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The dynamics of oil prices and valuation of oil derivatives

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Abstract

Over the last three decades financial derivatives, such as futures and options, have become increasingly important to financial institutions for the purposes of trading and risk management. In particular, commodity markets have undergone significant growth in terms of volumes and diversity of traded contracts. The most significant development since 2000 has been in the trading of commodity options. The London International Petroleum Exchange[1] (IPE) and the New York Mercantile Exchange (NYMEX), as well as other exchanges, regularly introduce futures and options contracts on different commodity products. Further, the growth of over-the-counter trading in physical commodity options, such as oil, is increasing rapidly.

The values of most financial derivatives are based on the movement of the underlying assets on which the derivatives are written. Consequently, it is not a trivial task to quantify their price, although mathematics provides a powerful tool in order to do so.

Over the last two decades, the behaviour of oil prices (which is one of the most important of the world's commodities) has become progressively more complex with many factors, such as interest rates, net demand, Middle Eastern conflicts and economic crises influencing its behaviour. The ability to capture the behaviour of oil prices affects the subsequent accurate pricing of derivatives on oil, thereby enabling the creation of successful portfolio investments and hedging strategies. Hence, studying the uncertainty linked to future movements of oil prices and identifying the affective factors on those movements, play an important role in financial decision making. While existing models for underlying assets may be appropriate for certain classes of assets at certain times, they may not be applicable to commodities such as oil.

[1]In 2005 its name was changed to the International Commodity Exchange (ICE).

As such, new models and pricing formulae are needed which consider the special characteristics of commodity prices.

The aim of this thesis, as its title suggests, is to study the dynamics of oil prices and value a variety of financial derivatives on oil. In particular, the performance of various one-factor stochastic models (which include well-known and proposed models) in their ability to explain the behaviour of crude oil prices is compared. Moreover, based on various proposed models for oil, analytic and analytic approximation formulae for the pricing of a variety of financial derivatives are derived and, where possible, empirically examined.

Studying the dynamics of oil prices is the main objective of Chapter 2. In this chapter, we summarise some of the most popular one-factor models found in the literature and identify some of their statistical properties. As well, based on empirical tests, we propose new models for oil price behaviour which have a three-quarters exponent in the diffusion term. The results of a Generalized Method of Moments analysis, which we use to compare one-factor stochastic models in their ability to capture the behaviour of Brent crude oil prices, show that our new proposed models are not only acceptable in describing the behaviour of Brent crude oil prices but perform better than current popular models.

In Chapter 3 we price futures contracts under one-factor models. In this chapter, two analytic formulae for futures prices under the new proposed models are derived and then used to calibrate market prices. Results from the calibration show that one of our proposed models outperforms other popular models in fitting and forecasting futures prices.

In Chapter 4, we continue to price futures contracts on oil but now incorporate two sources of randomness, as all futures returns are not perfectly correlated. In this chapter, net demand, interest rate and convenience yield are considered as additional factors to oil prices. Further, a regime switching model (which provides an alternative way to determine the impact of the cycles of booms and busts in the commodity market) is also considered. Analytic formulae for futures contract prices are derived for each proposed model.

The focus of Chapter 5 is on the accurate pricing of options on oil. In particular, we price European option contracts based on our proposed one-factor models, which were considered improvements in describing the behaviour of Brent crude oil prices. In this chapter, analytic formulae for pricing European option contracts and simple analytic approximation formulae for pricing European call option contracts are derived and then used to calibrate market prices. Results from the calibration show that one of our analytic approximation formulae outperforms one of the most well-known models in capturing market data.

In Chapter 6 we focus on pricing two types of popular exotic options on oil, that of European crack spread and quotient option contracts. In this chapter, new univariate and explicit (constant elasticity of variance) models are assumed. Under each model, new analytic approximation formulae for pricing European crack spread and quotient call option contracts are derived. These formulae are then used to calibrate market prices for European crack spread call option contracts. Results from the calibration show that our univariate (explicit) proposed formula outperforms other popular univariate (explicit) formulae in capturing market prices. In Chapter 7 we present our conclusion.

Certification

I, Mohammed AbdulAziz Aba Oud, declare that this thesis, submitted in partial fulfilment of the requirements for the award of Doctor of Philosophy, in the School of Mathematics and Applied Statistics, University of Wollongong, is wholly my own work unless otherwise referenced or acknowledged. The document has not been submitted for qualifications at any other academic institution.

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List of Publications

[**1**] M. Abaoud and J. Goard. Stochastic models for oil prices and the pricing of futures on oil. *Applied Mathematical Finance (submitted, 2012).*

[**2**] M. Abaoud and J. Goard. Analytic approximation formulae for European crack spread options. *Quantitative Finance (submitted, 2013).*

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Chapter 1

Introduction

Petroleum is a fossil fuel formed after millions of years from the remains of sea plants and animals. In its natural state when it is collected it is called 'crude oil' and provides one of the world's most crucial sources of energy. In 2012, oil provided about 33.10% of the world's energy needs[1] and in the future, oil is expected to continue to provide a leading component of the world's energy. Prices for crude oil often experience high levels of volatility. This has led to the need for producers and consumers to hedge and trade risk associated with variations in oil prices. In response to this need, commodity markets have grown significantly, especially since 2000 in terms of volumes and variety of traded financial contracts called '*derivatives*' which include futures, options and exotic options on oil. The price of a financial derivative depends centrally upon finding suitable models to describe the movement of the underlying asset on which the derivative is written. Broadly, this thesis focusses on modelling and valuing oil derivatives. However, before we elaborate on this, in this chapter we highlight some essential mathematics and financial information related to the modelling and pricing of financial derivatives on oil. The information provided is by no means exhaustive, but aims to provide the minimum background necessary for the reader of this thesis. Further references will be given throughout the thesis.

[1] According to the British Petroleum Statistical Review of World Energy (June 2013).

1.1 Stochastic Processes

1.1.1 Wiener process

A Wiener process (or Brownian Motion) is a continuous-time stochastic process. More precisely, a stochastic process $Z(= Z_t)$ at times $t \geq 0$ is a Wiener process under a given probability measure $\mathbb{P}$ if

- $Z_0 = 0$, by convention.
- Z_t has stationary independent increments, i.e. if $0 \leq s < t < u < v$, then $Z_t - Z_s$ and $Z_v - Z_u$ are independent random variables.
- $Z_t - Z_s \sim N(0, t-s)$, for $0 \leq s < t$,

where $N(\mu, \sigma)$ denotes the normal distribution under $\mathbb{P}$, with a mean of μ and a variance of σ^2. A basic Wiener process for a process Z has an expected change per unit time, or *drift rate*, of zero and a variance per unit time, i.e. *variance rate*, of 1.00.

1.1.2 Stochastic Differential Equations

We generally assume that a stochastic process P $(= P_t)$ at times $t \geq 0$ is driven by a stochastic differential equation (SDE) or Itô process which describes its evolution. A SDE typically consists of two components. The first component relates to the expected change in P in a short time interval dt and the second describes the uncertainty of the change in P. In this thesis we assume that the SDE has the following general form

$$dP = \mu(P, t)dt + \sigma(P, t)dZ \tag{1.1}$$

where, here and in the rest of this thesis, dZ represents an increment in a Wiener process Z under a probability measure $\mathbb{P}$. The terms $\mu(P, t)$ and $\sigma(P, t)$ in equation

(1.1) are called the drift and volatility of the process respectively. Equation (1.1) can also be written as a stochastic integral equation as

$$P_t - P_0 = \int_0^t \mu(P,u)du + \int_0^t \sigma(P,u)dZ_u \tag{1.2}$$

where $\int_0^t \sigma(P,u)dZ_u$ is an Itô integral (see Section 1.1.3).

1.1.3 Stochastic Integral

The second integral in equation (1.2), i.e. $\int_0^t \sigma(P,u)dZ_u$, is an integral with respect to increments in the Wiener process Z. The stochastic integral can be defined as the mean square limit namely

$$\int_0^t \sigma(P,u)dZ_u = \lim_{n\to\infty} \sum_{j=1}^{n} \sigma(P,k_j)(Z_{t_{j+1}} - Z_{t_j}),$$
$$\text{where} \quad 0 = t_1 < t_2 < ... < t_n = t \quad \text{and} \quad k_j \in [t_j, t_{j+1}]$$

subject to the boundedness of

$$\mathbb{E}\left[\int_0^t \sigma^2(P,u)du\right] < \infty \ . \tag{1.3}$$

In this thesis $\mathbb{E}$ denotes expectation. Different choices of k_j lead to different stochastic integrals:

- Choosing $k_j = t_j$ leads to an *Itô integral*, and we write the integral as $\int_0^t \sigma(P,u)dZ_u$.
- Choosing $k_j = \frac{t_j + t_{j+1}}{2}$ leads to a *Stratonovich integral*, and we write the integral as $\int_0^t \sigma(P,u) \circ dZ_u$.

A primary result from stochastic calculus (see for example Mikosch (1998)) is that under the assumption (1.3) then an Itô integral has an equivalent Stratonovich integral representation which is given by the transformation formula:

$$\int_0^t \sigma(P,u)dZ_u = \int_0^t \sigma(P,u) \circ dZ_u - \frac{1}{2}\int_0^t \sigma(P,u)\frac{\partial\sigma(P,u)}{\partial P}du \ . \tag{1.4}$$

Here and in the rest of this thesis the Itô SDE has the form (1.1). Now by writing (1.1) as a stochastic integral equation, then we can use (1.4) to replace the Itô integral with the Stratonovich integral to get the Stratonovich SDE equivalent to the Itô SDE, i.e.

$$dP = \bar{\mu}(P,t)dt + \sigma(P,t) \circ dZ, \quad \text{where } \bar{\mu}(P,t) = \mu(P,t) - \frac{1}{2}\sigma(P,t)\frac{\partial\sigma(P,t)}{\partial P} \ .$$

An advantage of a Stratonovich representation is that the calculus associated with the Stratonovich integral coincides with ordinary calculus, i.e. the usual chain rule for differentiation $df(x) = f'(x)dx$ holds for Stratonovich differentiation. Hence, the Stratonovich representation can be useful for the solution of Itô differential equations (DEs).

1.1.4 Itô's Lemma

Itô's lemma can be defined as the stochastic version of the chain rule for a deterministic variable. It relates small changes in a function of a random variable to small changes in the random variable itself.

Itô's Lemma: Let $f(P,t)$ be a function of t and a stochastic process P where

$$dP = \mu(P,t)dt + \sigma(P,t)dZ \ . \tag{1.5}$$

The variation of the dependent process $df(P,t)$ can be described by

$$df(P,t) = \left[\mu(P,t)\frac{\partial f(P,t)}{\partial P} + \frac{\partial f(P,t)}{\partial t} + \frac{\sigma(P,t)^2}{2}\frac{\partial^2 f(P,t)}{\partial P^2}\right]dt + \sigma(P,t)\frac{\partial f(P,t)}{\partial P}dZ \ . \tag{1.6}$$

The proof of Itô's Lemma can be found in Hull (2012).

1.2 Financial Contracts

1.2.1 Forward and Futures Contracts

A forward contract (or simply 'forward') is an agreement between two parties whereby one party agrees to buy an underlying asset from the other party, on a certain specified future date (called the *maturity date*) for a certain specified price (called the *delivery price*). The buyer is said to hold the long position, while the seller holds the short position. The value of the contract V, is a function of the underlying spot price, P, and the current time t; and also depends on the delivery price, K, and the maturity date, T. The price of a forward is determined by way of insuring that the value of the contract to both parties at the beginning of the contract is zero i.e. $V(P, 0; K, T) = 0$.

It should be clarified that the delivery price, or K, is held constant during the life of the contract. The values of the forward contract to the long and short positions fluctuate according to new information in the market. For example, the change on net demand of the commodity or the spot price of the commodity will change the value of the forward contract to both parties. The profit or loss of the forward contract will be realised only at the maturity date, whereupon the long position party pays K dollars to the short position who simultaneously delivers the underlying asset. The real profit (or loss) for the long position party can be calculated by $V(P_T, T; K, T) = P_T - K$ and $V(P_T, T; K, T) = K - P_T$ for the short position party, where P_T is the price of the underlying asset at the maturity date.

Forwards are usually traded over-the-counter (OTC) between a supplier and consumer or between two financial institutions in order to take a position on the relative performance of the underlying asset.

A futures contract (or 'futures') is similar to a forward contract in its nature. This means that both futures and forward contracts are used to buy (or sell) an underlying on a certain specified future date for a certain specified price. However, technical differences between the contracts exist with the main differences being that with futures contracts:

- They are traded on organized exchanges and their prices can be observed at any moment in time.
- There is no credit risk, as the exchange market guarantees the contract.
- Both positions can close out their positions prior to the maturity date. This can be done by taking an opposite position to the original position.
- They are marked-to-market[2], so the futures price is updated daily according to new information in the market.
- Their values differ from forwards only when the interest rate is stochastic[3].

Most investors use forward and futures contracts to hedge the risk that they face from fluctuations in futures prices. In particular, the uncertainty about the futures price of a commodity can be removed by trading forward or futures contracts. It also allows producers and consumers to determine their expected profits or expenses.

As an example, suppose that it is 01/01/2014, and the ABC company knows that it will have to buy one million barrels of oil in June 2014 in order to produce its goods. The ABC company can wait until June 2014 and then buy one million barrels of oil from the spot market at the selling price on that day. Another strategy is that the ABC company can hedge the price volatility risk by buying a 6-months forward contract from the market now at $100 per barrel. The benefit of the forward contract is that the company knows in advance the price they will pay for the million barrels

[2]The gains (or losses) are received (paid) at the end of each day during the life of the futures contract to reset the value of the futures contract to both parties to zero.

[3]See Cox *et al* (1985).

of oil. If the spot price on June 2014 is \$98 per barrel and the ABC company decides not to hedge, then one million barrels of oil will cost \$98 million (which is less than the \$100 million). However, if the spot price is \$104 per barrel, the one million barrels of oil will cost \$104 million (which is significantly more than the \$100 million). In summary, if the spot price on June 2014 is more than \$100, the ABC company will wish they had hedged. On the other hand, if the spot price is less than \$100, the ABC company will wish they had not hedged. From this example we must note that the outcome with hedging may be better (or worse) than the outcome without hedging, depending on the difference between the delivery price and the spot price at maturity. Most importantly, the use of a forward contract avoids uncertainty and possible disaster to the company should there be catastrophe such as a war, which drives oil prices to a high level.

1.2.2 Options

A ***European*** option is a contract between two parties (a holder and a writer) which gives the holder the right to buy or sell an underlying asset (or another financial derivative) at a specified future date (called the *expiry date*) for a price agreed upon at the opening of the contract (called the *strike or exercise price*). In particular, a call option gives the holder the right to buy an underlying asset at the expiry date (T) for the set exercise price (K), while a put option gives the holder the right to sell an underlying asset at the expiry date for the set exercise price. If the price of an underlying asset at expiry T of the option contract is P_T, then the value of a call option at the expiry date (called '*the payoff*') is given by $\max(P_T - K, 0)$ and by $\max(K - P_T, 0)$ for a put option. It is necessary to clarify that European options can be exercised only at the expiry date. An ***American option*** is another type of option contract which is similar to a European option but has the additional feature that is can be exercised at any time prior to the expiry date.

An option whose structure differs from standard calls and puts is referred to as ***an***

exotic option. A variety of exotic option contracts are available in the commodities markets and allow the investor to take a position on the relative performance of the underlying assets. Barrier options, Asian options and Digital options are examples of exotic options (see for example Hull (2012)). The ***crack spread option***, which provides an important risk management tool to industrial consumers of oil, is another example of an exotic option. It has a payoff which is based on the price difference between two underlying assets. Given that the prices of the underlying assets at expiry T of the option contract are P_{1T} and P_{2T} respectively, then the value of the crack spread call option with exercise price K at the expiry option is given by $\max(P_{1T} - P_{2T} - K, 0)$ and by $\max(K - (P_{1T} - P_{2T}), 0)$ for the crack spread put option. Another example of an exotic option which will be considered in this thesis is the ***quotient option***. It has a payoff which is based on the ratio of two underlying assets, namely its payoff is given by $\max\left(\frac{P_{2T}}{P_{1T}} - K, 0\right)$ for a quotient call option and by $\max\left(K - \frac{P_{2T}}{P_{1T}}, 0\right)$ for a quotient put option.

1.3 Fundamental Pricing Theorems

In this section we introduce the essential principles and theorems that are related to pricing financial derivatives. Without loss of generality, we make the following assumptions:

- The risk-free rate, r, is assumed constant. Hence, the discount factor for the present value at time t for one currency unit at time $T \geq t$ is given by $e^{-r(T-t)}$.
- The spot price of an underlying asset P is driven by the SDE

$$dP = \mu(P, t)dt + \sigma(P, t)dZ \ . \tag{1.7}$$

- The uncertainty in any financial market in the world can be defined on a probability space $(\Omega, \mathcal{F}_t, \mathbb{P})$ where Ω includes all possible events, $\mathcal{F}_t$ represents the filtration, that consists of all information about the underlying asset up to time t and $\mathbb{P}$ is a risky probability measure.

A special type of probability measure is the ***risk-neutral measure*** $\mathbb{Q}$ under which the underlying price process $P(=P_t)$ adapted to $\mathcal{F}_t$ satisfies the following martingale properties:

$$\mathbb{E}_t^{\mathbb{Q}}(|P_t|) < \infty \ , \quad \forall t \ . \tag{1.8a}$$

$$\mathbb{E}_t^{\mathbb{Q}}(e^{-r(T-t)}P_T) = P_t \quad 0 \leq t \leq T \ . \tag{1.8b}$$

In this thesis the t subscript in $\mathbb{E}_t^{\mathbb{Q}}$ (with any probability measure) denotes an expectation conditional on the filtration $\mathcal{F}_t$. Hence, a risk-neutral measure is a probability measure under which the current value of the financial asset at time t is equal to the expected future payoff of the asset, discounted at the risk-free rate, given the information structure available at time t. In order to derive fair values of derivative contracts, the idea of 'no-arbitrage' is used, i.e. contract values are found that preclude arbitrage opportunities. Mathematically, by a 'no-arbitrage value' of a derivative claim, refers to its fair value under the risk-neutral measure $\mathbb{Q}$. Buying the derivative claim for its fair value implies that the expected returns on the two investment strategies of buying the derivative claim and replicating it by trading in the underlying asset and money market account, are equal to the risk-free rate. If the derivative claim is not sold for its fair value, then arbitrage opportunities exist and smart investors can earn riskless profits. In that case it can be said that the efficient market hypothesis is violated. This is why no-arbitrage pricing methods are used to price financial derivatives.

In the following, we state the first fundamental theorem of asset pricing, which establishes a relationship between the no-arbitrage principle and the risk-neutral measure.

1.3.1 The First Fundamental Theorem of Asset Pricing

Theorem 1.3.1 *The existence of a risk-neutral measure $\mathbb{Q}$, that satisfies (1.8a,b), implies the absence of arbitrage opportunity in the market.*

1.3.2 Futures Prices on Investment Assets

By arbitrage arguments we can show that the current price of a futures contract $F(P,t)$, on an underlying *investment* asset with spot price P, that expires at time T is given by $F(P,t) = Pe^{r(T-t)}$.

At time t, suppose that $F(P,t) > Pe^{r(T-t)}$, an investor could:

- Borrow P for $T-t$ years at the risk-free rate r.
- Buy one quantity of the underlying asset.
- Take a short position in the futures contract.

At expiry (i.e. after $T-t$ years), $Pe^{r(T-t)}$ is repaid to the bank and one quantity of the underlying asset is sold for $F(P,t)$. With this strategy, the investor will have a profit of $F(P,t) - Pe^{r(T-t)}$.

On the other hand suppose that at time t, $F(P,t) < Pe^{r(T-t)}$, an investor could:

- Short sell one quantity of the underlying asset for P.
- Take a long position in the futures contract.
- Invest P for $T-t$ years in a risk-free rate environment.

At expiry, the value of the initial investment P will have grown to $Pe^{r(T-t)}$. Also one quantity of the underlying asset can be bought for $F(P,t)$ and then used to close the short position. With this strategy, the investor will have a profit of $Pe^{r(T-t)} - F(P,t)$. Hence, to eliminate any arbitrage opportunity we should have $F(P,t) = Pe^{r(T-t)}$.

1.3.3 Futures Prices on Commodity Assets

In the previous section, futures prices for investment assets were derived by arbitrage arguments. In the case of commodities, additional factors need to be considered in their pricing. These factors include storage costs and convenience yields.

Storage Costs and Convenience Yields

Unlike investment assets, producers and consumers of commodity assets normally hold inventories as they can provide a number of services. For example, producers and refiners (which are industrial consumers) hold inventories to regulate production volume and avoid delays in delivery. Costs associated with storing a physical commodity are known simply as ***storage costs***, and include both storage and insurance costs. However, holders of a physical commodity can generate incomes, known as ***convenience yields***, that decrease as inventory increases. The convenience yield can be simply defined as the benefits that are associated with holding physical commodities. An example of these benefits is the ability to profit when the market experiences shortages. Adjusting the storage costs by adding financing costs and subtracting the convenience yield is known as ***the cost of carrying*** a physical commodity. This concept can be used to explain the term structure of the commodity futures market. When the cost of carry is positive (which means that storage costs plus financing cost are higher than the convenience yield), then holding the physical commodity would be costly. As a result, the futures price tends to be above the spot price (the commodity market is in *contango*[4]). Conversely, when the cost of carry is negative then the futures price tends to be below the spot price (the commodity market is in *backwardation*[5]).

In order to price futures contracts on commodity assets we use the first fundamental theorem of asset pricing, which states that the existence of a risk-neutral measure

[4]The commodity market is in contango when the futures price is higher than the spot price.
[5]The commodity market is in backwardation when the futures price is lower than the spot price.

$\mathbb{Q}$ implies that the *discounted* no-arbitrage price processes of all financial claims are martingales under the risk-neutral measure $\mathbb{Q}$. As it costs nothing to enter into a futures contract, oil prices (and commodity prices in general) are martingales with respect to a risk-neutral measure $\mathbb{Q}$, i.e. $F(P,t) = \mathbb{E}_t^{\mathbb{Q}}(P_T)$. This also follows from arbitrage arguments. If $F(P,t) < \mathbb{E}_t^{\mathbb{Q}}(P_T)$ then an investor could on average profit by routinely holding $F(P,t)$; while if $F(P,t) > \mathbb{E}_t^{\mathbb{Q}}(P_T)$ the investor could profit by routinely selling $F(P,t)$.

1.3.4 Finding The Risk-Neutral Measure

Complete and Incomplete Markets

In order to understand the second fundamental theorem of asset pricing, it is necessary to define complete and incomplete markets. A market is complete when every financial derivative on an underlying asset with value P, can be uniquely replicated by holding positions in the underlying asset and in a money market account. However, in an incomplete market, replicating portfolios are not possible and so perfect risk transfer is not possible. The weather derivatives market is an example of an incomplete market; this is because the underlying asset (i.e. temperature) is not tradable. In such a situation, the market price of risk allows us to assess the market prices of the derivative dependent on the underlying process, and then we can express our source of uncertainty in terms of monetary value. In general, the market price of risk $\lambda(P,t)$ for holding a risky asset with value P that follows SDE (1.7) can be defined by[6]

$$\mu(P,t) - rP = \sigma(P,t)\lambda(P,t) \ . \tag{1.9}$$

Equation (1.9) interprets the market price of risk as the excess return over the risk-free rate per unit of volatility that we expect to receive for holding a risky asset. It

[6] See Hull (2012) for the derivation.

is necessary to clarify that in a complete market, the market price of risk is unique and completely specified. However, in an incomplete market the market price of risk is unspecified and expresses different attitudes towards risk.

In the following, we state the second fundamental theorem of asset pricing, which relates the uniqueness of the risk-neutral measure with the completeness of the market.

The Second Fundamental Theorem of Asset Pricing

Theorem 1.3.2 *A financial market is complete if and only if there exists a unique risk-neutral measure $\mathbb{Q}$ that is equivalent to the risky measure $\mathbb{P}$.*

The search for the risk-neutral measure $\mathbb{Q}$ involves moving to a world where all investors are risk-neutral. Then the expected return on every underlying asset is the risk-free rate and the expected return earned by holding any financial derivative based on the underlying asset is also equal to the risk-free rate. To find such a measure, we define $d\tilde{Z} = dZ + \lambda(P,t)dt$, where $\lambda(P,t)$ can be obtained from (1.9). Substituting $d\tilde{Z}$ into our SDE (1.7) yields $dP = rPdt + \sigma(P,t)d\tilde{Z}$. Now it is clear that the expected return on P will be the risk-free rate if $\tilde{Z}$ is a standard Wiener process. Hence, the search for the risk-neutral measure $\mathbb{Q}$ can be replaced by the search for a probability measure $\mathbb{Q}^*$ under which $\tilde{Z}$ is a standard Wiener process. The following Girsanov theorem states that for an appropriate choice of $\lambda(P,t)$, there is a probability measure $\mathbb{Q}^*$ under which $\tilde{Z}$ is a standard Wiener process.

Girsanov Theorem

Theorem 1.3.3 *If Z is a Wiener process with probability measure $\mathbb{P}$ and $\lambda^*(P,t)$ is a function that satisfies the Novikov condition of boundedness i.e.*

$$\mathbb{E}^{\mathbb{P}}\left[\exp(\frac{1}{2}\int_0^t |\lambda^*(P,u)|^2 du)\right] < \infty \ ,$$

then there is an equivalent probability measure $\mathbb{Q}^*$ *such that* $d\tilde{Z} = dZ + \lambda^*(P,t)dt$ *where* $\tilde{Z}$ *is a Wiener process under* $\mathbb{Q}^*$.

It is important to clarify that the equivalent probability measure $\mathbb{Q}^*$ is not necessarily the risk-neutral measure $\mathbb{Q}$. In particular, choosing $\lambda^*(P,t)$ to be the market price of risk, which makes the expected return on P equal to the risk-free rate, ensures that the probability measure $\mathbb{Q}^*$ is a risk-neutral measure. In other words, $\lambda^*(P,t) = \lambda(P,t)$ implies that $\mathbb{Q}^* = \mathbb{Q}$.

1.3.5 Connections Between Partial Differential Equation and SDE

The Feynman-Kac theorem is an important tool in this thesis as it establishes a link between the expected value of the derivative payoff under the risk-neutral measure $\mathbb{Q}$ with the solution of a particular form of partial differential equation (PDE). If the stated PDE in the Feynman-Kac theorem is the governing PDE for a derivative price, then the solution of the PDE (which can found analytically or numerically) is the expected value of the derivative payoff under the risk-neutral measure $\mathbb{Q}$.

Feynman-Kac Theorem (with risk-neutral measure)

Theorem 1.3.4 *Consider a stochastic process* P $(= P_t)$ *driven by the risk-neutral SDE*

$$dP = \mu(P,t)dt + \sigma(P,t)d\tilde{Z} \ , \tag{1.10}$$

where $\mu(P,t)$ *and* $\sigma(P,t)$ *are known functions and* $\tilde{Z}$ *is a Wiener process under the risk-neutral measure* $\mathbb{Q}$. *Then the following PDE*

$$\frac{\partial V(P,t)}{\partial t} + \mu(P,t)\frac{\partial V(P,t)}{\partial P} + \frac{1}{2}\sigma^2(P,t)\frac{\partial^2 V(P,t)}{\partial P^2} - u(P,t)V(P,t) = 0 \tag{1.11}$$

subject to the final condition $V(P,T) = \varphi(P)$, where $u(P,t)$ and $\varphi(P)$ are known functions, has a solution that can be written as a conditional expectation under the risk-neutral measure $\mathbb{Q}$, namely

$$V(P,t) = \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T u(P_s,s)ds}\varphi(P_s)\right] . \tag{1.12}$$

Note that in this theorem the probability measure need not be the risk-neutral measure. However, it is of particular importance in the pricing of derivatives when the probability measure is taken to be the risk-neutral measure.

1.4 Approaches to Pricing Financial Derivatives

From the Feynman-Kac theorem, we see that no-arbitrage prices for derivative contracts, that are based on an underlying asset whose risk-neutral dynamics follow (1.10) can be found by either using risk-neutral expectations (1.12) under a risk-neutral measure $\mathbb{Q}$ or by solving the appropriate PDE (1.11). These two approaches are employed in this thesis in order to price financial derivatives.

1.4.1 Solving the Governing PDEs

Once a differential equation is formulated to describe a particular problem, it is then a matter of obtaining solutions to the equation, given certain boundary and initial or final conditions. These solutions can be exact (analytic), approximate analytic or numerical. In this section, we highlight the Black and Scholes (1973) PDE for pricing European options and the associated formulae for the values of call and put options which are based on solving the governing PDE. Then, we describe two of the most popular techniques for finding exact and approximate analytical solutions of differential equations.

The Black-Scholes Formula

The Black-Scholes formulae for pricing European vanilla option contracts can be considered as the most famous result in the area of financial mathematics. The work of Black and Scholes (1973) has provided a new way for pricing and hedging financial derivatives. In this section, we briefly highlight the derivation of the Black-Scholes PDE and state the analytic solutions for call and put options.

Firstly, the Black-Scholes model makes a number of assumptions:

- The price of the underlying asset, P, that provides a dividend yield at rate D can by described by the following risk-neutral SDE

$$dP = (r - D)Pdt + \sigma P d\tilde{Z} \ . \tag{1.13}$$

- Borrowing and lending of any amount of cash, at the risk-free rate, is possible.
- Borrowing and lending of any amount of the underlying asset is possible (this includes short selling).
- Investors can continuously trade in the underlying asset without transaction costs.
- Arbitrage opportunities are precluded.

Now construct a portfolio (with value π) consisting of a long position of a European call option contract with value C and Δ short positions in the underlying asset. Hence,

$$\pi = C - \Delta P \ .$$

In a short time step dt, the change in the portfolio's value ($d\pi$) can be found by applying Itô's Lemma (see Section 1.1.4). The central idea of the Black-Scholes argument is that with a judicious choice of $\Delta = \frac{\partial C}{\partial P}$ the risk in the portfolio is eliminated, so by arbitrage the portfolio should earn the risk-free interest rate r, i.e. $d\pi = r\pi$. This leads to the PDE governing the price of a European call option contract, $C(P,t)$, with strike price K and expiry T as

$$\frac{\partial C}{\partial t} + (r-D)P\frac{\partial C}{\partial P} + \frac{\sigma^2 P^2}{2}\frac{\partial^2 C}{\partial P^2} - rC = 0 \tag{1.14a}$$

subject to the final condition

$$C(P,T) = \max(P-K,0)\ . \tag{1.14b}$$

Equation (1.14a) is known as the Black-Scholes PDE and its solution provides a closed-form formula for the option's price at time t, namely

$$C(P,t) = Pe^{-D(T-t)}\ N(d_1) - Ke^{-r(T-t)}\ N(d_2)\ , \tag{1.15a}$$

$$\text{where}\ \ d_1 = \frac{\ln(\frac{P}{K}) + (r - D + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}\ , \tag{1.15b}$$

$$d_2 = d_1 - \sigma\sqrt{T-t}\ . \tag{1.15c}$$

where $N(\cdot)$ is the cumulative standard normal distribution function.

Similarly, the Black-Scholes PDE governs the price of a European put option contract, $U(P,t)$, with strike price K and expiry T but subject to the final condition

$$U(P,T) = \max(K-P,0)\ .$$

The solution for the European put option is given by

$$U(P,t) = Ke^{-r(T-t)}\ N(-d_2) - Pe^{-D(T-t)}\ N(-d_1)\ . \tag{1.16}$$

where d_1 and d_2 are given in (1.15b,c).

Lie Classical Symmetry Method

The Lie Classical Symmetry Method is one of the most popular techniques for finding exact solutions of differential equations and was initiated by Sophus Lie in 1881 (Lie (1881)). This method transforms every solution of a differential equation to another solution of the differential equation. In this thesis the Lie Classical Symmetry Method is used to find a reduction of a second order PDE in one dependent variable and two (or three) independent variables. Suppose we have a governing PDE

$$\Lambda\big(x, t, \frac{\partial V}{\partial x}, \frac{\partial V}{\partial t}, \frac{\partial^2 V}{\partial x^2}, \frac{\partial^2 V}{\partial x \partial t}, \frac{\partial^2 V}{\partial t^2}) = 0 \tag{1.17}$$

where the dependent variable is V and independent variables are x and t. We seek a one parameter ε Lie group of transformations, in infinitesimal form

$$x^* = x + \varepsilon X(x, t, V) + O(\varepsilon^2) \tag{1.18a}$$

$$t^* = t + \varepsilon \bar{T}(x, t, V) + O(\varepsilon^2) \tag{1.18b}$$

$$V^* = V + \varepsilon \nu(x, t, V) + O(\varepsilon^2) \tag{1.18c}$$

which leaves (1.17) invariant. The coefficients X, $\bar{T}$ and ν of the infinitesimal symmetry are often referred to as 'infinitesimals'. Each one parameter Lie group (x^*, t^*, V^*) is obtained by exponentiating its infinitesimal generator given by

$$\Psi = X(x, t, V)\frac{\partial}{\partial x} + \bar{T}(x, t, V)\frac{\partial}{\partial t} + \nu(x, t, V)\frac{\partial}{\partial V} \ . \tag{1.19}$$

This is equivalent to solving

$$\frac{dx^*}{d\varepsilon} = X(x^*, t^*, V^*) \ , \ \frac{dt^*}{d\varepsilon} = \bar{T}(x^*, t^*, V^*) \ , \ \frac{dV^*}{d\varepsilon} = \nu(x^*, t^*, V^*) \tag{1.20}$$

subject to the initial condition

$$(x^*, t^*, V^*)|_{\varepsilon=0} = (x, t, V) \ . \tag{1.21}$$

The invariant requirement of (1.17) under (1.18a-c) is given by

$$\Psi^{(2)}\Lambda|_{\Lambda=0} = 0 \tag{1.22}$$

where $\Psi^{(2)}$ is the second extension of Ψ extended to the second jet space, coordinatised by $x, t, \frac{\partial V}{\partial x}, \frac{\partial V}{\partial t}, \frac{\partial^2 V}{\partial x^2}, \frac{\partial^2 V}{\partial x \partial t}, \frac{\partial^2 V}{\partial t^2}$. Equation (1.22) is a polynomial equation in a set of independent functions of the derivatives of V. As the equation must be true for arbitrary values of these independent functions, their coefficients must vanish, leading to an over-determined linear system of equations known as the determining equations for the coefficients $X(x, t, V)$, $\bar{T}(x, t, V)$ and $\nu(x, t, V)$. Then for known functions $X(x, t, V)$, $\bar{T}(x, t, V)$, $\nu(x, t, V)$, invariant solutions V corresponding to (1.18) satisfy

$$\nu(x, t, V) = X(x, t, V)\frac{\partial V}{\partial x} + \bar{T}(x, t, V)\frac{\partial V}{\partial t} \ . \tag{1.23}$$

Equation (1.23) is called the invariant surface condition (ISC) and can be solved as a first order PDE by the method of characteristics to yield the functional form of the similarity solution in terms of an arbitrary function, $\phi(z)$

$$V = g(x, t, \phi(z)), \quad z = z(x, t) \ . \tag{1.24}$$

Substituting (1.24) into (1.17) yields an ordinary differential equation (ODE) in $\phi(z)$. However, not all generators that reduce a given PDE might be appropriate to solve the problem when there is a given imposed initial (or final) condition. Given an initial condition $V(x, 0) = f(x)$, one way to determine appropriate generators is to ensure that the initial condition is invariant under the symmetry i.e. $\Psi(t) = 0|_{t=0}$

and $\Psi(V - f(x)) = 0|_{V=f(x),\ t=0}$ (e.g. see Bluman and Kumei (1989)). This is a sufficient but not a necessary condition. Alternatively, a less restructure condition for a diffusion equation of the form $\frac{\partial V}{\partial t} = Y(x, t, V, \frac{\partial V}{\partial x}, \frac{\partial^2 V}{\partial x^2})$ is that

$$X\big(x, 0, f(x)\big)f'(x) + \bar{T}\big(x, 0, f(x)\big)\ Y\big(x, 0, f(x), f'(x), f''(x)\big) = \eta\big(x, 0, f(x)\big)\ . \tag{1.25}$$

See Goard (2003), Ibragimov (1994) and Bluman and Kumei (1989) for a more detailed explanation of symmetries.

The Perturbation Method

In this thesis, perturbation methods are used to find analytic approximation solutions to PDEs which cannot be solved analytically. Generally with this method, we either have, or can introduce, a small parameter ε, $0 < \varepsilon \ll 1$ and assume that the solution of our PDE, say $V(x, t)$, can be written as a series in ε, i.e. $V(x, t) = \sum_{i=0}^{\infty} \varepsilon^i V_i(x, t)$. Substituting the series into our PDE yields a system of PDEs for $V_i(x, t)$ which can be obtained by equating powers of ε. Solving the system yields an approximate solution $V(x, t) = \sum_{i=0}^{N} \varepsilon^i V_i(x, t)$. If the approximated solution is valid over the whole domain to $O(\varepsilon^{N+1})$, then it can be said that our PDE is solved by the **regular perturbation method**.

However, the approximated solution obtained by the regular perturbation method might only be valid in a sub-domain and be invalid in other sub-domains, which consist of small areas in which that approximation is not accurate. These areas are referred to as transition, boundary or interior layers. In these cases, the **singular perturbation method** can be used to find an approximate solution over the whole domain. The singular perturbation method can be summarised in the following steps:

- We construct a regular perturbation expansion in the original variable (called the outer expansion). The outer expansion will be valid away from the boundary layer.

- We introduce an inner variable, which can be obtained by rescaling the original variable. Then, we construct a regular perturbation expansion in the inner variable (called the inner expansion). The inner expansion will be valid in the boundary layer.

- We match the outer and inner expansions to get a uniform expansion which is uniformly valid over the whole domain.

For a more detailed explanation, we refer the reader to Murdock (1999) and Paulsen (2013).

1.4.2 Risk-Neutral Expectation

The transition density function of the underling process $dP = \mu(P,t)dt + \sigma(P,t)dZ$, denoted $Tr(P,t;y,T)$, can be defined as

$$\Pr(a < P < b \text{ at time } T \mid P \text{ at time } t) = \int_a^b Tr(P,t;y,T)dy \qquad (1.26)$$

and satisfies the backward Kolmogorov equation, i.e.

$$\frac{\partial Tr}{\partial t} + \frac{1}{2}\sigma^2(P,t)\frac{\partial^2 Tr}{\partial P^2} + \mu(P,t)\frac{\partial Tr}{\partial P} = 0 \ . \qquad (1.27)$$

subject to $Tr(P,T;y,T) = \delta(P-y)$ and $\int_0^\infty Tr(P,t;y,T)\ dy = 1\ \forall t$, where δ is the Dirac delta function (see e.g. Spiegel (1967)).

From the Feynman-Kac theorem, as the current price of any financial derivative (say $V(P,t)$) with expiry T, given the current asset price P can be found by computing the expected payoff of the financial derivative at time T discounted for a period $T-t$ at the risk-free rate r, we can use the transition density function to compute the expected value i.e.

$$V(P,t) = e^{-r(T-t)}\ \mathbb{E}_t^{\mathbb{Q}}[h(P_T)] = e^{-r(T-t)} \left[\int_0^{\infty} h(y)\ Tr(P,t;y,T)\ dy \right] ,$$

where $h(\cdot)$ is the payoff function of the financial derivative.

1.4.3 Numerical Approaches

In this section we summarise two numerical methods used to price option contracts in this thesis.

Monte Carlo

As mentioned at the beginning of Section 1.4, the arbitrage price for derivative contracts that are based on an underlying asset whose risk-neutral dynamics follow (1.10) can be found by either using risk-neutral expectations (1.12) or by solving the appropriate PDE (1.11). In many cases, analytic solutions for the appropriate PDE and the transition density function do not exist. In these cases, the use of the Monte Carlo method becomes attractive. Valuing a financial derivative by the Monte Carlo method basically involves simulating paths of the stochastic process (or multiple processes) used to describe the evolution of the underlying asset prices. Generally, the use of the Monte Carlo method to price a financial contract can be summarised in three steps:

- Simulate N sample paths of the underlying asset price over the life of the financial contract (usually $[t, T]$, i.e. [current time, expiry of the financial contract]).

- At a given time t_0 ($t \leq t_0 \leq T$), calculate the payoff of the financial contract for each sample path.

- The value of the financial contract at time t_0 can be approximated by averaging the discounted payoffs.

The main advantage of the Monte-Carlo method is that it is an easy technique that can be used in the pricing of a variety of complicated types of financial contracts. For a more detailed explanation, we refer the reader to Glasserman (2003), McLeish (2005) and Wang (2012).

Finite-Difference Method

The finite-difference method (FDM) is one of the most popular methods for finding numerical solution of DEs. As such it is a useful technique to use to obtain accurate numerical solutions to a variety of problems arising in pricing financial derivatives. The use of the FDM to find the solution of a parabolic PDE in one dependent variable, V, and two independent variables, (x, t), requires setting up a 'grid' in space x and time t. So we discretise over the region $[0, X] \times [0, T]$:

- In the x direction with grid spacing $\Delta x = \frac{X}{m+1}$, where X is the largest value of the variable x. Hence $x_i = i\Delta x$ where $0 \leq i \leq m + 1$.

- In the t direction with grid spacing $\Delta t = \frac{T}{n+1}$, where T is the largest value of the variable t. Hence $t_j = j\Delta t$ where $0 \leq j \leq n + 1$.

The $(i, j)^{th}$ point on our grid represents $V(i\Delta x, j\Delta t)$ and we write $V(i\Delta x, j\Delta t) = V_{i,j}$.

The main idea of the FDM is to use Taylor series expansions to represent the derivatives that arise in the PDE. The forward, backward and central difference approximations are the most common finite-difference approximations, which are respectively given by

$$\frac{\partial V}{\partial x} = \frac{V(x+\Delta x,t)-V(x,t)}{\Delta x} + O(\Delta x) = \frac{V_{i+1,j}-V_{i,j}}{\Delta x} + O(\Delta x) \tag{1.28a}$$

$$\frac{\partial V}{\partial x} = \frac{V(x,t)-V(x-\Delta x,t)}{\Delta x} + O(\Delta x) = \frac{V_{i,j}-V_{i-1,j}}{\Delta x} + O(\Delta x) \tag{1.28b}$$

$$\frac{\partial V}{\partial x} = \frac{V(x+\Delta x,t)-V(x-\Delta x,t)}{2\Delta x} + O(\Delta x^2) = \frac{V_{i+1,j}-V_{i-1,j}}{2\Delta x} + O(\Delta x^2)\ . \tag{1.28c}$$

Equations (1.28a-c) can be used to approximate the second derivative, for example the backward difference on $\frac{\partial V(x+\Delta x,t)}{\partial x}$ and $\frac{\partial V(x,t)}{\partial x}$ can be used to get an approximation to $\frac{\partial^2 V}{\partial x^2}$, namely

$$\frac{\partial^2 V}{\partial x^2} = \frac{V_{i+1,j} - 2V_{i,j} + V_{i-1,j}}{(\Delta x)^2} + O(\Delta x^2)\ . \tag{1.29}$$

Depending on which combination of differences we use in discretising the PDE we will obtain either an *explicit* or *implicit* approach. Each of these approaches results in equations which can be solved for the dependent variable, $V_{i,j}$. Now with the known values V at X and T we can find the values of our dependent variable at the boundary and final conditions, i.e.

$$V(0, j\Delta t),\ 0 \le j \le n+1 \tag{1.30a}$$

$$V(X, j\Delta t),\ 0 \le j \le n+1 \tag{1.30b}$$

$$V(i\Delta x, T),\ 0 \le i \le m+1 \tag{1.30c}$$

With the explicit approach, the value of each point on our grid (say $V_{i,j}$) is found by using the known values at later times, namely $V_{i-1,j+1}$, $V_{i,j+1}$ and $V_{i+1,j+1}$. However, with the implicit approach, the value of each point in our grid is found by using the known value at the later time, namely $V_{i,j+1}$, and the unknown values at the same time, namely $V_{i-1,j}$ and $V_{i+1,j}$. Hence, this method requires solving a system of linear equations for all values at time step j. For a more detailed explanation, we refer the reader to Morton and Mayers (1994) and Wilmott (1998).

The numerical solution is expected to tend to the exact solution of the PDE as the grid spacings tend to zero. The simplicity and accuracy are the main advantages of the FDM. However, the stability of the approximate solution could be a problem (especially the explicit method) and care needs to be taken when using the method.

1.5 Literature review

1.5.1 Models for Oil Prices

One of the earliest one-factor models used to describe the behaviour of commodities prices was the Geometric Brownian Motion (GBM) model, which assumes that the change in commodity prices can be described by $dP = \mu P dt + \sigma P dZ$, where μ and σ are constant.

Based on this model, Brennan and Schwartz (1985) established the relationship between the spot and futures prices that incorporated a convenience yield. In their model, the interest rate and convenience yield were assumed to be known and constant. Similarly, Gabillon (1991) derived a closed form solution for futures prices of oil assuming futures prices depended only on the spot price of oil and the cost of carry of physical oil. However, he observed that the term structure in backwardation could not be described by his formula. Gabillon then extended his formula by assuming that the convenience yield also affected the futures price. Under this assumption Gabillon obtained a formula for futures prices that could describe backwardation and contango states. However, he observed that there was a discontinuity when changing from backwardation to contango and vice versa, and that using the GBM model to value financial derivatives could lead to unreasonable over- or undervaluations.

Other authors argue that the effect of supply and demand in the commodity, results in a mean-reversion property for its prices. Various mean-reverting one-factor models can be found in the literature. These include those of Dixit and Pindyk (1994), Ross (1995), Bjerksund and Ekern (1995) and Schwartz (1997). Schwartz (1997) is renowned for one of the most well-known mean-reverting one-factor models for oil prices. He assumed that the spot price followed the mean reverting model $dP = \eta P(\mu - \ln(P))dt + \sigma P dZ$ and derived an analytic solution for futures prices. Advantages of the use of one-factor models include their simplicity and tractability, i.e. it can be easier to derive closed and simple formulae for futures (and other derivative) prices under these models. It is these types of formulae that are preferred by traders and can easily describe the behaviour of futures prices. However, empirical studies by Barren (1991) and Schwartz (1997) indicate that one-factor models are often too restrictive to explain derivative prices. This indicates the need for an extra state variable to take into account, which produces richer shapes of curves compared with one-factor models. Fama and French (1987) and Miltersen and Schwartz (1998) provided evidence that the convenience yield should be specified by a stochastic process. This belief was supported by Ribeiro and Hodges (2004) who agreed that one-factor models were inappropriate as they did not take into account the inventory-dependence property of the convenience yield. Certainly a not-very-desirable implication of one-factor models is that all futures returns are perfectly correlated.

In order to achieve a more realistic stochastic behaviour of oil prices, various two-factor models have been introduced since 1990. The second factor is usually taken to be either the convenience yield or long-run mean. One of the most well-known models for commodity prices with convenience yield as a second factor was originally introduced by Gibson and Schwartz (1990). They assumed a risk-neutral process where the spot price of the commodity, P, follows a GBM and the convenience yield, δ, follows a mean-reverting Ornstein-Uhlenbeck (OU) process and is correlated with the spot price; namely the risk-neutral joint process

$$dP = (r - \delta)Pdt + \sigma_1 P d\tilde{Z}_1 \tag{1.31a}$$

$$d\delta = k(\alpha - \delta)dt + \sigma_2 d\tilde{Z}_2 \tag{1.31b}$$

$$\text{where} \quad Corr(dZ_1, dZ_2) = \rho dt \ . \tag{1.31c}$$

The authors used a numerical method to solve the governing PDE for the futures price. Then in 1997, Schwartz presented an analytic solution for futures contract prices[7] under (1.31a-c). Schwartz empirically showed that the suggested model (1.31a-c) clearly outperformed the one-factor model, i.e. $dX = k(\alpha - X)dt + \sigma dZ$ where $X = \ln(P)$, in fitting oil, gold and copper futures prices. Ribeiro and Hodges (2004) replaced the OU process for the convenience yield in the Schwartz (1997) model by the Cox-Ingersoll-Ross (CIR) model and assumed that the volatility of spot price was proportional to the square root of the convenience yield i.e.

$$dP = (\mu - \delta)Pdt + \sigma_1 P\sqrt{\delta} dZ_1 \tag{1.32a}$$

$$d\delta = k(\alpha - \delta)dt + \sigma_2 \sqrt{\delta} dZ_2 \tag{1.32b}$$

$$\text{where} \quad Corr(dZ_1, dZ_2) = \rho dt \ . \tag{1.32c}$$

Under (1.32a-c) Ribeiro and Hodges derived a closed form solution for futures prices and empirically compared their model with the Gibson and Schwartz (1990) model. They concluded that their proposed model slightly outperformed the Gibson and Schwartz (1990) model in fitting light crude oil futures prices. However, Ribeiro and Hodges indicated that the results of the empirical comparison was affected by the peculiarity[8] of the sample.

One of the earliest models for commodity prices with long-run price as a second

[7] The solution was derived by Jamshidian and Fein (1990) and Bjerksund (1991).

[8] The peculiarity of the data set used was that no contango states were observed for a significant period of time.

factor was introduced by Gabillon (1991). He assumed that there was a long-run price of oil, L, for delivery at infinite time which was correlated with the process of the spot price and could be described by a stochastic process. He suggested that the spot price of oil and the long-run price follow the risk-neutral joint process

$$dP = (r - \delta(P, L))Pdt + \sigma_1 Pd\tilde{Z}_1 \tag{1.33a}$$

$$dL = \sigma_2 Ld\tilde{Z}_2 \tag{1.33b}$$

$$\text{where} \quad Corr(dZ_1, dZ_2) = \rho dt \ . \tag{1.33c}$$

Under a special form of convenience yield ($\delta(P, L) = a\ln(\frac{P}{L}) + b$), where a and b are constant), Gabillon derived a closed-form for the price of futures contracts. Moreover, Pilipovic (1997) derived a closed form for pricing futures contracts under the assumption that the spot price reverts to its long-term equilibrium price, θ, and the long-term equilibrium follows a GBM process namely the risk-neutral joint process

$$dP = \alpha(\theta - P)Pdt + \sigma_1 Pd\tilde{Z}_1 \tag{1.34a}$$

$$d\theta = \alpha\theta dt + \sigma_2 \theta d\tilde{Z}_2 \ . \tag{1.34b}$$

Schwartz and Smith (2000) modelled the short (ξ) and long (υ) term variations of commodity prices with stochastic processes. The long term variation is assumed to follow the Arithmetic Brownian Motion (ABM), while the short term variation is assumed to revert to zero and follow the OU process i.e.

$$\ln(P) = \xi + \upsilon \tag{1.35a}$$

$$d\xi = -\eta\xi dt + \sigma_1 dZ_1 \tag{1.35b}$$

$$d\upsilon = \mu dt + \sigma_2 dZ_2 \tag{1.35c}$$

$$\text{where} \quad Corr(dZ_1, dZ_2) = \rho dt \ . \tag{1.35d}$$

Based on this model Schwartz and Smith derived a formula for futures prices. Korn (2005) generalised the Schwartz and Smith (2000) model by allowing two mean-reverting stochastic factors, thereby implying spot and futures prices to be stationary.

Various extensions to two-factor models can be found in the literature. These include three-factor models (see for example Schwartz (1997), Miltersen and Schwartz (1998), Hilliard and Reis (1998), Cortazar and Schwartz (2003)) and jump models (see for example Dias and Rocha (1999)).

1.5.2 Pricing Options on Oil

One of the earliest works on pricing options was due by Bachelier (1900), who derived a closed formula for valuing European option contracts. He assumed that the change in stock prices followed an ABM, i.e. $dP = \mu dt + \sigma dZ$. The main disadvantages of Bachelier's work, as noted by Merton (1973) and Smith (1976), were ignoring the time value of money and using the ABM for the underlying prices movements, which can yield negative values for their prices. Subsequently, in order to address the second disadvantage, in most of the works on pricing options the change in the underlying price is usually assumed to follow a GBM. Sprenkle (1961) derived a closed formula for valuing option contracts under the assumption that investors had risk aversion tendencies. However, the use of Sprenkle's formula requires estimation of two arbitrary parameters, namely the degree of risk aversion and the average rate of growth of the underlying price. Boness (1964) improved Sprenkle's formula by taking into account the time value of money. In the Boness formula the expected rate of return of the underlying is used as a discounting rate. Samuelson (1965) derived a closed formula for valuing option contracts under the assumption that the option and underlying asset have different levels of risk. The main disadvantages of the formulae of Sprenkle, Boness and Samuelson is that each of their formulae include one or two unobservable parameters. Samuelson and Merton (1969) suggested that the option value should be a function of the underlying price and that the discount rate should satisfy a hedging strategy where investors can hold an option with a certain amount of the underlying asset. Samuelson and Merton derived a formula for pricing option contracts based on a utility function approach. Thorp and Kassouf (1967) derived an empirical formula for pricing warrants[9]. In this formula the ratio of the underlying asset in order to have a hedging position was determined. Black and Scholes (1973) derived the well-known closed formula for pricing option contracts. Schwartz (1977) suggested the finite difference method to solve a special case of the Black-Scholes PDE in which the underlying pays discrete dividend yields and also derived the optimal strategy for exercising American options. Brennan and Schwartz (1978) noted that the approximate solution of the Black-Scholes PDE by using the finite difference method was equivalent to approximating the diffusion process of the underlying asset by a jump process.

