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THE PRIMITIVE IDEALS OF SOME ÉTALE GROUPOID C^*-ALGEBRAS

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Abstract. Consider the Deaconu–Renault groupoid of an action of a finitely generated free abelian monoid by local homeomorphisms of a locally compact Hausdorff space. We catalogue the primitive ideals of the associated groupoid C^*-algebra. For a special class of actions we describe the Jacobson topology.

1. Introduction

Describing the primitive-ideal space of a C^*-algebra is typically quite difficult, but for crossed products of $C_0(X)$ by abelian groups G, a very satisfactory description is available: for each point $x \in X$ and for each character χ of G there is an irreducible representation of the crossed product on $L^2(G \cdot x)$. The map which sends (x, χ) to the kernel of this representation is a continuous open map from $X \times \widehat{G}$ to the primitive-ideal space of $C_0(X) \rtimes G$, and it carries (x, χ) and (y, ρ) to the same ideal precisely when $\overline{G \cdot x} = \overline{G \cdot y}$ and χ and ρ restrict to the same character of the stability subgroup $G_x = \{ g : g \cdot x = x \}$ [29, Theorem 8.39].

Regarding $C_0(X) \rtimes G$ as a groupoid C^*-algebra leads to a natural question: what can be said about the primitive-ideal spaces of C^*-algebras of Deaconu–Renault groupoids of semigroup actions by local homeomorphisms? Examples of groupoids of this sort arise from the $\mathbf{N}$-actions by the shift map on the infinite-path spaces of row-finite directed graphs E with no sources. The primitive-ideal spaces of the associated graph C^*-algebras were described by Hong and Szymański [10] building on Huef and Raeburn's description of the primitive-ideal space of a Cuntz–Krieger algebra [11]. The description given in [10] is in terms of the graph rather than its groupoid. Recasting their results in groupoid terms yields a map from $E^\infty \times \mathbf{T}$ to the primitive-ideal space of $C^*(E)$ along more or less the same lines as described above for group actions. But this map is not necessarily open, and the equivalence relation it induces on $E^\infty \times \mathbf{T}$ is complicated by the fact that orbits with the same closure need not have the same isotropy in $\mathbf{Z}^k$.

The complications become greater still when $\mathbf{N}$ is replaced with $\mathbf{N}^k$, and the resulting class of C^*-algebras is substantial. For example, it contains the C^*-algebras of graphs [14] and k-graphs [13] and their topological generalisations [30, 31]. However, the results of [4] for higher-rank graph algebras suggest that a satisfactory description of the primitive-ideal spaces of Deaconu–Renault groupoids of $\mathbf{N}^k$ actions

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might be achievable. Here we take a substantial first step by producing a complete catalogue of the primitive ideals of the C^*-algebra $C^*(G_T)$ of the Deaconu–Renault groupoid associated to an action T of $\mathbf{N}^k$ by local homeomorphisms of a locally compact Hausdorff space X. Specifically, there is a surjection $(x, z) \mapsto I_{x,z}$ from $X \times \mathbf{T}^k$ to $\operatorname{Prim}(C^*(G_T))$. Moreover, $I_{x,z}$ and $I_{x',z'}$ coincide if and only if the orbits of x and x' under T have the same closure and z and z' determine the same character of the interior of the isotropy of the reduction of G_T to this orbit closure. For a very special class of actions T we are also able to describe the topology of the primitive-ideal space of $C^*(G_T)$, but in general we can say little about it. Indeed, graph-algebra examples show that any general description will require subtle adjustments to the "obvious" quotient topology.

The paper is organised as follows. In Section 2 we establish our conventions for groupoids, and prove that if G is an étale Hausdorff groupoid and the interior $\operatorname{Iso}(G)^\circ$ of its isotropy subgroupoid is closed as well as open, then the natural quotient $G/\operatorname{Iso}(G)^\circ$ is also a Hausdorff étale groupoid and there is a natural homomorphism of $C^*(G)$ onto $C^*(G/\operatorname{Iso}(G)^\circ)$.

In Section 3 we consider the Deaconu–Renault groupoids G_T associated to actions T of $\mathbf{N}^k$ by local homeomorphisms of locally compact spaces X. We state our main theorem about the primitive ideals of $C^*(G_T)$, and begin its proof. We first show that G_T is always amenable. We then consider the situation where $\mathbf{N}^k$ acts irreducibly on X. We show that there is then an open $\mathbf{N}^k$-invariant subset $Y \subset X$ on which the isotropy in $\mathbf{N}^k \times \mathbf{N}^k$ is maximal. For this set Y, $\operatorname{Iso}(G_T|_Y)^\circ$ is closed. We finish Section 3 by showing that restriction gives a bijection between irreducible representations of $C^*(G_T)$ that are faithful on $C_0(X)$ and irreducible representations of $C^*(G_T|_Y)$ that are faithful on $C_0(Y)$. Our arguments in this section are special to $\mathbf{N}^k$, and make use of techniques developed in [4].

In Section 4 we show that if the subspace Y from the preceding paragraph is all of X, then $C^*(G_T)$ is an induced algebra—associated to the canonical action of $\mathbf{T}^k$ on $C^*(G_T)$—with fibres $C^*(G_T/\operatorname{Iso}(G_T)^\circ)$. We use this description to give a complete characterisation of $\operatorname{Prim}(C^*(G_T))$ as a topological space under the rather strong hypothesis that the reduction of $G_T/\operatorname{Iso}(G_T)^\circ$ to any closed G_T-invariant subset of Y is topologically principal. In Section 5 we complete the proof of our main theorem. The fundamental idea is that for every irreducible representation ρ of $C^*(G_T)$ there is a set $Y = Y_\rho$ as above and an element $z = z_\rho \in \mathbf{T}^k$ for which ρ factors through an irreducible representation of $C^*(G_T|_Y)$ that is faithful on $C_0(Y)$ and which in turn factors through evaluation (in the induced algebra) at z.

Standing assumptions. Throughout this paper, all topological spaces (including topological groupoids) are second countable, and all groupoids are Hausdorff. By a homomorphism between C^*-algebras, we mean a $*$-homomorphism, and by an ideal of a C^*-algebra we mean a closed, 2-sided ideal. We take the convention that $\mathbf{N}$ is a monoid under addition, so it includes 0.

2. Preliminaries

Let G be a locally compact second-countable Hausdorff groupoid with a Haar system. For subsets $A, B \subset G$, we write

$$AB := \{\alpha\beta \in G : (\alpha, \beta) \in (A \times B) \cap G^{(2)}\}.$$

We use the standard groupoid conventions that $G^x = r^{-1}(x)$, $G_x = s^{-1}(x)$, and $G_x^x = G^x \cap G_x$ for $x \in G^{(0)}$. If $K \subset G^{(0)}$, then the restriction of G to K is the subgroupoid $G|_K = \{\gamma \in G : r(\gamma), s(\gamma) \in K\}$. We will be particularly interested in the *isotropy subgroupoid*

$$\mathrm{Iso}(G) = \{\gamma \in G : r(\gamma) = s(\gamma)\} = \bigcup_{x \in G^{(0)}} G_x^x.$$

This $\mathrm{Iso}(G)$ is closed in G and is a group bundle over $G^{(0)}$.

A groupoid G is *topologically principal* if the units with trivial isotropy are dense in $G^{(0)}$. That is, $\overline{\{x \in G^{(0)} : G_x^x = \{x\}\}} = G^{(0)}$. It is worth pointing out that the condition we are here calling topologically principal has gone under a variety of names in the literature and that those names have not been used consistently (see [3, Remark 2.3]).

Recall that $G^{(0)}$ is a left G-space: $\gamma \cdot s(\gamma) = r(\gamma)$. If $x \in G^{(0)}$, then $G \cdot x = r(G_x)$ is called the *orbit* of x and is denoted by $[x]$. A subset A of $G^{(0)}$ is called *invariant* if $G \cdot A \subset A$. The quotient space $G\backslash G^{(0)}$ (with the quotient topology) is called the orbit space. The quasi-orbit space $\mathcal{Q}(G)$ of a groupoid G is the quotient of $G\backslash G^{(0)}$ in which orbits are identified if they have the same closure. Alternatively it is the T_0-ization of orbit space $G\backslash G^{(0)}$ (see [29, Definition 6.9]). In particular, the quasi-orbit space has the quotient topology coming from the quotient map $q : G^{(0)} \to \mathcal{Q}(G)$.

An ideal $I \lhd C_0(G^{(0)})$ is called *invariant* if the corresponding closed set

$$C_I := \{x \in G^{(0)} : f(x) = 0 \text{ for all } f \in I\}$$

is invariant. If M is a representation of $C_0(G^{(0)})$ with kernel I, then C_I is called the *support of* M. We say C_I is G-irreducible if it is not the union of two proper closed invariant sets. For example, orbit closures, $\overline{[x]}$, are always G-irreducible.

Lemma 2.1. *Let G be a second-countable locally compact groupoid. A closed invariant subset C of $G^{(0)}$ is G-irreducible if and only if there exists $x \in G^{(0)}$ such that $C = \overline{[x]}$.*

Proof. It suffices to see that every closed G-invariant set is an orbit closure. This is a straightforward consequence of the lemma preceding [9, Corollary 19] and the observation that the orbit space $G\backslash G^{(0)}$ is the continuous open image of G and hence totally Baire. $\square$

Remark 2.2. We say that $C_0(G^{(0)})$ is *G-simple* if it has no nonzero proper invariant ideals. So $C_0(G^{(0)})$ is G-simple exactly when $G^{(0)}$ has a dense orbit. This is much weaker than the notion of minimality, which requires that *every* orbit is dense.

We also want to refer to a couple of old chestnuts. Recall that there is a nondegenerate homomorphism

$$V : C_0(G^{(0)}) \to M(C^*(G))$$

such that for $f \in C_c(G)$ and $\varphi \in C_0(G^{(0)})$, we have $\big(V(\varphi)f\big)(\gamma) = \varphi(r(\gamma))f(\gamma)$. In particular, if L is a nondegenerate representation of $C^*(G)$, then we obtain an associated representation M of $C_0(G^{(0)})$ by extension: $M(\varphi) = \bar{L}(V(\varphi))$. The next result is standard. A proof in the case where G is principal can be found in [5, Lemma 3.4 and Proposition 3.2], and the proof goes through in general *mutatis mutandis*.

Proposition 2.3. *Let G be a second-countable locally compact groupoid with a Haar system. Let L be a nondegenerate representation of $C^*(G)$ with associated representation M of $C_0(G^{(0)})$ as above. Then $\ker M$ is invariant. If L is irreducible, then the support of M is G-irreducible.*

Proposition 2.4. *Let G be a second-countable locally compact groupoid with a Haar system. Let L be a nondegenerate representation of $C^*(G)$ with associated representation M of $C_0(G^{(0)})$. If F is the support of M, then L factors through $C^*(G|_F)$. In particular, if L is irreducible, then L factors through $C^*(G|_{\overline{[x]}})$ for some $x \in G^{(0)}$.*

Proof. Since F is a closed invariant set, $U := G^{(0)} \setminus F$ is open and invariant. We have a short exact sequence

$$0 \longrightarrow C^*(G|_U) \xrightarrow{\iota} C^*(G) \xrightarrow{R} C^*(G|_F) \longrightarrow 0$$

of C^*-algebras with respect to the natural maps coming from extension (by 0) and restriction of functions in $C_c(G)$ [17, Lemma 2.10]. Since M has support F, the kernel of L contains the ideal corresponding to $C^*(G|_U)$, so L factors through $C^*(G|_F)$.

