(-1, -1)$, $(0, 1)$ or $(0, -1)$ at any time $t \geq T$. Consequently, we deduce that

1. if an edge is labelled either $(1, 0)$, $(-1, 0)$ or $(0, 0)$ at some time $t \geq T$, then it must be labelled $(0, 0)$ at time $t + 1$, and consequently, it must be labelled $(0, 0)$ at each time $t' \geq t + 1$;

2. if an edge is labelled $(-1, 1)$ at some time $t \geq T$, then it must be labelled either $(1, -1)$ or $(1, 0)$ at time $t + 1$; and

3. if an edge is labelled $(1, -1)$ at some time $t \geq T$, then it must be labelled either $(-1, 1)$ or $(-1, 0)$ at time $t + 1$.

If an edge is labelled either $(1, 0)$, $(-1, 0)$ or $(0, 0)$ at time T, then it is labelled $(0, 0)$ at each time $t \geq T + 1$. If an edge is labelled either $(1, -1)$ or $(-1, 1)$ at time T, then there are two possibilities: either the label of this edges alternates between $(1, -1)$ and $(-1, 1)$ for the rest of all time, or the label of this edge changes to either $(1, 0)$ or $(-1, 0)$ at some time $t \geq T + 1$, and is then labelled $(0, 0)$ at each time $t' \geq t + 1$. Since G has finitely many edges, it is now clear that there exists a time $T' \geq T$ at which the label of each edge belongs to the set $\{(1, -1), (-1, 1), (0, 0)\}$.

Finally, note that if the label of each edge belongs to $\{(1, -1), (-1, 1), (0, 0)\}$ at some time t, then we must have $w_G(t) = w_G(t + 2)$; indeed, at that time one of two things happens across each edge: either there is no transfer of chips across the edge in question in either of the next two steps, or a chip travels back and forth across the edge in question in the next two steps. Consequently, we have $w_G(t + 2) = w_G(t)$ for all $t \geq T'$, proving the result. $\square$

8.3 Concluding remarks

It is natural to ask if Theorem 8.1 holds under more general conditions. First, we remark that our proof runs essentially as described even when the underlying graph G is allowed to contain parallel edges (so that each vertex fires a chip along *each* edge to each of its neighbours with fewer chips), and when the initial configuration $w_G(0)$ is a vector of real numbers rather than integers. To deal with real-valued labels, one requires a small additional observation, which is that while the potential is no longer integer-valued, it can take only finitely many distinct values between the lower bound given by Lemma 8.2 and its initial value.

Next, while it is easy to see that diffusion on an infinite graph need not be periodic, one may wonder if anything can be said for, say, infinite graphs of bounded degree: does diffusion on an infinite graph of bounded degree from an initial configuration where the vertex labels are also bounded result in a process where the vertex labels remain bounded for all time? In fact, even the answer to this question is no – we are indebted to Joshua Erde for the following example. One may consider the infinite 4-regular tree in which every vertex has one neighbour above and three neighbours below. This separates the vertex set into 'levels' of the infinite tree (where all the levels are also infinite). We then pick some level arbitrarily, and give all the vertices on this level 0 chips. Finally, we assign chips to vertices based on their level with the repeating pattern $0, 3, 4, 5$. It is then easy to see that after a single step, the number of chips on each level has the repeating pattern $4, 5, 6, 1$. But since the graph is transitive, this is the same as having the repeating pattern $1, 4, 5, 6$ and so we have effectively just added one chip to each vertex. At each future step we will also effectively add one chip to each vertex, and so the diffusion process is clearly unbounded.

Duffy, Lidbetter, Messinger and Nowakowski [23] raise various other questions about diffusion that are not addressed here, and we conclude by mentioning a result in a similar vein. Note that the dynamics of diffusion are unchanged if we initially add a fixed number of chips to each vertex. Since we have shown that diffusion is eventually periodic (and consequently bounded), it is natural to ask whether, for each $n \in \mathbb{N}$, there exists an integer $f(n) \geq 0$ with the property that in diffusion on any n-vertex graph where each initial vertex label is at least $f(n)$, all the vertex labels are non-negative at all subsequent times. Recently, Carlotti and Herrman [17] have shown that $f(n) = n - 2$ is sufficient.

References

[1] R. Ahlswede and A. A. Yudin, *On Partitions of a Rectangle into Rectangles with Restricted Number of Cross Sections*, in: R. Ahlswede et al. (eds), *General Theory of Information Transfer and Combinatorics*, Lecture Notes in Computer Science, vol. 4123, Springer, 2006.

[2] N. Alon, *Subset sums*, J. Number Theory **27**(2) (1987), 196–205.

[3] N. Alon, T. Bohman, R. Holzman and D.J. Kleitman, *On partitions of discrete boxes*, Discrete Math. **257** (2002), 255–258.