Empirical studies on the performance of the Black-Scholes model showed that the model could be seriously mispricing real market prices (see for example Duan (1999)). Consequently, various extensions to the Black-Scholes model have been developed. These include models with stochastic volatility (see for example, Hull and White (1987), Johnson and Shanno (1987), Scott (1987), Wiggins (1987) and Heston (1993)), models with jump-diffusions (see for example, include Naik and Lee (1990) and Bates (1996)) or models with both stochastic volatility and jumps (see for example, Scott (1997) and Bakshi *et al* (1997)).

For pricing options on futures, Black (1976) indicated that the Black-Scholes formula could be used to price such options, by simply taking into account the facts that it costs nothing to enter into a futures contract and that futures contracts could be considered as an underlying asset that pays a dividend yield equal to the interest rate. However, various studies noted that the mean-reverting property of commodity markets was not considered in the Black and Scholes (1973) approach (see for exam-

[9]Warrants can be considered as call options, but they are issued and guaranteed by a company.

ple Ross (1995) and Schwartz (1997)). Ramaswamy and Sundaresan (1985) studied the pricing of options on futures with constant and stochastic interest rates. Further, Turnbull (1991) derived closed-form solutions for European option contracts written on interest rate forward and futures contracts. Hilliard and Reis (1998) considered stochastic convenience yields, stochastic interest rates, and jumps in the commodity spot price and derived a closed-form solution for pricing European options on commodity futures. Similarly, Miltersen and Schwartz (1998) considered a three-factor stochastic model (which included interest rates, convenience yields and spot price of the underlying commodity as sources of randomness) and obtained closed-form solutions for pricing options on futures and forwards. Recently, based on the mean-reverting model $dP = \eta P(\mu - \ln(P))dt + \sigma P dZ$ (which was suggested by Schwartz (1997)), Skorodumov (2008) derived a closed-form formula (based on the Black and Scholes (1973) framework) for pricing vanilla call and put options on oil futures.

1.5.3 Pricing Spread Options

One of the earliest works on pricing spread options is due to Margrabe (1978), who derived a closed-form formula for valuing spread options with zero strike (known as exchange options). This formula is based on the assumption that the change in underlying prices follows two correlated GBMs. Similarly, Fu (1996) derived a closed-form formula for valuing interest rate exchange options. In the case of non-zero strike, the resulting spread does not have a closed-form solution for the density function (see Carmona and Durrleman (2003)). This is problematic when adapting the direct approach in pricing spread option contracts, i.e. finding the expected payoff under the risk-neutral measure $\mathbb{Q}$, which involves solving the following double-integral problem

$$\begin{aligned} C(F_1, F_2, t) =& e^{-r(T-t)} E_t^{\mathbb{Q}} \big[\max(F_{1T} - F_{2T} - K, 0) \big] \\ =& e^{-r(T-t)} \left[\int_0^{\infty} \int_0^{\infty} \max(F_{1T} - F_{2T} - K, 0) \;\; f(F_{1T} \mid F_{2T}) \, dF_{1T} \, dF_{2T} \right] \\ =& e^{-r(T-t)} \left[\int_0^{\infty} \int_{\max(F_{2T}+K,0)}^{\infty} (F_{1T} - (F_{2T} + K)) \;\; f(F_{1T} \mid F_{2T}) \, dF_{1T} \, dF_{2T} \right] \end{aligned} \tag{1.36}$$

to find the price $C(F_1, F_2, t)$ at time t. In (1.36) $f(F_{1T} \mid F_{2T})$ is the joint density function of F_{1T} given F_{2T}. Various solution techniques can be found in the literature to approximate spread option values (see for example Ravindran (1993), Shimko (1994), Bjerksund and Stensland (1994), Pearson (1995), Kirk (1995) and Carmona and Durrleman (2003)).

Wilcox (1990) derived a formula for valuing call spread options with the assumption that under the risk-neutral measure the spread itself follows the ABM. However, the Wilcox pricing formula does not satisfy the relevant PDE for the dynamics of hedging. Poitras (1998) assumed that the changes in both underlying prices follow ABM and derived the Bachelier (1900) options pricing formulae for underlyings with equal and unequal dividends and for spread options on futures. One of the main disadvantages of his assumption was that it allowed each underlying asset price to become negative. Mahringer and Prokopczuk (2010) empirically showed that using univariate modelling (i.e. to model the movement of the spread itself) for pricing spread options can outperform explicit modelling (i.e. to model the movement of both underlyings).

Chapter 2

Comparison of One Factor Models for The Spot Price of Oil via GMM

Over the last two decades, the behaviour of oil prices, one of the world's most important commodities, has become progressively more complex. Various models have been proposed and examined by different authors to try to track the movement of oil and other commodity prices. Typically, these models have some common features such as mean reversion. However, they may vary in their diffusion structure and / or drift structure and number of sources of randomness. The ability of the models to capture the behaviour of oil prices affect the subsequent pricing of derivatives on oil. This chapter will be divided into three sections. The first section summarises some of the most popular one-factor models found in the literature and identify some of their statistical properties. In the second section we will introduce the estimation technique Generalized Method of Moments (GMM), and in the third section using GMM we will compare the performance of the stochastic models presented in the first section, in explaining the behaviour of Brent crude oil prices.

Existing Models	Drift Term	Diffusion Term
Model 1	μP	σP
Model 2	$\eta(\mu - P)$	σ
Model 3	$\eta(\mu - P)$	σP
Model 4	$\eta P(\mu - \ln(P))$	σP
Model 5	$\eta P(\mu - P)$	σ
Model 6	$\eta P(\mu - P)$	σP
Model 7	$\eta P(\mu - P)$	$\sigma P^{\frac{3}{2}}$
Proposed Models	**Drift Term**	**Diffusion Term**
Model 8	aP	$\sigma P^{\frac{3}{4}}$
Model 9	$a\sqrt{P}$	$\sigma P^{\frac{3}{4}}$
Model 10	$a\sqrt{P} + bP$	$\sigma P^{\frac{3}{4}}$

Table 2.1: Drift and diffusion terms of the most popular one-factor models.

2.1 One Factor Stochastic Processes

The aim of this section is to provide an overview of one-factor models found in the literature that can be used to model oil prices. Where possible we will derive the analytic expression for the price based on each model and identify some statistical properties of the model. In general, the dynamics of the spot oil price under a one-factor model is represented by the stochastic differential equation:

$$dP = \mu(P,t)dt + \sigma(P,t)dZ \tag{2.1.1}$$

where $\mu(P,t)$ and $\sigma(P,t)$ are the drift and diffusion terms respectively and Z is the standard Wiener process under a probability measure $\mathbb{P}$. The drift and diffusion terms of the most popular one-factor models found in the literature as well as some proposed models for the pricing of oil are provided in Table 2.1. Model 1 is the GBM, with instantaneous expected growth rate μ and instantaneous standard deviation σ. The model assumes that the expected percentage change in prices and the volatility of percentage change of prices are constant. Model 1 was considered to price oil by Brennan and Schwartz (1985), McDonald and Siegel (1985) and Gabillon (1991). Model 2 is a mean-reverting process, called an Ornstein-Uhlenbeck (OU) process and was used by Bjerksund and Ekern (1995). It assumes that the price reverts to a constant μ with the rate of reversion determined by η. A higher (lower) value of the rate of reversion indicates that the price pulls more quickly (slowly) back to the long-run price level μ. The volatility of P is constant in absolute terms.

Model 3 is a mean-reverting process and was used by Barone-Adesi *et al* (2005) and Sabanis (2003). Model 3 has the same drift term as Model 2, but unlike Model 2 it assumes that the instantaneous volatility of percentage changes in price is constant. Model 4 was considered by Ross (1995) and Schwartz (1997) for modelling oil prices. By Itô's Lemma (see Section 1.1.4), we can show that the model implies that the logarithm of prices follows the Ornstein-Uhlenbeck process. Models 5-7 have a similar drift term where the rate of reversion is determined by ηP, a linear function of P. This can generate a balancing effect of a stronger mean reversion for a higher level of P. Dixit and Pindyck (1994) considered Model 6 and Model 7 was considered by Heston (1997), Lewis (2000) and Goard and Mazur (2013).

Models 8-10, are our newly-proposed 'three-quarters models' with a diffusion term of the form $\sigma P^{\frac{3}{4}}$ which can reduce the variability of oil prices. With $b < 0$, Model 10 is a non-linear mean-reverting process, and assumes that the spot price reverts to a constant $\frac{a^2}{b^2}$ with reversion rate proportional to $\sqrt{P}$. Thus this model also generates a balancing effect of a stronger mean-reversion for higher prices. Models 8 and 9 are included for comparison with Model 10.

2.1.1 Model 1: The Geometric Brownian Motion Model (GBM)

$$dP = \mu P dt + \sigma P dZ \tag{2.1.2}$$

Equation (2.1.2) is known as the Geometric Brownian Motion (GBM), with instantaneous expected growth rate μ and an instantaneous standard deviation σ. The model assumes that the expected percentage change in prices and the volatility of percentage change of prices are constant. Hence if prices increase (or decrease) more than predicted in a given instant, all future forecasts are increased (or decreased) at the same ratio. The GBM implies an exponential trend in P with random fluctuations around this trend.

Analytic Expression for P_t

To find an analytic expression for $P(= P_t)$, where the subscript, t, denotes time dependence, we find the corresponding Stratonovich SDE (see Section 1.1.3) of (2.1.2), namely

$$dP = (\mu - \frac{\sigma^2}{2})Pdt + \sigma P \circ dZ \ . \tag{2.1.3}$$

The solution of equation (2.1.3) can be obtained by solving the following deterministic differential equation

$$dx = (\mu - \frac{\sigma^2}{2})xdt + \sigma x \ dc(t) \tag{2.1.4}$$

where $c(t)$ is an arbitrary differentiable function. Equation (2.1.4) can be written as

$$\frac{dx}{x} = (\mu - \frac{\sigma^2}{2} + \sigma c'(t))dt \tag{2.1.5}$$

Then integrating (2.1.5) from u to t where $u \leq t$, we get:

$$\ln(x_t) - \ln(x_u) = (\mu - \frac{\sigma^2}{2})(t-u) + \sigma(c(t) - c(u)) \ . \tag{2.1.6}$$

Finally, we can then replace x_t with P_t and $c(t)$ with Z_t to get

$$P_t = P_u \exp\{(\mu - \frac{\sigma^2}{2})(t-u) + \sigma(Z_t - Z_u)\} \ . \tag{2.1.7}$$

Statistical Properties

$$
\begin{aligned}
\forall\ u \leq t, \quad \mathbb{E}_u(P_t) &= P_u e^{(\mu-\frac{\sigma^2}{2})(t-u)}\mathbb{E}_u(e^{\sigma(Z_t-Z_u)}) \\
&= P_u e^{(\mu-\frac{\sigma^2}{2})(t-u)}\mathbb{E}_0(e^{\sigma Z_{t-u}}) \\
&= P_u e^{(\mu-\frac{\sigma^2}{2})(t-u)} e^{\frac{\sigma^2}{2}(t-u)} \\
\Rightarrow \mathbb{E}_u(P_t) &= P_u e^{\mu(t-u)} \qquad (2.1.8)
\end{aligned}
$$

$$
\begin{aligned}
\forall\ u \leq t, \quad Var_u(P_t) &= \mathbb{E}_u(P_t^2) - (\mathbb{E}_u(P_t))^2 \\
&= \mathbb{E}_u(P_u^2 e^{\{2(\mu-\frac{\sigma^2}{2})(t-u)+2\sigma(Z_t-Z_u)\}}) - P_u^2 e^{2\mu(t-u)} \\
&= P_u^2 e^{2(\mu-\frac{\sigma^2}{2})(t-u)}\mathbb{E}_0(e^{2\sigma Z_{t-u}}) - P_u^2 e^{2\mu(t-u)} \\
&= P_u^2 e^{2(\mu-\frac{\sigma^2}{2})(t-u)} e^{2\sigma^2(t-u)} - P_u^2 e^{2\mu(t-u)} \\
\Rightarrow Var_u(P_t) &= P_u^2 e^{2\mu(t-u)}(e^{\sigma^2(t-u)} - 1)\ . \qquad (2.1.9)
\end{aligned}
$$

2.1.2 Model 2: Ornstein-Uhlenbeck Model

$$
dP = \eta(\mu - P)dt + \sigma dZ \qquad (2.1.10)
$$

Model 2 is a mean reverting process, and assumes that the price reverts to a constant μ with the rate of reversion determined by η. A higher (lower) value of the rate of reversion indicates that the price pulls more quickly (slowly) back to the long-run price level μ. The volatility of P is constant in absolute terms. For example, if the standard deviation of the movement in P is one when $P = 100$, it is also one when $P = 200$. The disadvantage of Model 2 is that we cannot ensure that the value of P remains positive in this random walk.

Analytic Expression for P_t

To find an analytic expression for P_t we find the corresponding Stratonovich SDE of (2.1.10)

$$dP = \eta(\mu - P)dt + \sigma \circ dZ \ . \tag{2.1.11}$$

The solution of equation (2.1.11) can be obtained by solving the following deterministic differential equation

$$dx = \eta(\mu - x)dt + \sigma \ dc(t) \tag{2.1.12}$$

where $c(t)$ is an arbitrary differentiable function. Equation (2.1.12) can be written as

$$\frac{dx}{dt} + \eta x_t = \eta\mu + \sigma c'(t) \ . \tag{2.1.13}$$

Multiplying equation (2.1.13) by the integration factor $e^{\eta t}$ we get

$$\begin{aligned} e^{\eta t}(\frac{dx}{dt} + \eta x) &= e^{\eta t}(\eta\mu + \sigma c'(t)) \\ \Rightarrow \frac{d(xe^{\eta t})}{dt} &= e^{\eta t}(\eta\mu + \sigma c'(t)) \ . \end{aligned} \tag{2.1.14}$$

Then integrating (2.1.14) from u to t where $u \le t$, we get:

$$x_t e^{\eta t} - x_u e^{\eta u} = \mu e^{\eta t} - \mu e^{\eta u} + \sigma \int_u^t e^{\eta s} \grave{c}(s) ds \ . \tag{2.1.15}$$

Finally replacing x_t with P_t and $c(t)$ with Z_t we get

$$P_t = P_u e^{-\eta(t-u)} + \mu(1 - e^{-\eta(t-u)}) + \sigma e^{-\eta t} \int_u^t e^{\eta s} dZ_s \ . \tag{2.1.16}$$

Statistical Properties

$$\begin{aligned}\forall\ u \le t, \quad \mathbb{E}_u(P_t) &= P_u e^{-\eta(t-u)} + \mu(1 - e^{-\eta(t-u)}) + \sigma e^{-\eta t}\ \mathbb{E}_u\Big(\int_u^t e^{\eta s} dZ_s\Big) \\ \Rightarrow \mathbb{E}_u(P_t) &= P_u e^{-\eta(t-u)} + \mu(1 - e^{-\eta(t-u)})\ . \end{aligned} \qquad (2.1.17)$$

$$\begin{aligned}\forall\ u \le t, \quad Var_u(P_t) &= \sigma^2 e^{-2\eta t} Var\Big(\int_u^t e^{\eta s} dZ_s \mid \mathcal{F}_u\Big) \\ &= \sigma^2 e^{-2\eta t} \int_u^t e^{2\eta s} ds \\ \Rightarrow Var_u(P_t) &= \frac{\sigma^2}{2\eta}(1 - e^{-2\eta(t-u)})\ . \end{aligned} \qquad (2.1.18)$$

2.1.3 Model 3

$$dP = \eta(\mu - P)dt + \sigma P dZ \qquad (2.1.19)$$

With a similar drift to Model 2, Model 3 is a mean reverting process which assumes that the price reverts to a constant μ with rate reversion determined by η. However, unlike Model 2, the diffusion term shows that the instantaneous volatility of percentage changes in price is constant, σ.

Analytic Expression for P_t

To find an analytic expression for P_t we find the corresponding Stratonovich SDE of (2.1.19)

$$dP = [\eta(\mu - P) - \frac{\sigma^2}{2} P]dt + (\sigma P) \circ dZ\ . \qquad (2.1.20)$$

The solution of equation (2.1.20) can be obtained by solving the following deterministic differential equation

$$dx = [\eta(\mu - x) - \frac{\sigma^2}{2}x]dt + \sigma x \; dc(t) \qquad (2.1.21)$$

where $c(t)$ is an arbitrary differentiable function. Equation (2.1.21) can be written as

$$\frac{dx}{dt} + (\eta + \frac{\sigma^2}{2} - \sigma c'(t))x = \eta\mu \; . \qquad (2.1.22)$$

Multiplying equation (2.1.22) by the integration factor $e^{(\eta+\frac{\sigma^2}{2})t-\sigma c(t)}$, we get

$$\begin{aligned} e^{(\eta+\frac{\sigma^2}{2})t-\sigma c(t)}[\frac{dx}{dt} + (\eta + \frac{\sigma^2}{2} - \sigma c'(t))x] &= \eta\mu e^{(\eta+\frac{\sigma^2}{2})t-\sigma c(t)} \\ \Rightarrow \frac{d(xe^{(\eta+\frac{\sigma^2}{2})t-\sigma c(t)})}{dt} &= \eta\mu e^{(\eta+\frac{\sigma^2}{2})t-\sigma c(t)} \; . \end{aligned} \qquad (2.1.23)$$

Integrating (2.1.23) from u to t where $u \leq t$, we get:

$$x_t e^{(\eta+\frac{\sigma^2}{2})t-\sigma c(t)} - x_u e^{(\eta+\frac{\sigma^2}{2})u-\sigma c(u)} = \eta\mu \int_u^t e^{(\eta+\frac{\sigma^2}{2})s-\sigma c(s)} ds \; . \qquad (2.1.24)$$

We then replace x_t with P_t and $c(t)$ with Z_t to get

$$P_t = P_u e^{-(\eta+\frac{\sigma^2}{2})(t-u)+\sigma(Z_t - Z_u)} + \eta\mu e^{-(\eta+\frac{\sigma^2}{2})t} \int_u^t e^{(\eta+\frac{\sigma^2}{2})s+\sigma(Z_t - Z_s)} ds \; . \qquad (2.1.25)$$

Statistical Properties

$$
\begin{aligned}
\forall\ u \leq t,\ \mathbb{E}_u(P_t) =& P_u e^{-(\eta+\frac{\sigma^2}{2})(t-u)}\mathbb{E}_u(e^{\sigma(Z_t-Z_u)}) \\
&+ \eta\mu e^{-(\eta+\frac{\sigma^2}{2})t}\int_u^t e^{(\eta+\frac{\sigma^2}{2})s}\mathbb{E}_u(e^{\sigma(Z_t-Z_s)})ds \\
=& P_u e^{-(\eta+\frac{\sigma^2}{2})(t-u)}\mathbb{E}_0(e^{\sigma Z_{t-u}}) \\
&+ \eta\mu e^{-(\eta+\frac{\sigma^2}{2})t}\int_u^t e^{(\eta+\frac{\sigma^2}{2})s}\mathbb{E}_0(e^{\sigma Z_{t-s}})ds \\
=& P_u e^{-(\eta+\frac{\sigma^2}{2})(t-u)}e^{\frac{\sigma^2}{2}(t-u)} + \eta\mu e^{-(\eta+\frac{\sigma^2}{2})t}\int_u^t e^{(\eta+\frac{\sigma^2}{2})s}e^{\frac{\sigma^2}{2}(t-s)}ds \\
=& P_u e^{-\eta(t-u)} + \eta\mu e^{-\eta t}\int_u^t e^{\eta s}ds \\
\Rightarrow \mathbb{E}_u(P_t) =& P_u e^{-\eta(t-u)} + \mu(1-e^{-\eta(t-u)})\ .
\end{aligned}
\tag{2.1.26}
$$

2.1.4 Model 4: Schwartz Model

$$
dP = \eta P(\mu - \ln P)dt + \sigma P dZ \tag{2.1.27}
$$

Model 4 is a mean reverting process. By Itô's Lemma (see Section 1.1.4) with $f = \ln P$ we get

$$
df = \eta(\alpha - f)dt + \sigma dZ\ , \tag{2.1.28}
$$

where $\alpha = \mu - \frac{\sigma^2}{2\eta}$. Hence, the logarithm of prices reverts to a constant α and the rate of reversion is determined by η. The diffusion term indicates that the instantaneous volatility of percentage changes in price is constant, σ.

Analytic Expression for P_t

As above, with $f = \ln P$, and then applying Itô's Lemma we get

$$df = \eta(\alpha - f)dt + \sigma dZ \ , \tag{2.1.29}$$

where $\alpha = \mu - \frac{\sigma^2}{2\eta}$. Equation (2.1.29) is of the same form as equation (2.1.10), and so we get the analytic expression for f_t from (2.1.16) namely

$$\begin{aligned} f_t &= (\mu - \frac{\sigma^2}{2\eta})(1 - e^{-\eta(t-u)}) + e^{-\eta(t-u)} f_u + \sigma \int_u^t e^{-\eta(t-s)} dZ_s \\ \Rightarrow P_t &= \exp\{(\mu - \frac{\sigma^2}{2\eta})(1 - e^{-\eta(t-u)}) + e^{-\eta(t-u)} \ln P_u + \sigma \int_u^t e^{-\eta(t-s)} dZ_s\} \ . \end{aligned} \tag{2.1.30}$$

Statistical Properties

$$\begin{aligned} &\forall\ u \leq t, \\ &\mathbb{E}_u(P_t) = \exp\{(\mu - \frac{\sigma^2}{2\eta})(1 - e^{-\eta(t-u)}) + e^{-\eta(t-s)} \ln P_u\} \ \ \mathbb{E}_u(e^{\sigma \int_u^t e^{-\eta(t-s)} dZ_s}) \\ &\qquad = \exp\{(\mu - \frac{\sigma^2}{2\eta})(1 - e^{-\eta(t-u)}) + e^{-\eta(t-s)} \ln P_u\} \ \ e^{\frac{\sigma^2}{2} \int_u^t e^{-2\eta(t-s)} ds} \\ &\Rightarrow \mathbb{E}_u(P_t) = \exp\{(\mu - \frac{\sigma^2}{2\eta})(1 - e^{-\eta(t-u)}) + e^{-\eta(t-s)} \ln P_u + \frac{\sigma^2}{4\eta}(1 - e^{-2\eta(t-u)})\} \ . \end{aligned} \tag{2.1.31}$$

2.1.5 Model 5

$$dP = \eta P(\mu - P)dt + \sigma dZ \tag{2.1.32}$$

Model 5 is a mean reverting process and assumes that the price reverts to a constant μ with rate of reversion determined by ηP, a linear function of P. The diffusion term shows that the instantaneous volatility of absolute changes in price is constant, σ.

2.1.6 Model 6

$$dP = \eta P(\mu - P)dt + \sigma P dZ \tag{2.1.33}$$

Model 6 has a similar drift term to Model 5. However, with Model 6 the instantaneous volatility of percentage changes (rather than absolute changes) is constant, σ.

Analytic Expression for P_t

To find an analytic expression for P_t we find the corresponding Stratonovich SDE of (2.1.33), namely

$$dP = \left(\eta P(\mu - P) - \frac{\sigma^2}{2}P\right) dt + (\sigma P) \circ dZ\ . \tag{2.1.34}$$

The solution of equation (2.1.34) can be obtained by solving the following deterministic differential equation

$$dx = \left(\eta x(\mu - x) - \frac{\sigma^2}{2}x\right) dt + \sigma x\ dc(t) \tag{2.1.35}$$

where $c(t)$ is an arbitrary differentiable function. Now letting $x = \frac{1}{f} \Rightarrow \frac{dx}{dt} = \frac{-1}{f^2}\frac{df}{dt}$ then on substituting in (2.1.35) we get

$$\frac{df}{dt} + (\eta\mu - \frac{\sigma^2}{2} + \sigma c'(t))f = \eta\ . \tag{2.1.36}$$

Multiplying equation (2.1.36) by the integration factor $e^{(\eta\mu - \frac{\sigma^2}{2})t + \sigma c(t)}$ we get

$$e^{(\eta\mu - \frac{\sigma^2}{2})t + \sigma c(t)}\left(\frac{df}{dt} + \left(\eta\mu - \frac{\sigma^2}{2} + \sigma c'(t)\right)f\right) = \eta e^{(\eta\mu - \frac{\sigma^2}{2})t + \sigma c(t)}$$

$$\Rightarrow \frac{d(e^{(\eta\mu - \frac{\sigma^2}{2})t + \sigma c(t)}f)}{dt} = \eta e^{(\eta\mu - \frac{\sigma^2}{2})t + \sigma c(t)}\ . \tag{2.1.37}$$

Then integrating (2.1.37) from u to t where $u \leq t$, we get:

$$f_t e^{(\eta\mu - \frac{\sigma^2}{2})t + \sigma c(t)} - f_u e^{(\eta\mu - \frac{\sigma^2}{2})u + \sigma c(u)} = \eta \int_u^t e^{(\eta\mu - \frac{\sigma^2}{2})s + \sigma c(s)} ds$$

$$\Rightarrow f_t = f_u e^{-(\eta\mu - \frac{\sigma^2}{2})(t-u) - \sigma(c(t) - c(u))} + \eta e^{-(\eta\mu - \frac{\sigma^2}{2})t - \sigma c(t)} \int_u^t e^{(\eta\mu - \frac{\sigma^2}{2})s + \sigma c(s)} ds$$

$$\Rightarrow x_t = \frac{x_u}{e^{-(\eta\mu - \frac{\sigma^2}{2})(t-u) - \sigma(c_t - c_u)} + \eta x_u e^{-(\eta\mu - \frac{\sigma^2}{2})t - \sigma c_t} \int_u^t e^{(\eta\mu - \frac{\sigma^2}{2})s + \sigma c(s)} ds}$$

Finally replacing x_t with P_t and $c(t)$ with Z_t we get

$$P_t = \frac{P_u}{e^{-(\eta\mu - \frac{\sigma^2}{2})(t-u) - \sigma(Z_t - Z_u)} + \eta P_u e^{-(\eta\mu - \frac{\sigma^2}{2})t - \sigma Z_t} \int_u^t e^{(\eta\mu - \frac{\sigma^2}{2})s + \sigma Z_s} ds} \ . \tag{2.1.38}$$

2.1.7 Model 7

$$dP = \eta P(\mu - P)dt + \sigma P^{\frac{3}{2}} dZ \tag{2.1.39}$$

Model 7 is a mean reverting process with a similar drift term to Model 5 and Model 6. However, in Model 5 the volatility of absolute changes in price is constant and in Model 6 the volatility of percentage changes in price is a constant(σ), while in Model 7 the volatility of percentage changes in price , $\sigma P^{\frac{1}{2}}$, is an increasing function of P.

Analytic Expression for P_t

Letting $f = \frac{1}{P}$, and then applying Itô's Lemma (see Section 1.1.4) we get

$$df = \alpha(\beta - f)dt + \bar{\sigma}\sqrt{f} dZ \tag{2.1.40}$$

where $\alpha = \eta\mu$, $\beta = \frac{2\eta - \sigma^2}{2\eta\mu}$ and $\bar{\sigma} = -\sigma$. Equation (2.1.40) is known as the Cox-Ingersoll-Ross (CIR) model (Cox *et al* (1985)) which is often used to model interest rates. It does not have a closed form expression for f. However, it can be shown that if $\alpha\beta \geq \frac{\bar{\sigma}^2}{2}$ then if the price $f \geq 0$, then all future prices will remain positive.

Statistical Properties

In this part we will prove that if $f = f_t = \dfrac{1}{P_t}$ follows the CIR model (2.1.40), then its conditional expectation and conditional variance are given respectively by

$$(i) \qquad \mathbb{E}_u(f_t) = f_u e^{-\alpha(t-u)} + \beta(1 - e^{-\alpha(t-u)}) \tag{2.1.41a}$$

$$(ii) \qquad Var_u(f_t) = f_u \frac{\bar{\sigma}^2(e^{-\alpha(t-u)} - e^{-2\alpha(t-u)})}{\alpha} + \frac{\beta\bar{\sigma}^2(1 - e^{-\alpha(t-u)})^2}{2\alpha} \tag{2.1.41b}$$

For (i), on integrating (2.1.40) from 0 to t we get:

$$f_t = f_0 + \alpha \int_0^t (\beta - f_s)ds + \bar{\sigma} \int_0^t \sqrt{f_s} dZ_s \ . \tag{2.1.42}$$

Then, the unconditional mean of (2.1.42) is

$$\mathbb{E}(f_t) = f_0 + \alpha\beta t - \alpha \int_0^t \mathbb{E}(f_s)ds \tag{2.1.43}$$

Letting $\varphi_1(t) = \mathbb{E}(f_t)$ then from (2.1.43) we get the equation $\varphi_1(t) = f_0 + \beta\alpha t - \alpha \int_0^t \varphi_1(s)\ ds$, which on differentiating we get $\varphi_1'(t) = \alpha\beta - \alpha\varphi_1(t)$. Solving this subject to the initial condition $\varphi_1(0) = f_0$ gives the unconditional mean as

$$\mathbb{E}(f_t) = \beta + (f_0 - \beta)e^{-\alpha t} \ , \tag{2.1.44a}$$

or more gernerally for $u \leq t$, we get

$$\mathbb{E}_u(f_t) = f_u e^{-\alpha(t-u)} + \beta(1 - e^{-\alpha(t-u)}) \ . \tag{2.1.44b}$$

For (ii), we apply Itô's Lemma (see Section 1.1.4) to find $d(f^2)$

$$d(f^2) = [(2\alpha\beta + \sigma^2)f - 2\alpha f^2]dt + 2\bar{\sigma} f^{\frac{3}{2}} dZ \ . \tag{2.1.45}$$

Now integrating (2.1.45) from 0 to t we get:

$$f_t^2 = f_0^2 + (2\alpha\beta + \bar{\sigma}^2)\int_0^t f_s ds - 2\alpha\int_0^t f_s^2 ds + 2\bar{\sigma}\int_0^t f_s^{\frac{3}{2}} dZ_s \ . \tag{2.1.46}$$

The unconditional mean of f_t^2 can be found by substituting (2.1.44a) into (2.1.46), and taking the expected value, giving

$$\mathbb{E}(f_t^2) = f_0^2 + (2\alpha\beta + \bar{\sigma}^2)\int_0^t (\beta + (f_0 - \beta)e^{-\alpha s})ds - 2\alpha\int_0^t \mathbb{E}(f_s^2)ds \tag{2.1.47}$$

Letting $\varphi_2(t) = \mathbb{E}(f_t^2)$ equation (2.1.47) becomes

$$\varphi_2(t) = f_0^2 + (2\alpha\beta + \bar{\sigma}^2)\int_0^t (\beta + (f_0 - \beta)e^{-\alpha s})ds - 2\alpha\int_0^t \varphi_2(s)ds \ . \tag{2.1.48}$$

Differentiating (2.1.48) with respect to t we get

$$\varphi_2'(t) = (2\alpha\beta + \bar{\sigma}^2)(\beta + (f_0 - \beta)e^{-\alpha t}) - 2\alpha\varphi_2(t) \ ,$$

which needs to be solved subject to the initial condition $\varphi_2(0) = f_0^2$. The solution of $\varphi_2(t)$ is given by:

$$\varphi_2(t) = \frac{(\alpha\beta^2 - 2\alpha\beta f_0 + \beta\bar{\sigma}^2 - \bar{\sigma}^2 f_0 + \alpha f_0^2)e^{-2\alpha t}}{\alpha} + \frac{(2\alpha\beta + \bar{\sigma}^2)(\beta e^{\alpha t} + 2(f_0 - \beta))e^{-\alpha t}}{2\alpha} \ . \tag{2.1.49}$$

Now, we can use (2.1.44a) and (2.1.49) to obtain the variance as follows:

$$Var(f_t) = \mathbb{E}(f_t^2) - (\mathbb{E}(f_t))^2 = \frac{\bar{\sigma}^2}{\alpha}(1 - e^{-\alpha t})(f_0 e^{-\alpha t} + \frac{\alpha}{2}(1 - e^{-\alpha t})) \ .$$

or more gernerally for $u \leq t$, we find

$$Var_u(f_t) = f_u \frac{\bar{\sigma}^2(e^{-\alpha(t-u)} - e^{-2\alpha(t-u)})}{\alpha} + \frac{\beta\bar{\sigma}^2(1 - e^{-\alpha(t-u)})^2}{2\alpha} \ . \tag{2.1.50}$$

2.1.8 Model 8

$$dP = aPdt + \sigma P^{\frac{3}{4}} dZ \tag{2.1.51}$$

Model 8 has a similar drift term to the GBM model, in which the expected rate of return for the spot price is constant (a). However, in the GBM model the volatility of percentage changes in price is constant (σ), while in Model 8 the volatility of percentage changes in price, $\sigma P^{\frac{-1}{4}}$, is a decreasing function of P.

Analytic Expression for P_t

Letting $f = \sqrt{P}$, and then applying Itô's Lemma (see Section 1.1.4) we get

$$df = \alpha(\beta - f)dt + \bar{\sigma}\sqrt{f}dZ, \tag{2.1.52}$$

where $\alpha = \frac{-a}{2}$, $\beta = \frac{\sigma^2}{2a}$ and $\bar{\sigma} = \frac{\sigma}{2}$. Hence, Model 8 is reduced to the CIR model which does not have a known closed form.

Statistical Properties

We can use equations (2.1.41a) and (2.1.41b) to find the expected value and variance respectively of $f = \sqrt{P}$ respectively as

$$\forall u \leq t, \quad \mathbb{E}_u(f_t) = f_u e^{-\alpha(t-u)} + \beta(1 - e^{-\alpha(t-u)}) \tag{2.1.53}$$

$$Var_u(f_t) = f_u \frac{\bar{\sigma}^2(e^{-\alpha(t-u)} - e^{-2\alpha(t-u)})}{\alpha} + \frac{\beta\bar{\sigma}^2(1 - e^{-\alpha(t-u)})^2}{2\alpha} \,. \tag{2.1.54}$$

2.1.9 Model 9

$$dP = a\sqrt{P}dt + \sigma P^{\frac{3}{4}} dZ \tag{2.1.55}$$

Model 9 is non-mean-reverting and expects prices to grow in absolute terms in proportion to $\sqrt{P}$. The expected percentage change in prices, $aP^{\frac{-1}{2}}$, and the volatility of percentage change of prices, $\sigma P^{\frac{-1}{4}}$, are decreasing functions of P.

The transition density function of P that follows (2.1.55) is given by

$$Tr(P,t;P_T,T) = \frac{4\sqrt{P}e^{\frac{-8(\sqrt{P_T}+\sqrt{P_t})}{\sigma^2(T-t)}} I_\nu(z) \left(\frac{P_T}{P}\right)^{\frac{a}{\sigma^2}}}{P_T\sigma^2(T-t)} \tag{2.1.56a}$$

$$\text{where } \nu = \frac{4a}{\sigma^2} - 2, \; z = \frac{16(P_T P)^{\frac{1}{4}}}{\sigma^2(T-t)}$$

(see Goard(2006)) where P_T is the price at a future time T, and where $I_\nu(\cdot)$ is the modified Bessel function of the first kind of order ν (see e.g. Abramowitz and Stegun (1964)).

2.1.10 Model 10

$$dP = (a\sqrt{P} + bP)dt + \sigma P^{\frac{3}{4}} dZ, \quad b < 0 \tag{2.1.57}$$

Model 10 is a mean reverting process, that assumes that the spot price reverts to a constant $\frac{a^2}{b^2}$. The rate of reversion is determined by $\mid b \mid \sqrt{P}$. As with Model 9, the volatility of percentage changes in price, $\sigma P^{\frac{-1}{4}}$, is a decreasing function of P.

The transition density function of P that follows (2.1.57) is given by

$$Tr(P,t;P_T,T) = Q_1(P;P_T)\ Q_2(P;P_T)\ P_T^{\frac{a}{c^2}-1}\ P^{\frac{1}{2}-\frac{a}{c^2}}\ I_\nu(z) \tag{2.1.58a}$$

$$\text{where } Q_1(P;P_T) = \frac{2\sqrt{b^2}e^{\frac{2(\sqrt{P}-\sqrt{P_T})(\sqrt{b^2}-b)}{\sigma^2}+\frac{T-t}{4}(\sqrt{b^2}+b-\frac{4ab}{\sigma^2})}}{\sigma^2(e^{\frac{\sqrt{b^2}(T-t)}{2}}-1)}, \tag{2.1.58b}$$

$$Q_2(P;P_T) = \exp\left\{\frac{-4\sqrt{b^2}(\sqrt{P}e^{\frac{\sqrt{b^2}(T-t)}{2}}+\sqrt{P_T})}{\sigma^2(e^{\frac{\sqrt{b^2}(T-t)}{2}}-1)}\right\} \tag{2.1.58c}$$

$$\nu = \frac{4a}{c^2}-2,\ z = \frac{8(P_TP)^{\frac{1}{4}}\sqrt{b^2}e^{\frac{\sqrt{b^2}(T-t)}{4}}}{\sigma^2(e^{\frac{\sqrt{b^2}(T-t)}{2}}-1)} \tag{2.1.58d}$$

(see Goard(2006)) where P_T is the price at a future time T, and where $I_\nu(\cdot)$ is the modified Bessel function of the first kind of order ν (see e.g. Abramowitz and Stegun (1964)).

Analytic Expression for P_t

Letting $f = \sqrt{P}$, and then applying Itô's Lemma (see Section 1.1.4) we get

$$df = \alpha(\beta - f)dt + \bar{\sigma}\sqrt{f}dZ\ , \tag{2.1.59}$$

where $\alpha = \frac{-b}{2}$, $\beta = \frac{\sigma^2-4a}{4b}$ and $\bar{\sigma} = \frac{\sigma}{2}$. Hence Model 10 is reduced to the CIR model, which does not have a known closed form.

Statistical Properties

We can use equations (2.1.41a) and (2.1.41b) to find the expected value and variance respectively of $f = \sqrt{P}$:

$$\forall u \leq t,\ \ \mathbb{E}_u(f_t) = f_u e^{-\alpha(t-u)} + \beta(1 - e^{-\alpha(t-u)}) \tag{2.1.60}$$

$$Var_u(f_t) = f_u \frac{\bar{\sigma}^2(e^{-\alpha(t-u)} - e^{-2\alpha(t-u)})}{\alpha} + \frac{\beta\bar{\sigma}^2(1 - e^{-\alpha(t-u)})^2}{2\alpha}\ . \tag{2.1.61}$$

2.2 The Generalized Method of Moments

In 1982, Hansen (1982) introduced the modern form of the Generalized Method of Moments (GMM) as an extension of the work by Pearson (1936) who introduced the Method of Moments (MOM). GMM has become one of the most widely used methods of estimation for single factor random walk models in finance and economics. This method requires specified population moment conditions and does not require knowledge of the distribution of the data. An important feature of GMM is that it provides a way to test the specification of a proposed model, when the number of population moment conditions is higher than the number of parameters.

2.2.1 Estimation via GMM

GMM is a methodology that utilizes known moment conditions to estimate the unknown parameters of a specified model. The number of the unknown parameters must be lower than or equal to the number of the moment conditions. Consider a regression model with n observations and p independent variables of the form

$$\underset{\sim}{y} = f(\underset{\sim}{x}; \theta) + \underset{\sim}{\varepsilon} \tag{2.2.62}$$

where $\underset{\sim}{y}$ is a vector of n observations of endogenous[1] variables, f is the model function, $\underset{\sim}{x}$ is an $n \times p$ matrix of the exogenous[2] variables, θ is a vector of the unknown population parameters of order p and $\underset{\sim}{\varepsilon}$ is a vector of random error terms. GMM aims to estimate θ based on a number of specified population moment conditions, which can be prior information about the population. For a given population, suppose that $g(y_t;\theta)$ is a vector, of order j such that

$$\mathbb{E}[g(y_t;\theta)] = 0 \ , \ \forall t \ , \tag{2.2.63}$$

where y_t refers to the t^{th} element in $\underset{\sim}{y}$. Under the assumption that the specified population moment conditions are true, the expectation in (2.2.63) holds. The technique of GMM replaces (2.2.63) with its sample counterpart, which is given by

$$m(\theta) = \frac{1}{n}\sum_{t=1}^{n} g(y_t;\theta) \tag{2.2.64}$$

and then estimates the parameters in the vector θ that minimize the quadratic form

$$q(\hat{\theta}) = m(\hat{\theta})^T W m(\hat{\theta}) \tag{2.2.65}$$

where W is a positive definite, weighting matrix with the sample estimate adjusted for serial correlation and heteroscedasticity using the method of Newey and West (1987) with Bartlett weights. Hansen(1982) found that the inverse of the covariance matrix of the moment conditions, $\left[Var(m(\hat{\theta}))\right]^{-1}$ was the optimal choice for the weighting matrix, W. If the number of the unknown parameters is equal to the the number of specified population moment conditions, $p = j$, the system is exactly identified. This implies that the choice of W is irrelevant. In other words, there is a unique solution for θ for any choice of W. However, if the number of the unknown

[1]A variable in a model whose value is determined by the states of other variables in the model.
[2]A variable in a model whose value is independent from the states of other variables in the model.

parameters is lower than the number of specified population moment conditions, $p < j$, the system is over-identified i.e. there is no solution for θ. However, to minimize q we can differentiate $q(\hat{\theta})$ with respect to $\hat{\theta}$, and then set the derivative equal to zero and solve for $\hat{\theta}$,

$$\begin{aligned}
\frac{\partial q(\hat{\theta})}{\partial \hat{\theta}} &= \frac{\partial}{\partial \hat{\theta}}[m(\hat{\theta})^T W m(\hat{\theta})] \\
&= 2[\frac{\partial m(\hat{\theta})^T}{\partial \hat{\theta}}]^{-1} W m(\hat{\theta}) \\
&= 2G^T W m(\hat{\theta})
\end{aligned}$$

where G is the Jacobian matrix and is given by

$$G = \begin{bmatrix} \frac{\partial m_1}{\partial \theta_1} & \frac{\partial m_2}{\partial \theta_1} & \cdots & \frac{\partial m_j}{\partial \theta_1} \\ \frac{\partial m_1}{\partial \theta_2} & \frac{\partial m_2}{\partial \theta_2} & \cdots & \frac{\partial m_j}{\partial \theta_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial m_1}{\partial \theta_p} & \frac{\partial m_2}{\partial \theta_p} & \cdots & \frac{\partial m_j}{\partial \theta_p} \end{bmatrix} .$$

Since the population moment conditions are continuous functions of θ and the estimated parameters vector are consistent then

$$G(\hat{\theta}) \to G(\theta) \ .$$

According to the Slutsky's Theorem and Central Limit Theorem (see e.g. Gut (2005))

$$\hat{\theta} \to N(\theta, \Sigma)$$

where Σ is the asymptotic covariance matrix of the GMM estimator of θ, given by

$$\Sigma = [G^T W G]^{-1} \ .$$

2.2.2 Algorithm for Computing the Weighting Matrix W

As mentioned in Section 2.2.1, if the number of the unknown parameters is equal to the number of the population moment conditions, then the system is exactly identified and the choice of the weight matrix, W, is irrelevant. In contrast, if the number of the unknown parameters is lower than the number of the population moment conditions, then the system is over-identified and the choice of the weight matrix, W, affects the efficiency of the estimated vector θ. GMM uses the following iterative algorithm to estimate the covariance matrix of population moment conditions, W^{-1}.

1. Assume $W = I = A$ and estimate the model via GMM. This assumption provides a consistent but inefficient estimation of θ.

2. By using the estimated vector θ, calculate the residuals for each observation and moment condition.

3. Calculate the covariance matrix $\hat{A}$.

4. Using $W = \hat{A}^{-1}$, estimate the vector θ again via GMM.

5. Repeat steps 2 through 4 until $\hat{A}$ converges.

2.2.3 Testing the Validity of the Particular Models

Suppose that we have a general model (or unrestricted model), and other particular models as special cases of the general model. The particular models can be differentiated from the general model by assuming one or more of the parameters in the general model to be relevant constants. The aim of this section is to test whether the general model (with no restrictions) describes the data, and whether the assumed restrictions on the particular models are reasonable. As mentioned in Section 2.2.1, if the number of the unknown parameters is equal to the the number of moment conditions, $p = j$, then the expression

$$\min_{\theta} \quad q(\hat{\theta}) = m(\hat{\theta})^T W m(\hat{\theta})$$

will have a unique solution, which is zero. However, if we impose a restriction on the parameter θ_1, say $\theta_1 = 0$, then as θ_1 is known, the system is over-identified and q will not reach zero. For the restricted model (when $\theta_1 = 0$), if the value of q is significantly large, this indicates that the parameter restriction is unreasonable. The validity of the imposed parameter restriction can be tested through the following null hypothesis test.

$H_0: \quad R(\theta) = R(\theta_1)$ (i.e. the model is valid)

$H_1: \quad R(\theta) \neq R(\theta_1)$ (i.e. the model is invalid)

where $R(\theta)$ is a function of the true parameters, and $R(\theta_1)$ is a vector of constants. In our case, when $\theta_1 = 0$, $R(\theta)$ and $R(\theta_1)$ are given by

$$\begin{aligned} R(\theta) =& [\theta_1 \ \theta_2 \ \cdots \ \theta_p] \\ R(\theta_1) =& [0 \ \theta_2 \ \cdots \ \theta_p] \ . \end{aligned}$$

Now suppose $\hat{\theta}$ is the estimated value of the parameter vector θ of the general (unrestricted) model, and let $\hat{\theta_0}$ be the estimated value of the parameter vector of the particular (restricted) model. If the estimated value of q is far away from zero, then the null hypothesis must be rejected. Mullin (2009) indicates that "the metric that determines what is sufficiently far away is determined by the distribution of the deviation in the random samples". Therefore, we need to derive the distribution of $q(\hat{\theta})$ starting with the derivation of the distribution of $q(\theta)$. With $W = A^{-1}$, $q(\theta)$ can be written as

$$q(\theta) = \min_{\theta} \quad \left(m(\theta)^T A^{\frac{-1}{2}} A^{\frac{-1}{2}} m(\theta) \right)$$

so that

$$q(\theta) = \min_{\theta} \quad \left((A^{-\frac{1}{2}}m(\theta)\right)^T \left(A^{-\frac{1}{2}}m(\theta))\right) .$$

Following Mullin (2000), the distribution of the sample moment conditions is

$$\begin{aligned} \sqrt{n}m(\theta) &\sim N(0, A) \\ \text{so} \quad \sqrt{n}(A^{-\frac{1}{2}}m(\theta)) &\sim N(0, I) . \end{aligned}$$

So the transformed moment conditions are independently distributed standard normal random variable. As the sum of such independently distributed squared standard normal random variables is a Chi-square variable we have

$$q(\hat{\theta}) = (\sqrt{n}A^{-\frac{1}{2}}m(\hat{\theta}))^T(\sqrt{n}A^{-\frac{1}{2}}m(\hat{\theta})) \sim \chi^2_j .$$

This is the distribution of $q(\hat{\theta})$, where j is the number of given moment conditions. If there are k free parameters to be estimated, each parameter to be estimated uses one degree of freedom. Now for each particular (restricted) model, we can check whether the nested model's moment conditions match the data or not. This check can be done by using the over-identification test developed by Newey and West (1987), with the D statistic

$$D = n\Bigg(q(\hat{\theta}_0) - q(\hat{\theta})\Bigg) \sim \chi^2_{j-k} .$$

where $j - k$ is the number of the known parameters in the particular (restricted) model. If $\chi^2_{j-k;\alpha}$ is lower than the calculated D, then we should reject the null hypothesis. Therefore, it can be concluded that the restricted model is invalid (i.e. the restriction is unreasonable) at $100(1-\alpha)\%$ level of significance. In other words, if the p-value is lower than the required level of significance α, then it can be concluded that the restricted model is invalid at $100\alpha\%$ level of significance.

2.3 Comparison of One Factor Models for The Spot Price of Brent Crude Oil.

The aim of this section is to compare the performance of the stochastic models listed in Table 2.1 in their ability to capture the behaviour of Brent crude oil prices. We use the GMM for this analysis. This section is divided into three parts. In the first part, we establish a general unrestricted model for the GMM in which all our models can be embedded and note the required restrictions for each particular model. In the second part, we describe the data sets used in the GMM analysis and finally in the third part the results are presented.

2.3.1 Establishment of The General Model and The Required Restrictions for The Particular Models

Each of the models in Table 2.1 can be nested within the general (unrestricted) model

$$dP = (C_1 + C_2 P + C_3 P \ln P + C_4 P^2 + C_5 \sqrt{P})dt + \sigma P^{\gamma} dZ \tag{2.3.66}$$

where C_1, C_2, C_3, C_4, C_5 and γ are constants. The particular models can be obtained by setting relevant constants in (2.3.66) to zero (Table 2.2 shows the parameter restrictions). The Generalized Method of Moments is used to estimate the parameters of the continuous-time model (2.3.66) by using the corresponding discrete-time econometric specification:

$$P_{t+1} - P_t = C_1 + C_2 P_t + C_3 P_t \ln P_t + C_4 P_t^2 + C_5 \sqrt{P_t} + \varepsilon_{t+1} \tag{2.3.67a}$$

$$\mathbb{E}(\varepsilon_{t+1}) = 0 \tag{2.3.67b}$$

$$\mathbb{E}(\varepsilon_{t+1}^2) = \sigma^2 P^{2\gamma} \Delta t \ . \tag{2.3.67c}$$

Model	C_1	C_2	C_3	C_4	C_5	γ
1	0		0	0	0	1
2			0	0	0	0
3			0	0	0	1
4	0			0	0	1
5	0		0		0	0
6	0		0		0	1
7	0		0		0	1.5
8	0		0	0	0	0.75
9	0	0	0	0		0.75
10	0		0	0		0.75

Table 2.2: The parameter restrictions

We let θ be the parameter vector with elements $C_1, C_2, C_3, C_4, C_5, \sigma$ and γ, and let $f_t(\theta)$ be the vector:

$$f_t(\theta) = \begin{bmatrix} \varepsilon_{t+1} \otimes [1, P_t, P_t \ln P_t, P_t^2, \sqrt{P_t}]^T \\ (\varepsilon_{t+1}^2 - \sigma^2 P^{2\gamma} \Delta t) \otimes [1, P_t]^T \end{bmatrix} \tag{2.3.68}$$

As mentioned in Section 2.2.1, under the null-hypothesis that the moment conditions in (2.3.67) are true, $E[f_t(\theta)] = 0$. GMM replaces $E[f_t(\theta)]$ with its sample counterpart, $g(\theta)$, using n observations where

$$g(\theta) = \frac{1}{n} \sum_{t=1}^{n} f_t(\theta) \ ,$$

and GMM chooses θ that minimizes $g(\theta)^T W g(\theta)$, where W is a positive definite weighting matrix, as described in Section 2.2.2.

2.3.2 The Data

Brent crude oil prices between the years of 1987 and 2011 collected from the U.S. Energy Information Administration were used in our GMM analysis. The prices are plotted in Figure 2.1. From this figure, it can be seen that the behaviour of oil prices differed significantly in the period 1987-2000 to that in the period 2001-2011.

In the first period (1987-2000), oil prices exhibited constant fluctuations about an apparently stationary mean of approximately US $20 per barrel, indicating a mean-reverting nature, while in the second period (2001-2011) which includes the Global Financial Crisis (GFC) period of 2007/2008, oil prices generally appear to have a steady increasing trend. Table 2.3 presents the standard statistics for Brent crude oil prices.

Our data is divided into four sets:

- Data set 1: from 1987 to 2011

  This set includes all the available data of the Brent crude oil spot prices at the time of our analysis.

- Data set 2: from 1991 to 2011

  This set starts from 1991 to avoid the price spike that resulted from the Gulf War.

- Data set 3: from 2000 to 2011

  This set includes data from the 21^{st} century, a period in which steady growth in prices is observed (apart from the GFC period), in contrast to the mean-reverting character evidenced in the 90's.

- Data set 4: from 2005 to 2011

  Our shortest data set contains six years of recent data. It avoids the price spike that resulted from the 9/11 attacks and the 2003 invasion of Iraq, but includes the GFC period.