The last assertion follows from Proposition 2.3 and Lemma 2.1. □

When the range and source maps in a groupoid G are open maps (in particular, when G is étale), the multiplication map is also open: Fix open $A, B \subseteq G$ and composable $(\alpha, \beta) \in A \times B$, and suppose that $\gamma_i \to \alpha\beta$. Since r is open, the $r(\gamma_i)$ eventually lie in $r(A)$; say $r(\gamma_i) = r(\alpha_i)$ with $\alpha_i \in A$. Now $\alpha_i^{-1}\gamma_i \to \beta$, and since B is open, the $\alpha_i^{-1}\gamma_i$ eventually belong to B, so that $\gamma_i = \alpha_i(\alpha_i^{-1}\gamma_i)$ eventually belongs to AB; so AB is open.

For the remainder of this note, we specialize to the situation where G is étale. Since G is Hausdorff, this means that $G^{(0)}$ is clopen in G and that $r : G \to G^{(0)}$ is a local homeomorphism. Hence counting measures form a continuous Haar system for G. The I-norm on $C_c(G)$ is defined by

$$\|f\|_I = \sup_{x \in G^{(0)}} \max\Big\{ \sum_{\gamma \in G_x} |f(\gamma)|, \sum_{\gamma \in G^x} |f(\gamma)| \Big\}.$$

The groupoid C^*-algebra $C^*(G)$ is the completion of $C_c(G)$ in the norm $\|a\| = \sup\{\, \pi(a) : \pi \text{ is an } I\text{-norm bounded } *\text{-representation} \,\}$. For $x \in G^{(0)}$ there is a representation $L^x : C^*(G) \to B(\ell^2(G_x))$ given by $L^x(f)\delta_\gamma = \sum_{s(\alpha)=r(\gamma)} f(\alpha)\delta_{\alpha\gamma}$. This is called the (left-)regular representation associated to x. The reduced groupoid C^*-algebra $C^*_r(G)$ is the image of $C^*(G)$ under $\bigoplus_{x \in G^{(0)}} L^x$.

A *bisection* in a groupoid G, also known as a G-set, is a set $U \subset G$ such that r, s restrict to homeomorphisms on U. An important feature of étale groupoids is that they have plenty of open bisections: Proposition 3.5 of [8] together with local compactness implies that the topology on an étale groupoid has a basis of precompact open bisections.

If G is étale, then the homomorphism $V : C_0(G^{(0)}) \to MC^*(G)$ takes values in $C^*(G)$ and extends the inclusion $C_c(G^{(0)}) \hookrightarrow C_c(G)$ given by extension of functions (by 0). We regard $C_0(G^{(0)})$ as a $*$-subalgebra of $C^*(G)$. If L is a representation of $C^*(G)$, then the associated representation M of $C_0(G^{(0)})$ is just the restriction of L to $C_0(G^{(0)})$. Thus $\ker M = \ker L \cap C_0(G^{(0)})$.

We write $\mathrm{Iso}(G)^\circ$ for the interior of $\mathrm{Iso}(G)$ in G. Since G is étale, $G^{(0)} \subset \mathrm{Iso}(G)^\circ$ and $\mathrm{Iso}(G)^\circ$ is an open étale subgroupoid of G.

Proposition 2.5. *Suppose that G is a second-countable locally compact Hausdorff étale groupoid such that $\mathrm{Iso}(G)^\circ$ is closed in G.*

(a) *The subgroupoid $\mathrm{Iso}(G)^\circ$ acts freely and properly on the right of G, and the orbit space $G/\mathrm{Iso}(G)^\circ$ is locally compact and Hausdorff.*
(b) *For each $\gamma \in G$, the map $\alpha \mapsto \gamma\alpha\gamma^{-1}$ is a bijection from $\mathrm{Iso}(G)^\circ_{s(\gamma)}$ onto $\mathrm{Iso}(G)^\circ_{r(\gamma)}$.*
(c) *For each $x \in G^{(0)}$, the set $\mathrm{Iso}(G)^\circ_x$ is a normal subgroup of G^x_x.*
(d) *The set $G/\mathrm{Iso}(G)^\circ$ is a locally compact Hausdorff étale groupoid with respect to the operations $[\gamma]^{-1} = [\gamma^{-1}]$ for $\gamma \in G$, and $[\gamma][\eta] = [\gamma\eta]$ for $(\gamma, \eta) \in G^{(2)}$. The corresponding range and source maps are given by $r'([\gamma]) = r(\gamma)$ and $s'([\gamma]) = s(\gamma)$.*
(e) *The groupoid $G/\mathrm{Iso}(G)^\circ$ is topologically principal.*
(f) *If G is amenable, then so is $G/\mathrm{Iso}(G)^\circ$.*

Proof. (a) Since $\mathrm{Iso}(G)^\circ$ is closed in G, it acts freely and properly on the right of G. Hence the orbit space is locally compact and Hausdorff by [19, Corollary 2.3].

(b) Conjugation by γ is a multiplicative bijection of $\mathrm{Iso}(G)_{s(\gamma)}$ onto $\mathrm{Iso}(G)_{r(\gamma)}$. So it suffices to show that

$$\gamma\,\mathrm{Iso}(G)^\circ\gamma^{-1} \subset \mathrm{Iso}(G)^\circ \quad \text{for all } \gamma \in G. \tag{2.1}$$

Take $\alpha \in \mathrm{Iso}(G)^\circ$ such that $s(\gamma) = r(\alpha)$ and let U be an open neighborhood of α in $\mathrm{Iso}(G)^\circ$. Let V be an open neighborhood of γ. Since G is étale, we can assume that U and V are bisections with $s(V) = r(U)$. Since the product of open subsets of G is open, VUV^{-1} is an open neighborhood of $\gamma\alpha\gamma^{-1}$. Since U and V are bisections and U consists of isotropy, VUV^{-1} is contained in $\mathrm{Iso}(G)$. Hence $\gamma\alpha\gamma^{-1} \in \mathrm{Iso}(G)^\circ$.

(c) Follows from (b) applied with $\gamma \in \mathrm{Iso}(G)_x$.

(d) The maps r' and s' are clearly well defined. Suppose that $(\gamma, \eta) \in G^{(2)}$ and that $\gamma' = \gamma\alpha$ and $\eta' = \eta\beta$ with $\alpha, \beta \in \mathrm{Iso}(G)^\circ$. Then $\gamma'\eta' = \gamma\eta(\eta^{-1}\alpha\eta\beta)$. But $\eta^{-1}\alpha\eta\beta \in \mathrm{Iso}(G)^\circ$ by (*b*). Hence $[\gamma'\eta'] = [\gamma\eta]$. This shows that multiplication is well-defined. A similar argument shows that inversion is well-defined. Since the quotient map is open [18, Lemma 2.1], it is not hard to see that these operations are continuous. For example, suppose that $[\gamma_i] \to [\gamma]$ and $[\eta_i] \to [\eta]$ with $(\gamma_i, \eta_i) \in G^{(2)}$. It suffices to see that every subnet of $[\gamma_i\eta_i]$ has a subnet converging to $[\gamma\eta]$. But after passing to a subnet, relabeling, and passing to another subnet and relabeling, we can assume that there are $\alpha_i, \beta_i \in \mathrm{Iso}(G)^\circ$ such that $\gamma_i\alpha_i \to \gamma$ and $\eta_i\beta_i \to \eta$ in G (see [29, Proposition 1.15]). But then $\gamma_i\alpha_i\eta_i\beta_i \to \gamma\eta$, and so $[\gamma_i\eta_i] \to [\gamma\eta]$.

We still need to see that $G/\mathrm{Iso}(G)^\circ$ is étale. Its unit space is the image of $G^{(0)}$ which is open since the quotient map is open. So it suffices to show that r' is a local homeomorphism. Given $[\gamma] \in G/\mathrm{Iso}(G)^\circ$, choose a compact neighborhood K of γ in G such that $r|_K$ is a homeomorphism. Let $q : G \to G/\mathrm{Iso}(G)^\circ$ be the quotient map. Then $q(K)$ is a compact neighborhood of $[\gamma]$ and r' is a continuous bijection, and hence a homeomorphism, of $q(K)$ onto its image.

(e) Take $b \in G/\mathrm{Iso}(G)^\circ$ such that $r'(b) = s'(b)$ but $b \neq r'(b)$. (That is, $b \in \mathrm{Iso}(G/\mathrm{Iso}(G)^\circ) \setminus q(G^{(0)})$, but the notation is a bit overwhelming.) It follows that $b = q(\gamma)$ for some $\gamma \in \mathrm{Iso}(G) \setminus \mathrm{Iso}(G)^\circ$. Let U be a open neighborhood of b. Then $q^{-1}(U)$ is an open neighborhood of γ, so meets $G \setminus \mathrm{Iso}(G)$. Take $\delta \in q^{-1}(U) \setminus \mathrm{Iso}(G)$;

so $s(\delta) \neq r(\delta)$. Then $q(\delta) \in U$ and $r'(q(\delta)) \neq s'(q(\delta))$. In particular, $q(\delta)$ does not belong to the interior of the isotropy of the groupoid $G/\operatorname{Iso}(G)^\circ$. Thus the interior of the isotropy of $G/\operatorname{Iso}(G)^\circ$ is just $q(G^{(0)})$. Now (e) follows from [3, Lemma 3.1].

(f) To see that $G/\operatorname{Iso}(G)^\circ$ is amenable, we need to see that r' is an amenable map (see [1, Definition 2.2.8]). If G itself is amenable, then $r = r' \circ q$ is amenable. Thus r' is amenable by [1, Proposition 2.2.4]. □

Our analysis of primitive ideals in C^*-algebras of Deaconu–Renault groupoids G will hinge on realising $C^*(G)$ as an induced algebra with fibres $C^*(G/\operatorname{Iso}(G)^\circ)$. The first step towards this is to construct a homomorphism $C^*(G) \to C^*(G/\operatorname{Iso}(G)^\circ)$, which can be done in much greater generality.

Proposition 2.6. *Let G be a locally compact Hausdorff étale groupoid such that $\operatorname{Iso}(G)^\circ$ is closed in G. There is a C^*-homomorphism $\kappa : C^*(G) \to C^*(G/\operatorname{Iso}(G)^\circ)$ such that*

$$\kappa(f)(b) = \sum_{q(\gamma)=b} f(\gamma) \quad \text{for } f \in C_c(G) \text{ and } b \in G/\operatorname{Iso}(G)^\circ.$$

Proof. Lemma 2.9(b) of [16] implies that κ defines a surjection of $C_c(G)$ onto $C_c(G/\operatorname{Iso}(G)^\circ)$. It clearly preserves involution, and

$$\begin{aligned}\kappa(f) * \kappa(g)(b) &= \sum_{s'(a)=r'(b)} \kappa(f)(a^{-1})\kappa(g)(ab) = \sum_{s'(a)=r'(b)} \sum_{\substack{q(\gamma)=a \\ q(\delta)=b}} f(\gamma^{-1})g(\gamma^{-1}\delta) \\ &= \sum_{q(\delta)=b} \sum_{s(\gamma)=r(\delta)} f(\gamma^{-1})g(\gamma\delta) = \sum_{q(\delta)=b} f * g(\delta) = \kappa(f * g)(b).\end{aligned}$$

It is not hard to see that κ is continuous in the inductive-limit topology (see [20, Corollary 2.17]). Since the $\|\cdot\|_I$-norm dominates the C^*-norm, the inductive-limit topology is stronger than the C^*-norm topology. Hence κ extends to a C^*-homomorphism from $C^*(G)$ to $C^*(G/\operatorname{Iso}(G)^\circ)$ as claimed. □

Remark 2.7. It is fairly unusual for $\operatorname{Iso}(G)^\circ$ to be closed in a general étale groupoid G (but see Proposition 3.10 and [15, Proposition 2.1]). For example, let X denote the union of the real and imaginary axes in $\mathbf{C}$, and let $T : X \to X$ be the homeomorphism $z \mapsto \overline{z}$. Regarding T as the generator of an action of $\mathbf{N}$ by local homeomorphisms, we form the associated groupoid

$$G_T = \{(t, m, t) : t \in \mathbf{R}, m \in \mathbf{Z}\} \cup \{(z, 2m, z), (z, 2m+1, \overline{z}) : z \in i\mathbf{R}, m \in \mathbf{Z}\}.$$

Then

$$\operatorname{Iso}(G)^\circ = \{(z, 2m, z) : z \in X, m \in \mathbf{Z}\} \cup \{(t, 2m+1, t) : t \in \mathbf{R} \setminus \{0\}, m \in \mathbf{Z}\}$$

is not closed: for example, $(0, 1, 0) \in \overline{\operatorname{Iso}(G)^\circ} \setminus \operatorname{Iso}(G)^\circ$.

