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The time dependent net maternity function

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THE TIME DEPENDENT NET MATERNITY FUNCTION

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ABSTRACT

This thesis examines the resultant behaviour of a population in response to changes of the age-specific birth and death rates with time. The deterministic one-sex population model of Sharpe and Lotka is used as the basis for the analysis. In particular, the asymptotic behaviour is determined for a population with a time dependent net maternity function. Thus, the present study may be looked upon as representing a generalisation of stable population theory to include models of time dependent vital rates of birth and death. Laplace transform techniques are used extensively throughout the present work.

The problem of Keyfitz on the momentum of population growth is generalised to contain a gradual exponential scaling (at a rate λ*) of the age-specific birth rate to the level of bare replacement. An algorithm for obtaining the asymptotic total birth rate for general initial net maternity functions is outlined. The method is evaluated by comparing known analytic asymptotic values for two simple initial net maternity functions, to the approximations obtained through the algorithm. The converse problem is also examined: given a prescribed asymptotic population level, it is desired to determine the transition rate* λ*, which characterises the change of the age-specific birth rate. The converse problem is important in the planning and management of populations.*

An extension of the recurrence relation method on which the above algorithm was based, enables a description of the transient behaviour of the population. Short of using a strictly numerical method for solving the integral equation governing the total birth rate, the transient behaviour may also be obtained by a stepping procedure when the age structure of the time dependent net maternity function is

defined in a piecewise fashion.

Models are also proposed which allow for a time dependent change of the initial net maternity function more general than the simple exponential. The asymptotic behaviour of the ensuing population is evaluated.

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DEMOGRAPHIC SYMBOLS

$a(x)$	Stable age density.
$a(x,t)$	Age density.
$A(x)$	Stable age distribution.
$A(x,t)$	Age distribution.
α	Minimum age of childbearing
b	Intrinsic (or Crude) birth rate.
$B(t)$	Total birth rate.
β	Maximum age of childbearing.
$\overset{o}{e}_o$	Expectation of life at birth.
$F(t)$	Total birth rate due to the initial or parent population.
$G(t)$	Total number due to the initial population.
κ	Expected age of childbearing in the stable population.
$\ell(x)$	Survivor function.
$L(x,t)$	Time dependent survivor function.
$m(x)$	Age-specific birth rate.
$M(x,t)$	Time dependent age-specific birth rate.
$\mu(x)$	Age-specific death rate.
$N(t)$	Total number (or simply: numbers) in population.
P	Stable equivalent (numbers).
$\phi(x)$	Net maternity function.
$\Phi(x,t)$	Time dependent net maternity function.
$\psi(t)$	Scaling factor in $\Phi(x,t)$.
Q	Stable equivalent births.
r	Intrinsic rate of change.
R	Net reproduction rate.
$R(t)$	Time dependent net reproduction rate.
t	Time.

T_o	Generation time.
τ_o	Doubling time.
$U(x,t)$	Time dependent age-specific death rate.
ω	Oldest possible age.
x	Age.
$\xi(t)$	Time dependence in $\Phi(x,t)$. $\xi(0) = 1$, $\lim_{t\to\infty} \xi(t) = 0$.
$\zeta(t)$	Time dependence in $\Phi(x,t)$. $\zeta(0) = 0$, $\lim_{t\to\infty} \zeta(t) = 1$.

OTHER NOTATION

$\mathbb{R}_+$	Positive real numbers.
$f \in C^{(n)}(\mathbb{R}_+)$	The function f, defined on $\mathbb{R}_+$, and all its derivatives up to order n, are continuous.
$a \ll b$	a is much less than b.
(Section)[a.b]	a refers to the chapter in which the section, characterised by b, is located.
(Subsection)[a.b.c]	Similar to section except that c characterises the subsection.
(a.b)	Equation b in Chapter a.
1st equation 2nd equation } (a.b)	1st and 2nd equations are referenced as $(a.b)_1$ and $(a.b)_2$ respectively. Similar referencing is made if the equations are on the same line.
$\sim$	Asymptotically approaches.
$\approx$	Approximately equals.
$x \in (a,b]$	$\{x : a < x \leq b\}$.

1. Introduction.

Wide use has been made of the single sex deterministic population model developed by Sharpe and Lotka (1911) in which the population is assumed to be closed to migration and the vital rates of birth and death are assumed to be unchanging with time. Under such conditions the population approaches an asymptotic state known as "stable" (Sharpe & Lotka (1911), Feller (1941) and Lopez (1961)) in which the age distribution is independent of time.

In this work, stable population theory will be extended to allow for time dependent models of changes in the age-specific birth and death rates. These changes constitute a time dependent net maternity function.

Stable population theory has been studied by a number of authors, for example Bourgeois-Pichat (1968), with pioneering work being done by Lotka (1939) and Rhodes (1940). Related concepts of semi-stable and quasi-stable populations have been introduced to describe, respectively, populations whose age distribution is constant, and those whose mortality only is changing with time so that the age distribution is almost constant (Bourgeois-Pichat (1968), (1971) and Coale (1973)).

It was conjectured by Coale (1957) that the same forces that cause the effects of the initial age distribution to be transient for time independent vital rates would also operate if these rates changed with time. These two properties were given the names (following Hajnal (1956), (1958)) of strong and weak ergodicity in Lopez (1961) where he proved Coale's conjecture with the discrete formulation, and later, in Lopez (1967) showing it to be true for the continuous model.

There are basically two formulations of the one-sex population model,

namely the continuous model of Sharpe and Lotka (with which we are concerned) and, the discrete model pioneered by Bernardelli(1941), Lewis (1942), which is analysed in detail in Leslie (1945). In this thesis the main concern lies in the determination of parameters which describe the asymptotic behaviour of the total birth rate, total number and age distribution due to a variety of time dependent changes in the vital rates.

The sensitivity of the intrinsic rate of change r to perturbations of the vital rates was studied by Demetrius (1969). Goodman (1971), Keyfitz (1971a) and (1975) examined the extent to which alterations in the age-specific birth and death rates affect stable population parameters. Compensating changes, between those in the age-specific birth rate to changes in the age-specific death rate, having no effect on r, have been reported in Goodman (1971) and, Espenshade and Chan (1976). A related concept of neutral change in either fertility or mortality in which the stable age distribution is unaffected has been examined in Keyfitz (1968a), Coale (1972) and Preston (1974). Espenshade and Chan (1976) give a wider definition of neutrality as a state in which either the intrinsic rate of change r, or, the stable age distribution, is unaffected through changes of either of the vital rates. A neutral change in the age-specific birth rate is neutral with respect to both.

The asymptotic quantities will be obtained which characterise the eventual stable population resulting from models depicting time dependent changes in the age-specific birth and death rates. Besides determining the asymptotic behaviour, the transient behaviour may also be examined by three methods described in this work, using the basic Sharpe-Lotka model.

Firstly, a strictly numerical solution of the integral equation governing the total birth rate may be used, which is based on a modified block-by-block method of Campbell and Day (1971) (described in Chapter 2). Secondly, a stepping procedure is developed in Chapter 3 where the time dependent net maternity function is defined in a piecewise fashion with respect to age. Finally, the transient behaviour may be obtained by an extension of a method presented in Cerone and Keane (1978a) in which the time dependence is exponential. This extension is given in Section [4.4].

Keyfitz (1971b) determined the asymptotic results due to an abrupt constant scaling of the age-specific birth rate to replacement level. Assuming the population to be initially stable, he obtained elegant expressions for the asymptotic total birth rate and the asymptotic total number in terms of entities pertaining to the initial population. Keyfitz was able to demonstrate what he termed the momentum of population growth, in that even though the age-specific birth rate is abruptly scaled down to replacement level, the asymptotic total number is greater than the initial.

Frejka (1968) first called attention to the phenomenon of residual growth, however, as in a later study (Frejka (1973)) his analysis was based on projecting populations under different linear paths of change, to replacement level fertility. The growth potential of a particular age distribution has been studied by a number of authors (Vincent (1945), Keyfitz (1969), Bourgeois-Pichat (1968) and (1971), Preston (1970), and, Espenshade and Campbell (1977)). Keyfitz (1969) called attention to the stable equivalent population which when compared with the observed age distribution would demonstrate the potential for growth. The stable equivalent which is closely related to R.A. Fisher's "reproductive

value of a woman" (Fisher (1930)), would give the level to which a population would tend if the total numbers were discounted at the intrinsic rate of change. Abrupt changes in the age distribution at particular ages representing migration, were analysed by Keyfitz (1971c). Instantaneous changes of the age distribution representing a catastrophe were examined by both Le Bras (1969) and Tognetti (1976a). Preston (1970) demonstrated the major role played by age composition towards growth of a population. Keyfitz (1971b) and (1971d) stated that the age distribution which is more favourable to population increase, affects the momentum to a greater extent. The asymptotic total birth rate resulting from an abrupt Keyfitz change to replacement level, is smaller than the initial, and it is the age redistribution to stationary levels which more than compensates, resulting in a tendency for continued growth of the population.

Since the pioneer article of Keyfitz (1971b) a number of generalisations and extensions have appeared in the literature. Frauenthal (1975), Tognetti (1976b) and Mitra (1976) present models which consist of abrupt changes of the age-specific birth rate, while Cerone and Keane (1978a), and (1978b) give gradual models.

Frauenthal (1975) obtained the asymptotic stationary total birth rate and numbers by considering a model which was also mentioned in Keyfitz (1971b). Frauenthal allowed the parent population to continue with the old regime while those born after the origin $t = 0$, adopted replacement age-specific birth rate brought about by scaling that of the initial population by the net reproductive rate, R. Such a model results in the asymptotic total birth rate and hence numbers, R times those of Keyfitz (1971b). Frauenthal obtained approximations to these asymptotic values in terms of R, by assuming that all births occur at

the mean age of childbearing, thus showing that continued asymptotic growth is largely due to age redistribution. Frauenthal's model is discontinuous at the net maternity function level like the Keyfitz (1971b) model. However, unlike the Keyfitz solution, the total birth rate, the total number, and the time dependent net reproduction rate are continuous.

Keyfitz (1975) scaled the age-specific birth rate so that a population would eventuate with smaller intrinsic rate of change. Tognetti (1976b) scaled the age-specific birth rate by a general constant, thus resulting in an eventual stable population.

A model consisting of a gradual exponential scaling, at a rate λ to replacement level age-specific birth rate, was presented in Cerone and Keane (1978a) and an algorithm was developed for obtaining the asymptotic total birth rate. The analysis is given here in Chapter 4. The algorithm is tested against known analytic solutions developed in Chapter 3 with two simple initial net maternity functions. More realistic initial net maternity functions are also used and residual growth occurs which is greater than that of Keyfitz (1971b) since the transition to replacement is now monotonically decreasing in a gradual fashion. Simple extensions of the method on which the algorithm was based, provide for exponential time dependent generalisations of the Frauenthal (1975) and Tognetti (1976b) problems, which are given in Chapter 4. The all important converse problem of determining the transition rate λ (characterising the variation with time of the age-specific birth rate), which is needed when given a desired asymptotic goal, is also treated in Chapter 4. The solution of such a problem is not always possible since some goals may be unachievable using only the proposed change.

The above models which provide a scaling with time (whether abrupt or gradual) of the initial age-specific birth rate, will be known as separable since all age-groups are affected in a proportionate manner. Such models have also appeared in Coale (1956), (1970) and, Keyfitz (1969), amongst others. A number of authors (Ryder (1975) and Potter, Wolowyna and Kulkarni (1977)) have emphasised the need for models in which the initial age structure of the age-specific birth rate is allowed to vary. Even Keyfitz (1971b) states that a fall is likely to be greater for older women than for younger and later reiterates in Keyfitz (1975) that a disproportionate change in the age-specific birth rate is more realistic. Mitra (1976) gives a model in which the initial age-specific birth rate is abruptly altered to change exponentially with age giving replacement. Mitra in concluding, states that the abrupt change can be from any initial age-specific birth rate to any other which causes replacement and obtains an expression for the asymptotic total number. Thus Mitra allows for a non-separable or disproportionate change of the initial age-specific birth rate.

Cerone and Keane (1978b) developed a model in which the time dependent net maternity function changed exponentially from the initial net maternity function towards any arbitrary function. Thus the model is non-separable. The eventual stable equivalent births is obtained using the methods of Cerone and Keane (1978a) and hence the asymptotic behaviour of the total number and age distribution is evaluated.

When the eventual net maternity function is a scalar multiple of the initial, then the separable models are a special case of the non-separable. Both types of models are treated in this thesis since they each have their advantages and their disadvantages. The separable models, although less realistic, are more amenable to analytic investigation.

For example, the error analysis used in the algorithm developed in Cerone and Keane (1978a) can not in general be used for the non-separable model of Cerone and Keane (1978b). The non-separable model has its disadvantage in its generality - the eventual age structure of the net maternity function needs to be specified in advance.

Frejka (1973) discussed the problem of an increase followed by a decrease to replacement level. Keyfitz (1975) maintained that it may become necessary to hold a population at its initial numbers and thus a decrease of the age-specific birth rate at first well below replacement is essential - all other factors being equal. Ruzicka (1977) saw a shift to replacement level fertility being like "an inverted logistic curve" - gradual at first, then rapid, and ultimately gradual again. These articles and a natural extension of the work that has preceded provide the impetus and need for improved time variation of the net maternity function. This is done in Chapter 5.

Using extensions of the method presented in Cerone and Keane (1978a) and developed in Chapter 4, generalised models are given for the time path of change of the initial age-specific birth rate and the asymptotic behaviour is determined. These models allow, for example, a more gradual transition than exponential, towards a set goal, and thus may represent an initial reluctance of a population to change by adopting new policies aimed at altering its present age-specific birth rate. The time path of change depends to a large extent on whether the change is a voluntary response to stimuli generated by a government family planning programme or whether it is a direct consequence of a planned and perhaps forced nature. A model is given in Section [5.2] which allows for different age-groups, under diverse time paths, to tend towards a prescribed goal, thus allowing differential effects of certain policies by age.

For realistic models, the age-specific birth rate and hence the net maternity function, is non-zero over a finite interval. Then if we allow the age-specific birth rate to change explicitly with time only for $0 < t < \tau \leq \alpha$ (α, the minimum age of childbearing), the convolution integral is unaffected and we may determine the eventual stationary population. In fact, if the net maternity function changes explicitly with time only for the parent population then the convolution integral for the total birth rate is not disrupted and stable population theory without time dependent vital rates, is still applicable. The present work also allows for continuation of time variation beyond α, when the time dependence affecting the convolution integral is in terms of exponentials. Although, for $t > \alpha$ only exponential (or a combination of exponentials) paths can be handled by the methods of the present work, a great variety of paths covering a wide range of possibilities, can be obtained. A piecewise defined net maternity function is given in Section [5.5] taking these points into consideration.

Throughout the literature, changes in the existing population are assumed at various levels or tiers. Rhodes (1940) assumed the total birth rate and total number to be given by various expressions. Ryder (1975) assumed the rate of change of a population to vary in a linear manner over a period of 40 years, without explicitly specifying the change at the net maternity function level. Potter et al. (1977) went to the other extreme and analysed the amount of residual growth caused by a set of sterilisation policies (within marriage) resulting in an eventual stationary population. The age-specific birth rate was represented in terms of expressions for the proportion currently married, together with the marital fertility models developed by Coale and co-workers (Coale (1971), Coale and Lesthaeghe (1971), Coale and Trussell (1974), and, Coale, Hill and Trussell (1975)) covering the full

range of human experience. Potter et al. considered changes in the age-specific birth rate caused by changes to the marital fertility of the initial population. In the present work all changes are made at the net maternity function level. Hence the impact of changes of the initial age-specific birth rate and of the initial survivor function are considered. All changes of the age-specific death rate are made through the survivor function.

A time dependent survivor function is given in Chapter 6 which changes from an initial to an eventual survival behaviour. The methods developed in Chapter 4 and Chapter 5 are also used. However, some of the models (the separable models in particular) used for the time dependent age-specific birth rate cannot be used for the time dependent survivor function. It should be noted that the models of the time dependent survivor function presented here cause the age-specific death rate to change gradually with time.

With the aim of obtaining a stationary population, it should be noted that the asymptotic values obtained in Chapters 4 and 5, in which the survivor function is assumed not to change, are under-estimates with increased healthcare, which reduces mortality.

The tables of Demographic Symbols and Other Notation are provided following the Table of Contents. The nomenclature differs widely throughout the literature but the terminology of the Table will be used here except perhaps in Chapter 3 where analytic net maternity functions are discussed.

Offprints of published papers are given in Appendix B as supporting evidence.

2. Mathematical Models and Numerical Methods.

The basic Sharpe-Lotka one-sex deterministic population model is reviewed using Laplace transform techniques, the vital rates of birth and death being assumed independent of time. The models for the time dependent net maternity function are presented and a numerical method (the modified block-by-block method of Campbell and Day (1971)) to solve the integral equation for the total birth rate is introduced.

By presenting preliminary information, definitions and equations this chapter sets the foundation upon which the work in the later chapters can proceed.

2.1 The Deterministic One-Sex Population Model of Sharpe and Lotka.

In order to extend the deterministic one-sex model of Sharpe and Lotka to include time dependent vital rates, the original formulation of Sharpe and Lotka (1911) will be presented.

The method of solution for the time independent vital rates will differ from that of the pioneers, and, to some extent from current users of the model, in that formal Laplace transform techniques will be used to solve the linear Volterra integral equation of the second kind with a difference kernel, for the total birth rate.

The rigorous exposé of Feller (1941), to some extent a direct consequence of the controversy surrounding the initial complex exponential series solution of Sharpe and Lotka (1911), will not be examined in too much detail here. It suffices to present an outline of the method of Lopez (1961) showing that Lotka's solution is valid if, as always happens in a demographic context, fertility rates are continuous over a finite span of the female life time. Lopez (1961) proved the validity of the complex exponential series solution, for the total birth rate, by using Laplace transform methods, with contour integration and residue theory being used for the inversion process. However, care must be taken when using models of graduation of the net maternity function, such as the normal curve, the incomplete gamma function and the Malthusian function, each often used by demographers (see Keyfitz (1968b) and Pollard (1973)), and which do not satisfy the postulates of Lopez (1961).

2.1.1 Development of the Sharpe-Lotka Model.

The main assumptions of the deterministic Sharpe-Lotka model are:

(i) The population is isolated, namely, it is closed to migration.

(ii) The study can be applied to either sex. We shall apply the one-sex model to the female sex since females have a shorter and better defined reproductive life-span. Sharpe and Lotka (1911) used the male sex, but, Demographers today usually apply the model to the female sex.

(iii) The vital rates, that is, the birth and death rates, are age-specific and independent of time (see definition below).

In order to proceed with the formulation of the Sharpe-Lotka model we need some definitions.

Definitions.

Survivor Function $(\ell(x))$.

$\ell(x)$ is the fraction of newborn females that will survive to age x.

Age-Specific Birth Rate $(m(x))$.

$m(x)dx$ is the probability that a woman of age x will give birth to a female child between ages x to $x+dx$.

Age-Specific Death Rate $(\mu(x))$.

$\mu(x)dx$ is the probability of a woman who has survived x years of age dying in the interval x to $x+dx$.

$\ell(x)$ is related to $\mu(x)$ (see for example Keyfitz (1977)) by

$$\ell(x) = \exp\left[-\int_0^x \mu(u)du\right] .$$

Net Maternity Function $(\phi(x))$.

$\phi(x) = m(x)\ell(x)$. This is also known as the net fertility

schedule (Coale (1972)) and the maternity function (Lopez (1961)).

Vital Rates $(m(x)\&\mu(x))$.

The age-specific birth and death rates are known collectively as vital rates.

It should be noted that the condition that a person will die implies

$$\left(-\int_0^\infty \ell'(x)dx =\right) \int_0^\infty \mu(x)\ell(x)dx = 1 \quad . \tag{2.1}$$

Now, if $B(t)$ is the total birth rate at time t due to all mothers, then the birth rate of mothers alive at time t, of age x to $x+dx$, is

$$B(t-x)\ell(x)m(x)dx ,$$

and so integrating (summing) over all ages,

$$B(t) = \int_0^\infty B(t-x)\phi(x)dx . \tag{2.2}$$

It can be seen that (2.2) may be written in the form of a Volterra integral equation of the second kind with a difference kernel, namely

$$\left.\begin{aligned} B(t) &= F(t) + \int_0^t B(t-x)\phi(x)dx \\ \text{where,} \\ F(t) &= \int_t^\infty B(t-x)\phi(x)dx = \int_0^\infty B(-x)\phi(x+t)dx , \end{aligned}\right\} \tag{2.3}$$

is the birth rate at time t due to the females already alive at the origin, that is, due to the parent or initial population.

An expression for $B(-x)$, in terms of known entities at the origin, will be developed subsequently.

Let $N(t)$ be the total number in the population at time t and $a(x,t)$ be the age density at time t then,

$N(t)a(x,t)dx$, is the number of females of age x to $x+dx$ at time t . Females alive at time t , of age x to $x+dx$, must have been born $t-x$ to $t-(x+dx)$ and have survived a period of at least x years. Thus,

$$N(t)a(x,t)dx = B(t-x)\ell(x)dx \ . \tag{2.4}$$

Consequently, using the fact that $a(x,t)$ is a density function with respect to x , and integrating over all ages we obtain

$$N(t) = \int_0^\infty B(t-x)\ell(x)dx \ . \tag{2.5}$$

Also, from (2.4), we have

$$N(t)a(x,t) = B(t-x)\ell(x) \ , \tag{2.6}$$

from which we obtain the total births at the origin $B(-x)$ to be given by

$$B(-x) = N(0)a(x,0)/\ell(x) \ ,$$

and hence, from $(2.3)_2$,

$$F(t) = N(0) \int_0^\infty \frac{a(x,0)}{\ell(x)} \phi(x+t)dx \ . \tag{2.7}$$

Now, (2.6) must be modified to account for the lack of knowledge about the population prior to our chosen origin. Equation (2.6) only holds for $t \geq x$, but, for $t \leq x$ the number of mothers in the age group x to $x+dx$ is

$$N(t)a(x,t)dx = N(0) \frac{a(x-t,0)}{\ell(x-t)} \ell(x)dx \ ,$$

which can most easily be obtained from the use of (2.6) or by consulting a Lexis diagram as used, for example, by Keyfitz (1968b).

Hence if $H(x)$ is the Heaviside unit function defined by

$$H(x) = \begin{cases} 1 \ , & x > 0 \\ 0 \ , & \text{otherwise} \ , \end{cases}$$

then

$$A(x,t) = N(t)a(x,t) = N(0)\frac{a(x-t,0)}{\ell(x-t)}\ell(x)H(x-t)+B(t-x)\ell(x)H(t-x). \tag{2.8}$$

Integrating (2.8) with respect to x, using the fact that $a(x,t)$ is a density, gives the total number in the population at time t,

$$\left.\begin{aligned} N(t) &= G(t) + \int_0^t B(t-x)\ell(x)dx , \\ \text{where,} & \\ G(t) &= N(0)\int_0^\infty \frac{a(x,0)}{\ell(x)}\ell(x+t)dx , \end{aligned}\right\} \tag{2.9}$$

is the number of female children due to those mothers that were alive at the origin.

F and G from $(2.3)_1$ and $(2.9)_1$, are known as the forcing functions of the integral equation.

Thus, if we can solve for the total birth rate $B(t)$, as given by the Volterra integral equation $(2.3)_1$ with (2.7), given the age distribution at the origin and the age-specific vital rates, then, we can (theoretically at least) calculate the total number in the population, $N(t)$, and the age distribution, $A(x,t)$, from (2.6) [or, (2.9)] and (2.8) respectively.

2.1.2 The Transient and Asymptotic Solution of the Sharpe-Lotka Model.

The solution for the total birth rate $B(t)$ has been obtained from (2.2) by Sharpe and Lotka (1911), Lotka (1939), Rhodes (1940) and others. Once the solution is known for $t > 0$ a matching process with some initial arbitrary function for $t < 0$ is necessary. In spite of the warning given by, for example, Feller (1941) and Lopez (1961), current literature is still treating (2.2) rather than $(2.3)_1$.

We shall restrict our attention to the solution of $(2.3)_1$ where, as in the general theory of self-renewing aggregates, both ϕ and F are non-negative functions. The Volterra integral equation $(2.3)_1$ has been extensively treated in the literature, for example Hochstadt (1973) and, Bellman and Cooke (1963), in which the existance and and uniqueness of the solution are guaranteed. Also, characteristics of the solution of $(2.3)_1$ depending on properties of both ϕ and F are amply treated in the literature.

Now, $(2.3)_1$ is a linear Volterra integral equation of the second kind with a difference kernel, and thus, is amenable to Laplace transform techniques. The methodology of solving $(2.3)_1$ using Laplace transforms will now be demonstrated.

Formally taking the Laplace transform of $(2.3)_1$ we obtain, upon using the convolution theorem,

$$B^*(p) = F^*(p) + \phi^*(p)B^*(p)$$

or
$$B^*(p) = \frac{F^*(p)}{1-\phi^*(p)} \quad , \tag{2.10}$$

where $*$ denotes the one-sided Laplace transform viz.

$$u^*(p) = \int_0^\infty e^{-pt}u(t)dt \quad ,$$

and the complex variable p is chosen in such a way as to ensure convergence of the integrals. This can be done under very general conditions on the functions since if ϕ and F are of exponential order, then so is B (Bellman and Cooke (1963)).

Feller (1941) states that, in order that the solution to $(2.3)_1$ can be represented in the standard form

$$B(t) = \sum_{p_j} Q_j e^{p_j t} \quad , \quad t > 0 \ , \tag{2.11}$$

where convergence is absolute for $t \geq 0$,

and the sum is over all p_j the roots of the characteristic equation

$$\phi^*(p) = 1 \ , \tag{2.12}$$

it is necessary and sufficient that $B^*(p)$, of (2.10), admit an expansion of the form

$$B^*(p) = \sum_{p_j} \frac{Q_j}{p-p_j} \ , \tag{2.13}$$

and that $\sum_{p_j} |Q_j|$ converges absolutely. The coefficients Q_j are given by

$$Q_j = \frac{F^*(p_j)}{-\left[\frac{d}{dp}\phi^*(p)\right]_{p=p_j}} \ . \tag{2.14}$$

Also Feller (1941) states that it is necessary that $B^*(p)$ be a one-valued function.

As noted by both Doetsch (1950) and Lopez (1961), Feller's (1941) condition requiring a partial fraction decomposition for the solution to be written in the form (2.11), is difficult to apply since even though (2.10) and (2.13) have the same singularities, they may differ by a non-zero integer (or entire) function.

The method of solving for $B(t)$ using contour integration and residue theory will now be outlined following Doetsch (1950). However, we will firstly discuss the location of the roots of (2.12) since their influence on the solution is paramount.

When ϕ is a non-negative function, it can be established (Bellman and Cooke (1963) or Pollard (1973)) that there exists a unique real root r , of the characteristic equation (2.12), such that $Re(p_j) < r$. That is, the real root r has the greatest real part with

$r \left\{ \begin{matrix} < \\ = \\ > \end{matrix} \right\} 0$ depending on whether the net reproduction rate $R \left\{ \begin{matrix} < \\ = \\ > \end{matrix} \right\} 1$ where,

$$R = \phi^*(0) = \int_0^\infty \phi(x)dx \quad . \tag{2.15}$$

Further, the complex roots of (2.12) appear in conjugate pairs (see Pollard (1973)) which is to be expected since the solution we are seeking is real.

Now, the inversion of the Laplace transform (2.10) involves, as outlined by Doetsch (1950), constructing a sequence of simple closed contours Γ_n uniformly bounded away from the roots of (2.12). We note that the Γ_n can be taken as contours joining the points $\gamma \pm i\tau_n$ only to the left of $Re(p) = \gamma$ since $B^*(p)$ is an analytic function for $Re(p) > \gamma > r$ and so by Cauchy's residue theorem (see Levinson and Redheffer (1970)) we obtain no contribution from a contour enclosing the region of the plane $Re(p) > \gamma$. Now, by traversing the remaining enclosed region in an anti-clockwise direction we have, using Cauchy's residue theorem

$$\begin{aligned} &\frac{1}{2\pi i} \int_{\gamma - i\tau_n}^{\gamma + i\tau_n} e^{pt} \frac{F^*(p)}{1-\phi^*(p)} \, dp + \frac{1}{2\pi i} \int_{\Gamma_n} e^{pt} \frac{F^*(p)}{1-\phi^*(p)} \, dp \\ &\qquad = \sum_{p_j} Res_j \left\{ e^{pt} \frac{F^*(p)}{1-\phi^*(p)} \right\} \\ &\qquad = \sum_{p_j} e^{p_j t} \, Res_j \left\{ \frac{F^*(p)}{1-\phi^*(p)} \right\} , \end{aligned} \tag{2.16}$$

where the sum is over the residue Res_j from the poles contained within some simple closed contour. In (2.16) different subscripts, n and j, are used to denote the fact that we may have a different number of poles within a particular contour.

Lopez (1961) has shown, using rectangular contours that, for ϕ continuous and of compact support (that is, non-zero over a finite interval), the contribution from integrating around the contour Γ_n tends uniformly to zero and hence by the inversion theorem of Laplace transforms (see Bellman and Cooke (1963)) he obtains, from (2.16) and using (2.10)

$$B(t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{pt}B^*(p)dp$$

$$= \sum_{p_j} Q_j e^{p_j t} , \quad t > 0 ,$$

where, assuming that the poles are simple,

$$Q_j = Res_j\left\{\frac{F^*(p)}{1-\phi^*(p)}\right\} = \lim_{p\to p_j} (p-p_j)\,\frac{F^*(p)}{1-\phi^*(p)} . \tag{2.17}$$

Consequently

$$Q_j = \frac{F^*(p_j)}{\kappa_j} , \tag{2.18}$$

where

$$\kappa_j = -\left[\frac{d}{dp}\,\phi^*(p)\right]_{p=p_j} = \int_0^\infty e^{-p_j x} x\phi(x)dx , \tag{2.19}$$

$$F^*(p_j) = N(0)\int_0^\infty a(x,0)v(p_j,x)dx , \tag{2.20}$$

and $$v(p_j,x) = \frac{e^{p_j x}}{\ell(x)}\int_x^\infty e^{-p_j u}\phi(u)du . \tag{2.21}$$

Here we have assumed the poles of the meromorphic function (2.10), at least those in any finite region, are the roots p_j of (2.12). Lopez (1961) further states that, for ϕ continuous and of compact support, (2.10) is a quotient of two non-rational entire functions so that (2.12) will have a countably infinite number of roots with $-\infty$ being an essential singular point.

Of particular interest in renewal theory, and of any process in general, is the asymptotic behaviour of the solution. There are classes of Tauberian theorems in general, and, as applied to Laplace transforms in particular, (Bellman and Cooke (1963) or Widder (1941)) which determine the behaviour of the original problem knowing that of the transform. As stated earlier, the real root r, of (2.12), is the root with greatest real part, and we may write

$$B(t) = Qe^{rt} + O(e^{rt}) , \tag{2.22}$$

where $O(e^{rt})$ represents terms which approach zero as $t \to \infty$ so that

$$\lim_{t\to\infty} e^{-rt}B(t) = Q . \tag{2.23}$$

Q represents the stable equivalent births and is given by (2.18) with $p_j = r$. Equations (2.19) and (2.21), with $p_j = r$, represent the mean age of childbearing in the stable population and Fisher's reproductive value of a woman, respectively.

Using (2.22) and (2.23) in (2.5) and (2.6) we obtain the asymptotic behaviour of the numbers and the age distribution given by

$$N(t) \sim Pe^{rt} ,$$

where

$$P = Q\int_0^\infty e^{-rx}\ell(x)dx , \tag{2.24}$$

and,

$$a(x,t) \sim \frac{Q}{P} e^{-rx}\ell(x) = be^{-rx}\ell(x) , \tag{2.25}$$

where $b = \dfrac{1}{\int_0^\infty e^{-rx}\ell(x)dx}$ is the intrinsic birth rate and " $\sim$ " denotes "asymptotically approaches". We have further assumed, in deriving (2.24) and (2.25), that $O(e^{rt})$ terms tend uniformly to zero. Hence we have that a closed one-sex population, subject to unchanging fertility amd mortality schedules, asymptotically attains a fixed age composition, (2.25), and a constant rate of change r. This fixed state towards which a population tends, is known as <u>stable</u>.

2.2 The Model With Time Dependent Vital Rates.

If the age-specific vital rates of birth and death, presented in Subsection [2.1.1], are allowed to change with time, then, it is possible to proceed in a similar manner and obtain expressions for the total birth rate, the numbers in the population and the age distribution.

We let $\Phi(x,t)$ denote the time dependent net maternity function which is given by

$$\Phi(x,t) = M(x,t)L(x,t) \quad , \tag{2.26}$$

where $M(x,t)$ is the time dependent age-specific birth rate and $L(x,t)$ is the time dependent survivor function, representing the probability that a female born at time $t-x$ survives to age x. $L(x,t)$ is related to $U(x,t)$, the time dependent age-specific death rate, by (see Hoppensteadt (1975))

$$L(x,t) = \begin{cases} \exp\left[-\int_0^t U(x-t+u,u)du\right] & , \quad t \leq x \\ \exp\left[-\int_0^x U(u,t-x+u)du\right] & , \quad t > x \ . \end{cases} \tag{2.27}$$

The time dependent net reproduction rate $R(t)$, is given by

$$R(t) = \int_0^\infty \Phi(x,t)dx \ . \tag{2.28}$$

With $\Phi(x,t)$ in place of $\phi(x)$ in the discussion of the previous section it may be seen that the total birth rate $B(t)$ is given by the generalised renewal equation

$$B(t) = N(0)\int_0^\infty \frac{a(x,0)}{L(x,0)}\,\Phi(x+t,t)dx + \int_0^t B(t-x)\Phi(x,t)dx \ . \tag{2.29}$$

Further, the total number $N(t)$ and the age density $a(x,t)$ may be represented respectively by

$$N(t) = N(0)\int_0^\infty \frac{a(x,0)}{L(x,0)} L(x+t,t)dx + \int_0^t B(t-x)L(x,t)dx \ , \qquad (2.30)$$

and,

$$N(t)a(x,t) = N(0)\ \frac{a(x-t,0)}{L(x-t,0)}\ L(x,t)H(x-t)+B(t-x)L(x,t)H(t-x) \ , \qquad (2.31)$$

where H is the Heaviside unit function. It should be noted that we obtain (2.30) by integrating the age distribution with respect to x and using the fact that $a(x,t)$ is a density. Equations (2.30) and (2.31) correspond to generalisations of (2.9) and (2.8) respectively with time dependent vital rates.