[4] N. Alon and P. Frankl, *The maximum number of disjoint pairs in a family of subsets*, Graphs Combin. **1** (1985), 13–21.

[5] N. Alon and G. Freiman, *On sums of subsets of a set of integers*, Combinatorica **8** (4) (1988), 297–306.

[6] B. Aronov, V. Dujmović, P. Morin, A. Ooms and L. F. S. X. da Silveira, *More Turán-type theorems for triangles in convex point sets*, Electron. J. Comb. **26 (1)** (2019), P1.8.

[7] A. Balog and E. Szemerédi, *A statistical theorem of set addition*, Combinatorica **14** (1994), 263–268.

[8] J. Balogh, M. Lavrov, G. Shakan and Z. A. Wagner, *Monochromatic Hilbert cubes and arithmetic progressions*, preprint (2018), `arXiv:1805.08938 [math.CO]`.

[9] F. Behrend, *On sets of integers which contain no three terms in arithmetic progression*, Proc. Nat. Acad. Sci. **32** (1946), 331–332.

[10] J. Bitar and E. Goles, *Parallel chip firing games on graphs*, Theoret. Comput. Sci. **92** (1992), 291–300.

[11] A. Björner, L. Lovász and P. W. Shor, *Chip-firing games on graphs*, European J. Combin. **12** (1992), 283–291.

[12] B. Bollobás. *Combinatorics: Set Systems, Hypergraphs, Families of Vectors and Combinatorial Probability*, Cambridge University Press, 1986.

[13] H. Brandt, *Verallgemeinierung des Gruppenbegriffs*, Math. Ann. **96**, pp. 360–366, 1927.

[14] E. Breuillard, B. J. Green and T. C. Tao, *The structure of approximate groups*, Publ. Math. IHES **116** (2012), 115–221.

[15] W. G. Brown, P. Erdős and V. T. Sós, *Some extremal problems on r-graphs*, New Directions in the Theory of Graphs (Proc. Third Ann Arbor Conference on Graph Theory, Univ. Michigan, Ann Arbor, Michigan, 1971). Academic Press, New York. 1973.

[16] M. Bucic, B. Lidický, J. Long and Z. A. Wagner, *Partition problems in high dimensional boxes*, European J. Combin., to appear.

[17] A. Carlotti and R. Herrman, *Uniform Bounds for Non-negativity of the Diffusion Game*, preprint.

[18] D. Conlon, H. Hàn, Y. Person and M. Schacht *Weak quasi-randomness for uniform hypergraphs*, Random Struct. Algor., **40** (2011), 1–38.

[19] R. Cori, P. Petrullo and D. Senato, *Hall–Littlewood symmetric functions via the chip-firing game*, European J. Combin. **58** (2016), 225–237.

[20] J. Denés and A. D. Keedwell, *Latin Squares and their Applications*, Akadémiai Kiadó, Budapest, Hungary, 1974.

[21] A. Diaz-Lopez, *Po-Shen Loh interview*, Notices of the AMS **63** (8) (2016), 905–908.

[22] P. Dodos, V. Kanellopoulos and K. Tyros, *A simple proof of the density Hales-Jewett theorem*, Int. Math. Res. Not. **12** (2014), 3340–3352.

[23] C. Duffy, T. F. Lidbetter, M. E. Messinger and R. J. Nowakowski, *A Variation on Chip-Firing: the diffusion game*, Discrete Math. Theor. Comput. Sci. **20** (1) (2018), DOI: 10.23638/DMTCS-20-1-4.

[24] P. Erdős, *On a lemma of Littlewood and Offord*, Bull. Amer. Math. Soc. **51** (1945), 898–902.

[25] P. Erdős, *Problem sessions*, in: I. Rival (Ed.), *Ordered Sets*, Proc. NATO Adv. Study, Reidel, Dordrecht (1981), 860–861.

[26] P. Erdős and G. Freiman, *On two additive problems*, J. Number Theory **34** (1) (1990), 1–12.

[27] M. Fekete, *Über die Verteilung der Wurzeln bei gewissen algebraischen Gleichungen mit ganzzahligen Koeffizienten*, Math. Zeit. **17** (1) (1923), 228–249.

[28] J. Fox, *A new proof of the graph removal lemma*, Ann. of Math. **174** (2011), 561–579.

[29] G. A. Freiman, *Addition of finite sets*, Sov. Math., Dokl. **5** (1964), 1366–1370.

[30] Z. Füredi and P. Hajnal, *Davenport–Schinzel theory of matrices*, Discrete Math. **103** (1992), 233–251.

[31] H. Furstenberg and Y. Katznelson, *An ergodic Szemerédi theorem for commuting transformations*, J. Analyse Math. **34** (1978), 275–291.