However, we do not have an example of an étale groupoid G which acts minimally on its unit space and in which $\operatorname{Iso}(G)^\circ$ is not closed; and [15, Proposition 2.1] implies that no such example exists amongst the Deaconu–Renault groupoids of $\mathbf{N}^k$ actions that we consider for the remainder of the paper.

3. Deaconu–Renault Groupoids

Given k commuting local homeomorphisms of a locally compact Hausdorff space X, we obtain an action of $\mathbf{N}^k$ on X written $n \mapsto T^n$ (we do not assume that the T^n are surjective—cf., [7]). The corresponding *Deaconu–Renault Groupoid* is the set

$$G_T := \bigcup_{m,n\in\mathbf{N}^k} \{(x, m-n, y) \in X \times \mathbf{Z}^k \times X : T^m x = T^n y\} \tag{3.1}$$

with unit space $G_T^{(0)} = \{(x, 0, x) : x \in X\}$ identified with X, range and source maps $r(x, n, y) = x$ and $s(x, n, y) = y$, and operations $(x, n, y)(y, m, z) = (x, n + m, z)$ and $(x, n, y)^{-1} = (y, -n, x)$. For open sets $U, V \subseteq X$ and for $m, n \in \mathbf{N}^k$, we define

$$Z(U, m, n, V) := \{(x, m-n, y) : x \in U,\ y \in V \text{ and } T^m x = T^n y\}. \tag{3.2}$$

Lemma 3.1. *Let X be a locally compact Hausdorff space and let T be an action of $\mathbf{N}^k$ on X by local homeomorphisms. The sets* (3.2) *are a basis for a locally compact Hausdorff topology on G_T. The sets $Z(U, m, n, V)$ such that $T^m|_U$ and $T^n|_V$ are homeomorphisms and $T^m(U) = T^n(V)$ are a basis for the same topology. Under this topology and operations defined above, G_T is a locally compact Hausdorff étale groupoid.*

Proof. When X is compact and the T^m are surjective, this result follows immediately from [7, Propositions 3.1 and 3.2]. Their proof is easily modified to show that the $Z(U, m, n, V)$ form a basis for a topology on G_T when X is assumed only to be locally compact and the T^n are not assumed to be surjective. It is not hard to see that the groupoid operations are continuous in this topology.

Since the T^m are all local homeomorphisms, each $Z(U, m, n, V)$ is a union of sets $Z(U', m, n, V')$ such that $T^m|_{U'}$ and $T^n|_{V'}$ are local homeomorphisms. Given U, V, we have

$$Z(U, m, n, V) = Z\big(U \cap (T^m)^{-1}(T^m U \cap T^n V), m, n, V \cap (T^n)^{-1}(T^m U \cap T^n V)\big).$$

So the sets $Z(U, m, n, V)$ such that $T^m|_U$ and $T^n|_V$ are homeomorphisms with $T^m U = T^n V$ form a basis for the same topology as claimed.

To see that this topology is locally compact, let K_1 and K_2 be compact subsets of X. Then just as in [7, Proposition 3.2], the map $(x, y) \mapsto (x, p-q, y)$ is continuous from the compact set $\{(x, y) \in K_1 \times K_2 : T^p x = T^q y\}$ onto $Z(K_1, p, q, K_2)$. Hence the latter is compact in G_T. It now follows easily that G_T is locally compact. It is étale because the source map restricts to a homeomorphism on any set of the form described in the preceding paragraph. □

We now state our main theorem, which gives a complete listing of the primitive ideals of $C^*(G_T)$; but we need to establish a little notation first. Recall that for $x \in X$, the orbit $r((G_T)_x)$ is denoted $[x]$. So

$$[x] = \{y \in X : T^m x = T^n y \text{ for some } m, n \in \mathbf{N}^k\}.$$

We write

$$H(x) := \bigcup_{\substack{\emptyset \neq U \subset \overline{[x]} \\ U \text{ relatively open}}} \{m - n : m, n \in \mathbf{N}^k \text{ and } T^m y = T^n y \text{ for all } y \in U\}.$$

We write $H(x)^\perp := \{z \in \mathbf{T}^k : z^g = 1 \text{ for all } g \in H(x)\}$. We shall see later that $H(x)$ is a subgroup of $\mathbf{Z}^k$, so this usage of $H(x)^\perp$ is consistent with the usual notation for the annihilator in $\mathbf{T}^k$ of a subgroup of $\mathbf{Z}^k$. Our main theorem is the following.

Theorem 3.2. *Suppose that T is an action of $\mathbf{N}^k$ on a locally compact Hausdorff space X by local homeomorphisms. For each $x \in X$ and $z \in \mathbf{T}^k$, there is an irreducible representation $\pi_{x,z}$ of $C^*(G_T)$ on $\ell^2([x])$ such that*

$$\pi_{x,z}(f)\delta_y = \sum_{(u,g,y)\in G_T} z^g f(u,g,y)\delta_u \quad \text{for all } f \in C_c(G_T). \tag{3.3}$$

The relation on $X \times \mathbf{T}^k$ given by

$$(x,z) \sim (y,w) \text{ if and only if } \overline{[x]} = \overline{[y]} \text{ and } \overline{z}w \in H(x)^\perp$$

is an equivalence relation, and $\ker(\pi_{x,z}) = \ker(\pi_{y,w})$ if and only if $(x,z) \sim (y,w)$. The map $(x,z) \mapsto \ker \pi_{x,z}$ induces a bijection from $(X \times \mathbf{T}^k)/\sim$ to $\operatorname{Prim}(C^*(G_T))$.

Remark 3.3. A warning is in order. Theorem 3.2 lists the primitive ideals of $C^*(G_T)$, but it says nothing about the Jacobson topology. Example 3.4 below shows that neither the map $(x,z) \mapsto \ker \pi_{x,z}$ nor the induced map from $\mathcal{Q}(G_T)\times\mathbf{T}^k$ to $\operatorname{Prim}(C^*(G_T))$ is open in general.

Example 3.4. Consider the directed graph E with two vertices v and w and three edges e, f, g where e is a loop at v, g is a loop at w and f points from w to v. We use the conventions of [10], so the infinite paths in E are e^∞, g^∞ and $\{g^n f e^\infty : n = 0, 1, 2, \dots\}$. There are two orbits: $[g^\infty]$ and $[e^\infty]$. The latter is dense (because $\lim_{n\to\infty} g^n f e^\infty = g^\infty$), while the former is a singleton and is closed. As shown in [14], $C^*(E)$ is isomorphic to $C^*(G_T)$ where T is the shift operator on the infinite path space E^∞. Hence we can apply [10] to conclude that each $\ker \pi_{e^\infty,z} \subset \ker \pi_{g^\infty,w}$, and if $I_{x,z} := \ker \pi_{x,z}$ for $x \in E^\infty$ and $z \in \mathbf{T}$, we have $\overline{\{I_{g^\infty,z}\}} = \{I_{g^\infty,z}\} \cup \{I_{e^\infty,w} : w \in \mathbf{T}\}$. So, for example, the set $E^\infty \times \{w \in \mathbf{T} : \operatorname{Re}(w) > 0\}$ is open in $E^\infty \times \mathbf{T}$, but its image is not open in $\operatorname{Prim}(C^*(E))$; and likewise the set $\mathcal{Q}(E) \times \{w : \operatorname{Re}(w) > 0\}$ is open in $\mathcal{Q}(G_E) \times \mathbf{T}$, but its image is not open in $\operatorname{Prim}(C^*(E))$.

The proof of Theorem 3.2 occupies this and the next two sections, culminating in Section 5. Our first order of business is to show that G_T is always amenable.

Lemma 3.5. *Let G_T be the locally compact Hausdorff étale groupoid arising from an action of T of $\mathbf{N}^k$ on X by local homeomorphisms as above. Let $c : G_T \to \mathbf{Z}^k$ be the cocycle $c(x,k,y) = k$. Then both $c^{-1}(0)$ and G_T are amenable.*

Proof. For each $n \in \mathbf{N}^k$, let $F_n := \{(x,0,y) : T^n x = T^n y\}$. Then each F_n is a closed subgroupoid containing $G^{(0)}$, and

$$c^{-1}(0) = \bigcup_{n\in\mathbf{N}^k} F_n.$$

In fact, each F_n is also open in G: for $(x,0,y) \in F_n$ and any neighborhoods U of x and V of y, we have $(x,0,y) \in Z(U,n,n,V) \subset F_n$.

Since N^k acts by local homeomorphisms, for $x \in X$ the set $\{y \in X : T^n y = T^n x\}$ is discrete and therefore countable. So the Mackey–Glimm–Ramsay Dichotomy [22, Theorem 2.1] implies the orbit space is standard. It then follows from [1, Example 2.1.4(2)] that F_n is a properly amenable Borel groupoid, and hence Borel amenable as in [25, Definition 2.1]. Since F_n is open in G_T, it has a continuous Haar system (by restriction). Hence it is amenable by [25, Corollary 2.15]. It then follows from [1, Proposition 5.3.37] that $c^{-1}(0)$ is measurewise amenable. Since $c^{-1}(0)$ is open in G_T, it too is étale. Hence $c^{-1}(0)$ is amenable due to [1, Theorem 3.3.7].

The amenability of G_T now follows from [28, Proposition 9.3]. □

Our next task is to understand the interior of the isotropy in G_T. By definition of the topology on G_T this is the union of all the sets $Z(U, m, n, U)$ such that $U \subset X$ is open and $T^m x = T^n x$ for all $x \in U$. Our approach is based on that of [4, Section 4].

Lemma 3.6. *Let T be an action of $\mathbf{N}^k$ on X by local homeomorphisms. For each nonempty open set $U \subset X$, let*

$$\Sigma_U := \{(m, n) \in \mathbf{N}^k \times \mathbf{N}^k : T^m x = T^n x \text{ for all } x \in U\}. \tag{3.4}$$

Then

(a) *Σ_U is a submonoid of $\mathbf{N}^k \times \mathbf{N}^k$.*
(b) *Σ_U is an equivalence relation on $\mathbf{N}^k$.*
(c) *If $U \subset V$, then $\Sigma_V \subset \Sigma_U$.*
(d) *For $p \in \mathbf{N}^k$ and U open and nonempty, we have $\Sigma_U \subset \Sigma_{T^p U}$.*

Proof. Clearly $(0, 0) \in \Sigma_U$. Suppose that $(m, n), (p, q) \in \Sigma_U$. For $x \in U$ we have

$$T^{m+p} x = T^m T^p x = T^m T^q x = T^q T^m x = T^q T^n x = T^{n+q} x. \tag{3.5}$$

This proves (a). Statements (b) and (c) are immediate, and (d) follows from the special case of (3.5) where $p = q$. □

Since our aim is to identify the primitive ideals of $C^*(G_T)$, and since Lemma 2.1 shows that every irreducible representation of $C^*(G_T)$ factors through the restriction of G_T to some $\mathbf{N}^k$-irreducible subset, we will often assume that X itself (viewed as $G^{(0)}$) is $\mathbf{N}^k$-irreducible. In this case, we will say that T *acts irreducibly.* Lemma 2.1 then implies that X has a dense orbit: $X = \overline{[x]}$ for some $x \in X$.