For simplicity, and as is often done in practice, we may assume the population to be initially stable so that, from (2.29) and (2.25), we have the total birth rate $B(t)$ satisfying

$$B(t) = Q_1\int_0^\infty e^{-r_1 x}\phi(x+t,t)dx + \int_0^t B(t-x)\Phi(x,t)dx \ . \qquad (2.32)$$

Basically two broad classes of time dependent net maternity functions, namely separable and non-separable, are presented in this thesis.

For the separable model $\Phi(x,t)$ is such that all age groups have the same time dependence, so that

$$\Phi(x,t) = \psi(t)\phi(x) \ , \qquad (2.33)$$

where $\phi(x)$ can be regarded as the age-shape function and $\psi(t)$ as the time scaling function. If $\psi(0) = 1$ then $\phi(x)$ is the initial net maternity function.

To allow for the final shape of the net maternity function to differ from the initial shape we consider a non-separable $\Phi(x,t)$. In particular we will study a model of the form

$$\Phi(x,t) = \xi(t)\phi_1(x) + \zeta(t)\phi_2(x) \ , \tag{2.34}$$

where the functions ξ and ζ are such that

$$\left.\begin{aligned} &\Phi(x,0) = \phi_1(x) \\ \text{and} \quad &\lim_{t\to\infty} \Phi(x,t) = \phi_2(x) \ . \end{aligned}\right\} \tag{2.35}$$

Especially we shall have $\zeta(t) = 1 - \xi(t)$.

Thus, $\phi_1(x)$ and $\phi_2(x)$ are the initial and eventual net maternity functions and constitute the starting and final shapes of $\Phi(x,t)$. $\phi_1(x)$ and $\phi_2(x)$ will also be known as the extreme net maternity functions and together with $\phi(x)$, as the inherent age-shape functions.

We note that the non-separable model (2.34) contains both change with time and with age, whereas, the separable model (2.33) represents only a change, with time, of the initial net maternity function.

The convention will be used that the initial population parameters relating to the non-separable model will be denoted with a subscript of 1 and with no subscript for the separable model. A subscript of 2 will be used in either case for eventual population parameters. Hence for the non-separable model, for example,

$$\phi_i^*(r_i) = 1 \quad \text{and} \quad \phi_i^*(0) = R_i \ , \quad i = 1,2 \ , \tag{2.36}$$

where $*$ denotes the one sided Laplace transform and $\{r_i\}$ and $\{R_i\}$ are the intrinsic rates of change and the net reproduction rates, respectively, of $\{\phi_i(x)\}$. The braces $\{\ \}$ are used to denote the whole set over the subscript i .

2.3 Numerical Solution and Spline Interpolation.

The time dependent deterministic Lotka model presented in the previous section cannot readily be solved, by Laplace transform techniques described in Subsection [2.1.2], as could the model with time independent net maternity function.

Besides the classical iterative method proposed by Volterra (see, for example, Hochstadt (1973)) there is, for a general continuous time dependent net maternity function, no other avenue known to the author, of solving the integral equation (2.29), other than a strictly numerical one. If (2.29) is solved for the total birth rate $B(t)$ then the total number and age density can be obtained from (2.30) and (2.31) respectively.

There is a vast number of numerical methods for solving general Volterra integral equations. However the modified block-by-block method, as described by Campbell and Day (1971), was chosen because of, among other reasons, the merits stated for general block-by-block methods by Linz (1969). The main advantages of block-by-block methods, as stated by Linz (1969), are due to the fact that no special starting procedures are needed and switching step-size, h, presents no problem.

The block-by-block methods in general use numerical quadrature but the calculations are arranged so that several values of the unknown function are obtained simultaneously. The basic idea of the block-by-block approach was first suggested by Young (1954) with regards to product integration techniques. O'Neill and Byrne (1968) use what is essentially the block-by-block approach to develop a starting procedure, but their method requires the use of values of the kernel outside the range of integration. Linz (1969) presents a modified block-by-block method of $O(h^4)$ that does not require the use of values outside the range of integration. Campbell and Day (1971) utilize the method of

Linz to develop an $O(h^6)$ algorithm for the solution of non-linear Volterra integral equations of the second kind.

A FORTRAN programme was written, using the modified block-by-block method of Campbell and Day (1971), to solve general linear Volterra integral equations of the form

$$B(t) = F(t) + \int_0^t B(x)K(x,t)dx \quad . \tag{2.37}$$

In particular (2.32) is solved in order to demonstrate the total birth rate $B(t)$ converging towards the known asymptotic behaviour which is determined by, the use of an algorithm developed in Chapter 4 and presented in Cerone and Keane (1978a).

In order to solve integral equations of the form (2.37), it is necessary that F be continuous and bounded and K be continuous and uniformly bounded (see Linz (1969)). Data for the net maternity function is often only available as average rates over standard 5-year age-groups (see for example Keyfitz and Flieger (1971)). Usually the smoothing or interpolation of this data is accomplished by fitting a polynomial or some model curve such as the normal curve.

Fitting a single polynomial to the discrete data of the initial net maternity function is not satisfactory since it may produce difficulties such as violation of the non-negativity condition, especially near the external points (see for example McNeil, Trussell and Turner (1977)). The attractiveness of fitting a piecewise smooth polynomial, a spline, to discrete data is stated widely in the literature. For example, (Greville (1969)) it may be shown that given $f(x)$ defined at $\{x_n\} \in (a,b)$ then a spline of degree $2k-1$ is the unique function, passing through $\{x_n, f(x_n)\}$, which is the smoothest in the sense that

it minimises

$$\int_a^b \left[f^{(k)}(x)\right]^2 dx = 0 \quad .$$

Hence only splines of odd order should be considered if the smoothness property is desired.

Cubic spline interpolation was used to fit the discrete data of the net maternity function. A cubic spline $S(x)$ on an interval (a,b):

(i) passes through all the discrete points $\{x_n, f(x_n)\}$,

(ii) is a cubic $S_n(x)$ on each interval $(x_n, x_{n+1}) \subset (a,b)$,

and (iii) possesses continuous first and second derivatives at the nodes $\{x_n\}$.

For an excellent presentation of spline functions in general and cubic splines in particular the reader is referred to Ahlberg, Nilson and Walsh (1967).

Often the end data points of the net maternity function are comparatively small and thus the spline fitted curve $S(x)$ is inclined to take negative values. A method of overcoming this was suggested by McNeil, Trussell and Turner (1977) which entails taking a higher order spline. However instead of introducing the unnecessary complexity of taking a higher order spline we fit ordinary cubic polynomials over the two end intervals.

The discrete data from Keyfitz and Flieger (1971) is taken as occurring at the mid-points of the 5-year age-groups so that the nodes occur at

$$\left.\begin{aligned} & x_1 = \alpha(= 10) \text{ , the minimum age of childbearing,} \\ & x_{n+1} - x_n = h_n = \begin{cases} 2.5, & n = 1, \\ 5, & n = 2,\ldots,N-2, \\ 2.5, & n = N-1, \end{cases} \\ & x_N = \beta(= 50) \text{ , the maximum age of childbearing, } (N=10). \end{aligned}\right\} \quad (2.38)$$

Hence $\phi(x)$ can be written in the form

$$\phi(x) = \sum_{n=1}^{N-1} S_n(x) \quad , \tag{2.39}$$

where $S_1(x)$ and $S_{N-1}(x)$ are ordinary cubics with $\phi(x_1) = \phi(x_N) = 0$, and $S_2(x), \ldots, S_{N-2}(x)$ are cubic splines.

In any event $\{S_n(x)\}$ are of the form

$$S_n(x) = \sum_{m=0}^{3} a_{n,m} x^m H(x-x_n) H(x_{n+1}-x) \quad , \tag{2.40}$$

where H is the Heaviside unit function.

With $\phi(x)$ given by (2.39) and (2.40) the forcing term in (2.32) with (2.33) can be evaluated analytically by writing it in a form similar to that of $f_n(t)$ in (3.51). The resulting integrals like

$$I_{m+1}(t) = \int_{x_n}^{t} e^{-rx} x^m dx \quad , \quad t \in (x_n, x_{n+1}) \quad , \tag{2.41}$$

can be evaluated analytically using the recurrence relation

$$\left.\begin{aligned} rI_{m+1}(t) &= -\left[x^m e^{-rx}\right]_{x_n}^{t} + mI_m(t) \quad , \quad m = 1,2,3 \quad , \\ \text{and} \quad rI_1(t) &= e^{-rx_n} - e^{-rt} \quad , \end{aligned}\right\} \tag{2.42}$$

which is obtained by integrating (2.41) by parts.

In general however the forcing term in (2.29) and (2.32) has to be evaluated numerically by using, for example, Gaussian quadrature. Solving the equations (2.29) or (2.32) numerically is slow and time consuming since we have to proceed in small steps of time t in order to obtain a sufficiently accurate solution. The transient solution for the total birth rate from (2.32) is obtained in Chapter 3, when the age structure of the time dependent net maternity function is piecewise defined with the nodal points being integer multiples of some constant. The methods of Chapter 3 do however become complicated for non-simple time dependence.

3. The Total Birth Rate Resulting from Specific Representations of the Extreme Net Maternity Functions.

In this chapter, time dependent net maternity functions, $\Phi(x,t)$, are considered and their resultant effect, on the total birth rate $B(t)$, is examined for special forms of inherent age-shape functions (or extreme net maternity functions). Both separable and non-separable time dependent net maternity functions are considered where the age-shape functions are Malthusian, histogram, point form, and, defined in a piecewise fashion.

In Sections [3.1], [3.2] and [3.3] the population is assumed, for simplicity, and as is often done in the literature, to be initially stable. Thus $B(t) = Qe^{rt}$ for $t \leq 0$, where r is the intrinsic rate of change. The transient total birth rate is also obtained in Section [3.4], for general initial age distribution represented by a histogram.

The birth and death rates are assumed to be constant, in Section [3.1], and hence the age-shape functions are Malthusian. With a Malthusian initial net maternity function (when $\Phi(x,t)$ is separable) the integral equation for the total birth rate, is transformed into a first order differential equation which is easily solved. Further, when $\Phi(x,t)$ is <u>not</u> separable, we obtain a second order differential equation which is <u>not</u> easily solved, but certain solutions are extracted which would otherwise not have been possible. Some of these solutions are given in Appendix A.

Although a Malthusian net maternity function is not a demographically realistic model, it has in the past been used by a number of authors (for example Kendall (1949), Hoppensteadt (1975) and Tognetti (1975)). More specifically it was the starting point of population analysis and

it seems fitting to return to this point for time dependent net maternity functions. Further, the Malthusian provides insights into the effects of a time dependent net maternity function on the population. It also provides analytic results which may be used to compare numerical solutions for more realistic age-shape functions.

In Section [3.2] use is made of the discrete nature of the available data (see for example Keyfitz and Flieger (1971)) to transform the integral equation for $B(t)$ into an integro-difference or difference equation depending on whether a histogram or point representation of the age-shape function is made. A stepping procedure developed in Cerone (1978) is then used to solve for the transient total birth rate.

The use of the stepping procedure for the total birth rate is also discussed in Section [3.3] where the age-shape functions are piece-wise defined (for example by splines).

Using a similar process to that in Sections [3.2] and [3.3] the transient total birth rate is obtained when both the initial, age distribution and net maternity function are represented by histograms. As a consequence, the solution for $B(t)$ with a time independent net maternity function is obtained as a polynomial on each interval of length γ. Such a solution is believed to be novel. Rhodes (1940) also used a stepping procedure although he assumed the population to consist initially of individuals of the same age. Further, Rhodes' net maternity function was independent of time.

We restate here that extreme population parameters regarding the non-separable model will be denoted by subscripts of 1 and 2 depending on whether the parameter relates to the initial or the eventual population, respectively.

3.1 The Malthusian Function as the Age-Shape Function.

In the present section the time dependent net maternity function is considered where the vital rates are assumed to be constant with age. Thus, the age-shape functions are taken to be Malthusian and the effect on the total birth rate is determined.

3.1.1 The Separable Time Dependent Net Maternity Function with Malthusian Shape Function.

The problem, for the total birth rate $B(t)$ in the form of (2.32) will be examined where the time dependent net maternity function is separable (given by (2.33)) with Malthusian age-shape function, $\phi(x)$.

Theorem: Let the time dependent net maternity function be separable with Malthusian time independence, so that from (2.33)

$$\Phi(x,t) = \psi(t)\gamma e^{-\mu x} \quad , \tag{3.1}$$

and hence only the birth rate γ is changing with time.

Further let $\psi(t)$ satisfy the following conditions

(i) $\psi(0) = 1$ and $\psi(t) > 0$, for all $t \in \mathbb{R}_+$, and

(ii) $\psi \in C^{(1)}(\mathbb{R}_+)$.

Then the solution to (2.32) is given by

$$B(t) = Q\psi(t)\exp\left\{\gamma\int_0^t\left(\psi(s) - \frac{1}{R}\right)ds\right\} \quad , \tag{3.2}$$

where $Q = B(0)$ and $R = \frac{\gamma}{\mu}$.

Proof: With $\Phi(x,t)$ as given by (3.1), (2.32) becomes, after some rearrangement and upon using condition (i),

$$\frac{e^{\mu t}B(t)}{\gamma\psi(t)} = \frac{Q}{\gamma} + \int_0^t B(x)e^{\mu x}dx \quad . \tag{3.3}$$

Differentiating (3.3) with respect to t and using condition (ii) the first order linear differential equation,

$$B'(t) = [\gamma\psi(t) - \mu + \psi'(t)/\psi(t)]B(t) \tag{3.4}$$

is obtained, subject to the initial condition, from (3.3) and condition (i),

$$B(0) = Q \quad . \tag{3.5}$$

Hence (3.4)-(3.5) has, on noting that $R = \frac{\gamma}{\mu}$, solution given by (3.2).

Corollary: With the conditions as stated in the above Theorem and further if

(iii) $\lim_{t\to\infty} \psi(t) = \psi_\infty < \infty$, and

(iv) $\lim_{t\to\infty} \int_0^t \left(\psi(s) - \frac{1}{R}\right) ds = A < \infty$, then

$B(t)$, from (2.32), admits an asymptotic value Q_2 given by,

$$Q_2 = Q\psi_\infty e^{\gamma A} \quad . \tag{3.6}$$

In particular, let $\psi(t)$ satisfy the conditions of the Corollary and be of the form

$$\psi(t) = \psi_\infty + (1-\psi_\infty)\xi(t) \quad , \tag{3.7}$$

where $\xi(t)$ is such that $\xi(0) = 1$ and $\lim_{t\to\infty} \xi(t) = 0$, then the solution (3.2) is of the form

$$B(t) = \frac{Q}{\gamma} e^{r_2 t}[\gamma_2 + \omega\xi(t)]\exp\left\{\omega\int_0^t \xi(s)ds\right\} , \tag{3.8}$$

$\omega = \gamma-\gamma_2$ and we have used the fact, from (2.36), that

$$R_2 = \int_0^\infty \phi_2(x)dx = \frac{\gamma_2}{\mu_2} \quad ,$$

where $\phi_2(x) = \psi_\infty\phi(x)$ and hence, $\psi_\infty = \frac{R_2}{R} = \frac{\gamma_2}{\gamma}$, since $\mu_2 = \mu$.

Hence $b(t) = e^{-r_2 t}B(t)$, as given by (3.8), is the solution of

$$b'(t) = [\gamma\psi(t) - (\mu+r_2) + \psi'(t)/\psi(t)]b(t) \ , \quad b(0) = Q \qquad (3.9)$$

where, $\psi(t)$ is given by (3.7). Further, $b(t)$ has an asymptotic value provided that $\lim_{t\to\infty} \int_0^t \xi(s)ds < \infty$ (is finite).

3.1.2 A Non-Separable Time Dependent Net Maternity Function with Malthusian Shape Functions.

Consider the time dependent net maternity function to be given by (2.34) with Malthusian extreme net maternity functions $\{\phi_i(x)\}$ so that

$$\Phi(x,t) = \xi(t)\gamma_1 e^{-\mu_1 x} + \zeta(t)\gamma_2 e^{-\mu_2 x} \ . \qquad (3.10)$$

From (2.34), (3.10) and (2.36) $\{r_i\}$ and $\{R_i\}$, the intrinsic rates of change and the net reproduction rates respectively, are given by

$$\left.\begin{aligned} r_i &= \gamma_i - \mu_i \\ \text{and} \quad R_i &= \gamma_i/\mu_i \ , \quad i = 1,2 \ . \end{aligned}\right\} \qquad (3.11)$$

Further, $\xi(t)$ and $\zeta(t)$, besides satisfying conditions (2.35), are such that $\xi,\zeta \in C^{(2)}(\mathbb{R}_+)$, namely, the functions and their first and second derivatives are continuous.

Substitution of (3.10) into (2.32) results, after some manipulation, in $B(t)$ being given by

$$B(t) = \gamma_1 e^{-\mu_1 t}\xi(t)F_1(t) + \gamma_2 e^{-\mu_2 t}\zeta(t)F_2(t) \ , \qquad (3.12)$$

where,

$$F_i(t) = \frac{Q_1}{r_1+\mu_i} + \int_0^t B(x)e^{\mu_i x}dx \ , \quad i = 1,2 \ .$$

In order to obtain $B(t)$ explicitly from (3.12) we divide both sides of (3.12) by $e^{-\mu_1 t}\xi(t)$ and differentiate, thus eliminating the

integral within the expression for $F_1(t)$ and so, on rearrangement, we get

$$B'(t) + \beta(t)B(t) = \gamma_2 e^{-\mu_2 t}\alpha(t)F_2(t) , \tag{3.13}$$

where,

$$\left.\begin{aligned} &\alpha(t) = \zeta(t)[\theta - \xi'(t)/\xi(t)] + \zeta'(t) , \quad \theta = \mu_1 - \mu_2 , \\ \text{and} \quad &\beta(t) = \mu_1 - \xi'(t)/\xi(t) - \gamma_1\xi(t) - \gamma_2\zeta(t) . \end{aligned}\right\} \tag{3.14}$$

Dividing both sides of (3.13) by $e^{-\mu_2 t}\alpha(t)$ and differentiating again we obtain, after some algebra, the second order homogeneous linear differential equation

$$\alpha(t)B''(t) + P(t)B'(t) + Q(t)B(t) = 0 , \tag{3.15}$$

where,

$$\left.\begin{aligned} &P(t) = \alpha(t)[\beta(t) + \mu_2] - \alpha'(t) , \\ \text{and} \quad &Q(t) = \alpha(t)[\beta'(t) + \mu_2\beta(t) - \gamma_2\alpha(t)] - \alpha'(t)\beta(t) . \end{aligned}\right\} \tag{3.16}$$

Now, the conditions (2.35) imply that

$$\left.\begin{aligned} &\xi(0) = 1 , \quad \lim_{t\to\infty} \xi(t) = 0 , \\ \text{and} \quad &\zeta(0) = 0 , \quad \lim_{t\to\infty} \zeta(t) = 1 , \end{aligned}\right\} \tag{3.17}$$

so that, from (3.12) and (3.13)-(3.14) respectively, the initial conditions

$$\left.\begin{aligned} &B(0) = Q_1 \\ \text{and} \quad &B'(0) + \left[\beta(0) - \frac{\gamma_2}{r_1+\mu_2} . \alpha(0)\right]B(0) = 0 , \end{aligned}\right\} \tag{3.18}$$

are obtained. Here, (3.11) has been used.

Thus the problem of solving the integral equation (3.12) has been changed to that of solving the differential equation (3.15) with (3.14), (3.16) and, with initial conditions (3.18), about which a great deal

more is known and from which a number of interesting results will be obtained.

In the special situation in which $\theta = 0$ $(\mu_1 = \mu_2 \ (= \mu))$ the time dependent net maternity function is separable (and hence only the birth rate is being changed with time). That is,

$$\Phi(x,t) = \psi_o(t)\phi(x) \ ,$$

where

$$\psi_o(t) = \xi(t) + \frac{\gamma_2}{\gamma_1}\zeta(t) \quad , \tag{3.19}$$

for which the solution has been previously obtained and is given by (3.2). Thus,

$$B_o(t) = Q\psi_o(t)\exp\left\{\int_o^t (\gamma_1\psi_o(s)-\mu)ds\right\} \quad , \tag{3.20}$$

where $\psi_o(t)$ is given by (3.19), is the solution of

$$\alpha_o(t)B_o''(t) + P_o(t)B_o'(t) + Q_o(t)B_o(t) = 0 \ , \tag{3.21}$$

where

$$\left.\begin{aligned} P_o(t) &= \alpha_o(t)[\beta_o(t)+\mu] - \alpha_o'(t) \ , \\ Q_o(t) &= \alpha_o(t)[\beta_o'(t) + \mu\beta_o(t) - \gamma_2\alpha_o(t)] - \alpha_o'(t)\beta_o(t) \ , \\ \alpha_o(t) &= \zeta'(t) - \zeta(t)\xi'(t)/\xi(t) \ , \\ \text{and} \quad \beta_o(t) &= \mu - \xi'(t)/\xi(t) - \gamma_1\psi_o(t) \ , \end{aligned}\right\} \tag{3.22}$$

with initial conditions, from (3.18) (with $\theta = 0$) given by

$$\left.\begin{aligned} B_o(0) &= Q_1 \ , \\ \text{and} \quad B_o'(0) &+ \left[\beta_o(0) - \frac{\gamma_2}{\gamma_1}\alpha_o(0)\right]B_o(0) = 0 \ . \end{aligned}\right\} \tag{3.23}$$

Using variation of parameters (see for example Boyce and Di Prima (1969)) the general solution $W(t)$, of (3.21)-(3.22) is found to be

$$W(t) = U(t)\left[A + C\int_0^t \exp\left\{-\gamma_1\int_0^s \psi_o(v)dv\right\} \frac{\xi(s)\alpha_o(s)}{[\psi_o(s)]^2}\, ds\right]$$

where,

$U(t) = \psi_o(t)\exp\left\{\int_0^t (\gamma_1\psi_o(s)-\mu)ds\right\}$ is a known solution of (3.21)-(3.22),

$\psi_o(t)$ and $\alpha_o(t)$ are given by (3.19) and (3.22) respectively, and, both A and C are arbitrary constants.

Consider now the problem (2.32) with (3.10) where $\zeta(t)$ has the special form

$$\zeta(t) = 1 - \xi(t) , \tag{3.24}$$

in order that the system (3.15), with (3.14) and (3.16), and with initial conditions (3.18), may be simplified somewhat and may allow also a change in the survivor function (see Theorem in Section [6.1]).

In addition, if we let,

$$B(t) = e^{r_2 t}b(t) , \tag{3.25}$$

so that $b(t)$ admits an asymptotic value (normally) then (3.15), (3.16) with (3.14); and (3.18) are transformed into, with $\zeta(t)$ given by (3.24),

$$u(t)b''(t) + p_o(t)b'(t) + q_o(t)b(t) = 0 , \tag{3.26}$$

where

$$\left.\begin{aligned} &p_o(t) = u(t)[v(t) + r_2 + \gamma_2] - u'(t) , \\ &q_o(t) = u(t)\left[v'(t) + \gamma_2[v(t) + r_2 - u(t)]\right] - u'(t)[v(t) + r_2], \\ &u(t) = \theta\left(1-\xi(t)\right) - \xi'(t)/\xi(t) , \quad \theta = \mu_1 - \mu_2 , \\ &\text{and,}\quad v(t) = \mu_1 - \gamma_2 - \omega\xi(t) - \xi'(t)/\xi(t) , \quad \omega = \gamma_1 - \gamma_2 , \end{aligned}\right\} \tag{3.27}$$

with initial conditions

$$\left.\begin{aligned} & b(0) = Q_1 \ , \\ \text{and} \quad & b'(0) + \left[v(0) + r_2 - \frac{\gamma_2}{r_1+\mu_2}\, u(0)\right] b(0) = 0 \ , \end{aligned}\right\} \tag{3.28}$$

respectively.

Further, we can eliminate $v(t)$ by noting that

$$v(t) + r_2 = \theta - \omega\xi(t) - \frac{\xi'(t)}{\xi(t)} = u(t) + (\theta-\omega)\xi(t) \quad \text{and hence}$$

(3.26), (3.27) and (3.28) become

$$u(t)b''(t) + p(t)b'(t) + q(t)b(t) = 0 \ , \tag{3.29}$$

where

$$\left.\begin{aligned} & p(t) = u(t)[\gamma_2 + u(t) + (\theta-\omega)\xi(t)] - u'(t) \ , \\ & q(t) = (\theta-\omega)[u(t)\left(\xi'(t) + \gamma_2\xi(t)\right) - \xi(t)u'(t)] \ , \\ \text{and} \quad & u(t) = \theta\left(1 - \xi(t)\right) - \xi'(t)/\xi(t) \ , \end{aligned}\right\} \tag{3.30}$$

with initial conditions,

$$\left.\begin{aligned} & b(0) = Q_1 \ , \\ \text{and} \quad & b'(0) + (\theta-\omega)\left[1 - \frac{u(0)}{r_1+\mu_2}\right] b(0) = 0 \ . \end{aligned}\right\} \tag{3.31}$$

A number of special and interesting solutions exist for the system (3.29)-(3.31). Six different possibilities, A, B, ..., F, are considered in Appendix A.

3.2 The Transient Total Birth Rate Resulting from Discrete Shape Functions in an Initially Stable Population.

Here we discuss the transient total birth rate resulting from time dependent net maternity functions where the extreme net maternity functions are discrete. The population is assumed to be initially stable.

3.2.1 The Single Delta Shape Function.

The delta function has been used extensively in the literature, for example by Keyfitz et al. (1967), Coale (1972) and Tognetti (1976a), as the shape function or as the net maternity function. Let the average age of mothers at the birth of their daughters be κ, then assuming that all births occur at age κ we have

$$\phi(x) = R\delta(x-\kappa) , \tag{3.32}$$

where R is the net reproduction rate and,

$\delta(x-\kappa)$ is the Dirac delta function defined as zero everywhere except at $x = \kappa$ and is such that

$$\int f(x)\delta(x-\kappa)dx = f(\kappa) , \tag{3.33}$$

provided κ belongs to the interval of integration.
With such a net maternity function, (3.32), Keyfitz (1968b) points out that the value $r = \frac{\ell n\ R}{\kappa}$, will rarely be in error by as much as 5%. Also, with $\phi(x)$ as given by (3.32), $\kappa = T_o$ where T_o is the mean length of a generation in the stable population defined by $e^{rT_o} = R$.

With (3.32) in the separable time dependent net maternity function (2.33) the integral equation (2.32), for the total birth rate, becomes (on using (3.33)) a difference equation with non-constant coefficients, namely

$$\frac{B(t)}{R\psi(t)} = \begin{cases} Qe^{r(t-\kappa)} , & 0 < t < \kappa \\ B(t-\kappa) , & t > \kappa , \end{cases} \tag{3.34}$$

where $Re^{-r\kappa} = 1$ and r is the intrinsic rate of change, for $t < 0$.

We solve (3.34) by proceeding in steps of length κ. Since we known the solution on $0 < t < \kappa$ we can obtain the solution on $\kappa < t < 2\kappa$, and so on. That is, using the stepping procedure we have the solution of (3.34) given by

$$B(t) = Q \prod_{n=0}^{N} R^{n+1}\psi(t-n\kappa)e^{r[t-(n+1)\kappa]}, \quad N\kappa < t < (N+1)\kappa,$$

$$= Qe^{rt} \prod_{n=0}^{N} \psi(t-n\kappa), \tag{3.35}$$

since $R = e^{r\kappa}$. The substitution $\tau = t - N\kappa$ in (3.35) results in

$$B(\tau+N\kappa) = Qe^{r(\tau+N\kappa)} \prod_{n=0}^{N} \psi\big(\tau + (N-n)\kappa\big)$$

$$= Qe^{r(\tau+N\kappa)} \prod_{n=0}^{N} \psi(\tau+n\kappa), \quad 0 < \tau < \kappa. \tag{3.36}$$

If we let $\psi(t) = \sum_{j=0}^{J} \alpha_j e^{-\beta_j t}$, $\beta_j > 0$, then (3.36) may be written simply as

$$B(\tau+N\kappa) = Qe^{r(\tau+N\kappa)} \sum_{j=0}^{J} \alpha_j \prod_{n=0}^{N} e^{-\beta_j(\tau+n\kappa)},$$

$$= Qe^{r(\tau+N\kappa)} \sum_{j=0}^{J} \alpha_j \exp\{\beta_j[\tau+N(N-1)\kappa/2]\}, \quad 0<\tau<\kappa. \tag{3.37}$$

In particular with,

$$\psi(t) = \frac{1}{R} + \left(1 - \frac{1}{R}\right)e^{-\lambda t}, \quad \lambda > 0,$$

so that the population tends gradually towards an average stationary state, (3.36) becomes

$$B(\tau+N\kappa) = \frac{Q}{R} e^{r\tau} \prod_{n=0}^{N} (1 + sq^n), \quad 0 < \tau < \kappa, \tag{3.38}$$

where $s = (R-1)e^{-\lambda\tau}$ and $q = e^{-\lambda\kappa}$.

The asymptotic standing wave $Q_2(\tau)$ (see Bourgeois-Pichat (1971)) is

obtained by taking the limit, as $N \to \infty$, in (3.38). Thus,

$$Q_2(\tau) = \frac{Q}{R} e^{r\tau} \lim_{N\to\infty} \prod_{n=0}^{N} (1 + sq^n) ,$$

which exists (from Bellman and Cooke (1963) or Andrews (1971)) since $|q| < 1$. Hence,

$$Q_2(\tau) = \frac{Q}{R} e^{r\tau} \prod_{n=0}^{\infty} (1 + sq^n) , \quad 0 < t < \kappa . \tag{3.39}$$

In order to obtain the asymptotic average stationary value, Q_2, which is given by

$$Q_2 = \frac{1}{\kappa} \int_0^{\kappa} Q_2(\tau) d\tau , \tag{3.40}$$

we write the product expression for $Q_2(\tau)$, from (3.39), as a series. Namely, also from Andrews (1971),

$$\prod_{n=0}^{\infty} (1 + sq^n) = 1 + \sum_{n=1}^{\infty} \frac{q^{n(n-1)/2} s^n}{\prod_{k=1}^{n} (1-q^k)} ,$$

and hence from (3.38), (3.39) becomes

$$Q_2(\tau) = \frac{Q}{R} e^{r\tau} \left\{ 1 + \sum_{n=0}^{\infty} \frac{e^{-\frac{n}{2}(n-1)\lambda\kappa} (R-1)^n e^{-n\lambda\kappa}}{\prod_{k=1}^{n} \left[1-e^{-k\lambda\kappa}\right]} \right\} , \quad 0 < \tau < \kappa . \tag{3.41}$$

Substituting (3.41) into (3.40) and integrating we obtain the asymptotic average stationary value Q_2 from

$$Q_2 = \frac{Q}{R\kappa} \left\{ \frac{R-1}{r} + \sum_{n=1}^{\infty} \frac{e^{-\frac{n}{2}(n-1)\lambda\kappa} (R-1)^n}{\prod_{k=1}^{n} \left[1-e^{-k\lambda\kappa}\right]} \cdot \frac{1-Re^{-n\lambda\kappa}}{n\lambda-r} \right\} . \tag{3.42}$$

The above asymptotic value, given by (3.42), will be used for comparison purposes to test the numerical algorithm developed in Chapter 4. The algorithm allows for the calculation of the asymptotic total birth rate for general age-shape functions with, in particular, the above mentioned single exponential time dependence.