2.3.3 The Results

The results of our GMM analysis are listed in Tables 2.4-2.7. From these tables we note that

Standard Statistic \ Period	1987-2000	2000-2011	1987-2011
Mean	18.85	55.79	34.77
Standard deviation	4.82	27.16	25.80
Minimum	9.10	16.51	9.10
Maximum	41.45	143.95	143.95

Table 2.3: Standard Statistics of Brent crude oil prices

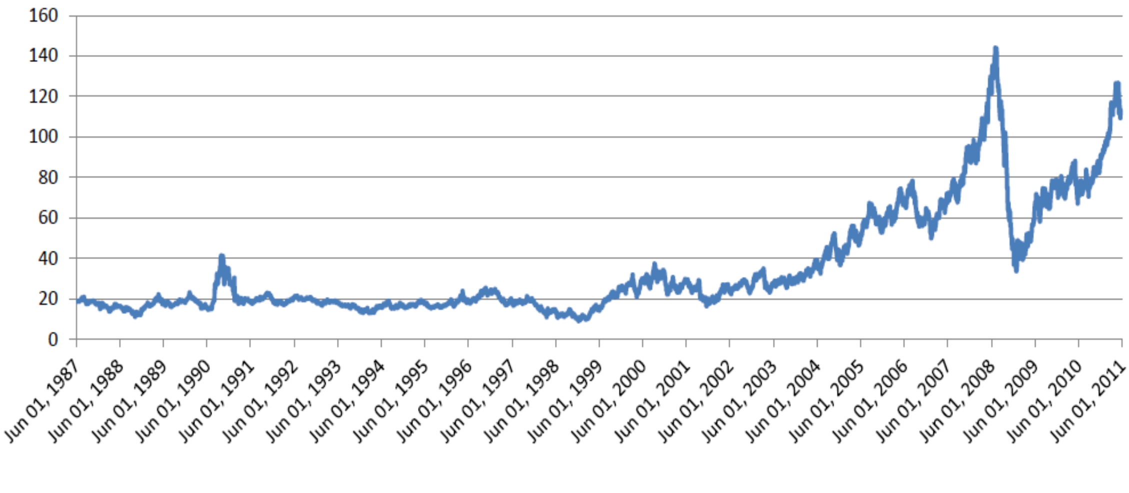


Figure 2.1: Brent Crude Oil Spot Prices 1987-2012

- The χ^2 values for the models with $\gamma = 0, 1.5$ imply that these models are rejected at the 1% level of significance. Therefore, these models are misspecified, at the 1% level of significance, in terms of their over-identifying restrictions.

- The χ^2 values for the models with $\gamma = 1$ imply that these models are generally not rejected at the 10% level of significance. Therefore, these models are specified (at the 10% level of significance) in terms of their over-identifying restrictions.

- The χ^2 values for the models with $\gamma = 0.75$ imply that these models are generally not rejected at the 20% level of significance (with the exception of data set 2, although in data set 2 these models are not rejected at the 7% level of significance). Therefore, these models are specified (at the 20% level of significance in three of the four data sets and at the 7% level of significance in one data set) in terms of their over-identifying restrictions.

- In the unrestricted models for all the data sets, the parameters σ and γ are statistically significantly different from zero. However, only in data set 4 are any of the other parameters statistically significantly different from zero in the unrestricted models. These are the parameters C_2, C_3 and C_4 which are not statistically significantly different from zero in any nested model where they are not set to zero.

- With the exception of data set 2 the highest p-values of the over-identification tests were achieved by models with $\gamma = 0.75$ (with model 9 attaining the highest). In data set 2, the highest p-values of the over-identification tests were achieved by the models with $\gamma = 1$ (with model 1 attaining the highest).

From the above we can conclude that the value of γ is the most important parameter differentiating the models. Further, models with $\gamma = 0, 1.5$ are rejected at the 1% level of significance, while models with $\gamma = 1$ or 0.75 are acceptable to describe the behaviour of Brent crude oil prices at the standard levels of significance (1% and 5%) and the three-quarters models are acceptable at even higher levels of significance.

Model	C_1	C_2	C_3	C_4	C_5	σ	γ	χ^2	DF
Unrestricted	-403.66 [0.599]	-141.087 [0.560]	25.305 [0.552]	-0.105 [0.514]	392.376 [0.574]	0.651 < [0.001]	0.863 <[0.001]	N/A	N/A
1	0	0.172 [0.020]	0	0	0	0.362 < [0.001]	1	8.890 [0.114]	5
2	2.879 [0.423]	0.024 [0.887]	0	0	0	11.468 < [0.001]	0	141.498 < [0.001]	4
3	0.784 [0.827]	0.138 [0.413]	0	0	0	0.362 < [0.001]	1	8.843 [0.065]	4
4	0	0.226 [0.555]	-0.016 [0.885]	0	0	0.362 < [0.001]	1	8.869 [0.064]	4
5	0	0.248 [0.040]	0	-0.003 [0.306]	0	11.464 < [0.001]	0	140.850 < [0.001]	4
6	0	0.178 [0.140]	0	-0.0002 [0.944]	0	0.362 < [0.001]	1	8.885 [0.064]	4
7	0	0.156 [0.197]	0	0.002 [0.467]	0	0.040 < [0.001]	1.5	122.032 < [0.001]	4
8	0	0.149 [0.045]	0	0	0	0.963 < [0.001]	0.75	6.059 [0.300]	5
9	0	0	0	0	0.753 [0.036]	0.964 < [0.001]	0.75	5.755 [0.331]	5
10	0	0.007 [0.978]	0	0	0.718 [0.585]	0.964 < [0.001]	0.75	5.755 [0.218]	4

Table 2.4: Parameter estimates for Models 1-10 from nesting the models within (2.3.66) with Data set 1.

Model	C_1	C_2	C_3	C_4	C_5	σ	γ	χ^2	DF
Unrestricted	-387.39 [0.629]	-131.055 [0.603]	23.420 [0.597]	-0.966 [0.563]	368.44 [0.613]	0.525 $<$ [0.001]	0.899 $<$[0.001]	N/A	N/A
1	0	0.163 [0.036]	0	0	0	-0.353 $<$ [0.001]	1	3.694 [0.594]	5
2	2.271 [0.557]	0.052 [0.765]	0	0	0	-11.692 $<$ [0.001]	0	134.240 $<$ [0.001]	4
3	0.458 [0.905]	0.145 [0.395]	0	0	0	-0.353 $<$ [0.001]	1	3.680 [0.451]	4
4	0	0.197 [0.905]	-0.0098 [0.934]	0	0	-0.353 $<$ [0.001]	1	3.688 [0.450]	4
5	0	0.249 [0.058]	0	-0.0026 [0.337]	0	-11.700 $<$ [0.001]	0	133.424 $<$ [0.001]	4
6	0	0.170 [0.195]	0	-0.0002 [0.949]	0	0.353 $<$ [0.001]	1	3.690 [0.450]	4
7	0	0.063 [0.630]	0	0.003 [0.236]	0	-0.041 $<$ [0.001]	1.5	106.755 $<$ [0.001]	4
8	0	0.152 [0.052]	0	0	0	-0.962 $<$ [0.001]	0.75	8.694 [0.122]	5
9	0	0	0	0	0.781 [0.046]	-0.932 $<$ [0.001]	0.75	8.553 [0.128]	5
10	0	0.044 [0.872]	0	0	0.564 [0.687]	-0.931 $<$ [0.001]	0.75	8.526 [0.074]	4

Table 2.5: Parameter estimates for Models 1-10 from nesting the models within (2.3.66) with Data set 2.

Model	C_1	C_2	C_3	C_4	C_5	σ	γ	χ^2	DF
Unrestricted	-1661.77 [0.579]	-440.775 [0.547]	75.172 [0.542]	-0.245 [0.514]	1359.32 [0.559]	0.795 < [0.001]	0.806 <[0.001]	N/A	N/A
1	0	0.217 [0.022]	0	0	0	-0.357 < [0.001]	1	7.781 [0.169]	5
2	12.244 [0.216]	-0.083 [0.746]	0	0	0	-17.077 < [0.001]	0	75.408 < [0.001]	4
3	7.866 [0.432]	0.046 [0.861]	0	0	0	-0.357 < [0.001]	1	7.112 [0.130]	4
4	0	0.826 [0.336]	0.154 [0.471]	0	0	-0.357 < [0.001]	1	7.283 [0.122]	4
5	0	0.575 [0.030]	0	-0.006 [0.120]	0	-17.092 < [0.001]	0	73.995 < [0.001]	4
6	0	0.353 [0.180]	0	-0.002 [0.577]	0	-0.357 < [0.001]	1	7.464 [0.133]	4
7	0	0.166 [0.528]	0	0.0016 [0.687]	0	-0.040 < [0.001]	1.5	81.835 < [0.001]	4
8	0	0.197 [0.080]	0	0	0	-1.00 < [0.001]	0.75	2.424 [0.778]	5
9	0	0	0	0	1.500 [0.039]	-1.000 < [0.001]	0.75	1.654 [0.859]	5
10	0	-0.229 [0.619]	0	0	3.011 [0.322]	-1.000 < [0.001]	0.75	1.370 [0.849]	4

Table 2.6: Parameter estimates for Models 1-10 from nesting the models within (2.3.66) with Data set 3.

Model	C_1	C_2	C_3	C_4	C_5	σ	γ	χ^2	DF
Unrestricted	-1491.6 [0.246]	-941.17 <[0.001]	167.66 <[0.001]	-0.602 [0.005]	2445.75 [0.423]	1.087 < [0.123]	0.737 <[0.001]	N/A	N/A
1	0	0.171 [1.00]	0	0	0	-0.35747 < [0.001]	1	7.210 [0.206]	5
2	47.001 [0.129]	-0.542 [0.028]	0	0	0	-23.856 < [0.001]	0	22.680 < [0.001]	4
3	31.999 [0.655]	-0.276 [0.841]	0	0	0	-0.347 < [0.001]	1	6.654 [0.155]	4
4	0	1.582 [0.455]	0.328 [0.456]	0	0	-0.347 < [0.001]	1	6.895 [0.142]	4
5	0	0.700 [0.268]	0	-0.008 [0.500]	0	23.850 < [0.001]	0	22.615 < [0.001]	4
6	0	0.397 [0.427]	0	-0.003 [1.00]	0	-0.347 < [0.001]	1	7.045 [0.134]	4
7	0	0.238 [0.806]	0	0.0001 [0.996]	0	0.038 < [0.001]	1.5	39.078 < [0.001]	4
8	0	0.142 [0.814]	0	0	0	-1.022 < [0.001]	0.75	4.214 [0.519]	5
9	0	0	0	0	1.303 [0.831]	1.022 < [0.001]	0.75	4.032 [0.545]	5
10	0	-0.766 [1.00]	0	0	7.761 [1.00]	1.021 < [0.001]	0.75	3.606 [0.462]	4

Table 2.7: Parameter estimates for Models 1-10 from nesting the models within (2.3.66) with Data set 4.

2.4 Conclusion

In this chapter, some of the most popular one-factor models found in the literature are reviewed and new models with a diffusion term of the form $\sigma P^{\frac{3}{4}}$, are proposed. Then, the GMM technique was used to compare the performance of various one-factor stochastic models in their ability to capture the behaviour of Brent crude oil prices. The results of the GMM analysis showed that the three-quarters models had the highest p-values of the over-identification tests so that the restrictions on these models on the unrestricted model were reasonable.

Chapter 3

Pricing and Calibrating Futures Contracts with One Factor Models

3.1 Introduction

Describing commodity price movements plays a central role in the valuing of futures contracts. Most of the early studies in this area assumed that the behaviour of commodities prices could be described by the GBM. One of the earliest works on pricing futures contracts is due to Brennan and Schwartz (1985). They assumed that the spot prices of commodities followed the GBM, and then established the relationship between the spot and futures prices that incorporated a convenience yield. Similarly, Gabillon (1991) assumed that futures prices depended on the spot price of oil and the cost of carry of the physical oil and then derived a closed-form solution for futures prices. Other authors argue that the effect of supply and demand in the commodity, results in a mean-reversion property for its prices. Consequently, various mean-reverting one-factor models for commodity prices have been proposed. These include for example, those of Dixit and Pindyk (1994) and Schwartz (1997). The model of Schwartz (1997) is one of the most well-known among these models and assumes that the logarithm of the spot price follows the OU process. Schwartz derived a closed-form solution for futures prices under this model. For further details, we refer the reader to Section 1.5.1.

In this chapter we derive and examine futures price under the one-factor oil price models that were identified by the GMM test of the previous chapter as being acceptable. In particular, new and simple formulae for futures price under Model 9 (2.1.55) and Model 10 (2.1.57) will be found. In addition, we will compare the performance of these new formulae against known popular formulae for fitting and forecasting market prices. The remainder of this chapter is organised into three sections. In Section 3.2, we derive the closed forms for the prices of futures contracts under Model 1 (2.1.2), Model 4 (2.1.27), Model 9 (2.1.55) and Model 10 (2.1.57) and describe their features. The results of empirical tests, which compare the performance of the various pricing formulae in their ability to capture market prices, are presented in Section 3.3. In Section 3.4 we present our conclusion.

3.2 Deriving Closed Forms for Futures Contract

In this section we derive four closed-form solutions for the prices of futures contracts, which are associated with the four stochastic models for oil that were deemed acceptable from our GMM analysis in Chapter 2. The first closed form was obtained by Gabillon(1990), who assumed that the spot price of oil followed the GBM. Moreover, he assumed the futures price depended on the spot price of oil, the cost of carry of the physical oil, the convenience yield and the time to maturity of the futures contract. The second futures pricing model based on Model 4 (2.1.27), where the logarithm of the spot price follows a mean-reverting process of the OU type, was derived by Schwartz (1997). The third and fourth futures pricing models are new pricing models derived in this thesis, in which we assume that the spot price of oil follows Model 9 (2.1.55) and Model 10 (2.1.57) respectively.

3.2.1 Gabillon Pricing Model

Here we follow and elaborate the mathematical derivation as given by Gabillon (1990) for futures prices. Gabillon firstly assumes that the spot price of oil follows a general stochastic process

$$dP = \mu(P,t)dt + \sigma(P,t)dZ \ , \tag{3.2.1}$$

where $\mu(P,t)$ and $\sigma(P,t)$ are arbitrary functions of the spot price P and time t. Denoting the futures price, that matures at time T, by $F(P,t)$ the instantaneous change in the futures price, dF, can be found by applying Itô's lemma (see Section 1.1.4) to get

$$dF = [\mu(P,t)\frac{\partial F}{\partial P} + \frac{\partial F}{\partial t} + \frac{\sigma^2(P,t)}{2}\frac{\partial^2 F}{\partial P^2}]dt + \sigma(P,t)\frac{\partial F}{\partial P}dZ \ . \tag{3.2.2}$$

Equation (3.2.2) can be written as

$$dF = A(P,t)dt + B(P,t)dZ \tag{3.2.3a}$$

where

$$A(P,t) = \mu(P,t)\frac{\partial F}{\partial P} + \frac{\partial F}{\partial t} + \frac{\sigma^2(P,t)}{2}\frac{\partial^2 F}{\partial P^2} \tag{3.2.3b}$$

and

$$B(P,t) = \sigma(P,t)\frac{\partial F}{\partial P} \ . \tag{3.2.3c}$$

Now construct a portfolio that consists of one futures contract with value V_1 and maturity T_1 and x futures contracts with value V_2 and maturity T_2. So the portfolio has value

$$\pi_1 = V_1(P, t; T_1) + xV_2(P, t; T_2) \tag{3.2.4}$$

and its instantaneous return is

$$\begin{aligned} d\pi_1 &= dV_1(P, t; T_1) + xdV_2(P, t; T_2) \\ &= (A(P, t; T_1)dt + B(P, t; T_1)dZ) + x\,(A(P, t; T_2)dt + B(P, t; T_2)dZ) \\ &= (A(P, t; T_1) + xA(P, t; T_2))\,dt + x\,(B(P, t; T_1) + B(P, t; T_2))\,dZ. \end{aligned} \tag{3.2.5}$$

The value of x can be chosen to eliminate the risk in the portfolio so that the instantaneous return of the portfolio should equal zero. The zero-risk and zero-return conditions lead to the following system of equations

$$A(P, t; T_1) + xA(P, t; T_2) = 0 \tag{3.2.6a}$$

$$B(P, t; T_1) + xB(P, t; T_2) = 0. \tag{3.2.6b}$$

Hence

$$\frac{A(P, t; T_1)}{A(P, t; T_2)} = \frac{B(P, t; T_1)}{B(P, t; T_2)} \Rightarrow A(P, t; T_1) = \frac{A(P, t; T_2)}{B(P, t; T_2)} B(P, t; T_1)\ . \tag{3.2.7}$$

This implies that for any maturity T

$$A(P, t) = \lambda(P, t)B(P, t) \tag{3.2.8}$$

for some arbitrage function $\lambda(P, t)$, which can be interpreted as the market price of risk (see Section 1.3.4). Now consider another portfolio, with value π_2, that consists of one barrel of oil, with price P and x futures contracts maturing at time T. Hence, the portfolio has value

$$\pi_2 = P + xF(P,t) \ . \tag{3.2.9}$$

Again, the value of x can be chosen to eliminate risk. However, the cost of carrying the physical oil needs to be paid by the holder of this portfolio at a marginal cost C_c. Gabillon assumed that the cost of carry was constant and positive, as the cost of carry would include at least a cost of storage. Moreover, a convenience yield is received by the holder of the portfolio at a marginal cost, C_y, and it is also assumed constant. Under these circumstances the instantaneous change in the value of the portfolio can be written as

$$\begin{aligned} d\pi_2 &= dP + xdF(P,t) \\ &= (\mu(P,t)dt + \sigma(P,t)dZ) + x\,(A(P,t)dt + B(P,t)dZ) + (C_y - C_c)Pdt \\ &= (\mu(P,t) + xA(P,t) + \delta P)\,dt + (\sigma(P,t) + xB(P,t))\,dZ \ . \end{aligned}$$

where $\delta = C_y - C_c$ is often referred to as the net convenience yield. By arbitrage the instantaneous return of the portfolio, π_2, should equal to the risk-free interest rate, r. The zero-risk and r-return conditions lead to the following system of equations

$$\sigma(P,t) + xB(P,t) = 0 \tag{3.2.10a}$$

$$\frac{d\pi_2}{\pi_2} = \frac{\mu(P,t) + \delta P + xA(P,t)}{P} = r \ . \tag{3.2.10b}$$

Note that the denominator in (3.2.10b) is P rather than $P + xF(P,t)$ as it costs nothing to enter into a futures contract. Solving (3.2.10a) and (3.2.10b) with (3.2.8) we get

$$\lambda(P,t) = \frac{\mu(P,t) - (r-\delta)P}{\sigma(P,t)} \ . \tag{3.2.11}$$

Now assume that the distribution of the spot price of oil is lognormal-stationary, so that $\sigma(P,t) = \sigma P$. Then, substituting (3.2.11) into (3.2.8), and using (3.2.3 b,c) we get the PDE for the futures price as

$$(r-\delta)P\frac{\partial F}{\partial P}+\frac{\partial F}{\partial t}+\frac{\sigma^2 P^2}{2}\frac{\partial^2 F}{\partial P^2}=0. \tag{3.2.12}$$

Equation (3.2.12) needs to be solved subject to the final condition $F(P,T)=P$. The required solution is

$$F(P,t)=Pe^{(r-\delta)(T-t)}\ . \tag{3.2.13}$$

Formula (3.2.13) for futures prices has the advantage of being simple, and can describe contango and backwardation states. The value of $r-\delta=r+C_c-C_y$ represents the difference between the cost of carry of oil (physically and financially) and the convenience yield. From the solution (3.2.13) if the cost of carry of oil is lower (greater) than the convenience yield, the market must be in backwardation (contango). This is what we would expect. See Figure 3.1 for a comparison of futures prices when $r-\delta>0$ and $r-\delta<0$. However, the formula (3.2.13) has some obvious shortcomings as noted by Gabillon:

- The behaviour of futures prices of infinite maturity is not in agreement with having a fixed long-term oil price and involves a discontinuity when changing from one state to the other.

$$\begin{aligned}\text{Contango}:\ & r+C_c-C_y>0\Rightarrow\lim_{T-t\to\infty}F(P,t)=\infty\\ \text{Backwardation}:\ & r+C_c-C_y<0\Rightarrow\lim_{T-t\to\infty}F(P,t)=0\ .\end{aligned}$$

- The volatility of the futures price $B(P,t)$ is equal to the volatility of the spot price as

$$B(P,t)=\sigma P\frac{\partial F}{\partial P}=\sigma F.$$

  This is not in agreement with evidence that shows that volatility of futures prices decreases with time.

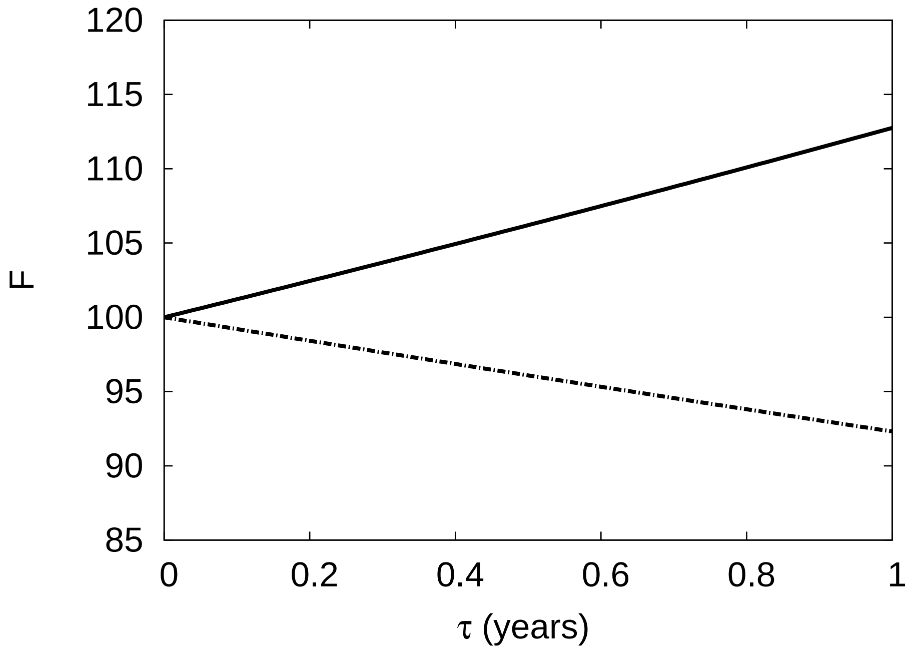


Figure 3.1: Futures prices (F) under Model 1 for various maturities ($\tau = T - t$) with $P_0 = 100$, $r = 0.02$, $\delta = 0.1$ (dashed line) and $\delta = -0.1$ (solid line).

- In reality, the effect of factors that influence futures contracts decreases as maturity increases. However, when in contango the first derivative of the futures prices (3.2.13) with respect to the spot price is always greater than one.

3.2.2 Schwartz Pricing Model (1997)

In the one-factor model considered by Schwartz (1997), the spot price of oil was assumed to follow the mean-reverting stochastic process

$$dP = \eta P(\mu - \ln P)dt + \sigma P dZ. \tag{3.2.14}$$

By letting $X = \ln P$ and applying Itô's Lemma (see Section 1.1.4), the instantaneous change in the log spot price becomes

$$dX = \eta(\alpha - X)dt + \sigma dZ \ , \quad \text{where } \alpha = \mu - \frac{\sigma^2}{2\eta}. \tag{3.2.15}$$

Equation (3.2.15) indicates that the log price follows the OU stochastic process. The parameter α can be interpreted as the long-term mean log price and the speed of reversion to the long-term mean price is determined by the parameter η. The corresponding risk-neutral process is given by

$$dX = \eta(\alpha^* - X)dt + \sigma d\tilde{Z} \ , \tag{3.2.16}$$

where $\alpha^* = \alpha - \frac{\sigma\lambda}{\eta}$, λ is market price of risk (assumed constant) and $\tilde{Z}$ is the equivalent Wiener process under the risk-neutral measure $\mathbb{Q}$ under which X becomes a martingale. Assuming that the interest rate, r, is constant and the oil spot price at time t, P, is known; the conditional mean and variance of $X = \ln P$ at time T under the risk-neutral measure $\mathbb{Q}$ (see Section 2.1.2) are given by

$$\forall\ t \leq T, \quad \mathbb{E}_t^{\mathbb{Q}}(X_T) = X_t e^{-\eta(T-t)} + \alpha^*(1 - e^{-\eta(T-t)}) \ , \tag{3.2.17a}$$

$$Var_t^{\mathbb{Q}}(X_T) = \frac{\sigma^2}{2\eta}(1 - e^{-2\eta(T-t)}) \ . \tag{3.2.17b}$$

Therefore, we can find the price of the futures contract that matures at time T, $F(P,t)$, as the expected price of oil under the risk-neutral measure $\mathbb{Q}$:

$$\begin{aligned} \forall\ t \leq T, \quad F(P,t) &= \mathbb{E}_t^{\mathbb{Q}}(P_T) = \mathbb{E}_t^{\mathbb{Q}}(e^{X_T}) \\ &= \exp\{\mathbb{E}_t^{\mathbb{Q}}(X_T) + \frac{1}{2}Var_t^{\mathbb{Q}}(X_T)\} \\ &= \exp\{e^{-\eta(T-t)}\ln P + \alpha^*(1 - e^{-\eta(T-t)}) + \frac{\sigma^2}{4\eta}(1 - e^{-2\eta(T-t)})\} \ . \end{aligned} \tag{3.2.18}$$

Sample plots of the futures prices (3.2.18) are given in Figure 3.2.

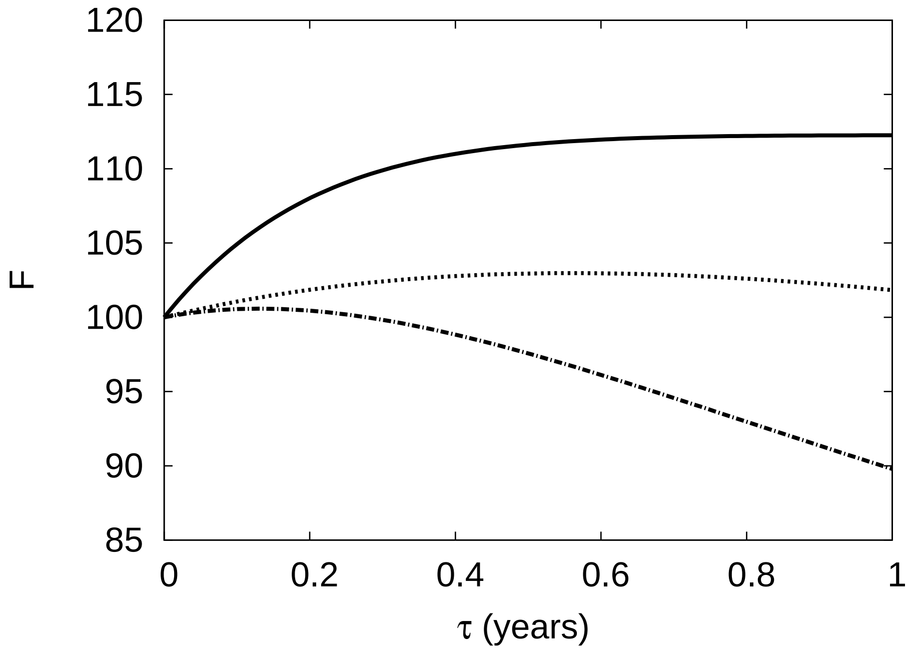


Figure 3.2: Futures prices under (F) Model 4 various maturities $(\tau = T - t)$ with $P_0 = 100$, $r = 0.02$, $(\alpha^* = 4.6,\ \eta = 3.5\ ,\ \sigma = 1.3$ (solid line)), $(\alpha^* = 4,\ \eta = 0.6,\ \sigma = 1$ (dotted line)) and $(\alpha^* = 3.85,\ \eta = 1.1\ ,\ \sigma = 1.4$ (dashed line)).

3.2.3 Futures Prices Under Stochastic Model 9

We now suppose that the spot price of a commodity follows the stochastic process given by Model 9 i.e.

$$dP = a\sqrt{P}dt + \sigma P^{\frac{3}{4}} dZ. \tag{3.2.19}$$

Here the expected rate of return from the spot price of oil, $aP^{-\frac{1}{2}}$ is a decreasing function of P. The volatility of return $\sigma P^{-\frac{1}{4}}$, is also a decreasing function of P. The results of the GMM analysis of Chapter 2 show that models with a three-quarters power in the diffusion term outperform many other models of different powers, in explaining the behaviour of Brent crude oil prices.

From (3.2.19) the risk-neutral process followed by P is

$$dP = \left(a\sqrt{P} - \sigma\lambda(P,t)P^{\frac{3}{4}}\right)dt + \sigma P^{\frac{3}{4}} d\tilde{Z}$$

where $\lambda(P,t)$ is the market price of risk and $\tilde{Z}$ is a Wiener process under an equivalent risk-neutral measure $\mathbb{Q}$ under which P becomes a martingale. Here, like many authors such as Stein and Stein (1991) and Gr$\ddot{u}$nbichler and Longstaff (1995) we will assume that the market price of risk is such that the risk-neutral process and the real process for P have the same form. Hence we assume here that $\lambda(P,t) = cP^{-\frac{1}{4}}$, c constant and so the risk-neutral process is the same form as (3.2.19) (but with a different constant a).

The transition density function of P follows the process (3.2.19) is given by

$$Tr_9(P,t;P_T,T) = \frac{4\sqrt{P}e^{\frac{-8(\sqrt{P_T}+\sqrt{P})}{\sigma^2(T-t)}} I_\nu(z) \left(\frac{P_T}{P}\right)^{\frac{a}{\sigma^2}}}{P_T\sigma^2(T-t)} \tag{3.2.20a}$$

where

$$\nu = \frac{4a}{\sigma^2} - 2, \; z = \frac{16(P_T P)^{\frac{1}{4}}}{\sigma^2(T-t)} \tag{3.2.20b}$$

see Goard(2006), and where $I_\nu(.)$ is the modified Bessel function of the first kind of order ν (see e.g. Abramowitz and Stegun (1964)). Hence we can find the price of a futures contract that matures at time T, $F(P,t)$, as the expected price of the commodity under the measure $\mathbb{Q}$ i.e.

$$\begin{aligned} \forall\, t \le T, \quad F(P,t) &= \mathbb{E}_t^{\mathbb{Q}}(P_T) \\ &= \int_0^\infty yTr_9(P,t;y,T)\; dy \\ &= P + a(T-t)\sqrt{P} + \frac{(a(T-t))^2}{4}\left(1 - \frac{\sigma^2}{4a}\right). \end{aligned} \tag{3.2.21}$$

Formula (3.2.21) for futures prices has the advantage of being simple. However, the behaviour of the model as $T - t \to \infty$ is not realistic as it precludes the existence of a fixed long-term price of oil. See Figure 3.3 for sample plots of futures prices

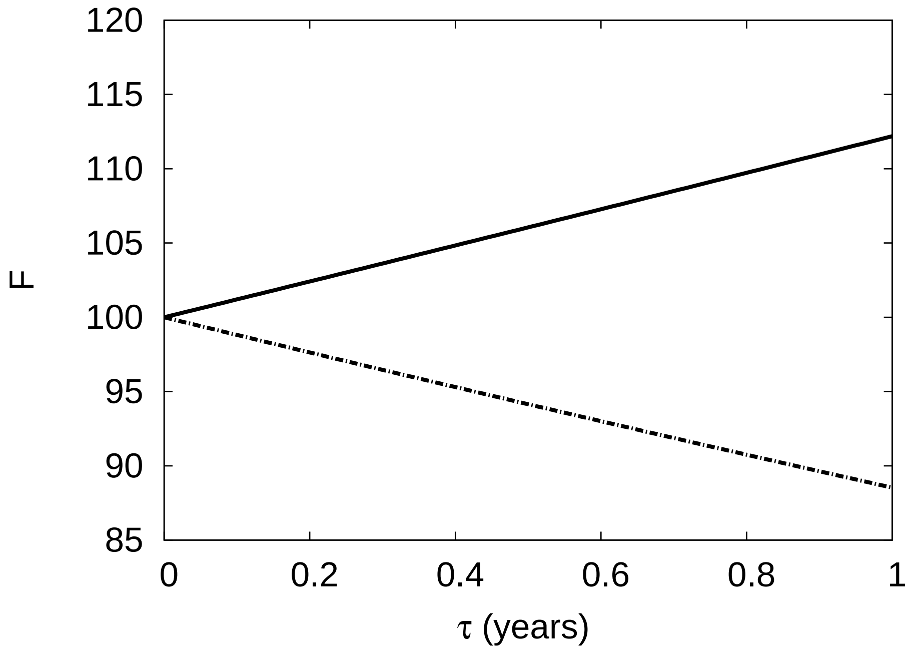


Figure 3.3: Futures prices (F) under Model 9 for various maturities ($\tau = T - t$) with $P_0 = 100$, $\sigma = 1.5$, $a = 1.2$ (solid line) and $a = -1.2$ (dashed line).

(3.2.21) for various maturities and parameters.

3.2.4 Futures Prices Under Stochastic Model 10

We now suppose that the spot price of a commodity follows the mean-reverting stochastic process given by Model 10 i.e.

$$dP = (a\sqrt{P} + bP)dt + \sigma P^{\frac{3}{4}} dZ \ , \quad b < 0 \ . \tag{3.2.22}$$

We can rewrite (3.2.22) as follows

$$dP = |b|\sqrt{P}\left(\frac{a}{|b|} - \sqrt{P}\right) dt + \sigma P^{\frac{3}{4}} dZ, \quad b < 0 \tag{3.2.23}$$

from which it is easier to see that the model assumes that the spot price reverts to a constant $\frac{a^2}{b^2}$, and the rate of reversion is determined by $|b|\sqrt{P}$. The corresponding risk-neutral process followed by P is then given by

$$dP = \left(a\sqrt{P} + bP - \lambda(P,t)\sigma P^{\frac{3}{4}}\right)dt + \sigma P^{\frac{3}{4}} d\tilde{Z} \ .$$

As in the previous section, we assume here that $\lambda(P,t) = c_1 P^{-\frac{1}{4}} + c_2 P^{\frac{1}{4}}$ (c_1 and c_2 are constants) giving the risk-neutral process the same form as (3.2.22) (with different constants a, b).

The transition density function for P that follows the process (3.2.22) is given by

$$Tr_{10}(P,t;P_T,T) = Q_1(P;P_T)\ Q_2(P;P_T)\ P_T^{\frac{a}{c^2}-1}\ P^{\frac{1}{2}-\frac{a}{c^2}}\ I_\nu(z) \qquad (3.2.24a)$$

where

$$Q_1(P;P_T) = \frac{2\sqrt{b^2}e^{\frac{2(\sqrt{P}-\sqrt{P_T})(\sqrt{b^2}-b)}{\sigma^2}+\frac{T-t}{4}(\sqrt{b^2}+b-\frac{4ab}{\sigma^2})}}{\sigma^2(e^{\frac{\sqrt{b^2}(T-t)}{2}}-1)} \qquad (3.2.24b)$$

$$Q_2(P;P_T) = \exp\{\frac{-4\sqrt{b^2}(\sqrt{P_t}e^{\frac{\sqrt{b^2}(T-t)}{2}}+\sqrt{P_T})}{\sigma^2(e^{\frac{\sqrt{b^2}(T-t)}{2}}-1)}\} \qquad (3.2.24c)$$

and

$$\nu = \frac{4a}{c^2}-2,\ z = \frac{8(P_T P)^{\frac{1}{4}}\sqrt{b^2}e^{\frac{\sqrt{b^2}(T-t)}{4}}}{\sigma^2(e^{\frac{\sqrt{b^2}(T-t)}{2}}-1)} \qquad (3.2.24d)$$

see Goard(2006), and where $I_\nu(.)$ is the modified Bessel function of the first kind of order ν (see e.g. Abramowitz and Stegun (1964)). Hence, we can find the price of the futures contract that matures at time T, $F(P,t)$, as the expected price of the commodity under the risk-neutral measure $\mathbb{Q}$ i.e.

$$\begin{aligned} \forall\ t \le T,\quad F(P,t) &= \mathbb{E}_t^{\mathbb{Q}}(P_T) \\ &= \int_0^\infty yTr_{10}(P,t;y,T)\ dy \\ &= Pe^{b(T-t)} + \frac{2a\sqrt{P}}{b}\left(e^{b(T-t)} - e^{\frac{b(T-t)}{2}}\right) + \frac{a(4a-\sigma^2)}{4b^2}\left(e^{\frac{b(T-t)}{2}} - 1\right)^2. \end{aligned} \tag{3.2.25}$$

An important feature of the futures price (3.2.25) is that it implies the existence of a long-term price of oil, namely [1]

$$\lim_{T-t\to\infty} F(P,t) = \frac{a(4a-\sigma^2)}{4b^2} > 0$$

See Figure 3.4 for sample plots of futures prices (3.2.25) for a various maturities and parameters.

3.3 Empirical Test

In this section we examine our new futures prices (3.2.21) and (3.2.25) under stochastic oil models Model 9 (3.2.19) and Model 10 (3.2.22) respectively, for fitting and forecasting market prices. In addition, we compare their performance with the Gabillon (3.2.13) and Schwartz (3.2.18) models. This section will be divided into three parts. In the first part we provide a description of the data used in our empirical test. In the second part, we describe the methodology used to estimate the parameters in the models and in the third part we present our results.

[1] As mentioned in Section 2.1.10 Model 10 can be transformed to the CIR model by the transformation $f = \sqrt{P}$ to get $df = (\alpha + \frac{b}{2}f)dt + \frac{\sigma}{2}\sqrt{f}dZ$, where $\alpha = \frac{a}{2} - \frac{\sigma^2}{8}$.
It is known (see Wilmott (1998)) that for the CIR model if $\frac{8\alpha}{\sigma^2} > 1 \Rightarrow \frac{8}{\sigma^2}\left(\frac{a}{2} - \frac{\sigma^2}{8}\right) > 1 \Rightarrow 2a > \sigma^2 \Rightarrow 4a > \sigma^2$ then f (and so P) remains positive.

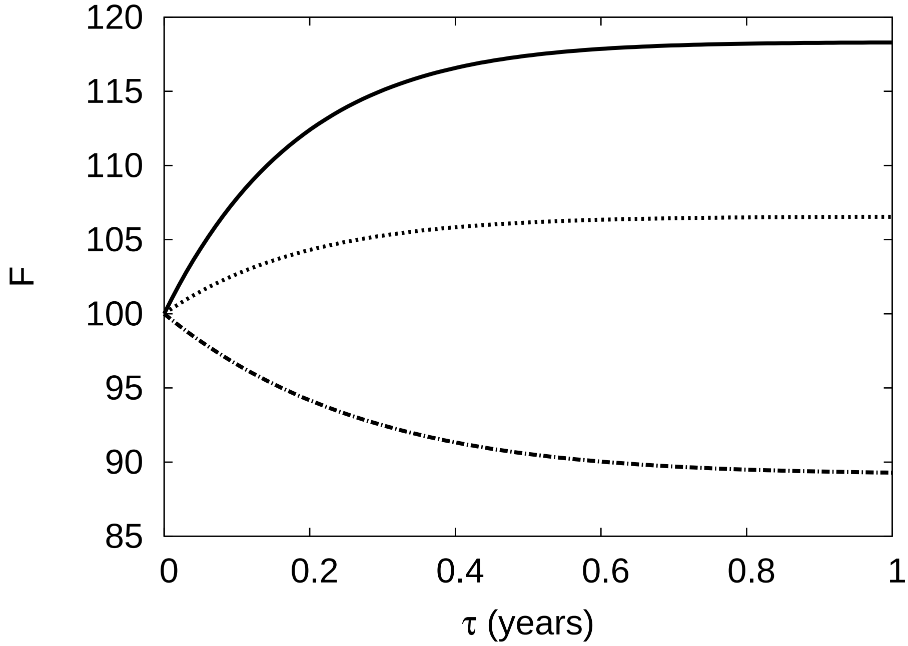


Figure 3.4: Futures prices under Model 10 (F) for various maturities ($\tau = T - t$) with $P_0 = 100$, ($a = 150$, $b = -13.5$, $\sigma = 5$ (solid line)), ($a = 125$, $b = -12$, $\sigma = 3$ (dotted line)) and ($a = 90$, $b = -9.5$, $\sigma = 1.5$ (dashed line)).

3.3.1 Data Description

The data used in this empirical work consists of daily observations of futures prices for Brent crude oil. We used 60 monthly contracts $\{F_1, F_2, \cdots F_{60}\}$ selected from the International Commodity Exchange (ICE). From these contracts, we construct a sequence of 55 data sets $\{G_1, G_2, \cdots G_{55}\}$, in which each data set has six sequential contracts i.e. $G_i = \{F_i, F_{i+1}, \cdots F_{i+5}\}$. For each data set, G_i, the first contract F_i expires in one month and the second contract, F_{i+1}, expires in two months and so on.

3.3.2 Estimation of Parameters

Table 3.1 lists the closed forms of the futures prices that will be calibrated to market prices, with $\tau = T - t$ representing time to maturity. It also lists the parameters in the models to be estimated. For the Gabillon price (3.2.13) we need to estimate the difference between the cost of carry and the convenience yield parameter, i.e. δ.

Oil Price Model	Futures Price	Parameters to be estimated
Model 1 (Gabillon) $dP = \mu P dt + \sigma P dZ$	$Pe^{(r-\delta)\tau}$	δ
Model 4 (Schwartz) $dP = \eta P(\mu - \ln P)dt + \sigma P dZ$	$\exp\{e^{-\eta\tau}\ln P + \alpha^*(1-e^{-\eta\tau}) + \frac{\sigma^2}{4\eta}(1-e^{-2\eta\tau})\}$	η, α^*, σ
Model 9 $dP = a\sqrt{P}dt + \sigma P^{\frac{3}{4}}dZ$	$P + a\tau\sqrt{P} + \frac{(a\tau)^2}{4}(1-\frac{\sigma^2}{4a})$	a, σ
Model 10 $dP = (a\sqrt{P} + bP)dt + \sigma P^{\frac{3}{4}}dZ$	$Pe^{b\tau} + \frac{2a\sqrt{P}}{b}(e^{b\tau} - e^{\frac{b\tau}{2}}) + \frac{a(4a-\sigma^2)}{4b^2}(e^{\frac{b\tau}{2}} - 1)^2$	a, b, σ

Table 3.1: The closed forms of futures price

For the Schwartz price (3.2.18) we need to estimate the speed of reversion (η), the long run-log price (α^*) and the volatility(σ). For the new pricing forms, (3.2.21) and (3.2.25) corresponding to Model 9 (3.2.19) and Model 10 (3.2.22) respectively, we need to estimate respectively a, σ and a, b, σ. For each pricing equation, we select the model parameters that produce a model curve as close as possible in some sense to the market curve. We choose to do this as follows:

We define F_{ijt} ($\hat{F}_{ijt}$) to be the market (estimated) futures price, at day t, of the contract i that belongs to data set j. Now let e_{ijt} to be the error, on day t, of the contract i that belongs to data set j. Define e_{ijt} as the difference between the market futures price and the estimated futures price

$$e_{ijt} = F_{ijt} - \hat{F}_{ijt}.$$

Let $\theta(j)$ be the parameter vector for the data set j. For each pricing form and for each data set j, we use the market's futures prices on the initial day, $t = 0$, and minimize the sum of squares of errors (SSE) i.e.

$$\min SSE(\theta(j)) = \sum_{i=1}^{6} e_{ij0}^2. \tag{3.3.26}$$

This results in the parameters vector ($\theta(j)$) for each pricing equation and for each data set j.

Under the assumption that the behaviour of the futures prices is stable for the next five business days, we can update the spot price and use the estimated parameters vector to forecast the futures prices for the next five business days.

3.3.3 Performance of Futures Models

The following measures are used to compare errors in the performance of the futures pricing models:

- The daily sum of squared errors

$$SSE_t = \sum_{i=1}^{6}\sum_{j=1}^{55} e_{ijt}^2 \ , \ t = 0, 1, .., 5 \tag{3.3.27}$$

- The daily root mean squared errors

$$RMSE_t = \sqrt{\frac{1}{N-q}\sum_{i=1}^{6}\sum_{j=1}^{55} e_{ijt}^2} \ , \quad t = 0, 1, .., 5 \tag{3.3.28}$$

- The total sum of squared errors

$$SSE_{total} = \sum_{t=0}^{5}\sum_{i=1}^{6}\sum_{j=1}^{55} e_{ijt}^2 \tag{3.3.29}$$

- The total root mean squared errors

$$RMSE_{total} = \sqrt{\frac{1}{N-q}\sum_{t=0}^{5}\sum_{i=1}^{6}\sum_{j=1}^{55} e_{ijt}^2} \tag{3.3.30}$$

where N and q are the number of observations and parameters, respectively. Tables 3.2-3.4 list the results of our analysis. In particular, we note that:

- *For fitting futures prices at* $t = 0$ ***Model 10 fits best the futures prices in 53 data sets and has the smallest SSE of 14.68***, while Model 4 fits best

t	Model 1 (3.2.13)	Model 4 (3.2.18)	Model 9 (3.2.21)	Model 10 (3.2.25)
0	274.77	15.16 (best in 2 sets)	76.32	14.68 (best in 53 sets)
1	1446.20	1173.46	1473.47	1174.11
2	2048.90	1387.49	2247.15	1387.48
3	2810.19	2307.17	3046.40	2302.85
4	1722.91	1675.02	1913.64	1649.31
5	2292.84	1820.72	2591.83	1794.45

Table 3.2: Comparison of daily SSE

t	Model 1 (3.2.13)	Model 4 (3.2.18)	Model 9 (3.2.21)	Model 10 (3.2.25)
0	\$1.000	\$0.303	\$0.589	\$0.298
1	\$2.093	\$1.886	\$2.113	\$1.886
2	\$2.492	\$2.050	\$2.610	\$2.050
3	\$2.918	\$2.644	\$3.038	\$2.642
4	\$2.285	\$2.253	\$2.408	\$2.236
5	\$2.636	\$2.349	\$2.803	\$2.332

Table 3.3: Comparison of daily RMSE

in two data sets with the next smallest SSE of 15.16. Model 9 and Model 1 have the highest values of SSE, 76.3 and 274.77 respectively. This is perhaps not a surprising result as Model 3 and Model 10 include three parameters whereas Model 1 and Model 9 have one parameter and two parameters respectively. However from Table 3.3, comparison of RMSE indicates that Model 10 has also the lowest RMSE \$ 0.289 per contract, followed closely by Model 3 then Model 9 and finally Model 1.

- *For forecasting futures prices on the next business day* ($t = 1$) the lowest SSE, 1173.46, is achieved by Model 4, followed by Model 10 (with an extra 0.69). Moreover, the lowest RMSE, \$ 1.886 per contract, is achieved by Model 4 and Model 10. The highest values of SSE (1473.47) and RMSE (\$ 2.113 per contract) are achieved by Model 9.

- *For forecasting futures prices on the following second business day* ($t = 2$)

	Model 1 (3.2.13)	Model 4 (3.2.18)	Model 9 (3.2.21)	Model 10 (3.2.25)
SSE_{total}	10595.73	8379.02	11348.8	8322.89
$RMSE_{total}$	\$ 2.346	\$ 2.149	\$ 2.465	\$ 2.141

Table 3.4: Comparison of total SSE and RMSE

Model 10 and Model 4 have the same lowest values of SSE and RMSE (1387.48 and \$ 2.050 per contract). In addition, Model 9 still has the highest value of SSE and RMSE compared with the other models.

- *For forecasting futures prices on the following third, fourth and fifth business days* ($t = 3, 4, 5$) Model 10 has the lowest value of SSE, followed by Model 4 with excess values of 4.13, 25.7 and 26.26 respectively.
- *Comparison of total SSE and total RMSE* ($t = 0, \cdots, 5$) indicates that the lowest value of total SSE and total RMSE are reached by Model 10 (8322.89 and \$ 2.141 per contract). Model 9 has the highest values for total SSE and total RMSE (1348.8 and \$ 2.465 per contract).

In summary we can infer from our given data and empirical analysis, that Model 10 outperforms the other models in fitting and forecasting futures prices. As an illustrative example of a typical fit on a particular day, Figure 3.5 displays a comparison of futures prices using (3.2.25) with the calibrated values for a, b and σ, and market prices on June 18, 2009. It can be seen that market and model prices are very close.

3.4 Conclusion

In this chapter, simple analytic solutions for futures prices under two of the three-quarters models presented in Chapter 2 were found. The new analytic solutions were calibrated to market data and compared with calibrations of futures prices for the GBM model and the Schwartz model. Our calibration results show that the

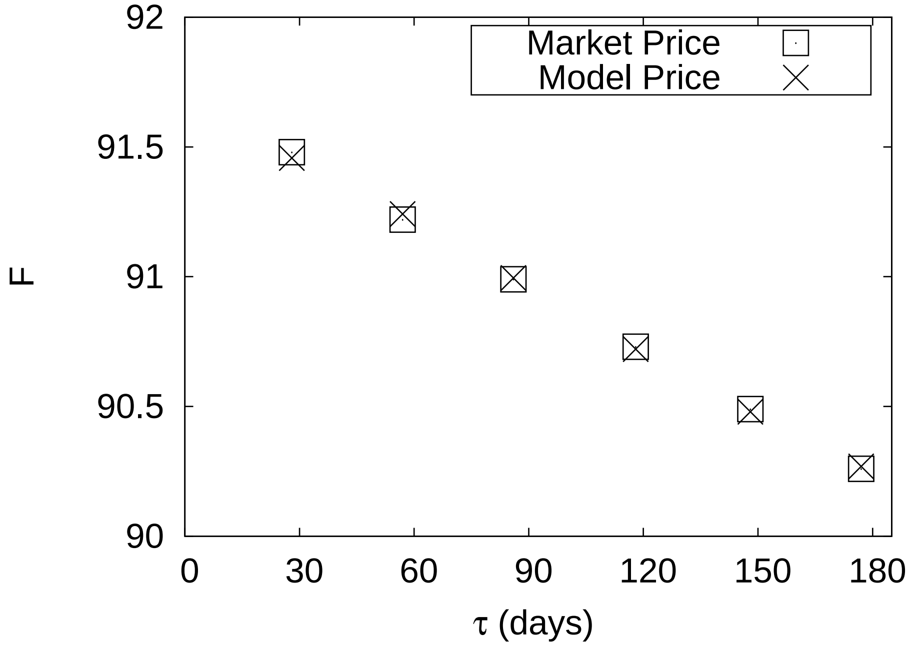


Figure 3.5: Comparison of futures prices using formula (3.2.25) with market data on 18 Jun 2009.

three-quarters model with the mean-reverting property, Model 10, outperforms all the other models in its ability for fitting market data and forecasting futures prices over the next five days, suggesting it to be a useful guide to traders.

Chapter 4

Extensions to Single One-Factor Models for Pricing Futures Contracts

4.1 Introduction

In the previous chapter we assumed that futures price depended only on one source of uncertainty related to the price of oil. An important advantage of the use of one-factor models is their tractability i.e. it can be easier to derive closed and simple formulae for futures prices under these models. This in turn facilitates calibration of futures contracts. However, empirical studies on pricing derivatives, see for example Barren (1991), indicate that the use of one-factor models is inappropriate in explaining adequately derivative prices. This is an indication for the need of an extra state variable. Fame and French (1987) showed that the convenience yield should be specified by a stochastic process. This belief is also supported by Miltersen and Schwartz (1998) and Ribeiro and Hodges (2004). Consequently, various two-factor models have been introduced, and the convenience yield or long-run price is assumed as the second factor in most of those models. Gibson and Schwartz (1990) introduced one of the most well-known models for commodity prices with convenience yield as a second factor. They assumed that the spot price of the commodity followed a GBM and the convenience yield followed an OU stochastic process and was correlated with the spot price. Further, Gabillon (1991) introduced one of the most well-known models for commodity prices with long-run price as a second factor. He assumed that there was a long-run price of oil L for delivery at infinite time which was correlated with the process of the spot price and could be described by a stochastic process. He suggested that the spot price of oil and the long run price follow two correlated GBMs and derived a closed-form solution for pricing futures contracts. For further details, we refer the reader to Section 1.5.1.

In this chapter, three two-factor models are proposed by adding a second state variable to the spot price of oil; and a regime-switching model, where the price of oil follows a one-factor stochastic model in each of two regimes, is also proposed. Analytic formulae for futures contract prices are derived for each proposed model. The remainder of this chapter is organised into three sections. In the following Section 4.2, the governing equation for pricing futures contracts under a two-factor model is derived. Analytic formulae for futures prices under our proposed models are derived in Section 4.3 and in Section 4.4 we present our conclusion.