Lemma 3.7. *Let T be an $\mathbf{N}^k$-irreducible action on X by local homeomorphisms. For all open subsets $U, V \subseteq X$, there exists a nonempty open set W such that $\Sigma_U \cup \Sigma_V \subset \Sigma_W$*

Proof. Fix x with $\overline{[x]} = X$. Choose $y \in U$ and $z \in V$ such that $T^r y = T^s z$ and $T^p z = T^l x$. Then $T^{r+l} y = T^{p+s} z$, so $m = r + l$ and $n = s + p$ satisfy $T^n U \cap T^m V \neq \emptyset$. Since T^m and T^n are local homeomorphisms, and therefore open maps, $W := T^m U \cap T^n V$ is open. Parts (c) and (d) of Lemma 3.6 show that $\Sigma_U \subset \Sigma_{T^m U} \subset \Sigma_W$ and $\Sigma_V \subset \Sigma_{T^n V} \subset \Sigma_W$. □

Given X and T as in Lemma 3.7, let

$$\Sigma := \bigcup_{\substack{\emptyset \neq U \subset X \\ U \text{ open}}} \Sigma_U. \tag{3.6}$$

We give $\mathbf{N}^k \times \mathbf{N}^k$ the usual partial order as a subset of $\mathbf{N}^{2k}$:

$$\big((n_i)_{i=1}^k, (n'_i)_{i=1}^k\big) \leq \big((m_i)_{i=1}^k, (m'_i)_{i=1}^k\big) \quad \text{if } n_i \leq m_i \text{ and } n'_i \leq m'_i \text{ for all } i.$$

We let $\Sigma^{\min}$ denote the collection of minimal elements of $\Sigma \setminus \{(0, 0)\}$ with respect to this order.

Lemma 3.8. *Let T be an irreducible action of $\mathbf{N}^k$ by local homeomorphisms on a locally compact space X, and let Σ and $\Sigma^{\min}$ be as above. Then Σ is a submoniod of $\mathbf{N}^k \times \mathbf{N}^k$ and an equivalence relation on $\mathbf{N}^k$. We have $\Sigma = (\Sigma - \Sigma) \cap (\mathbf{N}^k \times \mathbf{N}^k)$. Furthermore, $\Sigma^{\min}$ is finite and generates Σ as a monoid.*

Proof. We have $(0,0) \in \Sigma_X \subset \Sigma$. If $(m,n),(p,q) \in \Sigma$, then there are nonempty open sets U and V such that $(m,n) \in \Sigma_U$ and $(p,q) \in \Sigma_V$. Lemma 3.7 yields an open set W with $(m,n),(p,q) \in \Sigma_W$. Now $(m+p, n+q) \in \Sigma_W \subset \Sigma$ by Lemma 3.6(a), so Σ is a monoid.

To see that Σ is an equivalence relation, observe that it is reflexive and symmetric because each Σ_U is. Consider $(m,n),(n,p) \in \Sigma$; say $(m,n) \in \Sigma_U$ and $(n,p) \in \Sigma_V$. By Lemma 3.7, there is open set W with $(m,n),(n,p) \in \Sigma_W$. Hence $(m,p) \in \Sigma_W \subset \Sigma$ by Lemma 3.6(b).

The containment $\Sigma \subset (\Sigma - \Sigma) \cap (\mathbf{N}^k \times \mathbf{N}^k)$ is trivial because $(0,0) \in \Sigma$ and $\Sigma \subset \mathbf{N}^k \times \mathbf{N}^k$. For the reverse containment, suppose that $(m,n),(p,q) \in \Sigma$ and $m-p, n-q \in \mathbf{N}^k$. By Lemma 3.7 we may choose an open W such that $(m,n),(p,q) \in \Sigma_W$. Fix $x \in T^{p+q}W$, say $x = T^{p+q}y$. Lemma 3.6 implies first that $(q,p) \in \Sigma_W$, and then that $(m+q, n+p) \in \Sigma_W$. Hence

$$T^{m-p}x = T^{m-p}(T^{p+q}y) = T^{m+q}y = T^{n+p}y = T^{n-q}(T^{q+p}y) = T^{n-q}x.$$

So $(m-p, n-q) \in \Sigma_{T^{p+q}W} \subset \Sigma$.

Now we argue as in [4, Proposition 4.4].[1] Dickson's Lemma [26, Theorem 5.1] implies that $\Sigma^{\min}$ is finite. We must show that each $(m,n) \in \Sigma$ is a finite sum of elements of $\Sigma^{\min}$. We argue by induction on $|(m,n)| := \sum_{i=1}^k m_i + n_i$. If $|(m,n)| = 0$, the assertion is trivial. Now take $(m,n) \in \Sigma \setminus \{0\}$, and suppose that each $(p,q) \in \Sigma$ such that $|(p,q)| < |(m,n)|$ can be written as a finite sum of elements of $\Sigma^{\min}$. Since $(m,n) \neq 0$, by definition of $\Sigma^{\min}$ there exists $(a,b) \in \Sigma^{\min}$ such that $(a,b) \leq (m,n)$. The preceding paragraph shows that $(p,q) = (m,n) - (a,b) \in (\Sigma - \Sigma) \cap (\mathbf{N}^k \times \mathbf{N}^k) = \Sigma$. The induction hypothesis implies that (p,q) is a finite sum of elements of $\Sigma^{\min}$, and then so is $(m,n) = (p,q) + (a,b)$. □

We let

$$\begin{aligned} H(T) &:= \{\, m - n : (m,n) \in \Sigma \,\} \quad \text{and} \\ Y^{\max} &:= \bigcup \{\, Y \subset X : Y \text{ is open and } \Sigma_Y = \Sigma \,\}. \end{aligned} \tag{3.7}$$

Lemma 3.9. *Let T be an irreducible action of $\mathbf{N}^k$ by local homeomorphisms of a locally compact Hausdorff space X. With Σ as in (3.6), we have*

$$\Sigma = \{(m,n) \in \mathbf{N}^k \times \mathbf{N}^k : m - n \in H(T)\}. \tag{3.8}$$

The set $Y^{\max}$ is nonempty and open, and is the maximal open set in X such that $\Sigma_{Y^{\max}} = \Sigma$. We have $T^m Y^{\max} \subset Y^{\max}$ for all $m \in \mathbf{N}^k$.

Proof. By definition, $\Sigma \subset \{(m,n) : m - n \in H(T)\}$. For the reverse inclusion, suppose that $m - n = p - q$ with $(p,q) \in \Sigma$. Let $g = m - p \in \mathbf{Z}^k$. Fix $a, b \in \mathbf{N}^k$ such that $g = a - b$. Then both $(p+a, q+a)$ and (b,b) belong to Σ. Hence Lemma 3.8 implies that

$$(m,n) = (p+g, q+g) = (p+a, q+a) - (b,b) \in (\Sigma - \Sigma) \cap (\mathbf{N}^k \times \mathbf{N}^k) = \Sigma.$$

[1] Though in [4, Proposition 4.4], the crucial use, in the induction, of the fact that $\Sigma = (\Sigma - \Sigma) \cap (\mathbf{N}^k \times \mathbf{N}^k)$ is not made explicit.

This establishes (3.8).

Now $|\Sigma^{\min}|-1$ applications of Lemma 3.7 give a nonempty open set Y such that $\Sigma^{\min} \subset \Sigma_Y$. Since Σ_Y is monoid by Lemma 3.6, we have $\Sigma_Y = \Sigma$ by Lemma 3.8.

It now follows that $Y^{\max}$ is open and nonempty. It is clearly maximal. Each $T^m Y^{\max} \subset Y^{\max}$ by Lemma 3.6(d) and the definition of $Y^{\max}$. □

Proposition 3.10. *Let T be an irreducible action of $\mathbf{N}^k$ by local homeomorphisms of a locally compact Hausdorff space X, and let G_T be the associated Deaconu–Renault groupoid (as in* (3.1)*). The set $H(T)$ of* (3.7) *is a subgroup of $\mathbf{Z}^k$. Let Σ be as in* (3.6)*, and let $Y \subset X$ be an open set such that $\Sigma_Y = \Sigma$ and $T^p Y \subset Y$ for all $p \in \mathbf{N}^k$. Then* $\operatorname{Iso}(G_T|_Y)^\circ = \{(x, g, x) : x \in Y$ *and* $g \in H(T)\}$*, and* $\operatorname{Iso}(G_T|_Y)^\circ$ *is closed in $G_T|_Y$.*

Proof. Since Σ is an equivalence relation, 0 belongs to $H(T)$, and $g \in H(T)$ implies $-g \in H(T)$. Suppose that $m, n \in H(T)$, say $m = p_1 - q_1$ and $n = p_2 - q_2$ with $(p_i, q_i) \in \Sigma$. Lemma 3.8 implies that $(p_1 + p_2, q_1 + q_2) \in \Sigma$, and therefore that $m + n = p_1 + p_2 - q_1 - q_2$ belongs to $H(T)$. So $H(T)$ is a subgroup of $\mathbf{Z}^k$.

Take $x \in Y$ and $g \in H(T)$. By Lemma 3.9, there exists $(p, q) \in \Sigma$ such that $g = p - q$. Choose an open neighbourhood U of x in Y on which T^p and T^q are homeomorphisms. By choice of Y we have $T^p y = T^q y$ for all $y \in U$, and hence $\{(y, g, y) : y \in U\} = Z(U, p, q, U)$ is an open neighbourhood of (x, g, x) contained in $\{(y, g, y) : y \in Y, g \in H(T)\}$. So $\{(y, g, y) : y \in Y, g \in H(T)\} \subset \operatorname{Iso}(G_T)^\circ$. For the reverse inclusion, suppose that $(z, h, z) \in \operatorname{Iso}(G)^\circ$. By Lemma 3.1, there exist $m, n \in \mathbf{N}^k$ and open sets $U, V \subset Y$ such that $(z, h, z) \in Z(U, m, n, V) \subset \operatorname{Iso}(G_T)$ with $T^m U = T^n V$. So $T^m x = T^n x$ for all $x \in U$, and then $(m, n) \in \Sigma_U \subset \Sigma$. Thus $h \in H(T)$ as required.

We now have

$$\begin{aligned}\operatorname{Iso}(G_T|_Y)^\circ &= \{(x, g, x) : x \in Y, g \in H(T)\}\\ &= G_T \setminus \Big(\bigcup_{m-n \notin H(T)} Z(U, m, n, V) \cup \bigcup_{U \cap V = \emptyset} Z(U, m, n, V) \Big),\end{aligned}$$

and so $\operatorname{Iso}(G_T|_Y)^\circ$ is closed. □

Remark 3.11. We have an opportunity to fill a gap in the literature. The penultimate paragraph of the proof of [4, Theorem 5.3], appeals to [4, Corollary 2.8]. But unfortunately, the authors forgot to verify the hypothesis of [4, Corollary 2.8] that Γ should be aperiodic. We rectify this using our results above. Using the definition of aperiodicity of Γ [4, page 2575] and of the groupoid G_Γ of Γ [4, page 2573] as in the proof of [4, Corollary 2.8], we see that Γ is aperiodic if and only if G_Γ is topologically principal. In the situation of [4, Theorem 5.3], the groupoid $G_{H\Lambda T}$ discussed there is the restriction of the Deaconu–Renault groupoid $G_{\Lambda T}$ to $Y = H\Lambda^\infty$ which has the properties required of Y in Proposition 3.10 (see [4, Theorem 4.2(2)]), and so Proposition 3.10 shows that $\operatorname{Iso}(G_{H\Lambda T})^\circ$ is closed. It is easy to check that G_Γ is isomorphic to $G_{H\Lambda T} / \operatorname{Iso}(G_{H\Lambda T})^\circ$. So Proposition 2.6(e) shows that G_Γ is topologically principal and hence that Γ is aperiodic as required.