3.2.2 The Single Step (Rectangular) Shape Functions.

We will consider the problem for the total birth rate $B(t)$, in an initially stable population, with separable time dependent net maternity function. That is, $B(t)$ is given by (2.32)-(2.33), with $\phi(x)$ being represented by

$$\phi(x) = AH(x-\alpha)H(\beta-x) , \quad 0 < \alpha < \beta , \tag{3.43}$$

where H is the Heaviside unit function defined by

$$H(u) = \begin{cases} 1 , & u > 0 , \\ 0 , & u < 0 , \end{cases} \tag{3.44}$$

and $A = R/(\beta-\alpha)$, R being the net reproduction rate. Thus, $B(t)$ is given by

$$B(t) = A\psi(t) \left\{ Q\, e^{rt} \int_t^\infty e^{-rx} H(x-\alpha)H(\beta-x)dx + \int_0^t B(t-x)H(x-\alpha)H(\beta-x)dx \right\} , \tag{3.45}$$

or, on using the definition, (3.44), of the Heaviside unit function,

$$\frac{B(t)}{A\psi(t)} = \begin{cases} Q.e^{rt} & , \quad 0 \leq t \leq \alpha \\ Q.e^{rt}\left[1 - \int_\alpha^t e^{-rx}dx\right] + \int_\alpha^t B(t-x)dx & , \quad \alpha \leq t \leq \beta \\ \int_\alpha^\beta B(t-x)dx & , \quad t \geq \beta . \end{cases}$$

Making the substitution $u = t - x$ in the above integrands we obtain the following system of integro-difference equations

$$\frac{B(t)}{A\psi(t)} = \begin{cases} Q.e^{rt} & , \quad 0 \leq t \leq \alpha \\ \frac{Q}{r} . \left[1-e^{r(t-\beta)}\right] + \int_0^{t-\alpha} B(u)du & , \quad \alpha \leq t \leq \beta \\ \int_{t-\beta}^{t-\alpha} B(u)du & , \quad t \geq \beta . \end{cases} \tag{3.46}$$

From (3.46), we know the solution on $[0,\alpha]$ so that we can proceed in steps of length α unless we come to straddle β at which stage we would require the solution interval to be further subdivided. If however β is some integer multiple of α then we could proceed, in steps of length α, unhindered. Let $\beta = K\alpha$, K some positive integer, and $B_n(t)$ be the solution for $t \in [(n-1)\alpha, n\alpha]$, $n = 1,2,\ldots$ then we have from (3.46),

$$\frac{B_s(t)}{A\psi(t)} = \begin{cases} Q.e^{rt} & , \quad s = 1 \\ \frac{Q}{r} \cdot \left[1-e^{r(t-K\alpha)}\right] + \int_0^{t-\alpha} B_{s-1}(u)\,du & , \quad s = 2,3,\ldots,K \\ \int_0^{t-\alpha} B_{s-1}(u)\,du - \int_0^{t-K\alpha} B_{s-K}(u)\,du & , \quad s = K+1,K+2,\ldots . \end{cases} \tag{3.47}$$

We note that Tognetti (1976a) considered (3.43) with $\beta = 2\alpha$, that is, with $K = 2$, and used Laplace transform techniques. We cannot readily use Laplace transforms here because of the time dependence $\psi(t)$. Tognetti did not have any time dependence.

If both α and β are integer multiples of some constant γ (viz. $\alpha = k\gamma$, $\beta = K\gamma$) then the solution to (3.46) can most easily be obtained by proceeding in steps of length γ. That is, if we let $B_n(t)$ be the solution for $t \in [(n-1)\gamma, n\gamma]$, $n = 1,2,3,\ldots$ then,

$$\frac{B_n(t)}{A\psi(t)} = \begin{cases} Q.e^{rt} & , \quad n = 1,2,\ldots,k \\ \frac{Q}{r}\left[1-e^{r(t-K\gamma)}\right] + \int_0^{t-k\gamma} B_{n-1}(u)\,du & , \quad n = k+1,k+2,\ldots,K \\ \int_0^{t-k\gamma} B_{n-k}(u)\,du - \int_0^{t-K\gamma} B_{n-K}(u)\,du & , \quad n = K+1,K+2,\ldots . \end{cases} \tag{3.48}$$

3.2.3 The Separable Time Dependent Net Maternity Function with Discrete Shape Function.

With a time dependent net maternity function the integral equation for the total birth rate $B(t)$ is no longer readily amenable to the Laplace transform (or equivalent) method, since the convolution theorem cannot be used. It is common however to tabulate the data, for shape functions, in discrete form (see for example, Keyfitz and Flieger (1971)). With such a discrete representation the problem for the total birth rate will be solved by an extension of the stepping procedure discussed in [3.2.2].

Rhodes (1940) used a stepping procedure for the total birth rate with the net maternity function having no time dependence and the population having no age distribution, consisting only of the newborn.

Discrete data may be represented in either histogram or in concentrated form. The histogram may be represented by

$$\phi(x) = \sum_{n=1}^{N-1} a_n H(x-b_n)H(b_{n+1}-x) \quad , \quad N \geq 2 \quad , \tag{3.49}$$

where H is the Heaviside unit function defined by (3.44). The concentrated or point form, for the initial net maternity function is represented by

$$\phi(x) = \sum_{n=1}^{N} a_n \delta(x-b_n) \quad , \tag{3.50}$$

where δ is the Dirac delta function defined by (3.33).

We note that in (3.49) and (3.50) $\{a_n\}$ and $\{b_n\}$ are such that $a_n > 0$ for $n = 1,2,\ldots,$ and, $0 < b_1 < b_2 < \ldots < b_N$, where, $\{\ \}$ denotes the whole possible set. The $\{b_n\}$ will hence forth be called nodal points. Also, $b_1 = \alpha$ and $b_N = \beta$ where α and β are the youngest and oldest ages of childbearing in the population.

With the shape functions in discrete form, as given by (3.49) and (3.50), the integral equation, (2.32) with (2.33), for the total birth rate can be written as a system of difference equations. If the $\{b_n\}$ in (3.49) and (3.50) are commensurable, that is, from Bellman and Cooke (1963), if

$$b_n = \gamma k_n ,$$

where γ is some constant and $\{k_n\}$ are non-negative integers, then the system of difference equations can be solved more conveniently, as may be seen from [3.2.2], by advancing the solution in steps of γ .

We shall firstly examine the solution of (2.32) with (2.33) for $B(t)$, with the initial net maternity function $\phi(x)$ represented as a histogram, in the form of (3.49). Substituting (3.49) into (2.32) and using the definition of the Heaviside unit function, (3.44), we can rewrite the problem (2.32) with (2.33) in the form of a system of integro-difference equations

$$\frac{B(t)}{\psi(t)} = \begin{cases} Qe^{rt} & , \ t\epsilon[0,b_1] \\ f_n(t) + \sum_{m=1}^{n-1} a_m \int_{t-b_{m+1}}^{t-b_m} B(u)du + a_n \int_0^{t-b_n} B(u)du, & t\epsilon[b_n,b_{n+1}], \ n=1,2,\ldots,N-1 \\ \sum_{n=1}^{N-1} a_n \int_{t-b_{n+1}}^{t-b_n} B(u)du & , \ t \geq b_N \end{cases} \quad (3.51)$$

where,

$$f_n(t) = Qe^{rt}\left[1 - \sum_{m=1}^{n-1} a_m \int_{b_m}^{b_{m+1}} e^{-ru}du - a_n \int_{b_n}^{t} e^{-ru}du\right] ,$$

$$= Qe^{rt}\left[\sum_{m=n}^{N-1} a_m \int_{b_m}^{b_{m+1}} e^{-ru}du - a_n \int_{b_n}^{t} e^{-ru}du\right] ,$$

and we have used the fact that $\phi^*(r) = 1$, where $*$ denotes the one-sided Laplace transform. We have further made the substitution $u = t - x$ in the integrands of (3.51).

As we saw in [3.2.2], marching in steps of b_1 (or α) produces difficulties when we come to straddle one of the nodal points $\{b_n\}$. Let $\{b_n\}$ be such that

$$b_n = \gamma k_n \ , \quad \text{for} \quad n = 1,2,\ldots,N \ , \tag{3.52}$$

where, now $\{k_n\}$ are positive integers and $0 < b_1 < b_2 < \ldots < b_N$. We note that the above is a slightly more general problem than that discussed by demographers since (3.52) allows for irregular spacing of the $\{b_n\}$.

The solution to problem (3.51) with (3.52) can most easily be represented if we let $B_n(t)$ be the solution for $t \in [(n-1)\gamma, n\gamma]$, $n = 1,2,\ldots$. Hence proceeding systematically in steps of length γ we have from (3.51) that,

$$\frac{B_s(t)}{\psi(t)} = \begin{cases} Qe^{rt} & , \ s = 1,2,\ldots,k_1, \\ f_n(t) + \sum_{m=1}^{n-1} a_m \left[\int_0^{t-k_m\gamma} B_{s-k_m}(u)du - \int_0^{t-k_{m+1}\gamma} B_{s-k_{m+1}}(u)du \right] & \\ \quad + a_n \int_0^{t-k_n\gamma} B_{s-k_n}(u)du \ , \quad s = k_n+1, k_n+2, \ldots, k_{n+1} \ , & \\ \quad \text{for} \quad n=1,2,\ldots,N-1 \ , & \\ \sum_{m=1}^{N-1} a_m \left[\int_0^{t-k_m\gamma} B_{s-k_m}(u)du - \int_0^{t-k_{m+1}\gamma} B_{s-k_{m+1}}(u)du \right], & s = k_N+1, k_N+2, \ldots \end{cases} \tag{3.53}$$

Thus,

$$B_s(t) = Qe^{rt}\psi(t) \ , \quad (s-1)\gamma \leq t \leq s\gamma \ , \quad s = 1,2,\ldots,k_1 \ ,$$

$$B_{k_n+s}(t) = \psi(t)\left\{ f_n(t) + \sum_{m=1}^{n} \tilde{a}_m \int_0^{t-k_m\gamma} B_{k_n-k_m+s}(u)du \right\} \ ,$$

$$(k_n+s-1)\gamma \leq t \leq (k_n+s)\gamma \ , \tag{3.54}$$

$$\text{with} \quad s = 1,2,\ldots,k_{n+1}-k_n \ ; \quad n = 1,2,\ldots,N-1 \ ,$$

$$\text{and} \quad B_{k_N+s}(t) = \psi(t) \sum_{m=1}^{N} \tilde{a}_m \int_0^{t-k_m\gamma} B_{k_N-k_m+s}(u)du \ , \quad (k_N+s-1)\gamma \leq t \leq (k_N+s)\gamma,$$

$$\text{for} \quad s = 1,2,\ldots \ ,$$

where $\tilde{a}_1 = a_1$,

$\tilde{a}_m = a_m - a_{m-1}$, $m = 2,3,\ldots,N-1$,

and $\tilde{a}_N = -a_{N-1}$.

We note that with $\psi(t) \equiv 1$ then the solution (3.53) or (3.54) reduces to the initial stable population.

If we take $N = 2$ in the above problem then we obtain the results of [3.2.2] .

In practice the data is often given in five year age-groups from 10 to 55 years (see Keyfitz and Flieger (1971)). Hence in problem (3.51), with (3.52), and in the solution (3.53), or (3.54), we have $k_1 = 2$, $\gamma = 5$, $k_n - k_{n-1} = 1$ that is, $b_n - b_{n-1} = \gamma$ for $n = 2,3,\ldots,N(=10)$. A range from 10 to 55 years is the largest interval of reproduction encountered, but intervals of 15 to 45 (or 50) years are not uncommon allowing for differing cultures and customs. Cultural and social pressures, besides the biological constraints, play an important role in determining the length of the reproductive period. To conform to this possible variation in the length of the reproductive period we will keep the end nodal points general, but we will have regular spacing of length γ of the internal nodal points. Hence the solution, (3.54), with $k_n - k_{n-1} = 1$ namely $b_n - b_{n-1} = \gamma$ for $n = 2,\ldots,N$, may be written as

$$B_s(t) = Qe^{rt}\psi(t) \quad , \quad (s-1)\gamma \leq t \leq s\gamma \ , \quad s = 1,2,\ldots,k_1 \ ,$$

$$B_{k_{n+1}}(t) = \psi(t)\left\{f_n(t) + \sum_{m=1}^{n} \tilde{a}_m \int_0^{t-k_m\gamma} B_{k_{n+1}-k_m}(u)\,du\right\} \ ,$$

$$k_n\gamma \leq t \leq k_{n+1}\gamma \ , \quad \text{for} \quad n = 1,2,\ldots,N-1 \ ,$$

and $$B_{k_{N+1}}(t) = \psi(t) \sum_{m=1}^{N} \tilde{a}_m \int_0^{t-k_m\gamma} B_{k_{N+1}-k_m}(u)\,du \ , \quad k_N\gamma \leq t \leq k_{N+1}\gamma \ ,$$

$$\text{for} \quad n = 1,2,\ldots \ ,$$

where we have used the fact that $k_{n+1} = k_n + 1$.

In the solution to problem (3.51), as given by (3.53) or (3.54), we have to integrate the solutions over previous intervals in order to obtain the solution on the present interval. This can be done systematically and analytically for certain forms of time dependence $\psi(t)$. For example if $\psi(t)$ is a strict sum of exponentials then the solution $B_n(t)$, from (3.54), on successive intervals of length γ , can be done analytically using a recurrence relation of the form (2.42). The sum does however become more complicated as we proceed to higher intervals. For certain exponential forms of $\psi(t)$ we can use (3.54) to obtain the transient solution, and, the methods of Chapters 4 and 5 for the asymptotic behaviour of the solution.

We will now examine the solution of (2.32) with (2.33) where the initial net maternity function is in concentrated or point form, as given by (3.50). Using the properties of the Dirac delta function, explicit integration is avoided.

Substituting (3.50) into (2.32) with (2.33) and using (3.33) we obtain a system of difference equations namely,

$$\frac{B(t)}{\psi(t)} = \begin{cases} Qe^{rt} & , \quad t \in (0, b_1) \\ g_n(t) + \sum_{m=1}^{n} a_m B(t-b_m) & , \quad t \in (b_n, b_{n+1}) , \quad n = 1,2,\ldots,N-1 \\ \sum_{n=1}^{N} a_n B(t-b_n) & , \quad t > b_N \end{cases} \tag{3.55}$$

where,

$$g_n(t) = Qe^{rt}\left[1 - \sum_{m=1}^{n} a_m e^{-rb_m}\right] ,$$

and we have used the fact that $\phi^*(r) = 1$. Thus from (3.50), and

using (3.33),

$$\sum_{n=1}^{N} a_n e^{-rb_n} = 1 \ . \tag{3.56}$$

Again, if we let $\{b_n\}$ be such that $b_n = \gamma k_n$ where $\{k_n\}$ are positive integers, and let $B_n(t)$ be the solution for $t \in \big((n-1)\gamma, n\gamma\big)$ $n = 1,2,\ldots$, then, by proceeding in steps of γ, the solution to (3.55) may be given by

$$B_s(t) = Qe^{rt}\psi(t) \qquad ; \quad (s-1)\gamma < t < s\gamma, \quad s = 1,2,\ldots,k_1,$$

$$B_{k_n+s}(t) = \psi(t)\left\{g_n(t) + \sum_{m=1}^{n} a_m B_{k_n-k_m+s}(t-k_m\gamma)\right\}; \quad (k_n+s-1)\gamma < t < (k_n+s)\gamma,$$

$$\text{where} \quad s = 1,2,\ldots,k_{n+1}-k_n \ ; \quad n = 1,2,\ldots,N-1 \ , \tag{3.57}$$

$$\text{and} \quad B_{k_N+s}(t) = \psi(t) \sum_{m=1}^{N} a_m B_{k_N-k_m+s}(t-k_m\gamma); \quad (k_N+s-1)\gamma < t < (k_N+s)\gamma \ ,$$

$$\text{for} \quad s = 1,2,\ldots \ .$$

We note that if $N = 1$ we obtain the results of Subsection [3.2.1].

In a practical situation one might model the concentrations at the mid-points of the five-year intervals discussed earlier with regards to the histogram model. Such a procedure has been used by Lotka (1948), and by Cole (1954) as noted in Keyfitz (1968b). Hence we may have $k_1 = 5$, $\gamma = 2.5$, $k_n - k_{n-1} = 2$, that is, $b_n - b_{n-1} = 2\gamma$ for $n = 2,3,\ldots,N(=9)$. The solution (3.57) is thus now defined over smaller intervals than for the histogram formulation. Since larger intervals are more desirable we may interpolate the nodes, at which the concentrations occur, to multiples of 5 years as shown by Lotka (1948). Further, Lotka realised that we obtain a polynomial in the characteristic equation (3.56) and hence there are a finite number of roots. The polynomial is obtained because the $\{b_n\}$ are integer multiples of some constant γ.

3.2.4 A Non-Separable Time Dependent Net Maternity Function with Discrete Shape Functions.

In this subsection we will discuss the solution for the total birth rate $B(t)$, in an initially stable population in which the time dependent net maternity function is given by (2.34).

We will firstly consider the discrete shape functions $\{\phi_i(x)\}$ represented by a histogram, of the form given by (3.49), so that

$$\phi_i(x) = \sum_{n=1}^{N-1} a_{n,i} H(x-b_n)H(b_{n+1}-x) \quad , \quad 0 < b_1 < b_2 < \ldots < b_N \quad , \tag{3.58}$$

where $\{b_n\}$ are commensurable. That is, $\{b_n\}$, are as given by (3.52). Substituting (3.58) into (2.34) we obtain

$$\Phi(x,t) = \sum_{n=1}^{N-1} \psi_n(t)H(x-b_n)H(b_{n+1}-x) \quad , \tag{3.59}$$

where,

$$\psi_n(t) = \xi(t)a_{n,1} + \zeta(t)a_{n,2} \quad . \tag{3.60}$$

We note that (3.59) is a sum of separable functions and is similar to the separable net maternity function

$$\Phi(x,t) = \psi(t) \sum_{n=1}^{N-1} a_n H(x-b_n) \quad ,$$

if we have

$$a_n \cdot \psi(t) = \psi_n(t) \quad \text{for} \quad n = 1,2,\ldots,N-1 \quad .$$

Now, substituting (3.59) into (2.32) and using (3.44) we obtain a system of integro-difference equations similar to (3.51) with the substitutions

$$\left.\begin{aligned} &\psi(t) = \chi(t) \\ \text{where} \quad &\chi(t) = \xi(t) + \zeta(t)\phi_2^*(r_1) \quad , \\ \text{and} \quad &a_n\, \psi(t) = \psi_n(t) \quad , \end{aligned}\right\} \tag{3.61}$$

with $\psi_n(t)$ as given by (2.60). We have further used the fact that $\phi_1^*(r_1) = 1$. Hence the solution to (2.32) with (3.59) is given by

(3.54) with the substitutions (3.61). Consequently if, $B_n(t)$ denotes the solution for $t \in [(n-1)\gamma, n\gamma]$, $n = 1,2,\ldots$, then

$$B_s(t) = Q_1 \cdot e^{r_1 t}\, \chi(t); \quad (s-1)\gamma \leq t \leq s\gamma \ , \quad s = 1,2,\ldots,k_1 \ ,$$

$$B_{k_n+s}(t) = F_n(t) + \sum_{m=1}^{n} \tilde{\psi}_m(t) \int_0^{t-k_m\gamma} B_{k_n-k_m+s}(u)\,du \ ;$$

$$(k_n+s-1)\gamma \leq t \leq (k_n+s)\gamma \ ,$$

$$\text{with} \quad s = 1,2,\ldots,k_{n+1}-k_n \ , \quad n = 1,2,\ldots,N-1 \ , \tag{3.62}$$

and

$$B_{k_N+s}(t) = \sum_{m=1}^{N} \tilde{\psi}_m(t) \int_0^{t-k_m\gamma} B_{k_N-k_m+s}(u)\,du \ ; \ (k_N+s-1)\gamma \leq t \leq (k_N+s)\gamma \ ,$$

$$\text{for} \quad s = 1,2,\ldots \ ,$$

where $\tilde{\psi}_1(t) \equiv \psi_1(t)$,

$$\tilde{\psi}_m(t) \equiv \psi_m(t) - \psi_{m-1}(t) \ , \quad m = 2,3,\ldots,N-1 \ ,$$

$$\tilde{\psi}_N(t) \equiv -\psi_{N-1}(t) \ .$$

Further, $F_n(t)$ is given by

$$F_n(t) = Q_1 e^{r_1 t}\left[\chi(t) - \sum_{m=1}^{n-1} \psi_m(t) \int_{b_m}^{b_{m+1}} e^{-r_1 u}\,du - \psi_n(t) \int_{b_n}^{t} e^{-r_1 u}\,du\right] \ .$$

Similarly, we obtain the solution to (2.32), with (3.34) and $\{\phi_i(x)\}$ given by

$$\phi_i(x) = \sum_{n=1}^{N} a_{n,i}\,\delta(x-b_n) \ , \tag{3.63}$$

from (3.57) by making the substitutions (3.61).

We note that in (3.58) and (3.63), $\{b_n\}$ are independent of i and hence are the same for both the initial and final net maternity functions. If the $\{b_n\}$ do differ between the two shape functions then the positivity condition on the time dependent net maternity function may be violated if, for example, the support of the age-shape functions changes with time. Although the case with $\{b_{n,i}\}$, relating

to $\{\phi_i(x)\}$, is of importance it will not be examined further here. It is enough to say that if $\{b_{n,i}\}$ are integer multiples of the same constant then we may advance the solution in steps of that constant.

3.3 Shape Functions Piecewise Defined Over a Finite Interval with Commensurable Nodal Points.

We now discuss the resultant total birth rate, $B(t)$, due to shape functions defined in a piecewise fashion where the nodal points are commensurable.

We assume the time dependent net maternity function to be separable, of the form (2.33), where

$$\phi(x) = \sum_{n=1}^{N-1} s_n(x)H(x-b_n)H(b_{n+1}-x) , \tag{3.64}$$

with $\{b_n\}$ being given by (3.52).

$s_n(x) \equiv a_n$ has been previously considered in Section [3.2]. Thus proceeding in a similar fashion we obtain the solution to (2.32)-(2.33) with (3.64) which contains inherent difficulties, similar to those of (3.54), due to the initial net maternity function being represented by a histogram. Here the difficulties are greater because of the $\{s_n(x)\}$ since we need to integrate explicitly, over previous intervals, to obtain the solution on the current interval. However, for simple time dependence $\psi(t)$ such as exponential, we may make some progress. The solution is further enhanced if $\gamma = \alpha$, the minimum age of child-bearing, thus allowing for the largest possible step length in the stepping procedure.

A simple example of (3.64) is provided in Rhodes (1940) with $N = 2$, $b_1 = \alpha$, $b_2 = \beta$ and $s_1(x) = A \sin a(x-\alpha)$.

We note that most of the curves used for graduation (see Keyfitz (1968b) or Pollard (1973)) such as the normal curve, do not have compact support. Even if these curves were truncated we would need, short of numerical quadrature, to be able to integrate the solution over

successive steps of length γ .

The attraction of fitting piecewise smooth polynomials was discussed in Chapter 2. McNeil, Trussell and Turner (1977) advocate the use of splines for a piecewise fit of the discrete data. With $\phi(x)$ of the form (2.39)-(2.40) and if $\psi(t)$ is given by a sum of exponentials then a sum of integrals of the form (2.41) result, which may be evaluated by using (2.42). The solution to (2.32)-(2.33) with (2.39)-(2.40) and (3.52) does however become complicated as we proceed to higher intervals.

3.4 The Transient Total Birth Rate Resulting from Histogram Shape Functions and Age Distributions.

In the present section we will discuss the resultant transient total birth rate where the initial net maternity function as well as the age distribution, are represented by histograms. Namely we will not assume the population to be initially stable, as we have done in the previous sections of this Chapter, but will contend with a general age distribution represented by a histogram.

From (2.29), the total birth rate with an arbitrary initial age distribution is given by

$$B(t) = \int_0^\infty \tilde{A}(x)\Phi(x+t,t)dx + \int_0^t B(t-x)\Phi(x,t)dx \ , \tag{3.65}$$

where $\Phi(x,t)$ is assumed separable - of the form (2.33), and $\tilde{A}(x) = \frac{N(0)a(x,0)}{\ell(x)} = B(-x)$ $\left(\text{from } (2.3)_2 \text{ and } (2.7)\right)$, the total births at $t = 0$.

We consider the solution to (3.65) where $\tilde{A}(x)$, and $\phi(x)$, the initial net maternity function, are given by

$$\tilde{A}(x) = \sum_{n=1}^{M-1} c_n \, H(x-b_n)H(b_{n+1}-x) \ , \tag{3.66}$$

and (3.49) respectively.

For simplicity we will only consider the nodal points $\{b_n\}$ to be regularly distributed, as they often are in practice. Thus, with $b_1 = 0$, we have

$$b_{n+1} = n\gamma \quad \text{for} \quad n = 1,2,\ldots,N-1 \text{ (or } M-1) \ . \tag{3.67}$$

We note that $b_1 = 0$,

$b_M = \omega$, the maximum possible age of an individual,

$b_k = \alpha$, the minimum age of childbearing, and hence,

$$a_1 = a_2 = \ldots = a_{k-1} = 0 \ ,$$

and $b_N = \beta$, the maximum age of childbearing.

In practice k will be 2 or 3 (with $\gamma = 5$ years) so that α is 10 or 15 years.

Proceeding in a manner similar to that of Subsection [3.2.3] we obtain the solution $B_n(t)$ on $[(n-1)\gamma, n\gamma]$ for $n = 1,2,\ldots,$ of (3.65) with (2.33), (3.49) and (3.66), using (3.44), as

$$\frac{B_n(t)}{\psi(t)} = \begin{cases} F_n(t) + \sum_{m=1}^{n} \tilde{a}_m \int_0^{t-m\gamma} B_{n-m+1}(u)\,du \ , & n = 1,2,\ldots,N-1 \\ \sum_{m=1}^{N} \tilde{a}_m \int_0^{t-m\gamma} B_{n-m+1}(u)\,du \ , & n = N,N+1,\ldots \ , \end{cases} \tag{3.68}$$

where,

$$\tilde{a}_1 = a_1 \ ,$$

$$\tilde{a}_m = a_m - a_{m-1} \ , \quad m = 2,3,\ldots,N-1 \ ,$$

and $$\tilde{a}_N = -a_{N-1} \ .$$

Further, using (3.49) and (3.66) we have

$$\begin{aligned} F_n(t) &= \sum_{m=1}^{M-1} c_m \int_{(m-1)\gamma}^{m\gamma} \phi(x+t)\,dx \ , \\ &= \sum_{m=1}^{M-1} c_m \int_{t+(m-1)\gamma}^{t+m\gamma} \phi(u)\,du \ , \\ &= \sum_{m=1}^{M-1} c_m \left[\left(t-(n-1)\gamma\right) a_{m+n} + (n\gamma - t) a_{m+n-1} \right] \ , \end{aligned} \tag{3.69}$$

and $a_1 = a_2 = \ldots = a_{k-1} = 0$ with $F_n(t) = 0$ for $t \geq N\gamma$ (or $n \geq N$).

We note that if $\psi(t) \equiv 1$ then, (3.68) represents the solution for the total birth rate, (3.65), where the net maternity function is independent of time and is defined by a histogram. The solution, to the

time independent problem on each interval of length γ, is given simply by a polynomial, as can be seen from (3.68) with (3.69). It is believed, by the author, that such a solution as (3.68) has not previously been presented.

Rhodes (1940) used a stepping procedure for the total birth rate where the solution was assumed to consist of equi-aged individuals. The net maternity function was assumed by Rhodes to be continuous and unchanging with time.

In evaluating the solution (3.68) with (3.69), similar problems to those encountered in Section [3.2] present themselves. If $\psi(t)$ is a sum of exponentials, with $F_n(t)$ as defined by (3.69) it can be easily seen that we need to evaluate integrals of the form (2.41) which can be done using (2.42).

The problem for the total birth rate, with a non-separable time dependent net maternity function and with nodal points $\{b_n\}$ being the same for the extreme net maternity functions, can be solved in a similar manner to that outlined in Subsection [3.2.4].

4. The Asymptotic Effects of General Extreme Net Maternity Functions With Exponential Time Dependence.

The effects of both abrupt and gradual changes of the age-specific birth rate on the ensuing population will be discussed in the present chapter.

The first analytic study demonstrating what Keyfitz termed "the momentum of population growth" was presented in Keyfitz (1971b). He showed by abruptly scaling the age-specific birth rate down to replacement level, that the population would have a tendency towards continued growth. Keyfitz did this by demonstrating for various data that under such a change, the eventual stationary population would be greater than the initial.

For a number of populations Frejka (1968), (1973) has noted the phenomenon of residual growth resulting from allowing the age-specific birth rate to change along different paths of time. Frejka (1973) however projects the population whereas Keyfitz (1971b) produces an elegant closed-form expression for both the asymptotic total birth rate and the asymptotic total number of the eventual stationary population.

Since the foundation article of Keyfitz (1971b) a number of generalisations and extensions have appeared in the literature.

Firstly Frauenthal (1975) produces a "gradual" change by assuming that only the new-born population scales its age-specific birth rate by a constant, to replacement level.

An extension by Keyfitz (1975) and also Tognetti (1976b) assumes that the age-specific birth rate is instantaneously scaled by a constant so that a stable rather than stationary population results. In an

adjacent paper to Tognetti's, Mitra (1976) generalised the Keyfitz (1971b) model by allowing the net maternity function to change from the initial to any other without being necessarily scaled by a constant as in previous models. Mitra's model allows for the age structure of the net maternity function to change.

As Keyfitz has noted in Keyfitz (1971b), an abrupt change to replacement level fertility is "unrealistic" and hence gradual changes should be considered. A model which allows a gradual exponential change of the age-specific birth rate, to bare replacement, was presented in Cerone and Keane (1978a). A numerical method for obtaining the asymptotic total birth rate, and hence, the asymptotic numbers and age density function, was outlined and is given here in Subsection [4.1.2].

Extending the ideas of Frauenthal (1975), the asymptotic effects of an exponential time dependent differential scaling towards replacement level fertility rates is studied in Subsection [4.1.3]. With this model, the parent population is allowed to change its age-specific birth rate at a different transition rate to that of those born after the origin.

The stable birth rate resulting from an exponential time dependent change between any two net maternity functions was analysed in Cerone and Keane (1978b) and is presented here in Subsection [4.1.4]. The model allows for change with both age and time, and, represents a non-separable time dependent net maternity function. As stated earlier, the initial and final net maternity functions will be referred to as the extreme net maternity functions or as the inherent age-shape functions of the time dependent net maternity function.

The converse problem is discussed in Section [4.3] and consists of determining the transition rate that will result in a given

asymptotic behaviour, with the initial net maternity function being assumed to change in a certain exponential fashion. Such a problem is very important in the planning and management of populations as pointed out by Nortman and Bongaarts (1975).

The method of Cerone and Keane (1978a), to obtain the asymptotic behaviour, is generalised in Section [4.4] where the transient total birth rate is obtained.

It should be noted that, in this chapter, all changes of the net maternity function are via the age-specific birth rate.

4.1 The Separable Time Dependent Net Maternity Function and its Effect on the Momentum of Population Growth.

The momentum of population growth problem of Keyfitz is generalised to contain a gradual exponential change, at a rate λ , of the age-specific birth rate to the level of bare replacement. It is shown that for a Malthusian initial net maternity function, the asymptotic total birth rate for the gradual change is the Keyfitz value multiplied by $\exp(r/\lambda)$ where, r is the rate of increase of the population before $t = 0$. All age-groups experience the same time dependent scaling and hence the model is separable.

A numerical algorithm is presented for obtaining the asymptotic total birth rate for general initial net maternity functions with exponential time dependence. The numerical method is demonstrated by comparing it to known analytic solutions, found in Chapter 3, for the model with, Dirac delta and Malthusian initial net maternity functions. The method is also demonstrated for demographically more realistic data.

4.1.1 The Asymptotic Effects of an Instantaneous Scaling, to Replacement Level, of the Age-Specific Birth Rate.

Keyfitz (1971b) analysed the potential or momentum of a growing population, for further growth. He showed that even if a growing population reduced its age-specific birth rate abruptly to replacement level, there would be a tendency for further growth, with the extent of the residual growth after the change depending on the age distribution and the number of people of reproductive age. Frejka (1968) and (1973) has studied the phenomenon of residual population growth by projection. For present high fertility countries in particular, Keyfitz (1971b) showed that it was unwarranted to hesitate

in making contraception available merely because the population had not yet reached the desired level. Even if high-fertility countries were to drop immediately to replacement level age-specific birth rates, Keyfitz maintained that the ultimate stationary population would be approximately two thirds higher than the present total.

We shall present the analysis of Keyfitz, but, we will use Laplace transform techniques.

Let the population be initially stable, changing such that the total birth rate $B(t) = Qe^{rt}$, $r > 0$. Keyfitz abruptly alters the age-specific birth rate, at time $t = 0$, from $m(x)$ to $m(x)/R$, R being the net reproductive rate. Thus, the population will eventually become stationary. That is, $B(t)$ will aysmptotically tend to Q_2 .

Recall that $\phi(x) = m(x)\ell(x)$, where $\ell(x)$ is the probability of living to age x . Further let $\phi^*(p) = \int_0^\infty e^{-px}\phi(x)dx$ denote the one-sided Laplace transform of $\phi(x)$, and, hence

$$\phi^*(0) = R \quad \text{and} \quad \phi^*(r) = 1 \ . \tag{4.1}$$

For $t > 0$, the total birth rate $B(t)$, for a females-only population closed to migration, is thus given by the renewal equation

$$B(t) = Q\int_0^\infty e^{-rx}\,\frac{\phi(x+t)}{R}\,dx + \int_0^t B(t-x)\,\frac{\phi(x)}{R}\,dx \ . \tag{4.2}$$

Taking Laplace transforms of (4.2) we obtain

$$B^*(p) = \frac{Q}{R} \cdot \frac{\phi^*(r) - \phi^*(p)}{(p-r)\left[1 - \frac{\phi^*(p)}{R}\right]} \ , \tag{4.3}$$

which may be inverted using the residue theorem.

We note that $p = r$ is not a pole but is merely a removable singularity; and hence, the only contribution to the solution results

from the roots of

$$\frac{\phi^*(p)}{R} = 1 \quad . \tag{4.4}$$

Using the results of Chapter 2 we know that the real root of the characteristic equation (4.4) has the greatest real part. Hence by the Tauberian theorem an asymptotic value Q_2 exists; since, on using (4.1), the real root of (4.4) can be seen to be zero. Thus, we have

$$Q_2 = \lim_{p\to 0} pB^*(p) = \frac{Q}{rR}\,[\phi^*(0)-\phi^*(r)] \lim_{p\to 0} \frac{p}{1 - \frac{\phi^*(p)}{R}} \quad .$$

Therefore using L'Hopital's rule and (4.1), the asymptotic total birth rate,

$$Q_2 = Q \cdot \frac{R-1}{rR\kappa} \quad , \tag{4.5}$$

where

$$\kappa = -\frac{1}{R}\left[\frac{d}{dp}\,\phi^*(p)\right]_{p=0} = \int_0^\infty x\,\frac{\phi(x)}{R}\,dx \quad ,$$

is the expected age of childbearing.