4.2 The Governing Equation for Pricing Futures Contracts Under a Two-Factor Model

We assume that the price of a futures contract depends on the spot price of oil P, time t and an additional factor ϑ, where the spot price of oil and ϑ follow the correlated stochastic process

$$dP = \mu_1(P,\vartheta,t)dt + \sigma_1(P,\vartheta,t)dZ_1 \tag{4.2.1a}$$

$$d\vartheta = \mu_2(\vartheta,t)dt + \sigma_2(\vartheta,t)dZ_2 \tag{4.2.1b}$$

$$Corr(dZ_1, dZ_2) = \rho dt\ . \tag{4.2.1c}$$

The instantaneous change of the futures price, $F(P,\vartheta,t)$, can be found by applying Itô's Lemma (see Section 1.1.4) to get

$$dF = \frac{\partial F}{\partial P}dP + \frac{1}{2}\frac{\partial^2 F}{\partial P^2}(dP)^2 + \frac{\partial F}{\partial \vartheta}d\vartheta + \frac{1}{2}\frac{\partial^2 F}{\partial \vartheta^2}(d\vartheta)^2 + \frac{\partial^2 F}{\partial P\partial \vartheta}(dPd\vartheta) + \frac{\partial F}{\partial t}\ . \tag{4.2.2}$$

Substituting (4.2.1a-c) into (4.2.2), we get

$$\begin{aligned} dF = \Bigg[&\mu_1(P,t)\frac{\partial F}{\partial P} + \frac{\sigma_1(P,\vartheta,t)^2}{2}\frac{\partial F^2}{\partial P^2} + \mu_2(\vartheta,t)\frac{\partial F}{\partial \vartheta} + \frac{\sigma_2(\vartheta,t)^2}{2}\frac{\partial^2 F}{\partial \vartheta^2} \\ &+ \rho\ \sigma_1(P,t)\sigma_2(\vartheta,t)\frac{\partial^2 F}{\partial P\partial \vartheta} + \frac{\partial F}{\partial t}\Bigg]dt + \sigma_1(P,\vartheta,t)\frac{\partial F}{\partial P}dZ_1 + \sigma_2(\vartheta,t)\frac{\partial F}{\partial \vartheta}dZ_2\ . \end{aligned} \tag{4.2.3}$$

Equation (4.2.3) can be written as

$$dF = \mathcal{L}(F)dt + \sigma_1(P,\vartheta,t)\frac{\partial F}{\partial P}dZ_1 + \sigma_2(\vartheta,t)\frac{\partial F}{\partial \vartheta}dZ_2 \tag{4.2.4a}$$

where

$$\begin{aligned} \mathcal{L}(F) = &\mu_1(P,\vartheta,t)\frac{\partial F}{\partial P} + \frac{\sigma_1(P,\vartheta,t)^2}{2}\frac{\partial^2 F}{\partial P^2} + \mu_2(\vartheta,t)\frac{\partial F}{\partial \vartheta} \\ &+ \frac{\sigma_2(\vartheta,t)^2}{2}\frac{\partial^2 F}{\partial \vartheta^2} + \rho\ \sigma_1(P,\vartheta,t)\sigma_2(\vartheta,t)\frac{\partial^2 F}{\partial P\partial \vartheta} + \frac{\partial F}{\partial t}\ . \end{aligned} \tag{4.2.4b}$$

Now we construct a portfolio with value π that consists of one long futures contract on P with value F_1 maturing at time T_1, x short futures contracts each with value F_2 maturing at time T_2 and y short futures contracts each with value F_3 maturing at time T_3. So the portfolio has value

$$\pi = F_1 - xF_2 - yF_3 \tag{4.2.5}$$
$$\text{where} \quad F_i = F_i(P, \vartheta, t; T_i) \ ,$$

and its instantaneous return is

$$d\pi = \Big[\mathcal{L}(F_1) - x\mathcal{L}(F_2) - y\mathcal{L}(F_3)\Big]dt + \left[\frac{\partial F_1}{\partial P} - x\frac{\partial F_2}{\partial P} - y\frac{\partial F_3}{\partial P}\right]\sigma_1(P, \vartheta, t)dZ_1 + \left[\frac{\partial F_1}{\partial \vartheta} - x\frac{\partial F_2}{\partial \vartheta} - y\frac{\partial F_3}{\partial \vartheta}\right]\sigma_2(\vartheta, t)dZ_2 \tag{4.2.6}$$

where $\mathcal{L}(F_i)$ is given in (4.2.4b). The values of x and y can be chosen to make the coefficients of dZ_1 and dZ_2 in (4.2.6) equal to zero, so the corresponding portfolio will have zero-risk. Hence the instantaneous return of the portfolio, $d\pi$, should equal zero as it costs nothing to enter into a futures contract. The zero-risk and zero-return conditions lead to the following system of equations

$$\frac{\partial F_1}{\partial P} - x\frac{\partial F_2}{\partial P} - y\frac{\partial F_3}{\partial P} = 0 \tag{4.2.7a}$$
$$\frac{\partial F_1}{\partial \vartheta} - x\frac{\partial F_2}{\partial \vartheta} - y\frac{\partial F_3}{\partial \vartheta} = 0 \tag{4.2.7b}$$
$$\mathcal{L}(F_1) - x\mathcal{L}(F_2) - y\mathcal{L}(F_3) = 0 \ . \tag{4.2.7c}$$

Conditions (4.2.7a-c) imply that there is a linear relationship between the functions $\mathcal{L}(F)$, $\sigma_1(P, \vartheta, t)\frac{\partial F}{\partial P}$ and $\sigma_2(\vartheta, t)\frac{\partial F}{\partial \vartheta}$ which is independent of T. Therefore, we can write

$$\mathcal{L}(F) = \sigma_1(P, \vartheta, t)\ \lambda_P(P, \vartheta, t)\frac{\partial F}{\partial P} + \sigma_2(\vartheta, t)\ \lambda_\vartheta(P, \vartheta, t)\frac{\partial F}{\partial \vartheta} \tag{4.2.8}$$

for some arbitrary functions $\lambda_P(P, \vartheta, t)$ and $\lambda_\vartheta(P, \vartheta, t)$. These two functions can be interpreted as the market price per unit of spot price risk and market price per unit of the additional factor ϑ risk respectively. Finally, substituting (4.2.4b) into (4.2.8) we get the PDE governing the price of futures contracts as

$$\frac{\sigma_1^2(P,\vartheta,t)}{2}\frac{\partial F^2}{\partial P^2} + \big(\mu_1(P,\vartheta,t) - \lambda_P(P,\vartheta,t)\big)\frac{\partial F}{\partial P} + \frac{\sigma_2^2(\vartheta,t)}{2}\frac{\partial^2 F}{\partial \vartheta^2}$$
$$+ \big(\mu_2(\vartheta,t) - \lambda_\vartheta(P,\vartheta,t)\big)\frac{\partial F}{\partial \vartheta} + \rho\ \sigma_1(P,\vartheta,t)\sigma_2(\vartheta,t)\frac{\partial^2 F}{\partial P \partial \vartheta} + \frac{\partial F}{\partial t} = 0\ . \qquad (4.2.9)$$

Equation (4.2.9) needs to be solved subject to the final condition $F(P,\vartheta,T) = P$ and can be written as

$$\frac{\sigma_1^2(P,\vartheta,t)}{2}\frac{\partial F^2}{\partial P^2} + \tilde{\mu_1}(P,\vartheta,t)\frac{\partial F}{\partial P} + \frac{\sigma_2^2(\vartheta,t)}{2}\frac{\partial^2 F}{\partial \vartheta^2} \qquad (4.2.10)$$
$$+ \tilde{\mu_2}(P,\vartheta,t)\frac{\partial F}{\partial \vartheta} + \rho\ \sigma_1(P,\vartheta,t)\sigma_2(\vartheta,t)\frac{\partial^2 F}{\partial P \partial \vartheta} + \frac{\partial F}{\partial t} = 0$$

where $\tilde{\mu_1}(P,\vartheta,t)$ and $\tilde{\mu_2}(P,\vartheta,t)$ are the risk-adjusted drifts for (4.2.1a) and (4.2.1b) respectively.

4.3 Deriving New Closed Form Solutions for Futures Contracts

In this section we consider four new models for commodity prices. Two-factor models are assumed in three of these models and we use net demand, interest rate and convenience yield respectively as additional state variables in the models. The fourth model is a regime-switching model where the price of oil follows a one-factor stochastic model in each regime.

Any shock in supply or demand has an influence on spot prices. When the demand is higher than supply, traders expect an increasing trend in spot prices. Conversely, when the demand is lower than supply, traders expect a decreasing trend in spot prices. Hence, we expect that spot price is correlated with the difference between the supply and demand (net demand) and consider net demand as an additional variable in our first model.

Stochastic interest rates have been considered as a third factor to the spot price and convenience yield by Schwartz (1997), Hilliard and Reis (1998), Miltersen and Schwartz (1998) for the pricing of commodity derivatives. Schwartz (1997) empirically shows that the three-factor model which assumes stochastic convenience yields and interest rates outperforms the two- and one-factor models for describing oil and copper futures prices. Hence, we consider the interest rate as an additional variable in our second model, and use the stochastic interest rate model ($dr = ar(G(t) - r)dt + \sigma_2 r^{\frac{3}{2}} d\tilde{Z}$), which has been proven empirically by a number of authors (see e.g. Campbell *et al* (1996) and Goard (2008)) as the most successful model in capturing the dynamics of the interest rate.

The convenience yield, which comes from the theory of storage, can be defined as the benefit of holding physical commodities. When inventories are low then holding physical commodities becomes more valuable which hence increases the value of the convenience yield. Conversely, when inventories are high then the value of the convenience yield decreases. Therefore, the spot price is positively correlated with the convenience yield. The assumption of a constant convenience yield is equivalent to assuming that the level of inventories is also constant, which is not realistic. Hence, we consider stochastic convenience yield as an additional variable in our third model.

Finally, in the fourth model we assume a regime-switching situation where the price of oil follows a one-factor stochastic model, but with a different convenience yield in each regime.

While considering a second state variable can lead to more reasonable and accurate models compared with one-factor models, there are some shortcomings. For example, the convenience yield is not observable and is not easy to estimate. Moreover, the estimated net demand of oil can be observed only on a monthly basis with daily observations of net demand not available.

4.3.1 Spot Price and Net Demand Model

In this model we assume that the price of a futures contract depends on the spot price of oil P, time t and net demand q. The net demand of a product at a given time can be defined as the difference between the demand, D, and supply, S, of the product i.e. $q = D - S$. We assume that the spot price of oil and net demand follow the risk-neutral correlated stochastic process:

$$dP = mqPdt + \sigma_1 P d\tilde{Z}_1 \tag{4.3.11a}$$

$$dq = -(nq + \lambda\sigma_2)dt + \sigma_2 d\tilde{Z}_2 \tag{4.3.11b}$$

$$Corr(dZ_1, dZ_2) = \rho dt \tag{4.3.11c}$$

where $m,\ n,\ \rho,\ \sigma_1, \sigma_2$ are constant and λ represents the market price of risk per unit of net demand and is also assumed constant. Equation (4.3.11b) indicates for the real net demand process, net demand reverts to zero (in equilibrium) with rate of reversion determined by n, but a random shock can move the net demand from zero. From equation (4.3.11a), the expected relative change in the spot price is proportional to net demand. When net demand is positive ($D > S$), we expect the spot price to increase and conversely, when net demand is negative we expect the spot price to decrease. From Section 4.2, the PDE governing the futures contract is given by

$$\frac{\sigma_1^2 P^2}{2}\frac{\partial F^2}{\partial P^2} + mqP\frac{\partial F}{\partial P} + \frac{\sigma_2^2}{2}\frac{\partial^2 F}{\partial q^2} - (nq + \lambda\sigma_2)\frac{\partial F}{\partial q} + \sigma_1\sigma_2\rho P\frac{\partial^2 F}{\partial P \partial q} - \frac{\partial F}{\partial \tau} = 0 \tag{4.3.12}$$

where $\tau = T - t$. Equation (4.3.12) needs to be solved subject to the initial condition $F(P, q, 0) = P$. The initial condition suggests a solution to (4.3.12) of the form $F(p, q, \tau) = P\varphi(q, \tau)$. Substituting this form into (4.3.12) we get

$$\frac{\sigma_2^2}{2}\frac{\partial^2 \varphi}{\partial q^2} + (\sigma_1\sigma_2\rho - nq - \lambda\sigma_2)\frac{\partial \varphi}{\partial q} + mq\varphi = \frac{\partial \varphi}{\partial \tau} \tag{4.3.13}$$

to be solved subject to the initial condition $\varphi(q,0) = 1$.

Assuming the solution of (4.3.13) can be written as $\varphi(q,\tau) = e^{A(\tau)+qB(\tau)}$, then by substituting this form into (4.3.13) we get

$$\frac{\sigma_2^2}{2}B(\tau)^2 + (\sigma_1\sigma_2\rho - nq - \lambda\sigma_2)B(\tau) + mq = \frac{\partial A(\tau)}{\partial\tau} + q\frac{\partial B(\tau)}{\partial\tau} \; . \tag{4.3.14}$$

Equation (4.3.14) needs to be solved subject to the initial conditions $A(0) = B(0) = 0$. Equating coefficients of q to zero in (4.3.14) we get

$$\frac{\partial B(\tau)}{\partial\tau} + nB(\tau) = m \tag{4.3.15a}$$

$$\frac{\sigma_2}{2}B(\tau)^2 + \sigma_2(\sigma_1\rho - \lambda)B(\tau) - \frac{\partial A(\tau)}{\partial\tau} = 0 \; . \tag{4.3.15b}$$

Solving (4.3.15a) for $B(\tau)$ subject to $B(0) = 0$ and substituting this into (4.3.15b) to solve for $A(\tau)$ subject to $A(0) = 0$ we get

$$B(\tau) = \frac{m}{n}(1 - e^{-n\tau}) \tag{4.3.16a}$$

$$\begin{aligned} A(\tau) = &\frac{m\sigma_2}{2n^2}\big\{2(\lambda - \sigma_1\rho) + 2e^{-n\tau}(\frac{\sigma_2 m}{n} + \sigma_1\rho - \lambda) - \frac{\sigma_2 m}{2n}(3 + e^{-2n\tau}) \\ &+ \tau(\sigma_2 m + 2n\sigma_1\rho - 2n\lambda)\big\} \; . \end{aligned} \tag{4.3.16b}$$

Hence, undoing our change of variables $F(p,q,\tau) = P\varphi(q,\tau) = Pe^{A(\tau)+qB(\tau)}$ we get the futures prices as

$$\begin{aligned} F(P,q,\tau) = P\exp\Bigg\{&\frac{m\sigma_2}{2n^2}\big\{2(\lambda - \sigma_1\rho) + 2e^{-n\tau}(\frac{\sigma_2 m}{n} + \sigma_1\rho - \lambda) - \frac{\sigma_2 m}{2n}(3 + e^{-2n\tau}) \\ &+ \tau(\sigma_2 m + 2n\sigma_1\rho - 2n\lambda)\big\} + q\big[\frac{m}{n}(1 - e^{-n\tau})\big]\Bigg\} \; . \end{aligned} \tag{4.3.17}$$

See Figure 4.1 for sample plots of futures prices (4.3.17) with $P = 100,\ m = 0.6,\ n = 2,\ \sigma_1 = 1,\ \sigma_2 = 3$ and $\lambda = 0.1$ for various correlation coefficients and expiries.

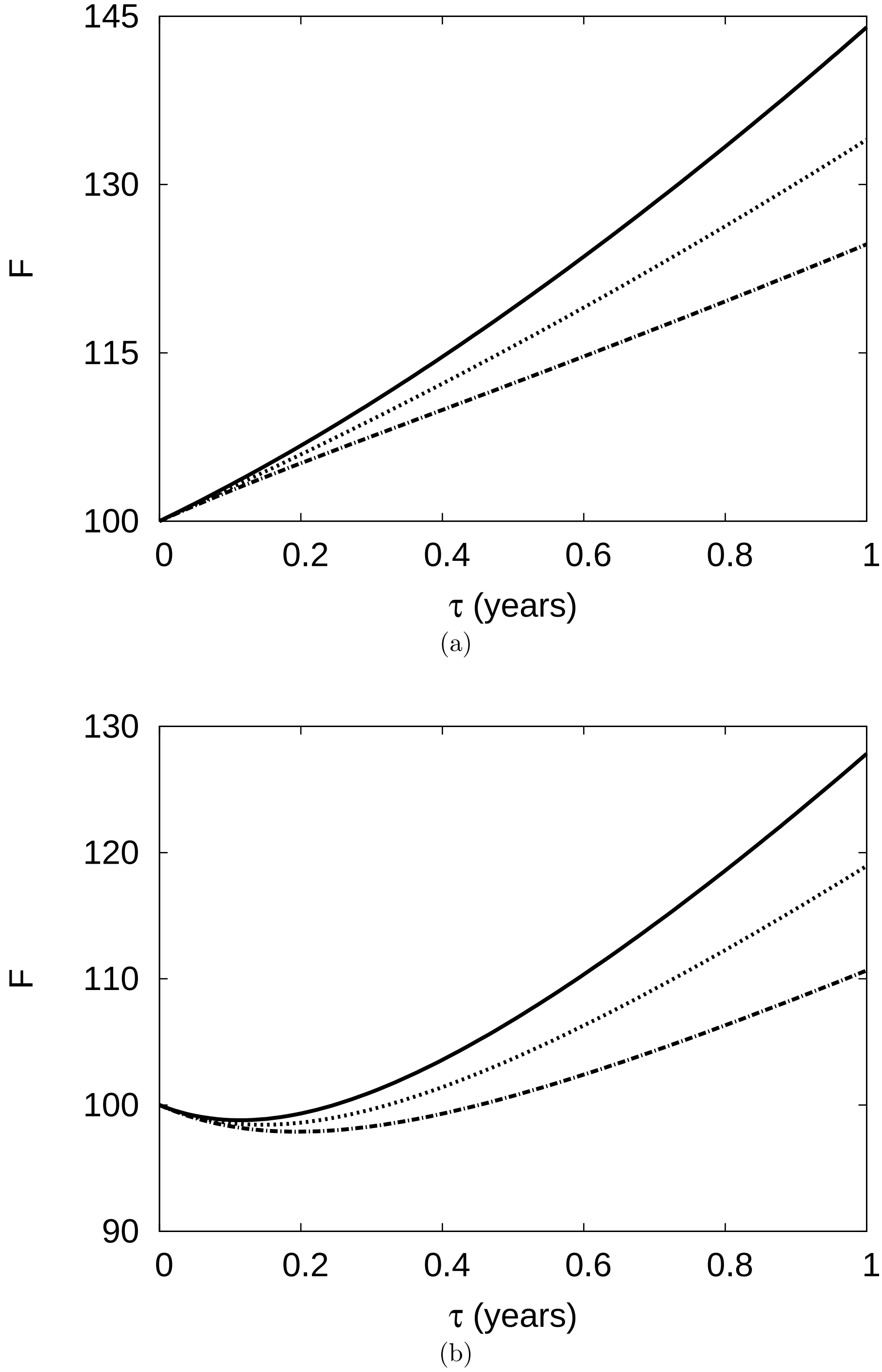


Figure 4.1: Futures prices (F) with (a) $q = 0.5$ (b) $q = -0.5$ under (4.3.11) for various expiries (τ) and correlation coefficients (solid line $\rho = 1$, dotted line $\rho = 0.75$ and dashed line $\rho = 0.5$) with $P = 100,\ m = 0.6,\ n = 5,\ \sigma_1 = 1,\ \sigma_2 = 3$ and $\lambda = 0.1$.

From Figure 4.1 it can be observed that for futures contracts with the given parameter values for the case $q = 0.5$, futures prices increase as time to expiry increases. This is to be expected at least initially, as under (4.3.11) with positive net demand, oil prices are expected to increase with time. For the case $q = -0.5$, it can be observed that futures prices decrease as time to expiry increases for expiries less than about 0.1. For expiries greater than this, futures prices increase as time to expiry increases. This again is not surprising as under (4.3.11) with a negative net demand, oil prices are expected to decrease initially. However, with net demand reverting to zero the trend changes for longer expiries. For both cases ($q = 0.5$ and $q = -0.5$), the higher the correlation coefficient the more valuable the futures contract is and the effect of the correlation coefficient on the futures contracts increases as time to expiry increases.

4.3.2 Spot Price and Interest Rate Model

We now assume that oil prices and interest rates follow the risk-neutral correlated stochastic model

$$dP = (r - \delta)Pdt + \sigma_1 P d\tilde{Z}_1 \tag{4.3.18a}$$

$$dr = ar(G(t) - r)dt + \sigma_2 r^{\frac{3}{2}} d\tilde{Z}_2 \tag{4.3.18b}$$

$$Corr(d\tilde{Z}_1, d\tilde{Z}_2) = \rho dt \ . \tag{4.3.18c}$$

This model assumes that oil is traded as an asset that pays a constant convenience yield δ, and the interest rate r reverts to a free function of time $G(t)$ with speed of reversion determined by ar. The choice of (4.3.18b) is supported by many studies. For example, Chan *et al* (1992) used GMM to compare the performance of various interest rate models. They found that models with a diffusion term $\sigma_2 r^\gamma$, where the power γ was higher than one were the most successful models in capturing the dynamics of the interest rate, and their estimated value for γ was $\frac{3}{2}$. This result was also supported by Campbell *et al* (1996) and Goard (2008). Schwartz (1997) used stochastic interest rates, namely $dr = \alpha(m - r)dt + \sigma d\tilde{Z}$, as a third factor in addition to commodity price and convenience yield. He empirically found that the correlation coefficients between commodities prices (oil, gold and copper) and interest rates were very close to zero. Hence, we expect that assuming $\rho = 0$ in (4.3.18) will not have a major impact on futures prices. From Section 4.2 the PDE governing the futures price is given by

$$\frac{\sigma_1^2 P^2}{2}\frac{\partial^2 F}{\partial P^2} + (r-\delta)P\frac{\partial F}{\partial P} + \frac{\sigma_2^2 r^3}{2}\frac{\partial^2 F}{\partial r^2} + ar(g(\tau)-r)\frac{\partial F}{\partial r} - \frac{\partial F}{\partial \tau} = 0 \qquad (4.3.19)$$

where $g(\tau) = G(T-\tau)$. Equation (4.3.19) needs to be solved subject to the initial condition $F(P,r,0) = P$ and boundary conditions $F(P,r,\tau) \to Pe^{-\delta\tau}$ as $r \to 0$, $F(0,r,\tau) = 0$, $F(P,r,\tau) \to \infty$ as $P \to \infty$ and $F(P,r,\tau) \to 0$ as $r \to \infty$. Letting $F(P,r,\tau) = Pv(r,\tau)$, from equation (4.3.19) we get the following PDE for $v(r,\tau)$

$$\frac{\sigma_2^2 r^3}{2}\frac{\partial^2 v}{\partial r^2} + ar(g(\tau)-r)\frac{\partial v}{\partial r} - \frac{\partial v}{\partial \tau} + (r-\delta)v = 0 \qquad (4.3.20)$$

and the conditions become $v(r,0) = 1$, $v(r,\tau) \to e^{-\delta\tau}$ as $r \to 0$ and $v(r,\tau) \to 0$ as $r \to \infty$. The computer package DIMSYM (see Sherring (1993)) was used to find the Lie symmetry generators of (4.3.20). The most general finite-dimensional generator is given by

$$\Psi = v\left[f(\tau) + \frac{aW(\tau)g'(\tau)}{r\sigma_2^2} - \frac{W''(\tau)}{r\sigma_2^2} + \frac{ag(\tau)W'(\tau)}{r\sigma_2^2}\right]\frac{\partial}{\partial v} + W(\tau)\frac{\partial}{\partial \tau} - rW'(\tau)\frac{\partial}{\partial r} \qquad (4.3.21a)$$

where $f(\tau)$ and $W(\tau)$ satisfy the following conditions

$$(a^2\delta + a\delta\sigma_2^2)W(\tau)g'(\tau) + (a+\sigma_2^2)f''(\tau) + (a^2g(\tau) + ag(\tau)\sigma_2^2 - \delta\sigma_2^2)f'(\tau)$$
$$+ (2a^2\delta g(\tau) + 2a\delta\sigma_2^2 g(\tau) - \delta^2\sigma_2^2)W'(\tau) = 0 \tag{4.3.21b}$$
$$(a^2 + a\sigma_2^2)W(\tau)g'(\tau) - \sigma_2^2 f'(\tau) - (a+\sigma_2^2)W''(\tau)$$
$$+ (a^2g(\tau) + a\sigma_2^2 g(\tau) - \delta\sigma_2^2)W'(\tau) = 0 \tag{4.3.21c}$$
$$W'''(\tau) - a^2g^2(\tau)W'(\tau) = 0 \ . \tag{4.3.21d}$$

Further, for the initial condition $v(r,0) = 1$ to be invariant under the symmetry, we require $\Psi(\tau) = 0|_{\tau=0}$ and $\Psi(v-1) = 0|_{\tau=0,\ v=1}$. This leads to

$$\Psi = -\delta W(\tau)v\frac{\partial}{\partial v} + W(\tau)\frac{\partial}{\partial \tau} - rW'(\tau)\frac{\partial}{\partial r} \tag{4.3.22a}$$
$$\text{where} \quad W(\tau) = e^{-a\int_\tau^T g(s)ds}\left\{1 - \frac{\int_\tau^T e^{a\int_u^T g(s)ds}du}{\int_0^T e^{a\int_u^T g(s)ds}du}\right\} . \tag{4.3.22b}$$

Using the method of characteristics to solve the corresponding invariant surface condition, we have

$$\frac{dr}{d\tau} = -\frac{rW'(\tau)}{W(\tau)} \qquad \Rightarrow \qquad rW(\tau) = c_1 \tag{4.3.23a}$$
$$\frac{dv}{d\tau} = -\delta v \qquad \Rightarrow \qquad v = c_2e^{-\delta\tau} \tag{4.3.23b}$$

so that two invariants are $rW(\tau)$ and $ve^{\delta\tau}$. Hence, we let $v(r,\tau) = e^{-\delta\tau}\phi(z);\ z = rW(\tau)$ and substitute this form into equation (4.3.20). We then get that ϕ needs to satisfy

$$\sigma_2^2 z^2\phi'' + 2(\beta - az)\phi' + 2\phi = 0 \ , \tag{4.3.24a}$$

where

$$\beta = \frac{-1}{\int_0^T e^{a\int_u^T g(s)ds} du} \ , \tag{4.3.24b}$$

subject to the initial condition $\phi(0) = 1$. The solution of (4.3.24a) is given by

$$\phi(z) = e^{\frac{2\beta}{\sigma_2^2 z}} z^{-k} \left[C_1 M\left(A, B, \frac{-2\beta}{\sigma_2^2 z}\right) + C_2 \ U\left(A, B, \frac{-2\beta}{\sigma_2^2 z}\right)\right] \tag{4.3.25a}$$

where k satisfies

$$k^2 + \left(1 + \frac{2a}{\sigma_2^2}\right)k - \frac{2}{\sigma_2^2} = 0 \tag{4.3.25b}$$

$$\text{and} \quad A = k + 2 + \frac{2a}{\sigma_2^2} \tag{4.3.25c}$$

$$B = 2k + 2 + \frac{2a}{\sigma_2^2} \tag{4.3.25d}$$

where $M(a, b, x)$ and $U(a, b, x)$ are the Kummer-M and Kummer-U functions respectively and where C_1 and C_2 are arbitrary constants. To find the values of C_1 and C_2, we use the following results for the Kummer functions (see Abramowitz and Stegun (1964)):

As $x \to \infty$

$$M(a, b, x) = \frac{\Gamma(b)}{\Gamma(a)} e^x x^{a-b}[1 + O(|x|^{-1})] \tag{4.3.26a}$$

$$U(a, b, x) = x^{-a}[1 + O(|x|^{-1})] \ . \tag{4.3.26b}$$

Hence, we require

$$C_1 = \frac{\Gamma(A)}{\Gamma(B)}\left(\frac{-2\beta}{\sigma_2^2}\right)^k \quad \text{and} \quad C_2 = 0 \tag{4.3.27}$$

in (4.3.25a) to satisfy the appropriate conditions. Finally, we get the futures price by substituting the values of C_1 and C_2 into (4.3.25a) and undoing our change of variables, so $F(P, r, \tau) = Pv(r, \tau) = Pe^{-\delta\tau}\phi(z), \quad z = rW(\tau)$ giving

$$F(P, r, \tau) = Pe^{-\delta\tau + \frac{2\beta}{\sigma_2^2 rW(\tau)}} \frac{\Gamma(A)}{\Gamma(B)}\left(-\frac{2\beta}{\sigma_2^2 rW(\tau)}\right)^k M\left(A, B, -\frac{2\beta}{\sigma_2^2 rW(\tau)}\right) \tag{4.3.28}$$

where A, B, $W(\tau)$ and β are given in (4.3.25c,d), (4.3.22b) and (4.3.24b) respectively. See Figure 4.2 for sample plots of futures prices (4.3.28) with $P = 100$, $a = 1$, $\sigma_1 = 1$, $\sigma_2 = 6$ and $G(t) = 0.03$ for various interest rates, convenience yields and expiries. From Figure 4.2 it can be observed that for futures contracts with the given parameter values, for the case $\delta = 0.05$ futures prices decrease as time to expiry increases (with the exception of the case when $r = 0.07$). This is to be expected as under (4.3.18a) with $r - \delta < 0$ oil prices are expected (at least initially) to decrease with time. For the case $\delta = 0.05$ with $r = 0.07$, it can be observed that the futures prices increase as time to expiry increases for expiries less than about 0.6. This again is not surprising as under (4.3.18a) with $r - \delta \geq 0$ oil prices are expected (at least initially) to increase with time. However, with interest rate reverting to 0.03, $r - \delta$ becomes negative and the trend changes for longer expiries. For the cases $\delta = 0, \ -0.05$, it can be observed that futures prices increase as time to expiry decreases. This is to be expected as under (4.3.18a) with $r - \delta \geq 0$ oil prices are expected (at least initially) to increase with time. For all cases, the higher the interest rate, the more valuable the futures contract is, and the effect of the interest rate on the futures contracts increases as the convenience yield decreases.

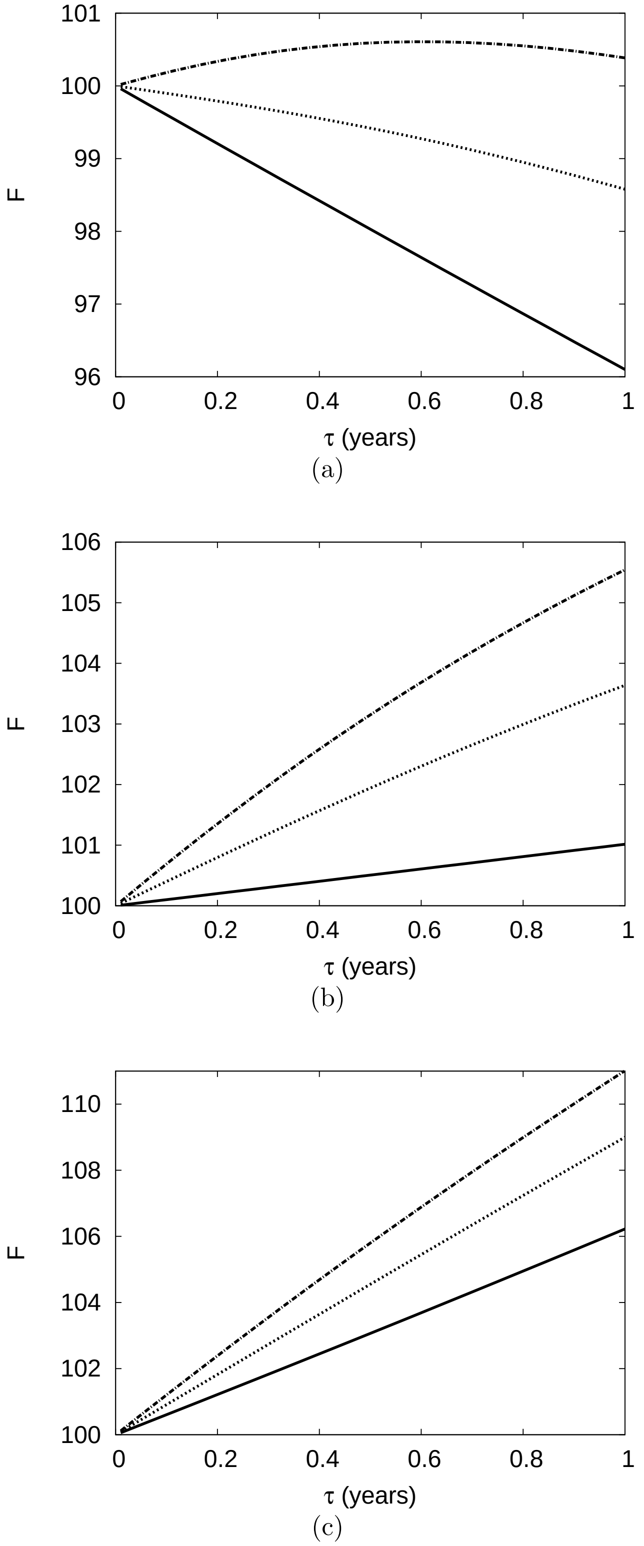


Figure 4.2: Futures prices (F) (a) $\delta = 0.05$ (b) $\delta = 0$ (c) $\delta = -0.05$ under (4.3.11) for various expiries (τ) and interest rates (solid line $r = 0.01$, dotted line $r = 0.04$ and dashed line $r = 0.07$) with $P = 100,\ a = 1,\ G(t) = 0.03,\ \sigma_1 = 1,$ and $\sigma_2 = 6$.

4.3.3 Spot Price and Convenience Yield Model

In this section the convenience yield δ is considered as an additional state variable. From Section 4.3.2, we see that if δ followed a similar dynamics to that of the interest rate in (4.3.18b), then a closed form solution for futures price might be possible. However, a zero correlation between spot price and convenience yield is not feasible. Hence, we assume that the oil price and convenience yield follow the risk-neutral correlated stochastic model

$$dP = (r - \delta)Pdt + \sigma_1 P\sqrt{\delta}d\tilde{Z}_1 \tag{4.3.29a}$$

$$d\delta = a\delta(G(t) - \delta)dt + \sigma_2 \delta^{\frac{3}{2}} d\tilde{Z}_2 \tag{4.3.29b}$$

$$Corr(d\tilde{Z}_1, d\tilde{Z}_2) = \rho dt\ . \tag{4.3.29c}$$

This model makes two further assumptions and so is only appropriate given that the assumptions hold. The first assumption is that $\delta > 0$, so that it is always beneficial to hold the commodity. The second assumption is that the volatility of the oil price is proportional to the square root of the convenience yield. Hence, a higher value of convenience yield (which is affected by net demand, inventory or seasonality) produces a higher volatility in the oil price. From Section 4.2 the PDE governing for the futures price is given by

$$\begin{aligned}&\frac{\sigma_1^2 \delta P^2}{2}\frac{\partial^2 F}{\partial P^2} + (r - \delta)P\frac{\partial F}{\partial P} + \frac{\sigma_2^2 \delta^3}{2}\frac{\partial^2 F}{\partial \delta^2} + a\delta(g(\tau) - \delta)\frac{\partial F}{\partial \delta}\\ &+ \rho\sigma_1\sigma_2 P\delta^2 \frac{\partial^2 F}{\partial P \partial \delta} - \frac{\partial F}{\partial \tau} = 0\end{aligned} \tag{4.3.30}$$

where $g(\tau) = G(T - \tau)$. Equation (4.3.30) needs to be solved subject to the initial condition $F(P, \delta, 0) = P$ and boundary conditions $F(P, \delta, \tau) \to Pe^{r\tau}$ as $\delta \to 0$, $F(0, \delta, \tau) = 0$, $F(P, \delta, \tau) \to \infty$ as $P \to \infty$ and $F(P, \delta, \tau) \to 0$ as $\delta \to \infty$. Letting

$F(P, \delta, \tau) = Pv(\delta, \tau)$, equation (4.3.30) gives the following PDE for $v(\delta, \tau)$

$$\frac{\sigma_2^2 \delta^3}{2} \frac{\partial^2 v}{\partial \delta^2} + \big(a\delta(g(\tau) - \delta) + \rho\sigma_1\sigma_2\delta^2\big)\frac{\partial v}{\partial \delta} - \frac{\partial v}{\partial \tau} + (r - \delta)v = 0 \ , \tag{4.3.31}$$

and the conditions become $v(r, 0) = 1$, $v(\delta, \tau) \to e^{r\tau}$ as $\delta \to 0$ and $v(\delta, \tau) \to 0$ as $\delta \to \infty$. Comparing (4.3.31) with (4.3.20) indicates the substitution $v(\delta, \tau) = e^{r\tau}\phi(z)$; $z = \delta W(\tau)$ where $W(\tau)$ is given in (4.3.22b). Following the same procedure as in Section 4.3.2, we get the futures prices as

$$F(P, \delta, \tau) = Pe^{r\tau + \frac{2\beta}{\sigma_2^2 \delta W(\tau)}} \frac{\Gamma(A)}{\Gamma(B)} \left(-\frac{2\beta}{\sigma_2^2 \delta W(\tau)}\right)^k M\left(A, B, -\frac{2\beta}{\sigma_2^2 \delta W(\tau)}\right) \quad \text{(4.3.32a)}$$

where

$$k^2 + \left(1 + \frac{2(a - \rho\sigma_1\sigma_2)}{\sigma_2^2}\right)k - \frac{2}{\sigma_2^2} = 0 \ , \quad \text{(4.3.32b)}$$

$$A = k + 2 + \frac{2(a - \rho\sigma_1\sigma_2)}{\sigma_2^2} \ , \quad \text{(4.3.32c)}$$

$$B = 2k + 2 + \frac{2(a - \rho\sigma_1\sigma_2)}{\sigma_2^2} \ , \quad \text{(4.3.32d)}$$

$$\beta = \frac{-1}{\int_0^T e^{a\int_u^T g(s)ds} du} \ . \quad \text{(4.3.32e)}$$

See Figure 4.3 for sample plots of futures prices (4.3.32a) with $P = 100$, $a = 3$, $\sigma_1 = 1.5$, $\sigma_2 = 5$, $r = 0.04$ and $G(t) = 0.03$ for various convenience yields, correlation coefficients and expiries. From Figure 4.3 it can be observed that for futures contracts with the given parameter values, for the case $\delta = 0.02$ the futures prices increase as time to expiry increases. This is to be expected as under (4.3.32a) with $r - \delta > 0$ oil prices are expected (at least initially) to increase with time. For the cases $\delta = 0.04,\ 0.06$, it can be observed that the futures prices decrease as time to expiry increases for expiries less than about 1.2 and 2 respectively. This again is not surprising as under (4.3.32a) with $r - \delta \leq 0$ oil prices are expected (at least initially) to decrease with time. However, with convenience yield reverting to 0.03, $r - \delta$ becomes positive and the trend changes for longer expiries. For all cases, the lower the correlation coefficient the more valuable the futures contract is and the effect of the correlation coefficient on the futures contracts increases as time to expiry increases.

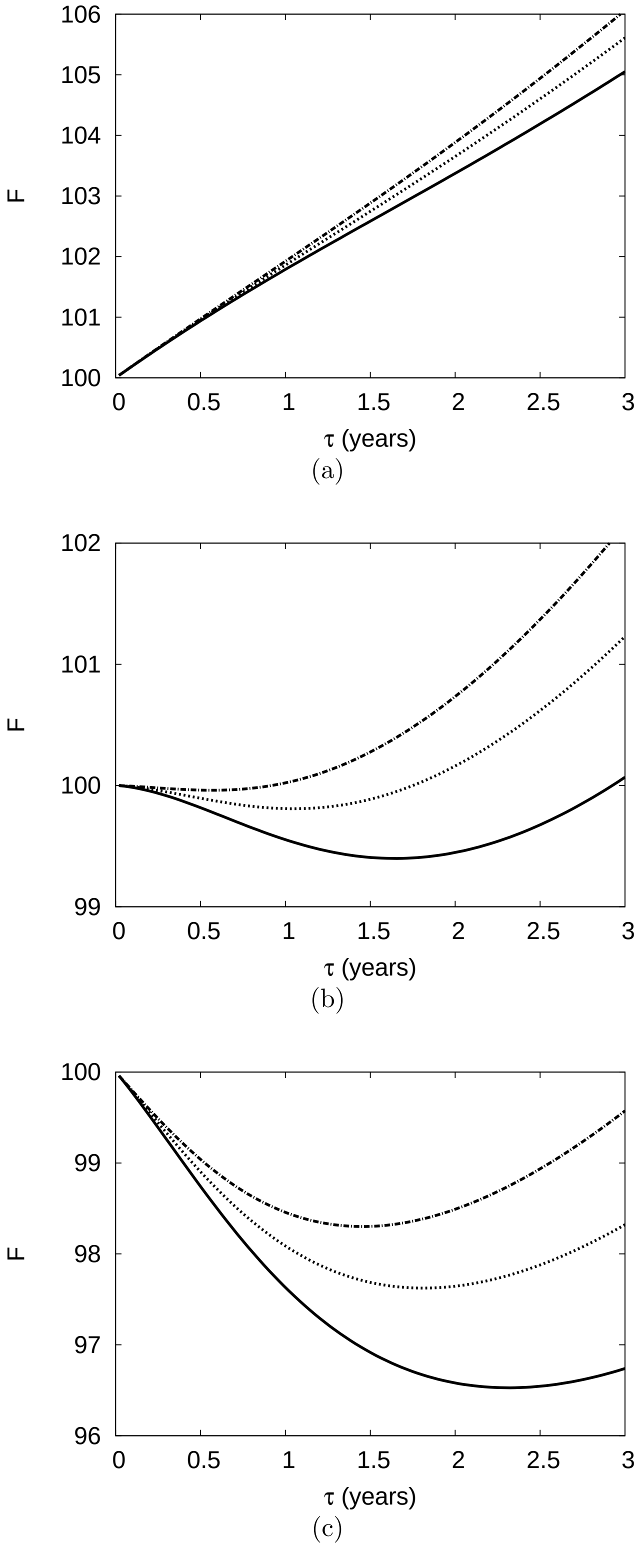


Figure 4.3: Futures prices (F) (a) $\delta = 0.02$ (b) $\delta = 0.04$ (c) $\delta = 0.06$ under (4.3.18) for various expiries (τ) and correlation coefficients (solid line $\rho = 1$, dotted line $\rho = 0.75$ and dashed line $\rho = 0.5$) with $P = 100$, $a = 3$, $G(t) = 0.03$, $\sigma_1 = 1.5$, $\sigma_2 = 5$ and $r = 0.04$.

4.3.4 The Regime Switching Model

Convenience yield can be considered as a function of the commodity inventory, which in turn can be considered as a function of net demand. The assumption of a constant convenience yield is equivalent to assuming that the market has only one state all the time. This assumption also ignores the cycles of booms and busts in the commodity market. Hence, it is improbable to assume that the commodity market has only one state all of the time. In this section we assume that the commodity market has two states $I = \{1, 2\}$. $X(t) = 1$ represents the state when the supply of oil is in shortage while $X(t) = 2$ represents the state when there is no shortage in supply. A different convenience yield is assumed in each state. Hence, when $X(t) = i, \quad i = 1,\ 2$, then $\delta = \delta_i$ and the change of spot price of oil is assumed to follow

$$dP = (r - \delta_i)Pdt + \sigma P d\tilde{Z}\ . \tag{4.3.33}$$

We assume that the transition between states

- jumps from $X(t) = 1$ to $X(t) = 2$ and occurs as a Poisson process with rate λ_2,
- jumps from $X(t) = 2$ to $X(t) = 1$ and occurs as a Poisson process with rate λ_1.

Hence, the value of a futures contract $F(P, t)$ will take one of two values $F_1(P, t)$ or $F_2(P, t)$ depending on the current supply state $X(t) = 1,\ 2$ respectively.

Suppose that at time t, $X(t) = 1$, so the value of a futures contract is $F_1(P, t)$. In a short time step dt, the value of the futures contract is given by

$$F(P_{t+dt}, t + dt) = \begin{cases} F_1(P_{t+dt}, t + dt) & \text{with probability } \ 1 - \lambda_2 dt, \\ F_2(P_{t+dt}, t + dt) & \text{with probability } \ \lambda_2 dt\ . \end{cases} \tag{4.3.34}$$

Hence, the change in the futures contract can be written as

$$dF = \begin{cases} F_1(P_{t+dt}, t+dt) - F_1(P_t, t) & \text{with probability } \ 1 - \lambda_2 dt, \\ F_2(P_{t+dt}, t+dt) - F_1(P_t, t) & \text{with probability } \ \lambda_2 dt\ . \end{cases} \tag{4.3.35}$$

This is equivalent to

$$dF = \begin{cases} dF_1 & \text{with probability } \ 1 - \lambda_2 dt, \\ F_2(P, t) - F_1(P, t) + dF_2 & \text{with probability } \ \lambda_2 dt\ . \end{cases} \tag{4.3.36}$$

We construct a portfolio with value π that consists of one long futures contract with expiry $\dot{T}$ and value $\dot{F}$, and Δ short futures contracts with expiry $\ddot{T}$ and each with value $\ddot{F}$ so $\pi = \dot{F} - \Delta\ddot{F}$. The change in our portfolio can be written as

$$d\pi = d\dot{F} - \Delta d\ddot{F} = \begin{cases} d\dot{F}_1 - \Delta d\ddot{F}_1 & \text{with probability } \ 1 - \lambda_2 dt, \\ \dot{F}_2 - \dot{F}_1 + d\dot{F}_2 - \Delta(\ddot{F}_2 - \ddot{F}_1 + d\ddot{F}_2) & \text{with probability } \ \lambda_2 dt\ . \end{cases} \tag{4.3.37}$$

The change in futures prices, dF, can be found by applying Itô's Lemma (see Section 1.1.4) so $d\pi$ can be written as

$$d\pi = \begin{cases} \mathcal{L}(\dot{F}_1) - \Delta\mathcal{L}(\ddot{F}_1) & \text{with probability } \ 1 - \lambda_2 dt, \\ \dot{F}_2 - \dot{F}_1 + \mathcal{L}(\dot{F}_2) - \Delta\Big(\ddot{F}_2 - \ddot{F}_1 + \mathcal{L}(\ddot{F}_2)\Big) & \text{with probability } \ \lambda_2 dt\ , \end{cases} \tag{4.3.38}$$

$$\text{where } \ \mathcal{L}(F) = \big((r - \delta_1)P\frac{\partial F}{\partial P} + \frac{\sigma^2 P^2}{2}\frac{\partial^2 F}{\partial P^2} + \frac{\partial F}{\partial t}\big)dt + \sigma P\frac{\partial F}{\partial P}d\tilde{Z} \tag{4.3.39}$$

The expected value of the change in our portfolio is given by

$$\mathbb{E}(d\pi) = \mathcal{L}(\dot{F}_1) - \Delta\mathcal{L}(\ddot{F}_1) + \lambda_2 dt\Big(\dot{F}_2 - \dot{F}_1 - \Delta(\ddot{F}_2 - \ddot{F}_1)\Big) + O(dt^2)\ . \tag{4.3.40}$$

With the judicious choice of $\Delta = \frac{\partial \dot{F}_1}{\partial P} / \frac{\partial \ddot{F}_1}{\partial P}$ the risk in our portfolio is eliminated and the change in the value can be simplified to

$$\begin{aligned}\mathbb{E}(d\pi) = \Bigg[&\frac{\sigma^2 P^2}{2}\frac{\partial^2 \dot{F}_1}{\partial P^2} + \frac{\partial \dot{F}_1}{\partial t} + \lambda_2(\dot{F}_2 - \dot{F}_1) \\ &- \frac{\partial \dot{F}_1}{\partial P} / \frac{\partial \ddot{F}_1}{\partial P}\left(\frac{\sigma^2 P^2}{2}\frac{\partial^2 \ddot{F}_1}{\partial P^2} + \frac{\partial \ddot{F}_1}{\partial t} + \lambda_2(\ddot{F}_2 - \ddot{F}_1)\right)\Bigg] dt\ .\end{aligned} \tag{4.3.41}$$

As it costs nothing to enter into a futures contract $E(d\pi) = 0$, and (4.3.41) can be rewritten as

$$\frac{\frac{\sigma^2 P^2}{2}\frac{\partial^2 \dot{F}_1}{\partial P^2} + \frac{\partial \dot{F}_1}{\partial t} + \lambda_2(\dot{F}_2 - \dot{F}_1)}{\frac{\partial \dot{F}_1}{\partial P}} = \frac{\frac{\sigma^2 P^2}{2}\frac{\partial^2 \ddot{F}_1}{\partial P^2} + \frac{\partial \ddot{F}_1}{\partial t} + \lambda_2(\ddot{F}_2 - \ddot{F}_1)}{\frac{\partial \ddot{F}_1}{\partial P}}\ . \tag{4.3.42}$$

The left and right-hand sides in the equation (4.3.42) are functions of $\dot{F}_1$ and $\ddot{F}_1$ respectively. This means that both sides must be independent of the expiry date. Hence, we can drop the accents and write

$$\frac{\frac{\sigma^2 P^2}{2}\frac{\partial^2 F_1}{\partial P^2} + \frac{\partial F_1}{\partial t} + \lambda_2(F_2 - F_1)}{\frac{\partial F}{\partial P}} = \xi(P,t) \tag{4.3.43}$$

for some arbitrary function $\xi(P,t)$. If we let $\xi(P,t) = (r - \delta_1)P$ then we get

$$\frac{\sigma^2 P^2}{2}\frac{\partial^2 F_1}{\partial P^2} + \frac{\partial F_1}{\partial t} + (r - \delta_1)P\frac{\partial F_1}{\partial P} = \lambda_2(F_1 - F_2) \tag{4.3.44}$$

which needs to be solved subject to $F_1(P,T) = P$.

Similarity, if we start with state 2 we get

$$\frac{\sigma^2 P^2}{2}\frac{\partial^2 F_2}{\partial P^2} + \frac{\partial F_2}{\partial t} + (r - \delta_2)P\frac{\partial F_2}{\partial P} = \lambda_1(F_2 - F_1) \tag{4.3.45}$$

which needs to be solved subject to $F_2(P,T) = P$. The initial conditions suggest solutions to (4.3.44) and (4.3.45) of the forms $F_i(P,t) = Pv_i(t)$ where $v_i(T) = 1$ for $i = 1,\ 2$. Letting $\tau = T - t$ then substituting this form into (4.3.44) and (4.3.45) we get

$$\frac{dv_1(\tau)}{d\tau} = av_1(\tau) + bv_2(\tau) \qquad \text{subject to} \quad v_1(0) = 1 \tag{4.3.46a}$$

$$\frac{dv_2(\tau)}{d\tau} = cv_2(\tau) + dv_1(\tau) \qquad \text{subject to} \quad v_2(0) = 1 \tag{4.3.46b}$$

$$\text{where} \quad a = r - \delta_1 - \lambda_2,\ b = \lambda_2,$$

$$c = r - \delta_2 - \lambda_1,\ d = \lambda_1\ .$$

Equations (4.3.46a,b) are a system of two linear homogeneous first-order ODEs with constant coefficients, which can be written in matrix form as

$$V' = AV \tag{4.3.47a}$$

where

$$V' = \begin{pmatrix} \frac{dv_1(\tau)}{d\tau} \\ \frac{dv_1(\tau)}{d\tau} \end{pmatrix},\ A = \begin{pmatrix} a & b \\ d & c \end{pmatrix} \text{ and } V = \begin{pmatrix} v_1(\tau) \\ v_2(\tau) \end{pmatrix}\ . \tag{4.3.47b}$$

The general solution of (4.3.47a) (see e.g. Polyanin and Manzhirov (2007)) is given by

$$V = \begin{pmatrix} v_1(\tau) \\ v_1(\tau) \end{pmatrix} = c_1 e^{x_1 \tau} V_1 + c_2 e^{x_2 \tau} V_2 \tag{4.3.48}$$

where c_1 and c_2 are arbitrary constants, x_1 and x_2 are the eigenvalues of A (i.e. x_1 and x_2 satisfy $x^2 - (a+c)x + ac - bd = 0$), V_1 and V_2 are the corresponding eigenvectors of x_1 and x_2 respectively[1]. The values of V_1, V_2, c_1 and c_2 are easily found to be $V_1 = \binom{b}{x_1-a}$, $V_2 = \binom{b}{x_2-a}$, $c_1 = \frac{1-\frac{x_2-a}{b}}{x_1-x_2}$ and $c_2 = \frac{1}{b} - c_1$. Hence, substituting these values into (4.3.48) and using our change of variables $F_i(p,t) = Pv_i(\tau)$ where $\tau = T - t$ we get the price of the futures contract as

$$F_1(P,t) = \frac{P}{2\sqrt{H}}\left(A_1 e^{(T-t)x_1} + A_2 e^{(T-t)x_2}\right) \tag{4.3.49a}$$

$$F_2(P,t) = \frac{P}{4\lambda_2\sqrt{H}}\left(B_1 e^{(T-t)x_1} + B_2 e^{(T-t)x_2}\right) \tag{4.3.49b}$$

where

$$x_1 = \frac{1}{2}\left(2r - (\delta_1 + \delta_2 + \lambda_1 + \lambda_2) + \sqrt{H}\right) \tag{4.3.49c}$$

$$x_2 = \frac{1}{2}\left(2r - (\delta_1 + \delta_2 + \lambda_1 + \lambda_2) - \sqrt{H}\right) \tag{4.3.49d}$$

$$A_1 = \sqrt{H} - \delta_1 + (\delta_2 + \lambda_1 + \lambda_2) \tag{4.3.49e}$$

$$A_2 = \sqrt{H} + \delta_1 - (\delta_2 + \lambda_1 + \lambda_2) \tag{4.3.49f}$$

$$B_1 = A_1\left((\delta_1 - \delta_2) + (\lambda_2 - \lambda_1) + \sqrt{H}\right) \tag{4.3.49g}$$

$$B_2 = -A_2\left((\delta_2 - \delta_1) + (\lambda_1 - \lambda_2) + \sqrt{H}\right) \tag{4.3.49h}$$

$$H = (\lambda_1 + \lambda_2)^2 + (\delta_1 - \delta_2)^2 + 2\lambda_1(\delta_2 - \delta_1) + 2\lambda_2(\delta_1 - \delta_2)\ . \tag{4.3.49i}$$

[1]Note that for the case $(a-c)^2 + 4bd = 0$, the general solution of (4.3.46a,b) has a different form from (4.3.48) (further details can be found in Polyanin and Manzhirov (2007)). However, we ignore this case here as here the determinant is always positive.