Corollary 3.12. *Let T be an irreducible action of $\mathbf{N}^k$ by local homeomorphisms on a locally compact Hausdorff space X. Let Σ and $H(T)$ be as in* (3.6) *and* (3.7)*. Suppose that Y is an open subset of X such that $T^p Y \subset Y$ for all p and such that $\Sigma_Y = \Sigma$.*

(a) *Regard $C_c(G_T|_Y)$ as a subalgebra of $C_c(G_T)$. The identity map extends to a monomorphism $\iota : C^*(G_T|_Y) \to C^*(G_T)$, and $\iota(C^*(G_T|_Y))$ is a hereditary subalgebra of $C^*(G_T)$.*

(b) *The map $\pi \mapsto \pi \circ \iota$ is a bijection from the collection of irreducible representations of $C^*(G_T)$ that are injective on $C_0(X)$ to the space of irreducible representations of $C^*(G_T|_Y)$ that are injective on $C_0(Y)$. Moreover, the map $\ker \pi \mapsto \ker(\pi \circ \iota)$ is a homeomorphism from $\{I \in \operatorname{Prim} C^*(G_T) : I \cap C_0(X) = \{0\}\}$ onto $\{J \in \operatorname{Prim} C^*(G_T|_Y) : J \cap C_0(Y) = \{0\}\}$.*

Proof. The inclusion $C_c(G_T|_Y) \hookrightarrow C_c(G_T)$ is a $*$-homomorphism and continuous in the inductive-limit topology. Hence we get a homomorphism ι. Fix $x \in Y$. Let L^x be the regular representation of $C^*(G_T)$ on $\ell^2((G_T)_x)$. Then $L^x \circ \iota$ leaves the subspace $\ell^2\{(y, g, x) \in G_T : y \in Y\})$ invariant. Hence $L^x \circ \iota$ is equivalent to $L^x_Y \oplus 0$ where L^x_Y is the corresponding regular representation of $C^*(G_T|_Y)$. Since G_T and $G_T|_Y$ are both amenable by Lemma 3.5, ι is isometric and hence a monomorphism.

Let $\{f_i\}$ be an approximate identity for $C_0(Y)$. For $f \in C_c(G_T)$ we have $f_i f f_i \in C_c(G_T|_Y)$. Thus $\iota(C^*(G_T|_Y))$ is the closure of $\bigcup_i f_i C^*(G_T) f_i$. It follows easily that the image of ι is a hereditary subalgebra of $C^*(G_T)$ as claimed.

If π is an irreducible representation of $C^*(G_T)$ that is injective on $C_0(X)$, then it does not vanish on the ideal I_Y in $C^*(G_T)$ generated by $C_0(Y)$. Clearly, $\iota(C^*(G_T|_Y))$ is Morita equivalent to I_Y, and restriction of representations implements Rieffel induction from I_Y to $\iota(C^*(G_T|_Y))$. Since Rieffel induction between Morita equivalent C^*-algebras takes irreducibles to irreducibles ([21, Corollary 3.32]) and since $\pi|_{I_Y}$ is irreducible ([2, Theorem 1.3.4]), $\pi \circ \iota$ is irreducible and clearly injective on $C_0(Y)$. If ρ is an irreducible representation of $C^*(G_T|_Y)$, then it extends to an irreducible representation of I_Y. Since I_Y is an ideal, this representation extends to a (necessarily irreducible) representation π of $C^*(G_T)$ such that $\rho = \pi \circ \iota$. The kernel of $\pi|_{C_0(X)}$ is proper and has $\mathbf{N}^k$-invariant support. Since T acts irreducibly, $C_0(X)$ is G_T-simple, and so $\ker(\pi|_{C_0(X)}) = \{0\}$ and we obtain the required bijection.

The remaining assertion follows from this bijection and the Rieffel correspondence (see [21, Corollary 3.33(a)]). □

4. The primitive ideals of the C^*-algebra of an irreducible Deaconu–Renault groupoid

In this section we specialize to the situation where T is an irreducible action of $\mathbf{N}^k$ on a locally compact Hausdorff space Y with the property that, in the notation of (3.6), $\Sigma_Y = \Sigma$. We then have $\Sigma = \Sigma_U$ for all nonempty open subsets U of Y by Lemma 3.6. Lemma 3.9 says that $m - n \in H(T)$ implies $T^m x = T^n x$ for all $x \in Y$; and Proposition 3.10 gives

$$\operatorname{Iso}(G_T)^\circ = \{(x, g, x) : x \in Y \text{ and } g \in H(T)\}.$$

We show that under these hypotheses, the primitive ideals of $C^*(G_T)$ with trivial intersection with $C_0(Y)$ are indexed by characters of $H(T)$. More precisely, we show that the irreducible representations of $C^*(G_T)$ that are faithful on $C_0(Y)$ are indexed by pairs (π, χ) where π is an irreducible representation of $C^*(G_T/\operatorname{Iso}(G_T)^\circ)$ and χ is a character of $H(T)$. Our approach is to exhibit $C^*(G_T)$ as an induced algebra. Recall from Proposition 3.10 that $\operatorname{Iso}(G_T)^\circ$ is closed in G_T, so Proposition 2.6 gives a homomorphism $\kappa : C^*(G_T) \to C^*(G_T/\operatorname{Iso}(G_T)^\circ)$.

Lemma 4.1. *Suppose that T is an irreducible action of $\mathbf{N}^k$ on a locally compact space Y such that $\Sigma_Y = \Sigma$. There is an action α of $\mathbf{T}^k$ on $C^*(G_T)$ such that $\alpha_z(f)(x,g,y) = z^g f(x,g,y)$ for $f \in C_c(G_T)$. Let $\kappa : C^*(G_T) \to C^*(G_T/\operatorname{Iso}(G_T)^\circ)$ be the homomorphism of Proposition 2.6. There is an action $\tilde\alpha$ of $H(T)^\perp$ on $C^*(G_T/\operatorname{Iso}(G_T)^\circ)$ such that $\tilde\alpha_z \circ \kappa = \kappa \circ \alpha_z$ for all $z \in H(T)^\perp \subset \mathbf{T}^k$.*

If $\bar z w \notin H(T)^\perp$, then $\big(\ker(\kappa\circ\alpha_z) + \ker(\kappa\circ\alpha_w)\big) \cap C_0(Y) \neq \{0\}$. We have $\ker(\kappa\circ\alpha_z) = \ker(\kappa\circ\alpha_w)$ if and only if $\bar z w \in H(T)^\perp$.

Proof. Let $c : G_T \to \mathbf{Z}^k$ be the canonical cocycle $c(x,g,y) = g$. The formula $\alpha_z(f)(\gamma) = z^{c(\gamma)} f(\gamma)$ defines a $*$-homomorphism $\alpha_z : C_c(G_T) \to C_c(G_T)$. This α_z is trivially I-norm preserving, so extends to $\alpha_z : C^*(G_T) \to C^*(G_T)$. Since $\alpha_{\bar z}$ is an inverse for α_z, we have $\alpha_z \in \operatorname{Aut}(C^*(G_T))$. The map $z \mapsto \alpha_z$ is a homomorphism because α_{zw} and $\alpha_z \circ \alpha_w$ agree on each $C_c(c^{-1}(g))$. To see that $z \mapsto \alpha_z$ is strongly continuous, first note that if $f \in C_c(G_T)$ is supported on $c^{-1}(g)$, then each $\alpha_z(f) = z^g f$, so $z \mapsto \alpha_z(f)$ is continuous. Since each $f \in C_c(G_T)$ is a finite linear combination $f = \sum_{\operatorname{supp}(f)\cap c^{-1}(g)\neq\emptyset} f|_{c^{-1}(g)}$ of such functions, $z \mapsto \alpha_z(f)$ is continuous for each $f \in C_c(G_T)$. Now an $\varepsilon/3$ argument shows that $z \mapsto \alpha_z$ is strongly continuous.

Let $q : \mathbf{Z}^k \to \mathbf{Z}^k/H(T)$ be the quotient map. We have,

$$\operatorname{Iso}(G_T)^\circ = \{(x,g,x) : x \in Y \text{ and } g \in H(T)\}.$$

Identify $G_T/\operatorname{Iso}(G_T)^\circ$ with $\{(x,q(g),y) : (x,g,y) \in G_T\} \subset Y \times (\mathbf{Z}^k/H(T)) \times Y$.

Proposition 2.5 implies that the quotient map from G_T onto $G_T/\operatorname{Iso}(G_T)^\circ$ is continuous and open, so the sets

$$\underline{Z}(U,q(m),q(n),V) = \{(x,q(m-n),y) : x \in U,\ y \in V \text{ and } T^m x = T^n y\}$$

are a basis for the topology on $G_T/\operatorname{Iso}(G_T)^\circ$ (this makes sense because $T^m x = T^n y$ if and only if $T^{m+a}x = T^{n+b}y$ whenever $a - b \in H(T)$). Arguing as in the first paragraph, we get an action $\tilde\alpha$ of $H(T)^\perp$ on $C^*(G_T/\operatorname{Iso}(G_T)^\circ)$ such that $\tilde\alpha_z(f)(x,q(g),y) = z^g f(x,q(g),y)$ for $f \in C_c(G_T/\operatorname{Iso}(G_T)^\circ)$. For $f \in C_c(G_T)$, it is easy to check that $\tilde\alpha_z \circ \kappa(f) = \kappa\circ\alpha_z(f)$ for $z \in H(T)^\perp$. This identity then extends by continuity to all of $C^*(G_T)$.

Suppose that $\bar z w \notin H(T)^\perp$. Choose $n \in H(T)$ such that $z^n \neq w^n$. Fix a nonzero function $f \in C_c(Y)$ and define $f_n \in C_c(\{(x,n,x) : x \in Y\} \subset C_c(G_T)$ by $f_n(x,n,x) = f(x,0,x)$ for all $x \in Y$. Then $w^n f - f_n \in \ker(\kappa\circ\alpha_w)$ and $z^n f - f_n \in \ker(\kappa\circ\alpha_z)$. Hence $(z^n - w^n) f \in \big(\ker(\kappa\circ\alpha_z) + \ker(\kappa\circ\alpha_w)\big) \cap C_0(Y) \setminus \{0\}$ by choice of n. This proves the second-last statement of the lemma.

Since each of $\kappa\circ\alpha_z$ and $\kappa\circ\alpha_w$ is injective on $C_0(Y)$, this also proves the (contrapositive of the) implication $\implies$ in the final statement of the lemma. For the reverse implication, suppose that $\bar z w \in H(T)^\perp$. Then

$$\ker(\kappa\circ\alpha_w) = \ker(\kappa\circ\alpha_{\bar z w}\circ\alpha_z) = \ker(\tilde\alpha_{\bar z w}\circ\kappa\circ\alpha_z) = \ker(\kappa\circ\alpha_z). \qquad \square$$

The final assertion of Lemma 4.1 ensures that we can form the induced algebra $\operatorname{Ind}_{H(T)^\perp}^{\mathbf{T}^k}\big(C^*(G_T/\operatorname{Iso}(G_T)^\circ), \tilde\alpha\big)$, namely

$$\{s \in C(\mathbf{T}^k, C^*(G_T/\operatorname{Iso}(G_T)^\circ) : s(wz) = \tilde\alpha_z(s(w)) \text{ for all } w \in \mathbf{T}^k \text{ and } z \in H(T)^\perp\}.$$

Induced algebras have a well-understood structure. Some of their elementary properties (in particular, the ones that we rely upon) are discussed in [21, §6.3].

Before proving the next result, we recall some basic results from abelian harmonic analysis. We write $C_c(H(T))$ for the set of finitely supported functions on $H(T)$. If $\varphi \in C_c(H(T))$, then its Fourier transform $\hat{\varphi} \in C(\mathbf{T}^k)$ is given by

$$\hat{\varphi}(z) = \sum_{n \in H(T)} \varphi(n) z^n,$$

and is constant on $H(T)^\perp$ cosets. Taking a few liberties with notation and terminology, we regard $\hat{\varphi}$ as an element of $C(\mathbf{T}^k/H(T)^\perp)$. The general theory implies that $\{\hat{\varphi} : \varphi \in C_c(H(T))\}$ is a (uniformly) dense subalgebra of $C(\mathbf{T}^k/H(T)^\perp)$.