Now, since the population is assumed to be initially stable, the total number, $N(t)$, is initially of the form $N(t) = P.e^{rt}$ and will tend asymptotically to the stationary state $N(t) = P_2$. Thus from (2.6) we obtain

$$P = Q\int_0^\infty e^{-rx}\ell(x)dx$$

and $$P_2 = Q_2\int_0^\infty \ell(x)dx \quad .$$

Hence the asymptotic total number of females P_2 is given by

$$\frac{P_2}{P} = \frac{Q_2}{Q} \cdot \frac{\int_0^\infty \ell(x)dx}{\int_0^\infty e^{-rx}\ell(x)dx} \quad .$$

That is, using (4.5),

$$P_2 = P \cdot \frac{\overset{o}{e}_o b(R-1)}{rR_K} \quad , \tag{4.6}$$

where,

$$b = \frac{Q}{P} = \frac{1}{\int_o^\infty e^{-rx} \ell(x)dx} \quad ,$$

the crude birth rate in the stable population,

and $\overset{o}{e}_o = \int_o^\infty \ell(x)dx \left(= \frac{1}{b_2} = \frac{P_2}{Q_2} \right)$,

the life expectancy at birth.

Keyfitz (1971b) shows by using a number of examples that a population has the tendency to continue to grow even after an abrupt change to stationary reproductive rates. That is, he shows that, (from (4.6))

$$\frac{P_2}{P} = \overset{o}{e}_o b \, \frac{(R-1)}{rR_K} > 1 \quad .$$

The task of showing that $P_2 > P$ analytically for general parameters, is thought by the author to be an impossible one. This is so because we need to know the <u>amount</u> by which $\overset{o}{e}_o b > 1$ and $\frac{Q_2}{Q} < 1$. However, we can show that $Q_2 < Q$ by simply using (4.1) in (4.5) to obtain

$$\frac{Q_2}{Q} = \frac{\int_o^\infty (1-e^{-rx})\phi(x)dx}{\int_o^\infty rx\phi(x)dx} \quad .$$

Now, since $1-e^{-rx} < rx$ for all $rx > 0$ then the result follows because $\phi(x)$ is a non-negative function.

The initial stable age density is given by (from (2.6))

$$a(x) = be^{-rx}\ell(x)$$

and, the eventual stationary age density by

$$a^{(2)}(x) = b_2 \ell(x) = \frac{\ell(x)}{\overset{o}{e}_o} \quad . \tag{4.7}$$

Hence, assuming an initially stable population, an abrupt scaling of the age-specific birth rate by the net reproductive rate resulting eventually in a stationary population, then the asymptotic total birth rate Q_2, the asymptotic numbers P_2 and the eventual stationary age density $a^{(2)}(x)$ are given by (4.5), (4.6) and (4.7) respectively. The abrupt scaling of the age-specific birth rate by the net reproductive rate will henceforth be known as, the Keyfitz change to replacement level fertility rates.

As a simple example, consider $m(x) = \gamma$ and $\ell(x) = e^{-\mu x}$. That is, to illustrate the results of Keyfitz (1971b) we will consider a Malthusian net maternity function $\phi(x) = \gamma e^{-\mu x}$. Thus we have $\phi^*(p) = \frac{\gamma}{p+\mu}$, $R = \frac{\gamma}{\mu}$, $r = \gamma - \mu > 0$, $\overset{o}{e}_o = \frac{1}{\mu} = \kappa$ and $b = \gamma$ so that,

$$Q_2 = \frac{Q}{R}, \quad P_2 = P \quad \text{and} \quad a^{(2)}(x) = \mu e^{-\mu x}, \tag{4.8}$$

obtained from (4.5), (4.6) and (4.7) respectively. We notice from (4.8) that, the asymptotic numbers in the eventual stationary population equals the present numbers, and, since the Malthusian does not have a transient solution the asymptotic behaviour is attained immediately and for all $t > 0$.

Using the data from Keyfitz and Flieger (1971), Table 4.1 shows the intrinsic birth rate, b, the intrinsic rate of change, r, the expectation of life at birth, $\overset{o}{e}_o$, the mean age of childbearing, κ, and the net reproduction rate R, for five countries. The values differ from those presented in Keyfitz and Flieger (1971) since spline interpolation was used for both the net maternity function and survivor

function data. Spline interpolation is used in order to solve the integral equation numerically where the method is dependent, amongst other conditions, on the integrands being continuous. The modified block-by-block method as described by Campbell and Day (1971) is used to solve the integral equation. The method is further discussed in Chapter 2.

The asymptotic, total birth rate and total numbers, under the assumption of replacement level age-specific birth rates, are presented for five countries in Table 4.2. We observe that countries of highest present growth have the greatest ability to grow further. The tendency to remain on the present path of growth occurs, as noted by Keyfitz (1971b), because a history of high fertility has brought about a high proportion of women in the reproductive ages. High crude birth rates result long after the age-specific birth rates have dropped to replacement level. We see, from Tables 4.1 and 4.2, that the total birth rate decreases most for Honduras and, for Trinidad and Tobago, that is, for countries of highest present age-specific birth rates, as examplified by the intrinsic rate of change r and the net reproductive rate R. The decrease to replacement level fertility, naturally, has the least impact on those countries which are closest to a stationary state, for example, England and Wales. The greatest potential for growth comes from the size of the intrinsic birth rate b, as governed by the intrinsic rate of change r (since the survivor function is not assumed to change). From equation (4.6) we see that as r increases so does b and hence so does the asymptotic numbers in the population.

Since the pioneer work of Keyfitz (1971b) a number of extensions and generalisations have appeared in the literature.

	$b\times10^2$	r	$\overset{o}{e}_o$	κ	R
AUST-67	2.05177	1.09537×10^{-2}	71.1806	27.256	1.3451
E & W-67	1.85845	8.11145×10^{-3}	71.7249	27.029	1.2437
HOND-66	4.42545	3.57580×10^{-2}	59.7896	29.558	2.7698
T & T-67	1.84109	2.14292×10^{-2}	66.3965	27.450	1.7821
U.S.-67	2.93059	7.40129×10^{-3}	70.5488	26.273	1.2134

TABLE 4.1

Intrinsic birth rate, b , intrinsic rate of change, r , expectation of life at birth, $\overset{o}{e}_o$, mean age of childbearing, κ and net reproduction rate, R , for five countries. The discrete data from Keyfitz and Flieger (1971) together with cubic spline interpolation was used for both the net maternity function and the survivor function for the determination of the tabulated values. The countries are respectively; Australia, England and Wales, Honduras, Trinidad and Tobago, and, The United States.

	Q_2/Q	P_2/P	$Q_2/Q = R^{-\frac{1}{2}}$	$P_2/P = \frac{\overset{o}{e}_o b}{\sqrt{R}}$
AUST-67	0.859355	1.25505	0.862225	1.25924
E & W-67	0.893781	1.19138	0.896685	1.19525
HOND-66	0.604553	1.59962	0.600858	1.58985
T & T-67	0.746053	1.45168	0.749095	1.45760
U.S.-67	0.904495	1.17482	0.907807	1.17912

TABLE 4.2

Asymptotic total birth rate Q_2 and asymptotic total number P_2 , as given by (4.5) and (4.6), resulting from an abupt Keyfitz change. The last two columns give Frauenthal's approximation, (4.11). The values from Table 4.1 are used as data. The countries are respectively; Australia, England and Wales, Honduras, Trinidad and Tobago, and, The United States.

Frauenthal (1975) produces a gradual change to a stationary level by using a technique also noted by Keyfitz (1971b). That is, Frauenthal allows only the females born since time $t = 0$ to be subjected to the new regime of fertility and those already alive at the origin (the parent population) to continue with the initial stable regime. It is the age structure of the population which produces the gradual change from one net reproductive rate to another since individuals of the parent population age out of, and those born after the origin age into, the reproductive period.

The effect of Frauenthal's assumptions is that, for $t > 0$, the total birth rate $B(t)$ is now given by

$$B(t) = Q\int_0^\infty e^{-rx}\phi(x+t)dx + \int_0^t B(t-x)\,\frac{\phi(x)}{R}\,dx\,, \tag{4.9}$$

which, unlike the Keyfitz model, is continuous at the origin. Following the analysis performed for the Keyfitz model we obtain the asymptotic total birth rate

$$Q_2 = Q \cdot \frac{R-1}{r\kappa}\,. \tag{4.10}$$

We note that the asymptotic total birth rate and hence the asymptotic numbers obtained for the Frauenthal model is R times the results obtained using the Keyfitz model. Hence since $R > 1$ the ratio of subsequent to initial births seems to be greater than one although this has not been shown analytically. Thus there is an increase, rather than a decrease as with the Keyfitz model, from the initial to the asymptotic total birth rate. This serves to accentuate the effects of the age redistribution on the momentum of population growth demonstrated for the Keyfitz model.

Frauenthal obtains an approximation to the Keyfitz formula (4.5)

(and hence his own (4.10)) by assuming that all births occur at the mean age of childbearing. That is, he assumes that (in our notation)

$$\phi(x) = R\delta(x-\kappa) \quad ,$$

where, $\delta(u)$ is the Dirac delta function defined in (3.33). With this net maternity function, making the further assumption that $r\kappa$ is small, Frauenthal obtains

$$Q_2 \approx Q \cdot R^{\frac{1}{2}} \quad . \tag{4.11}$$

The values using Frauenthal's approximation for the Keyfitz problem are also presented in Table 4.2, and agreement is quite good (within 0.4% relative error for Q_2/Q) compared with the simplicity of the calculation of (4.11).

Frauenthal also obtains an approximation for be_o^o as

$$be_o^o \approx R \quad , \tag{4.12}$$

and notes it to be accurate within 7%. The combined effect of Frauenthal's approximations is to produce a relative error, for the asymptotic numbers, in the range of 4 to 8% for the 5 populations examined. Such approximations, in particular (4.12), are not satisfactory since results for a gradual exponential change (to be outlined presently) can differ by a comparable amount. The approximations do however illustrate the relative contribution towards the momentum of the population due to a shift to bare replacement reproductive behaviour - which was the foremost intention of Frauenthal.

4.1.2 The Asymptotic Effects of a Gradual Time Dependent Scaling to Replacement Fertility.

The momentum of population growth problem of Keyfitz will be generalised to include an exponential time dependent change from the initial age-specific birth rate to replacement level fertility. Thus the model will constitute a gradual Keyfitz change resulting in an eventual stationary population.

Let the time dependent net maternity function $\Phi(x,t)$ be separable such that

$$\Phi(x,t) = \psi(t)\phi(x) = M(x,t)\ell(x) \quad , \tag{4.13}$$

where $\psi(0) = 1$ and $\lim_{t\to\infty} \psi(t) = \frac{1}{R}$,

which implies that all age-groups have the same time dependence. It is assumed, as may be seen from (4.13), that the change occurs only in the age-specific birth rate. Hence only the total birth rate will be directly affected while the total number and age density function will be affected indirectly and will tend asymptotically to

$$P_2 = Pb\overset{o}{e}_o Q_2 \quad \text{and} \quad a^{(2)}(x) = \frac{\ell(x)}{\overset{o}{e}_o} \quad ,$$

respectively. From (4.13) and (2.28) we see that

$$R(t) = R\psi(t) \quad ,$$

which is continuous, unlike the Keyfitz model.

Hence with $\Phi(x,t)$ given by (4.13), we have the generalised renewal equation for $B(t)$, the total birth rate, as

$$B(t) = \psi(t)\left\{Q\int_0^\infty e^{-rx}\phi(x+t)dx + \int_0^t B(t-x)\phi(x)dx\right\} \quad , \tag{4.14}$$

where the population is assumed to be initially stable and growing, so that $r > 0$. In particular we will consider

$$\psi(t) = \frac{1}{R} + \left(1 - \frac{1}{R}\right)e^{-\lambda t} \quad , \quad \lambda > 0 \quad . \tag{4.15}$$

Then, instead of a discontinuity as with the Keyfitz model (4.2), we have from (4.13) and (4.15), a gradual change in the time dependent net maternity function from $\phi(x)$ at $t = 0$, decreasing exponentially to $\frac{\phi(x)}{R}$. The model (4.14)-(4.15) was first presented in Cerone and Keane (1978a). We note that, if $\lambda = 0$ in (4.15) then from (4.13) there would be no change and the population would continue with its initial parameters.

We now consider some values of λ that describe realistic changes in the net maternity function. Let the population have a doubling time τ_o, then $e^{r\tau_o} = 2$. Further let us assume that the effectiveness of the contraception method is such that the total birth rate will be within 0.1% of the stationary level in the time τ_o. Then $e^{\lambda\tau_o} \approx 2^{10}$. Thus $\lambda \approx 10r$. A reduction to only 1% of the desired total birth rate in the same time would give $\lambda \approx 7r$. One would expect such reductions to occur over one or two generations.

Figure 4.1 shows the behaviour of (4.15) for $\lambda = 0, r, 4r, 7r, 10r$ and the abrupt Keyfitz change corresponding to letting $\lambda \to \infty$ is represented by the broken line. The values of r and R are those of the 1967 Australian females given in Table 4.1.

Now, we have already obtained an analytic solution to (4.14) for $\phi(x) = \gamma e^{-\mu x}$ - the Malthusian initial net maternity function - as given by (3.2). Thus with the particular time dependence (4.15), we have

$$B(t) = \frac{Q}{R}\left[1 + (R-1)e^{-\lambda t}\right]\exp\left[\frac{r}{\lambda}(1-e^{-\lambda t})\right] , \tag{4.16}$$

where we have used the fact that $r = \gamma - \mu$ and $R = \frac{\gamma}{\mu}$. From (4.16) or directly from (3.6), the asymptotic total birth rate is given by

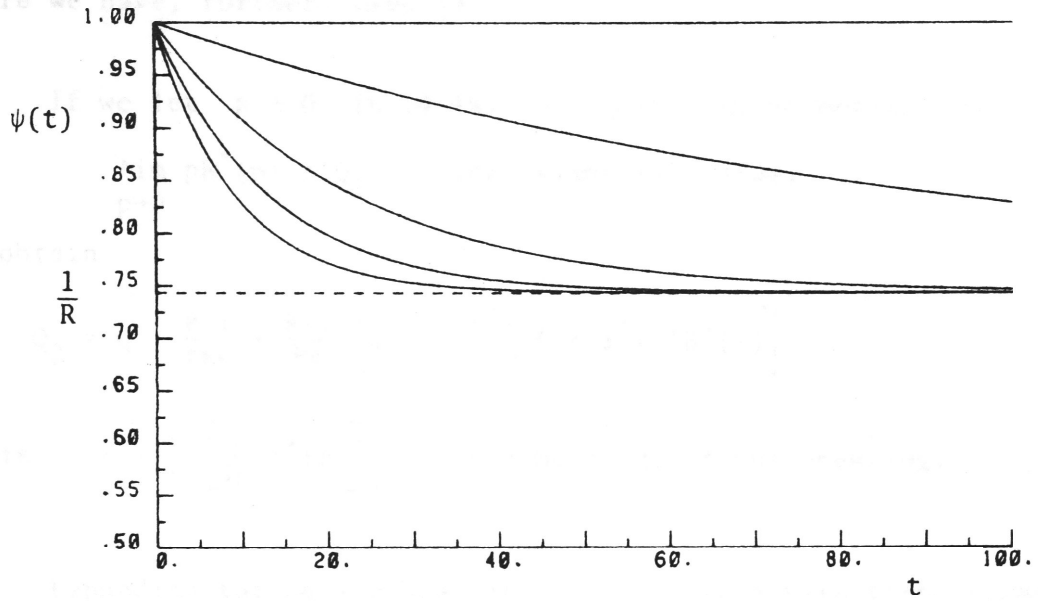

FIGURE 4.1

Diagram showing $\psi(t)$ as given by (4.15), versus time. From top to bottom the graphs represent $\psi(t)$ for $\lambda = 0, r, 4r, 7r, 10r$, and the broken line is the abrupt Keyfitz change to replacement level (corresponding to allowing $\lambda \to \infty$ in (4.15)). R and r are the values for the Australian Female's data of Table 4.1.

* * * *

$$Q_2 = \lim_{t\to\infty} B(t) = \frac{Q}{R} e^{r/\lambda}. \qquad (4.17)$$

Letting $\lambda \to \infty$ in (4.17) we obtain the asymptotic value, as given by (4.8), for the Keyfitz change and Malthusian net maternity function. We note that our gradual change to replacement level with an initial Malthusian net maternity function results in asymptotic total birth rate and numbers, $e^{r/\lambda}$ times those obtained under an abrupt Keyfitz change.

Now, to obtain the asymptotic total birth rate for the model (4.14)-(4.15) with <u>general</u> $\phi(x)$, we proceed in the following manner.

Taking Laplace transforms of (4.14), with equation (4.15), and using the results of Section [2.1], we obtain

$$B^*(p) = \frac{1}{R-\phi^*(p)} \cdot \left\{ Q \cdot \frac{1-\phi^*(p)}{p-r} + Q(R-1)\frac{1-\phi^*(p+\lambda)}{p+\lambda-r} + (R-1)\phi^*(p+\lambda)B^*(p+\lambda) \right\} , \qquad (4.18)$$

where we have, further, used (4.1).

If we let $p \to 0$ in (4.18) and recall the Tauberian result that

$$\lim_{p\to 0} pB^*(p) = Q_2 , \quad \text{the asymptotic value,}$$

we obtain

$$Q_2 = Q \cdot \frac{R-1}{rR\kappa} + \frac{R-1}{R\kappa}\left[Q \cdot \frac{1-\phi^*(\lambda)}{\lambda-r} + \phi^*(\lambda)B^*(\lambda) \right] , \qquad (4.19)$$

where $\kappa = -\frac{1}{R}\left[\frac{d}{dp}\phi^*(p)\right]_{p=0}$, the mean age of childbearing.

Expanding the square bracket in (4.19) we obtain three terms which are the contributions to the asymptotic total birth rate from an abrupt change, and, from a gradual change relating to the parent and subsequent populations respectively.

In equation (4.19) we have two unknowns, Q_2 and $B^*(\lambda)$. Although we have no specific knowledge of the form of $B(t)$, and hence $B^*(\lambda)$, we do know that

$$B(t) < Qe^{rt} ,$$

since $\psi(t) < 1$ for all $t > 0$. Hence for $\lambda > r$ we have that

$$B^*(\lambda) < \frac{Q}{\lambda-r} . \qquad (4.20)$$

Substituting (4.20) into (4.19), we have an upper bound for Q_2 given by

$$Q_2 < Q\frac{R-1}{rR\kappa} \cdot \frac{\lambda}{\lambda-r} , \quad \lambda > 0 . \qquad (4.21)$$

We note that for $\lambda = 10r$ and $\lambda = 7r$, likely values as discussed earlier, the asymptotic value Q_2 under a gradual exponential change

can differ at most by $\frac{1}{9}$ and $\frac{1}{7}$ (11.11% and 14.28%) relative error respectively from that obtained as a result of the abrupt Keyfitz change. Further, a lower bound is, of course, given by (4.5), the value obtained by Keyfitz under the assumption of an instantaneous change in the age-specific birth rate at $t = 0$.

Calculation of Q_2 .

It would be possible to find the asymptotic total birth rate Q_2, from (4.19), if we knew $B^*(\lambda)$. To this end let $p = n\lambda$, $n \neq 0$ in equation (4.18), thus obtaining the following backward recurrence relation

$$B^*(n\lambda) = \delta_n + \varepsilon_n B^*\big((n+1)\lambda\big) \quad , \tag{4.22}$$

where,

$$\delta_n = \frac{Q}{R-\phi^*(n\lambda)} \left[\frac{1-\phi^*(n\lambda)}{n\lambda-r} + (R-1) \cdot \frac{1-\phi^*\big((n+1)\lambda\big)}{(n+1)\lambda-r}\right] ,$$

and

$$\varepsilon_n = (R-1) \frac{\phi^*\big((n+1)\lambda\big)}{R-\phi^*(n\lambda)} .$$

Now, the error $E_N^{(N)}$ in $B^*(N\lambda)$ produced from assuming that $B^*(N\lambda) = 0$, with $\lambda > r$, is such that

$$E_N^{(N)} < \frac{Q}{N\lambda-r} ,$$

and from equation (4.22) the resulting error $E_N^{(n)}$ in $B^*(n\lambda)$ is given by

$$E_N^{(n)} = E_N^{(N)} \prod_{i=n}^{N-1} \varepsilon_i < \frac{Q}{N\lambda-r} \prod_{i=n}^{N-1} \varepsilon_i \quad . \tag{4.23}$$

However, $\phi^*(q\lambda)$, $q > 0$, decreases as q increases, so that $E_N^{(\ell)} < E_N^{(m)}$ for $\ell < m$. Hence, $E_N^{(1)}$, the error in $B^*(\lambda)$ when we assume $B^*(N\lambda) = 0$, can be made as small as we wish by taking N

large enough. In fact, $E_N^{(1)}$ decreases so rapidly as N increases that small values of N lead to very accurate results for $B^*(\lambda)$ and, hence, Q_2 .

The absolute error e_N in Q_2 produced by assuming that $B^*(N\lambda) = 0$ can be seen, on using (4.19) and (4.23) (with n = 1), to satisfy the inequality

$$e_N < (R-1)\frac{\phi^*(\lambda)}{R\kappa} E_N^{(1)} = \frac{Q}{N\lambda - r}(R-1)\frac{\phi^*(\lambda)}{R\kappa}\prod_{i=1}^{N-1}\varepsilon_i = b_N . \qquad (4.24)$$

An <u>algorithm</u> to calculate $B^*(\lambda)$ and hence Q_2 , follows.

Calculate

$$\tilde{Q}_2 = Q\frac{R-1}{R\kappa}\left[\frac{1}{r} + \frac{1-\phi^*(\lambda)}{\lambda - r}\right] . \qquad (4.25)$$

Evaluate the upper bound, b_N , on the absolute error e_N , as given by (4.24) for $N = 2,3,\ldots,$ until b_N is considered small enough. Note that the magnitude of $\tilde{Q}_2$ must be taken into consideration in order to obtain a bound on the relative error.

Then, with $B^*(N\lambda) = 0$ for some N found above, we can calculate from (4.22)

$$B^*(n\lambda) \quad \text{for} \quad n = N-1,\ldots,2,1 ,$$

and hence, from (4.19),

$$Q_2 = \tilde{Q}_2 + \frac{R-1}{R\kappa}\phi^*(\lambda)B^*(\lambda) , \qquad (4.26)$$

with an upper bound on the absolute error b_N given by (4.24).

We note that $\tilde{Q}_2$ is the approximation to Q_2 in assuming $B^*(\lambda) = 0$, with absolute error

$$e_1 < (R-1)\frac{\phi^*(\lambda)}{R\kappa}\cdot\frac{Q}{\lambda - r} = b_1 ,$$

where $\lambda > r$.

For $0 < \lambda \leq r$, N must be chosen so that $N\lambda > r$ in order that the bound, on the absolute error, b_N, as given by (4.24), is defined and decreases with increasing N.

If $r < 0$, so that $0 < R < 1$, then we have from (4.14), and (4.15), a gradual change in the time dependent net maternity function from $\phi(x)$ at $t = 0$, increasing exponentially at a rate $\lambda > 0$, to $\frac{\phi(x)}{R}$. The above algorithm cannot be used since we do not have an upper bound on $B(t)$ and hence on $B^*(\lambda)$. We can however take successive approximations to Q_2 and compare, stopping when the relative error is considered small enough.

In order to observe the operation of the numerical method it is instructive to consider a number of examples of $\phi(x)$ for which we have already obtained analytic results.

Firstly consider $\phi(x) = \gamma e^{-\mu x}$ for which we have already obtained the asymptotic value analytically as given by (4.17). With $\gamma = 2\mu$, $\mu = 1$ and $\lambda = 10r$ we have the analytic asymptotic total birth rate $Q_2 = 0.552585Q$, which is to be compared with numerical values of Q_2 given in Table 4.3. It is obvious from the table that six decimal place accuracy is obtained from assuming that $B^*(4\lambda) = 0$ in the algorithm.

N	Q_2/Q	b_N
1	0.545455	1.010×10^{-2}
2	0.552381	2.506×10^{-4}
3	0.552581	5.562×10^{-6}
4	0.552585	1.042×10^{-7}
5	0.552585	1.667×10^{-9}

TABLE 4.3

Approximations to the asymptotic value Q_2 of (4.14)-(4.15) where, $\phi(x) = \gamma e^{-\mu x}$, $\gamma = 2\mu$, $\mu = 1$, $\lambda = 10r$. b_N is the bound on the error, (4.24), from assuming $B^*(N\lambda) = 0$.

* * * *

It is also of interest to evaluate the upper bound given by equation (4.21) as $Q_2 < 0{\cdot}5\dot{Q}$ and the value $Q_2 = 0{\cdot}5Q$, given by (4.8), resulting from the abrupt Keyfitz change. There is a 10·5% relative difference between the Keyfitz value for Q_2 and that resulting from a gradual change at a rate $\lambda = 10r$.

From (4.16) - the analytic solution for the problem with a Malthusian $\phi(x)$ - if we evaluate $B^*(\lambda)$ and substitute into (4.19) we correctly obtain, after some algebra, the asymptotic value (4.17) which was obtained directly from the solution.

Secondly we have already obtained in Subsection [3.2.1] the average asymptotic total birth rate for the problem (4.14)-(4.15), with

$$\phi(x) = R\delta(x-\kappa) ,$$

as given by (3.42). With $R = 2$, $\kappa = 27$, $r = \frac{1}{\kappa} \ell n\, R$ and $\lambda = 10r$ we have, from (3.42), $Q_2 = 0.801456Q$, which is to be compared with numerical values of Q_2, (obtained from using the algorithm) given in Table 4.4.

N	Q_2/Q	b_N
1	0.801341	1.565×10^{-4}
2	0.801456	7.079×10^{-11}

TABLE 4.4

Approximations to the asymptotic value Q_2 of (4.14)-(4.15) where, $\phi(x) = R\delta(x-\kappa)$, $R = 2$, $\kappa = 27$, $r = \frac{1}{\kappa} \ell n\, R$, $\lambda = 10r$. b_N is the bound on the error, (4.24), from assuming $B^*(N\lambda) = 0$.

* * * *

The upper bound from (4.21), and the asymptotic value, as given in (4.8), resulting from an abrupt change to replacement fertility, are given by 0.801497Q and 0.721347Q respectively. Hence there is a relative difference of 11·1% between, the asymptotic total birth rate resulting from an abrupt Keyfitz change to that due to a gradual change

at a rate $\lambda = 10r$.

It may be seen from (4.22) and (4.24) that, the speed of convergence of the above numerical method, depends mainly on how fast $\phi^*(n\lambda)$ decreases for increasing n . For the Malthusian $\phi(x)$, $\phi^*(n\lambda)$ decreases like $\frac{1}{n\lambda}$ while with the delta function formulation the decrease, for increasing n , is like $e^{(r-n\lambda)\kappa}$. With a more realistic representation of $\phi(x)$, that is one which is bounded and of compact support, $\phi^*(n\lambda)$ would behave like $\frac{Ae^{-n\lambda\alpha}}{n\lambda} > Ae^{-n\lambda(\alpha+1)}$ where $A = \max_{\alpha<x<\beta} \{\phi(x)\}$, and, α and β are the minimum and maximum age of childbearing, respectively. Hence we expect the convergence, to the asymptotic total birth rate Q_2 , with a realistic initial net maternity function $\phi(x)$ to be faster than that with the Malthusian function, and slower than that with the delta function since $\kappa > \alpha + 1$. We note that the magnitude of λ plays an important role in the speed of convergence of the method. The larger λ becomes, and hence the more abrupt the change, the more the error decreases and; from (4.19), the asymptotic value Q_2 , tends towards the Keyfitz value, (4.5).

In order that we may appreciate the convergence of the numerical method with a more realistic initial net maternity function we will consider mainly the data of 1967 Australian Females obtained from Keyfitz and Flieger (1971). That is the initial and final net maternity functions are as given by Figure 4.2 where interpolation has been necessary so that (4.14) could be solved numerically. In what follows, unless specifically stated otherwise, the interpolated Australian data will be used.

The convergence of the numerical method to the stationary asymptotic total birth rate Q_2 is demonstrated in Table 4.5 with varying rates of decrease, λ . For $\lambda = 10r$ and $\lambda = 7r$, which are

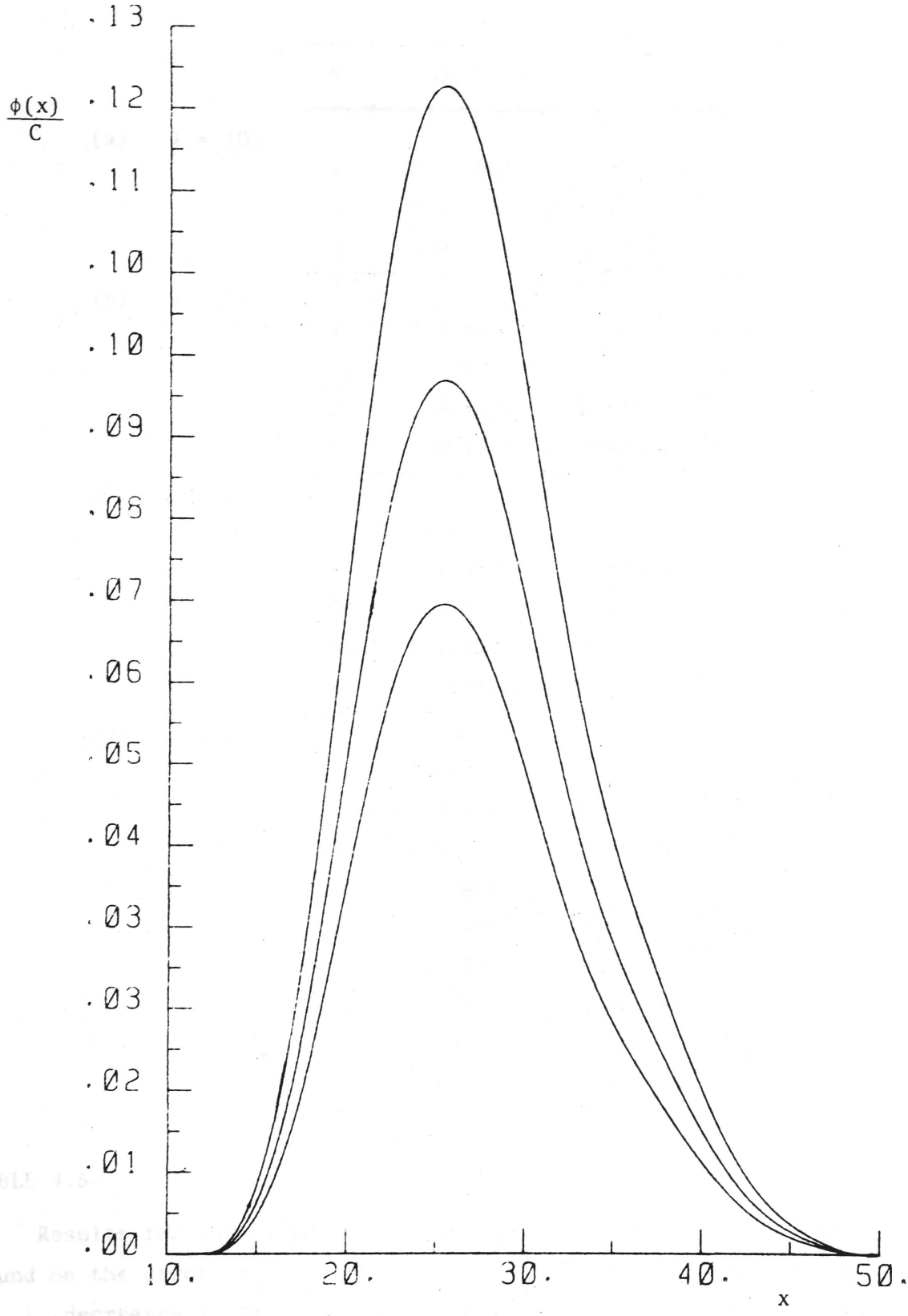

FIGURE 4.2
Diagram of $\phi(x)/C$ where $\phi(x)$ is the net maternity function of 1967 Australian females. From top to bottom, C = 0.75, 1.0, R .