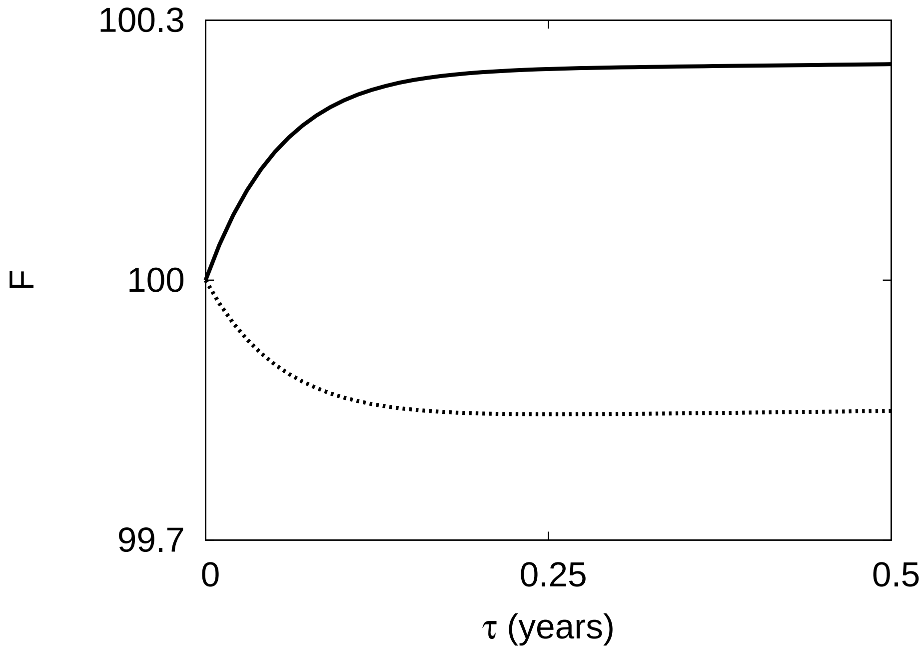


Figure 4.4: Futures prices (F) under (4.3.33) for various expiries (τ) (dotted line $F_1(P,\tau)$ and solid line $F_2(P,\tau)$) with $P = 100,\ r = 0.02,\ \delta_1 = 0.1,\ \delta_2 = -0.1,\ \lambda_1 = 30$ and $\lambda_2 = 20$.

See Figure 4.4 for sample plots of futures prices under (4.3.33) with $P = 100,\ r = 0.02,\ \delta_1 = 0.1,\ \delta_2 = -0.1,\ \lambda_1 = 30$ and $\lambda_2 = 20$ for various expiries. From Figure 4.4 it can be observed that for futures contracts with the given parameter values, for the first state with $\delta_1 = 0.1$ the futures prices decrease as time to expiry increases. This is to be expected as under (4.3.33) with $r - \delta < 0$ oil prices are expected (at least initially) to decrease with time. For the second state with $\delta_2 = -0.1$, it can be observed that the futures prices increase as time to expiry decreases. This is because the transition to the second state yields $r - \delta > 0$ in (4.3.33) and so oil prices are expected (at least initially) to increase with time.

4.4 Conclusion

In this chapter we extended our approach to pricing futures contracts by proposing four different models. In particular, net demand, interest rate and convenience yield are considered as additional factors to oil prices. Further, we considered a regime-switching model, whereby the price of oil was assumed to follow an one-factor stochastic in each regime. For each proposed model, an analytic formula for futures contract prices was derived.

Chapter 5

Pricing European Option Contracts on Futures with One Factor Models

5.1 Introduction

Trading in oil occurs mostly on futures and forward contracts rather than in oil itself. In this thesis we concentrate on futures rather than forward contracts as futures trade on futures exchanges and offer greater liquidity. Options on futures contain two maturities, the maturity of the option, T_O, and the maturity of the underlying futures, T_F, where $T_O \leq T_F$. An overview of the notation used in this chapter is provided in Table 5.1. Given that the value of the underlying futures contract at time t is $F(P,t)$, then at the maturity of the option contract with strike price K, its value is given by $\max(F(P,T_O)-K,0)$ for a call option and by $\max(K-F(P,T_O),0)$ for a put option. Hence for a call option if $F(P,T_O) > K$ then the holder receives the futures contract for the strike price K and gains \$ $(F(P,T_O) - K)$. In contrast, for a put option if $F(P,T_O) < K$ then the holder sells the futures contract for the strike price K and gains \$ $(K - F(P,T_O))$. For times other than at the expiry time, there are a number of ways that options can be valued. Some of these are discussed in Section 1.4 and include finding risk-neutral expectations and solving governing equations. Both of these particular methods will be used in this chapter.

The story of options pricing begins with the work of Bachelier (1900), who assumed that the underlying asset prices follow the ABM and derived a formula for pricing option contracts. More than six decades later, Samuelson (1965) replaced Bachelier's assumption on the underlying asset price with the GBM. Then, Black and Scholes (1973) used Samuelson's assumption and studied the risk-neutral valuation to derive a formula for pricing European option contracts. Consequently, research into option pricing theory was spurred. For further details, we refer the reader to Section 1.5.2.

In this chapter we derive and examine European option prices on oil futures under the one-factor price models that were identified by the GMM analysis (see Chapter 2) as being acceptable in explaining the behaviour of Brent crude oil prices. In particular, exact and approximate formulae for option prices under Model 9 (2.1.55) and Model 10 (2.1.57) are found. In addition, we compare the performance of these new formulae against a known popular formula for describing market prices of options on oil futures. The remainder of this chapter is organised into four sections. In Section 5.2 the exact solutions for the prices of European option contracts under the Schwartz model (2.1.27), Model 9 (2.1.55) and Model 10 (2.1.57) are derived. Then in Section 5.3, analytic approximation formulae for option prices under Model 9 and Model 10 are derived and then the accuracy of these formulae is investigated in Section 5.4. The results of empirical tests which compare the performance of the exact and approximate pricing formulae in their ability to describe market prices are presented in Section 5.5 and in Section 5.6 we present our conclusion.

Notation	Definition
$P \ (= P_t)$	The spot price of oil at time t. Subscripts will be used only when necessary to avoid confusion.
$F \ (= F(P,t))$	The value of a futures contract at time t.
$V \ (= V(F,t))$	The value of the option contract at time t. This value is a function of the current underlying futures price F and time t. As $F = F(P,t)$ it may be convenience to consider the oil as the underlying and so use $V(P,t)$.
$C \ (= C(F,t))$	The value of the call option contract at time t.
$U \ (= U(F,t))$	The value of the put option contract at time t.
K	The strike price of the option contract.
T_F	The expiry time of the futures contract.
τ_F	The time to expiry of the futures contract i.e. $T_F - t$.
T_O	The expiry time of the option contract.
τ_O	The time to expiry of the option contract i.e. $T_O - t$.
$T_D \ (= \tau_D)$	The difference between the expiries of the futures and options contracts i.e. $T_F - T_O (= \tau_F - \tau_O)$.

Table 5.1: Notation used in Chapter 5.

5.2 Deriving Exact Solutions for European Option Contracts

5.2.1 Preliminaries

In this section, we firstly show that futures prices have zero drift in a risk-neutral world.

Suppose that the risk-neutral process followed by the oil price P is given by

$$dP = \tilde{\mu}(P,t)dt + \sigma(P,t)d\tilde{Z}. \tag{5.2.1}$$

By Itô's Lemma (see Section 1.1.4) the change in the futures price in a small time step dt is

$$dF = \left[\frac{\partial F}{\partial t} + \tilde{\mu}(P,t)\frac{\partial F}{\partial P} + \frac{\sigma(P,t)^2}{2}\frac{\partial^2 F}{\partial P^2}\right]dt + \sigma(P,t)\frac{\partial F}{\partial P}d\tilde{Z} \ . \tag{5.2.2}$$

Comparison of the drift term in (5.2.2) with the PDE[1] followed by F indicates that the drift term in (5.2.2) is zero and so

$$dF = \sigma(P,t)\frac{\partial F}{\partial P}d\tilde{Z} \ . \tag{5.2.3}$$

Hence, in general the futures price has zero drift in a risk-neutral world where the numeraire is the money-market account. This mean that futures prices are martingales, i.e. their expected value at a future time is equal to their value today (see Hull (2012)). This is to be expected as it costs nothing to enter into a futures contract.

This section is concerned with deriving exact solutions for options on oil futures and is sub-divided into four subsections. In the following subsections, we derive exact solutions for the prices of European option on futures contracts, which are associated with the three stochastic models for oil that were deemed acceptable from our GMM analysis in Chapter 2. The first stochastic model is the Schwartz (1997) model, which assumes that the logarithm of the spot price follows a mean-reverting OU process, namely (2.1.10). The second and third stochastic models are new models suggested in this thesis, in which we assume that the spot price of oil follows Model 9 (2.1.55) and Model 10 (2.1.57) respectively. For each of the three stochastic models we assume that the risk-neutral processes associated with them have the same forms as the real processes. That is, like many authors such as Stein and Stein (1991) and Grünbichler and Longstaff (1995) we will assume that the market price of risk is such that the risk-neutral process and the real process for P have the same form.

[1] By the Feynman-Kac theorem (see Section 1.3.5) this PDE is given by $\frac{\partial F}{\partial t} + \tilde{\mu}(P,t)\frac{\partial F}{\partial P} + \frac{\sigma(P,t)^2}{2}\frac{\partial^2 F}{\partial P^2} = 0$.

5.2.2 European Option Prices Under the Schwartz Model

The risk-neutral Schwartz model (Model 4 in Chapter 2) is given by

$$dP = \eta P(\mu - \ln P)dt + \sigma P d\tilde{Z} \ . \tag{5.2.4}$$

The futures prices with expiry T_F under this risk-neutral process is given by

$$F(P,t) = \exp\{\ e^{-\eta(T_F-t)}\ln P + \alpha^*(1-e^{-\eta(T_F-t)}) + \frac{\sigma^2}{4\eta}(1-e^{-2\eta(T_F-t)})\} \tag{5.2.5}$$

where $\alpha^* = \mu - \frac{\sigma^2}{2\eta}$. Using (5.2.3) the risk-neutral process followed by the futures price under (5.2.4) is given by

$$\begin{aligned} dF &= \sigma P \times F \times \frac{e^{-\eta(T_F-t)}}{P} d\tilde{Z} \\ &= \sigma F e^{-\eta(T_F-t)} d\tilde{Z} \ . \end{aligned} \tag{5.2.6}$$

As an option contract is a function of the futures price F and time t, the instantaneous change in the option price, dV, can be found by applying Itô's Lemma (see Section 1.1.4) to get

$$dV = \left[\frac{\partial V}{\partial t} + \frac{e^{-2\eta(T_F-t)}\sigma^2 F^2}{2}\frac{\partial^2 V}{\partial F^2}\right]dt + \sigma e^{-\eta(T_F-t)} F \frac{\partial V}{\partial F} d\tilde{Z} \ . \tag{5.2.7}$$

Now construct a portfolio with value π, that consists of $\frac{\partial V}{\partial F}$ futures contracts and a short position in the options contract. Hence $\pi = -V$ as there is no cost incurred when entering into a futures contract. The change in value of our portfolio in time dt can be written as

$$d\pi = \frac{\partial V}{\partial F}dF - dV = -\left[\frac{\partial V}{\partial t} + \frac{\sigma^2 e^{-2\eta(T_F-t)} F^2}{2}\frac{\partial^2 V}{\partial F^2}\right]dt \ . \tag{5.2.8}$$

As the portfolio is riskless, by arbitrage the instantaneous return of the portfolio, $d\pi$, should equal the risk-free interest rate r i.e.

$$\frac{d\pi}{\pi} = rdt \; . \tag{5.2.9}$$

Substituting (5.2.8) into (5.2.9) we get the PDE for the options price as

$$\frac{\partial V}{\partial t} + \frac{e^{-2\eta(T_F-t)}\sigma^2 F^2}{2}\frac{\partial^2 V}{\partial F^2} - rV = 0 \; . \tag{5.2.10}$$

Equation (5.2.10) needs to be solved subject to a final condition: $V(F, T_O) = \max(F(P, T_O) - K, 0)$ for a call option and $V(F, T_O) = \max(K - F(P, T_O), 0)$ for a put option. We note that equation (5.2.10) also follows from the the Feynman-Kac theorem (see Section 1.3.5).

Equation (5.2.10) is a special case of the Black-Scholes PDE (1.14a) with time-dependent volatility $\sigma F e^{-\eta(T_F-t)}$ and dividend yield D equal to the interest rate r. The Black-Scholes formula then is still valid with the averaged future volatility w, see Wilmott (1998), where

$$w^2(t, T_F, T_O) = \frac{1}{T_O - t}\int_t^{T_O} \sigma^2 e^{-2\eta(T_F-s)} ds = \frac{\sigma^2}{2\eta(T_O - t)}(e^{-2\eta(T_F-T_O)} - e^{-2\eta(T_F-t)}) \; . \tag{5.2.11}$$

Hence the price at time t, of a European call option contract with expiry T_O and strike price K on a futures contract that matures at time T_F is given by

$$C(F, t) = e^{-r(T_O-t)}\left[FN(d_1) - KN(d_2)\right] \tag{5.2.12}$$

$$\text{where} \quad d_1 = \frac{\ln(\frac{F}{K}) + \frac{w^2}{2}(T_O - t)}{w\sqrt{T_O - t}} \quad \text{and} \quad d_2 = d_1 - w\sqrt{T_O - t}$$

and the formula for a European put option contract is given by

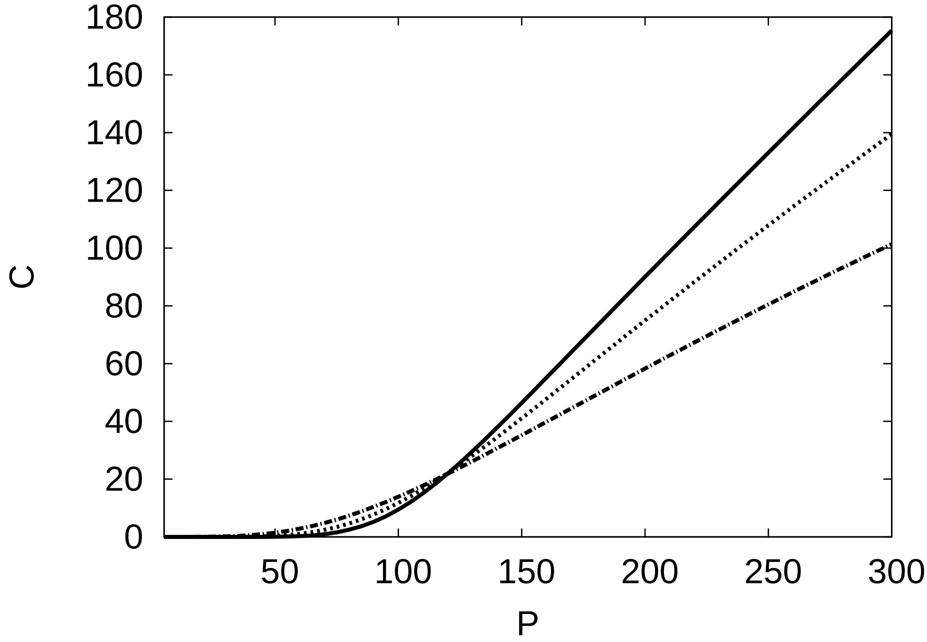
$$U(F,t) = e^{-r(T_O - t)} \left[-FN(-d_1) - KN(-d_2) \right] . \tag{5.2.13}$$

See Figure 5.1 for sample plots of options prices (5.2.12) and (5.2.13) with $K = 100, r = 1\%,\ \alpha^* = 4.6,\ \eta = 1$ and $\sigma = 0.5$ for various spot prices and expiries. From Figure 5.1 it can be observed that for call option contracts with the given parameter values, the option is more valuable with longer expiries for underlying prices less than just above \$ 112.73, the mean spot price. For spot prices greater than this, option prices decrease with time to expiry. This is to be expected as under (5.2.4) oil prices revert to the mean price; so that if the option is in-the-money there is a high probability that the spot price will decrease with time. In contrast, for put option contracts the option is more valuable with shorter expiries for smaller values of the underlying prices where the option is in-the-money. However, for near-at-the-money options and larger values of P where the option is out-of-the-money the value increases with time to expiry. For both call and put option contracts, the effect of time to expiry on the option value starts to decrease as the options get near at-the-money and is eliminated as the options get deep out-of-the-money.

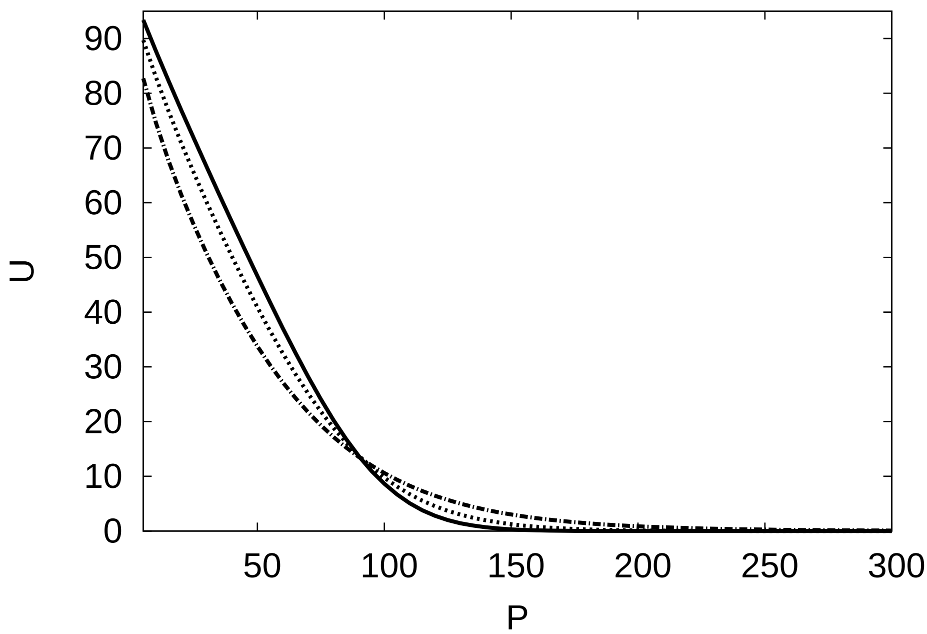
5.2.3 European Option Prices Under Stochastic Model 9

Here we assume that P follows the risk-neutral process corresponding to Model 9 i.e.

$$dP = a\sqrt{P}dt + \sigma P^{\frac{3}{4}} d\tilde{Z} \ . \tag{5.2.14}$$



(a)



(b)

Figure 5.1: (a) Call option prices (C), and (b) Put option prices (U) under the Schwartz model (5.2.4) for various spot prices (P) and expiries (solid line $\tau_O = 30$ days, dotted line $\tau_O = 90$ days and dashed line $\tau_O = 180$ days) with $K = 100, r = 1\%$, $\alpha^* = 4.6$, $\eta = 1$ and $\sigma = 0.5$.

Then the transition density function of P is given by

$$Tr_9(P,t;P_T,T) = \frac{4\sqrt{P}e^{\frac{-8(\sqrt{P_T}+\sqrt{P})}{\sigma^2(T-t)}} I_\nu(z)\left(\frac{P_T}{P}\right)^{\frac{a}{\sigma^2}}}{P_T\sigma^2(T-t)} \tag{5.2.15a}$$

$$\text{where } \nu = \frac{4a}{\sigma^2} - 2, \;\; z = \frac{16(P_T P)^{\frac{1}{4}}}{\sigma^2(T-t)} \tag{5.2.15b}$$

(see Goard(2006)) and where $I_\nu(.)$ is the modified Bessel function of the first kind of order ν (see e.g. Abramowitz and Stegun (1964)). Hence we can find the price of a call option contract that expires at time T_O with strike price K, on a futures contract that matures at time T_F , as the present value of the expected payoff under the risk-neutral measure $\mathbb{Q}$ i.e.

$$\begin{aligned} C(P,t) &= e^{-r(T_O-t)}\mathbb{E}_t^{\mathbb{Q}}\left(\max[F(P,T_O)-K,0]\right) \\ &= e^{-r(T_O-t)}\left[\int_0^\infty \max[F(y,T_O)-K,0]\; Tr_9(P,t;y,T_O)\; dy\right] \\ &= e^{-r(T_O-t)}\left[\int_{H^2}^\infty \max[F(y,T_O)-K,0]\; Tr_9(P,t;y,T_O)\; dy\right] \\ &= e^{-r(T_O-t)}\left[F-K-\int_0^{H^2} [F(y,T_O)-K]\; Tr_9(P,t;y,T_O)\; dy\right] \end{aligned} \tag{5.2.16a}$$

$$\text{where} \quad F(P,t) = P + a(T_F-t)\sqrt{P} + \frac{(a(T_F-t))^2}{4}(1-\frac{\sigma^2}{4a}) \; , \tag{5.2.16b}$$

$$H = \frac{-\alpha + \sqrt{\alpha^2 - 4(\beta-K)}}{2} \tag{5.2.16c}$$

$$\text{and} \;\; \alpha = aT_D, \quad \beta = \frac{(aT_D)^2}{4}(1-\frac{\sigma^2}{4a}). \tag{5.2.16d}$$

Similarly the price of put option contracts is given by

$$U(P,t) = e^{-r(T_O-t)}\left[\int_0^\infty \max[K-F(y,T_O),0]\; Tr_9(P,t;y,T_O)\; dy\right] \; , \tag{5.2.17}$$

where F is given in (5.2.16b).

See Figure 5.2 for sample plots of option prices (5.2.16a) and (5.2.17) with $K = 100$, $r = 1\%$, $a = 2$ and $\sigma = 1$ for various spot prices and expiries. From Figure 5.2 it can be observed that for call option contracts with the given parameter values the longer the time to expiry the more valuable the option is. This is to be expected as under Model 9 oil prices are expected to increase with time. In contrast for put option contracts, the shorter the time to expiry the more valuable the option is except for large values of P where the option is out-of-the-money. For both call and put option contracts, the effect of time to expiry on the option value starts to decrease as the options get near at-the-money and is eliminated as the options get deep out-of-the-money.

5.2.4 European Option Prices Under Stochastic Model 10

Here we assume that P follows the risk-neutral process associated with Model 10 i.e.

$$dP = (a\sqrt{P} + bP)dt + \sigma P^{\frac{3}{4}} d\tilde{Z}, \quad b < 0\ . \tag{5.2.18}$$

The transition density function of P is given by

$$Tr_{10}(P, t; P_T, T) = Q_1(P; P_T)\ Q_2(P; P_T)\ P_T^{\frac{a}{c^2}-1}\ P^{\frac{1}{2}-\frac{a}{c^2}}\ I_\nu(z) \tag{5.2.19a}$$

$$\text{where}\quad Q_1(P; P_T) = \frac{2\sqrt{b^2} e^{\frac{2(\sqrt{P}-\sqrt{P_T})(\sqrt{b^2}-b)}{\sigma^2} + \frac{T-t}{4}(\sqrt{b^2}+b-\frac{4ab}{\sigma^2})}}{\sigma^2(e^{\frac{\sqrt{b^2}(T-t)}{2}} - 1)}, \tag{5.2.19b}$$

$$Q_2(P; P_T) = \exp\left\{\frac{-4\sqrt{b^2}(\sqrt{P}e^{\frac{\sqrt{b^2}(T-t)}{2}} + \sqrt{P_T})}{\sigma^2(e^{\frac{\sqrt{b^2}(T-t)}{2}} - 1)}\right\} \tag{5.2.19c}$$

$$\nu = \frac{4a}{c^2} - 2 \quad\text{and}\quad z = \frac{8(PP_T)^{\frac{1}{4}}\sqrt{b^2}e^{\frac{\sqrt{b^2}(T-t)}{4}}}{\sigma^2(e^{\frac{\sqrt{b^2}(T-t)}{2}} - 1)}. \tag{5.2.19d}$$

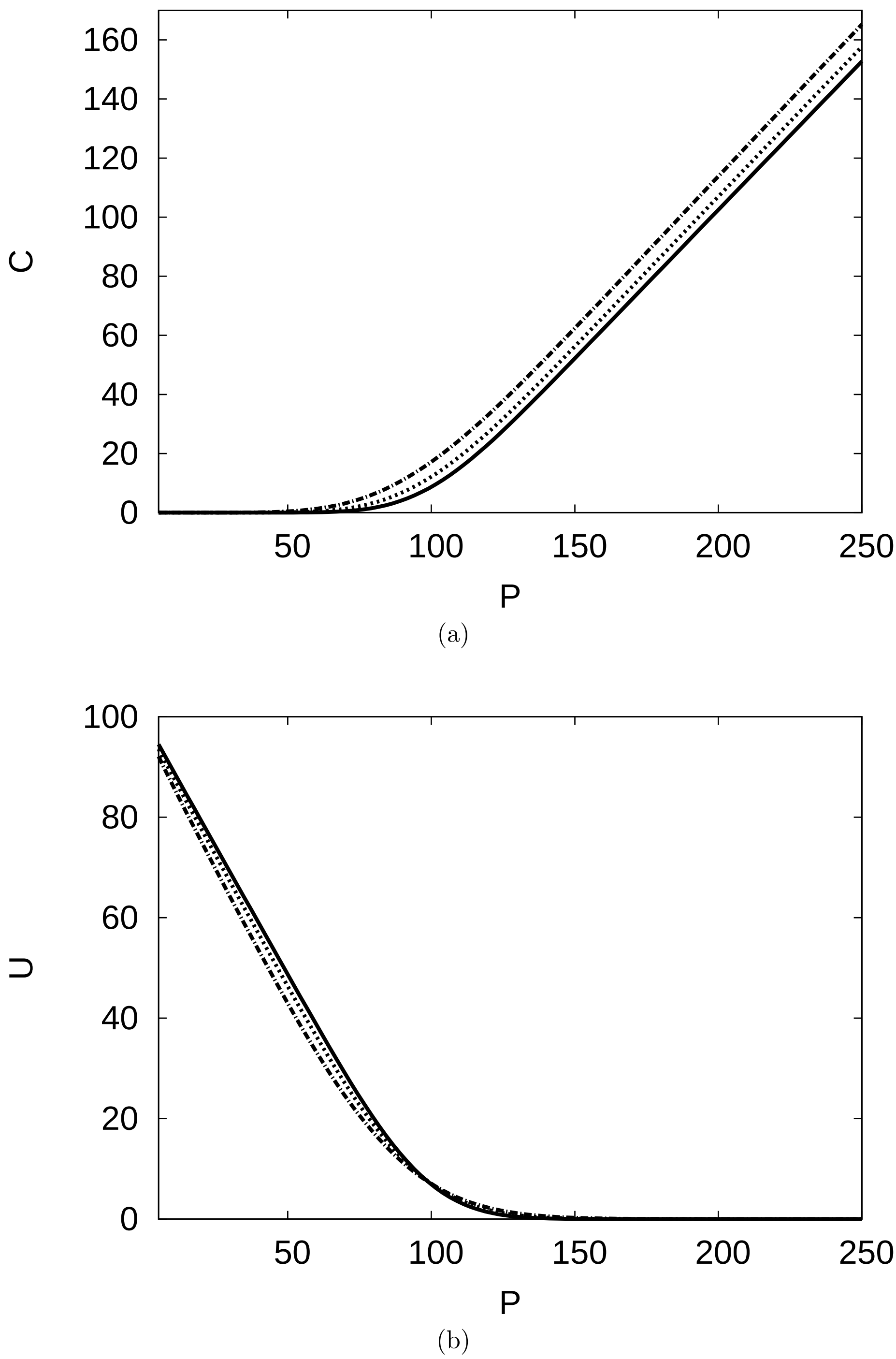


Figure 5.2: (a) Call option prices (C), and (b) Put option prices (U) under Model 9 (5.2.14) for various spot prices prices (P) and expiries (solid line $\tau_O = 30$ days, dotted line $\tau_O = 90$ days and dashed line $\tau_O = 180$ days) with $K = 100, r = 1\%,\ a = 2$ and $\sigma = 1$.

(see Goard(2006)) and where $I_\nu(.)$ is the modified Bessel function of the first kind of order ν (see e.g. Abramowitz and Stegun (1964)). Hence we can find the price of a call option contract that expires at time T_O with a strike price K, on a futures contract that matures at time T_F, as the present value of the expected payoff under the risk-neutral measure $\mathbb{Q}$ i.e.

$$\begin{aligned} C(P,t) &= e^{-r(T_O-t)}\mathbb{E}_t^{\mathbb{Q}}\left(\max[F(P,T_O)-K,0]\right) \\ &= e^{-r(T_O-t)}\left[\int_0^{\infty}\max[F(y,T_O)-K,0]\; Tr_{10}(P,t;y,T_O)\;dy\right] \\ &= e^{-r(T_O-t)}\left[\int_{H^2}^{\infty}\max[F(y,T_O)-K,0]\; Tr_{10}(P,t;y,T_O)\;dy\right] \\ &= e^{-r(T_O-t)}\left[F-K-\int_0^{H^2}[F(y,T_O)-K]\; Tr_{10}(P,t;y,T_O)\;dy\right] \end{aligned} \tag{5.2.20a}$$

where $$F(P,t) = Pe^{b(T_F-t)} + \frac{2a\sqrt{P}}{b}(e^{b(T_F-t)} - e^{\frac{b}{2}(T_F-t)}) + \frac{a(4a-\sigma^2)}{4b^2}(e^{\frac{b}{2}(T_F-t)}-1)^2\ , \tag{5.2.20b}$$

$$H = \frac{-\alpha+\sqrt{\alpha^2-4e^{bT_D}(\beta-K)}}{2e^{bT_D}}\ , \tag{5.2.20c}$$

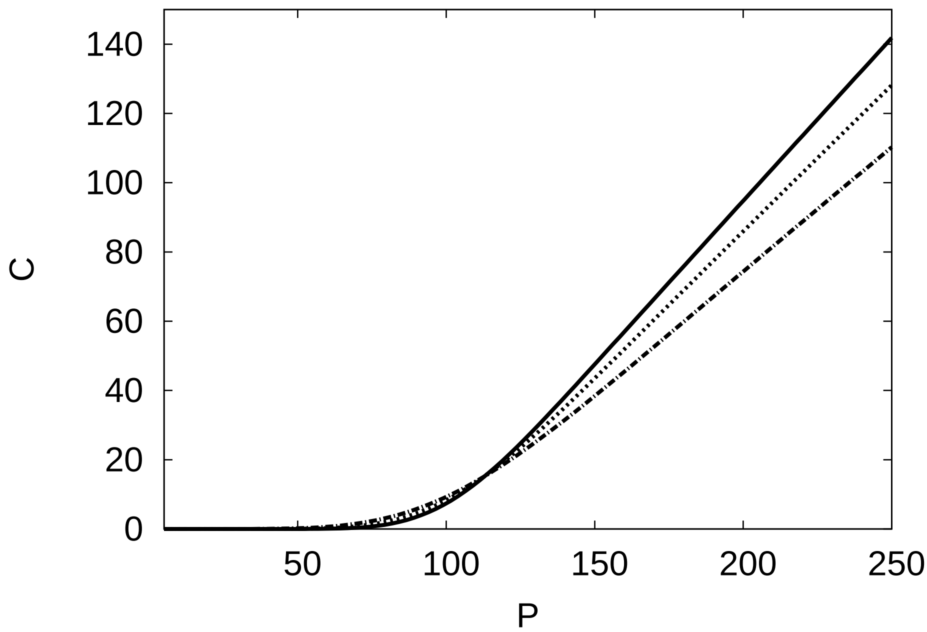
$$\alpha = \frac{2a}{b}(e^{bT_D}-e^{\frac{b}{2}T_D}) \quad \text{and} \quad \beta = \frac{a(4a-\sigma^2)}{4b^2}(e^{\frac{b}{2}T_D}-1)^2. \tag{5.2.20d}$$

The price of a put option contract is given by

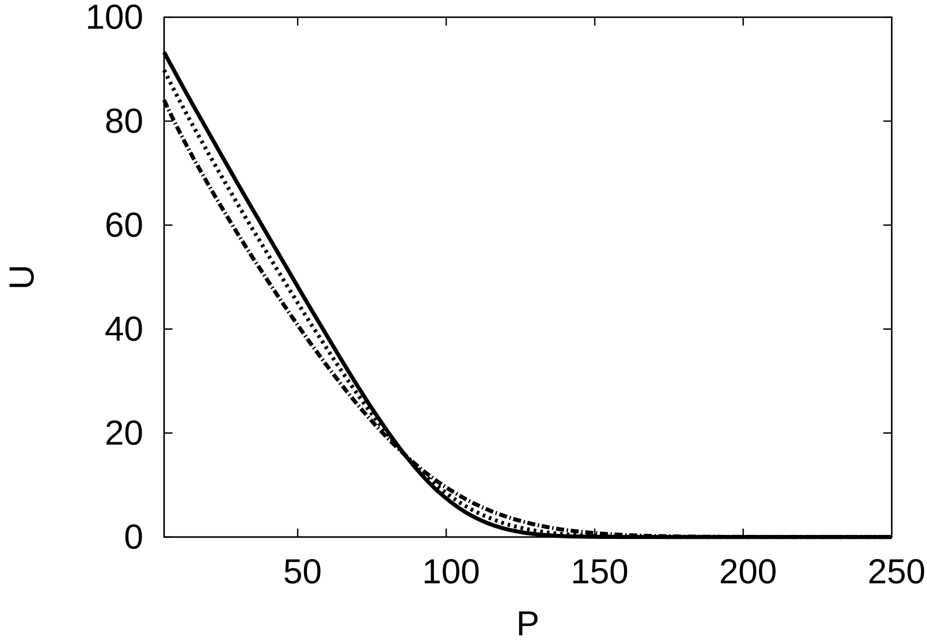
$$U(P,t) = e^{-r(T_O-t)}\left[\int_0^{\infty}\max[F(y,T_O)-K,0]\; Tr_{10}(P,t;y,T_O)\;dy\right]. \tag{5.2.21}$$

where F is given in (5.2.20b).

See Figure 5.4 for sample plots of options prices (5.2.20a) and (5.2.21) with $K = 100,\ r = 1\%,\ a = 10,\ b = -1$ and $\sigma = 1$ for various spot prices and expiries. From Figure 5.4 it can be observed that for call option contracts with the given parameter values, the option is more valuable with longer expiries for underlying prices less than just above \$ 118.4 , the mean spot price. For spot prices greater than this, option prices decrease with time to expiry. This is to be expected as under (5.2.18)



(a)



(b)

Figure 5.3: (a) Call option prices (C), and (b) Put option prices (P) under Model 10 (5.2.18) for various spot prices (P) and expiries (solid line $\tau_O = 30$ days, dotted line $\tau_O = 90$ days and dashed line $\tau_O = 180$ days) with $K = 100$, $r = 1\%$, $a = 10$, $b = -1$ and $\sigma = 1$.

oil prices revert to the mean price; so that if the option is in-the-money there is a high probability that the spot price will decrease with time. In contrast, for put option contracts the option is more valuable with shorter expiries for smaller values of the underlying prices where the option is in-the-money. However, for near-at-the-money options and larger values of P, where the option is out-of-the-money, the value increases with time to expiry. For both call and put option contracts, the effect of time to expiry on the option value starts to decrease as the options get near at-the-money and is eliminated as the options get deep out-of-the-money.

5.3 Analytic Approximation Formulae for Option Prices

The formulae derived in the previous section for the options prices under Models 9 and 10 still involve the evaluation of integrals. We now wish to provide simple analytic approximation formulae for the option prices which only involve standard functions and possibly special functions which are built-in to most mathematical software packages such as Maple (see Maplesoft (2008)), and which are fast and easy to implement. The new analytic approximations we find are suitable for pricing options with short time to expiry (i.e. short tenor) which is a common feature of most options in the traded market. This section is sub-divided into three parts. First we derive the governing equations for pricing option contracts under Models 9 and 10. In the second part, we derive an analytic approximation for the price of European call option contracts under Model 10. This is done firstly with the option as a function of P and t and then as a function of F and t. Finally in the third part, an analytic approximation for the price of European call option contracts under Model 9 is presented.

5.3.1 Deriving Governing Equations for Option Pricing

In Section 5.2.2, we derived the PDE (5.2.10) to price options on a futures contract when the futures model was based on the Schwartz model for the underlying price. In a similar way we can show that if futures prices follow the zero drift stochastic process $dF = \sigma(F,t)d\tilde{Z}$, then the PDE for pricing options on futures is given by

$$\frac{\partial V}{\partial t} + \frac{\sigma(F,t)^2}{2}\frac{\partial^2 V}{\partial F^2} - rV = 0 \tag{5.3.22}$$

which needs to be solved subject to the final condition $V(F,T_O) = \max(F-K,0)$ for the call option and $V(F,T_O) = \max(K-F,0)$ for the put option. This again follows from the Feynman-Kac theorem (see Section 1.3.5).

As the volatility term in the process for the dynamics of futures prices under Models 9 and 10 are not simple functions of F and t, we now derive the governing equation for option prices in terms of the oil price, P, given that P follows the stochastic process $dP = \mu(P,t)dt + \sigma P^{\frac{3}{4}}dZ$. However, if we consider oil (which is not a traded security) as the underlying asset, then we cannot use it to hedge with the options, like the role of the futures contracts in options on futures. Instead we now hedge options of different expiries.

We construct a portfolio with value π that consists of two option contracts, with values V_1 and V_2 that have different expiries T_{O_1} and T_{O_2} respectively. Now let our portfolio have one long position in an option with value V_1 and Δ short positions in options each with value V_2 so that

$$\pi = V_1 - \Delta V_2\ . \tag{5.3.23}$$

The option contract is a function of time t and the futures price $F(P,t)$, which is in turn a function of the spot price P and time t. So the instantaneous change in our portfolio, $d\pi$, can be found by applying Itô's Lemma (see Section 1.1.4) to functions of P and t giving

$$d\pi = \frac{\partial V_1}{\partial t}dt + \frac{\partial V_1}{\partial P}dP + \frac{P^{\frac{3}{2}}\sigma^2}{2}\frac{\partial^2 V_1}{\partial P^2}dt - \Delta\left[\frac{\partial V_2}{\partial t}dt + \frac{\partial V_2}{\partial P}dP + \frac{P^{\frac{3}{2}}\sigma^2}{2}\frac{\partial^2 V_2}{\partial P^2}dt\right] . \tag{5.3.24}$$

With the judicious choice of $\Delta = \frac{\partial V_1}{\partial P}/\frac{\partial V_2}{\partial P}$, the risk in our portfolio is eliminated and the change in value will be

$$d\pi = \left[\frac{\partial V_1}{\partial t} + \frac{\sigma^2 P^{\frac{3}{2}}}{2}\frac{\partial^2 V_1}{\partial P^2}\right]dt - \frac{\partial V_1}{\partial P}/\frac{\partial V_2}{\partial P}\left[\frac{\partial V_2}{\partial t} + \frac{\sigma^2 P^{\frac{3}{2}}}{2}\frac{\partial^2 V_2}{\partial P^2}\right]dt . \tag{5.3.25}$$

By arbitrage, the instantaneous return on our portfolio, $d\pi$, should equal the risk-free interest rate r i.e.

$$\frac{d\pi}{\pi} = rdt . \tag{5.3.26}$$

Substituting (5.3.25) into (5.3.26), and rearranging we get

$$\frac{\frac{\partial V_1}{\partial t} + \frac{\sigma^2 P^{\frac{3}{2}}}{2}\frac{\partial^2 V_1}{\partial P^2} - rV_1}{\frac{\partial V_1}{\partial P}} = \frac{\frac{\partial V_2}{\partial t} + \frac{\sigma^2 P^{\frac{3}{2}}}{2}\frac{\partial^2 V_2}{\partial P^2} - rV_2}{\frac{\partial V_2}{\partial P}} . \tag{5.3.27}$$

The left and right-hand sides in equation (5.3.27) are functions of T_{O_1} and T_{O_2} respectively. This means that both sides are independent of the expiry date. Hence, we can drop the subscript from V and write

$$\frac{\frac{\partial V}{\partial t} + \frac{\sigma^2 P^{\frac{3}{2}}}{2}\frac{\partial^2 V}{\partial P^2} - rV}{\frac{\partial V}{\partial P}} = \xi(P,t) , \tag{5.3.28}$$

for some arbitrary function $\xi(P,t)$. If we let $\xi(P,t) = \sigma P^{\frac{3}{4}}\lambda(P,t) - \mu(P,t)$ then we get

$$\frac{\partial V}{\partial t} + \frac{\sigma^2 P^{\frac{3}{2}}}{2}\frac{\partial^2 V}{\partial P^2} - \left(\sigma P^{\frac{3}{4}}\lambda(P,t) - \mu(P,t)\right)\frac{\partial V}{\partial P} - rV = 0 . \tag{5.3.29}$$

Equation (5.3.29) is the PDE governing the price of an option contract. Note that $\lambda(P,t)$ represents the market price of risk for oil. To see why, we consider $dV - rV dt$. From Itô's Lemma (see Section 1.1.4) and (5.3.29) we get

$$\begin{aligned} dV - rV dt = &\left[\frac{\partial V}{\partial t} + \frac{\sigma^2 P^{\frac{3}{2}}}{2}\frac{\partial^2 V}{\partial P^2} + \mu(P,t)\frac{\partial V}{\partial P}\right] dt + \sigma P^{\frac{3}{4}}\frac{\partial V}{\partial P} dZ \\ &- \left[\frac{\partial V}{\partial t} + \frac{\sigma^2 P^{\frac{3}{2}}}{2}\frac{\partial^2 V}{\partial P^2} - \left(\sigma P^{\frac{3}{4}}\lambda(P,t) - \mu(P,t)\right)\frac{\partial V}{\partial P}\right] dt \\ =&\sigma P^{\frac{3}{4}}\frac{\partial V}{\partial P}(\lambda(P,t)dt + dZ)\ . \end{aligned} \tag{5.3.30}$$

Equation (5.3.30) implies that our portfolio is not riskless and can be interpreted as meaning that in return for taking the extra risk associated with oil, the portfolio profits $\lambda(P,t)dt$ per unit of extra risk dZ.

PDE for pricing option contracts under Model 9

We assume that $\lambda(P_t,t) = cP^{-\frac{1}{4}}$, ($c$ constant), so that $\xi(P,t)$ and $\mu(P,t)$ have the same form, namely $a\sqrt{P}$, where a is constant. Then, substituting the value of $\xi(P,t)$ into (5.3.29) we get the PDE governing option prices as

$$\frac{\partial V}{\partial t} + a\sqrt{P}\,\frac{\partial V}{\partial P} + \frac{P^{\frac{3}{2}}\sigma^2}{2}\frac{\partial^2 V}{\partial P^2} - rV = 0 \tag{5.3.31}$$

which needs to be solved subject to the final condition

$$V(P,T_O) = \begin{cases} \max(F(P,T_O) - K, 0) = \max[P + \alpha\sqrt{P} + \beta - K, 0], & \text{for a call option} \\ \max(F(P,T_O) - K, 0) = \max[K - (P + \alpha\sqrt{P} + \beta), 0], & \text{for a put option} \end{cases} \tag{5.3.32}$$

where F is given in (5.2.16b) and α and β are given in (5.2.16d).
We note that $P + \alpha\sqrt{P} + \beta - K = 0$ when $\sqrt{P} = \frac{-\alpha + \sqrt{\alpha^2 - 4(\beta - K)}}{2}$. As $\sqrt{P} \geq 0$, we assume that $\beta \leq K$.

PDE for pricing option contracts under Model 10

We suppose that $\lambda(P,t) = c_1 P^{-\frac{1}{4}} + c_2 P^{\frac{1}{4}}$, c_1, c_2 constants, so that $\xi(P,t)$ and $\mu(P,t)$ have the same form namely, $a\sqrt{P} + bP$, where a, b are constants. Then substituting the value of $\xi(P,t)$ into (5.3.29) we get the PDE governing call option prices as

$$\frac{\partial V}{\partial t} + (a\sqrt{P} + bP)\,\frac{\partial V}{\partial P} + \frac{P^{\frac{3}{2}}\sigma^2}{2}\frac{\partial^2 V}{\partial P^2} - rV = 0\ , \tag{5.3.33}$$

which needs to be solved subject to the final condition

$$V(P,T_O) = \begin{cases} \max(F(P,T_O) - K, 0) = \max[Pe^{bT_D} + \alpha\sqrt{P} + \beta - K, 0] & \text{for a call option,} \\ \max(F(P,T_O) - K, 0) = \max[K - (Pe^{bT_D} + \alpha\sqrt{P} + \beta), 0] & \text{for a put option} \end{cases} \tag{5.3.34}$$

where F is given in (5.2.20b) and α and β are given in (5.2.20d).
We note that $Pe^{bT_D} + \alpha\sqrt{P} + \beta - K = 0$ when $\sqrt{P} = \frac{-\alpha + \sqrt{\alpha^2 - 4e^{bT_D}(\beta - K)}}{2e^{bT_D}}$. As $\sqrt{P} \geq 0$, we assume that $\beta \leq K$.

5.3.2 Analytic Approximation Formula for Option Prices Under Model 10

In this section we derive an analytic approximation for call option prices, $C(P,t)$, under Model 10 when the time to expiry is small. From the previous section, the governing equation for call option prices under Model 10 with expiry T_O and strike price K is given by

$$\frac{\partial C}{\partial t} + (a\sqrt{P} + bP)\,\frac{\partial C}{\partial P} + \frac{P^{\frac{3}{2}}\sigma^2}{2}\frac{\partial^2 C}{\partial P^2} - rC = 0 \tag{5.3.35}$$

where $b < 0$, and needs to be solved subject to the final condition

$$\begin{aligned} C(P,T_O) =& \max(F(P,T_O) - K, 0) \\ =& \max(Pe^{bT_D} + \frac{2a\sqrt{P}}{b}(e^{bT_D} - e^{\frac{b}{2}T_D}) + \frac{a(4a-\sigma^2)}{4b^2}(e^{\frac{b}{2}T_D} - 1)^2) - K, 0) \end{aligned}$$

and boundary conditions

$C(0,t) = 0$ and $C(P,t) \sim Pe^{bT_D} + \frac{2a\sqrt{P}}{b}(e^{bT_D} - e^{\frac{b}{2}T_D}) + \frac{a(4a-\sigma^2)}{4b^2}(e^{\frac{b}{2}T_D} - 1)^2) - K$ as $P \to \infty$. Letting $\tau_O = T_O - t$ and $\tau_D = \tau_F - \tau_O$ we write the futures price under Model 10 as

$$\begin{aligned} F(P,\tau_F) =& Pe^{b\tau_F} + \frac{2a\sqrt{P}}{b}(e^{b\tau_F} - e^{\frac{b}{2}\tau_F}) + \frac{a(4a-\sigma^2)}{4b^2}(e^{\frac{b}{2}\tau_F)} - 1)^2 \\ =& Pe^{b(\tau_O+\tau_D)} + \frac{2a\sqrt{P}}{b}(e^{b(\tau_O+\tau_D)} - e^{\frac{b}{2}(\tau_O+\tau_D)}) + \frac{a(4a-\sigma^2)}{4b^2}(e^{\frac{b}{2}(\tau_O+\tau_D)} - 1)^2 \ . \end{aligned}$$

Equation (5.3.35) can be written as

$$\frac{\partial C}{\partial \tau_O} = (a\sqrt{P} + bP)\ \frac{\partial C}{\partial P} + \frac{P^{\frac{3}{2}}\sigma^2}{2}\frac{\partial^2 C}{\partial P^2} - rC \tag{5.3.36}$$

which needs to be solved subject to initial condition

$$C(P,0) = \max(F(P,\tau_D) - K, 0) = \max(Pe^{b\tau_D} + \alpha\sqrt{P} + \beta - K, 0)$$

where α and β are given in (5.2.16d), and boundary conditions

$$C(0,\tau_O) = 0 \ \text{ and } \ C(P,\tau_O) \to F - K \ \text{ as } \ P \to \infty \ .$$

For a small time to expiry τ_O we let $\tau_O = \varepsilon\hat{\tau}$ where ε is a small parameter $0 < \varepsilon \ll 1$. Then equation (5.3.32) becomes

$$\frac{\partial C}{\partial \hat{\tau}} = \varepsilon(a\sqrt{P} + bP)\ \frac{\partial C}{\partial P} + \varepsilon\frac{P^{\frac{3}{2}}\sigma^2}{2}\frac{\partial^2 C}{\partial P^2} - \varepsilon rC \ . \tag{5.3.37}$$

We assume that the solution can be written as a series in ε i.e.

$$C(P,\grave{\tau}) = \sum_{i=0}^{\infty} \varepsilon^i C_i(P,\grave{\tau}) \ . \tag{5.3.38}$$

Substituting (5.3.38) into (5.3.37) and collecting terms of $O(1)$, we get an equation for $C_0(P,\grave{\tau})$, namely

$$\frac{\partial C_0}{\partial \grave{\tau}} = 0 \ , \tag{5.3.39}$$

which solved subject to the initial condition $C_0(P,0) = \max(F(P,\tau_D) - K, 0)$, gives

$$C_0(P,\grave{\tau}) = \max(F(P,\tau_D) - K, 0) = \begin{cases} F(P,\tau_D) - K & \text{if } F(P,\tau_D) \geq K \ , \\ 0 & \text{otherwise.} \end{cases} \tag{5.3.40}$$

Collecting terms of $O(\varepsilon)$, we get an equation for $C_1(P,\grave{\tau})$:

$$\frac{\partial C_1}{\partial \grave{\tau}} = (a\sqrt{P} + bP)\frac{\partial C_0}{\partial P} + \frac{P^{\frac{3}{2}}\sigma^2}{2}\frac{\partial^2 C_0}{\partial P^2} - rC_0 \ . \tag{5.3.41}$$

Solving (5.3.41) with initial condition $C_1(P,0) = 0$, we get

$$C_1(P,\grave{\tau}) = \begin{cases} \Big((a\sqrt{P} + bP)(e^{b\tau_D} + \frac{\alpha}{2\sqrt{P}}) - \frac{\alpha\sigma^2}{8} - r(F(P,\tau_D) - K)\Big)\grave{\tau} & \text{if } F(P,\tau_D) \geq K \ , \\ 0 & \text{otherwise.} \end{cases} \tag{5.3.42}$$

Hence, the first two terms of our series solution are given by

$$C_0(P,\grave{\tau})+\varepsilon C_1(P,\grave{\tau}) = \begin{cases} (F(P,\tau_D)-K)(1-\varepsilon r\grave{\tau}) & \\ +\varepsilon\grave{\tau}[(a\sqrt{P}+bP)(e^{b\tau_D}+\frac{\alpha}{2\sqrt{P}})-\frac{\alpha\sigma^2}{8}] & F(P_t,\tau_D)\geq K\ , \\ 0 & \text{otherwise.} \end{cases} \tag{5.3.43}$$

From equation (5.3.43) it can be observed that the expansion is continuous, but not differentiable at $F(P,\tau_D)-K$ (this situation is called a *corner layer*). Hence we expect a corner layer, or derivative layer in the vicinity of P where $F(P,\tau_D)-K = Pe^{b\tau_D}+\alpha\sqrt{P}+\beta-K = 0$. Solution (5.3.43) is therefore invalid in this region and is thus termed our "outer" solution. We now analyse the solution in the inner region by introducing a stretching variable

$$x = \frac{F(P,\tau_D)-K}{\sqrt{\varepsilon}K} = \frac{Pe^{b\tau_D}+\alpha\sqrt{P}+\beta-K}{\sqrt{\varepsilon}K} \tag{5.3.44a}$$

and rescale

$$C(P,\grave{\tau}) = K\sqrt{\varepsilon}W(P,\grave{\tau})\ . \tag{5.3.44b}$$

The choice of power $\frac{1}{2}$ in $\sqrt{\varepsilon}$ is a well-balanced choice and ensures that the coefficient of the second-order derivative term is not small compared to the other coefficients. Equation (5.3.37) then becomes

$$\begin{aligned}\frac{\partial W}{\partial\grave{\tau}} &= \frac{c^2}{2K^2}\left(P^{\frac{3}{2}}e^{2b\tau_D}+\alpha Pe^{b\tau_D}+\frac{\alpha^2}{4}\sqrt{P}\right)\frac{\partial^2 W}{\partial x^2}\\ &+\frac{\sqrt{\varepsilon}}{K}\left(bPe^{b\tau_D}+\frac{2ae^{b\tau_D}+\alpha b}{2}\sqrt{P}+\frac{\alpha(4a-\sigma^2)}{8}\right)\frac{\partial W}{\partial x}-r\varepsilon W\end{aligned} \tag{5.3.45}$$

to be solved subject to the initial condition $W(x,0) = \max(x,0)$, and

$$W(x,\grave{\tau}) \sim x + \sqrt{\varepsilon}\theta_1\grave{\tau} \quad \text{as} \quad x \to +\infty \quad \text{and} \quad W(x,\grave{\tau}) \to 0 \quad \text{as} \quad x \to -\infty$$
$$\text{where} \quad \theta_1 = \frac{(4a\sqrt{B} - \alpha\sigma^2)e^{b\tau_D} + 2b(B - \alpha\sqrt{B})}{8Ke^{b\tau_D}} \; . \tag{5.3.46}$$

Now we expand $W(x,\grave{\tau})$ in terms of $\sqrt{\varepsilon}$ i.e.

$$W(x,\grave{\tau}) \; = \sum_{i=0}^{\infty} (\sqrt{\varepsilon})^i W_i(x,\grave{\tau}) \tag{5.3.47}$$

which we substitute into (5.3.45) and then aim to collect terms in powers of $\sqrt{\varepsilon}$. Firstly the terms P, $\sqrt{P}$ and $P^{\frac{3}{2}}$ in (5.3.45) are expanded in powers of $\sqrt{\varepsilon}$ as

$$\sqrt{P} = \frac{-\alpha + \sqrt{B}}{2e^{b\tau_D}} + \sqrt{\varepsilon}\,\frac{xK}{\sqrt{B}} + \; O(\varepsilon) \tag{5.3.48a}$$
$$P = \left(\frac{-\alpha + \sqrt{B}}{2e^{b\tau_D}}\right)^2 + \sqrt{\varepsilon}\,\frac{(-\alpha + \sqrt{B})xK}{\sqrt{B}e^{b\tau_D}} + \; O(\varepsilon) \tag{5.3.48b}$$
$$P^{\frac{3}{2}} = \left(\frac{-\alpha + \sqrt{B}}{2e^{b\tau_D}}\right)^3 + \sqrt{\varepsilon}\,\frac{3(-\alpha + \sqrt{B})^2 xK}{4\sqrt{B}e^{2b\tau_D}} + \; O(\varepsilon) \tag{5.3.48c}$$
$$\text{where } B = \alpha^2 - 4e^{b\tau_D}(\beta - K) \; . \tag{5.3.48d}$$

Now by substituting (5.3.47) and (5.3.48 a-d) into (5.3.45) and collecting terms of $O(1)$, we find an equation in $W_0(x,\grave{\tau})$, namely

$$\frac{\partial W_0}{\partial \grave{\tau}} = \theta_2 \frac{\partial^2 W_0}{\partial x^2} \; , \tag{5.3.49}$$
$$\text{where } \theta_2 = \frac{\sigma^2(B^{\frac{3}{2}} - \alpha B)}{16e^{b\tau_D}K^2} \; . \tag{5.3.50}$$

Equation (5.3.49) needs to be solved subject to the initial condition $W_0(x,0) = \max(x,0)$, while the conditions at $x \to \pm\infty$ are given by

$$W_0(x,\grave{\tau}) \sim x \quad \text{as} \quad x \to +\infty \quad \text{and} \quad W_0(x,\grave{\tau}) \to 0 \quad \text{as} \quad x \to -\infty \; .$$

The PDE (5.3.49) has a symmetry with generator

$$\Psi = x\frac{\partial}{\partial x} + 2\grave{\tau}\frac{\partial}{\partial \grave{\tau}} + W_0\frac{\partial}{\partial W_0} \tag{5.3.51}$$

which leads to an invariant solution of the form

$$W_0(z,\grave{\tau}) = \sqrt{\grave{\tau}}\phi(z) \;\; \text{where } z = \frac{x}{\sqrt{\grave{\tau}}} \; . \tag{5.3.52}$$

Substituting (5.3.52) into (5.3.49) yields the reduced equation

$$2\theta_2\phi'' + z\phi' - \phi = 0 \tag{5.3.53}$$

which needs to be solved subject to the boundary conditions

$$\phi(z) \sim z \quad \text{as} \quad z \to +\infty \quad \text{and} \quad \phi(z) \to 0 \quad \text{as} \quad z \to -\infty \; .$$

The solution of (5.3.53) subject to above conditions is easily found to be

$$\phi(z) = \sqrt{\frac{\theta_2}{\pi}}\exp\{-\frac{z^2}{4\theta_2}\} + \frac{z}{2}\,\text{erfc}\left(-\frac{z}{2\sqrt{\theta_2}}\right) \tag{5.3.54}$$

so that in terms of x, $\grave{\tau}$ we get from (5.3.52)

$$W_0(x,\grave{\tau}) = \sqrt{\frac{\theta_2\grave{\tau}}{\pi}}\exp\{-\frac{x^2}{4\theta_2\grave{\tau}}\} + \frac{x}{2}\,\text{erfc}\left(-\frac{x}{2\sqrt{\theta_2\grave{\tau}}}\right) \; . \tag{5.3.55}$$

Now collecting terms of $O(\sqrt{\varepsilon})$ in (5.3.45) we get an equation for $W_1(x, \dot{\tau})$ namely

$$\frac{\partial W_1}{\partial \dot{\tau}} = \theta_2 \frac{\partial^2 W_1}{\partial x^2} + \theta_3 x \frac{\partial^2 W_0}{\partial x^2} + \theta_1 \frac{\partial^2 W_0}{\partial x^2} \tag{5.3.56}$$

$$\text{where } \theta_3 = \frac{\sigma^2(3\sqrt{B} - 2\alpha)}{8K}$$

and where θ_1, θ_2 are given in (5.3.46) and (5.3.50) receptively. Equation (5.3.56) needs to be solved subject to the initial condition $W_1(x, 0) = 0$, while the conditions at $x \to \pm\infty$ are given by

$$W_1(x, \dot{\tau}) \sim \theta_1 \dot{\tau} \quad \text{as} \quad x \to +\infty \quad \text{and} \quad W_1(x, \dot{\tau}) \to 0 \quad \text{as} \quad x \to -\infty \; .$$

To find the solution of (5.3.56) we use the following results[2] which are easily verified:

Result 5.3.1

$$\text{If} \quad \frac{\partial u}{\partial y} = \frac{1}{2}\frac{\partial u^2}{\partial x^2} \quad \text{and} \quad \frac{\partial v}{\partial y} = \frac{1}{2}\frac{\partial^2 v}{\partial x^2} + u, \quad \text{then} \quad v = yu \quad \text{is a particular solution.} \tag{5.3.57a}$$

$$\text{If} \quad \frac{\partial u}{\partial y} = \frac{1}{2}\frac{\partial u^2}{\partial x^2} \quad \text{and} \quad \frac{\partial v}{\partial y} = \frac{1}{2}\frac{\partial^2 v}{\partial x^2} + xu,$$

$$\text{then} \quad v = xyu + \frac{y^2}{2}\frac{\partial u}{\partial x} \quad \text{is a particular solution.} \tag{5.3.57b}$$

[2]See Howison (2005).