Lemma 4.2. *Let T be an irreducible action of $\mathbf{N}^k$ on a locally compact space Y by local homeomorphisms, and suppose that $\Sigma_Y = \Sigma$. If $(x, g, y) \in G_T$, then $(x, g + n, y) \in G_T$ for all $n \in H(T)$.*

Proof. Let $(x, g, y) = (x, p - q, y)$ with $T^p x = T^q y$. Fix $n \in H(T)$. Then $n = n_+ - n_-$ with $(n_+, n_-) \in \Sigma = \Sigma_Y$. Hence $T^{n_+} z = T^{n_-} z$ for all $z \in Y$, giving

$$T^{p+n_+} x = T^{n_+} T^p x = T^{n_+} T^q y = T^{n_-} T^q y = T^{q+n_-} y.$$

Hence $(x, g + n, y) = (x, (p + n_+) - (q + n_-), y) \in G_T$. □

Because of Lemma 4.2, we can define a left action of $C_c(H(T))$ on $C_c(G_T)$ by

$$\varphi \cdot f(x, g, y) := \sum_{n \in H(T)} \varphi(n) f(x, g - n, y). \tag{4.1}$$

Lemma 4.3. *Let T be an irreducible action of $\mathbf{N}^k$ on a locally compact space Y by local homeomorphisms such that $\Sigma_Y = \Sigma$, and let $\kappa : C^*(G_T) \to C^*(G_T/\operatorname{Iso}(G_T)^\circ)$ be as in Proposition 2.6. Then*

$$\kappa(\alpha_z(\varphi \cdot f)) = \hat{\varphi}(z) \kappa(\alpha_z(f))$$

for all $f \in C_c(G_T)$, all $z \in \mathbf{T}^k$, and all $\varphi \in C_c(H(T))$.

Proof. We compute:

$$\begin{aligned}
\kappa(\alpha_z(\varphi \cdot f))(x, q(g), y) &= \sum_{m \in H(T)} \alpha_z(\varphi \cdot f)(x, g + m, y) \\
&= \sum_{m \in H(T)} z^{g+m} \varphi \cdot f(x, g + m, y) \\
&= \sum_{m \in H(T)} \sum_{n \in H(T)} z^{g+m} \varphi(n) f(x, g + m - n, y).
\end{aligned}$$

Since both sums are finite and we can interchange the order of summations at will, we may continue the calculation:

$$\begin{aligned}
&= \sum_{m \in H(T)} \sum_{n \in H(T)} z^{g+m+n} \varphi(n) f(x, g + m, y) \\
&= \sum_{m \in H(T)} z^{g+m} \hat{\varphi}(z) f(x, g + m, y) \\
&= \hat{\varphi}(z) \kappa(\alpha_z(f))(x, q(g), y).
\end{aligned}$$

□

Proposition 4.4. *Let T be an irreducible action of $\mathbf{N}^k$ on a locally compact space Y by local homeomorphisms, and suppose that $\Sigma_Y = \Sigma$. Let*

$$\alpha : \mathbf{T}^k \to \operatorname{Aut} C^*(G_T) \quad \textit{and} \quad \tilde{\alpha} : H(T)^{\perp} \to \operatorname{Aut} C^*(G_T/\operatorname{Iso}(G_T)^{\circ})$$

be as in Lemma 4.1, and let $\kappa : C^*(G_T) \to C^*(G_T/\operatorname{Iso}(G_T)^{\circ})$ be as in Proposition 2.6. There is an isomorphism $\Phi : C^*(G_T) \to \operatorname{Ind}_{H(T)^{\perp}}^{\mathbf{T}^k}(C^*(G_T/\operatorname{Iso}(G_T)^{\circ}), \tilde{\alpha})$ such that $\Phi(a)(z) = \kappa(\alpha_z(a))$ for $a \in C^*(G_T)$ and all $z \in \mathbf{T}$.

Proof. For $a \in C^*(G_T)$, the map $z \mapsto \kappa(\alpha_z(a))$ is continuous by continuity of α. Take $f \in C_c(G_T)$, $w \in \mathbf{T}^k$ and $z \in H(T)^{\perp}$. Lemma 4.1 gives $\tilde{\alpha}_z \circ \kappa = \kappa \circ \alpha_z$. Hence

$$\Phi(f)(wz) = \kappa(\alpha_{wz}(f)) = \kappa(\alpha_z(\alpha_w(f))) = \tilde{\alpha}_z \kappa(\alpha_w(f)) = \tilde{\alpha}_z(\Phi(f)(w)).$$

Thus Φ takes values in $\operatorname{Ind}_{H(T)^{\perp}}^{\mathbf{T}^k}(C^*(G_T/\operatorname{Iso}(G_T)^{\circ}), \tilde{\alpha})$. It is not hard to check that Φ is a homomorphism.

To see that Φ is injective we use an averaging argument. Let $\mathbf{T}^k$ act on the left of $\operatorname{Ind}_{H(T)^{\perp}}^{\mathbf{T}^k}(C^*(G_T/\operatorname{Iso}(G_T)^{\circ}), \tilde{\alpha})$ by left translation: $\operatorname{lt}_z(c)(w) = c(\bar{z}w)$. We have $\Phi \circ \alpha_z = \operatorname{lt}_{\bar{z}} \circ \Phi$. So the standard argument involving the faithful conditional expectations obtained from averaging over $\mathbf{T}^k$ actions (see, for example, [27, Lemma 3.13]) shows that it is sufficient to check that Φ restricts to an injection on $C^*(G_T)^{\alpha}$.

If $f \in C_c(G_T)$, then arguing as in [29, Lemma 1.108], we have $\int_{\mathbf{T}^k} \alpha_z(f)\, dz \in C_c(G_T)$ and for $\gamma \in G_T$,

$$\begin{aligned}\Big(\int_{\mathbf{T}^k} \alpha_z(f)\, dz\Big)(\gamma) &= \int_{\mathbf{T}^k} \alpha_z(f)(\gamma)\, dz = \Big(\int_{\mathbf{T}^k} z^{c(\gamma)}\, dz\Big) f(\gamma)\\ &= \begin{cases} f(\gamma) & \text{if } \gamma \in c^{-1}(0)\\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

It follows that $C^*(G_T)^{\alpha} = \overline{C_c(c^{-1}(0))} \subset C^*(G_T)$. Thus the inclusion map induces a monomorphism $\rho : C^*(c^{-1}(0)) \to C^*(G_T)$ whose image is exactly $C^*(G_T)^{\alpha}$. To see that $\Phi|_{C^*(G)^{\alpha}}$ is injective, it suffices to show that $\Phi \circ \rho$ is injective. Since $c^{-1}(0)$ is amenable by Lemma 3.5 and principal by construction, [6, Theorem 4.4] implies that we need only show that $(\Phi \circ \rho)|_{C_0(Y)}$ is injective. As ρ restricts to the canonical inclusion $C_0(Y) \hookrightarrow C^*(G_T)^{\alpha}$, it is enough to verify that Φ is injective on $C_0(Y)$. The homomorphism $\kappa \circ \alpha_z$ restricts to the identity map of $C_0(Y) \subset C^*(G_T)$ onto $C_0(Y) \subset C^*(G_T/\operatorname{Iso}(G_T)^{\circ})$. So if $f \in C_0(Y)$, $z \in \mathbf{T}^k$ and $b \in G_T/\operatorname{Iso}(G_T)^{\circ}$, then

$$\Phi(f)(z)(b) = \kappa(\alpha_z(f))(b) = \begin{cases} f(x) & \text{if } b = (x, 0, x) \in (G_T/\operatorname{Iso}(G_T)^{\circ})^{(0)}\\ 0 & \text{otherwise.}\end{cases}$$

Thus if $\Phi(f) = 0$, then $f = 0$. This completes the proof that Φ is injective.

We still have to show that Φ is surjective. Lemma 4.3 implies that if $\varphi \in C_c(H(T))$ and $f \in C_c(G_T)$, then $\hat{\varphi} \cdot \Phi(f) = \Phi(\varphi \cdot f)$ for the obvious action of $C(\mathbf{T}^k/H(T))$ on the induced algebra. Since $\{\hat{\varphi} : \varphi \in C_c(H(T))\}$ is dense in $C(\mathbf{T}^k/H(T))$ it follows that the range of Φ is a $C(\mathbf{T}^k/H(T))$-submodule. So it suffices to show that the range of $\kappa \circ \alpha_z$ contains $C_c(G_T/\operatorname{Iso}(G_T)^{\circ})$.

For this, fix $g \in \mathbf{Z}^k$ and $f \in \tilde{c}^{-1}(q(g))$; it suffices to show that f is in the range of $\pi \circ \alpha_z$. Define $h \in C_c(G_T)$ by

$$h(\gamma) = \begin{cases} \overline{z}^g f(\tilde{q}(\gamma)) & \text{if } c(\gamma) = g \\ 0 & \text{otherwise.} \end{cases}$$

Then h is continuous because each $c^{-1}(g)$ is clopen in G_T; and $\kappa(\alpha_z(h)) = f$. $\square$

We now aim to apply [21, Proposition 6.6], which describes the primitive-ideal space of an induced algebra, to describe the topology of $\mathrm{Prim}(C^*(G_T))$ for a special class of $\mathbf{N}^k$-actions T. To achieve this we first describe, in Lemma 4.6, the Jacobson topology on $\mathrm{Prim}(C^*(G))$ when G is an amenable étale Hausdorff groupoid whose reduction to any closed invariant set is topologically principal. This topology is also described by [24, Corollary 4.9], but the statement given there is not quite the one we need.

Lemma 4.5. *Let G be a second-countable locally compact Hausdorff étale groupoid, and fix $x \in G^{(0)}$. There is an irreducible representation $\omega_{[x]} : C^*(G) \to \mathscr{B}(\ell^2([x]))$ satisfying $\omega_{[x]}(f)\delta_y = \sum_{s(\gamma)=y} f(\gamma)\delta_{r(\gamma)}$ for all $f \in C_c(G)$. If G is topologically principal and amenable and if $[x]$ is dense in $G^{(0)}$, then $\omega_{[x]}$ is faithful, and hence $C^*(G)$ is primitive.*

Proof. Let E_x denote the 1-dimensional representation of the group G_x^x. Then $\omega_{[x]} := \mathrm{Ind}_{\{x\}}^G E_x$ is a representation satisfying the desired formula.[2] Hence $\omega_{[x]}$ is irreducible by [12, Theorem 5].

Now suppose that G is amenable and topologically principal with $[x]$ dense in $G^{(0)}$. Then clearly $\omega_{[x]}$ is faithful on $C_0(G^{(0)})$. So [6, Theorem 4.4] says that it is faithful on $C^*(G)$, whence $C^*(G)$ is primitive. $\square$

Recall that the quasi-orbit space $\mathcal{Q}(G) = \{\, \overline{[x]} : x \in G^{(0)} \,\}$ carries the quotient topology for the map $q : G^{(0)} \to \mathcal{Q}(G)$ that identifies u with v exactly when $[u]$ and $[v]$ have the same closure in $G^{(0)}$. In particular, if $S \subset \mathcal{Q}(G)$, then $\overline{S} = \{\, q(x) : x \in \overline{q^{-1}(S)} \,\}$.