	N	Q_2/Q	b_N
(a) $\lambda = 10r$			
	1	0.947003	7.836×10^{-3}
	2	0.953705	6.871×10^{-6}
	3	0.953712	8.072×10^{-11}
(b) $\lambda = 7r$			
	1	0.976359	2.622×10^{-2}
	2	0.998282	1.055×10^{-4}
	3	0.998378	9.618×10^{-8}
	4	0.998378	1.983×10^{-11}
(c) $\lambda = 4r$			
	1	1.025257	1.205×10^{-1}
	2	1.117392	2.694×10^{-3}
	3	1.119781	2.395×10^{-5}
	4	1.119804	8.338×10^{-8}
	5	1.119804	1.160×10^{-10}
(d) $\lambda = r$			
	1	1.112364	?
	2	2.063362	6.415×10^{-1}
	3	2.416502	1.034×10^{-1}
	4	2.492048	1.275×10^{-2}
	5	2.502471	1.134×10^{-3}
	6	2.503456	7.340×10^{-5}
	7	2.503522	3.499×10^{-6}
	8	2.503525	1.244×10^{-7}
	9	2.503525	3.332×10^{-9}

TABLE 4.5

Results for the asymptotic value Q_2 of (4.14)-(4.15) and the bound on the error b_N, (4.21), in taking $B^*(N\lambda) = 0$ for varying λ. As λ decreases it may be seen that the number of iterations increases to obtain the same accuracy. Also, Q_2 increases with decreasing λ.

realistic values of transition rates as discussed earlier, 10 and 7 decimal place accuracy is obtained from assuming $B^*(3\lambda) = 0$. As λ increases it should be discernable that the value obtained by assuming $B^*(\lambda) = 0$ is closest to the "true" asymptotic value. In general, as λ increases the less gradual is the change and the faster is the method. Note that for $\lambda = r$ we do not have an upper bound, on the error, b_1 caused by taking $B^*(\lambda) = 0$ since, as stated earlier, we need $N\lambda$ to be greater than r in order that an upper bound on $B^*(N\lambda)$ exist. The upper bounds on Q_2 , as given by (4.21), for $\lambda = 10r$, $7r$ and $4r$ are 0.954839Q, 1.002581Q and 1.145807Q respectively. The asymptotic values of Table 4.5 are compared, in Table 4.6, with the asymptotic total birth rates obtained as a result of an abrupt Keyfitz change. The asymptotic total number, P_2 , are also given for the various λ values.

λ	Q_2/Q	P_2/P	Relative % Difference From the Keyfitz Values
∞	0.859355	1.25505	0
10r	0.953712	1.39286	10.98
7r	0.998378	1.45809	16.18
4r	1.119800	1.63543	30.31
r	2.503525	3.65630	191.54

TABLE 4.6

Asymptotic total birth rates Q_2 for the model (4.14)-(4.15). The asymptotic total number P_2 is also given, and the values are compared to those of Table 4.2 resulting from an abrupt Keyfitz change ($\lambda \to \infty$).

* * * *

We note here, that, the asymptotic values, with $\lambda = 10r$, for the countries whose essential data is presented in Table 4.1, are all approximately 11% different from the values obtained in Table 4.2 resulting from the abrupt Keyfitz change.

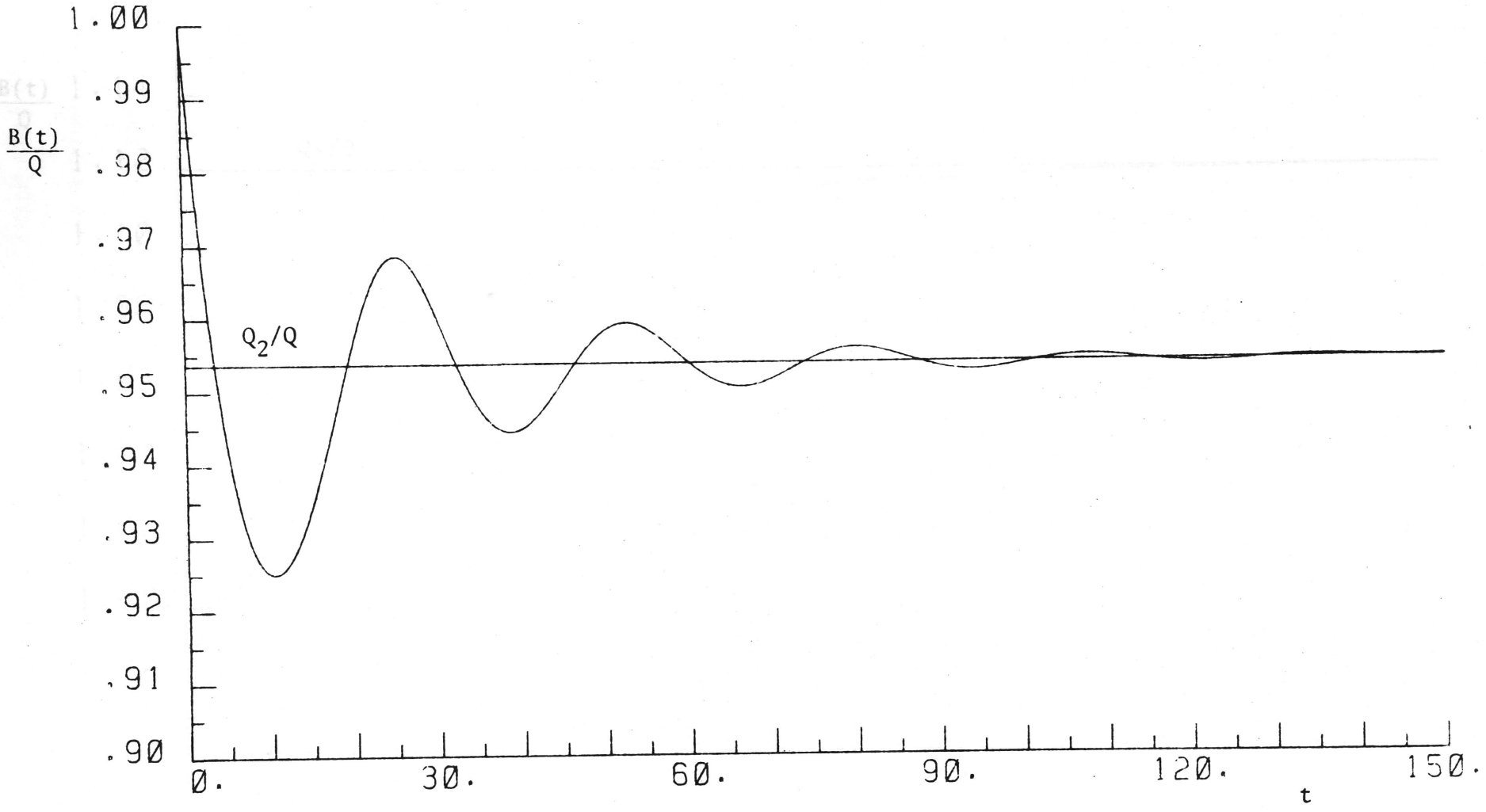

FIGURE 4.3

B(t) resulting from an exponential change of the net maternity function from $\phi(x)$ to $\phi(x)/R$ at a rate $\lambda = 10r$. $\phi(x)$ is given in Figure 4.2 with $C = 1$. The graph is that of $B(t)/Q$ versus time t. The straight line represents the asymptotic value Q_2/Q as determined using the algorithm and as given in Table 4.5.

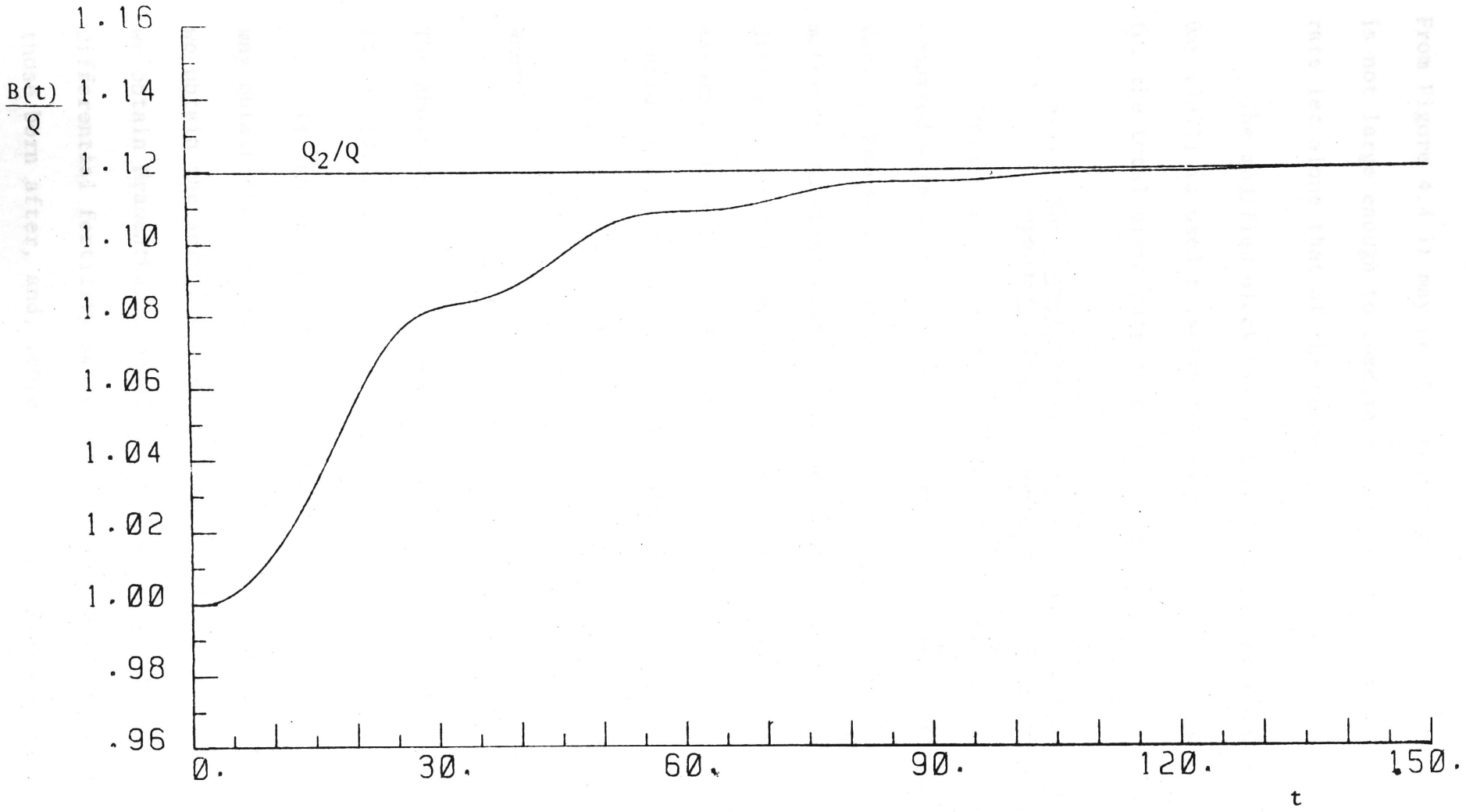

FIGURE 4.4

B(t) resulting from an exponential change of the net maternity function from $\phi(x)$ to $\phi(x)/R$ at a rate $\lambda = 4r$. $\phi(x)$ is given in Figure 4.2 with $C = 1$. B(t) approaches the asymptotic value Q_2/Q (the straight line). Q_2 is obtained using the algorithm and is given in Table 4.5.

The approach of the total birth rate to the asymptotic value is demonstrated in Figures 4.3 and 4.4 for $\lambda = 10r$ and $4r$ respectively. From Figure 4.4 it may be seen that the transition to replacement $\lambda = 4r$ is not large enough to immediately check the momentum of the total birth rate let alone that of the numbers.

The modified block-by-block method, as described by Campbell and Day (1971) is used to solve the Volterra integral equation (4.14)-(4.15) for the total birth rate given in Figures 4.3 and 4.4.

4.1.3 The Asymptotic Effects of a Differential, Gradual Time Dependent Scaling Towards Replacement Level Fertility Rates.

The potential growth, due to those already alive at the origin compared with that of those born after, will be demonstrated by taking various transition rates ν and λ. A separable time dependent net maternity function will be considered where the time dependence $\psi_k(t)$ differs between the parent and subsequent populations. That is, assuming an initially stable population then, the model to be considered for the total birth rate $B(t)$ is

$$B(t) = Q\psi_\nu(t)\int_0^\infty e^{-rx}\phi(x+t)dx + \psi_\lambda(t)\int_0^t B(t-x)\phi(x)dx \ , \qquad (4.27)$$

where,

$$\psi_k(t) = \frac{1}{R} + \left(1 - \frac{1}{R}\right)e^{-kt} \quad , \quad k > 0 \ . \qquad (4.28)$$

The above model has already been examined for $\nu = \lambda$ and is given by (4.14)-(4.15).

It can be seen that by taking various values of ν and λ we may obtain the previous models. For example if we let ν and $\lambda \to \infty$ we obtain the abrupt Keyfitz model. If $\nu = 0$ and we let $\lambda \to \infty$ then we obtain Frauenthal's model. The model (4.27) with (4.28) permits differential fertility schedules for those born before the origin to those born after, and, represents a single exponential time dependent generalisation, to its fullest extent, of Frauenthal's model.

From (4.28) we can see that $k = 0$ represents no change in the original net maternity function, while taking the limit as $k \to \infty$ gives an abrupt change from $\phi(x)$ to $\phi(x)/R$. For $0 < k < \infty$ there is a gradual change to replacement level fertility.

The asymptotic behaviour of (4.27)-(4.28) may be ascertained by again using Laplace transform techniques. If $\lambda = 0$ then there is no change in the net maternity function of the new-born population and hence the population will continue to grow at the same initial exponential rate r with the initial stable equivalent births only changing if ν is <u>not</u> also zero.

Taking Laplace transforms of (4.27) and using (4.28) we obtain upon isolating $B^*(p)$

$$\left[1 - \frac{\phi^*(p)}{R}\right]B^*(p) = \frac{Q}{R}\frac{1-\phi^*(P)}{p-r} + Q\frac{(R-1)}{R}\frac{1-\phi^*(p+\nu)}{p+\nu-r} + \frac{R-1}{R}\phi^*(p+\lambda)B^*(p+\lambda) \quad . \tag{4.29}$$

With $\lambda \neq 0$, the real root of the characteristic equation occurs at $p = 0$ so that we have the asymptotic value Q_2, using the Tauberian result, being given by

$$Q_2 = Q\frac{R-1}{r\kappa R} + \frac{R-1}{R\kappa}\left[Q \cdot \frac{1-\phi^*(\nu)}{\nu-r} + \phi^*(\lambda)B^*(\lambda)\right], \tag{4.30}$$

where $\kappa = -\frac{1}{R}\left[\frac{d}{dp}\phi^*(p)\right]_{p=0}$, the mean age of childbearing. We note that (4.30) is exactly the same as (4.19) with λ replaced by ν for the terms arising from the parent population. Table 4.7 shows the asymptotic total birth rate, given by (4.30), for λ and ν taking in turn the values $10r$, $7r$ and $4r$. The values for $\lambda = \nu$, presented also in Table 4.5, have been included for completeness. $B^*(\lambda)$ in (4.30) is obtained by putting $p = n\lambda$ in (4.29) and thus setting up a recurrence relation as in the previous section.

ν/r	λ/r	Q_2/Q
10	10	0.953712
	7	0.968411
	4	1.034882
7	10	0.983249
	7	0.998378
	4	1.066869
4	10	1.032397
	7	1.048167
	4	1.119804

TABLE 4.7

Values of Q_2 as a result of the generalised time dependent counterpart of the differential fertility model of Frauenthal, as given by (4.27)-(4.28). ν and λ are transition rates belonging to the parent and subsequent populations respectively. The data used is that of the Australian females of 1967.

* * * *

Perhaps the most realistic model as represented by (4.27)-(4.28) would be a gradual change, with the change for the initial population being slower than that for the subsequent population. Therefore, the model with $\nu = 7r$ and $\lambda = 10r$ would not be unreasonable. The extreme situation representing a realistic model would be a gradual change for those born before the origin with an abrupt change for the others, since, those born after the origin would have had α (the minimum age of childbearing) years to become accustomed to the new regime of fertility. The asymptotic total birth rate Q_2, for this latter situation, is obtained, from (4.30) by taking the limit as $\lambda \to \infty$, giving

$$\frac{Q_2}{Q} = \frac{R-1}{rR\kappa} + \frac{R-1}{R\kappa} \cdot \frac{1-\phi^*(\nu)}{\nu-r} \quad ,$$

where the first term is that obtained by Keyfitz under an abrupt change to bare replacement. We note that the above expression for Q_2 is the

same as (4.25) and hence the asymptotic values obtained are the values for the first approximations in Table 4.5. The contribution from the gradual change at a rate ν, the second term in the above expression, can be determined by comparing the first values of Table 4.5 with the asymptotic values, for the Australian females, obtained under an abrupt change to bare replacement (Table 4.2). It can be seen, after a small calculation, that the contribution to Q_2, from the gradual component, is 10·2% and 29·4% for $\nu = 10r$ and $\nu = r$ respectively.

For extreme ($\lambda = 0$ and $\lambda \to \infty$) values of λ, when $\psi_\lambda(t)$ does not explicitly involve time, the convolution in (4.27) is not violated and hence there is no need to use the algorithm described in [4.1.2]. Further generalisations capitalising on this fact will be presented in Chapter 5.

4.1.4 The Asymptotic Effects of a Gradual Time Dependent General Scaling of the Age-Specific Birth Rate.

Tognetti (1976b) extended the Keyfitz (1971b) momentum problem by considering an abrupt constant scaling of the age-specific birth rate so that the population would eventually become stable rather than stationary. Keyfitz (1975) examined the model where the age-specific birth rate is abruptly scaled down and thus resulting in a population with a lower intrinsic rate of change.

By allowing the age-specific birth rate to change abruptly from $m(x)$ to $m(x)/C$, $C = \frac{R}{R_2}$ Tognetti (1976b) obtains, assuming an initially stable population,

$$Q_2 = Q \cdot \frac{C-1}{(r-r_2)C\kappa} = Q \cdot \frac{R-R_2}{(r-r_2)R\kappa} \qquad (4.31)$$

where,

$\kappa = \int_0^\infty e^{-r_2 x} \times \frac{\phi(x)}{C}\, dx$, the expected age of giving birth for the stable population after the change.

Then, the total birth rate will asymptotically approach $B(t) = Q_2 e^{r_2 t}$.

To allow for a gradual time dependent change we will take

$$\psi(t) = \frac{1}{C} + \left(1 - \frac{1}{C}\right) e^{-\lambda t} \quad , \quad \lambda > 0 \quad , \tag{4.32}$$

in (4.14) and thus the stable equivalent births Q_2 may be obtained in a fashion similar to that of Subsection [4.1.2]. The Laplace transform of (4.14), with (4.32), yields on rearrangement

$$[C-\phi^*(p)]B^*(p) = Q\,\frac{1-\phi^*(p)}{p-r} + Q(C-1)\,\frac{1-\phi^*(p+\lambda)}{p+\lambda-r} + (C-1)\phi^*(p+\lambda)B^*(p+\lambda). \tag{4.33}$$

Hence with the real root of $\phi^*(p) = C$, r_2 , having the greatest real part we have, from (4.33), using the Tauberian result, that

$$\begin{aligned} Q_2 &= \lim_{p \to r_2} (p-r_2)B^*(p) \quad , \\ &= Q\,\frac{C-1}{(r-r_2)C\kappa} + \frac{C-1}{C\kappa}\left[Q\,\frac{1-\phi^*(r_2+\lambda)}{\lambda+r_2-r} + \phi^*(r_2+\lambda)B^*(r_2+\lambda)\right] , \end{aligned} \tag{4.34}$$

where

$$\kappa = -\frac{1}{C}\left[\frac{d}{dp}\,\phi^*(p)\right]_{p=r_2} ,$$

and Q_2 is such that the total birth rate will asymptotically approach $B(t) = Q_2 e^{r_2 t}$.

It would be possible to evaluate Q_2 from (4.34), and hence the asymptotic behaviour of the resulting population could be determined if, $B^*(r_2+\lambda)$ were known. In order to obtain $B^*(r_2+\lambda)$ we proceed as in Subsection [4.1.2] and set up a recurrence relation from (4.33). Thus putting $p = r_2 + n\lambda$ into (4.33) results in a recurrence relation from which we may obtain approximations to the stable equivalent births Q_2 by determining approximations to $B^*(r_2+\lambda)$, obtained from assuming

$B^*(r_2+N\lambda) = 0$ for successive values of integer N. If $r_2 = 0$ we have $C = R$ and thus we obtain the model treated in Subsection [4.1.2].

If $0 < C < 1$ then $\psi(t)$, as given by (4.32), is a monotonically increasing function of time approaching $\frac{1}{C}$. Thus the error analysis, and hence the algorithm of Subsection [4.1.2], cannot be used since we do not have an obvious upper bound on $B(t)$. Hence, comparison of successive approximations of Q_2 will have to be made stopping when the relative error is considered small enough. We note that the present model need no longer be one of contraception depending on C relative to R since if $C < R$ we have $r_2 > r$.

The effect of C on the stable equivalent births Q_2 is shown in Table 4.8 with the intrinsic rates of change and the net reproduction rates of $\phi(x)/C$ given in Table 4.7. For $C = 0.75$ we do not have an upper bound on the absolute error since $\psi(t)$, as given by (4.32), is now monotonically increasing. The approach to Q_2 for $C = 0.75$ is demonstrated in Figure 4.4 with $\lambda = 10r$.

C	r	R
1.0	1.09537×10^{-2}	1.3451

C	r_2	R_2
1.50	-3.98839×10^{-3}	0.89674
1.25	2.69517×10^{-2}	1.07609
R	0.0	1.0
0.75	2.17295×10^{-2}	1.79348

TABLE 4.7

Intrinsic rates of change r and net reproduction rates R of the net maternity functions $\phi(x)/C$ for varying C. $\phi(x)$ is the net maternity function for the Australian females (1967).

* * * *

	N	Q_2/Q	b_N
C = 1.50			
	1	0.931268	1.115×10^{-2}
	2	0.940661	1.384×10^{-5}
	3	0.940673	2.287×10^{-9}
C = 1.25			
	1	0.958691	5.746×10^{-3}
	2	0.963661	3.707×10^{-6}
	3	0.963664	3.217×10^{-10}

	N	Q_2/Q	RELATIVE ERROR
C = 0.75			
	1	1.067154	—
	2	1.061161	5.648×10^{-3}
	3	1.061165	4.240×10^{-6}

TABLE 4.8

Approach to the stable equivalent births Q_2 of (4.14) with (4.32) for varying C, at a rate $\lambda = 10r$. Approximations to Q_2 in taking $B^*(r_2+N\lambda) = 0$ with bound on the error, b_N , are presented. There is no known bound on the error for $C < 1$; hence the relative error is computed. The total birth rate will asymptotically behave like

$$B(t) = Q_2 e^{r_2 t} .$$

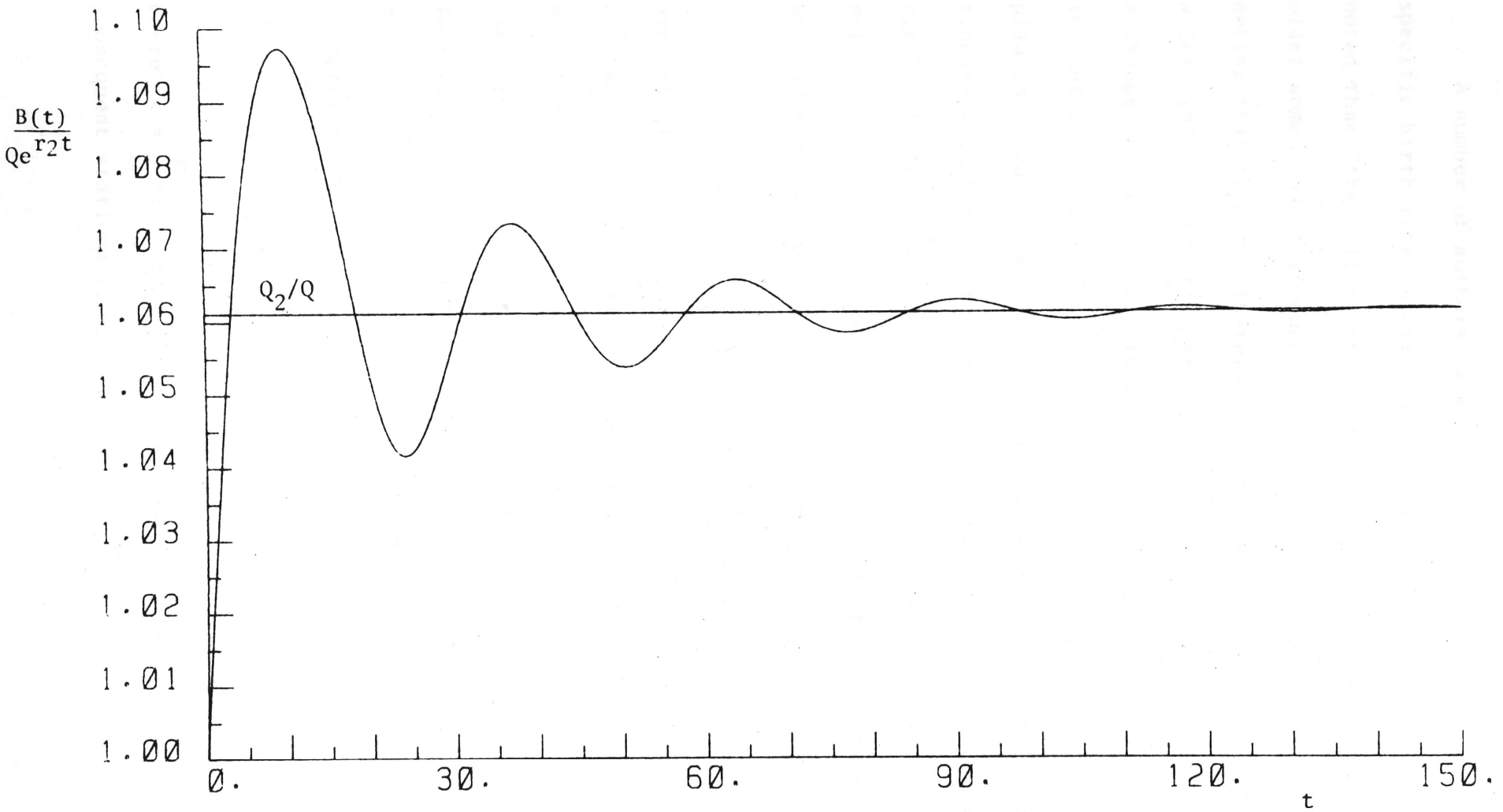

FIGURE 4.5

Scaled total birth rate resulting from an exponential change of the net maternity function from $\phi(x)$ to $\phi(x)/C$, $C = 0.75$, at a rate $\lambda = 10r$. The diagram shows the approach of $e^{-r_2 t}\frac{B(t)}{Q}$ towards the asymptotic value Q_2/Q given in Table 4.8. $B(t)$ is the solution to (4.14) with (4.32).

4.2 A Non-Separable Time Dependent Net Maternity Function and Its Effect on the Momentum of Population Growth.

A number of authors have stated that a uniform scaling of the age-specific birth rate is difficult to rationalize. Keyfitz (1971b) himself noted that "the fall in the birth rate is likely to be more rapid for older women than for younger", later reiterating this in Keyfitz (1975), noting that the drop is related to the method of contraception used. Ryder (1975) indirectly obtains a gradual change to replacement including a change in the age structure of the net maternity function, by allowing the intrinsic rate of natural increase to change. Ryder notes "it seems plausible that reductions in fertility will tend to appear disproportionately in births of higher order, which generally occur at higher ages". Potter et al. (1977) analyse the impact of certain sterilisation policies on the momentum which affects certain older age-groups and postulate a gradual coverage of the population.

To allow women of older ages to be more affected by a fall in the age-specific birth rates, Mitra (1976) abruptly changes $m_1(x)$ to $e^{-r_1 x} m_1(x)$ so that the population would eventually become stationary. Mitra states as a "final observation", what I consider to be the more important result (although suffering from lack of analytic investigation), that we can change from <u>any</u> age-specific birth rate $m_1(x)$ to any other $m_2(x)$. Thus with $\phi_2^*(0) = 1$, in our notation, Mitra obtains, the asymptotic total birth rate

$$Q_2 = \frac{Q_1}{r_1 \kappa} [1-\phi_2^*(r_1)] \quad , \tag{4.35}$$

where $\kappa = \int_0^\infty x\phi_2(x)dx$, the average age of childbearing in the subsequent stationary population.

Again, it is unrealistic that such a change can occur abruptly. Thus in Cerone and Keane (1978b) the non-separable time dependent net maternity functions

$$\Phi(x,t) = \phi_2(x) + e^{-\lambda t}[\phi_1(x)-\phi_2(x)] , \quad \lambda > 0 , \tag{4.36}$$

was presented, incorporating both a change with time and age. As discussed in Chapter 2, $\phi_1(x)$ and $\phi_2(x)$ are the initial and the eventual net maternity functions, respectively.

It should be noted that a subscript of 1 refers to population parameters of the initial population while a subscript of 2 refers to the ultimate value of a parameter after the change.

We will assume that the initial and final net maternity functions to be such that

$$\phi_i^*(r_i) = 1 \quad \text{and} \quad \phi_i^*(0) = R_i , \quad i = 1,2, , \tag{4.37}$$

and hence the population will, in general, eventually become stable with the stationary state being a special case ($r_2 = 0$ and $R_2 = 1$). Thus from (2.28), (4.36) and (4.37) we have the time dependent net reproduction rate

$$R(t) = R_2 + e^{-\lambda t}[R_1-R_2] .$$

Inserting (4.36) into the generalised renewal equation

$$B(t) = Q_1 e^{r_1 t} \int_t^\infty e^{-r_1 x}\Phi(x,t)dx + \int_0^t B(t-x)\Phi(x,t)dx , \tag{4.38}$$

where $B(t)$ is the total birth rate, gives the model to be solved. We have further assumed, for simplicity, that the population is initially stable.

Now, to obtain the asymptotic behaviour of (4.38) with (4.36) we proceed in a similar manner as in Section [4.2]. Taking Laplace

transforms of (4.38) and using (4.36) gives on rearrangement

$$[1-\phi_2^*(p)]B^*(p) = Q_1\left[\frac{\phi_2^*(r_1)-\phi_2^*(p)}{p-r_1} + \frac{1-\phi_1^*(p+\lambda)}{p+\lambda-r_1} - \frac{\phi_2^*(r_1)-\phi_2^*(p+\lambda)}{p+\lambda-r_1}\right]$$

$$+ B^*(p+\lambda)[\phi_1^*(p+\lambda) - \phi_2^*(p+\lambda)] \quad . \qquad (4.39)$$

We obtain the asymptotic behaviour by letting $p \to r_2$, in (4.39) where r_2 is given by $(4.37)_1$, and using the Tauberian result

$\lim_{p\to r_2} (p-r_2)B^*(p) = Q_2$, the stable equivalent births.

Hence on using L'Hopital's rule

$$\lim_{p\to r_2} \frac{1-\phi_2^*(p)}{p-r_2} = \int_0^\infty e^{-r_2 x} x\phi_2(x)dx = \kappa \ , \qquad (4.40)$$

the average age of childbearing, we have

$$\kappa Q_2 = Q_1\left[\frac{\phi_2^*(r_1)-1}{r_2-r_1} + \frac{1-\phi_1^*(r_2+\lambda)}{\lambda+r_2-r_1} - \frac{\phi_2^*(r_1)-\phi_2^*(r_2+\lambda)}{\lambda+r_2-r_1}\right]$$

$$+ B^*(r_2+\lambda)[\phi_1^*(r_2+\lambda) - \phi_2^*(r_2+\lambda)] \ , \qquad (4.41)$$

where the ultimate total birth rate will be of the form $B(t) = Q_2 e^{r_2 t}$.

If we let $\lambda \to \infty$ in (4.41) so that, using (4.36), the change in the net maternity function occurs instantaneously at $t = 0$, we obtain a generalisation of equation (4.35) as

$$Q_2 = \frac{Q_1}{(r_1-r_2)\kappa} \cdot [1-\phi_2^*(r_1)] \ ,$$

which is now the stable equivalent births rather than the asymptotic stationary value.

In order to find the stable equivalent births Q_2 from (4.41), we need to know $B^*(r_2+\lambda)$. $B^*(r_2+\lambda)$ can be found by using the technique developed in Section [4.1], with the separable time dependent net maternity function, which involves setting up a backward

recurrence relation by putting $p = r_2 + n\lambda$ in (4.39).

The error analysis and hence the algorithm developed in Subsection [4.1.2] may not readily be used, since, a meaningful upper bound on $B(t)$ is not at all obvious for the non-separable model. If, for example, $\phi_2(x) < \phi_1(x)$ for all values of x, then (4.36) is a monotonically decreasing function of time, t. Hence, now, $B(t) < Q_1 e^{r_1 t}$ and thus the algorithm of [4.1.2] may be used. In general, however, comparison of successive approximations of Q_2 will have to be made, stopping when the relative error is considered small enough.

Consider a numerical example where $\phi_1(x)$ and $\phi_2(x)$ are as given by Figure 4.6. The convergence of the numerical method, for $\lambda = 10r$, to the stable equivalent births is demonstrated in Table 4.9. The approach of the scaled total birth rate to Q_2/Q_1 is shown in Figure 4.7.