[32] H. Furstenberg and Y. Katznelson, *A density version of the Hales–Jewett Theorem for $k = 3$*, Discrete Math. **75** (1989), 227–241.

[33] D. Gerbner, B. Keszegh, N. Lemons, C. Palmer, D. Pálvölgyi and B. Patkós, *Saturating Sperner Families*, Graphs Combin. **29** (5) (2013), 1355–1364.

[34] W. T. Gowers, *Bipartite Graphs of Approximate Rank One*, preprint, `https://www.dpmms.cam.ac.uk/~wtg10/approxrankone3.pdf`.

[35] W. T. Gowers and J. Long, *The length of an s-increasing sequence of r-tuples*, Combin. Probab. Comput., to appear.

[36] W. T. Gowers and J. Long, *Partial associativity and rough approximate groups*, preprint.

[37] W. T. Gowers and J. Long, *Obtaining rough approximate abelian groups from Latin squares*, preprint.

[38] R. L. Graham, *Rudiments of Ramsey Theory*, Regional conference series in mathematics, No. 45, American Mathematical Society, 1981 (reprinted with corrections, 1983).

[39] R. L. Graham. and B. L. Rothschild, *A survey of finite Ramsey theorems*, in: Proc. 2nd Louisiana Conf. on Combinatorics, Graph Theory and Computing (1971), 21–40.

[40] B. Green, *On a conjecture of Gowers and Long*, B. Lond. Math. Soc., *to appear*, 2020.

[41] J. Grytczuk, A. P. Kisielewicz and K. Przesławski, *Minimal partitions of a box into boxes*, Combinatorica **24** (2004), 605–614.

[42] D. S. Gunderson and V. Rödl, *Extremal problems for affine cubes of integers*, Combin. Probab. Comput. **7** (1) (1998), 65–79.

[43] Gurobi Optimization, Inc., *Gurobi Optimizer Reference Manual*, (2018), `http://www.gurobi.com`.

[44] R. K. Guy, *A miscellany of Erdős problems*, Amer. Math. Monthly **90** (1983), 118–120.

[45] J. Guzmán and C. Klivans, *Chip-firing and energy minimization on M-matrices*, J. Combin. Theory Ser. A **132** (2015), 14–31.

[46] J. Guzmán and C. Klivans, *Chip firing on general invertible matrices*, SIAM J. Discrete Math. **30** (2016), 1115–1127.

[47] R. Holzman, *On 2-colored graphs and partitions of boxes*, preprint.

[48] E. Hrushovski, *Metrically approximate subgroups*, preprint, `http://www.ma.huji.ac.il/~ehud/metric-stabilizer.pdf`.

[49] T. Jiang, Z. Scully, and Y. X. Zhang, *Motors and impossible firing patterns in the parallel chip-firing game*, SIAM J. Discrete Math. **29** (2015), 615–630.

[50] K. A. Kearnes and E. W. Kiss, *Finite algebras of finite complexity*, Discrete Math. **207** (1999), 89–135.

[51] A. P. Kisielewicz and K. Przesławski, *Polyboxes, cube tilings and rigidity*, Discrete Comput. Geom. **40** (2008), 1–30.

[52] D. Kleitman, *A conjecture of Erdős–Katona on commensurable pairs of subsets of an n-set*, in: Theory of Graphs, Proceedings of the Colloquium Held at Tihany, Hungary, September 1966 (P. Erdős and G. Katona, eds.), Academic Press, New York (1968), 215–218.

[53] D. J. Kleitman, *Partitioning a rectangle into many sub-rectangles so that a line can meet only a few*, DIMACS Series in Discrete Math. Theor. Comput. Sci. **9** (1993), 95–107.

[54] I. Leader, L Milićević and T. S. Tan, *Decomposing the complete r-graph*, J. Combin. Theory Ser. A **154** (2018), 21–31.

[55] E. Levi, *Symmetric abstract independence relations and the group configuration theorem*, Master's thesis, 2012.

[56] P. Loh, *Directed paths: from Ramsey to Ruzsa and Szemerédi*, Combin. Probab. Comput., to appear.

[57] J. Long, *A note on the Brown–Erdős–Sós conjecture in finite groups*, preprint (2019), `arXiv:1902.07693 [math.CO]`.

[58] J. Long and B. Narayanan, *Diffusion on graphs is eventually periodic*, J. Combin. **10** (2) (2019), 235–241.

[59] J. Long and Z. A. Wagner, *The largest projective cube-free subsets of $\mathbb{Z}_{2^n}$*, J. Combin. Theory Ser. A, to appear.