Lemma 4.6. *Let G be an amenable, étale Hausdorff groupoid and suppose that $G|_X$ is topologically principal for every closed invariant subset X of the unit space*[3] *For $x \in G^{(0)}$, let ω_x be the irreducible representation of Lemma 4.5. The map $x \mapsto \ker \omega_x$ from $G^{(0)}$ to $\mathrm{Prim}(C^*(G))$ descends to a homeomorphism of the quasi-orbit space $\mathcal{Q}(G)$ onto $\mathrm{Prim}(C^*(G))$.*

Proof. For $x \in G^{(0)}$, we have $\ker \omega_x \cap C_0(G^{(0)}) = C_0(G^{(0)} \setminus \overline{[x]})$. Since $G|_X$ is topologically principal for every closed invariant subset $X \subset G^{(0)}$, [24, Corollary 4.9][4] therefore implies that $\ker \omega_x = \ker \omega_y$ if and only if $\overline{[x]} = \overline{[y]}$. Hence $x \mapsto \ker \omega_x$ descends to a well-defined injection $\overline{[x]} \mapsto \ker \omega_x$. To see that it is surjective, observe that if π is an irreducible representation of $C^*(G)$, then Proposition 2.4 implies that $\ker \pi \cap C_0(G^{(0)}) = C_0(G^{(0)} \setminus \overline{[x]})$ for some $x \in G^{(0)}$. That is,

[2] This is also the representation described in [3, Proposition 5.2].

[3] Although the term has been used inconsistently, in [24] for example, one says the G-action on $G^{(0)}$ is essentially free.

[4] Specifically, [24, Corollary 4.9] applied to the groupoid dynamical system $(G, \Sigma, \mathscr{A})$ where Σ is the bundle of trivial groups over $G^{(0)}$ and $\mathscr{A}$ is the trivial bundle $G^{(0)} \times \mathbf{C}$ of 1-dimensional C^*-algebras—see also [3, Corollary 5.9].

$\ker\pi \cap C_0(G^{(0)}) = \ker\omega_x \cap C_0(G^{(0)})$, and then [24, Corollary 4.9] again shows that $\ker\pi = \ker\omega_x$.

To show that $\overline{[x]} \mapsto \ker\omega_x$ is a homeomorphism, it suffices to take a set $S \subset \mathcal{Q}(G)$ and an element $x \in G^{(0)}$ and show that $\overline{[x]} \in \overline{S}$ if and only if $\ker\omega_x \in \overline{\{\ker\omega_y : q(y) \in S\}}$; for then $S \subset \mathcal{Q}(G)$ is closed if and only if its image $\{\ker\omega_y : q(y) \in S\}$ is closed in $\operatorname{Prim}(C^*(G))$.

Fix $S \subset \mathcal{Q}(G)$ and $x \in G^{(0)}$. We have

$$\overline{\{\, \ker\omega_y : q(y) \in S \,\}} = \{\ker\omega_z : \bigcap_{q(y)\in S} \ker\omega_y \subset \ker\omega_z\}.$$

Using [24, Corollary 4.9] again, we deduce that $\ker\omega_x \in \overline{\{\, \ker\omega_y : q(y) \in S \,\}}$ if and only if $\big(\bigcap_{q(y)\in S} \ker\omega_y\big) \cap C_0(G^{(0)}) \subset \ker\omega_x \cap C_0(G^{(0)})$. We have

$$\begin{aligned}\Big(\bigcap_{q(y)\in S} \ker\omega_y\Big) \cap C_0(G^{(0)}) &= \{\, f \in C_0(G^{(0)}) : f|_{q^{-1}(S)} = 0 \,\} \\ &= \{\, f \in C_0(G^{(0)}) : f|_{\overline{q^{-1}(S)}} = 0 \,\}.\end{aligned}$$

On the other hand,

$$\ker\omega_x \cap C_0(G^{(0)}) = \{\, f \in C_0(G^{(0)}) : f|_{\overline{[x]}} = 0 \,\}.$$

Hence $\ker\omega_x \in \overline{\{\ker\omega_y : y \in \bigcup S\}}$ if and only if $\overline{[x]} \subset \overline{q^{-1}(S)}$, and this is equivalent to $q(x) \in \overline{S} = \{\, q(x) : x \in \overline{q^{-1}(S)} \,\}$ since $\overline{q^{-1}(S)}$ is closed and invariant. □

For the next result, recall that the quotient map $q : G \to G/\operatorname{Iso}(G)^\circ$ restricts to a homeomorphism of unit spaces. Since q also preserves the range and source maps, it carries G-orbits bijectively to the corresponding $(G/\operatorname{Iso}(G)^\circ)$-orbits, and therefore carries orbit closures in G to the corresponding orbit closures in $G/\operatorname{Iso}(G)^\circ$. Hence the identification $G^{(0)} = (G/\operatorname{Iso}(G)^\circ)^{(0)}$ induces a homeomorphism $\mathcal{Q}(G) \cong \mathcal{Q}(G/\operatorname{Iso}(G)^\circ)$.

Theorem 4.7. *Let T be an irreducible action of $\mathbf{N}^k$ on a locally compact space Y by local homeomorphisms such that, in the notation of* (3.6), *$\Sigma_Y = \Sigma$. Suppose that for every $y \in Y$, the set*

$$\Sigma_{\overline{[y]}} := \{(m,n) \in \mathbf{N}^k \times \mathbf{N}^k : T^m x = T^n x \text{ for all } x \in \overline{[y]}\}$$

satisfies $\Sigma_{\overline{[y]}} = \Sigma$. Let $\alpha : \mathbf{T}^k \to \operatorname{Aut} C^*(G_T)$ be as in Lemma 4.1, and let $\kappa : C^*(G_T) \to C^*(G_T/\operatorname{Iso}(G_T)^\circ)$ be as in Proposition 2.6. For $y \in (G_T/\operatorname{Iso}(G_T)^\circ)^{(0)}$, let ω_x be the irreducible representation of $C^*(G_T)$ described in Lemma 4.5. Then the map $(y,z) \mapsto \ker(\omega_y \circ \alpha_z)$ from $Y \times \mathbf{T}^k$ to $\operatorname{Prim}(C^*(G_T))$ descends to a homeomorphism $\mathcal{Q}(G_T) \times H(T)^\wedge \cong \operatorname{Prim}(C^*(G_T))$.

Proof. Let $\Phi : C^*(G_T) \to \operatorname{Ind}_{H(T)^\perp}^{\mathbf{T}^k}(C^*(G_T/\operatorname{Iso}(G_T)^\circ), \tilde\alpha)$ be the isomorphism of Proposition 4.4. For each $y \in Y$, let $\tilde\omega_y$ be the irreducible representation of $C^*(G_T/\operatorname{Iso}(G_T)^\circ)$ obtained from Lemma 4.5. Observe that $\tilde\omega_y \circ \kappa = \omega_y$. We have

$$\Phi(\ker(\omega_y \circ \alpha_z)) = \{s \in \operatorname{Ind}_{H(T)^\perp}^{\mathbf{T}^k}(C^*(G_T/\operatorname{Iso}(G_T)^\circ), \tilde\alpha) : f(z) \in \ker\tilde\omega_y\}.$$

Write ε_z for the homomorphism of the induced algebra onto $C^*(G_T/\operatorname{Iso}(G_T)^\circ)$ given by evaluation at z. It now suffices to show that

$$(y,z) \mapsto \ker\big(\tilde\omega_y \circ \varepsilon_z\big) \tag{4.2}$$

induces a homeomorphism of $\mathcal{Q}(G_T) \times H(T)^\wedge$ onto the primitive ideal space of the induced algebra.

Proposition 3.10 combined with the hypothesis that each $\Sigma_{\overline{[y]}} = \Sigma$ ensures that $\mathrm{Iso}(G)^\circ|_{\overline{[y]}} = \mathrm{Iso}(G|_{\overline{[y]}})^\circ$ for each y. Hence Proposition 2.5 ensures that the reduction of $G_T/\mathrm{Iso}(G_T)^\circ$ to any orbit closure, and hence to any closed invariant set, is topologically principal. Now Lemma 4.6 implies that $\ker(\tilde{\omega}_y \circ \varepsilon_z) = \ker(\tilde{\omega}_x \circ \varepsilon_z)$ if and only if $\overline{[y]} = \overline{[x]}$. So the map (4.2) descends to a map $(\overline{[y]}, z) \mapsto \ker\big(\tilde{\omega}_y \circ \varepsilon_z\big)$. Composing this with the homeomorphism of Lemma 4.6 shows that (4.2) induces a well-defined map

$$M : (\ker \tilde{\omega}_y, z) \mapsto \ker\big(\tilde{\omega}_y \circ \varepsilon_z\big)$$

from $\mathrm{Prim}(C^*(G_T/\mathrm{Iso}(G_T)^\circ) \times \mathbf{T}^k$ to $\mathrm{Prim}\,\big(\mathrm{Ind}_{H(T)^\perp}^{\mathbf{T}^k}(C^*(G_T/\mathrm{Iso}(G_T)^\circ), \tilde{\alpha})\big)$. An application of Proposition 6.16 of [21]—or, rather, of the obvious primitive-ideal version of that result—shows that M induces a homeomorphism of the quotient of $(\mathrm{Prim}(C^*(G_T/\mathrm{Iso}(G_T)^\circ)) \times \mathbf{T}^k)$ by the diagonal action of $H(T)^\perp$ onto the primitive ideal space of the induced algebra. Since the action of $H(T)^\perp$ on $\mathbf{T}^k$ is by translation and has quotient $H(T)^\wedge$, it now suffices to show that the action of $H(T)^\perp$ on $\mathrm{Prim}\big(C^*(G_T/\mathrm{Iso}(G_T)^\circ)\big)$ is trivial. Since $\tilde{\alpha}_z$ fixes $C_0(G^{(0)}) \subset C^*(G_T/\mathrm{Iso}(G_T)^\circ)$ pointwise, for any ideal I of $C^*(G_T/\mathrm{Iso}(G_T)^\circ)$, we have $\tilde{\alpha}_z(I) \cap C_0(G^{(0)}) = I \cap C_0(G^{(0)})$, and then [24, Corollary 4.9] implies that $\tilde{\alpha}_z(I) = I$. So $H(T)^\perp$ acts trivially on $\mathrm{Prim}\big(C^*(G_T/\mathrm{Iso}(G_T)^\circ)\big)$. $\square$

5. The Primitive Ideals of the C^*-algebra of a Deaconu–Renault groupoid

In this section, our aim is to catalogue the primitive ideals of $C^*(G_T)$. We need to refine our notation from Section 4 to accommodate actions which are not necessarily irreducible.

Notation. Let T be an action of $\mathbf{N}^k$ on a locally compact space X by local homeomorphisms. Recall that for $x \in X$,

$$[x] = \{y \in X : T^m x = T^n y \text{ for some } m, n \in \mathbf{N}^k\}.$$

For $x \in X$ and $U \subset \overline{[x]}$ relatively open, let

$$\Sigma(x)_U := \{(m,n) \in \mathbf{N}^k \times \mathbf{N}^k : T^m y = T^n y \text{ for all } y \in U\},$$

and define

$$\Sigma(x) := \textstyle\bigcup_U \Sigma(x)_U.$$

Lemma 3.9 implies that

$$Y(x) := \bigcup\{Y \subset \overline{[x]} : Y \text{ is relatively open and } \Sigma(x)_Y = \Sigma(x)\}$$

is nonempty and is the maximal relatively open subset of $\overline{[x]}$ such that $\Sigma(x)_{Y(x)} = \Sigma(x)$. Proposition 3.10 implies that

$$H(x) := H(T|_{\overline{[x]}}) = \{m - n : (m,n) \in \Sigma(x)\}$$

is a subgroup of $\mathbf{Z}^k$. To lighten notation, set $\mathcal{I}(x) := \mathrm{Iso}(G_T|_{Y(x)})^\circ$. Proposition 3.10 says that

$$\mathcal{I}(x) = \{(y, g, y) : y \in Y(x) \text{ and } g \in H(x)\},$$

and is a closed subset of $G_T|_{Y(x)}$.