Now letting $y = 2\theta_2\grave{\tau}$, (5.3.49) and (5.3.56) becomes

$$\frac{\partial W_0}{\partial y} = \frac{1}{2}\frac{\partial^2 W_0}{\partial x^2} \tag{5.3.58a}$$

$$\frac{\partial W_1}{\partial y} = \frac{1}{2}\frac{\partial^2 W_1}{\partial x^2} + \frac{x\theta_3}{2\theta_2}\frac{\partial^2 W_0}{\partial x^2} + \frac{\theta_1}{2\theta_2}\frac{\partial^2 W_0}{\partial x^2}\ . \tag{5.3.58b}$$

Using the results (5.3.57 a,b), the solution of $W_1(x, y)$ is given by

$$W_1(x, y) = \frac{\theta_1 y}{2\theta_2}\frac{\partial W_0}{\partial x} + \frac{\theta_3}{2\theta_2}\left[xy\frac{\partial^2 W_0}{\partial x^2} + \frac{y^2}{2}\frac{\partial^3 W_0}{\partial x^3}\right]$$

Now we can rewrite $W_1(x, y)$ in terms of $W_1(x, \grave{\tau})$ as

$$W_1(x, \grave{\tau}) = \frac{\theta_3 x\sqrt{\grave{\tau}}}{4\sqrt{\pi\theta_2}}\exp\{-\frac{x^2}{4\theta_2\grave{\tau}}\} + \frac{\theta_1\grave{\tau}}{2}\text{erfc}\left(-\frac{x}{2\sqrt{\theta_2\grave{\tau}}}\right)\ . \tag{5.3.59}$$

This solution satisfies the necessary initial and boundary conditions. By substituting (5.3.55) and (5.3.59) into (5.3.47) we get the two-term inner expansion

$$\begin{aligned} W(x, \grave{\tau}) =& \sqrt{\frac{\theta_2\grave{\tau}}{\pi}}\exp\{-\frac{x^2}{4\theta_2\grave{\tau}}\} + \frac{x}{2}\text{erfc}\left(-\frac{x}{2\sqrt{\theta_2\grave{\tau}}}\right) \\ &+\sqrt{\varepsilon}\ \left[\frac{\theta_3 x\sqrt{\grave{\tau}}}{4\sqrt{\pi\theta_2}}\exp\{-\frac{x^2}{4\theta_2\grave{\tau}}\} + \frac{\theta_1\grave{\tau}}{2}\text{erfc}\left(-\frac{x}{2\sqrt{\theta_2\grave{\tau}}}\right)\right]. \end{aligned} \tag{5.3.60}$$

Equation (5.3.60) is valid in the inner region, while equation (5.3.43) is valid in the outer region. Now we can match the outer and inner expansions to get the uniform expansion which is uniformly valid in both outer and inner regions. The uniform expansion can be found by combining the outer and inner expansions and then subtracting the common part, i.e. '$W_{outer} + W_{inner} - W_{common}$'. In our solution as $\varepsilon \to 0$ the outer expansion coincides with the common part, so the inner expansion can be used to approximate the price of call option contracts.

Hence, we get the price of a call option contract by using our change of variables

(5.3.44) in (5.3.60) to get

$$C(P,t) = \frac{\sqrt{T_O - t}\,(4K\theta_2 + \theta_3(F(P,T_O) - K))}{4\sqrt{\pi\theta_2}} \exp\left\{-\frac{(F(P,T_O)-K)^2}{4K^2\theta_2(T_O-t)}\right\} + \frac{F(P,T_O) - K + \theta_1 K(T_O - t)}{2} \operatorname{erfc}\left(-\frac{F(P,T_O)-K}{2K\sqrt{\theta_2(T_O-t)}}\right) \tag{5.3.61}$$

where
$$F(P,t) = Pe^{b(T_F-t)} + \frac{2a\sqrt{P}}{b}(e^{b(T_F-t)} - e^{\frac{b}{2}(T_F-t)}) + \frac{a(4a-\sigma^2)}{4b^2}(e^{\frac{b}{2}(T_F-t)} - 1)^2) \ ,$$
$$\theta_1 = \frac{(4a\sqrt{B} - \alpha\sigma^2)e^{b\tau_D} + 2b(B - \alpha\sqrt{B})}{8Ke^{b\tau_D}} \ ,$$
$$\theta_2 = \frac{\sigma^2(B^{\frac{3}{2}} - \alpha B)}{16e^{b\tau_D}K^2} \ ,$$
$$\theta_3 = \frac{\sigma^2(3\sqrt{B} - 2\alpha)}{8K} \ ,$$
$$B = \alpha^2 - 4(\beta - K) \ ,$$
$$\alpha = \frac{2a}{b}(e^{bT_D} - e^{\frac{b}{2}T_D}) \ \text{ and } \ \beta = \frac{a(4a-\sigma^2)}{4b^2}(e^{\frac{b}{2}T_D} - 1)^2 \ .$$

Special Note

When the difference between times of expiry of the option and the underlying futures is small, then we can simplify the approximation to the option price under Model 10. Given that the spot price of oil follows Model 10, then from Section 5.2.1 the futures price has a zero drift in the risk-neutral world and from (5.2.3) its process can be written as

$$dF = \frac{\sigma}{P^{\frac{1}{4}}}(F - \frac{\alpha}{2}\sqrt{P} - \beta)d\tilde{Z},$$

where α and β are given in (5.2.20d). For an option contract with short time to expiry as noted earlier in this section we can write $\tau_O = \varepsilon\dot{\tau}$ where $0 < \varepsilon \ll 1$. In practice $\tau_D = \tau_F - \tau_O = T_F - T_O$ is often very small; possibly 3-5 days. In this case we could approximate τ_D by $\tau_D = \varepsilon^2\dot{\tau}$.

Then when $\varepsilon \to 0$, we have $\alpha,\ \beta \to 0$ and $P^{\frac{1}{4}} \to F^{\frac{1}{4}}$. Hence the change in futures price can be approximated by

$$dF = \sigma F^{\frac{3}{4}} d\tilde{Z}.$$

From (5.3.22) the PDE for pricing call option on futures is given by

$$\frac{\partial C}{\partial t} + \frac{F^{\frac{3}{2}}\sigma^2}{2}\frac{\partial^2 C}{\partial F^2} - rC = 0 \tag{5.3.62}$$

which needs to be solved subject to the final condition

$$C(F, T_O) = \max(F - K, 0)\ . \tag{5.3.63}$$

Following the method used in this section, the solution can be approximated for short times to expiry by

$$\begin{aligned} C(F,t) = & \frac{\sqrt{T_O - t}\,(4K\theta_1 + \theta_2(F-K))}{4\sqrt{\pi\theta_1}} \exp\left\{-\frac{(F-K)^2}{4K^2\theta_1(T_O - t)}\right\} \\ & + \frac{F-K}{2}\,\mathrm{erfc}\left(-\frac{F-K}{2K\sqrt{\theta_1(T_O - t)}}\right) \end{aligned} \tag{5.3.64}$$

$$\text{where}\quad \theta_1 = \frac{\sigma^2}{2\sqrt{K}}\ , \qquad \theta_2 = \frac{3\sigma^2}{4\sqrt{K}}\ .$$

5.3.3 Analytic Approximation Formula for Option Prices Under Model 9

Following the method used in Section 5.3.2, the PDE for valuing call options under Model 9 with expiry T_O and strike price K is given by

$$\frac{\partial C}{\partial t} + a\sqrt{P}\,\frac{\partial C}{\partial P} + \frac{P^{\frac{3}{2}}\sigma^2}{2}\frac{\partial^2 C}{\partial P^2} - rC = 0 \tag{5.3.65}$$

subject to the final condition

$$C(P,T_O) = \max(F(P,T_O) - K, 0) = \max[P + \alpha\sqrt{P} + \beta - K, 0] \ ,$$

where α and β are given in (5.2.16d). The solution is found to be

$$\begin{aligned} C(P,t) = &\frac{\sqrt{T_O - t}\,(4K\theta_2 + \theta_3(F(P,T_O) - K))}{4\sqrt{\pi\theta_2}} \exp\{-\frac{(F(P,T_O) - K)^2}{4K^2\theta_2(T_O - t)}\} \\ &+ \frac{F(P,T_O) - K + \theta_1 K(T_O - t)}{2} \operatorname{erfc}\left(-\frac{F(P,T_O) - K}{2K\sqrt{\theta_2(T_O - t)}}\right) \ . \end{aligned} \tag{5.3.66}$$

$$\begin{aligned} \text{where} \quad F(P,t) &= P + a(T_F - t)\sqrt{P} + \frac{(a(T_F - t))^2}{4}(1 - \frac{\sigma^2}{4a}) \ , \\ \theta_1 &= \frac{4a\sqrt{B} - \alpha\sigma^2}{8K} \ , \\ \theta_2 &= \frac{\sigma^2 B(\sqrt{B} - \alpha)}{16K^2} \ , \\ \theta_3 &= \frac{\sigma^2(3\sqrt{B} - 2\alpha)}{8K} \ , \\ B &= \alpha^2 - 4(\beta - K) \ , \\ \alpha &= aT_D \text{ and } \beta = \frac{(aT_D)^2}{4}(1 - \frac{\sigma^2}{4a}) \ . \end{aligned}$$

5.4 Accuracy of the Analytic Approximation Formulae

In this section, we compare our analytic approximation solutions (5.3.66) and (5.3.61) under stochastic Models 9 and 10 with their corresponding exact solutions (5.2.16a) and (5.2.20a). To do this we use three groups[3] of call option contracts selected from the International Commodity Exchange (ICE) from the years 2010-2012, with time

[3] A group of option contracts can be defined as observations of option contracts (observed at a given time t) that have the same type (i.e. call or put), expiration date but with different strike prices.

to expiry of one month. Then for each of these three groups and pricing formula, we fit three market price observations (we use deep out-, deep in-, and near at-the-money observations) to obtain the parameters that best fit the exact solutions. These parameters are then used to calculate exact and analytic approximation values for various strike prices. Moreover, we use the best parameters to compare the accuracy of the approximate solutions when the time to expiry is two months.

For each contract, strike price (K) and time to expiry (one and two months) we calculate the signed relative error which is given by

$$\text{Signed relative errors} = \frac{C_{approximate} - C_{exact}}{C_{exact}} \times 100\% \tag{5.4.67}$$

and the results are listed in Table 5.2. From Table 5.2 we note that:

- As expected, analytic approximation solutions under Model 10, i.e. equation (5.3.61), mostly yield better results than under Model 9, i.e. equation (5.3.66).

- For call option contracts with one month to expiry, the analytic approximation (5.3.61) yields absolute relative errors that are mostly less than 1% and in two of the three groups, slightly underpriced their values compared with the exact solution. In comparison, the analytic approximation (5.3.66) yields absolute relative errors that are mostly less than 2% and in two of the three groups slightly overpriced their values compared with the exact solution.

- For call option contracts with two months to expiry, the analytic approximation (5.3.61) under Model 10 yields absolute relative errors that are mostly less than 2% and in two of the three groups slightly underpriced their value compared with the exact solution. In comparison, the analytic approximation (5.3.66) under Model 9 yields absolute relative errors that are less than 4% and also tend to slightly underprice their value compared with the exact solution.

From the above it can be surmised that the analytic approximation solutions give good approximations to the exact solutions for small times to expiry, with the smaller times to expiry, the better. In particular, the analytic approximation (5.3.61) under stochastic Model 10 provides better approximations as compared to (5.3.66) and is deemed sufficient in pricing a large class of short tenor options.

K	signed relative error of (5.3.66)		signed relative error of (5.3.61)	
	$\tau_O = 30$ days	$\tau_O = 60$ days	$\tau_O = 30$ days	$\tau_O = 60$ days
76*	1.09	-2.16	-1.44	2.45
77	1.18	-2.27	-1.19	2.09
77.5	1.22	-2.33	-1.06	1.90
78	1.27	-2.38	-0.92	1.72
78.5*	1.31	-2.43	-0.79	1.53
79	1.34	-2.48	-0.66	1.35
79.5	1.38	-2.53	-0.52	1.16
80	1.40	-2.57	-0.39	0.98
85*	1.19	-2.64	0.73	-0.78
106*	-1.71	-3.17	-0.01	-0.02
106.5	-1.76	-3.23	-0.12	-0.16
107	-1.80	-3.30	-0.22	-0.30
107.5	-1.85	-3.36	-0.33	-0.43
108*	-1.90	-3.41	-0.44	-0.57
108.5	-1.94	-3.47	-0.55	-0.71
109	-1.99	-3.53	-0.65	-0.85
110	-2.06	-3.63	-0.86	-1.12
110.5*	-2.10	-3.68	-0.97	-1.25
123*	1.77	-3.17	0.04	-0.02
123.5	1.84	-3.25	0.17	-0.18
124	1.91	-3.34	0.31	-0.35
124.5	1.97	-3.42	0.44	-0.52
125*	2.04	-3.50	0.57	-0.69
125.5	2.10	-3.58	0.71	-0.86
126	2.17	-3.66	0.85	-1.03
127	2.29	-3.82	1.11	-1.37
128*	2.39	-3.96	1.37	-1.70

Table 5.2: Signed relative errors (%) of asymptotic solutions (5.3.66) and (5.3.61) using three groups of call option contracts, and for each group the three observations with strike prices K^* are used to estimate the parameters.

5.5 Empirical Test

In this section we examine our new exact call option prices (5.2.16a) and (5.2.20a) under stochastic oil models Model 9 (5.2.14) and Model 10 (5.2.18) respectively, in their ability to capture market prices. In addition, we compare their performance with call option prices under Model 4, namely (5.2.12). This section is sub-divided into four parts. In the first part we provide a description of the data used in our empirical tests, while in the second part an outline of the methodology used to estimate the parameters is presented. The results of our empirical tests are presented in the third part. In the fourth part the performance of the analytic approximation formula under Model 10, namely (5.3.64), is examined.

5.5.1 Data Description

The data used in this empirical work consists of daily observations of American call option contracts prices[4] with the underlying asset being futures contracts for Brent crude oil. We used 72 groups of call option contracts selected from the International Commodity Exchange (ICE). One of our aims in this section is to measure the effect of the time to expiry variable on our new call option prices. To measure this, we selected 36 groups which expired in one month and also another 36 groups which expired in six months.

5.5.2 Estimation of Parameters

In this section we explain the methodology used to estimate the parameters of the closed-forms of the call option prices. For the Schwartz price (5.2.12) we need to estimate the speed of reversion (η), the long run-log price (α^*) and the volatility(σ). For the new pricing forms, (5.2.16a) and (5.2.20a) we need to estimate a, σ and

[4]The call options are quoted as American style. However, as no dividends are paid the value of the American call options is the same as the value of European call options.

a, b, σ respectively.

To estimate the parameters (for each pricing formula and group) we fit three (and two for Model 9) near at-the-money market prices using pricing formulae, and we choose to do this as follows:

We define C_{ij} ($\hat{C}_{ij}$) to be the market (estimated) price of the call option contract with strike price K_i that belongs to the group of option contracts j ($j = 1, ..., 72$). We define e_{ij} to be the error of contract i that belongs to group j with strike price K_i. Hence e_{ij} is the difference between the market price and the estimated price of the call option contract C_{ij} with strike price K_i, i.e.

$$e_{ij} = C_{ij} - \hat{C}_{ij}.$$

For each pricing formula let $\theta(j)$ be the parameter vector for the group j. We use near at-the-money market prices (with strike prices K_{i^*}) and minimize the sum of squares of errors (SSE) i.e.

$$\min SSE(\theta(j)) = \sum_{i^*} e^2_{i^*j} \ . \tag{5.5.68}$$

This results in the parameter vector $\theta(j)$ for each pricing formula and and for each group j. The estimated parameter vector $\theta(j)$ is used to compute the values of call options for other strike prices that belong to the group of option contracts j.

5.5.3 Performance of Options Models

The following measures are used to compare errors in the performance of the call options pricing models:

- The total sum of squared errors

$$SSE = \sum_i (\hat{C}_{ij} - C_{ij})^2, \quad j = 1, .., 36 \tag{5.5.69}$$

- The total root mean squared errors

$$RMSE = \sqrt{\frac{1}{N-q}\sum_i (\hat{C}_{ij} - C_{ij})^2}, \quad j = 1, .., 36 \tag{5.5.70}$$

- Signed percentage errors

$$\text{Signed percentage errors} = \sum_i \frac{\hat{C}_{ij} - C_{ij}}{C_{ij}} \times 100, \quad j = 1, .., 36 \tag{5.5.71}$$

- Unsigned percentage errors

$$\text{Unsigned percentage errors} = \sum_i |\frac{\hat{C}_{ij} - C_{ij}}{C_{ij}}| \times 100, , \quad j = 1, .., 36 \tag{5.5.72}$$

where N is the number of observations and q is the number of the parameters. The results of signed and unsigned percentage errors will be presented in various ranges of moneyness M, defined as $M = \ln(\frac{F_j}{K_{ij}})$ where F_j is the spot price of the underlying futures of the group of option contracts j and the K_{ij} are various strike prices of the same group of option contracts j. The average of signed and unsigned percentage errors were calculated for each range of moneyness.

	$\tau_O = 30$ days			$\tau_O = 180$ days		
	Model 4 (5.2.12)	**Model 9** (5.2.16a)	**Model 10** (5.2.20a)	**Model 4** (5.2.12)	**Model 9** (5.2.16a)	**Model 10** (5.2.20a)
SSE	3.7709	3.8991	3.5941	0.8814	1.6623	0.8181
$RMSE$	0.0804	0.0793	0.0785	0.0427	0.0566	0.0412

Table 5.3: Comparison of SSE and RMSE.

M	Average signed percentage error **Model 4** (5.2.12)	 **Model 9** (5.2.16a)	 **Model 10** (5.2.20a)	Average unsigned percentage error **Model 4** (5.2.12)	 **Model 9** (5.2.16a)	 **Model 10** (5.2.20a)
(-0.20, -0.10)	-2.4987	-4.4659	-4.9003	2.6993	5.7119	4.9003
(-0.10, -0.05)	-2.7580	-3.4040	-3.4683	4.4866	5.8639	4.7927
(-0.05, -0.02)	-0.2863	-0.4782	-0.3943	0.8602	1.2710	0.8984
(-0.01, 0.01)	-0.0109	-0.0089	-0.0141	0.0957	0.1421	0.0959
(0.02,0.05)	0.1548	0.0482	0.0799	0.4308	0.5850	0.4189
(0.05,0.10)	0.6153	0.4131	0.3877	1.1283	1.2451	1.0544
(0.10,0.25)	1.2723	0.3447	0.9551	1.2723	0.8921	0.9551

Table 5.4: Percentage Errors in calibrations of call options on futures contracts with $\tau_O = 30$ days.

M	Average signed percentage error **Model 4** (5.2.12)	 **Model 9** (5.2.16a)	 **Model 10** (5.2.20a)	Average unsigned percentage error **Model 4** (5.2.12)	 **Model 9** (5.2.16a)	 **Model 10** (5.2.20a)
(-0.10, -0.05)	-0.2414	-0.4243	-0.4764	0.5252	0.8947	0.5887
(-0.05, -0.02)	-0.0263	-0.0013	-0.0623	0.1486	0.2983	0.1433
(-0.01, 0.01)	-0.0015	0.0033	-0.0033	0.0287	0.0562	0.0289
(0.02,0.05)	0.0102	-0.0011	-0.0229	0.1419	0.2530	0.1358
(0.05,0.10)	0.3220	0.1757	0.1824	0.4692	0.5737	0.4178

Table 5.5: Percentage Errors in calibrations of call options on futures contracts with $\tau_O = 180$ days.

Tables 5.3-5.5 list the results of our analysis. In particular, we note that:

- ***Comparison of SSE and RMSE*** indicates that the values of SSE and RMSE decrease as time to expiry increases, and the lowest values are reached by Model 10 followed by Model 4. The lowest value of SSE for contracts with expiry of one month (six months) is 3.5941 (0.8181), while the lowest value of RMSE is \$ 0.0758 (\$ 0.0411) per contract. See Figure 5.4 for a comparison of option prices under Model 10 to market data.

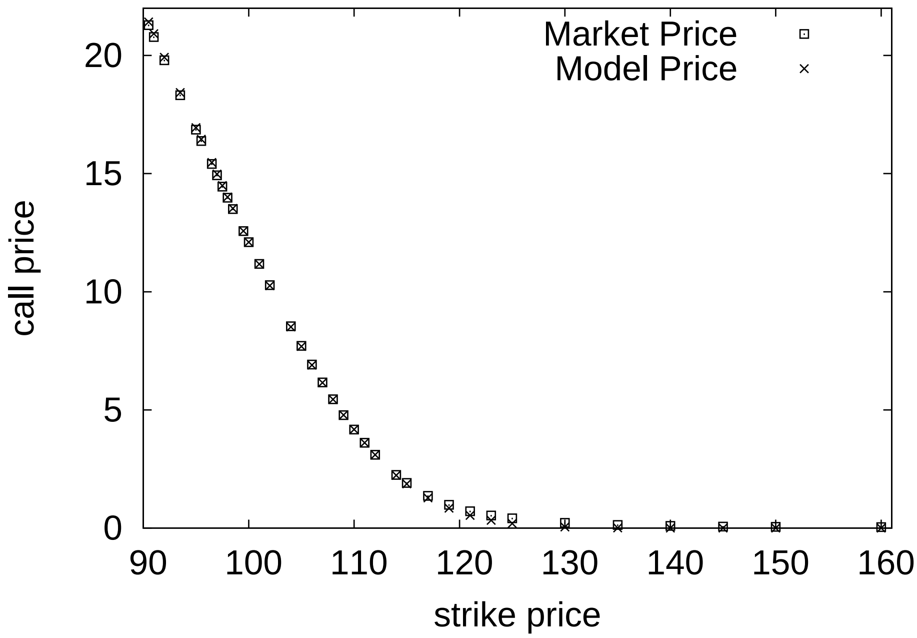
- ***Comparison of average signed percentage errors*** indicates that all models underprice out-the-money call options compared to market prices. As well for near at-the-money call options, all models prices (with the exception of Model 9 with six months to expiry) are slightly under those of market prices. Moreover, the extent of underpricing decreases as time to expiry and M

increase. For in-the-money call options, all models prices (with the exception of near in-the-money options with Model 9 and Model 10 with six months to expiry) are higher than market prices and the range of overpricing increases as time to expiry decreases and M increases.

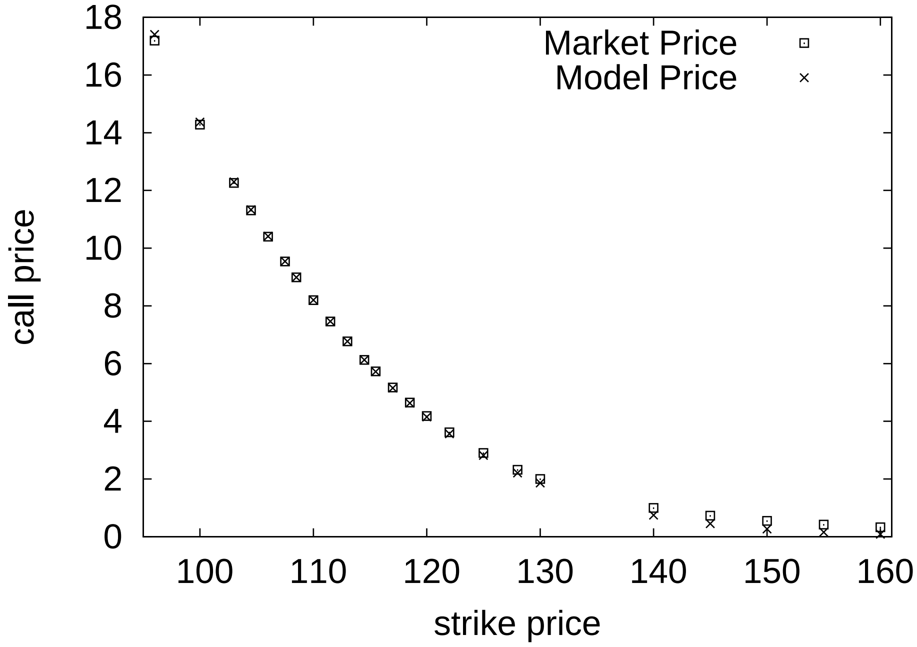
- ***Comparison of average unsigned percentage errors*** indicates that the lowest values are reached by Model 4 and Model 10. For near at- and out-the-money call options, the average unsigned percentage errors of Model 4 are slightly lower than those of Model 4 (with the exception of near out-the-money options with six months to expiry). However, Model 10 has the lowest value when the options are in-the-money, and for deep in-the-money call options with one month to expiry the lowest value is reached by Model 9.

Our empirical results lead to suggestions for practitioners on the use of the new three-quarters models. Given that the models correctly price at-the-money option contracts then for out- and near at-the-money call option contracts, option holders (writers) expect to pay (receive) premiums that are slightly higher (lower) than those predicted by the formulae. However, for in-the-money call option contracts, option holders (writers) expect to pay (receive) premiums that are slightly lower (higher) than those predicted by the formulae. These anomalies reduce as time to expiry increases.

In summary, we can infer from our given data and empirical analysis, that Model 10 outperforms other models in describing the prices of call option contracts and performs best in pricing in-the-money call options.



(a)



(b)

Figure 5.4: Comparison of call option prices using formula (5.2.20a) with market data (a) on 13 Aug 2012 ($\tau_O = 30$ days) and (b) on 11/5/2012 ($\dot{\tau} = 180$ days).

5.5.4 The Performance of the Analytic Approximation Formula (5.3.64)

In this section, we examine the performance of our analytic approximation solution (5.3.64) under Model 10 as a function of the futures price. The same data and methodology to obtain the best parameters is used as in Sections 5.5.1 and 5.5.2. The performance of formula (5.3.64) in modelling market prices is measured with two empirical tests:

- In the first empirical test, we compare the performance of (5.3.64), the analytic approximate solution under Model 10 (with one parameter) with (5.2.12), the exact solution under the Schwartz model (with two parameters) and in both formulae we use the market value for the futures price. The surprising result that was found was that the analytic approximate solution (5.3.64) fitted the data better in 71 groups, while the exact solution under the Schwartz model (5.2.12) fitted better in only one group. Formula (5.3.64) outperformed (5.2.12) even at the larger time to expiry of 180 days. The SSE and RMSE are listed in Table 5.6, from which it can be observed that the lowest values are reached by using (5.3.64). However, compared with the SSE and RMSE values in Table 5.3, the values in Table 5.6 are relatively high. This is largely due to the fact that only one parameter is fitted in (5.3.64).

- In the second empirical test, we substitute the futures price (5.2.20b) into (5.3.64) and compare the resultant formula with the exact solution (5.2.20a). Surprisingly, the lowest values of SSE and RMSE for both times of expiry 30 and 180 days (see Table 5.7) are reached by the analytic approximation solution (5.3.64).

The results of those two empirical tests indicate that the analytic approximation formula (5.3.64) outperforms the exact solution under the Schwartz model (5.2.12) and also outperforms the exact solution under Model 10 (5.2.20a) in describing the prices of call option contracts.

	$\tau_O = 30$ days		$\tau_O = 180$ days	
	Model 4 (5.2.12)	**Model 10** (5.3.64)	**Model 4** (5.2.12)	**Model 10** (5.3.64)
SSE	18.2985	15.5313	40.7893	27.3523
$RMSE$	0.1718	0.1539	0.2894	0.2287

Table 5.6: Comparison of SSE and RMSE (using market value for futures price).

	$\tau_O = 30$ days		$\tau_O = 180$ days	
	(5.2.20a)	(5.3.64)	(5.2.20a)	(5.3.64)
SSE	3.5941	3.5567	0.8181	0.7999
$RMSE$	0.0785	0.0780	0.0412	0.0404

Table 5.7: Comparison of SSE and RMSE (using (5.2.20b) for futures price).

5.6 Conclusion

In this chapter, analytic solutions for European option prices and analytic approximation solutions for European call option prices under two of the three-quarters models were derived, namely (5.2.16a) and (5.2.20a). These solutions were calibrated to market data and compared with calibrations of option prices under the Schwartz model. Our calibration results show that option prices under Model 10 i.e. (5.2.20a) describe our market option data more effectively than (5.2.12) and (5.2.16a). However, the analytic approximation solution (5.3.64) under Model 10, with futures prices given by (5.2.20b) performed the best overall. It is also a simple formula involving standard functions and may be a useful guide to traders.

Chapter 6

Pricing Correlation Options on Oil

6.1 Introduction

In this chapter we examine the pricing of options whose payoff depends on the prices of two correlated assets. Such options are termed correlation options. Unlike ordinary options, a *spread* option is an option whose payoff is based on the difference between two underlying assets. Given that the prices of the underlying assets at expiry T of the option are given by F_{1T} and F_{2T} respectively, then the value of a spread call option contract with strike price K is given by $\max(F_{1T} - F_{2T} - K, 0)$ and by $\max\big(K - (F_{1T} - F_{2T}), 0\big)$ for a spread put option. There are many different types of spread options available to the investor; some of which are traded on an exchange, but most of which are traded over-the-counter (OTC). In general spread options allow the investor to take a position on the relative performance of the underlying assets. As well they can be used to hedge the risk caused by the difference in the performance of the two underlying assets. In particular, spread options which are written on the difference between two underlyings of the same commodity but at two different geographical locations are called location spreads and can be used to hedge the risk caused by volatility in transportation costs. Spread options which are written on the difference between two underlyings of the same commodity but with different expiries are called calendar spreads and can be used to hedge the risk caused by volatility in future commodity prices. Spread options which are written on the difference between two underlyings of the same commodity but with different grades are called quality spreads and can be used to hedge the risk caused by volatility in future commodity grades. An example of a quality spread is the crack spread option which is the main focus of this chapter. The value of a crack spread option depends on the spread of crude oil futures and refined product futures. The NYMEX[1] offers crack spread options on futures contracts of crude oil, heating oil and unleaded gasoline. A 3:2:1 crack spread option is an option with a long position in two contracts on unleaded gasoline futures, one contract on heating oil futures and a short position in three contracts on crude oil futures. A 1:0:1 heating oil crack spread option is an option with a long position in one contract on heating oil futures and a short position in one contract on crude oil futures. A 1:1:0 gasoline crack spread option is an option with a long position in one contract on unleaded gasoline futures and a short position in one contract on crude oil futures. These types of options provide an important risk management tool to refineries. This is because refineries purchase crude oil, then convert it to other petroleum products such as heating oil, and finally sell it. Hence, the profit of any refinery depends on the uncertain price spread between heating oil (output) and crude oil (input). Any unexpected decrease in this spread may expose the refinery to a major risk which can be hedged by trading crack spread options.

Successful investment in crack spread options depends on the accuracy of their pricing. The pricing of these contracts differs from the pricing of vanilla options for a number of reasons:

- The value of a spread option depends on at least two underlying assets rather than a single underlying asset. This implies that a spread option value relies

[1]For further details, we refer the reader to the Crack Spread Handbook (Chicago Mercantile Exchange (2013)).

on at least two dynamics.

- The distribution of spread values between two underlying assets normally differs from the distribution of a single asset value as the spread value can attain negative values, which is normally not possible for a single asset value.
- The correlation between the price movements of the two underlying assets has an effect on the spread option value. However, the correlation does not play any role in the valuation of vanilla options.

Margrabe (1978) and Fu (1996) developed pricing formulae for spread options with zero strike (known as exchange options) in the Black-Scholes framework. In the case of non-zero strike, the resulting spread does not have a closed-form solution. The direct approach in pricing spread option contracts (i.e. finding the expected payoff under the risk-neutral measure $\mathbb{Q}$) involves solving a double-integral problem. Various solution techniques can be found in the literature to approximate spread option values, see for example Kirk (1995) and Carmona and Durrleman (2003). Assuming that the spread itself or both underlying prices follow the ABM can lead to closed-formulae for spread options, see for example Wilcox (1990) and Poitras (1998). However, considering both underlying prices follow the ABM has the disadvantage of allowing each underlying asset price to become negative. For further details, we refer the reader to Section 1.5.3.

An important consideration in the pricing of spread options is whether to model the movement of the two underlyings (Explicit Modelling) or to model the movement of the spread itself (Univariate Modelling). The main disadvantage of using univariate modelling is that the correlation between the price movement of the two underlyings, which has a major effect in the spread value is ignored. However, univariate modelling can outperform explicit modelling (as shown by Mahringer and Prokopczuk (2010) in empirical investigations on real data).

In this chapter we aim to price European crack spread call option contracts on futures of the types[2] 1:0:1 and 1:1:0. At expiry, these options have payoff[3] $\max(F_{CL_T} - F_{PP_T} - K, 0)$ where F_{CL_T} and F_{PP_T} are the futures prices at the expiry of the option contract, of crude light oil and another petroleum product (e.g. unleaded gasoline or heating oil) respectively. Both univariate and explicit models are used to derive analytic approximate formulae for options with short times to expiry. It is these type of options with short tenor that actually dominate options markets. Moreover, we empirically test our new formulae with various well-known formulae (derived using both univariate and explicit modelling) in order to determine whether univariate or explicit models behave best in capturing option prices for the given data.

Another correlation option we consider briefly in this chapter is the *quotient* option. The quotient (or ratio) option is related to the spread option in its functionality. The quotient option has a payoff based on the ratio between two underlying assets. Given that the price of the underlying assets at expiry T of the option are given by F_{1T} and F_{2T} respectively, then the value of a quotient call option contract with strike price K is given by $\max\left(\frac{F_{2T}}{F_{1T}} - K, 0\right)$ and by $\max\left(K - \frac{F_{2T}}{F_{1T}}, 0\right)$ for a quotient put option.

The remainder of this chapter is organised into seven sections. In Section 6.2, univariate modelling is assumed, and exact and analytic approximation formulae for the price of European crack spread call option contracts under the constant elasticity of variance (CEV) model, i.e.

$$dF = \mu(F,t)dt + \sigma F^{\gamma} dZ \text{ where } \gamma \in \mathbb{R} \ , \tag{6.1.1}$$

for the spread, are derived. In Section 6.3, explicit modelling is assumed and an analytic approximation formula for the price of European crack spread call option contracts with short tenor is derived under the assumption that each underlying

[2]In the remainder this chapter, reference to a Crack Spread Option will refer to an option of the type 1:0:1 or 1:1:0.

[3]The work in this chapter can be applied for any Crack Spread Call Option with similar payoff.

follows a CEV model, i.e.

$$dF_i = \mu(F_i, t)dt + \sigma F_i^{\gamma} dZ_i \text{ where } i = 1,\ 2 \text{ and } \gamma \in \mathbb{R}\ . \tag{6.1.2}$$

Popular existing formulae for pricing crack spread options are highlighted in Section 6.4. In Section 6.5, the value of the parameter γ i.e. under (6.1.1) and (6.1.2), in our proposed formulae, is estimated and numerical examples are provided in order to measure the accuracy of our proposed formulae. The results of empirical tests which compare the performance of our formulae with the performance of popular existing formulae in their ability to describe market prices are presented in Section 6.6. A brief look at pricing quotient options is provided in Section 6.7 and in Section 6.8 we present our conclusion.

6.2 Univariate Modelling

In this section we consider modelling the underlying spread directly. This is known as univariate modelling. It is known that in a risk-neutral world, futures prices have zero drift. Hence, in this section we assume that the change in the risk-neutral futures crack spread follows the CEV process

$$dF_{sp} = \sigma(t, T) F_{sp}^{\gamma} d\tilde{Z} \tag{6.2.3}$$

where $F_{sp} = F_1 - F_2$ for some $\gamma \in \mathbb{R}$ and $\tilde{Z}$ is a Wiener process under a risk-neutral probability measure $\mathbb{Q}$ under which F_{sp} becomes a martingale. Considering a portfolio, with value π, that consists of a long position of a crack spread call option contract and Δ short positions in futures spread contracts (i.e. $\pi = C - \Delta F_{sp}$). The instantaneous change in the portfolio, $d\pi$, can be found by applying Itô's Lemma (see Section 1.1.4). With a judicious choice of $\Delta = \frac{\partial C}{\partial F_{sp}}$ the risk in the portfolio is eliminated so by arbitrage the portfolio should earn the risk-free interest rate r.

This leads to the PDE governing the price of a crack spread call option contract, $C(F_{sp}, t)$, with strike price K and expiry T as

$$\frac{\partial C}{\partial t} + \frac{\sigma(t,T)^2 F_{sp}^{2\gamma}}{2}\frac{\partial^2 C}{\partial F_{sp}^2} - rC = 0 \tag{6.2.4a}$$

subject to the final condition

$$C(F_{sp}, T) = \max(F_{sp} - K, 0). \tag{6.2.4b}$$

We now consider the underlying model (6.2.3) with $\sigma(t,T) = \sigma$, constant and γ arbitrary in $0 \leq \gamma < 1$. The exact solution to (6.2.4a,b) can be found using the transition density function of F_{sp} which is given by

$$Tr(F_{sp}, t; F_{spT}, T) = \frac{\sqrt{F_{sp}}\; {F_{spT}}^{\frac{1}{2}-2\gamma}\; I_\nu(z)}{\sigma^2 \mid 1-\gamma \mid (T-t)} \quad \exp\left\{ -\frac{{F_{sp}}^{2-2\gamma} + {F_{spT}}^{2-2\gamma}}{2\sigma^2(1-\gamma)^2(T-t)} \right\} \tag{6.2.5}$$

where

$$\nu = \frac{1}{2}\sqrt{1 - \frac{\gamma(\gamma-2)}{(\gamma-1)^2}} \quad \text{and} \quad z = \frac{(F_{sp}F_{spT})^{1-\gamma}}{\sigma^2(1-\gamma)^2(T-t)} \tag{6.2.6}$$

(see Goard(2006)) and where $I_\nu(.)$ is the modified Bessel function of the first kind of order ν (see e.g. Abramowitz and Stegun (1964)). In this case the exact solution of (6.2.4a,b) can be found as the present value of the expected payoff under the risk-neutral measure $\mathbb{Q}$ i.e.

$$\begin{aligned} C(F_{sp}, t) &= e^{-r(T-t)}\mathbb{E}_t^{\mathbb{Q}}\left(\max[F_{spT} - K, 0]\right) \\ &= e^{-r(T-t)}\left[\int_K^\infty (y-K)\; Tr(F_{sp}, t; y, T)\; dy\right] . \end{aligned} \tag{6.2.7}$$

Equation (6.2.7) involves an integral which would normally need to be solved numerically. We now prove the following theorem that gives a simple approximation solution to (6.2.4a,b) valid for short times to expiry and which only involves simple mathematical functions. This would facilitate pricing and calibration of options with short tenor.

Theorem 6.2.1 *Given that the futures crack spread prices, F_{sp}, follow the risk-neutral process (6.2.3) where $\sigma(t,T) = \sigma$ constant, then an approximate solution for a European crack spread call option valid for short times to expiry T and strike price K is given by*

$$C(F_{sp}, t) = \frac{\sqrt{T-t}\,(4K\theta_1 + \theta_2(F_{sp} - K))}{4\sqrt{\pi\theta_1}} \exp\left\{-\frac{(F_{sp} - K)^2}{4K^2\theta_1(T-t)}\right\} + \frac{F_{sp} - K}{2} \operatorname{erfc}\left(-\frac{F_{sp} - K}{2K\sqrt{\theta_1(T-t)}}\right) \tag{6.2.8}$$

$$\text{where } \theta_1 = \frac{\sigma^2 K^{2\gamma-2}}{2} \text{ and } \theta_2 = \gamma\sigma^2 K^{2\gamma-2}.$$

Proof: Letting $\tau = T - t$, equation (6.2.4a) can be written as

$$\frac{\partial C}{\partial \tau} = \frac{\sigma^2 F_{sp}^{2\gamma}}{2} \frac{\partial^2 C}{\partial F_{sp}^2} - rC \tag{6.2.9}$$

which needs to be solved subject to the initial condition $C(F_{sp}, 0) = \max(F_{sp} - K, 0)$ and the boundary conditions

$$C(0, \tau) = 0 \text{ and } C(F_{sp}, \tau) \sim F_{sp} - K \text{ as } F_{sp} \to \infty .$$

For a small time to expiry τ we let $\tau = \varepsilon\grave{\tau}$ where ε is a small parameter $0 < \varepsilon \ll 1$. Then equation (6.2.9) becomes

$$\frac{\partial C}{\partial \grave{\tau}} = \varepsilon\left(\frac{\sigma^2 F_{sp}^{2\gamma}}{2} \frac{\partial^2 C}{\partial F_{sp}^2} - rC\right). \tag{6.2.10}$$

We assume that the solution can be written as a series in ε i.e.

$$C(F_{sp}, \grave{\tau}) = \sum_{i=0}^{\infty} \varepsilon^i C_i(F_{sp}, \grave{\tau}). \tag{6.2.11}$$

Substituting (6.2.11) into (6.2.10) and collecting terms of $O(1)$, we get an equation for $C_0(F_{sp}, \grave{\tau})$, namely

$$\frac{\partial C_0}{\partial \grave{\tau}} = 0, \tag{6.2.12}$$

which solved subject to the initial condition $C_0(F_{sp}, 0) = \max(F_{sp} - K, 0)$, gives

$$C_0(F_{sp}, \grave{\tau}) = \max(F_{sp} - K, 0) = \begin{cases} F_{sp} - K & \text{if } F_{sp} \geq K, \\ 0 & \text{otherwise.} \end{cases} \tag{6.2.13}$$

Collecting terms of $O(\varepsilon)$, we get an equation for $C_1(F_{sp}, \grave{\tau})$:

$$\frac{\partial C_1}{\partial \grave{\tau}} = \frac{\sigma^2 F_{sp}^{2\gamma}}{2} \frac{\partial^2 C_0}{\partial F_{sp}^2} - rC_0 \ . \tag{6.2.14}$$

Solving (6.2.14) with initial condition $C_1(F_{sp}, 0) = 0$, we get

$$C_1(F_{sp}, \grave{\tau}) = \begin{cases} -r\grave{\tau}(F_{sp} - K) & \text{if } F_{sp} \geq K \ , \\ 0 & \text{otherwise.} \end{cases} \tag{6.2.15}$$

Hence, the first two terms of our series solution are given by

$$C_0(F_{sp}, \grave{\tau}) + \varepsilon C_1(F_{sp}, \grave{\tau}) = \begin{cases} (F_{sp} - K)(1 - \varepsilon r\grave{\tau}) & \text{if } F_{sp} \geq K \ , \\ 0 & \text{otherwise.} \end{cases} \tag{6.2.16}$$

From equation (6.2.16) it can be observed that the expansion is continuous, but not differentiable at $F_{sp} = K$. This suggests a *corner layer* at $F_{sp} = K$ where very fast changes occur in the derivative of the solution, but not in the value of the solution. Solution (6.2.16) is therefore invalid in this region and is thus termed our "outer" solution. We now analyse the solution in the inner region by introducing a stretching variable

$$x = \frac{F_{sp} - K}{\sqrt{\varepsilon} K} \tag{6.2.17a}$$

and rescaling

$$C(F_{sp}, \grave{\tau}) = K\sqrt{\varepsilon}\, W(F_{sp}, \grave{\tau}) \ . \tag{6.2.17b}$$

The choice of power $\frac{1}{2}$ in $\sqrt{\varepsilon}$ is a well-balanced choice and ensures that the coefficient of the second-order derivative term is not small compared to the other coefficients. Equation (6.2.10) then becomes

$$\frac{\partial W}{\partial \grave{\tau}} = \frac{\sigma^2 K^{2\gamma-2}}{2}(1+\sqrt{\varepsilon}x)^{2\gamma}\frac{\partial^2 W}{\partial x^2} - r\varepsilon W \tag{6.2.18}$$

which needs to be solved subject to the initial condition $W(x, 0) = \max(x, 0)$, and

$$W(x, \grave{\tau}) \sim x \quad \text{as} \quad x \to +\infty \quad \text{and} \quad W(x, \grave{\tau}) \to 0 \quad \text{as} \quad x \to -\infty \ .$$

Now we expand $W(x, \grave{\tau})$ in terms of $\sqrt{\varepsilon}$ i.e.

$$W(x, \grave{\tau}) \ = \sum_{i=0}^{\infty} \varepsilon^{\frac{i}{2}} W_i(x, \grave{\tau}) \tag{6.2.19}$$

which we substitute into (6.2.18) and then aim to collect terms in powers of $\sqrt{\varepsilon}$. Firstly the term $(1+\sqrt{\varepsilon}x)^{2\gamma}$ in (6.2.18) is expanded in powers of $\varepsilon^{\frac{1}{2}}$ as

$$(1+\sqrt{\varepsilon}x)^{2\gamma} = 1+\sqrt{\varepsilon}\ 2\gamma x + O(\varepsilon)\ . \tag{6.2.20}$$

Now by substituting (6.2.19) and (6.2.20) into (6.2.18) and collecting terms of $O(1)$, we find an equation in $W_0(x,\grave{\tau})$, namely

$$\frac{\partial W_0}{\partial \grave{\tau}} = \theta_1 \frac{\partial^2 W_0}{\partial x^2}, \tag{6.2.21}$$

$$\text{where } \theta_1 = \frac{\sigma^2 K^{2\gamma-2}}{2}\ . \tag{6.2.22}$$

Equation (6.2.21) needs to be solved subject to the initial condition $W_0(x,0) = \max(x,0)$, while the conditions at $x \to \pm\infty$ are given by

$$W_0(x,\grave{\tau}) \sim x \quad \text{as} \quad x \to +\infty \quad \text{and} \quad W_0(x,\grave{\tau}) \to 0 \quad \text{as} \quad x \to -\infty\ .$$

The PDE (6.2.21) has a symmetry with generator

$$\Gamma = x\frac{\partial}{\partial x} + 2\grave{\tau}\frac{\partial}{\partial \grave{\tau}} + W_0\frac{\partial}{\partial W_0} \tag{6.2.23}$$

which leads to an invariant solution of the form

$$W_0(z,\grave{\tau}) = \sqrt{\grave{\tau}}\phi(z) \ \text{ where } z = \frac{x}{\sqrt{\grave{\tau}}}\ . \tag{6.2.24}$$

Substituting (6.2.24) into (6.2.21) yields the reduced equation

$$2\theta_1\phi'' + z\phi' - \phi = 0 \tag{6.2.25}$$

which needs to be solved subject to the boundary conditions

$$\phi(z) \sim z \quad \text{as} \quad z \to +\infty \quad \text{and} \quad \phi(z) \to 0 \quad \text{as} \quad z \to -\infty \; .$$

The solution of (6.2.25) subject to above conditions is easily found to be

$$\phi(z) = \sqrt{\frac{\theta_1}{\pi}} \exp\{-\frac{z^2}{4\theta_1}\} + \frac{z}{2} \operatorname{erfc}\left(-\frac{z}{2\sqrt{\theta_1}}\right) \tag{6.2.26}$$

so that in terms of x and $\grave{\tau}$ we get from (6.2.24)

$$W_0(x, \grave{\tau}) = \sqrt{\frac{\theta_1 \grave{\tau}}{\pi}} \exp\{-\frac{x^2}{4\theta_1 \grave{\tau}}\} + \frac{x}{2} \operatorname{erfc}\left(-\frac{x}{2\sqrt{\theta_1 \grave{\tau}}}\right), \tag{6.2.27}$$

where θ_1 is given in (6.2.22). Now collecting terms of $O(\sqrt{\varepsilon})$ we get an equation for $W_1(x, \grave{\tau})$ namely

$$\frac{\partial W_1}{\partial \grave{\tau}} = \theta_1 \frac{\partial^2 W_1}{\partial x^2} + \theta_2 x \frac{\partial^2 W_0}{\partial x^2} \tag{6.2.28}$$

$$\text{where } \theta_2 = \gamma \sigma^2 K^{2\gamma - 2} \tag{6.2.29}$$

and where θ_1 is given in (6.2.22). Equation (6.2.28) needs to be solved subject to the conditions $W_1(x, 0) = 0$ and $W_1(x, \grave{\tau}) \to 0$ as $x \to \pm\infty$. As $W_0(x, \grave{\tau})$ satisfies (6.2.21) we find the particular solution of (6.2.28) satisfying the given conditions using Result 5.3.1 as

$$W_1(x, \grave{\tau}) = \frac{\theta_2 x \sqrt{\grave{\tau}}}{4\sqrt{\pi \theta_1}} \exp\{-\frac{x^2}{4\theta_1 \grave{\tau}}\} \; . \tag{6.2.30}$$

From (6.2.19), (6.2.27) and (6.2.30) we get the two-term inner expansion

$$W(x,\grave{\tau}) = \sqrt{\frac{\theta_1\grave{\tau}}{\pi}}\exp\{-\frac{x^2}{4\theta_1\grave{\tau}}\} + \frac{x}{2}\,\mathrm{erfc}\left(-\frac{x}{2\sqrt{\theta_1\grave{\tau}}}\right) + \sqrt{\varepsilon}\,\left[\frac{\theta_2 x\sqrt{\grave{\tau}}}{4\sqrt{\pi}\theta_1}\exp\{-\frac{x^2}{4\theta_1\grave{\tau}}\}\right]. \tag{6.2.31}$$

Equation (6.2.31) is valid in the inner region, while equation (6.2.16) is valid in the outer region. Now we can match the outer and inner expansions to get the uniform expansion which is uniformly valid in both outer and inner regions. The uniform expansion can be found by combining the outer and inner expansions and then subtracting the common part, i.e. 'outer+inner-common'. In our solution as $\varepsilon \to 0$ the outer expansion coincides with the common part, so the inner expansion can be used to approximate the price of call option contracts. Hence, we get the price of a call option contract by substituting (6.2.17a,b) into (6.2.31) to get (6.2.8).
□

This formula is valid for arbitrary γ and can actually be more useful for pricing spread options with small times to expiry as compared to (6.2.7) as it involves no integration. For this reason also it more convenient for calibration purposes.