[60] A. Marcus and M. Klazar, *Extensions of the linear bound in the Füredi–Hajnal conjecture*, Adv. in Appl. Math. **38** (2006), 258–266.

[61] A. Marcus and G. Tardos, *Excluded permutation matrices and the Stanley–Wilf conjecture*, J. Combin. Theory Ser. A **107** (1) (2004), 153–160.

[62] C. Merino, *The chip-firing game*, Discrete Math. **302** (2005), 188–210.

[63] N. Morrison, J. A. Noel and A. Scott, *On Saturated k-Sperner Systems*, Electron. J. Comb. **21** (3) (2014), 3–22.

[64] R. Nenadov, B. Sudakov and M. Tyomkyn, *Proof of the Brown–Erdős–Sós conjecture in groups*, preprint (2019), `arXiv:1902.07614 [math.CO]`.

[65] H. J. Prömel, *Ramsey theory for discrete structures*, Springer, 2016.

[66] L. Pyber. *How abelian is a finite group?*, in: R. L. Graham and J. Nešetřil (eds), *The Mathematics of Paul Erdős I*, Algorithms and Combinatorics, vol 13. Springer, Berlin, Heidelberg (1997), 372–384.

[67] R. Rado, *Studien zur Kombinatorik*, Math. Zeit. **36** (1933), 425–480.

[68] I. Z. Ruzsa, *Solving a linear equation in a set of integers I*, Acta Arith. **65** (3) (1993), 259–282.

[69] I. Z. Ruzsa and E. Szemerédi, *Triple systems with no six points carrying three triangles*, Combinatorics (Keszthely, 1976), Coll. Math. Soc. J. Bolyai **18** (1978), 939–945.

[70] M. Saks, *Kleitman and combinatorics*, Discrete Math. **257** (2-3) (2002), 225–247.

[71] W. Samotij, *Subsets of posets minimising the number of chains*, preprint (2017), `arXiv:1708.02436 [math.CO]`.

[72] W. Samotij and B. Sudakov, *The number of additive triples in subsets of abelian groups*, Math. Proc. Camb. Philos. Soc. **160** (3) (2016), 495–512.

[73] J. H. Sanders, *A generalization of Schur's theorem*, Doctoral paper, Yale University, USA, 1968.

[74] G. N. Sárközy and S. Selkow, *An extension of the Ruzsa–Szemerédi theorem*, Combinatorica **25** (1) (2004), 77–84.

[75] J. Solymosi, *The* $(7,4)$*-conjecture in finite groups*, Combin. Probab. Comput. **24** (4) (2015), 680–686.

[76] J. Solymosi, C. Wong, *The Brown–Erdős–Sós conjecture in finite abelian groups*, preprint (2019), `arXiv:1901.09871 [math.CO]`.

[77] E. Sperner, *Ein Satz über Untermengen einer endlichen Menge*, Math. Zeit. **27** (1928), 544–548.

[78] S. Stein, *Factoring by subsets*, Pacific J. Math. **22** (1967), 523–541.

[79] S. Stein, *Combinatorial packing of* R^n *by certain error spheres*, IEEE Trans. Inform. Theory **30** (2) (1984), 364–368.

[80] S. Stein, *Packing tripods*, Math. Intelligencer **17** (2) (1995), 37–39.

[81] S. Stein and S. Szabó, *Algebra and Tiling: Homomorphisms in the Service of Geometry*, Vol. 25 of The Carus Mathematical Monographs, The Mathematical Association of America, 1994.

[82] T. Tao, *Metric entropy analogues of sum set theory*, blog post, `https://terrytao.wordpress.com/2014/03/19/metric-entropy-analogues-of-sum-set-theory/`

[83] T. Tao, *Product set estimates for non-commutative groups*, Combinatorica **28** (5), 547–594.

[84] T. Tao, *1% quasimorphisms and group cohomology*, blog post, `https://terrytao.wordpress.com/2018/07/07/1-quasimorphisms-and-group-cohomology/`

[85] G. Thomsen, *Topologische Fragen der Differentialgeometrie XII*, Schnittepunktsätze in ebenen Gewebe. Abh. Math. Seminar Univ. Hambg. **7** (1939), 99–106

[86] J. Tidor, V. Y. Wang and B. Yang, *1-color-avoiding paths, special tournaments, and incidence geometry*, preprint (2016), `arXiv:1608.04153 [math.CO]`.

[87] A. Tiskin, *Packing tripods: Narrowing the density gap*, Discrete Math. **307** (16) (2007), 1973–1981.

[88] Z. A. Wagner, *Large subgraphs in rainbow-triangle free colorings*, J. Graph Theory **86** (2) (2017), 141–148.

[89] C. Wong, *On the existence of dense substructures in finite groups*, preprint (2019), `arXiv:1902.07819 [math.CO]`.