Lemma 5.1. *Let T be an action of $\mathbf{N}^k$ on a locally compact Hausdorff space X by local homeomorphisms. For $x, y \in X$, we have $Y(x) = Y(y)$ if and only if $\overline{[x]} = \overline{[y]}$.*

Proof. The "if" direction is trivial. Suppose that $Y(x) = Y(y)$. By symmetry, it suffices to show that $y \in \overline{[x]}$. Since $Y(x) = Y(y)$ is open in $\overline{[y]}$, we have $Y(x) \cap [y] \neq \emptyset$. Since $Y(x) \subset [x]$, and $[x]$ is G_T-invariant, we deduce that $y \in \overline{[x]}$. $\square$

The key to the proof of our main theorem is the following result, which works at the level of irreducible representations.

Theorem 5.2. *Let T be an action of $\mathbf{N}^k$ on a locally compact Hausdorff space X by local homeomorphisms. Take $x \in X$ and $z \in \mathbf{T}^k$. Suppose that ρ is a faithful irreducible representation of $C^*(G_T|_{Y(x)}/\mathcal{I}(x))$. Let $\iota : C^*(G_T|_{Y(x)}) \to C^*(G_T)$ be the inclusion of Corollary 3.12. Let*

$$\Phi : C^*(G_T|_{Y(x)}) \to \operatorname{Ind}_{H(x)^\perp}^{\mathbf{T}^k}(C^*(G_T|_{Y(x)}/\mathcal{I}(x)), \tilde{\alpha})$$

be the isomorphism of Proposition 4.4, and let

$$\varepsilon_z : \operatorname{Ind}_{H(x)^\perp}^{\mathbf{T}^k}\big(C^*(G_T|_{Y(x)}/\mathcal{I}(x)), \tilde{\alpha}\big) \to C^*(G_T|_{Y(x)}/\mathcal{I}(x))$$

denote evaluation at z. Let $R_x : C^*(G_T) \to C^*(G_T|_{\overline{[x]}})$ be the homomorphism induced by restriction of compactly supported functions. There is a unique irreducible representation $\pi_{x,z,\rho}$ of $C^*(G_T)$ such that

- (a) *$\pi_{x,z,\rho}$ factors through R_x, and*
- (b) *the representation $\pi^0_{x,z,\rho}$ of $C^*(G_T|_{\overline{[x]}})$ such that $\pi_{x,z,\rho} = \pi^0_{x,z,\rho} \circ R_x$ satisfies $\pi^0_{x,z,\rho} \circ \iota = \rho \circ \varepsilon_z \circ \Phi$.*

Every irreducible representation of $C^*(G_T)$ has the form $\pi_{x,z,\rho}$ for some x, z, ρ.

Proof. The representation $\rho \circ \varepsilon_z \circ \Phi$ is an irreducible representation of $C^*(G_T|_{Y(x)})$, and is injective on $C_0(Y(x))$ because both Φ and ε_z restrict to injections on $C_0(Y(x))$. Corollary 3.12(b) applied to $Y(x) \subset \overline{[x]}$ yields a unique representation $\pi^0_{x,z,\rho}$ of $C^*(G_T|_{\overline{[x]}})$ such that $\pi^0_{x,z,\rho} \circ \iota = \rho \circ \varepsilon_z \circ \Phi$. The set $\overline{[x]}$ is a closed invariant set in X. As in Proposition 2.4, restriction of functions induces a homomorphism $R_x : C^*(G_T) \to C^*(G_T|_{\overline{[x]}})$. Now $\pi_{x,z,\rho} := \pi^0_{x,z,\rho} \circ R_x$ satisfies (a) and (b).

For uniqueness, take a representation φ of $C^*(G_T)$ satisfying (a) and (b). Then φ vanishes on the ideal generated by $C_0(X \setminus \overline{[x]})$ which is precisely the kernel of R_x by Proposition 2.4. So $\varphi = \varphi_0 \circ R_x$ for some irreducible representation φ_0 of $C^*(G_T|_{\overline{[x]}})$ satisfying $\varphi_0 \circ \iota = \rho \circ \varepsilon_z \circ \Phi$. We saw in the preceding paragraph that $\pi^0_{x,z,\rho}$ is the unique such representation, so $\varphi^0 = \pi^0_{x,z,\rho}$ and hence $\varphi = \pi_{x,z,\rho}$.

To see that every irreducible representation of $C^*(G_T)$ has the form $\pi_{x,z,\rho}$, fix an irreducible representation φ of $C^*(G_T)$. Since it is irreducible, Proposition 2.4 implies that $\varphi = \varphi^0 \circ R_x$ for some $x \in X$ and some irreducible representation φ^0 of $C^*(G_T|_{\overline{[x]}})$ that is faithful on $C_0(\overline{[x]})$. Since Φ is an isomorphism, Corollary 3.12(b) implies that φ^0 is uniquely determined by $\varphi^0 \circ \iota \circ \Phi^{-1}$, which is an irreducible representation of $\operatorname{Ind}_{H(x)^\perp}^{\mathbf{T}^k}(C^*(G_T|_{Y(x)}/\mathcal{I}(x)), \tilde{\alpha})$ that is faithful on $C_0(Y(x))$. By [21, Proposition 6.16], there exists z such that $\ker(\varepsilon_z) \subset \ker \varphi^0 \circ \iota \circ \Phi^{-1}$, and then $\varphi^0 \circ \iota \circ \Phi^{-1}$ descends to an irreducible representation ρ of $C^*(G_T|_{Y(x)}/\mathcal{I}(x))$. That is $\rho \circ \varepsilon_z = \varphi^0 \circ \iota \circ \Phi^{-1}$. Post-composing with Φ on both sides of this equation shows that $\varphi^0 \circ \iota = \rho \circ \varepsilon_z \circ \Phi$. So we now need only prove that ρ is faithful.

Since φ^0 is faithful on $C_0(\overline{[x]})$, the composition $\varphi^0\circ\iota\circ\Phi^{-1}$ is faithful on $C_0(Y(x))$, and hence ρ is faithful on $C_0(Y(x)) = C_0\big((G_T|_{Y(x)}/\mathcal{I}(x))^{(0)}\big)$. Proposition 2.5(e) implies that $G_T|_{Y(x)}/\mathcal{I}(x)$ is topologically principal, and Proposition 2.5(f) combined with Lemma 3.5 implies that $G_T|_{Y(x)}/\mathcal{I}(x)$ is amenable. So [6, Theorem 4.4] implies that ρ is faithful as claimed. □

Proof of Theorem 3.2. Fix $x \in G_T^{(0)}$ and $z \in \mathbf{T}^k$. Let $\alpha_z \in \operatorname{Aut}(C^*(G_T))$ be the automorphism of Lemma 4.1, and let $\omega_{[x]}$ be the irreducible representation of Lemma 4.5. Then $\pi_{x,z} := \omega_{[x]} \circ \alpha_z$ is an irreducible representation satisfying (3.3). Furthermore $\pi_{x,z}|_{C_0(G^{(0)})}$ has support $\overline{[x]}$.

It is clear that the relation $\sim$ is an equivalence relation. To see that $\ker \pi_{x,z} = \ker \pi_{y,w}$ if and only if $\overline{[x]} = \overline{[y]}$ and $\overline{z}w \in H(x)^\perp$, first suppose that $\overline{[x]} \neq \overline{[w]}$. Then $\ker \pi_{x,z} \cap C_0(X) \neq \ker \pi_{y,w} \cap C_0(X)$.

Second, suppose that $\overline{[x]} = \overline{[y]}$ but $\overline{z}w \notin H(x)$. Then $\pi_{x,z}$ and $\pi_{y,w}$ descend to representations $\pi^0_{x,z}$ and $\pi^0_{y,w}$ of $C^*(G_T|_{\overline{[x]}})$. Corollary 3.12(b) implies that their kernels are equal if and only if the kernels of $\pi^0_{x,z}\circ\iota$ and $\pi^0_{y,w}\circ\iota$ are equal. Lemma 5.1 shows that $Y(x) = Y(y)$, and for $f \in C_c(G_T|_{Y(x)}) = C_c(G_T|_{Y(y)})$, we have

$$\pi^0_{x,z} \circ \iota(f)\delta_y = \sum_{(u,g,y)\in G_T|_{Y(x)}} z^g f(u,g,y)\delta_u.$$

Lemma 4.2 shows that for $n \in H(x)$,

$$\sum_{(u,g,y)\in G_T|_{Y(x)}} z^g f(u,g,y)\delta_u = \sum_{(u,g+n,y)\in G_T|_{Y(x)}} z^g f(u,g,y)\delta_u.$$

As in Lemma 4.3, for $\varphi \in C_c(H(x))$ and $f \in C_c(G_T|_{Y(x)})$, we have $\pi_{x,z} \circ \iota(\varphi \cdot f) = \hat{\varphi}(z)(\pi_{x,z}\circ\iota)(f)$ and $\pi_{y,w}\circ\iota(\varphi\cdot f) = \hat{\varphi}(w)(\pi_{y,w}\circ\iota)(f)$. Choose φ such that $\hat{\varphi}(w) = 0$ and $\hat{\varphi}(z) \neq 0$, and choose $f \in C_c(Y(x))$ such that $f(x) = 1$. Then $\pi_{y,w}\circ\iota(\varphi\cdot f) = 0$ whereas $\pi_{x,z}(\varphi \cdot f)\delta_x = \hat{\varphi}(z)\delta_x \neq 0$. So the kernels are not equal.

Third, suppose that $\overline{[x]} = \overline{[y]}$ and $\overline{z}w \in H(x)^\perp$. Again Lemma 5.1 shows that $Y(x) = Y(y)$. Let $\pi_{[x]}$ and $\pi_{[y]}$ be the faithful irreducible representations of $C^*(G_T|_{Y(x)}/\mathcal{I}(x)) = C^*(G_T|_{Y(y)}/\mathcal{I}(y))$ described by Lemma 4.5. It is routine to check that $\pi^0_{x,z} \circ \iota = \omega_{[x]} \circ \varepsilon_z \circ \Phi$ and $\pi^0_{y,w} \circ \iota = \omega_{[y]} \circ \varepsilon_w \circ \Phi$. We have

$$\omega_{[x]} \circ \tilde{\alpha}_{\overline{z}w} \circ \varepsilon_z \circ \Phi = \omega_{[x]} \circ \varepsilon_w \circ \Phi.$$

Since $\tilde{\alpha}_{\overline{z}w}$ is an automorphism, we deduce that $\ker(\omega_{[x]}\circ\varepsilon_z\circ\Phi) = \ker(\omega_{[y]}\circ\varepsilon_w\circ\Phi)$. Thus $\ker(\pi^0_{x,z}\circ\iota) = \ker(\pi^0_{y,w}\circ\iota)$. Now Corollary 3.12(b) implies that $\pi^0_{x,z}$ and $\pi^0_{y,w}$ have the same kernel. Since $\overline{[x]} = \overline{[y]}$, we have $R_x = R_y$, and so

$$\ker \pi_{x,z} = R_x^{-1}(\ker \pi^0_{x,z}) = R_y^{-1}(\ker \pi^0_{y,w}) = \ker \pi_{y,w}.$$

It remains to show that $(x,z) \mapsto \ker \pi_{x,z}$ is surjective. Fix a primitive ideal $I \lhd C^*(G_T)$. Theorem 5.2 gives $I = \ker \pi_{x,z,\rho}$ for some x,z, ρ. Choose $y \in [x] \cap Y(x)$, and let $\tilde{\omega}_{[y]}$ be the faithful irreducible representation of $C^*(G_T|_{Y(x)}/\mathcal{I}(x))$ of Lemma 4.5. Since ρ is faithful on $C^*(G_T|_{Y(x)}/\mathcal{I}(x)))$, we have $\ker(\omega_{[y]}\circ\varepsilon_z\circ\Phi) = \ker(\rho\circ\varepsilon_z\circ\Phi)$. So Theorem 5.2 gives $\ker \pi_{x,z,\omega_{[y]}} = \ker(\pi_{x,z,\rho})$. As in the second step above, one checks on basis elements that $\pi_{x,z} = \pi_{x,z,\omega_{[y]}}$, completing the proof. □

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