* * * *

$r_1 = 1.095374\times10^{-2}$, $R_1 = 1.345111$
$r_2 = 1.210951\times10^{-2}$, $R_2 = 1.372250$

N	Q_2/Q_1	RELATIVE ERROR
1	1.006630	—
2	1.005075	1.547×10^{-3}
3	1.005075	1.580×10^{-7}

TABLE 4.9

Approach to the stable equivalent births Q_2 for the model (4.38) with (4.36) at transition rate $\lambda = 10r$. Approximations, to Q_2, are given from assuming $B^*(r_2+N\lambda) = 0$. The time dependent net maternity function varies exponentially from $\phi_1(x)$ to $\phi_2(x)$ which are given in Figure 4.6. The total birth rate will asymptotically behave like

$$B(t) = Q_2 e^{r_2 t} .$$

* * * *

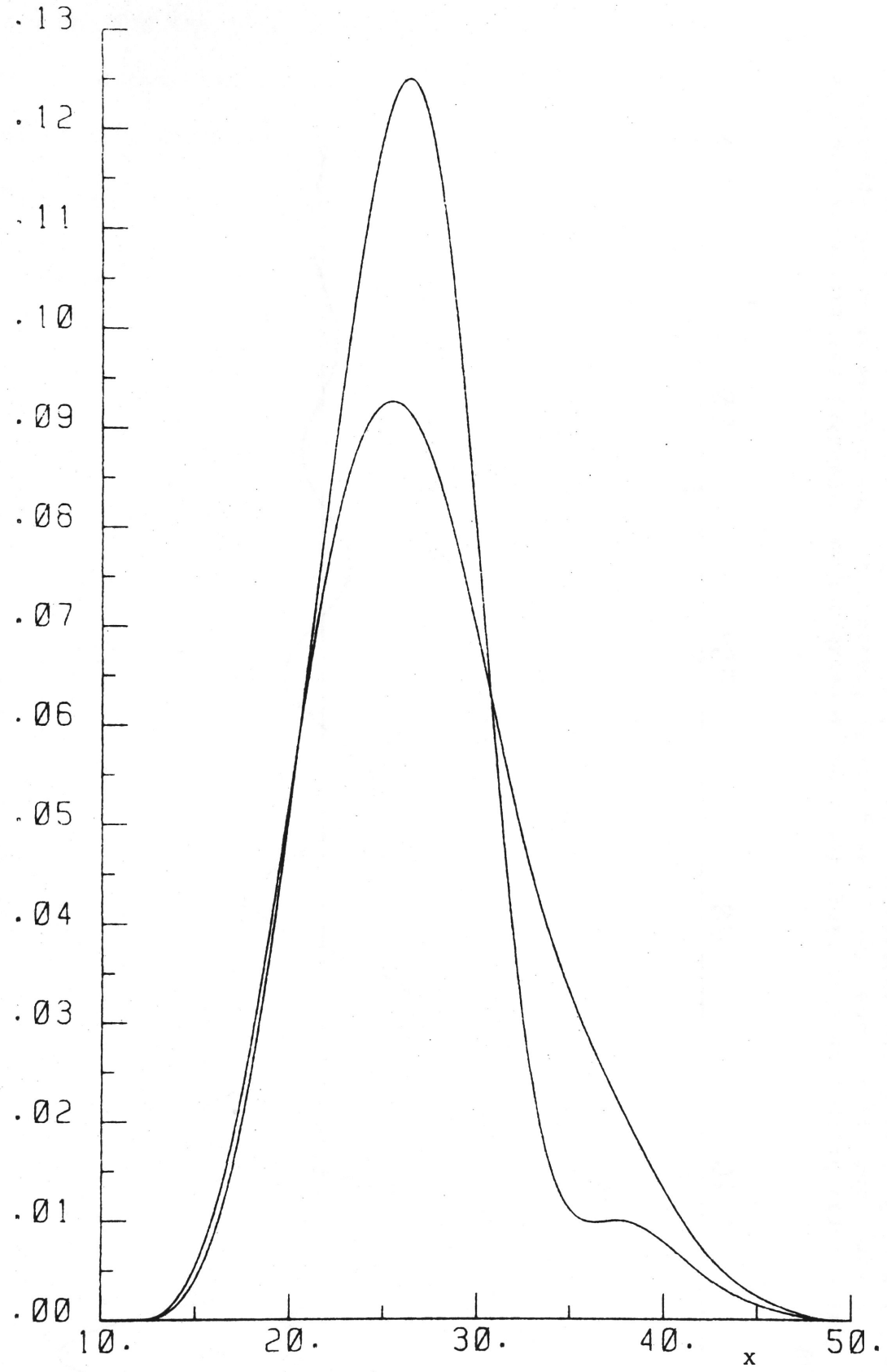

FIGURE 4.6

Extreme net maternity functions $\phi_1(x)$, $\phi_2(x)$ where $\phi_1(x)$ is the net maternity function of 1967 Australian females. The graph of $\phi_1(x)$ is that with the lower peak.

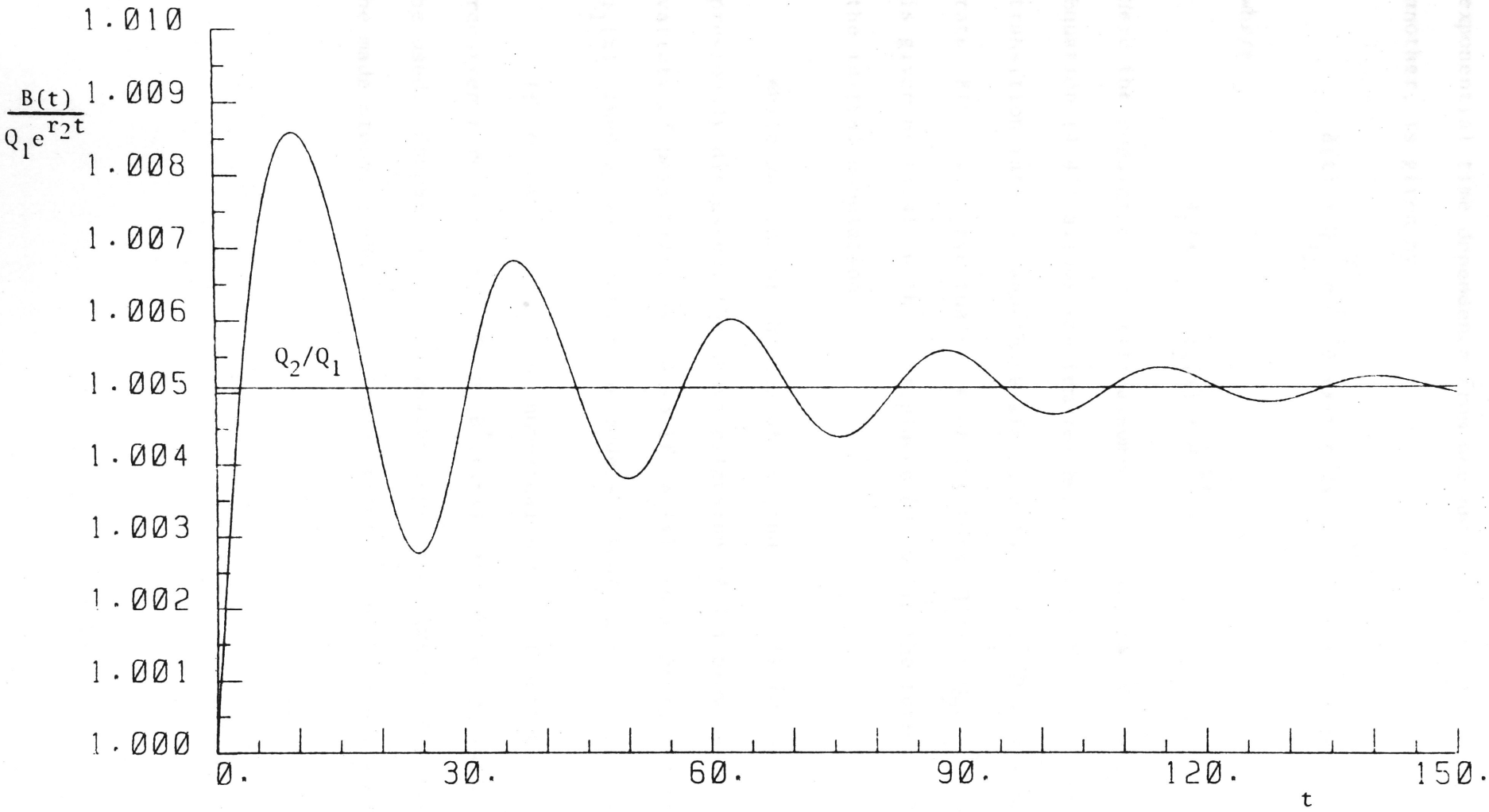

FIGURE 4.7

Scaled total birth rate resulting from a non-separable time dependent net maternity function with $\lambda = 10r_1$ and $\phi_1(x)$ and $\phi_2(x)$ as given by Figure 4.6. The diagram shows the approach of $e^{-r_2 t}\, B(t)/Q_1$ towards the asymptotic value Q_2/Q_1 given in Table 4.9. $B(t)$ is the solution of (4.38) with (4.36).

A model for the total birth rate $B(t)$; which allows for differential rates between the parent and subsequent population, with exponential time dependence from one net maternity function towards another, is given by

$$B(t) = Q_1\int_0^\infty e^{-r_1 x}\Phi_\nu(x+t,t)dx + \int_0^t B(t-x)\Phi_\lambda(x,t)dx \quad , \qquad (4.42)$$

where

$$\Phi_k(x,t) = \phi_2(x) + e^{-kt}[\phi_1(x)-\phi_2(x)] \ , \ k > 0. \qquad (4.43)$$

Here the population is still assumed to be initially stable. Equation (4.42) allows the females born before the origin to adopt a transition rate ν and those after, a rate λ . Thus the total birth rate $B(t)$ will eventually be of the form $B(t) = Q_2 e^{r_2 t}$, where Q_2 is given by (4.41) with λ replaced by ν in the terms arising from the initial population.

While particular choices of ν and λ will reproduce the models previously discussed, the above extension (4.42) provides for a further variety of possibilities. Also if $\phi_2(x)$ is a constant multiple of $\phi_1(x)$ then we will obtain the models discussed in Section [4.1].

If $0 < \lambda < \infty$ then the numerical method of setting up a recurrence relation to obtain $B^*(r_2+\lambda)$, and hence Q_2, will have to be used. Comparison of successive approximations to Q_2 will have to be made stopping when the relative error is deemed small enough.

4.3 The Transition Rate Needed to Approach a Given Asymptotic Behaviour - The Converse Problem.

In Sections [4.1] and [4.2] we have been concerned with obtaining the asymptotic behaviour of a population resulting from the net maternity function, in particular the age-specific birth rate, changing with time. A question of both theoretical and practical importance asks what changes, in the age-specific birth rate, are needed in order to approach a given asymptotic behaviour? In other words we have the converse problem. Nortman and Bongaarts (1975) determine the total annual number of contraceptive acceptors required to achieve a prescribed crude birth rate target path. Although our aims here are not as ambitious, we are able to find, using the results and models of Sections [4.1] and [4.2], the transition rate λ, given the stable equivalent births Q_2.

We will assume that the age-specific birth rate changes at a rate λ according to the model (4.13) and (4.15). Then, given the asymptotic total birth rate Q_2, (4.19) results in an equation in λ, where $B^*(\lambda)$ is found in a similar manner to that indicated in Subsection [4.1.2], using the backward recurrence relation. We can find λ by using any of a number of root finding procedures such as the Newton-Raphson method and the secant method (see for example Keyfitz (1968b)). The author has used the Modified Muller method, as presented in Blatt (1975), which is both rapid and stable. The method involves giving an estimate of an upper and lower bound on λ as initial values. Care must be taken in determining whether such a λ does exist. For example, if the asymptotic total birth rate Q_2 is less than that obtained under an abrupt Keyfitz change to bare replacement, then no such λ exists.

We can also determine a transition rate λ that will asymptotically result in a certain total number P_2, in the population. For, given $P_2 = \tilde{P}_2$ then we can find a λ, from (4.18) and (4.19), with which the total birth rate, of (4.14)-(4.15), will asymptotically tend to

$$Q_2 = \frac{\tilde{P}_2}{{}^{o}_{e_o}b}$$

and hence the total number will tend to $\tilde{P}_2$.

Table 4.10 presents values of λ given Q_2 and/or P_2 for the Australian data. We note that we can obtain a λ so that $Q_2 = Q$ asymptotically but no λ exists which will result in $P_2 = P$ since, we see from Table 4.2 that under an abrupt Keyfitz change (corresponding to $\lambda \to \infty$) $P_2 > P$.

Q_2/Q	P_2/P	$\lambda \times 10^2$
1.0	1.369432	7.587028
1.027074	1.5	6.468735
1.369432	2.0	2.502898
2.054147	3.0	1.343036

TABLE 4.10

The transition rates λ are obtained for the model (4.14) with (4.15) for given asymptotic total birth rates Q_2 or equivalently for given asymptotic numbers P_2. The data of the Australian females of 1967 is used (Table 4.1).

* * * *

The converse problem can also be solved for the models of Subsection [4.1.4]and Section [4.2] in a similar manner. However, given the desired asymptotic behaviour under the differential fertility models of Subsection [4.1.5], and also of [4.2], one of the transition rates must be specified, either in terms of the other or explicitly, in order that the other may be obtained.

4.4 The Transient Solution Resulting From an Exponential Time Dependent Change of the Age-Specific Birth Rate.

In the present chapter we have been concerned with obtaining the asymptotic behaviour of a population under various time dependent fertility behaviour. We will now consider obtaining the transient behaviour of the solution resulting from the time dependent net maternity function as given by (4.36) - the non-separable model. The asymptotic behaviour was determined in Section [4.2] by obtaining the contribution from the real root of the characteristic equation

$$\phi_2^*(p) = 1 \ . \tag{4.44}$$

Using residue theory, we can develop the transient solution of (4.38) with (4.36) by obtaining the contribution from the complex roots of (4.44).

Let p_j be a complex root of (4.44) which is assumed to be simple. Then proceeding in a formal manner, using the residue theorem, letting $p \to p_j$ in (4.39) gives

$$\kappa_j Q_j = F^*(p_j) + B^*(p_j+\lambda)[\phi_1^*(p_j+\lambda) - \phi_2^*(p_j+\lambda)] \ , \tag{4.45}$$

where

$$Q_j = \lim_{p \to p_j} (p-p_j)B^*(p) \ ,$$

$F^*(p_j)$ is the contribution from the parent population,

$$\kappa_j = -\left[\frac{d}{dp}\,\phi_2^*(p)\right]_{p=p_j} = \int_0^\infty e^{-p_j x} x \phi_2(x)\,dx \ ,$$

and $B_j(t) = Q_j e^{p_j t}$ is the contribution, to the total birth rate $B(t)$, from p_j a root of (4.44). Equating real and imaginary parts of (4.45) we obtain

$$K_j \underset{\sim}{q}_j = \underset{\sim}{f}(p_j) + C(p_j+\lambda)\underset{\sim}{b}(p_j+\lambda) \ , \tag{4.46}$$

where,

$$K_j = \begin{pmatrix} Re\ \kappa_j & -Im\ \kappa_j \\ Im\ \kappa_j & Re\ \kappa_j \end{pmatrix}, \quad \underset{\sim}{q}_j = \begin{pmatrix} Re\ Q_j \\ Im\ Q_j \end{pmatrix}, \quad \underset{\sim}{f}(s) = \begin{pmatrix} Re\ F^*(s) \\ Im\ F^*(s) \end{pmatrix},$$

$$\underset{\sim}{b}(s) = \begin{pmatrix} Re\ B^*(s) \\ Im\ B^*(s) \end{pmatrix}, \quad C(s) = \begin{pmatrix} Re\ \tilde{\phi}^*(s) & -Im\ \tilde{\phi}^*(s) \\ Im\ \tilde{\phi}^*(s) & Re\ \tilde{\phi}^*(s) \end{pmatrix}$$

and $\tilde{\phi}^*(s) = \phi_1^*(s) - \phi_2^*(s)$.

We do not know $\underset{\sim}{b}(p_j+\lambda)$ which is needed to obtain $\underset{\sim}{q}_j$. Hence proceeding as we did previously in similar circumstances, we let $p = p_j + n\lambda$ in (4.39), and equating real and imaginary parts we obtain the recurrence relation

$$\underset{\sim}{b}(p_j+n\lambda) = \underset{\sim}{d}_n + E_n\ \underset{\sim}{b}\big(p_j+(n+1)\lambda\big) \quad , \tag{4.47}$$

where,

$$\underset{\sim}{d}_n = A^{-1}(p_j+n\lambda)\underset{\sim}{f}(p_j+n\lambda) \quad ,$$

$$E_n = A^{-1}(p_j+n\lambda)C\big(p_j+(n+1)\lambda\big) \quad ,$$

$$A^{-1}(s) = \frac{1}{|A|}\begin{pmatrix} 1 - Re\ \phi_2^*(s) & Im\ \phi_2^*(s) \\ -Im\ \phi_2^*(s) & 1 - Re\ \phi_2^*(s) \end{pmatrix} ,$$

and $|A| = [1 - Re\ \phi_2^*(s)]^2 + [Im\ \phi_2^*(s)]^2$.

We let $\underset{\sim}{b}(p_j+N\lambda) = \underset{\sim}{0}$ for some N chosen so that each of the entries of the matrix $C(p_j+N\lambda)$ is numerically less than 1 . That is, we choose N so that

$$Re(p_j) + N\lambda > \max\{r_1, r_2\} ,$$

where r_1 and r_2 are given by (4.37). Such a choice of N is necessary to enable the method to converge and thus obtain a reasonably accurate approximation to $B^*(p_j+\lambda)$.

The numerical method, to obtain the contribution from the complex roots of the characteristic equation, is slower than that described earlier to obtain the asymptotic behaviour which results from the contribution from the real root. Thus the method is less efficient the more $Re(p_j)$ becomes negative. We can however obtain the solution resulting from a "few" of the right-most roots, which will be sufficient to characterise the solution for large time t, and use the methods of Chapter 3 to obtain the total birth rate for smaller values of time. In any event, the above procedure is still more efficient than a detailed projection obtained by solving the integral equation numerically.

It should be noted that the population need not be initially stable for the methods of the present chapter to be used. This assumption is only made for simplicity and invaluable only in obtaining the error analysis of Subsection [4.1.2]. If the assumption of initial stability is not made then, as in other sections, comparison of successive approximations to the stable equivalent would have to be made.

5. The Asymptotic Behaviour Resulting From a General Time Dependent Net Maternity Function.

In the present chapter the methods developed in Chapter 4 to obtain the asymptotic behaviour of a population, will be extended to include more general time dependent changes of the initial age-specific birth rate.

The time dependence is firstly extended to a sum of exponentials which provides scope for a great variety of paths of change from the single exponential of Chapter 4. Using the extension of Section [5.1] to the recurrence relation method developed in the previous chapter, we present a model in Section [5.2] which allows the various age groups of the initial net maternity function to change at different transition rates of time, towards the eventual net maternity function. Keyfitz (1975) notes that the dissemination of birth control information and materials strikes the various age-groups unequally and hence the model of Section [5.2] would amply cover this situation.

One could postulate that the effects of policies aimed at changing the age-specific birth rate are slow at first, then accelerate only to slow down as a set target is approached. A model which allows for a more gradual change with time than the exponential, is presented in Section [5.3]. The general sum of exponentials of Section [5.1] may also represent this type of a more gradual change.

Realistic net maternity functions are positive for age x such that $\alpha < x < \beta$ and zero elsewhere. Thus if the initial net maternity function is allowed to change with time over an interval less than or equal to the lowest age of childbearing α then, the convolution is not disrupted. In Section [5.4] the asymptotic effects of a general time

dependent scaling of the initial net maternity function for $0 < t < \tau \lesssim \alpha$ is examined. A non-separable model which allows for the age structure of the initial net maternity function to differ from that at $t = \tau$, is also presented and the asymptotic behaviour is obtained.

A change of the age-specific birth rate with time is likely however, especially in the less developed countries, to occur over a period of more than $\alpha = 10$ or 15 years. Frejka (1973) allows for a linear change to replacement level fertility over 0, 10, 30, 50, 70 years and Keyfitz (1971b) states that at best, such a change would occur over 30 or more years. Appreciating the need for the time dependence to occur over a larger period than α, and utilising the fact that we can now handle problems with exponential time dependence, we present models in Section [5.5] which take these points into consideration. The time dependence for $0 < t \lesssim \alpha$ can be general whereas for $t > \alpha$ any of the exponential-based models presented previously may be used to find the asymptotic behaviour of the population where the recurrence relation method will have to be utilised.

Only the asymptotic behaviour of the total birth rate is considered since that of the total number and the age density, information which is usually wanted, follow without too much difficulty.

5.1 The Time Dependence as a Sum of Exponentials.

We have discussed, in the previous chapter, the asymptotic behaviour of a population subjected to an exponential time dependent change of a general, initial age-specific birth rate. We will now consider the time dependence to be a sum of exponentials and thus allow a more general time path. The methods of the previous chapter can again be used here but require some modification. The non-separable model will be examined since the separable model is a special case. However, it is more enlightening to determine the time path for a separable net maternity function since the effect on the initial age structure is more obvious because the time dependence merely provides a scaling factor.

Let the time dependent net maternity function be given by

$$\Phi(x,t) = \phi_2(x) + \xi(t)[\phi_1(x) - \phi_2(x)] , \tag{5.1}$$

where $\xi(t)$ is a continuous non-negative function satisfying

$$\xi(0) = 1 \quad \text{and} \quad \lim_{t\to\infty} \xi(t) = 0 , \tag{5.2}$$

and, $\phi_1(x)$ and $\phi_2(x)$ are the initial and the final net maternity functions, respectively. The time dependence, in (5.1), is assumed only to affect the age-specific birth rate and not the survivor function (and hence, not the age-specific death rate). Then, with $\Phi(x,t)$ as given by (5.1), we have, assuming an initially stable population, the total birth rate $B(t)$ being given by

$$\begin{aligned} B(t) = {} & Q_1 \int_0^\infty e^{-r_1 x}\phi_2(x+t)dx + \int_0^t B(t-x)\phi_2(x)dx \\ & + \xi(t)\left\{Q_1\int_0^\infty e^{-r_1 x}[\phi_1(x+t)-\phi_2(x+t)]dx + \int_0^t B(t-x)[\phi_1(x)-\phi_2(x)]dx\right\} . \end{aligned} \tag{5.3}$$

Let $I_M = \{1,2,3,\ldots,M\}$ and,

$$\xi(t) = \sum_{m=1}^{M} \gamma_m e^{-\lambda_m t} , \tag{5.4}$$

where the conditions, on $\xi(t)$, (5.2) imply that; $\lambda_m \neq 0$ unless $\gamma_m = 0$ and hence, without loss of generality,

$$\left.\begin{aligned} &\lambda_m > 0 \quad \text{for all} \quad m \in I_M \\ \text{and} \quad &\sum_{m=1}^{M} \gamma_m = 1 \quad . \end{aligned}\right\} \tag{5.5}$$

The above time dependence, (5.4), while including the model of Chapter 4 as a special case (viz. $M = 1$), provides scope for a further variety of paths from the initial, to the eventual net maternity function. The above sum for $\xi(t)$, in (5.4), allows for $2M$ degrees of freedom in determining the $\{\gamma_m\}$ and $\{\lambda_m\}$. However, finding these values is a very difficult problem. A variety of shapes <u>can</u> be obtained even if we take $M = 2$. For example, Figures 5.1 and 5.2 show the types of time variations that can be obtained from (5.4) with (5.5), where $M = 2$ and $\lambda_2 = 2\lambda_1 = 2\lambda$. Figure 5.1 shows the effect of changes in λ by fixing $\gamma_1 = \gamma = 2.0$, and thus having $\xi'(0) = 0$. The response to variations in γ is demonstrated in Figure 5.2 with $\lambda = 0.05$.

To obtain the asymptotic behaviour of the total birth rate, for the model given by (5.3) with (5.4), we proceed in a formal fashion similar to that of the previous chapter viz. by Laplace transform techniques.

Taking the Laplace transform of (5.3) and using (5.4) we obtain, on rearrangement

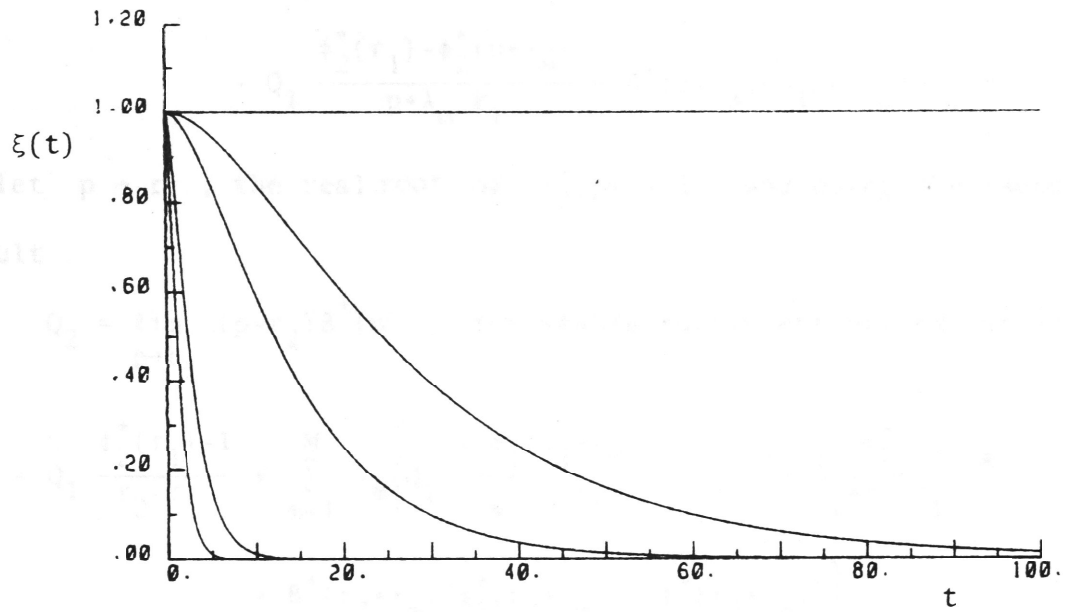

FIGURE 5.1
Diagram showing $\xi(t) = \gamma e^{-\lambda t} + (1-\gamma)e^{-2\lambda t}$, $(M = 2$ and $\lambda_2 = 2\lambda_1 = 2\lambda$ in (5.4)$)$ for $\gamma = 2$ and varying λ. From top to bottom $\lambda = 0.0, 0.05, 0.1, 0.5, 1.0$.

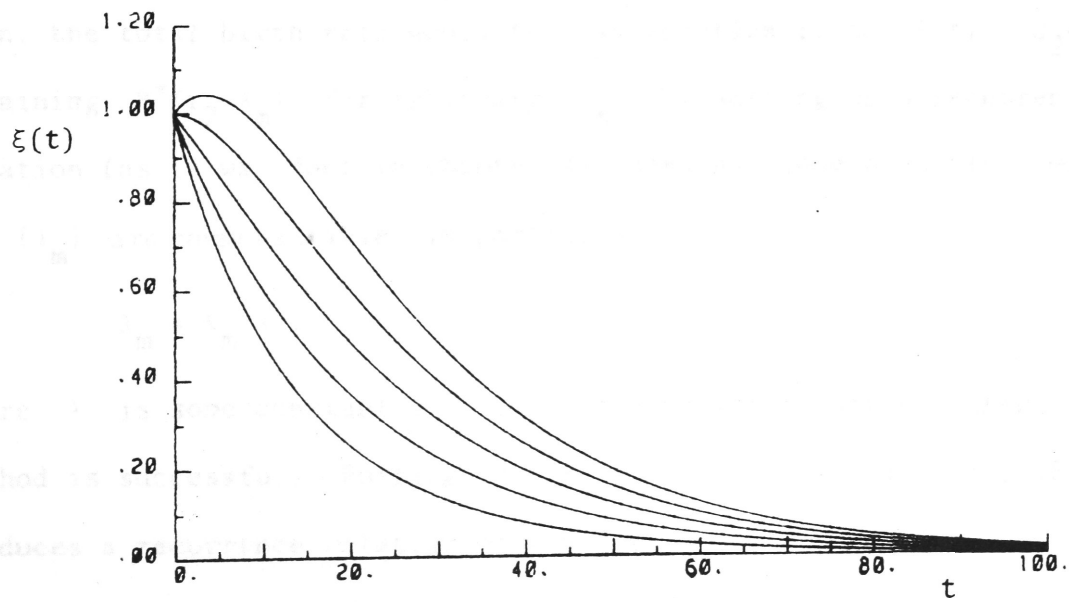

FIGURE 5.2
Diagram showing $\xi(t) = \gamma e^{-\lambda t} + (1-\gamma)e^{-2\lambda t}$, $(M = 2$ and $\lambda_2 = 2\lambda_1 = 2\lambda$ in (5.4)$)$ for $\lambda = 0.05$ and varying γ. From top to bottom $\gamma = 2.5, 2.0, 1.5, 1.0, 0.5$.

$$[1-\phi_2^*(p)]B^*(p) = Q_1 \frac{\phi_2^*(r_1)-\phi_2^*(p)}{p-r_1} + \sum_{m=1}^{M} \gamma_m \Bigg\{ Q_1 \frac{1-\phi_1^*(p+\lambda_m)}{p+\lambda_m-r_1} - Q_1 \frac{\phi_2^*(r_1)-\phi_2^*(p+\lambda_m)}{p+\lambda_m-r_1} + B^*(p+\lambda_m)[\phi_1^*(p+\lambda_m)-\phi_2^*(p+\lambda_m)] \Bigg\}. \tag{5.6}$$

We let $p \to r_2$, the real root of $\phi_2^*(p) = 1$, and using the Tauberian result

$Q_2 = \lim_{p \to r_2} (p-r_2)B^*(p)$, the stable equivalent births, we obtain

$$\kappa Q_2 = Q_1 \frac{\phi_2^*(r_1)-1}{r_2-r_1} + \sum_{m=1}^{M} \gamma_m \Bigg\{ Q_1 \frac{1-\phi_1^*(r_2+\lambda_m)}{\lambda_m+r_2-r_1} - Q_1 \frac{\phi_2^*(r_1)-\phi_2^*(r_2+\lambda_m)}{\lambda_m+r_2-r_1} + B^*(r_2+\lambda_m)[\phi_1^*(r_2+\lambda_m) - \phi_2^*(r_2+\lambda_m)] \Bigg\}, \tag{5.7}$$

where κ is the expected age of childbearing in the eventual stable population and is given by (4.40).

In order to obtain Q_2 from (5.7) we need to evaluate

$$B^*(r_2+\lambda_m) \quad \text{for} \quad m = 1,2,\ldots,M .$$

Then, the total birth rate would tend asymptotically to $B(t) = Q_2 e^{r_2 t}$. Obtaining $B^*(r_2+\lambda_m)$ for arbitrary λ_m, by setting up a recurrence relation (as it was done in Chapter 4), does not seem possible. However, if $\{\lambda_m\}$ are commensurable, in particular if

$$\lambda_m = k_m \lambda , \tag{5.8}$$

where λ is some constant and $\{k_m\}$ are positive integers then, the method is successful. Putting $p = r_2 + n\lambda$ in (5.6) and using (5.8) produces a recurrence relation of the form

$$B^*(r_2+n\lambda) = \delta_n + \sum_{m=1}^{M} \varepsilon_n^{(m)} B^*[r_2+(n+k_m)\lambda] , \quad n = 1,2,\ldots .$$

Choosing $B^*(r_2+N\lambda) = 0$ for some N we acquire, using the backward recurrence relation above, successive approximations to the $\{B^*(r_2+k_m\lambda)\}$

and hence to Q_2 .

When determining the unknown constants, in (5.4) with (5.8), a number of optimising conditions for the above process (using the recurrence relation) should be observed. Namely,

(i) The smaller M is the better since M determines the number of unknown quantities in (5.7).

(ii) The smaller $\{k_m\}$ are the better since they determine the relative size of N , in assuming $B^*(r_2+N\lambda) = 0$, for obtaining approximations to $\{B^*(r_2+k_m\lambda)\}$.

(iii) The convergence of the method is dependent on the speed with which the $\{\varepsilon_n^{(m)}\}$ decrease as n increases. Hence the bigger λ is the better; and, if $|\gamma_m| \leq 1$ for all m , then the convergence of the numerical method is fastest. The effect of λ on the convergence is greater than that of the $\{\gamma_m\}$, when the $\{\phi_i(x)\}$ are defined over a finite interval since, as demonstrated in Chapter 4, $\{\phi_i^*(n\lambda)\}$ would then decrease exponentially with increasing n .

Although it is difficult to determine the function (5.4) with (5.8), especially for "large" M , the above exercise does show that the recurrence relation method can still be applied if more than one exponential is present; provided, the exponents are integer multiples of some constant. This fact allows for a number of further generalities such as being able to allow the various age-groups, of the age-specific birth rate, to change at <u>different</u> rates of time. This model is presented in the next section.

5.2 Differential Time Dependence For the Various Age-Groups.

Problems in which the age-specific birth rate is scaled abruptly at the origin, by a constant, have been studied by Keyfitz (1971b) and, Tognetti (1976b). A gradual exponential time dependent scaling was developed and presented in Cerone and Keane (1978a). Mitra (1976) analysed the consequences of an abrupt change of the net maternity function to any other; and, hence allowing variation in the age structure of the maternity behaviour. A time dependent counterpart to Mitra's model was presented in Cerone and Keane (1978b).

We shall, here, present a model in which not only does, the age-specific birth rate change its age structure with time as in Cerone and Keane (1978b), but, the time dependence will differ with age. Thus we will have the various age-groups changing at different rates from one net maternity function to another.

The initial and final net maternity functions will be assumed to be represented in a discrete fashion since even if they were continuous then, with differential time dependence over the age-groups, $\Phi(x,t)$ would in general be discontinuous. We will consider the time dependent net maternity function to be of the form (5.1) where $\{\phi_i(x)\}$ are represented by histograms with the discontinuities occurring at the same points $\{b_m\}$. Hence, the model to be considered is

$$\Phi(x,t) = \sum_{m=1}^{M} \psi_m(t) S_m(x) \quad , \tag{5.9}$$

where,

$$\left.\begin{aligned} \psi_m(t) &= a_{m,2} + (a_{m,1} - a_{m,2})\xi_m(t) \quad , \\ \text{and} \quad S_m(x) &= H(x-b_m)H(b_{m+1} - x) \quad . \end{aligned}\right\} \tag{5.10}$$

The above model (5.9) was also presented in Chapter 2, and the transient solution was obtained, however the time dependence $\xi_m(t)$ in (5.10) was the same for all age-groups. The model (5.9) not only allows for a change in the age structure but also for the rate at which the eventual net maternity function is achieved.