6.3 Explicit Modelling of the Two Underlying Futures

An alternative approach to pricing crack spread options is to model the dynamics of each of the underlying futures prices explicitly. In this section we assume that each of the underlying futures prices follows a risk-neutral CEV process with zero drift namely

$$dF_i = \sigma_i F_i^{\gamma} d\tilde{Z}_i, \quad \text{where} \quad i = 1,2 \quad \text{and} \quad Corr(d\tilde{Z}_1, d\tilde{Z}_2) = \rho dt\ , \tag{6.3.32}$$

where σ_i, $-1 \leq \rho \leq 1$ and $\gamma \in \mathbb{R}$ are constant. In Section 5.3.1 we derived the PDEs to price options on a futures contract when the futures model was based on the three-quarters models for the underlying price. In a similar way we can show that if the underlying prices follow (6.3.32) then the PDE for pricing crack spread call options with expiry T and strike price K is given by

$$\frac{\partial C}{\partial t} + \frac{\sigma_1^2 F_1^{2\gamma}}{2}\frac{\partial^2 C}{\partial F_1^2} + \frac{\sigma_2^2 F_2^{2\gamma}}{2}\frac{\partial^2 C}{\partial F_2^2} + \rho\sigma_1\sigma_2 F_1^{\gamma} F_2^{\gamma}\frac{\partial^2 C}{\partial F_1 \partial F_2} - rC = 0 \qquad (6.3.33a)$$

subject to the final condition

$$C(F_1, F_2, T) = \max(F_1 - F_2 - K, 0)\ . \qquad (6.3.33b)$$

Theorem 6.3.1 *Given that the two underlying futures prices F_1 and F_2 follow the risk neutral process (6.3.32), then an approximate solution for a European crack spread call option valid for short times to expiry T and strike price K is given by*

$$\begin{aligned} C(F_1, F_2, t) = &\left(K\sqrt{\frac{\theta_1(T-t)}{\pi}} + \frac{\sqrt{\tau}(F_1 - F_2 - K)}{8\sqrt{\theta_1 \pi}}(2\theta_2 - \frac{\theta_3\theta_4}{\theta_1})\right) e^{-\frac{(F_1-F_2-K)^2}{4\theta_1 K^2 (T-t)}} \\ &+ \frac{F_1 - F_2 - K}{2}\ \text{erfc}\left(-\frac{F_1 - F_2 - K}{2K\sqrt{\theta_1(T-t)}}\right) \end{aligned} \qquad (6.3.34)$$

$$\begin{aligned} \text{where}\ \ \theta_1 &= \frac{K^{2\gamma-2}}{2^{2\gamma+1}}\left(\sigma_1^2(y+1)^{2\gamma} + \sigma_2^2(y-1)^{2\gamma} - 2\rho\sigma_1\sigma_2(y^2-1)^{\gamma}\right)\ , \\ \theta_2 &= \frac{\gamma K^{2\gamma-2}}{2^{2\gamma}}\left(\sigma_1^2(y+1)^{2\gamma-1} - \sigma_2^2(y-1)^{2\gamma-1} + 2\rho\sigma_1\sigma_2(y^2-1)^{\gamma-1}\right)\ , \\ \theta_3 &= \frac{K^{2\gamma-2}}{2^{2\gamma}}\left(\sigma_1^2(y+1)^{2\gamma} - \sigma_2^2(y-1)^{2\gamma}\right)\ , \\ \theta_4 &= \frac{\gamma K^{2\gamma-2}}{2^{2\gamma}}\left(\sigma_1^2(y+1)^{2\gamma-1} + \sigma_2^2(y-1)^{2\gamma-1} - 2\rho\sigma_1\sigma_2 y(y^2-1)^{\gamma-1}\right)\ . \\ y &= \frac{F_1+F_2}{K}\ . \end{aligned}$$

Proof: Letting $\tau = T - t$, equation (6.3.33a) can be written as

$$\frac{\partial C}{\partial \tau} = \frac{\sigma_1^2 F_1^{2\gamma}}{2}\frac{\partial^2 C}{\partial F_1^2} + \frac{\sigma_2^2 F_2^{2\gamma}}{2}\frac{\partial^2 C}{\partial F_2^2} + \rho\sigma_1\sigma_2 F_1^{\gamma} F_2^{\gamma}\frac{\partial^2 C}{\partial F_1 \partial F_2} - rC \tag{6.3.35}$$

subject to initial condition

$$C(F_1, F_2, 0) = \max(F_1 - F_2 - K, 0) \ . \tag{6.3.36}$$

For a small time to expiry τ we let $\tau = \varepsilon\grave{\tau}$ where ε is a small parameter $0 < \varepsilon \ll 1$. Then equation (6.3.35) becomes

$$\frac{\partial C}{\partial \grave{\tau}} = \varepsilon\left(\frac{\sigma_1^2 F_1^{2\gamma}}{2}\frac{\partial^2 C}{\partial F_1^2} + \frac{\sigma_2^2 F_2^{2\gamma}}{2}\frac{\partial^2 C}{\partial F_2^2} + \rho\sigma_1\sigma_2 F_1^{\gamma} F_2^{\gamma}\frac{\partial^2 C}{\partial F_1 \partial F_2} - rC\right) . \tag{6.3.37}$$

We assume that the solution can be written as a series in ε i.e.

$$C(F_1, F_2, \grave{\tau}) = \sum_{i=0}^{\infty} \varepsilon^i C_i(F_1, F_2, \grave{\tau}) \ . \tag{6.3.38}$$

Substituting (6.3.38) into (6.3.37) and collecting terms of $O(1)$, we get an equation for $C_0(F_1, F_2, \grave{\tau})$, namely

$$\frac{\partial C_0}{\partial \grave{\tau}} = 0 \ , \tag{6.3.39}$$

which solved subject to the initial condition $C_0(F_1, F_2, 0) = \max(F_1 - F_2 - K, 0)$, gives

$$C_0(F_1, F_2, \tau) = \max(F_1 - F_2 - K, 0) = \begin{cases} F_1 - F_2 - K & \text{if } F_1 - F_2 \geq K \ , \\ 0 & \text{otherwise.} \end{cases} \tag{6.3.40}$$

Collecting terms of $O(\varepsilon)$ in (6.3.37), we get an equation for $C_1(F_1, F_2, \check{\tau})$:

$$\frac{\partial C_1}{\partial \check{\tau}} = \frac{\sigma_1^2 F_1^{2\gamma}}{2}\frac{\partial^2 C_0}{\partial F_1^2} + \frac{\sigma_2^2 F_2^{2\gamma}}{2}\frac{\partial^2 C_0}{\partial F_2^2} + \rho\sigma_1\sigma_2 F_1^{\gamma} F_2^{\gamma}\frac{\partial^2 C_0}{\partial F_1 \partial F_2} - rC_0 \ . \tag{6.3.41}$$

Solving (6.3.41) with initial condition $C_1(F_1, F_2, 0) = 0$, we get

$$C_1(F_1, F_2, \check{\tau}) = \begin{cases} -r\check{\tau}(F_1 - F_2 - K) & \text{if } F_1 - F_2 \geq K \ , \\ 0 & \text{otherwise.} \end{cases} \tag{6.3.42}$$

Hence, the first two terms of our series solution are given by

$$C_0(F_1, F_2, \check{\tau}) + \varepsilon C_1(F_1, F_2, \check{\tau}) = \begin{cases} (F_1 - F_2 - K)(1 - \varepsilon r\check{\tau}) & \text{if } F_1 - F_2 \geq K \ , \\ 0 & \text{otherwise.} \end{cases} \tag{6.3.43}$$

From equation (6.3.43) it can be observed that the expansion is continuous, but not differentiable at $F_1 - F_2 - K = 0$. Solution (6.3.43) is therefore invalid in this region and is thus termed our "outer" solution. Now letting $\xi = F_1 - F_2$ and $\vartheta = F_1 + F_2$ equation (6.3.37) becomes

$$\begin{aligned}\frac{\partial C}{\partial \check{\tau}} =& \varepsilon\left[\frac{\sigma_1^2(\xi+\vartheta)^{2\gamma}}{2^{2\gamma+1}}\left(\frac{\partial^2 C}{\partial \xi^2} + 2\frac{\partial^2 C}{\partial\xi\partial\vartheta} + \frac{\partial^2 C}{\partial \vartheta^2}\right) + \frac{\sigma_2^2(\vartheta-\xi)^{2\gamma}}{2^{2\gamma+1}}\left(\frac{\partial^2 C}{\partial \xi^2} - 2\frac{\partial^2 C}{\partial\xi\partial\vartheta} + \frac{\partial^2 C}{\partial \vartheta^2}\right)\right. \\ &\left. + \frac{\rho\sigma_1\sigma_2(\vartheta^2-\xi^2)^{\gamma}}{2^{2\gamma}}\left(\frac{\partial^2 C}{\partial \vartheta^2} - \frac{\partial^2 C}{\partial \xi^2}\right) - rC\right] \ . \end{aligned} \tag{6.3.44}$$

We now analyse the solution in the inner region by introducing a stretching variable

$$x = \frac{\xi - K}{\sqrt{\varepsilon}K} \tag{6.3.45a}$$

and rescaling

$$y = \frac{\vartheta}{K} \ , \tag{6.3.45b}$$

$$C(\xi, \vartheta, \grave{\tau}) = K\sqrt{\varepsilon}W(x, y, \grave{\tau}) \ . \tag{6.3.45c}$$

The choice of power $\frac{1}{2}$ in $\sqrt{\varepsilon}$ is a well-balanced choice and ensures that the coefficient of the second-order derivative term is not small compared to the other coefficients. Equation (6.3.37) then becomes

$$\begin{aligned}\frac{\partial W}{\partial \grave{\tau}} = \varepsilon\Bigg[&\frac{\sigma_1^2 K^{2\gamma-2}(1+\sqrt{\varepsilon}x+y)^{2\gamma}}{2^{2\gamma+1}}\left(\frac{1}{\varepsilon}\ \frac{\partial^2 W}{\partial x^2}+\frac{2}{\sqrt{\varepsilon}}\frac{\partial^2 W}{\partial x \partial y}+\frac{\partial^2 W}{\partial y^2}\right)\\ &+\frac{\sigma_2^2 K^{2\gamma-2}(y-(1+\sqrt{\varepsilon}x))^{2\gamma}}{2^{2\gamma+1}}\left(\frac{1}{\varepsilon}\ \frac{\partial^2 W}{\partial x^2}-\frac{2}{\sqrt{\varepsilon}}\ \frac{\partial^2 W}{\partial x \partial y}+\ \frac{\partial^2 W}{\partial y^2}\right)\\ &+\frac{\rho\sigma_1\sigma_2 K^{2\gamma-2}(y^2-(1+\sqrt{\varepsilon}x)^2)^{\gamma}}{2^{2\gamma}}\left(\frac{\partial^2 W}{\partial y^2}-\frac{1}{\varepsilon}\ \frac{\partial^2 W}{\partial x^2}\right)-rW\Bigg] \ ,\end{aligned} \tag{6.3.46}$$

which needs to be solved subject to the initial condition $W(x, y, 0) = \max(x, 0)$, and

$$W(x, y, \grave{\tau}) \sim x \quad \text{as} \quad x \to +\infty \quad \text{and} \quad W(x, y, \grave{\tau}) \to 0 \quad \text{as} \quad x \to -\infty \ .$$

Now we expand $W(x, y, \grave{\tau})$ in terms of $\varepsilon^{\frac{1}{2}}$ i.e.

$$W(x, y, \grave{\tau}) \ = \sum_{i=0}^{\infty} \ \varepsilon^{\frac{i}{2}} W_i(x, y, \grave{\tau}) \tag{6.3.47}$$

which we substitute into (6.3.46) and then aim to collect terms in powers of $\sqrt{\varepsilon}$. Firstly the terms $(1 + \sqrt{\varepsilon}x + y)^{2\gamma}$, $(y - (1 + \sqrt{\varepsilon}x))^{2\gamma}$ and $(y^2 - (1 + \sqrt{\varepsilon}x)^2)^{\gamma}$ in

(6.3.46) are expanded in powers of $\varepsilon^{\frac{1}{2}}$ as

$$(1+\sqrt{\varepsilon}x+y)^{2\gamma} = (1+y)^{2\gamma} + \sqrt{\varepsilon}\ \ 2\gamma x(1+y)^{2\gamma-1} +\ O(\varepsilon) \tag{6.3.48a}$$

$$(y-(1+\sqrt{\varepsilon}x))^{2\gamma} = (y-1)^{2\gamma} - \sqrt{\varepsilon}\ \ 2\gamma x(y-1)^{2\gamma-1} +\ O(\varepsilon) \tag{6.3.48b}$$

$$(y^2-(1+\sqrt{\varepsilon}x)^2)^{\gamma} = (y^2-1)^{\gamma} - \sqrt{\varepsilon}\ \ 2\gamma x(y^2-1)^{\gamma-1} +\ O(\varepsilon)\ . \tag{6.3.48c}$$

Now by substituting (6.3.47) and (6.3.48 a-c) into (6.3.46) and collecting terms of $O(1)$, we find an equation in $W_0(x, y, \grave{\tau})$, namely

$$\frac{\partial W_0}{\partial \grave{\tau}} = \theta_1 \frac{\partial^2 W_0}{\partial x^2}\ , \tag{6.3.49}$$

$$\text{where } \theta_1 = \frac{K^{2\gamma-2}}{2^{2\gamma+1}}\left(\sigma_1^2(y+1)^{2\gamma} + \sigma_2^2(y-1)^{2\gamma} - 2\rho\sigma_1\sigma_2(y^2-1)^{\gamma}\right)\ . \tag{6.3.50}$$

We note that as there are no derivatives with respect to y in (6.3.49), so θ_1 can be treated as a constant. Equation (6.3.49) needs to be solved subject to the initial condition $W_0(x, y, 0) = \max(x, 0)$, while the conditions at $x \to \pm\infty$ are given by

$$W_0(x, y, \grave{\tau}) \sim x \quad \text{as} \quad x \to +\infty \quad \text{and} \quad W_0(x, y, \grave{\tau}) \to 0 \quad \text{as} \quad x \to -\infty\ .$$

As in the previous section, the solution of $W_0(x, y, \grave{\tau})$ is given by

$$W_0(x, y, \grave{\tau}) = \sqrt{\frac{\theta_1 \grave{\tau}}{\pi}}\exp\{-\frac{x^2}{4\theta_1 \grave{\tau}}\} + \frac{x}{2}\ \text{erfc}\left(-\frac{x}{2\sqrt{\theta_1 \grave{\tau}}}\right)\ , \tag{6.3.51}$$

where θ_1 is given in (6.3.50).

Now collecting terms of $O(\sqrt{\varepsilon})$ in (6.3.46) we get an equation for $W_1(x,y,\grave{\tau})$ namely

$$\frac{\partial W_1}{\partial \grave{\tau}} = \theta_1 \frac{\partial^2 W_1}{\partial x^2} + \theta_2 x \frac{\partial^2 W_0}{\partial x^2} + \theta_3 \frac{\partial^2 W_0}{\partial x \partial y} \tag{6.3.52}$$

$$\text{where} \quad \theta_2 = \frac{\gamma K^{2\gamma-2}}{2^{2\gamma}} \left(\sigma_1^2 (y+1)^{2\gamma-1} - \sigma_2^2 (y-1)^{2\gamma-1} + 2\rho\sigma_1\sigma_2 (y^2-1)^{\gamma-1} \right) , \tag{6.3.53}$$

$$\theta_3 = \frac{K^{2\gamma-2}}{2^{2\gamma}} \left(\sigma_1^2 (y+1)^{2\gamma} - \sigma_2^2 (y-1)^{2\gamma} \right) , \tag{6.3.54}$$

and θ_1 is given in (6.3.50). Equation (6.3.52) needs to be solved subject to the initial condition $W_1(x,y,0) = 0$, and $W_1(x,y,\grave{\tau}) = 0$ as $x \to \pm\infty$.

The solution of this problem is found by using Result 5.3.1 and is given by

$$\begin{aligned} W_1(x,y,\grave{\tau}) =& \grave{\tau} \left(\theta_2 (x \frac{\partial^2 W_0}{\partial x^2} + \theta_1 \grave{\tau} \frac{\partial^3 W_0}{\partial x^3}) \right) \\ &+ \int_0^{\grave{\tau}} \frac{1}{2\sqrt{\pi\theta_1(\grave{\tau}-\omega)}} \int_{-\infty}^{\infty} q(\zeta,y,\omega) e^{-\frac{(x-\zeta)^2}{4\theta_1(\grave{\tau}-\omega)}} d\zeta d\omega \end{aligned} \tag{6.3.55}$$

$$\text{where} \quad q(x,y,\grave{\tau}) = \theta_3 \frac{\partial^2 W_0}{\partial x \partial y} \; .$$

This can be simplified to get

$$W_1(x,y,\grave{\tau}) = \frac{\sqrt{\grave{\tau}}\, x e^{-\frac{x^2}{4\theta_1 \grave{\tau}}}}{8\sqrt{\theta_1 \pi}} (2\theta_2 - \frac{\theta_3\theta_4}{\theta_1}) \; , \tag{6.3.56}$$

$$\text{where} \quad \theta_4 = \frac{\gamma K^{2\gamma-2}}{2^{2\gamma}} \left(\sigma_1^2 (y+1)^{2\gamma-1} + \sigma_2^2 (y-1)^{2\gamma-1} - 2\rho\sigma_1\sigma_2 y (y^2-1)^{\gamma-1} \right) . \tag{6.3.57}$$

This solution satisfies the necessary initial and boundary conditions. By substituting (6.3.51) and (6.3.56) into (6.3.47) we get the two-term inner expansion as

$$W(x,y,\grave{\tau}) = \sqrt{\frac{\theta_1\grave{\tau}}{\pi}}\exp\{-\frac{x^2}{4\theta_1\grave{\tau}}\} + \frac{x}{2}\,\mathrm{erfc}\left(-\frac{x}{2\sqrt{\theta_1\grave{\tau}}}\right) + \sqrt{\varepsilon}\left[\frac{\sqrt{\grave{\tau}}xe^{-\frac{x^2}{4\theta_1\grave{\tau}}}}{8\sqrt{\theta_1\pi}}(2\theta_2 - \frac{\theta_3\theta_4}{\theta_1})\right]. \tag{6.3.58}$$

Equation (6.3.58) is valid in the inner region, while equation (6.3.43) is valid in the outer region. Now we can match the outer and inner expansions to get the uniform expansion which is uniformly valid in both outer and inner regions. The uniform expansion can be found by combining the outer and inner expansions and then subtracting the common part, i.e. '$W_{outer} + W_{inner} - W_{common}$'. In our solution as $\varepsilon \to 0$ the outer expansion coincides with the common part, so the inner expansion can be used to approximate the price of call option contracts.

Hence, we get the price of a call crack spread option contract by using our change of variables (6.3.45a-c) , $\xi = F_1 - F_2$ and $\vartheta = F_1 + F_2$ in (6.3.58) to get (6.3.34). □

6.4 Popular Existing Models

6.4.1 Popular Existing Univariate Models

Bachelier model (ABM)

With $\sigma(t,T) = \sigma$, constant and $\gamma = 0$ in (6.2.3), risk-neutral futures crack spread prices are assumed to follow an ABM with zero drift. The transition density function of F_{sp} is given by:

$$Tr(F_{sp},t;F_{spT},T) = \frac{\exp\left\{-\frac{(F_{sp}-F_{spT})^2}{2\sigma^2(T-t)}\right\}}{\sigma\sqrt{2\pi(T-t)}}. \tag{6.4.59}$$

In this case the price of a call spread option, known as the Bachelier (1900) formula, can be found as the present value of the expected payoff under the risk-neutral measure $\mathbb{Q}$ i.e.

$$\begin{aligned} C(F_{sp},t) &= e^{-r(T-t)}\mathbb{E}_t^{\mathbb{Q}}\left(\max[F_{spT}-K,0]\right) \\ &= e^{-r(T-t)}\left[\int_K^{\infty}(y-K)\ Tr(F_{sp},t;y,T)\ dy\right] \\ &= e^{-r(T-t)}\left[(F_{sp}-K)\int_{-u}^{\infty}\frac{e^{-\frac{\xi^2}{2}}}{\sqrt{2\pi}}\ d\xi+\sigma\sqrt{T-t}\int_{-u}^{\infty}\frac{\xi e^{-\frac{\xi^2}{2}}}{\sqrt{2\pi}}\ d\xi\right] \\ &= e^{-r(T-t)}\left[(F_{sp}-K)N(u)+\frac{\sigma\sqrt{T-t}\ e^{-\frac{u^2}{2}}}{\sqrt{2\pi}}\right]\ , \qquad (6.4.60) \\ &\text{where}\ \ \xi=\frac{F_{spT}-F_{sp}}{\sigma\sqrt{T-t}}\ \ \text{and}\ \ u=\frac{F_{sp}-K}{\sigma\sqrt{T-t}}\ , \end{aligned}$$

and where $N(\cdot)$ is the cumulative standard normal distribution function. The assumption that the spread, which in general can be defined as $F_{sp}=aF_1-bF_2$ for some constants a and b, follows $dF_{sp}=\sigma d\tilde{Z}$ implies that each underlying futures price, F_1 and F_2, follows an ABM with zero drift i.e.

$$dF_i=\sigma_i d\tilde{Z}_i\ \ \text{where}\ \ i=1,\ 2\ \ \text{and}\ \ Corr(d\tilde{Z}_1,d\tilde{Z}_2)=\rho dt\ . \qquad (6.4.61)$$

In this case σ^2 and $\tilde{Z}$ in (6.2.3) can be defined by $\sigma^2=\sqrt{(a\sigma_1)^2+-2ab\rho\sigma_1\sigma_2(b\sigma_2)^2}$ and $\tilde{Z}=\frac{a\sigma_1}{\sigma}\tilde{Z}_1-\frac{b\sigma_2}{\sigma}\tilde{Z}_2$.

Black-Scholes model (GBM)

With $\sigma(t,T)=\sigma$, constant and $\gamma=1$ in (6.2.3), risk-neutral futures crack spread prices are assumed to follow a GBM with zero drift. In this case the solution of the option pricing equation (6.2.4) is a special case of the Black-Scholes (1.15a-c) pricing formula (with dividend yield D equals to the risk-free rate r) and is given by:

$$C(F_{sp}, t) = e^{-r(T-t)} \left[F_{sp} N(d_1) - K N(d_2) \right] \qquad (6.4.62)$$

$$\text{where} \quad d_1 = \frac{\ln(\frac{F_{sp}}{K}) + \frac{\sigma^2}{2}(T-t)}{\sigma\sqrt{T-t}} \quad \text{and} \quad d_2 = d_1 - \sigma\sqrt{T-t}.$$

Schwartz model

With $\sigma(t,T) = \sigma e^{\eta(T-t)}$ and $\gamma = 1$ in (6.2.3), risk-neutral futures crack spread prices are assumed to follow the Schwartz model with zero drift. In this case the solution of (6.2.4) can be derived as in Section 5.2.2 (with $F = F_{sp}$ and $T = T_F = T_O$) and is given by:

$$C(F_{sp}, t) = e^{-r(T-t)} \left[F_{sp} N(d_1) - K N(d_2) \right] \qquad (6.4.63)$$

$$\text{where} \quad d_1 = \frac{\ln(\frac{F_{sp}}{K}) + \frac{w^2}{2}(T-t)}{w\sqrt{T-t}}, \quad d_2 = d_1 - w\sqrt{T-t} \qquad (6.4.64)$$

$$\text{and} \quad w^2 = \frac{\sigma^2}{2\eta(T-t)}(1 - e^{-2\eta(T-t)}) \; .$$

While using univariate modelling for the spread has the advantage of simplicity and tractability, there are obvious shortcomings. As mentioned in Section 6.1, the main disadvantage of using univariate modelling for the spread is that the correlation between the price movements of the two underlying assets, which has a major effect on the spread value, is ignored. As well, under some univariate models such as the GBM and Schwartz models, futures spread prices cannot attain negative values, even though it is possible in reality for futures spreads to become negative. Under the ABM, futures spread prices can become negative, but unfortunately the model implies that the individual underlying futures prices can also become negative, which is not realistic. So we anticipate that these univariate models might not perform as well as the explicit models and we test this assumption in Section 6.5.

6.4.2 Popular Existing Explicit Models

Kirk model

Kirk (1995) derived a well-known approximation formula for valuing European crack spread options. He assumed that when $K \ll F_2$, the dynamics of $\frac{F_1}{F_2+K}$ can be approximated by a GBM. Kirk wrote the payoff of a call spread option in the form $(F_2+K)\max(\zeta-1,0)$ where $\zeta = \frac{F_1}{F_2+K}$, so that a spread option could be considered as an option on ζ with strike price 1. Then the dynamics of ζ is given by[4]

$$d\zeta = \sigma\zeta d\tilde{Z} \tag{6.4.65}$$

where

$$\sigma^2 = \sigma_1^2 - 2\left(\frac{F_2}{F_2+K}\right)\rho\sigma_1\sigma_2 + \left(\frac{F_2}{F_2+K}\right)^2\sigma_2^2 \,. \tag{6.4.66}$$

This way the price of a European crack call spread option is a special case of the Black-Scholes pricing formula (1.15a-c) (with dividend yield D equal to the risk-free rate r) and is given by:

$$\begin{aligned} C(F_1,F_2,t) =& e^{-r(T-t)}\left[F_1\ N(d_1) - (F_2+K)\ N(d_2)\right] \\ \text{where}\ \ d_1 =& \frac{\ln(\frac{F_1}{F_2+K}) + \frac{\sigma^2}{2}(T-t)}{\sigma\sqrt{T-t}}\ , \\ d_2 =& d_1 - \sigma\sqrt{T-t}\ . \end{aligned} \tag{6.4.67}$$

[4]More details can be found in Alexander and Venkatramanan (2007).

Carmona and Durrleman model

Carmona and Durrleman (2003) assume that the two underlying risk-neutral futures prices follow two correlated GBMs with zero drift i.e.

$$dF_i = \sigma_i F_i d\tilde{Z}_i, \quad \text{where} \quad i = 1, 2 \quad \text{and} \quad Corr(d\tilde{Z}_1, d\tilde{Z}_2) = \rho dt \ , \tag{6.4.68}$$

where σ_i and $-1 \leq \rho \leq 1$ are constant. The analytic expression at time T for each of the underlying futures contracts is given by

$$F_{iT} = F_{it} \exp\left\{ -\frac{\sigma_i^2 (T-t)}{2} + \sigma_i (\tilde{Z}_{iT} - \tilde{Z}_{it}) \right\}, \quad \text{where} \quad i = 1, 2 \quad \text{and} \quad t < T. \tag{6.4.69}$$

If we define F_{spT} to be the value of the futures crack spread at expiry i.e. $F_{spT} = F_{1T} - F_{2T}$, then the distribution of F_{spT} can be approximated by the Gaussian distribution by matching their first two moments i.e.

$$F_{spT} \sim N(\mathbb{E}\{F_{1T} - F_{2T}\}, \ \mathrm{Var}\{F_{1T} - F_{2T}\}) \ . \tag{6.4.70}$$

From Section 2.1.1, $\mathbb{E}\{F_{1T} - F_{2T}\}$ and $\mathrm{Var}\{F_{1T} - F_{2T}\}$ are given by[5]

$$\mu = \mathbb{E}\{F_{1T} - F_{2T}\} = F_{1t} - F_{2t} \ , \tag{6.4.71a}$$

$$\sigma = \mathrm{Var}\{F_{1T} - F_{2T}\} = F_{1t}^2 (e^{\sigma_1^2 (T-t)} - 1) - 2F_{1t}F_{2t}(e^{\rho\sigma_1\sigma_2(T-t)} - 1) + F_{2t}^2(e^{\sigma_2^2(T-t)} - 1) \ . \tag{6.4.71b}$$

Then we can find the price of a European crack spread call option contract that expires at time T with strike price K as the present value of the expected payoff

[5] As the dynamics of futures prices are assumed to be martingales, their expected value at a future time is equal to their value today.

under the risk-neutral measure $\mathbb{Q}$ i.e.

$$\begin{aligned} C(F_1, F_2, t) =& e^{-r(T-t)}\mathbb{E}_t^{\mathbb{Q}}\left[\max\left(F_{spT} - K, 0\right)\right] \\ =& e^{-r(T-t)}\mathbb{E}_t^{\mathbb{Q}}\left[\max\left(\mu - K + \sigma\zeta, 0\right)\right], \quad \text{for some random variable} \quad \zeta \sim N(0,1) \\ =& \frac{e^{-r(T-t)}}{\sqrt{2\pi}} \int_d^\infty \left(\mu - K + \sigma u\right) \quad e^{-\frac{u^2}{2}}\, du \;, \;\text{ where }\; d = \frac{K-\mu}{\sigma} \\ =& e^{-r(T-t)}\left[(\mu - K)N(d) + \frac{\sigma\sqrt{T-t}\; e^{-\frac{d^2}{2}}}{\sqrt{2\pi}}\right] . \end{aligned} \tag{6.4.72}$$

6.5 Estimation of the Parameter γ and Accuracy of the Analytic Approximation Formulae

As exact solutions in terms of transcendental functions to crack spread options under univariate and explicit CEV models, i.e. to (6.2.4) and (6.3.33), do not exist for arbitrary γ, solutions (6.2.8) and (6.3.34) which are valid for arbitrary γ are useful for determining the best γ for valuing options with short times to expiry. In this section, we estimate the parameter γ which provides the best fit to our options data by using our new analytic approximation formulae (6.2.8) and (6.3.34), which approximates the price of spread options with short times to expiry. Moreover, numerical examples are provided in order to measure the accuracy of the analytic approximation formulae with the estimated values of γ.

6.5.1 Estimation of the Parameter γ

We now estimate the parameter γ in our new analytic approximation formulae i.e. (6.2.8) and (6.3.34) that provides the best fit to our options data. To do this, we use 955 observations of Heating Oil Crack Spread call option prices selected from the NYMEX with various strikes and expiries selected from the years 2010 and 2011. Based on the maximum time to expiry, we divide the data into four groups where the maximum time to expiry is chosen to be 60, 45, 30 and 15 days in the first, second, third and fourth groups respectively. For each pricing formula and group, we minimise the sum of squared errors i.e.

$$\min(SSE) = \sum_i (C_{approximate_i} - C_{market_i})^2 , \qquad (6.5.73)$$

in order to estimate the parameter γ and the other parameters. The results[6] are listed in Tables 6.1 and 6.2. From these tables we note that:

- For the analytic approximation formula (6.2.8) using univariate modelling, the estimated values of γ lie within a small range, i.e. $0.58 \leq \gamma \leq 0.62$, and this suggests that choosing $\gamma = 0.50,\ 0.60$ or 0.75 might be good choices for pricing crack spread call options.

- For the analytic approximation formula (6.3.34) using explicit modelling for all groups the estimated value of the correlation coefficient is (or is close to) one. This is to be expected as the futures prices of the two underlyings are highly correlated. In addition, the estimated values of γ lie within a small range, i.e. $0.60 \leq \gamma \leq 0.86$, and this suggests that choosing $\gamma = 0.75$ might be a good choice for pricing crack spread call options, especially with expiries up to two months.

- Comparison of SSE of the fourth group, indicates that the univariate pricing formula (6.2.8) has a lower SSE value compared to the explicit pricing formula (6.3.34). This is a surprising result as the univariate pricing formula includes two free parameters whereas the explicit pricing formula has four parameters. In general, for groups 1-3, the SSEs are smaller using the explicit pricing formula but possibly not significantly so, and this will be discussed in Section 6.6.

[6] We should note that the results of our estimation provide the best values of γ for our given data, and different values of γ could be estimated by using different samples. However, the estimated values of γ found here give us a good indication of the range of best γ.

group	number of observations	γ	σ	SSE
1	955	0.60	2.19	14.75
2	837	0.58	2.36	8.31
3	617	0.60	2.25	4.36
4	333	0.62	2.20	1.32

Table 6.1: The estimated parameters for analytic approximation formula (6.2.8).

group	number of observations	γ	ρ	σ_1	σ_2	SSE
1	955	0.75	0.99	1.88	1.88	12.21
2	837	0.86	0.99	0.83	0.74	7.63
3	617	0.71	1.00	2.10	1.87	4.24
4	333	0.60	1.00	4.40	4.05	1.42

Table 6.2: The estimated parameters for analytic approximation formula (6.3.34).

6.5.2 The Accuracy of the Analytic Approximation Formulae

In this part, we provide numerical examples to measure the accuracy of our new analytic approximation formulae i.e. (6.2.8) and (6.3.34). For the analytic approximation formula (6.2.8), we consider $\gamma = 0,\ \frac{1}{2},\ \frac{3}{4},\ 1$ and for the analytic approximation formula (6.3.34), we consider $\gamma = \frac{1}{2},\ \frac{3}{4}$.

The formula (6.2.8) is compared with its corresponding exact solution of (6.2.4), i.e. (6.4.60) when $\gamma = 0$, (6.2.7) when $\gamma = \frac{1}{2},\ \frac{3}{4}$, and (6.4.62) when $\gamma = 1$. For each value of γ we choose $F_{sp} = 13$, $T - t = \frac{1}{12}$, $r = 0.01$ and four values of strike and three values of volatility. The signed percentage error (SPE):

$$\text{SPE} = \frac{C_{approximate} - C_{exact}}{C_{exact}} \times 100\% \qquad (6.5.74)$$

and absolute error (AE) for each couple (K, σ) are calculated.

In the explicit modelling case there is no known exact solution to (6.3.33). Hence, we apply the Monte Carlo technique with 150,000 trials to price crack spread call options and then use the simulated values as proxy for the exact solutions. For each value of γ we chose $F_1 = 96$, $F_2 = 83$, $T - t = \frac{1}{12}$, $r = 0.01$ and three values of strike, volatility and correlation. For each case $(K, \sigma_1, \sigma_2, \rho)$, SPE and AE are calculated. Tables 6.3-6.5 list the results. In particular we note that:

- ***From Table 6.3 the analytic approximation formula (6.2.8)*** generally slightly overprices option contracts compared with the exact solution. For any given case (σ, K), the SPE values lie within a small range, namely $(0.08\%, 0.11\%)$, $(-2.54\%, 0.08\%)$, $(-3.42\%, 0.11\%)$ and $(-2.45\%, 0.15\%)$ using $\gamma = 0, \frac{1}{2}, \frac{3}{4}, 1$ respectively. The average absolute errors are less than 0.42×10^{-3}, 0.42×10^{-3}, 0.43×10^{-3} and 0.49×10^{-3} for the formulae with $\gamma = 0, \frac{1}{2}, \frac{3}{4}, 1$ respectively. This suggests that (6.2.8) provides an excellent approximation to the exact solution.

- ***From Tables 6.4 and 6.5, the prices using analytic approximation formula (6.3.34)*** are generally just under or just above exact solution prices. For any given case $(\sigma_1, \sigma_2, \rho, K)$, with the exceptions[7] of examples with $K = 14$, the SPE values lie within a small range, namely $(-0.69\%, 0.90\%)$ and $(-0.69\%, 0.53\%)$ using $\gamma = \frac{1}{2}$ and $\gamma = \frac{3}{4}$ respectively. Moreover, the average absolute errors are less than 0.72×10^{-3} and 1.33×10^{-3} for the formulae with $\gamma = \frac{1}{2}$ and $\gamma = \frac{3}{4}$ respectively. This suggests that (6.3.34) also provides an excellent approximation to the exact solution.

[7]When $K = 14$ the option is out-the-money and is thus worthless. Both exact and approximate solutions are close to zero, yielding high percentage errors.

σ	K	$\gamma = 0$ SPE	$\gamma = 0$ AE	$\gamma = \frac{1}{2}$ SPE	$\gamma = \frac{1}{2}$ AE	$\gamma = \frac{3}{4}$ SPE	$\gamma = \frac{3}{4}$ AE	$\gamma = 1$ SPE	$\gamma = 1$ AE
0.10	11.50	0.08	1.25	0.08	1.25	0.08	1.25	0.08	1.25
0.10	12.50	0.08	0.42	0.08	0.42	0.08	0.41	0.08	0.42
0.10	13.50	0.08	0.00	-2.54	0.00	-0.37	0.00	0.03	0.00
0.10	14.50	***	0.00	***	0.00	***	0.00	***	0.00
0.20	11.50	0.08	1.25	0.08	1.25	0.08	1.25	0.07	1.06
0.20	12.50	0.08	0.42	0.08	0.42	0.08	0.43	0.09	0.55
0.20	13.50	0.08	0.00	-0.04	0.00	0.08	0.02	0.12	0.14
0.20	14.50	0.11	0.00	***	0.00	***	0.00	-2.45	0.22
0.30	11.50	0.08	1.25	0.08	1.25	0.08	1.22	0.06	0.92
0.30	12.50	0.08	0.42	0.08	0.42	0.09	0.49	0.11	0.77
0.30	13.50	0.08	0.00	0.08	0.01	0.11	0.08	0.15	0.38
0.30	14.50	0.09	0.00	***	0.00	-3.42	0.05	-0.26	0.15
average			0.42		0.42		0.43		0.49

Table 6.3: Signed percentage error (SPE (%)) and absolute error (AE ($\times 10^{-3}$)) of analytic approximation formula (6.2.8) with $\tau = \frac{1}{12}$ (the missing values indicate both exact and approximation solutions are close to zero, yielding high percentage errors).

σ_1	σ_2	K	$\rho = 1$ SPE	$\rho = 1$ AE	$\rho = 0.9$ SPE	$\rho = 0.9$ AE	$\rho = 0.8$ SPE	$\rho = 0.8$ AE
0.35	0.35	12	0.10	0.98	0.11	1.14	0.34	3.45
0.35	0.40	12	0.09	0.89	0.32	3.18	0.04	0.42
0.35	0.45	12	0.05	0.55	0.05	0.47	0.03	0.30
0.40	0.35	12	0.16	1.62	0.17	1.70	-0.16	1.62
0.40	0.40	12	0.10	0.97	0.23	2.30	0.06	0.57
0.40	0.45	12	0.08	0.80	0.08	0.82	0.09	0.92
0.45	0.35	12	0.27	2.65	0.09	0.94	0.03	0.34
0.45	0.40	12	0.02	0.24	0.39	3.90	0.36	3.70
0.45	0.45	12	0.11	1.11	0.23	2.28	-0.10	1.01
0.35	0.35	13	0.49	0.13	-0.69	1.18	0.06	0.15
0.35	0.40	13	0.32	0.08	-0.22	0.40	0.58	1.48
0.35	0.45	13	-0.42	0.32	0.09	0.20	0.90	2.52
0.40	0.35	13	-0.24	0.20	-0.38	0.76	0.16	0.43
0.40	0.40	13	0.32	0.10	0.55	1.07	0.15	0.41
0.40	0.45	13	0.44	0.09	-0.01	0.02	-0.05	0.15
0.45	0.35	13	0.42	0.59	0.42	1.00	0.40	1.23
0.45	0.40	13	0.51	0.45	-0.07	0.15	0.02	0.08
0.45	0.45	13	0.14	0.05	0.46	1.01	-0.36	1.13
0.35	0.35	14	***	0.00	-2.23	0.03	2.38	0.29
0.35	0.40	14	***	0.00	6.06	0.13	1.25	0.20
0.35	0.45	14	***	0.00	-2.50	0.13	-1.35	0.34
0.40	0.35	14	***	0.00	-0.47	0.02	-1.53	0.32
0.40	0.40	14	***	0.00	1.65	0.06	-1.85	0.43
0.40	0.45	14	***	0.00	0.72	0.04	-0.89	0.25
0.45	0.35	14	-0.99	0.00	1.20	0.14	0.95	0.32
0.45	0.40	14	***	0.00	-2.46	0.21	1.81	0.60
0.45	0.45	14	***	0.00	-2.29	0.18	-0.74	0.27
average				0.44		0.87		0.85

Table 6.4: Signed percentage error $\big(\text{SPE } (\%)\big)$ and absolute error $\big(\text{AE } (\times 10^{-3})\big)$ of analytic approximation formula solution (6.3.34) with $\gamma = \frac{1}{2}$ and $\tau = \frac{1}{12}$ (the missing values indicate both exact and approximation solutions are close to zero, yielding high percentage errors).

σ_1	σ_2	K	$\rho = 1$ SPE	AE	$\rho = 0.9$ SPE	AE	$\rho = 0.8$ SPE	AE
0.20	0.20	12	0.14	1.37	0.51	5.21	0.17	1.86
0.20	0.25	12	0.06	0.55	-0.07	0.78	0.02	0.22
0.20	0.30	12	-0.13	1.30	-0.22	2.44	-0.33	4.02
0.25	0.20	12	0.05	0.48	-0.15	1.67	0.02	0.24
0.25	0.25	12	0.10	1.02	-0.12	1.28	0.36	4.25
0.25	0.30	12	0.13	1.28	0.21	2.29	0.25	2.99
0.30	0.20	12	0.32	3.49	-0.30	3.54	-0.43	5.49
0.30	0.25	12	-0.14	1.46	0.45	5.07	0.11	1.38
0.30	0.30	12	0.04	0.41	0.26	2.96	0.51	6.35
0.20	0.20	13	0.53	0.38	-0.69	2.14	-0.59	2.53
0.20	0.25	13	-0.12	0.10	0.44	1.49	-0.15	0.73
0.20	0.30	13	-0.22	0.53	-0.41	1.82	0.17	0.97
0.25	0.20	13	-0.28	0.70	0.11	0.47	-0.28	1.49
0.25	0.25	13	-0.23	0.21	-0.50	1.91	0.41	2.18
0.25	0.30	13	-0.30	0.20	0.02	0.07	-0.14	0.80
0.30	0.20	13	-0.17	0.71	-0.63	3.52	0.06	0.39
0.30	0.25	13	-0.09	0.25	0.09	0.44	0.39	2.46
0.30	0.30	13	-0.22	0.24	0.16	0.74	-0.39	2.50
0.20	0.20	14	***	0.00	1.99	0.69	-0.18	0.19
0.20	0.25	14	***	0.00	-0.74	0.38	0.99	1.31
0.20	0.30	14	-2.31	0.25	0.51	0.53	-0.28	0.57
0.25	0.20	14	1.39	0.21	0.82	0.77	-1.04	1.85
0.25	0.25	14	***	0.00	0.19	0.14	0.00	0.01
0.25	0.30	14	***	0.00	-1.07	0.98	0.15	0.31
0.30	0.20	14	-0.47	0.49	0.50	0.98	0.54	1.54
0.30	0.25	14	-1.15	0.25	-0.07	0.09	0.04	0.12
0.30	0.30	14	***	0.00	0.76	0.93	0.65	1.67
average				0.59		1.60		1.79

Table 6.5: Signed percentage error $\big(\text{SPE } (\%)\big)$ and absolute error $\big(\text{AE } (\times 10^{-3})\big)$ of analytic approximation formula (6.3.34) with $\gamma = \frac{3}{4}$ and $\tau = \frac{1}{12}$ (the missing values indicate both exact and approximation solutions are close to zero, yielding high percentage errors).

6.6 Empirical Test

In this section we examine our new analytic approximation formulae for crack spread call option prices (6.2.8) and (6.3.34) in their ability to capture market prices. In addition, we compare their performance with other well-known formulae. This section is sub-divided into three parts. In the first part we provide a description of the data used in our empirical tests, while in the second part an outline of the methodology used to estimate the parameters is presented. The results of our empirical tests are presented in the final part.

6.6.1 Data Description

The data used in this empirical work consists of daily observations of Heating Oil Crack Call Spread option contracts prices[8] selected from the NYMEX. Heating Oil Crack Call Spread options represent options with a long position in the underlying Heating Oil futures contract, F_{HO}, and a short position in the underlying light "sweet" crude oil futures contract, F_{CL}. We note that the price of a Heating Oil Crack Spread Option is quoted per barrel, and the price of a Heating Oil futures contract is quoted per gallon. Hence, the payoff is given by $\max(42F_{HO}-F_{CL}-K, 0)$. We used 42 groups[9] of call option contracts with $\frac{1}{365} \leq \tau \leq \frac{62}{365}$, where τ represents the time to expiry and is measured in years.

The crack spread prices (i.e. $F_{sp} = 42F_{HO} - F_{CL}$) of the nearest expiring futures contracts between the years of 1994 and 2013 are plotted in Figure 6.1. From this figure, it can be seen that the behaviour of the crack spread prices differed significantly in the period 1994-2004 from that of the period 2004-2013. In the first period (1994-2004), the crack spread prices exhibited constant fluctuations about an apparently stationary mean of approximately US \$3.41 per barrel, indicating a mean-reverting nature; while in the second period (2004-2013) the mean spread prices has mostly steadily increased (apart from the slump in 2008/2009) to reach about \$18 per barrel. Table 6.6 presents the standard statistics for the nearest expiring crude light sweet oil futures, the nearest expiring heating oil futures and

[8]These options are quoted as American style. However, as no dividends are paid the value of the American call options is the same as European call options.

[9]A group of option contracts can be defined as observations of option contracts (observed at a given time t) that have the same type (i.e. call or put), expiration date but with different strike prices.

	F_{HO}	F_{CL}	F_{sp}
average	58.40	48.57	9.84
standard deviation	39.38	31.10	9.90

Table 6.6: Standard statistics for the nearest expiring crude light sweet oil futures (F_{CL}), the nearest expiring heating oil futures (F_{HO}) and the crack spread prices ($F_{sp} = 42F_{HO} - F_{CL}$) ($ per barrel).

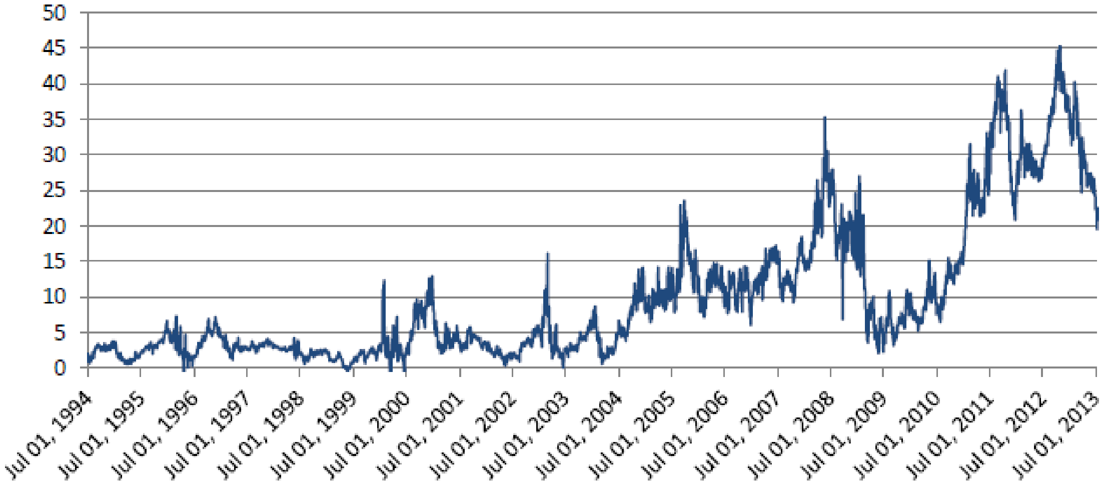


Figure 6.1: The crack spread prices ($F_{sp} = 42F_{HO} - F_{CL}$) between the years of 1994 and 2013 ($ per barrel).

the crack spread prices between these contracts.

6.6.2 Estimation of Parameters

In this part the formulae for crack spread call option prices are calibrated to market prices. For each pricing formula we estimate the volatility parameter(s) as well as the correlation parameter for the explicit models.

To estimate the parameters (for each pricing formula and group) we fit three near at-the-money market prices using the pricing formulae, and we choose to do this as follows:

We define C_{ij} ($\hat{C}_{ij}$) to be the market (estimated) price of the crack spread call option contract with strike price K_i that belongs to the group of option contracts j

($j = 1, ..., 42$). We define e_{ij} to be the error of contract i that belongs to group j with strike price K_i. Hence e_{ij} is the difference between the market price and the estimated price of the call option contract C_{ij} with strike price K_i, i.e.

$$e_{ij} = \hat{C}_{ij} - C_{ij} \ .$$

For each pricing formula we let $\theta(j)$ be the parameter vector for group j. We use three near at-the-money market prices ($i^* = 1, 2, 3$) and minimise the sum of squares of errors (SSE) i.e.

$$\min SSE(\theta(j)) = \sum_{i^*=1}^{3} e^2_{i^*j} \ . \tag{6.6.75}$$

This results in the parameter vector $\theta(j)$ for each pricing formula and for each group j. The estimated parameter vector $\theta(j)$ is used to compute the values of crack spread call options for other strike prices that belong to the group of option contracts j.

6.6.3 Performance of Options Models

The following measures are used to compare errors in the performance of the call options pricing models:

- The total sum of squared errors

$$SSE = \sum_i (\hat{C}_{ij} - C_{ij})^2, \quad j = 1, .., 42 \tag{6.6.76}$$

- The total root mean squared errors

$$RMSE = \sqrt{\frac{1}{N - 42q} \sum_i (\hat{C}_{ij} - C_{ij})^2}, \quad j = 1, .., 42 \tag{6.6.77}$$

- Signed percentage errors

$$SPE = \sum_i \frac{\hat{C}_{ij} - C_{ij}}{C_{ij}} \times 100\%, \quad j = 1, .., 42 \tag{6.6.78}$$

- Unsigned percentage errors

$$USPE = \sum_i |\frac{\hat{C}_{ij} - C_{ij}}{C_{ij}}| \times 100\%, \quad j = 1, .., 42 \qquad (6.6.79)$$

where N is the number of observations and q is the number of the parameters. The results of signed and unsigned percentage errors will be presented in various ranges of moneyness M, defined as $M = \ln(\frac{F_{sp_j}}{K_{ij}})$ where $F_{sp_j} = 42F_{HOj} - F_{CLj}$ is the crack spread price of the group of option contracts j and the K_{ij} are various strike prices of the same group of option contracts j. The average of signed and unsigned percentage errors are calculated for each range of moneyness.

Tables 6.7-6.9 list the results of our analysis. In particular, we note that:

- ***Comparison of SSE (Table 6.7)*** indicates that all univariate pricing models with the exception of (6.4.60) have lower SSE values compared to explicit pricing models. The lowest values of SSE are reached by our new pricing formulae. In particular, the lowest value of SSE (0.5308) is reached by our new analytic approximation formula (6.2.8) with $\gamma = \frac{3}{4}$ followed by the same formula with $\gamma = \frac{3}{5}$ and $\gamma = \frac{1}{2}$ receptively. See Figure 6.2 for a comparison of option prices under (6.2.8) with $\gamma = \frac{3}{4}$ to market data.

- ***Comparison of RMSE values (Table 6.7)*** indicates that all univariate pricing models produce lower RMSE values compared to explicit pricing models. In particular, the lowest values of RMSE are also reached by our new pricing formulae; the lowest value ($0.0470 per contract) reached by our new analytic approximation formula (6.2.8) with $\gamma = \frac{3}{4}$, followed by the same pricing model but with $\gamma = \frac{3}{5}$ and $\gamma = \frac{1}{2}$ respectively.

- ***Comparison of average signed percentage errors (Table 6.8)*** indicates that all univariate pricing models (with the exception of (6.4.60)) underprice in-the-money option contracts compared to market prices, and generally the extent of underpricing decreases as moneyness, M, increases. However, for near at-the-money option contracts all prices under univariate pricing models are above those of market prices. For out-the-money option contracts, all univariate models prices (with the exception of (6.4.60)) are higher than market prices, and all univariate models prices are lower than market prices when options become deep out-of-money. In contrast, all explicit pricing models overprice in-the-money option contracts compared to market prices and generally the extent of overpricing decreases as M increases. However, with (6.4.67) the extent of overpricing increases as M increases. For near at-the-money option contracts all prices under explicit models are slightly over those of market prices. Finally, all explicit pricing models underprice out-the-money options compared to market prices, and the extent of underpricing (with exception of (6.4.67)) increases as M decreases.

- ***Comparison of average unsigned percentage errors (Table 6.9)*** indicates that the lowest averages generally are reached using the Kirk formula (6.4.67). This formula produces the lowest averages when the options are near at-the-money, in-the-money and out-the-money. However, for deep in-the-money and deep out-the-money option contracts the lowest averages are reached by our new analytic approximation formula (6.2.8) with $\gamma = \frac{3}{5}$ and the Schwartz model (6.4.63) respectively.

In summary, firstly we can infer from our empirical analysis that for our given data, that univariate models perform better than explicit models. This is contrary to what we would expect but agrees with the results of Mahringer and Prokopczuk (2010). Further the best performing univariate model is the new CEV model (6.2.8) with $\gamma = \frac{3}{4}$ which outperforms other models in describing the prices of crack spread call option contracts and performs best in pricing in-the-money and deep out-the-money options. Moreover, our empirical results lead to suggestions for practitioners on the use of the new univariate CEV model. Given that the model correctly prices at-the-

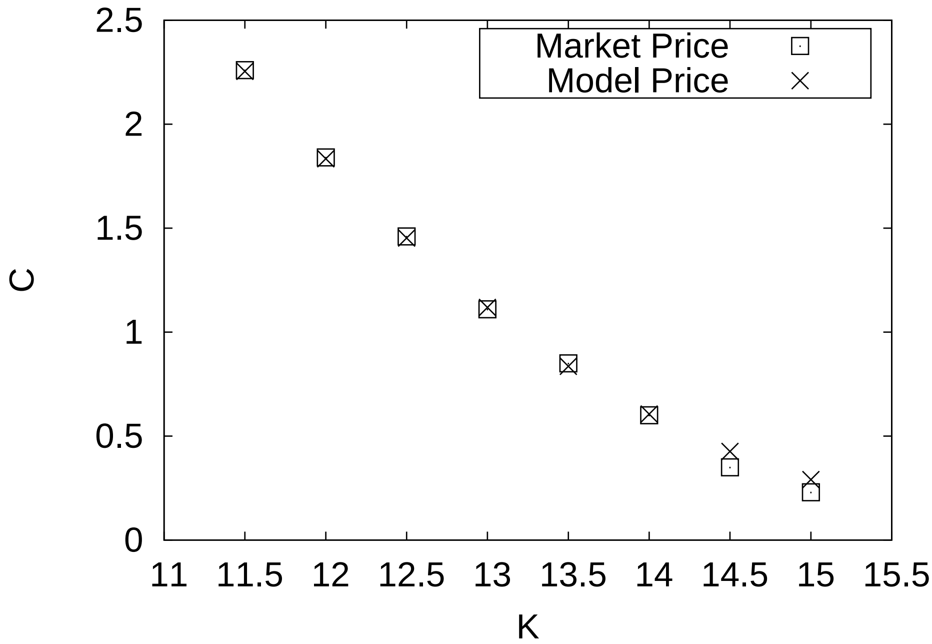


Figure 6.2: Comparison of crack spread call option prices using analytic approximation formula (6.2.8) with $\gamma = \frac{3}{4}$ with market data on 08 Oct 2010.

money option contracts then for out- and near at-the-money call option contracts, options holders (writers) expect to pay (receive) premiums that are slightly lower (higher) than those predicted by the formulae. However, for in-the-money and deep out-the money option contracts holders (writers) expect to pay (receive) premiums that are slightly higher (lower) than those predicted by the formulae.

	Univariate Models						Explicit Models			
	(6.4.60)	(6.4.62)	(6.4.63)	(6.2.8) $\gamma = \frac{1}{2}$	(6.2.8) $\gamma = \frac{3}{5}$	(6.2.8) $\gamma = \frac{3}{4}$	(6.4.67)	(6.4.72)	(6.3.34) $\gamma = \frac{1}{2}$	(6.3.34) $\gamma = \frac{3}{4}$
SSE	0.8922	0.5876	0.5942	0.5644	0.5402	0.5308	0.6781	0.8922	0.6679	0.6571
$RMSE$	0.0610	0.0495	0.0548	0.0485	0.0474	0.0470	0.0659	0.0756	0.0654	0.0649

Table 6.7: SSE and RMSE of Univariate and Explicit Models.

	Univariate Models						Explicit Models			
M	(6.4.60)	(6.4.62)	(6.4.63)	(6.2.8) $\gamma = \frac{1}{2}$	(6.2.8) $\gamma = \frac{3}{5}$	(6.2.8) $\gamma = \frac{3}{4}$	(6.4.67)	(6.4.72)	(6.3.34) $\gamma = \frac{1}{2}$	(6.3.34) $\gamma = \frac{3}{4}$
(-0.20, -0.15)	-22.20	-5.39	-5.34	-13.73	-12.03	-9.50	5.76	-22.20	-24.84	-20.48
(-0.15, -0.05)	-3.75	4.83	4.38	0.58	1.44	2.74	-0.08	-3.75	-5.75	-4.68
(-0.05, 0.05)	0.31	0.60	0.12	0.46	0.48	0.53	0.00	0.31	0.08	0.10
(0.05, 0.15)	1.12	-1.37	-1.38	-0.09	-0.34	-0.72	1.62	1.12	1.44	1.43
(0.15, 0.20)	0.83	-1.09	-1.07	-0.11	-0.30	-0.59	2.08	0.83	0.72	1.29

Table 6.8: Average Signed Percentage Errors of Univariate and Explicit Models.

M	Univariate Models						Explicit Models			
	(6.4.60)	(6.4.62)	(6.4.63)	(6.2.8) $\gamma = \frac{1}{2}$	(6.2.8) $\gamma = \frac{3}{5}$	(6.2.8) $\gamma = \frac{3}{4}$	(6.4.67)	(6.4.72)	(6.3.34) $\gamma = \frac{1}{2}$	(6.3.34) $\gamma = \frac{3}{4}$
(-0.20, -0.15)	22.20	8.32	8.27	13.76	12.52	10.67	12.00	22.20	24.84	20.48
(-0.15, -0.05)	8.86	8.21	8.63	7.57	7.47	7.55	6.02	8.86	8.71	7.57
(-0.05, 0.05)	1.90	2.10	2.48	1.84	1.85	1.91	0.53	1.90	0.99	0.78
(0.05, 0.15)	3.09	2.51	2.52	2.45	2.35	2.36	2.14	3.09	2.38	2.30
(0.15, 0.20)	1.60	1.21	1.20	1.03	0.95	0.98	2.35	1.60	1.15	1.72

Table 6.9: Average Unsigned Percentage Errors of Univariate and Explicit Models.

6.7 Quotient Options

In this section, we take a brief look at an exotic option that is related to spread options in its functionality. The quotient (or ratio) option has a payoff based on the ratio of two underlying assets. Given that the price of the underlying assets at expiry T of the option are given by F_{1T} and F_{2T} respectively, then the value of a ratio call option contract with strike price K is given by $\max\left(\frac{F_{2T}}{F_{1T}} - K, 0\right)$ and by $\max\left(K - \frac{F_{2T}}{F_{1T}}, 0\right)$ for a ratio put option. Quotient options, like spread options, can thus be used to take advantage of the relative performance of two underlying assets. When lognormal underlying models are assumed, the quotient options have the advantage over spread options in having a closed-form solution of the Black-Scholes type. Various solution techniques can be found in the literature to value quotient options. For example Zhang (1998), Zhu (2000) and Buchen (2012) assumed that the changes in both underlying prices follow two correlated GBMs and derived a closed-form formula for pricing European quotient option contracts.

6.7.1 Pricing Quotient Options

The aim of this section is to price European quotient call option contracts on futures. As in Sections 6.2 and 6.3, both univariate and explicit models can be considered for the pricing of quotient call options. When we assume that the change in the risk-neutral futures quotient price follows the CEV process with zero drift

$$dF_q = \sigma {F_q}^{\gamma} d\tilde{Z} \ , \tag{6.7.80}$$

where $F_q = \frac{F_2}{F_1}$, then an exact solution and an approximate solution valid for short times to expiry for a European quotient call option contract can be easily found[10] as in (6.2.7) and (6.2.8) respectively. Alternatively, we can model the dynamics of

[10]By replacing F_{sp} with F_q.

each of the underlying futures prices explicitly. We assume here that each of the risk-neutral underlying futures prices follows a CEV process with zero drift namely

$$dF_i = \sigma_i F_i^{\gamma} d\tilde{Z}_i, \quad \text{where} \quad i = 1,2 \quad \text{and} \quad Corr(d\tilde{Z}_1, d\tilde{Z}_2) = \rho dt \ , \tag{6.7.81}$$

where σ_i, $-1 \leq \rho \leq 1$ and $\gamma \in \mathbb{R}$ are constant. If the underlying prices follow (6.7.81) then the PDE for pricing quotient call options with expiry T and strike price K is given by

$$\frac{\partial C}{\partial t} + \frac{\sigma_1^2 F_1^{2\gamma}}{2}\frac{\partial^2 C}{\partial F_1^2} + \frac{\sigma_2^2 F_2^{2\gamma}}{2}\frac{\partial^2 C}{\partial F_2^2} + \rho\sigma_1\sigma_2 F_1^{\gamma} F_2^{\gamma}\frac{\partial^2 C}{\partial F_1 \partial F_2} - rC = 0 \tag{6.7.82a}$$

subject to the final condition

$$C(F_1, F_2, T) = \max\big(\frac{F_2}{F_1} - K, 0\big) \ . \tag{6.7.82b}$$

We now give an exact solution to (6.7.82a,b) when $\gamma = 1$ and an approximate solution for $\gamma \neq 1$.

Theorem 6.7.1 *Given that the two underlying futures prices F_1 and F_2 follow the risk neutral process (6.7.81), then an exact solution for a European quotient call option contract with expiry T and strike price K is given by*

$$C(F_1, F_2, t) = e^{-r(T-t)}\phi_0(z, w) \tag{6.7.83a}$$

when $\gamma = 1$ and an approximate solution for a European quotient call option contract for $\gamma = 1 - \varepsilon$, $0 < \varepsilon \ll 1$, is given by

$$C(F_1, F_2, t) = e^{-r(T-t)}\big[\phi_0(z, w) + (1-\gamma)\phi_1(z, w)\big] \tag{6.7.83b}$$

where

$$w = (T-t)F_1^{2\gamma-2} \ , \tag{6.7.83c}$$
$$z = \frac{F_2}{F_1} \ , \tag{6.7.83d}$$
$$\phi_0(z,w) = ze^{\theta_2 w} \ N(d_1) - K \ N(d_2) \ , \tag{6.7.83e}$$
$$\phi_1(z,w) = z^n e^{-\frac{\theta_1^2 n^2 w}{2}} \int_0^w \int_{-\infty}^{\infty} \frac{f(\xi,\vartheta)e^{-\frac{(h-\xi)^2}{2\theta_1^2(w-\vartheta)}}}{\theta_1\sqrt{2\pi(w-\vartheta)}} \ d\xi d\vartheta \ , \tag{6.7.83f}$$
$$f(h,w) = e^{-nh+\frac{\theta_1^2 n^2 w}{2}} \Bigg[\theta_3 h(\frac{\partial^2\phi_0(h,w)}{\partial h^2} - \frac{\partial\phi_0(h,w)}{\partial h}) + \theta_4 w \frac{\partial^2\phi_0(h,w)}{\partial h \partial w} + \theta_5 h e^{-h}\frac{\partial\phi_0(h,w)}{\partial h} + \sigma_1^2 \frac{\partial\phi_0(h,w)}{\partial w}\Bigg], \tag{6.7.83g}$$
$$\theta_1 = \sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2} \ , \tag{6.7.83h}$$
$$\theta_2 = \sigma_1(\sigma_1 - \rho\sigma_2) \ , \tag{6.7.83i}$$
$$\theta_3 = \sigma_2(\rho\sigma_1 - \sigma_2) \ , \tag{6.7.83j}$$
$$\theta_4 = 2\theta_2 \ , \tag{6.7.83k}$$
$$\theta_5 = \rho\sigma_1\sigma_2 \ , \tag{6.7.83l}$$
$$d_1 = \frac{\ln(\frac{z}{K}) + (\theta_2 + \frac{\theta_1^2}{2})w}{\theta_1\sqrt{w}} \ , \tag{6.7.83m}$$
$$d_2 = d_1 - \theta_1\sqrt{w} \ , \tag{6.7.83n}$$
$$n = \frac{\theta_1^2 - 2\theta_2}{2\theta_1^2} \ , \tag{6.7.83o}$$
$$h = \ln(z) \ . \tag{6.7.83p}$$

Proof: Letting $\tau = T-t$, equation (6.7.82a) can be written as

$$\frac{\partial C}{\partial \tau} = \frac{\sigma_1^2 F_1^{2\gamma}}{2}\frac{\partial^2 C}{\partial F_1^2} + \frac{\sigma_2^2 F_2^{2\gamma}}{2}\frac{\partial^2 C}{\partial F_2^2} + \rho\sigma_1\sigma_2 F_1^{\gamma}F_2^{\gamma}\frac{\partial^2 C}{\partial F_1 \partial F_2} - rC \tag{6.7.84}$$

subject to the initial condition $C(F_1,F_2,0) = \max\left(\frac{F_2}{F_1} - K, 0\right)$. The computer package DIMSYM (see Sherring (1993)) was used to find the following classical Lie symmetry generator of (6.7.84) which leaves invariant the initial condition:

$$\Psi = 2\tau r(\gamma-1)C\frac{\partial}{\partial C} - 2\tau(\gamma-1)\frac{\partial}{\partial \tau} + F_1\frac{\partial}{\partial F_1} + F_2\frac{\partial}{\partial F_2}\ . \tag{6.7.85}$$

We consider the two cases, $\gamma = 1$ and $\gamma \neq 1$.

For the case when $\gamma = 1$

When $\gamma = 1$ in (6.7.85), the functional form of the invariant solution is $C(F_1, F_2, \tau) = e^{-r\tau}\phi(z, \tau);\ z = \frac{F_2}{F_1}$. Substituting this form into equation (6.7.84) we get that ϕ needs to satisfy

$$\frac{\partial \phi}{\partial \tau} = \frac{\theta_1^2 z^2}{2}\frac{\partial^2 \phi}{\partial z^2} + \theta_2 z\frac{\partial \phi}{\partial z} \tag{6.7.86}$$

$$\text{where } \theta_1 = \sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}\ ,$$

$$\theta_2 = \sigma_1(\sigma_1 - \rho\sigma_2)\ .$$

Equation (6.7.86) needs to be solved subject to the initial condition $\phi(z, 0) = \max(z - K, 0)$, while the boundary conditions are given by

$$\phi(z,\tau) \sim z \quad \text{as} \quad z \to +\infty \quad \text{and} \quad \phi(z,\tau) \to 0 \quad \text{as} \quad z \to 0.$$

Comparing (6.7.86) with the the Black-Scholes PDE (1.14a), we see that (6.7.86) is a special case of (1.14) with $\tau = T - t$, $D = -\theta_2$, $r = 0$ and $\sigma = \theta_1$. Hence, we can write the solution to (6.7.86) as

$$\phi(z,\tau) = ze^{\theta_2\tau}\ N(d_1) - K\ N(d_2)\ , \tag{6.7.87a}$$

$$\text{where}\ \ d_1 = \frac{\ln(\frac{z}{K}) + (\theta_2 + \frac{\theta_1^2}{2})\tau}{\theta_1\sqrt{\tau}}\ , \tag{6.7.87b}$$

$$d_2 = d_1 - \theta_1\sqrt{\tau}\ . \tag{6.7.87c}$$

Hence with $C(F_1, F_2, t) = e^{-r\tau}\phi(z, \tau)$ we get the solution as given in (6.7.83a) with $\gamma = 1$ (this pricing formula agrees with the formula provided by Zhang (1998)).

For the case when $\gamma \neq 1$

Using the method of characteristics to solve the corresponding invariant surface condition (see Section 1.4.1) corresponding to (6.7.85), we have

$$\frac{dF_2}{dF_1} = \frac{F_2}{F_1} \quad \Rightarrow \quad c_1 = \frac{F_2}{F_1} \tag{6.7.88a}$$

$$\frac{dF_1}{d\tau} = -\frac{F_1}{2\tau(\gamma-1)} \quad \Rightarrow \quad c_2 = \tau F_1^{2\gamma-2} \tag{6.7.88b}$$

$$\frac{dC}{d\tau} = -rC \quad \Rightarrow \quad c_3 = e^{r\tau}C \tag{6.7.88c}$$

where c_1, c_2 and c_3 are constants, so that invariants are $\frac{F_2}{F_1}$, $\tau F_1^{2\gamma-2}$ and $e^{r\tau}C$. Hence, the solution can be written as $C(F_1, F_2, \tau) = e^{-r\tau}\phi(z, w)$; $z = \frac{F_2}{F_1}$ and $w = \tau F_1^{2\gamma-2}$. By substituting this form into equation (6.7.84) we get that ϕ needs to satisfy

$$(\sigma_1^2 z^2 + \sigma_2^2 z^{2\gamma} - 2\rho\sigma_1\sigma_2 z^{\gamma+1})\frac{\partial^2\phi}{\partial z^2} + 4\sigma_1^2 w^2(\gamma-1)^2\frac{\partial^2\phi}{\partial w^2} + 2\sigma_1(\sigma_1 z - \rho\sigma_2 z^\gamma)\frac{\partial\phi}{\partial z}$$
$$+ 2\big(\sigma_1^2 w(2\gamma-3)(\gamma-1) - 1\big)\frac{\partial\phi}{\partial w} + 4w\big(\sigma_1^2 z(1-\gamma) + \rho\sigma_1\sigma_2 z^\gamma(\gamma-1)\big)\frac{\partial^2\phi}{\partial z\partial w} = 0 \ . \tag{6.7.89}$$

For $\gamma = 1 - \varepsilon$ where ε is a small parameter $0 < \varepsilon \ll 1$, we assume that the solution (6.7.89) can be written as a series in ε i.e.

$$\phi(z, w) = \sum_{i=0}^{\infty} \varepsilon^i \phi_i(z, w) \ . \tag{6.7.90}$$

The term z^γ in (6.7.89) can be expanded in a series about $\varepsilon = 0$ as

$$z^\gamma = z - \varepsilon \ z\ln(z) + \ O(\varepsilon^2) \tag{6.7.91a}$$

and so

$$z^{1+\gamma} = z^2 - \varepsilon \ z^2\ln(z) + \ O(\varepsilon^2) \tag{6.7.91b}$$

$$z^{2\gamma} = z^2 - \varepsilon \ 2z^2\ln(z) + \ O(\varepsilon^2) \ . \tag{6.7.91c}$$

Substituting (6.7.90) and (6.7.91a-c) into (6.7.89) and collecting terms of $O(1)$, we get an equation for $\phi_0(z,w)$, namely

$$\frac{\partial \phi_0}{\partial w} = \frac{\theta_1^2 z^2}{2}\frac{\partial^2 \phi_0}{\partial z^2} + \theta_2 z \frac{\partial \phi_0}{\partial z} \tag{6.7.92a}$$

$$\text{where} \quad \theta_1 = \sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2} \tag{6.7.92b}$$

$$\theta_2 = \sigma_1(\sigma_1 - \rho\sigma_2) \tag{6.7.92c}$$

which needs to be solved subject to the initial condition $\phi_0(z,0) = \max(z-K,0)$, and boundary conditions

$$\phi_0(z,w) \sim z \quad \text{as} \quad z \to +\infty \quad \text{and} \quad \phi_0(z,w) \to 0 \quad \text{as} \quad z \to 0\ .$$

Hence, $\phi_0(z,w)$ is the exact solution we found for $\phi(z,w)$ when $\gamma = 1$, namely (6.7.87a) i.e.

$$\phi_0(z,w) = ze^{\theta_2 w}\ N(d_1) - K\ N(d_2) \tag{6.7.93a}$$

$$d_1 = \frac{\ln(\frac{z}{K}) + (\theta_2 + \frac{\theta_1^2}{2})w}{\theta_1\sqrt{w}}\ , \tag{6.7.93b}$$

$$d_2 = d_1 - \theta_1\sqrt{w}\ . \tag{6.7.93c}$$

Collecting terms of $O(\varepsilon)$ in (6.7.89), we get an equation for $\phi_1(z,w)$

$$\frac{\partial \phi_1}{\partial w} = \frac{\theta_1^2 z^2}{2}\frac{\partial^2 \phi_1}{\partial z^2} + \theta_2 z \frac{\partial \phi_1}{\partial z} + f_1(z,w) \tag{6.7.94a}$$

where

$$f_1(z,w) = \theta_3 z^2 \ln(z)\frac{\partial^2\phi_0(z,w)}{\partial z^2} + \theta_4 zw\frac{\partial^2\phi_0(z,w)}{\partial z\partial w} + \theta_5\frac{\partial\phi_0(z,w)}{\partial z} + \sigma_1^2\frac{\partial\phi_0(z,w)}{\partial w}\ , \tag{6.7.94b}$$

$$\theta_3 = \sigma_2(\rho\sigma_1 - \sigma_2)\ , \tag{6.7.94c}$$

$$\theta_4 = 2\theta_2\ , \tag{6.7.94d}$$

$$\theta_5 = \rho\sigma_1\sigma_2\ . \tag{6.7.94e}$$

Equation (6.7.94a) needs to be solved subject to the initial condition $\phi_1(z,0) = 0$, and the boundary conditions $\phi_1(z,w) \to 0$ as $z \to \infty$ and $z \to 0$. Now we let $\phi_1(z,\tau) = z^n e^{-\frac{\theta_1^2 n^2 w}{2}} u_1(h,w)$, where $h = \ln(z)$ and $n = \frac{\theta_1^2 - 2\theta_2}{2\theta_1^2}$. Substituting this form into (6.7.94a) yields an equation in $u_1(h,w)$

$$\frac{\partial u_1}{\partial w} = \frac{\theta_1^2}{2}\frac{\partial^2 u_1}{\partial h^2} + f(h,w) \tag{6.7.95a}$$

where

$$f(h,w) = e^{-nh + \frac{\theta_1^2 n^2 w}{2}}\Bigg[\theta_3 h(\frac{\partial^2\phi_0(h,w)}{\partial h^2} - \frac{\partial\phi_0(h,w)}{\partial h}) + \theta_4 w\frac{\partial^2\phi_0(h,w)}{\partial h\partial w} + \theta_5 h e^{-h}\frac{\partial\phi_0(h,w)}{\partial h} + \sigma_1^2\frac{\partial\phi_0(h,w)}{\partial w}\Bigg], \tag{6.7.95b}$$

to be solved subject to the initial condition $u_1(h,0) = 0$, and the boundary conditions $u_1(h,w) \to 0$ as $h \to \pm\infty$. The solution of (6.7.95a) (see e.g. Polyanin (2002)) is given by

$$u_1(h,w) = \int_0^w \int_{-\infty}^{\infty} \frac{f(\xi,\vartheta)\ e^{-\frac{(h-\xi)^2}{2\theta_1^2(w-\vartheta)}}}{\theta_1\sqrt{2\pi(w-\vartheta)}}\ d\xi d\vartheta \tag{6.7.96}$$

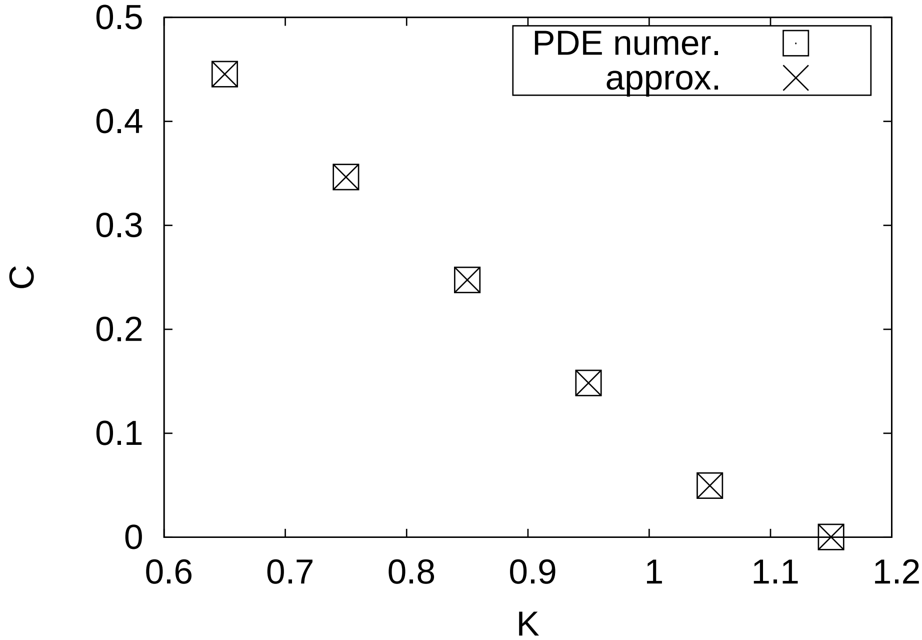
and so $\phi_1(z,w)$ is given by

$$\phi_1(z,w) = z^n e^{-\frac{\theta_1^2 n^2 w}{2}} u_1(\ln(z), w) \ . \tag{6.7.97}$$

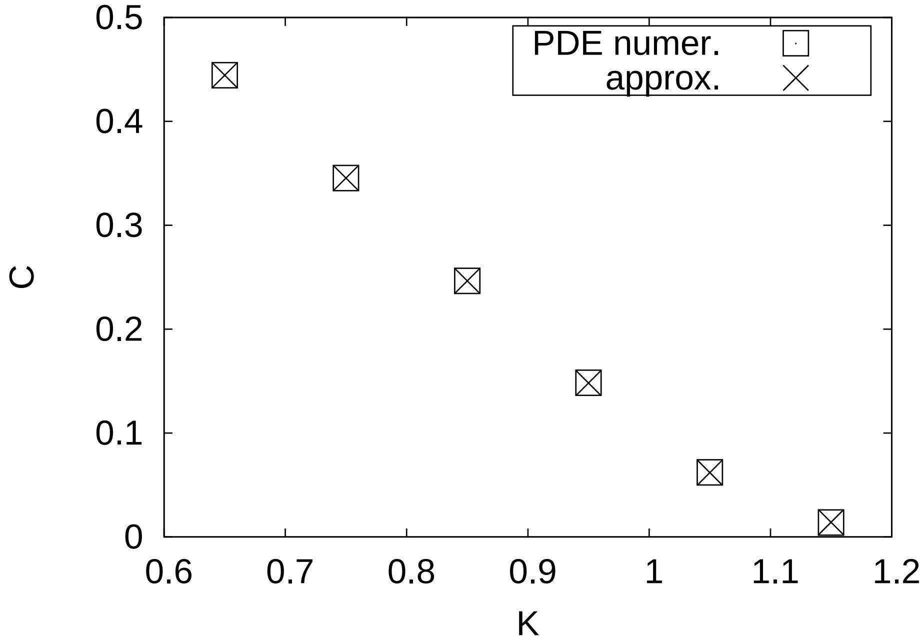
Hence, we get an approximate price for a quotient call option contract to $O(\varepsilon^2)$ as $\phi(z,w) = \phi_0(z,w) + \varepsilon\phi_1(z,w)$ where $\phi_0(z,w)$ and $\phi_1(z,w)$ are given in (6.7.93a) and (6.7.97) respectively and then using our change of variables $C(F_1, F_2, \tau) = e^{-r\tau}\phi(z,w)$ where $\tau = T - t$, $w = (T-t)F_1^{2\gamma-2}$ and $z = \frac{F_2}{F_1}$ to get (6.7.83b). □

Figure 6.3 displays a comparison of quotient call option prices using the approximate solution (6.7.83b) with the numerical solution[11] of (6.7.82a) for two values of γ namely 0.50 and 0.75, and various exercise prices. From this figure with the given parameter values, it can be seen that the approximate solution (6.7.83b) provides an excellent approximation to the numerical solution of (6.7.82a).

[11] The central implicit finite-difference method, with $\Delta z = \Delta\tau = 10^{-4}$, is used to obtain numerical solutions of (6.7.89). Then we use our change of variable (i.e. $C(F_1, F_2, \tau) = e^{-r\tau}\phi(z,w)$, where $z = \frac{F_2}{F_1}$, $w = \tau F_1^{2\gamma-2}$ and $\tau = T - t$) to obtain quotient call option prices.



(a)



(b)

Figure 6.3: Comparison of call option prices prices (C) using the approximate solution (6.7.83b) with the numerical solution of (6.7.82a) for various exercise prices (K) ((a) $\gamma = 0.5$, (b) $\gamma = 0.75$, $r = 0.02$, $F_1 = 100$, $F_2 = 110$, $T - t = 0.5$, $\sigma_1 = 0.5$, $\sigma_2 = 0.65$ and $\rho = 0.85$).

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6.8 Conclusion

In this chapter we proposed univariate and explicit CEV models for futures spread prices, and then derived analytic approximation formulae for European crack spread call option prices with small times to expiry. A sample of market data was used to estimate the power in the diffusion term of our proposed models, and numerical examples showed that our analytic approximation formulae provided an excellent approximation to the exact solution. The analytic approximation formulae were calibrated to market data and compared with calibrations of option prices for other well-known formulae. Our calibration results showed that the analytic approximation solution under the simple univariate CEV model outperforms all the other formulae in its ability for fitting market data. Furthermore, both univariate and explicit CEV models for futures quotient prices were proposed and then analytic and analytic approximation formulae for European quotient call option prices were derived.

Chapter 7

Summary

In this thesis we studied the dynamics of oil prices and priced a variety of financial derivatives on oil. In particular, various one- and two-factor models were proposed for oil price movements and then used to derive a set of analytic and analytic approximation formulae for pricing futures and option contracts on oil. Where possible, the performance of these formulae were empirically examined.

In Chapter 2, we reviewed the most popular one-factor models found in the literature for modelling commodity dynamics and proposed two new models ((2.1.55) and (2.1.57)). Then, the estimation technique GMM (Generalized Method of Moments) was used to compare the performance of various one-factor stochastic models in their ability to capture the behaviour of Brent crude oil prices. The results of the GMM analysis showed that our new proposed models are not only acceptable in describing the behaviour of Brent crude oil prices but perform better than current popular models.

Using our empirical results for oil price models we then focussed on pricing futures contracts on oil. In particular, using our proposed models (which are acceptable in describing the behavior of Brent crude oil prices) we derived simple analytic solutions for futures prices. These new analytic solutions were calibrated to market data and compared with calibrations of futures prices for two of the most well-known formulae (Gabillon(1990) and Schwartz (1997)). Our calibration results show that the derived formula (3.2.25) outperforms all the other models in its ability for fitting market data and forecasting futures prices over the next five days. Our approach to pricing futures contracts was then extended by considering two-factor models. Particularly, net demand, interest rate and convenience yield were in turn considered as additional factors to oil prices. For each model, an analytic formula for futures contract prices was derived. Moreover, an analytic formula for futures contract prices was derived under a regime-switching model, whereby the price of oil was assumed to follow a one-factor stochastic model in each regime.

Following on from the study of pricing futures contracts, we studied the pricing of another important financial derivative, namely a European option contract on oil futures. By considering our proposed one-factor models for oil prices, i.e. (2.1.55) and (2.1.57), we derived analytic solutions for European option prices in integral form. As the solution was in integral form, we then derived simple analytic approximations to the solution of European call option prices valid for short times to expiry. These approximations only involve known functions. All derived solutions were calibrated to market data and compared with calibrations of option prices under the Schwartz (1997) model. Our calibration results show that the analytic approximation solution under our proposed model with the mean-reverting property, namely (5.3.61), outperforms the other models in its ability for fitting market data.

In Chapter 6, we focussed on pricing correlation option contracts. In particular, univariate and explicit CEV (constant elasticity of variance) models were assumed and new analytic approximation formulae were derived for pricing European crack spread and quotient call options. Our proposed formulae for European crack spread options were calibrated to market data and compared with calibrations of option prices for other well-known formulae (Bachelier (1900), Black-Scholes (1973), Schwartz (1997), Kirk (1995) and Carmona and Durrleman (2003)). Our calibration results showed that our purposed formulae, namely (6.2.8) and (6.3.34), outperformed all the other well-known formulae.

In summary, the main empirical results (using the given data) of this thesis, which we hope will be a useful guide to traders and researchers, are reiterated in the following points:

- The results of the GMM analysis show that our new proposed models, which have a three-quarters exponent in the diffusion term, are acceptable in their ability to describe the movement of Brent crude oil prices and perform better than many well-known models.

- The analytic formula for futures prices, namely (3.2.25) under model (2.1.57), outperforms two of the most well-known models (Gabillon(1990) and Schwartz (1997)) in its ability for fitting market data and forecasting futures prices over the next five days.

- The analytic approximation formula for European call option prices, namely (5.3.61), outperforms one of the most well-known models (Schwartz (1997)) in describing market data.

- The univariate (explicit) analytic approximation formula for European crack spread call option prices, namely (6.2.8) and (6.3.34), outperforms other popular univariate (explicit) formulae (Bachelier (1900), Black-Scholes (1973), Schwartz (1997), Kirk (1995) and Carmona and Durrleman (2003)) in capturing market prices.

These results point to several interesting areas for future research. For instance, it would be interesting to measure the performance of our formulae in describing market prices of financial derivatives under other underlying assets or financial instruments (i.e. rather than oil). For example, univariate and explicit constant elasticity of variance models ((6.2.3) and (6.3.32)) with their analytic approximation formulae ((6.2.8) and (6.3.34)) could perhaps be applied to price exotic option contracts on grains.

Appendix A

GMM Code

```
|      1  OPTIONS MEMORY=6;
|      2  READ(FILE='file name') P ;
|      3  GENR LNP=LOG(P);
|      4  dt=1/252;
|      5  GENR P2=S^2;
|      6  GENR P3=S^3;
|      7  GENR P12=P^(0.5);
|      8  GENR PLNP=P*LOG(P);
|      9  LIST LAGXS C,P,PLNP,P2,P12;
|     10  LIST UEQS U1EQ U2EQ;
|     11  FRML U1EQ P(1)-P-(C1+C2*P+C3*PLNP+C4*S2+C5*P12)*dt;
|     12  FRML U2EQ (P(1)-P-(C1+C2*P+C3*PLNP+C4*P2+C5*P12)*dt)^2
|     12            -sigma^2*P^(2*gamma)*dt;
|     13  PARAM  C1  C2  C3  C4  C5  sigma  gamma;
|     14  READ(NROW=5,NCOL=2) SPEC;
|     14  1 1
|     14  1 1
|     14  1 0
|     14  1 0
|     14  1 0
|     14  ;
|     15  GMM(HET,INST=LAGXS,MASK=SPEC)UEQS;
|     16  MAT CM1=@COVOC;
|     17  PARAM C2;
|     18  CONST C1 0 C3 0 C4 0 C5 0 gamma 1;
|     19  GMM(HET,INST=LAGXS,MASK=SPEC,COVOC=CM1)UEQS; ? for model 1
|     20  MAT CM1=@COVOC;
|     21  PARAM C1 C2;
|     22  CONST C3 0 C4 0 C5 0 gamma 0;
|     23  GMM(HET,INST=LAGXS,MASK=SPEC,COVOC=CM1)UEQS; ? for model 2
```

```
MAT CM1=@COVOC;
PARAM C1 C2 ;
CONST C3 0 C4 0 C5 0 gamma 1;
GMM(HET,INST=LAGXS,MASK=SPEC,COVOC=CM1)UEQS; ? for model 3
MAT CM1=@COVOC;
PARAM C2 C3;
CONST C1 0 C4 0 C5 0 gamma 1;
GMM(HET,INST=LAGXS,MASK=SPEC,COVOC=CM1)UEQS; ? for model 4
MAT CM1=@COVOC;
PARAM C2 C4  ;
CONST C1 0 C3 0 C5 0 gamma 0;
GMM(HET,INST=LAGXS,MASK=SPEC,COVOC=CM1)UEQS; ? for model 5
MAT CM1=@COVOC;
PARAM  C2 C4 ;
CONST C1 0 C3 0 C5 0 gamma 1;
GMM(HET,INST=LAGXS,MASK=SPEC,COVOC=CM1)UEQS; ? for model 6
MAT CM1=@COVOC;
PARAM  C2 C4 ;
CONST C1 0 C3 0 C5 0 gamma 1.5;
GMM(HET,INST=LAGXS,MASK=SPEC,COVOC=CM1)UEQS; ? for model 7
MAT CM1=@COVOC;
PARAM  C2;
CONST C1 0 C3 0 C4 0 C5 0 gamma 0.75;
GMM(HET,INST=LAGXS,MASK=SPEC,COVOC=CM1)UEQS; ? for model 8
MAT CM1=@COVOC;
PARAM  C5;
CONST C1 0 C2 C3 0 C4 0 gamma 0.75;
GMM(HET,INST=LAGXS,MASK=SPEC,COVOC=CM1)UEQS; ? for model 9
MAT CM1=@COVOC;
PARAM  C2 C5;
CONST C1 0  C3 0 C4 0  gamma 0.75;
GMM(HET,INST=LAGXS,MASK=SPEC,COVOC=CM1)UEQS; ? for model 10
EXECUTION
```

Appendix B

DIMSYM Codes

B.1 For equation (4.3.20).

```
freeunknown(sigma2,a,g,delta);
loaddeq(u(1,1,1)=1/(sigma2^2*x(1)^3)*(-a*x(1)*(g-x(1))*u(1,1)
+u(1,2)-(x(1)-delta)*u(1)));
mkdets(point);
showdets();
solvedets(std);
showdets();
mkgens();
```

B.2 For equation (6.7.84).

```
freeunknown(sigma1,sigma2,gamma,rho);
loaddeq(u(1,1,1)=1/(sigma1^2*x(1)^(2*gamma))*(r*u(1)+u(1,3)
-rho*sigma1*sigma2*x(1)*x(2)*u(1,1,2)-sigma2^2/2*x(2)^(2*gamma)*u(1,2,2)));
mkdets(point);
showdets();
solvedets(std);
showdets();
mkgens();
```

References

[1] M. Abramowitz and I. Stegun. *Handbook of Mathematical Functions With Formulas, Graphs, and Mathematical Tables.* U.S. Govt. Print. Off, Washington, 1964.

[2] C. Alexander and A. Venkatramanan. Analytic approximations for spread options. *ICMA Centre (University of Reading)*, 2007.

[3] L. Bachelier. Theory of speculation, thesis for the doctorate in mathematical sciences (defended march 29,1900). In P. Cootner, editor, *The Random Character of Stock Market Prices.* MIT Press, Cambridge, 1964.

[4] G. Bakshi, C. Cao, and Z. Chen. Jumps and stochastic volatility: Exchange rate processes implicit in deutsche mark options. *Journal of Finance*, 52:2003–2049, 1997.

[5] G. Barone-Adesi, H. Rasmussen, and C. Ravanelli. An option pricing formula for the garch diffusion model. *Computational Statistics and Data Analysis*, 49:287–310, 2005.

[6] D. Bates. Jumps and stochastic volatility: Exchange rate processes implicit in deutsche mark options. *Review of Financial Studies*, 9:69–107, 1996.

[7] P. Bjerksund. Contingent claims evaluation when the convenience yield is stochastic: analytical results. *Norwegian School of Economics and Business Administration, Working Paper*, 1991.

[8] P. Bjerksund and S. Ekern. Contingent claims evaluation of mean-reverting cash flows in shipping. In TrigeorgisReal L., editor, *Real Options in Capital Investment: Models, Strategies, and Applications.* Praeger, Westport, 1995.

[9] P. Bjerksund and G. Stensland. An American call on the difference of two assets. *International Review of Economics and Finance*, 3:1–26, 1994.

[10] F. Black. The pricing of commodity contracts. *Journal of Financial Economics*, 3:167–179, 1976.

[11] F. Black and M. Scholes. The pricing of options and corporate liabilities. *Journal of Political Economy*, 81:637–654, 1973.

[12] G. Bluman and S. Kumei. *Symmetries and Differential Equations.* Springer-Verlag, New York, 1989.

[13] A. Boness. Elements of a theory of stock-option value. *The Journal of Political Economy*, 72:163–175, 1964.

[14] M. Brennan. The price of convenience and the valuation of commodity contingent claims. In D. Lund and B. Øksendal, editors, *Stochastic Models and Option Values*. Elsevier Science, Amsterdam, 1991.

[15] M. Brennan and E. Schwartz. Finite difference methods and jump processes arising in the pricing of contingent claims: A synthesis. *Journal of Financial and Quantitative Analysis*, 13:461–474, 1978.

[16] M. Brennan and E. Schwartz. Evaluating natural resource investments. *Journal of Business*, 58:135–157, 1985.

[17] British Petroleum. *The Statistical Review of World Energy (June 2013)*. Available at http://www.bp.com/content/dam/bp/pdf/statistical-review/statistical_review_of_world_energy_2013.pdf.

[18] P. Buchen. *An Introduction to Exotic Option Pricing*. Chapman and Hall/CRC, Boca Raton, 2012.

[19] J. Campbell, A. Lo, and A. MacKinlay. *The Econometrics of Financial Markets*. Princeton University Press, New Jersey, 1996.

[20] R. Carmona and V. Durrleman. Pricing and hedging spread options. *SIAM Review*, 45:627–685, 2003.

[21] K. Chan, G. Karolyi, F. Longstaff, and A. Sanders. An empirical comparison of alternative models of the short-term interest rate. *The Journal of Finance*, 47:1029–1227, 1992.

[22] Chicago Mercantile Exchange. *Crack Spread Handbook*, 2013. Available at http://www.cmegroup.com/trading/energy/crack-spread-handbook.html.

[23] G. Cortazar and E. Schwartz. Implementing a stochastic model for oil futures prices. *Energy Economics*, 25:215–238, 1997.

[24] J. Cox, J. Ingersoll, and S. Ross. A theory of the term structure of interest rates. *Econometrica*, 53:385–407, 1985.

[25] M. Dias and K. Rocha. Petroleum concessions with extendible options: timing and value using mean reversion and jump processes for oil prices. *IPEA, Rio de Janeiro, Working Paper*, 1999. Available at http://papers.ssrn.com/sol3/papers.cfm?abstract_id=159692.

[26] A. Dixit and R. Pindyck. *Investment Under Uncertainty*. Princeton University Press, New Jersey, 1994.

[27] J. Duan. Conditionally fat tailed distribution and the volatility smile in options. *Working Paper, University of Toronto*, 1999.

[28] E. Fama and K. French. Commodity futures prices: some evidence on forecast power, premiums, and the theory of storage. *The Journal of Business*, 60:55–73, 1987.

[29] Q. Fu. On the valuation of an option to exchange one interest rate for another. *Journal of Banking and Finance*, 20:645–653, 2003.

[30] J. Gabillon. The term structures of oil futures prices. *Oxford Institute for Energy Studies*, 1991.

[31] R. Gibson and E. Schwartz. Stochastic convenience yield and the pricing of oil contingent claims. *The Journal of Finance*, 45:959–976, 1990.

[32] P. Glasserman. *Monte Carlo Methods in Financial Engineering.* Springer, New York, 2003.

[33] J. Goard. Noninvariant boundary conditions. *Applicable Analysis*, 82:473–481, 2003.

[34] J. Goard. Fundamental solutions to Kolmogorov equations via reduction to canonical form. *Advances in Decision Sciences*, 3:1–24, 2006.

[35] J. Goard. Lie symmetry methods in finance - an example of the bond pricing equation. *Proceedings of the World Congress on Engineering*, 2008.

[36] J. Goard and M. Mazur. Stochastic volatility models and the pricing of vix options. *Mathematical Finance*, 23:439–458, 2013.

[37] A. Grünbichler and F. Longstaff. Valuing futures and options on volatility. *Journal of Banking and Finance*, 20:985–1001, 1996.

[38] A. Gut. *Probability: A Graduate Course.* Springer, New York, 2005.

[39] L. Hansen. Large sample properties of generalized method of moments estimators. *Econometrica*, 50:1029–1054, 1982.

[40] S. Heston. A close-form solution for options with stochastic volatility with applications to bond and currency options. *Review of Financial Studies*, 6:327–343, 1993.

[41] S. Heston. A simple new formula for options with stochastic volatility. *School of Business, Washington University*, 1997.

[42] J. Hilliard and J. Reis. Valuation of commodity futures and options under stochastic convenience yield, interest rates, and jump diffusions in the spot. *Journal of Financial and Quantitative Analysis*, 33:61–86, 1998.

[43] S. Howison. Matched asymptotic expansions in financial engineering. *Journal of Engineering Mathematics*, 53:385–406, 2005.

[44] J. Hull. *Options, Futures and Other Derivatives.* Peasrson Prentice Hall, New Jersey, 2012.

[45] J. Hull and A. White. The pricing of options on assets with stochastic volatilities. *Journal of Finance*, 42:281–300, 1987.

[46] N. Ibragimov. *CRC Handbook of Lie Group Analysis of Differential Equations*. CRC Press, Boca Raton, 1994.

[47] F. Jamshidian and M. Fein. Closed-form solutions for oil futures and european options in the gibson-schwartz model: A note. *Merrill Lynch Capital Markets, Working Paper*, 1990.

[48] R. Jarrow. Convenience yields. *Review of Derivatives Research*, 13:25–43, 2010.

[49] H. Johnson and D. Shanno. Option pricing when the variance is changing. *Journal of Financial and Quantitative Analysis*, 22:143–151, 1987.

[50] E. Kirk. Correlation in the energy markets. In R. Jameson, editor, *Managing Energy Price Risk*. Risk Publication, London, 1995.

[51] O. Korn. Drift matters: An analysis of commodity derivatives. *Journal of Futures Markets*, 25:211–241, 2005.

[52] A. Lewis. *Option Valuation Under Stochastic Volatility*. CA: Finance Press, Newport Beach, 2000.

[53] S. Lie. Über die integration durch bestimmte integrale von einer klasse linear partieller differentialgleic. *Archives of Mathematics*, 6:328–368, 1881.

[54] S. Mahringer and M. Prokopczuk. An empirical model comparison for valuing crack spread options. *ICMA Centre (University of Reading)*, 2010.

[55] Maplesoft. *Maple 12 Users Manual*. Maplesoft, Waterloo, Canada, 2008.

[56] W. Margrabe. The value of option to exchange one asset for another. *Journal of Finance*, 33:177–186, 1978.

[57] R. McDonald and D. Siegel. Investment and the valuation of firms when there is an option to shut down. *International Economic Review*, 26:331–349, 1985.

[58] D. McLeish. *Monte Carlo Simulation and Finance*. Wiley, New Jersey, 2005.

[59] C. Merton. Theory of rational option pricing. *Bell Journal of Economics*, 4:141–183, 1973.

[60] T. Mikosch. *Elementary Stochastic Calculus with Finance in View*. World Scientific, New Jersey, 1998.

[61] K. Miltersen and E. Schwartz. Pricing of options on commodity futures with stochastic term structures of convenience yields and interest rates. *The Journal of Financial and Quantitative Analysis*, 33:33–59, 1998.

[62] K. Morton and D. Mayers. *Numerical Solution of Partial Differential Equations: An Introduction*. Cambridge University Press, Cambridge, 1994.

[63] C. Mullin. Generalized method of moments. 2000. Available at http://www.sscnet.ucla.edu/econ/mullin/lecture_notes/gmm.pdf.

[64] J. Murdock. *Perturbations: Theory and Methods*. Society for Industrial and Applied Mathematics, Philadelphia, 1999.

[65] V. Naik and M. Lee. General equilibrium: Pricing of options on the market portfolio with discontinuous returns. *Review of Financial Studies*, 3:493–512, 1990.

[66] W. Newey and K. West. A simple, positive semi-definite, heteroskedasticity and autocorrelation consistent covariance matrix. *Econometrica*, 55:703–708, 1987.

[67] W. Paulsen. *Asymptotic Analysis and Perturbation Theory*. Taylor & Francis Group, Boca Raton, 2013.

[68] K. Pearson. Method of moments and method of maximum likelihood. *Biometrika*, 28:34–59, 1936.

[69] N. Pearson. An efficient approach for pricing spread option. *Journal of Derivatives*, 3:76–97, 1995.

[70] D. Pilipovic. *Energy Risk: Valuing and Managing Energy Derivatives*. McGraw-Hill, New York, 1997.

[71] G. Poitras. Spread options, exchange options and arithmetic brownian motion. *Journal of Futures Markets*, 18:487–517, 1998.

[72] A. Polyanin. *Handbook of Linear Partial Differential Equations for Engineers and Scientists*. Chapman and Hall/CRC, Boca Raton, 2002.

[73] A. Polyanin and A. Manzhirov. *Handbook of Mathematics for Engineers and Scientists*. Chapman and Hall/CRC, Boca Raton, 2007.

[74] K. Ramaswamy and S. Sundaresan. The valuation of options on futures contracts. *The Journal of Finance*, 40:1319–1340, 1985.

[75] K. Ravindran. Low-fat spreads. *Risk*, 6:56–57, 1993.

[76] D. Ribeiro and S. Hodges. A two-factor model for commodity prices and futures valuation. *Financial Options Research Centre, University of Warwick, Working Paper*, 2004. Available at http://papers.ssrn.com/sol3/papers.cfm?abstract_id=498802.

[77] S. Ross. Hedging long run commitments: Exercises in incomplete market pricing. In C. Culp and M. Miller, editors, *Corporate Hedging in Theory and Practice, Lessons from Metallgesellschaft*. Risk, Oxford, 1995.

[78] S. Sabanis. Stochastic volatility and the mean reverting process. *Journal of Futures Markets*, 23:33–47, 2003.

[79] P. Samuelson. Rational theory of warrant pricing. *Industrial Management Review*, 6:13–31, 1965.

[80] P. Samuelson and R. Merton. A complete model of warrant pricing that maximizes utility. *Industrial Management Review*, 10:17–46, 1969.

[81] E. Schwartz. The valuation of warrants: Implementing a new approach. *Journal of Financial Economics*, 4:79–94, 1977.

[82] E. Schwartz. The stochastic behaviour of commodity prices: implications for valuation and hedging. *The Journal of Finance*, 52:923–973, 1997.

[83] E. Schwartz and J. Smith. Short-term variations and long-term dynamics in commodity prices. *Management Science*, 46:893–911, 2000.

[84] L. Scott. Option pricing when the variance changes randomly: Theory, estimation, and an application. *Journal of Financial and Quantitative Analysis*, 22:419–438, 1987.

[85] L. Scott. Pricing stock options in a jump-diffusion model with stochastic volatility and interest rates: Applications of fourier inversion methods. *Mathematical Finance*, 7:345–358, 1997.

[86] J. Sherring. *DIMSYM Users Manual*. La Trobe University, 1993.

[87] D. Shimko. Options on future spreads: Hedging, speculation and valuation. *The Journal of Future Markets*, 14:183–213, 1994.

[88] E. Shwartz and J. Smith. Short-term variation and long-term dyanamics in commodity prices. *Management Science*, 46:893–911, 2000.

[89] B. Skorodumov. Analytical one-factor pricing model for energy vanilla options. Technical report, Mitsui Co. & and Energy Risk Management LTD., 1991.

[90] Jr. Smith and W. Clifford. Option pricing. a review. *Journal of Financial Economics*, 3:3–51, 1976.

[91] M. Spiegel. *Applied differential equations*. Prentice-Hall, New Jersey, 1967.

[92] C. Sprenkle. Warrant prices as indicators of expectations and preferences. *Yale Economic Essays*, 1:179–231, 1961.

[93] E. Stein and J. Stein. Stock price distributions with stochastic volatility: An analytic approach. *The Review of Financial Studies*, 4:727–752, 1991.

[94] E. Thorp and S. Kassouf. *Beta The Market*. Random House, New York, 1967.

[95] S. Turnbull. A simple approach to interest-rate option pricing. *The Review of Financial Studies*, 4:87–120, 1991.

[96] H. Wang. *Monte Carlo Simulation with Applications to Finance.* CRC Press, Boca Raton, 2012.

[97] J. Wiggins. Option values under stochastic volatility: Theory and empirical estimates. *Journal of Financial Economics*, 19:351–372, 1987.

[98] D. Wilcox. *Energy Futures and Options: Spread Options in Energy Markets.* Goldman Sachs & Co., New York, 1990.

[99] P. Wilmott. *Derivatives: The Theory and Practice of Financial Engineering.* John Wiley & Sons, Chichester, England, 1998.

[100] P. Zhang. *Exotic Options: A Guide to Second Generation Options.* World Scientific, New Jersey, 1998.

[101] J. Zhu. *Modular Pricing of Options: An Application of Fourier Analysis.* Springer, New York, 2000.