The simplest example of (5.9)-(5.10) is when particular policies influence <u>one</u> age-group at a different rate to the rest. Thus the time dependence $\xi_m(t)$ for $m = 1,2,\ldots,M$ would be the same except for the <u>one</u> age-group.

Hence, with $\Phi(x,t)$ as given by (5.9)-(5.10), and, assuming the population to be initially stable, the total birth rate $B(t)$ is given by

$$B(t) = \sum_{m=1}^{M} \psi_m(t)\left\{Q_1\int_0^{\infty} e^{-r_1 x}S_m(x+t)dx + \int_0^t B(t-x)S_m(x)dx\right\} . \qquad (5.11)$$

In order to obtain the asymptotic total birth rate we recall from Section [5.1] that the algorithm developed in Chapter 4 can be extended to handle problems where the time dependence is a sum of exponentials with commensurable exponents. Thus we will consider in particular

$$\xi_m(t) = e^{-k_m\lambda t} , \qquad (5.12)$$

where $\{k_m\}$ are positive integers and λ is a constant. We obtain the model of Section [4.2] if $k_m = 1$ for $m = 1,2,\ldots,M$.

The Laplace transform of (5.11) with (5.10) and (5.12) yields,

$$B^*(p) = \sum_{m=1}^{M} \left\{Q_1\, a_{m,2}\,\frac{S_m^*(r_1)-S_m^*(p)}{p-r_1} + Q_1(a_{m,1}-a_{m,2})\left[\frac{S_m^*(r_1)-S_m^*(p+k_m\lambda)}{p+k_m\lambda-r_1}\right]\right.$$

$$\left. + a_{m,2}S_m^*(p)B^*(p) + (a_{m,1}-a_{m,2})S_m^*(p+k_m\lambda)B^*(p+k_m\lambda)\right\} .$$

Hence, using the fact that

$$\phi_i^*(p) = \sum_{m=1}^{M} a_{m,i} S_m^*(p) \quad , \quad i = 1,2 \ ,$$

$$[1-\phi_2^*(p)]B^*(p) = Q_1 \frac{\phi_2^*(r_1)-\phi_2^*(p)}{p-r_1} + Q_1 \sum_{m=1}^{M} (a_{m,1}-a_{m,2}) \left[\frac{S_m^*(r_1)-S_m^*(p+k_m\lambda)}{p+k_m\lambda-r_1} \right]$$

$$+ \sum_{m=1}^{M} (a_{m,1}-a_{m,2}) S_m^*(p+k_m\lambda) B^*(p+k_m\lambda) \quad . \tag{5.13}$$

To obtain the stable equivalent births Q_2 we proceed by using the Tauberian result and letting $p \to r_2$ (the real root of $\phi_2^*(p) = 1$) then

$$\kappa Q_2 = Q_1 \frac{\phi_2^*(r_1)-1}{r_2-r_1} + Q_1 \sum_{m=1}^{M} (a_{m,1}-a_{m,2}) \left[\frac{S_m^*(r_1)-S_m^*(r_2+k_m\lambda)}{r_2+k_m\lambda-r_1} \right]$$

$$+ \sum_{m=1}^{M} (a_{m,1}-a_{m,2}) S_m^*(r_2+k_m\lambda) B^*(r_2+k_m\lambda) \ , \tag{5.14}$$

where $B(t)$ will asymptotically behave like $B(t) = Q_2 e^{r_2 t}$.

We need $\{B^*(r_2+k_m\lambda)\}$ in order that the stable equivalent births Q_2 may be evaluated using (5.14). To do this, we proceed in much the same manner as previously. Putting $p = r_2 + n\lambda$ in (5.13) results in a recurrence relation of the form

$$B^*(r_2+n\lambda) = \delta_n + \sum_{m=1}^{M} \varepsilon_n^{(m)} B^*[r_2+(n+k_m)\lambda] \ , \quad n = 1,2,\ldots \ . \tag{5.15}$$

Successive approximations to $\{B^*(r_2+k_m\lambda)\}$, and hence from (5.14), to Q_2 , are obtained by taking $B^*(r_2+N\lambda) = 0$ for some N . We compare successive approximations to Q_2 , stopping, when the relative error is considered small enough. It should be noted that N need not be taken as $1,2,3,\ldots$ but a sequential advancement in equal steps is advisable for the comparison of successive approximations to Q_2 , to be of value.

5.3 Time Dependence as a Generalised Sech Function.

In Chapter 4, a change in the age-specific birth rate was assumed to be either abrupt or exponential. However, one would postulate that the implementation of certain policies, which are to bring about a change in the age-specific birth rate, would at first be slow in their implementation, accelerating, then slowing down as the appropriate set target is approached. The model would also be able to represent an initial reluctance of a population to adopt the new policies.

We will consider the time dependent net maternity function $\Phi(x,t)$ to be given by (5.1) where,

$$\xi(t) = \frac{1}{\gamma e^{\lambda t} + (1-\gamma)e^{-\lambda t}} = \frac{e^{-\lambda t}}{\eta(t)} \quad , \quad \gamma,\lambda > 0 \ . \tag{5.16}$$

When $\gamma = 1$, (5.1) with (5.16) represents the model presented in Section [4.2] in which the change is exponential between two net maternity functions. If $\gamma = \frac{1}{2}$ in (5.16) then $\xi(t) = \text{Sech } \lambda t$. Furthermore (5.16) can be looked upon as an exponentially scaled logistic, where $\eta(t)$ is an upside down logistic (Keyfitz (1968b)).

The behaviour of (5.16), due to changes of λ , is demonstrated (for fixed γ) in Figure 5.3. The effect of γ on $\xi(t)$ is shown in Figure 5.4. It can be shown, by differentiating (5.16), that $\xi(t)$ has no turning point t_o for $\gamma \geq 1$ and $t_o \left\{\begin{matrix}<\\=\\>\end{matrix}\right\} 0$ depending on whether $\gamma \left\{\begin{matrix}>\\=\\<\end{matrix}\right\} \frac{1}{2}$. Hence t_o is to the left of the vertical axis for $\frac{1}{2} < \gamma < 1$ and thus with such a γ , $\xi(t)$ would represent a function which decreases to zero monotonically for $t > 0$. A possible way of determining the two unknowns γ and λ of (5.16) would be to specify the slope at the origin and the point of inflection (for $0 < \gamma < 1$).

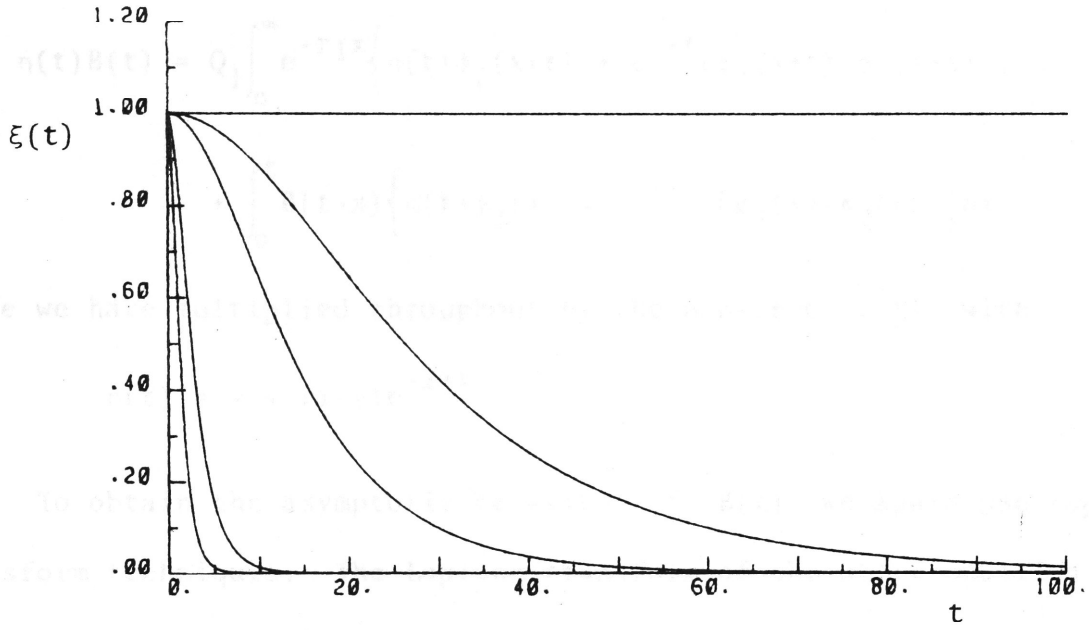

FIGURE 5.3

The effect of λ on $\xi(t)$ as given by (5.16) with $\gamma = 0.5$. The graphs represent $\xi(t)$ from top to bottom with $\lambda = 0.0, 0.05, 0.1, 0.5, 1.0$.

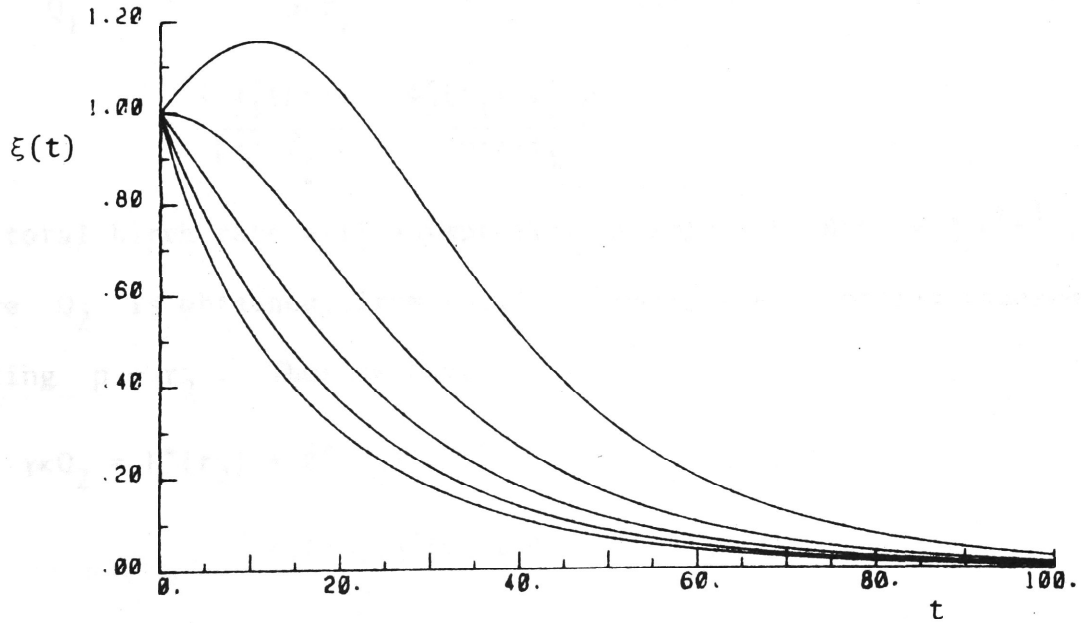

FIGURE 5.4

Diagram of the time dependence $\xi(t)$ given by (5.16) with $\lambda = 0.05$ and varying values of γ. From top to bottom $\gamma = 0.25, 0.5, 0.75, 1.0, 1.25$.

With $\Phi(x,t)$ given by (5.1) and (5.16) we have, assuming an initially stable population, the total birth $B(t)$ being given by (from (2.32))

$$\eta(t)B(t) = Q_1\int_0^\infty e^{-r_1x}\left\{\eta(t)\phi_2(x+t) + e^{-\lambda t}[\phi_1(x+t)-\phi_2(x+t)]\right\}dx$$

$$+ \int_0^t B(t-x)\left\{\eta(t)\phi_2(x) + e^{-\lambda t}\ [\phi_1(x)-\phi_2(x)]\right\}dx\ ,$$

where we have multiplied throughout by the non-zero $\eta(t)$ with

$$\eta(t) = \gamma + (1-\gamma)e^{-2\lambda t}\ .$$

To obtain the asymptotic behaviour of $B(t)$ we again use Laplace transform techniques. The Laplace transform of the above equation yields, upon rearrangement,

$$\gamma[1-\phi_2^*(p)]B^*(p) = F^*(p) + B^*(p+\lambda)[\phi_1^*(p+\lambda) - \phi_2^*(p+\lambda)]$$

$$- (1-\gamma)B^*(p+2\lambda)[1-\phi_2^*(p+2\lambda)]\ , \qquad (5.17)$$

where,

$$\frac{F^*(p)}{Q_1} = \gamma\ \frac{\phi_2^*(r_1)-\phi_2^*(p)}{p-r_1} + (1-\gamma)\ \frac{\phi_2^*(r_1)-\phi_2^*(p+2\lambda)}{p+2\lambda-r_1}$$

$$+ \frac{1-\phi_1^*(p+\lambda)}{p+\lambda-r_1} - \frac{\phi_2^*(r_1)-\phi_2^*(p)}{p+\lambda-r_1}\ .$$

The total birth rate will asymptotically approach $B(t) = Q_2e^{r_2t}$, where Q_2 is obtained, from (5.17) by using the Tauberian theorem and letting $p \to r_2$. Thus we have

$$\gamma\kappa Q_2 = F^*(r_2) + B^*(r_2+\lambda)[\phi_1^*(r_2+\lambda) - \phi_2^*(r_2+\lambda)]$$

$$- (1-\gamma)B^*(r_2+2\lambda)[1 - \phi_2^*(r_2+2\lambda)]\ . \qquad (5.18)$$

We need to find $B^*(r_2+2\lambda)$ and $B^*(r_2+\lambda)$ in order that the stable equivalent births Q_2 may be evaluated from (5.18). Letting $p = r_2 + n\lambda$ in (5.17) gives the following recurrence relation

$$B^*(r_2+n\lambda) = \delta_n + \varepsilon_n^{(1)} B^*[r_2+(n+1)\lambda] + \varepsilon_n^{(2)} B^*[r_2+(n+2)\lambda], \quad n=1,2,.., \qquad (5.19)$$

where,

$$\delta_n = \frac{F^*(r_2+n\lambda)}{\gamma[1-\phi_2^*(r_2+n\lambda)]} ,$$

$$\varepsilon_n^{(1)} = \frac{\phi_1^*[r_2+(n+1)\lambda] - \phi_2^*[r_2+(n+1)\lambda]}{\gamma[1-\phi_2^*(r_2+n\lambda)]} ,$$

$$\varepsilon_n^{(2)} = -\frac{(1-\gamma)}{\gamma} \frac{1-\phi_2^*[r_2+(n+2)\lambda]}{1-\phi_2^*(r_2+n\lambda)} .$$

As we saw in Section [5.1] the recurrence relation, now given by (5.19), can be used to obtain successive approximations to $B^*(r_2+2\lambda)$ and $B^*(r_2+\lambda)$ and hence, from (5.18), to Q_2 by assuming $B^*(r_2+N\lambda) = 0$ for some N .

We have stated previously that the speed of convergence of the method depends upon the coefficients ($\varepsilon_n^{(1)}$ and $\varepsilon_n^{(2)}$ in the present situation) of the B^* terms on the right hand side of the recurrence relation. Hence, the method converges fastest if γ is "close" to 1; that is, for $\frac{1}{2} \leq \gamma \leq \frac{3}{2}$. The method is slower than for the previous models since the error, produced in $B^*(r_2+2\lambda)$ and $B^*(r_2+\lambda)$ from assuming $B^*(r_2+N\lambda) = 0$, (for the separable model at least) is dependent on a product of $\varepsilon_n^{(2)}$ and $\varepsilon_n^{(1)}$ terms respectively. A product of $\varepsilon_n^{(2)}$ terms here decreases according to the coefficient in γ and not exponentially as would be the case for demographically realistic $\{\phi_i(x)\}$.

Comparable shapes (See Figures 5.1, 5.2, 5.3 and 5.4) to those of (5.16) can be obtained by representing $\xi(t)$, as in Section [5.1], by a sum of exponentials and thus the speed of convergence of the recurrence relation method would be faster if the latter were used. However the

difficulty in determining the various parameters rises sharply as the number of exponentials in the sum increases.

Various generalisations of (5.16) are possible, such as

$$\xi(t) = \frac{1}{\gamma e^{k_1 \lambda t} + (1-\gamma) e^{-k_2 \lambda t}} \quad ,$$

where k_1 and k_2 are positive integers and λ is a real constant. However these will not be pursued further here.

As shown in Section [5.1] any non-negative function $\xi(t)$ satisfying conditions (5.2) can be used in (5.1) and, the recurrence relation method is appropriate; provided, $\xi(t)$ is a strict function of exponentials with commensurable exponents. Hence we may have a wide variety of functions as the time dependence, as long as, they consist of exponentials with the exponents being integer multiples of some $\lambda > 0$.

5.4 General Time Dependence No Greater Than The Lowest Age of Childbearing, α.

We noted in Subsection [4.1.3] that the convolution, for the total birth rate, is not violated for extreme values of the transition rate λ. This is also the case if only the maternity behaviour of those already alive at the origin is allowed to change explicitly with time. Further, with a demographically realistic net maternity function - one that is non-zero on a finite interval (α,β) - the convolution is not violated if the time dependence is explicit only for $0 < t < \tau \leq \alpha$.

We present both separable and non-separable models for the time dependent net maternity function where the time dependence is explicit only for $0 < t < \tau \leq \alpha$.

It should be noted that the total number and age density are not affected explicitly if the time dependence, as here, refers only to the age-specific birth rate. A changing age-specific birth rate only affects the total birth rate explicitly and the change is transmitted to the total number and age density through its effect on the total birth rate. Hence we will only consider the effects of the time dependent net maternity function on $B(t)$.

Consider the time dependent net maternity function given by

$$\Phi(x,t) = \begin{cases} \chi(t)\phi(x) & , \quad 0 < t < \tau \leq \alpha \, , \\ \dfrac{\phi(x)}{R} & , \quad t > \tau \, , \end{cases}$$

which, on using the Heaviside unit function defined by (3.44), can be written conveniently as

$$\Phi(x,t) = \left[\chi(t)H(\tau-t) + \frac{H(t-\tau)}{R}\right]\phi(x) \quad , \quad t > 0 \quad , \quad \tau \leq \alpha \quad . \tag{5.20}$$

Hence (5.20) is a separable time dependent net maternity function of the form (2.33) which provides a time dependent scaling of the initial net maternity function. Figure 5.5 gives a diagrammatic representation of the time dependence in (5.20) with linear and exponential $\chi(t)$.

With $\Phi(x,t)$ from (5.20) we have, assuming an initially stable population, the total birth rate $B(t)$, from (4.13) with (4.14), given by

$$B(t) = \left[\chi(t)H(\tau-t) + \frac{H(t-\tau)}{R}\right]\left\{Q\int_0^\infty e^{-rx}\phi(x+t)dx + \int_0^t B(t-x)\phi(x)dx\right\} \quad , \quad \tau \leq \alpha \quad , \tag{5.21}$$

where $\phi(x) \neq 0$ for $\alpha < x < \beta$.

We proceed to solve (5.21) in the usual manner using Laplace transform techniques. In particular we are interested in determining the asymptotic behaviour of $B(t)$.

The Laplace transform of (5.21) yields

$$B^*(p) = F^*(p) + \frac{\phi^*(p)}{R} B^*(p) \quad , \tag{5.22}$$

with,

$$\frac{F^*(p)}{Q} = \int_0^\tau e^{-(p-r)t}\chi(t)dt + \frac{1}{R}\left\{\int_\tau^\alpha e^{-(p-r)t}dt + \int_\alpha^\beta e^{-(p-r)t}\int_t^\beta e^{-rx}\phi(x)dxdt\right\}$$

$$= \int_0^\tau e^{-(p-r)t}\chi(t)dt + \frac{e^{-(p-r)\tau} - \phi^*(p)}{R(p-r)} \quad ,$$

where we have used, the fact that $\phi^*(r) = 1$ and the definition (3.44) of the Heaviside unit function.

Hence, rearranging (5.21) and substituting for $F^*(p)$ we obtain

$$\frac{B^*(p)}{Q} = \frac{\int_0^\tau e^{-(p-r)t}\chi(t)dt}{1 - \frac{\phi^*(p)}{R}} + \frac{e^{-(p-r)\tau} - \phi^*(p)}{R(p-r)\left[1 - \frac{\phi^*(p)}{R}\right]} \quad . \tag{5.23}$$

Now $p = r$ is not a pole, in the second expression of (5.23), but is merely a removable singularity. Further, the integral in (5.23) is an entire function for all positive functions $\chi(t)$ so that it has no poles. Hence letting $p \to 0$, the real root of $\frac{\phi^*(p)}{R} = 1$ which is greater than the real part of all the other roots, and using the Tauberian result,

$$Q_2 = \lim_{p\to 0} pB^*(p) \quad , \quad \text{the asymptotic value,}$$

we obtain the eventual stationary total birth rate, from (5.23), as

$$Q_2 = \frac{Q}{\kappa}\int_0^\tau e^{rt}\chi(t)dt + \frac{Q}{rR\kappa}\left[R - e^{r\tau}\right] , \tag{5.24}$$

where κ is the average age of childbearing.

Note that if we let $\tau \to 0$ in (5.20) then we obtain the abrupt change to replacement level fertility model of Keyfitz and hence from (5.23) we correctly obtain the asymptotic value found by Keyfitz as given by (4.5). The difference D of (4.5) from (5.24) gives a comparison, between the asymptotic total birth rate Q_2, resulting from an abrupt Keyfitz change to replacement level fertility with that from a gradual change depicted by (5.20), where

$$D = \frac{Q}{\kappa}\int_0^\tau e^{rt}\chi(t)dt + Q\,\frac{1-e^{r\tau}}{rR\kappa} \quad . \tag{5.25}$$

Hence writing (5.24) as

$$Q_2 = Q_2(\text{Keyfitz}) + D \ ,$$

we can say that if for all t, $0 < t < \tau$, $\chi(t) \gtrless \frac{1}{R}$ then $D \gtrless 0$,

and for $0 < \chi(t) \lesssim \frac{1}{R}$ then $D \lesssim 0$.

However if $\chi(t)$ can take values on either side of $\frac{1}{R}$ then we cannot readily say whether, with the present model (5.20), we would have an asymptotic value higher or lower than that obtained by Keyfitz.

We now consider a number of models for $\chi(t)$.

If $\chi(t) \equiv 1$ then we obtain a model also discussed by Keyfitz (1971b) in that the abrupt change to replacement level fertility behaviour is adopted at time $t = \tau$ and not necessarily at the origin. Hence from (5.24) we obtain the asymptotic value

$$Q_2 = \frac{Q(R-1)}{rR\kappa} e^{r\tau} , \tag{5.26}$$

which is $e^{r\tau}$ times the value obtained under an abrupt change, to replacement level fertility, at the origin. Note that the time dependence affects the convolution if $\tau > \alpha$. In particular, an abrupt change at some time τ would have to be such that $\tau \leqslant \alpha$, the minimum age of childbearing.

Further models involve the separable time dependent net maternity function (5.20) with linear and exponential $\chi(t)$. Thus we can determine the asymptotic total birth rate due to the model (5.20) with

$$\left.\begin{aligned} &\chi(t) = 1 - \theta t , \\ \text{and} \quad &\chi(t) = e^{-\lambda t} , \end{aligned}\right\} \tag{5.27}$$

where, for continuity $\theta = \frac{1}{\tau}\left(1 - \frac{1}{R}\right)$ and $e^{\lambda\tau} = R$.

Figure 5.5 shows a diagram of $\chi(t)$ given by (5.27) which make $\psi(t)$, the coefficient of $\phi(x)$ in (5.20), continuous. A discontinuous $\psi(t)$ at $t = 0$ and $t = \tau$ is demonstrated in Figure 5.6 for linear and exponential $\chi(t)$ viz.,

$$\chi(t) = a_o\left(1 - \frac{t}{\tau}\right) + a_\tau \cdot \frac{t}{\tau} \quad ,$$

and $\chi(t) = a_o e^{-\lambda t}$, $e^{\lambda\tau} = \frac{a_o}{a_\tau}$,

respectively. The asymptotic total birth rate Q_2 may be obtained from (5.24).

* * * *

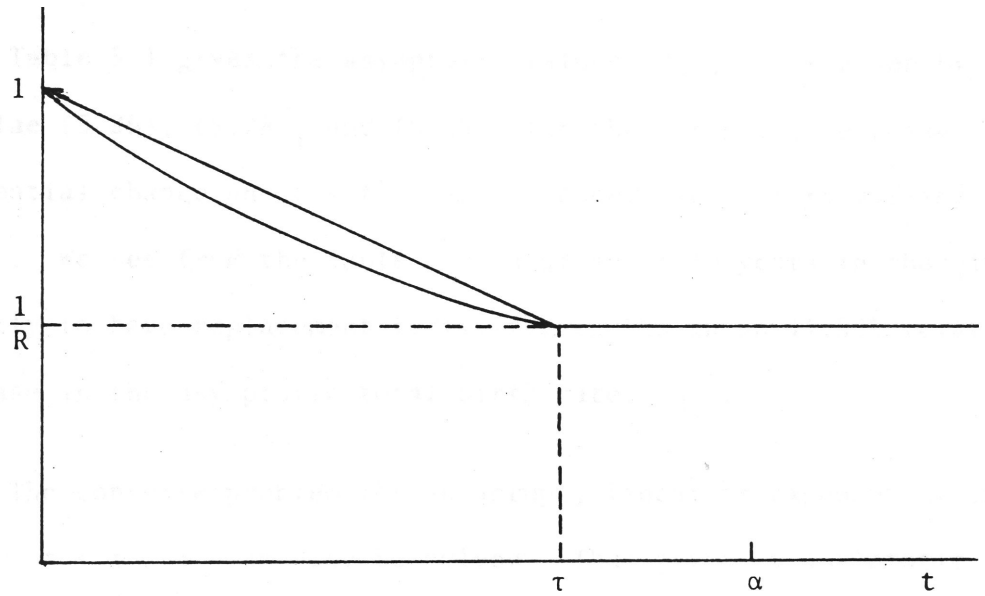

FIGURE 5.5
Linear and exponential time dependence over $0 < t < \tau \lesssim \alpha$. This diagram represents the continuous time scaling models of the age-specific birth rate to replacement level, as given by (5.20) with $\chi(0) = 1$ and $\chi(\tau) = 1/R$.

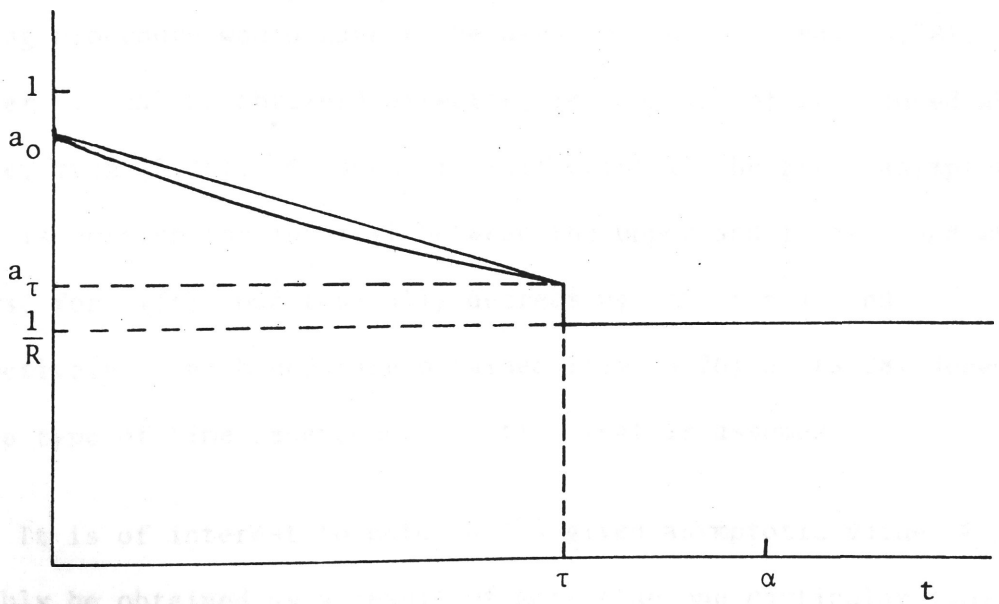

FIGURE 5.6
Sketch of linear and exponential time dependence over $0 < t < \tau \lesssim \alpha$ with discontinuities at $t = 0$ and $t = \tau$. The above sketch represents the coefficient of $\phi(x)$ in (5.20) with linear and exponential $\chi(t)$.

* * * *

With the time dependence (5.27) we obtain, from (5.24), the asymptotic total birth rate as

$$\left.\begin{aligned} Q_2 &= Q\,\frac{(R-1)}{rR\kappa}\left[\frac{e^{r\tau}-1}{r\tau}\right], \\ \text{and}\qquad Q_2 &= Q\,\frac{R-e^{r\tau}}{rR\kappa}\,\frac{\lambda}{\lambda-r}, \quad \lambda = \frac{1}{\tau}\,\ell n\, R, \end{aligned}\right\} \qquad (5.28)$$

respectively.

Table 5.1 gives the asymptotic values Q_2/Q, as given by the formulae (5.26), $(5.28)_1$ and $(5.28)_2$ for the abrupt, linear and exponential change on $0 < t < \tau \leq \alpha$ respectively, for, varying values of τ. We see from the table that a delay of 10 years in changing abruptly to bare replacement fertility, results in an 11.58% relative increase in the asymptotic total birth rate, Q_2.

The converse problem for an abrupt, linear or exponential change on $0 < t < \tau \leq \alpha$ can also be solved. Thus given an asymptotic total birth rate Q_2 we can determine τ (which characterises the particular time dependence $\chi(t)$) from (5.26), $(5.28)_1$ and $(5.28)_2$ depending on whether we assume an abrupt, linear or exponential change. Some root finding procedure would have to be used to find τ from (5.28). However τ can be obtained directly, in the case of an assumed abrupt change, from (5.26). No such τ will exist if the given asymptotic value is outside the interval between the upper and lower bound which occurs, for $\chi(t)$ monotonically decreasing, at $\tau = \alpha$ and $\tau = 0$ respectively. The bounds are obtained from (5.26) or (5.28) depending on the type of time dependence, $\chi(t)$, that is assumed.

It is of interest to note that a given asymptotic value Q_2 may possibly be obtained as a result of more than one particular type of time dependence. For example, given a linear change to bare replacement

Q_2/Q

τ	ABRUPT	LINEAR	EXPONENTIAL
0	0.859355 (0.0)	0.859355 (0.0)	0.859355 (0.0)
2	0.878389 (2.21)	0.868837 (1.10)	0.868368 (1.05)
4	0.897845 (4.48)	0.878459 (2.22)	0.877510 (2.11)
6	0.917731 (6.79)	0.888224 (3.36)	0.886784 (3.19)
8	0.938058 (9.16)	0.898132 (4.51)	0.896192 (4.29)
10	0.958836 (11.58)	0.908188 (5.68)	0.905735 (5.40)

TABLE 5.1

Asymptotic total birth rate Q_2 resulting from an abrupt change at $t = \tau$, and, a linear and an exponential change over $0 < t < \tau \leq \alpha$. Various values of τ are taken and Q_2 is given by (5.26), $(5.28)_1$ and $(5.28)_2$ respectively. The numbers in the brackets are the percent relative difference from the asymptotic value obtained under an abrupt change at the origin $(t = \tau = 0)$, to replacement level. The Australian 1967 female data is used.

for $0 < t < \tilde{\tau}$, then an abrupt change at

$$\tau = \frac{1}{r} \ell n \left(\frac{e^{r\tilde{\tau}} - 1}{r\tilde{\tau}} \right) ,$$

obtained from equating (5.26) with $(5.28)_1$, will produce the same asymptotic total birth rate Q_2. Care must be taken with such problems since we have the restriction on τ that $0 < \tau \lesssim \alpha$, and, no such τ may exist. For example, it can be easily seen from Table 5.1 that for $\tau = \tilde{\tau} = 10$ in the abrupt model, resulting in $\tilde{Q}_2 = 0.958836Q$, then neither a linear nor an exponential change can produce the same asymptotic value $\tilde{Q}_2$. We need to check that $\tilde{Q}_2$ is in the range of possible asymptotic values of the new model.

Another example of the time dependence $\chi(t)$ in equation (5.20) is

$$\chi(t) = \sum_{n=1}^{N} \sigma_n H(t-t_{n-1}) H(t_n-t) , \quad t_o = 0 , \quad t_N = \tau \lesssim \alpha , \tag{5.29}$$

which represents a histogram and thus allows great flexibility in the type of change involved. The $\{\sigma_n\}$ are all strictly positive since $\chi(t)$ is positive. The expression (5.29) allows for variable width $\{h_n\}$ of the rectangles, where

$$h_n = t_n - t_{n-1} , \quad n = 1,2,\ldots,N ,$$

and thus wider rectangles may be used where the change is gradual, while narrower rectangles are used when the change with time is rapid. The time dependence $\chi(t)$ can thus be used when the change is expected to be irregular enough for a simple mathematical curve to be able to represent it. Figure 5.7 gives a diagrammatical demonstration of an approximation of some $\chi(t)$ by (5.29). Keyfitz (1971b) used a two step time dependence.

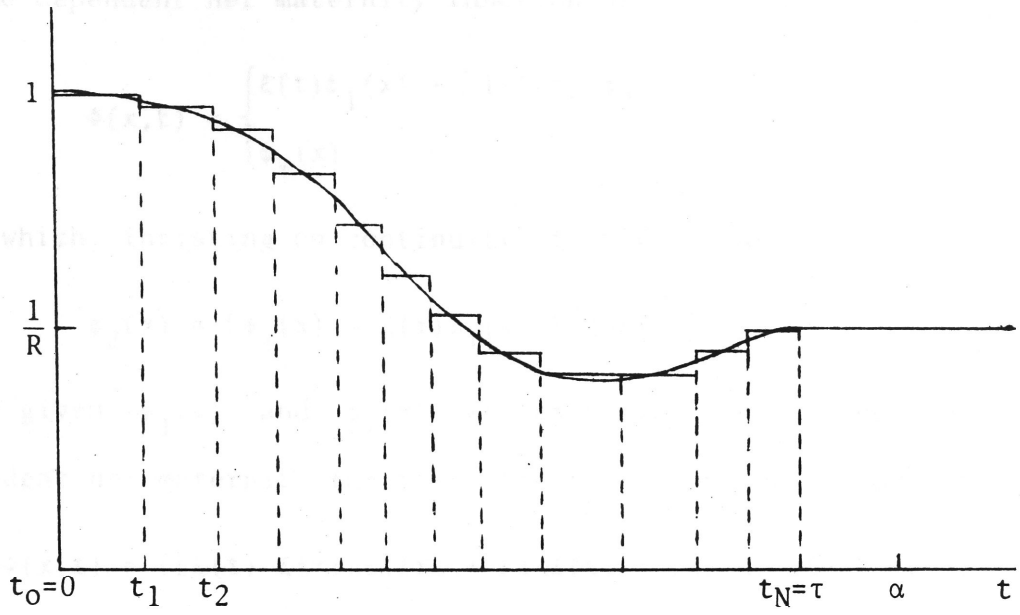

FIGURE 5.7
Diagrammatic representation showing the approximation of an arbitrary curve by a histogram ((5.29)). The function of time varies for $0 < t < \tau \leq \alpha$ until $1/R$ is reached at $t=\tau$. R is the net reproductive rate and α is the minimum age of childbearing.

* * * *

With the separable time dependent net maternity function (5.20) where $\chi(t)$ is represented by the histogram (5.29) we obtain, from (5.24), the asymptotic total birth rate Q_2, given by

$$Q_2 = \frac{Q}{r\kappa} \sum_{n=0}^{N} \tilde{\sigma}_n e^{rt_n} + \frac{Q}{rR\kappa} [R - e^{r\tau}] \quad , \tag{5.30}$$

where,

$$\tilde{\sigma}_o = -\sigma_1 \quad ,$$
$$\tilde{\sigma}_n = \sigma_n - \sigma_{n+1} \quad , \quad n = 1,2,\ldots,N-1 \quad ,$$
$$\tilde{\sigma}_N = \sigma_N \quad .$$

Consider now, a non-separable time dependent net maternity function, of the form (5.1), which allows the final net maternity function to be of a different age structure to that of the initial net maternity function. Let the time dependence only be explicitly stated for $0 < t \leq \tau \leq \alpha$ and let the age structure of the time dependent net maternity function at $t = \tau$ be $\phi_\tau(x)$ which is maintained for $t \geq \tau$. Thus we consider

a time dependent net maternity function of the form,

$$\Phi(x,t) = \begin{cases} \xi(t)\phi_1(x) + [1-\xi(t)]\phi_2(x) , & 0 < t < \tau \leq \alpha \\ \phi_\tau(x) , & t > \tau , \end{cases} \tag{5.31}$$

from which, insisting on continuity at $t = \tau$, we have

$$\phi_2(x) = [\phi_\tau(x) - \xi(\tau)\phi_1(x)]/(1-\xi(\tau)) . \tag{5.32}$$

Hence given $\phi_1(x)$ and $\phi_\tau(x)$ we can define the non-separable time dependent net maternity function $\Phi(x,t)$, using Heaviside notation, by

$$\Phi(x,t) = \{[\xi(t)-\xi(\tau)]\phi_1(x) + [1-\xi(t)]\phi_\tau(x)\}H(\tau-t)/[1-\xi(\tau)] + \phi_\tau(x)H(t-\tau) , \quad \tau \leq \alpha . \tag{5.33}$$

With $\Phi(x,t)$ as given by (5.33) and assuming an initially stable population we have the total birth rate $B(t)$ from (2.32), given by

$$B(t) = F(t) + \int_0^t B(t-x)\phi_\tau(x)dx , \quad \tau \leq \alpha , \tag{5.34}$$

where,

$$\frac{e^{-r_1 t}}{Q_1} F(t) = \{[\xi(t)-\xi(\tau)]\phi_1^*(r_1) + [1-\xi(t)]\phi_\tau^*(r_1)\} \frac{H(\tau-t)}{1-\xi(\tau)} + \phi_\tau^*(r_1)H(t-\tau)H(\alpha-t) + H(t-\alpha)\int_t^\beta e^{-r_1 x}\phi_\tau(x)dx .$$

Assume that $\phi_\tau^*(0) = 1$ so that a stationary population results. We obtain the asymptotic stationary birth rate Q_2 by the usual method of Laplace transforms. The Laplace transform of (5.34) yields, upon rearrangement and using the definition of the Heaviside unit function,

$$[1-\phi_\tau^*(p)]B^*(p) = F^*(p) = c_1\int_0^\tau e^{-(p-r_1)t}\xi(t)dt + c_2\int_0^\tau e^{-(p-r_1)t}dt + \phi_\tau^*(r_1)\int_t^\alpha e^{-(p-r)t}dt + \int_\alpha^\beta e^{-(p-r_1)t}\int_t^\beta e^{-r_1 x}\phi_\tau(x)dx\,dt , \tag{5.35}$$

where, using the fact that $\phi_1^*(r_1) = 1$,

$$c_1 = \frac{1-\phi_\tau^*(r_1)}{1-\xi(\tau)} \quad ,$$

and $$c_2 = \frac{\phi_\tau^*(r_1)-\xi(\tau)}{1-\xi(\tau)} \quad . \qquad (5.36)$$

From (5.35), evaluating the last three integral expressions and combining the result of the last two, yields

$$B^*(p) = c_1 \frac{\int_0^\tau e^{-(p-r_1)t}\xi(t)\,dt}{1-\phi_\tau^*(p)} + c_2 \frac{1-e^{-(p-r_1)\tau}}{(p-r_1)[1-\phi_\tau^*(p)]}$$

$$+ \frac{\phi_\tau^*(r_1)e^{-(p-r_1)\tau} - \phi_\tau^*(p)}{(p-r_1)[1-\phi_\tau^*(p)]} \quad . \qquad (5.37)$$

Now, since $\phi_\tau^*(0)$ was assumed to be 1, and using the fact (from Chapter 2) that the real root has the greatest real part we have the asymptotic value Q_2, using the Tauberian result, given by

$$\frac{Q_2}{Q_1} = \lim_{p\to 0} pB^*(p) = \frac{c_1}{\kappa}\int_0^\tau e^{r_1 t}\xi(t)\,dt + \frac{c_2}{r_1\kappa}[e^{r_1\tau}-1] + \frac{1-\phi_\tau^*(r_1)}{r_1\kappa} \quad ,$$

or, on using $(5.36)_2$ and combining the last two expressions we have,

$$\frac{Q_2}{Q_1} = \frac{c_1}{\kappa}\int_0^\tau e^{r_1 t}\xi(t)\,dt + \frac{1-\phi_\tau^*(r_1)}{r_1\kappa}\cdot\frac{1-\xi(\tau)e^{r_1\tau}}{1-\xi(\tau)} \quad . \qquad (5.38)$$

κ is the expected age of childbearing in the eventual stationary population and is given by,

$$\kappa = \int_0^\infty x\phi_\tau(x)\,dx \quad .$$

We note that $p = r_1$ is not a pole, in (5.37), and the numerator of the first term in (5.37) is an entire function and thus using residue theory there is no contribution from this term.

Letting $\tau \to 0$ in (5.38) results in the asymptotic total birth rate obtained by Mitra (of the form (4.35)) under an abrupt change from any net maternity function $\phi_1(x)$ to any other $\phi_\tau(x)$.

As an example, consider the time dependence $\xi(t)$ to be exponential so that, $\xi(t) = e^{-\lambda t}$ in (5.38) gives the stationary asymptotic total birth rate,

$$Q_2 = Q_1 \frac{1-\phi_\tau^*(r_1)}{r_1 \kappa} \cdot \frac{1-e^{r_1 \tau} \xi(\tau)}{1-\xi(\tau)} \cdot \frac{\lambda}{\lambda - r} \quad , \quad \lambda > 0 \ ,$$

where we have used $(5.36)_1$.

5.5 General Time Dependence Defined in a Piecewise Fashion.

The time dependent change to replacement level fertility over $0 < t < \tau \lesssim \alpha$ which was presented in the previous section, represents in many circumstances an unachievable goal. At best the change allowed is up to $t = \alpha$ (= 10 or 15 years) whereas a number of authors have indicated a longer period of time for the change of the age-specific birth rate to replacement level.

Frejka (1973) allows for a linear change over 0, 10, 30, 50, 70 years of the Gross Reproduction Rate to replacement level and thus presents a range of alternatives that would be meaningful for both the more and the less developed countries. Keyfitz (1971b) states

"About the best any country can hope for is a gradual drop of fertility over 30 or more years."

In Section [5.1] we considered the time dependence to be expressed as a sum of exponentials with commensurable exponents, but, determining the constants $\{\gamma_m\}$ and $\{\lambda_m\}$ is an extremely difficult problem. A model which allows for a more gradual (than exponential) change was presented in Section [5.2] but, the convergence of the backward recurrence relation method to obtain Q_2 is slower than that for the models of Chapter 4.

It is difficult to anticipate the effects of certain policies which would produce changes in the fertility behaviour of a population. However one can imagine that such a change would vary most in the short term before settling down. Thus, in the present section we introduce a time dependent net maternity function which is defined in a piecewise fashion. The model allows for a general time dependent change over $0 < t < \tau \lesssim \alpha$ with a simpler change (such as exponential) for

$t > \alpha$.

Consider the separable time dependent net maternity function given by

$$\Phi(x,t) = \{\chi(t)H(\tau-t) + \theta(t)H(t-\tau)\}\phi(x) \ , \qquad \tau \lesseqgtr \alpha \ , \tag{5.39}$$

where,

$$\left.\begin{aligned} &\chi(0) = a_o \ , \quad \chi(\tau) = a_\tau \ , \\ &\theta(t) = \frac{1}{R} + \left(a_\tau - \frac{1}{R}\right)\xi(t-\tau) \ , \\ &\xi(t) \text{ satisfies the conditions (5.2) and is strictly exponential,} \\ &\phi(x) \neq 0 \quad \text{for} \quad \alpha < x < \beta \ , \end{aligned}\right\} \tag{5.40}$$

and H is the Heaviside unit function.

With (5.39) and assuming an initially stable population the total birth rate $B(t)$, from (4.13) and (4.14), is given by

$$B(t) = F(t) + \theta(t)H(t-\tau)\int_o^t B(t-x)\phi(x)dx \ , \qquad \tau \lesseqgtr \alpha \ , \tag{5.41}$$

where,

$$\frac{e^{-rt}}{Q}F(t) = \chi(t)H(\tau-t) + \theta(t)\left\{H(t-\tau)H(\alpha-t) + H(t-\alpha)\int_t^\beta e^{-rx}\phi(x)dx\right\} \ ,$$

and, we have used the fact that $\phi^*(r) = 1$.

We will examine in some detail the above model with

$$\xi(t) = e^{-\lambda t} \ , \tag{5.42}$$

but, any of the previous exponential representations of $\xi(t)$, satisfying (5.2), could be taken.

In order to find the asymptotic total birth rate Q_2 we proceed in the usual manner using Laplace transform techniques.

The Laplace transform of (5.41), with (5.40) and (5.42), yields

$$B^*(p) = F^*(p) + \frac{\phi^*(p)}{R} B^*(p) + \left(a_\tau - \frac{1}{R}\right) e^{\lambda\tau} \phi^*(p+\lambda) B^*(p+\lambda) \ , \qquad (5.43)$$

where,

$$\left.\begin{aligned} \frac{F^*(p)}{Q} &= \int_0^\tau e^{-(p-r)t} \chi(t)dt + \int_\tau^\alpha e^{-(p-r)t} \theta_o(t)dt \\ &\quad + \int_\alpha^\beta e^{-(p-r)t} \theta_o(t) \int_t^\beta e^{-rx} \phi(x)dx\, dt \\ \text{and} \quad \theta_o(t) &= \frac{1}{R} + \left(a_\tau - \frac{1}{R}\right) e^{-\lambda(t-\tau)} \ . \end{aligned}\right\} \qquad (5.44)$$

Evaluating the integrals in $(5.44)_1$, and using $(5.44)_2$ we obtain, after some simplification,

$$\frac{F^*(p)}{Q} = \int_0^\tau e^{-(p-r)t} \chi(t)dt + \frac{f(p)}{R} + \left(a_\tau - \frac{1}{R}\right) e^{\lambda\tau} f(p+\lambda) \ , \qquad (5.45)$$

where,

$$f(p) = \frac{e^{-(p-r)\tau} - \phi^*(p)}{p-r} \ .$$

Isolating $B^*(p)$ in (5.43) and using the facts that

$$\frac{\phi^*(0)}{R} = 1 \ ,$$

and $\lim_{p\to 0} pB^*(p) = Q_2$, the asymptotic value,

we have, on letting $p \to 0$,

$$\kappa Q_2 = F^*(0) + \left(a_\tau - \frac{1}{R}\right) e^{\lambda\tau} \phi^*(\lambda) B^*(\lambda) \ , \qquad (5.46)$$

where from (5.45) and using $\phi^*(0) = R$

$$\frac{F^*(0)}{Q} = \int_0^\tau e^{rt} \chi(t)dt + \frac{R - e^{r\tau}}{rR} + \left(a_\tau - \frac{1}{R}\right) \frac{e^{r\tau} - e^{\lambda\tau}\phi^*(\lambda)}{\lambda - r} \ . \qquad (5.47)$$

In order to obtain the asymptotic value Q_2 from (5.46) we need to know $B^*(\lambda)$. Hence we proceed, as discussed on earlier occasions, by putting $p = n\lambda$ in (5.43) and using (5.45); thus resulting in a recurrence relation of the form

$$B^*(n\lambda) = \delta_n + \varepsilon_n B^*\big((n+1)\lambda\big) \ , \quad n = 1,2,\ldots, \qquad (5.48)$$

where,

$$\delta_n = \frac{RF^*(n\lambda)}{R-\phi^*(n\lambda)} \quad ,$$

and $$\varepsilon_n = (Ra_\tau - 1)\ \frac{e^{\lambda\tau}\phi^*((n+1))}{R-\phi^*(n\lambda)} \quad .$$

Successive approximations to $B^*(\lambda)$ and hence Q_2 are obtained from (5.48) and (5.46) respectively by assuming $B^*(N\lambda) = 0$ for some N . We note that

$$\tilde{Q}_2 = \frac{F^*(0)}{\kappa} \quad ,$$

is the approximation to Q_2 in assuming $B^*(\lambda) = 0$. Also, the error analysis of Subsection [4.1.2] may not readily be used here since a meaningful upper bound on $B(t)$, and hence $B^*(\lambda)$, is not at all obvious for general $\chi(t)$. However, comparison of successive approximations to Q_2 can be made, stopping when the relative error is considered small enough.

Frejka (1973) considers the problem of determining the transient effect on a population, of a 15% linear increase of the age-specific birth rate over 10 years followed by a decrease to replacement level, over a variety of time intervals. Keyfitz (1975) notes that; with an initially increasing population it is necessary, at first, to reduce the age-specific birth rate to well below replacement level in order that the population be kept at its present size. A time dependent net maternity function of the form (5.39) – (5.40) with (5.42) may be used to model the general initial increase followed by a decrease to replacement level fertility. The alternative problem of an initial gradual drop followed by an increase to replacement level fertility of Keyfitz may also be modelled in a similar fashion. Figures 5.8 and 5.9 represent general changes, of the age-specific birth rate, of the type described by Frejka (1973) and Keyfitz (1975).

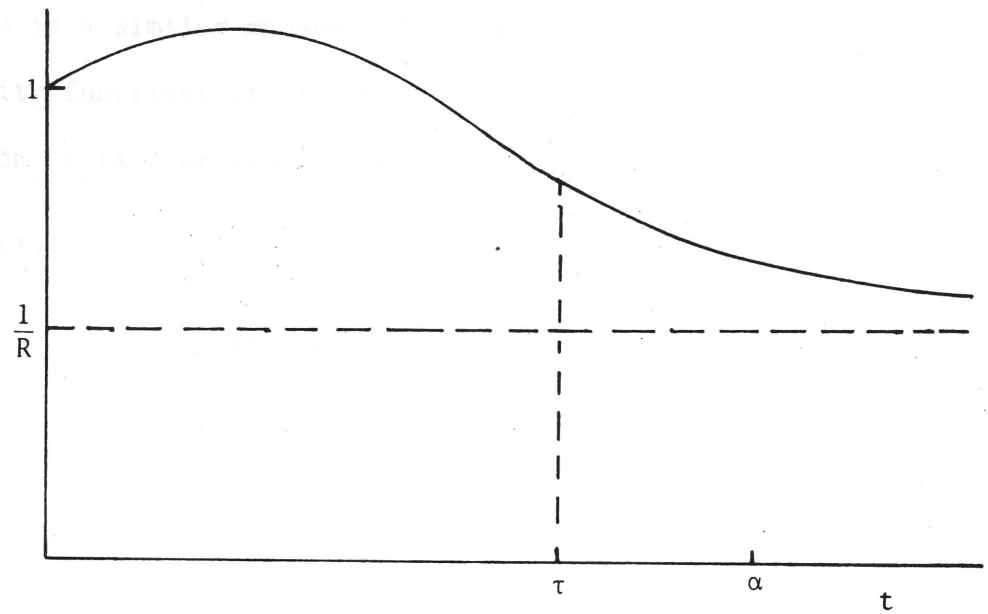

FIGURE 5.8
Diagrammatical representation of an initial increase, followed by a decrease to replacement as envisaged by Frejka (1973).

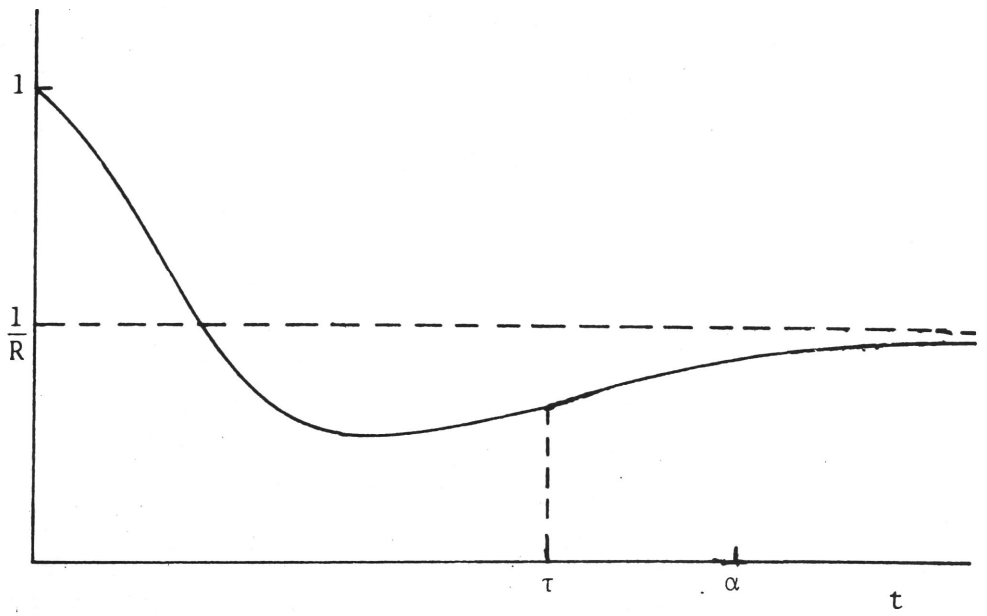

FIGURE 5.9
An initial decrease in the age-specific birth rate is necessary (Keyfitz (1975)) if the present numbers are to be asymptotically maintained. The above sketch shows such a change.

A non-separable time dependent net maternity function of the form (5.3) with exponential time dependence for t greater than α can be handled in a similar manner. Thus, given $\phi_1(x)$, $\phi_\tau(x)$ the net maternity functions at $t = 0$, $t = \tau$ and, the eventual net maternity function $\phi_2(x)$ we can define

$$\Phi(x,t) = \{[\xi(t)-\xi(\tau)]\phi_1(x) + [1-\xi(t)]\phi_\tau(x)\}H(\tau-t)/[1-\xi(\tau)]$$

$$+ \{\phi_2(x) + e^{-\lambda(t-\tau)}[\phi_\tau(x)-\phi_2(x)]\}H(t-\tau) \ , \quad \tau \leq \alpha \ .$$

6. The Asymptotic Effects when a Time Dependent Net Maternity Function Includes Changes in the Survivor Function.

In considering the asymptotic effects of a time dependent net maternity function, only the age-specific birth rate has so far been assumed to change. In the present chapter the time dependent net maternity function includes temporal changes in the survivor function and thus, in the age-specific death rate.

The effect of abrupt changes in the vital rates on stable population parameters in general, have been extensively treated by Coale (1956), Goodman (1971), Keyfitz (1968a) and (1971b), Preston (1974) and Tognetti (1976b), to name a few. Coale (1956) also determines the effect on the age distribution for $t < 2\alpha$ (where α is the youngest age of childbearing) when the age-specific death rate, and hence the survivor function, changes with time. The transient behaviour of a population was determined by Frejka (1973) using projection techniques, where he assumed the age-specific death rate to change linearly with time.

A non-separable time dependent survivor function is presented in Section [6.1] and the asymptotic behaviour is determined for the case of a simple exponential time dependence. This time dependent survivor function invokes a gradual change in the age-specific death rate.

The eventual stable population parameters due to time dependent, age-specific birth rate and survivor function are examined in Section [6.2]. Both the time dependent survivor function and the time dependent age-specific birth rate are taken as being non-separable and hence the time dependent net maternity function is non-separable.

The time dependence is taken for simplicity to be exponential, and thus, by taking extreme values of the exponent transition rates the results of previous models can be obtained.

Given certain restrictions on changes of the net maternity function parameters, the achievement of a given goal may not be possible unless the age distribution is allowed to change (through migration for example). Abrupt changes of the initial age distribution are discussed in Section [6.3].

6.1 A Gradual Change in the Survivor Function.

In the previous two chapters we have been concerned with finding the asymptotic behaviour of a population experiencing time dependent changes in the age-specific birth rate. We now consider the consequences of changes with time in the survivor function, and, hence of changes in the net maternity function.

In this section we show that the time dependent survivor function must not be separable. Although we only examine single exponential changes in the survivor function, the extensions of Chapter 5 can equally well be applied.

Theorem: Any time dependent function of the form

$$L(x,t) = \xi(t)\ell_1(x) + [1-\xi(t)]\ell_2(x) \quad , \tag{6.1}$$

with $L(x,t) \geq 0$ for all x and t, can be a survivor function, given that $\ell_1(x)$ and $\ell_2(x)$ are the initial and the eventual survivor functions respectively.

Proof: For $\{\ell_i(x)\}$ to be survivor functions we have, from (2.1)

$$\int_0^\infty \ell_i'(x)dx = -\int_0^\infty \mu_i(x)\ell_i(x)dx = -1 \quad , \quad i = 1,2, \tag{6.2}$$

where $\mu_1(x)$ and $\mu_2(x)$ are the initial and the eventual age-specific death rates respectively.

Differentiating (6.1) with respect to x yields

$$L_x(x,t) = \xi(t)\ell_1'(x) + [1-\xi(t)]\ell_2'(x) \quad , \tag{6.3}$$

where the subscript x denotes partial differentiation with respect to x. Thus, integrating (6.3) with respect to x from 0 to ∞ and using (6.2) gives

$$\int_0^\infty L_x(x,t)dx = -1 \quad ,$$

and hence we conclude that (6.1) is a permissible survival behaviour.

Corollary: Given that $\{\ell_i(x)\}$ are survivor functions then, a function of the form

$$L(x,t) = \sum_{i=1}^{I} \xi_i(t)\ell_i(x) \quad ,$$

is also a survivor function provided that

$$\sum_{i=1}^{I} \xi_i(t) = 1 \ .$$

The time dependent survivor function of the Corollary is at present mainly of theoretical, as opposed to practical, interest. Quite the opposite is true of (6.1), however. Coale (1972) states that

"There is a remarkable similarity in the age schedules of mortality among today's highly industrialized countries" and hence, there is a similarity in the age-specific survivor functions. The time dependent survivor function, as given by (6.1), represents a smooth transition from some initial survivor function $\ell_1(x)$ towards some eventual survival goal, $\ell_2(x)$. Hence, (6.1) may represent a temporal change of the mortality behaviour of a less developed country towards some standard mortality pattern of a more developed country.

We will now consider the effects, in particular the asymptotic effects, of a time dependent survivor function of the form (6.1). The age-specific birth rate $m(x)$ is assumed not to change with the passage of time.

Assuming an initially stable population we have, from (2.32),

$$B(t) = Q_1 e^{r_1 t}\int_t^\infty e^{-r_1 x} m(x)L(x,t)dx + \int_o^t B(t-x)m(x)L(x,t)dx \ , \qquad (6.4)$$

for the total birth rate, where $t > 0$.

The asymptotic behaviour of the population will be discussed where, from (6.1) with $\xi(t) = e^{-\lambda t}$, $L(x,t)$ is given by

$$L(x,t) = e^{-\lambda t}\ell_1(x) + [1 - e^{-\lambda t}]\ell_2(x) \ , \quad \lambda > 0 \ . \qquad (6.5)$$

It should be noted that; (6.1) in general and, (6.5) in particular, represent non-separable time dependent survivor functions. The question arises as to whether we can have $L(x,t)$ separable. $L(x,t)$, defined for all positive x and t , cannot be separable; for if $\ell_2(x) = c\ell_1(x)$ where c is a constant and $\ell_1(x)$ is a survivor function then $\ell_2(x)$ is necessarily <u>not</u> a survivor function except for the trivial case, $c = 1$.

It is of interest to find the asymptotic behaviour of (6.4), with (6.5). We do not have to proceed further however, if we note that the time dependent net maternity function from (2.26), (2.32), (6.4) and (6.5), is non-separable and is given by (4.36) where the initial and the eventual net maternity functions, $\phi_1(x)$ and $\phi_2(x)$, are <u>now</u> given by

$$\phi_i(x) = m(x)\ell_i(x) \ , \quad i = 1,2 \ . \qquad (6.6)$$

Hence, the total birth rate $B(t)$ will asymptotically behave like

$$B(t) = Q_2 e^{r_2 t} \ , \qquad (6.7)$$

where Q_2 is obtained from (4.41) , using the recurrence relation method discussed in Section [4.2]; and, r_2 is given by (4.37).

Once the asymptotic behaviour for the total birth rate ((6.7)) is determined then the asymptotic behaviour of the total number and hence the asymptotic stable age density can be found. The total number will asymptotically approach

$$N(t) = P_2 e^{r_2 t} \quad , \tag{6.8}$$

where,

$$P_2 = \frac{Q_2}{b_2} \quad , \quad \text{the eventual stable equivalent,}$$

$$\text{with} \quad b_2 = \frac{1}{\int_0^\infty e^{-r_2 x} \ell_2(x)\, dx} \quad , \quad \text{the eventual crude birth rate.}$$

The asymptotic stable age density is given by

$$a^{(2)}(x) = b_2 e^{-r_2 x} \ell_2(x) \quad . \tag{6.9}$$

It should be noted that, if we let $\lambda \to \infty$ in (6.5) then we obtain an abrupt or step change in the survivor function at time $t = 0$. Hence the stable equivalent births Q_2 may be obtained from (4.41) by letting $\lambda \to \infty$, where, the population is, of course, assumed to be initially stable.

Given a survivor function of the form (6.5), one may ask the important question as to what change is needed in the age-specific birth rate so that an asymptotically stationary population results. If the change in the age-specific birth rate is assumed to be abrupt then a change from $m(x)$ to any $m_2(x)$ so that $\phi_2^*(0) = 1$ would suffice. In particular if we abruptly scaled $m(x)$ to $\frac{m(x)}{R_2}$, in a manner similar to that of Keyfitz (1971b), then, a stationary population would eventuate. Temporal changes of the age-specific birth rate are discussed in the next section.

The converse problem discussed in Section [4.3] can also be analysed here. Thus; given the stable equivalent births Q_2 then, we can determine the transition rate λ needed for the total birth rate to asymptotically behave like $Q_2 e^{r_2 t}$. Similarly, given $P_2 = \tilde{P}_2$ we can find a λ which will produce $\tilde{Q}_2 = b_2 \tilde{P}_2$ and the population would then asymptotically behave like $\tilde{P}_2 e^{r_2 t}$.

It is of interest to determine the required transition rate, λ, for a population to have the eventual stable equivalent numbers equal to the initial. That is, we want to determine a λ which will produce

$$\frac{P_2}{P_1} = 1 = \frac{b_1}{b_2} \cdot \frac{Q_2}{Q_1} \quad . \tag{6.10}$$

Hence we seek λ, from (4.41) and using (4.39) with $p = r_2 + n\lambda$, which will give $Q_2 = \tilde{Q}_2 = \frac{b_2}{b_1} Q_1$. Of course, such a λ may not exist.

6.2 Changes in Both the Survivor and the Age-Specific Birth Rate with Time.

The problems of determining the asymptotic behaviour of a population resulting from time dependent changes in the age-specific birth rate and in the survivor function, have been discussed in Chapters 4 and 5, and, in Section [6.1] respectively. We now consider the asymptotic effects of the more realistic situation in which both the age-specific birth rate and the survivor, and hence the age-specific death rate, change with time. However, the previous studies are necessary in evaluating the relative contribution between a change in the age-specific birth rate with time while keeping the age-specific death rate constant, and visa versa.

Let the time dependent age-specific birth rate $M(x,t)$, and the time dependent survivor function $L(x,t)$, be given by

$$\left.\begin{aligned} M(x,t) &= \xi_1(t)m_1(x) + \xi_2(t)m_2(x) \ , \\ \text{and} \quad L(x,t) &= \zeta_1(t)\ell_1(x) + \zeta_2(t)\ell_2(x) \ . \end{aligned}\right\} \tag{6.11}$$

The time dependent net maternity function $\Phi(x,t)$ is thus given, from (2.26), by

$$\Phi(x,t) = \sum_{i=1}^{2}\sum_{j=1}^{2} \psi_{ij}(t)\phi_{ij}(x) \ , \tag{6.12}$$

where

$$\left.\begin{aligned} \psi_{ij}(t) &= \xi_i(t)\zeta_j(t) \ , \\ \text{and} \quad \phi_{ij}(x) &= m_i(x)\ell_j(x) \ , \end{aligned}\right\} \tag{6.13}$$

with $\zeta_2(t) = 1-\zeta_1(t)$ so that $L(x,t)$ in $(6.11)_2$ satisfies the Theorem of [6.1]. Further, $\{\xi_i(t)\}$ and $\{\zeta_j(t)\}$ are such that

$$\left.\begin{aligned} \Phi(x,0) &= \phi_{11}(x) \ , \\ \text{and} \quad \lim_{t\to\infty} \Phi(x,t) &= \phi_{22}(x) \ . \end{aligned}\right\} \tag{6.14}$$

Thus $\phi_{11}(x)$ and $\phi_{22}(x)$ are the initial and the eventual age-shapes of the time dependent net maternity function, (6.12).

We extend the notation used previously so that population parameters relating to the inherent age structure $\{\phi_{ij}(x)\}$ of the time dependent net maternity function (6.12) will be denoted by using a double subscript of ij. For example, the intrinsic rates of change $\{r_{ij}\}$ and the net reproduction rates $\{R_{ij}\}$ relating to $(6.13)_2$ are given by

$$\phi_{ij}^{*}(r_{ij}) = 1 \quad \text{and} \quad \phi_{ij}^{*}(0) = R_{ij} , \quad i = 1,2; \quad j = 1,2 . \tag{6.15}$$

Thus from (2.28), (6.12) and (6.15) we have the time dependent net reproduction rate

$$R(t) = \sum_{i=1}^{2} \sum_{j=1}^{2} R_{ij}\psi_{ij}(t) . \tag{6.16}$$

Substituting (6.12) into (2.32) and making use of the notation just outlined, we have the total birth rate, in an initially stable population, represented by

$$\left.\begin{aligned} B(t) &= \sum_{i=1}^{2} \sum_{j=1}^{2} \psi_{ij}(t)C_{ij}(t) , \\ \text{where,} \\ C_{ij}(t) &= Q_{11}e^{r_{11}t}\int_{t}^{\infty} e^{-r_1 x}\phi_{ij}(x)dx + \int_{0}^{t} B(t-x)\phi_{ij}(x)dx. \end{aligned}\right\} \tag{6.17}$$

Assuming that $\phi_{22}^{*}(0) = 1$, we proceed to find the asymptotic total birth rate Q_{22}, from (6.17) with (6.13), where

$$\left.\begin{aligned} &\xi_1(t) = e^{-\nu t} , \quad \nu > 0 , \quad \xi_2(t) = 1-\xi_1(t) , \\ \text{and} \quad &\zeta_1(t) = e^{-\sigma t} , \quad \sigma > 0 , \quad \zeta_2(t) = 1-\zeta_1(t) . \end{aligned}\right\} \tag{6.18}$$

From the results of Section [5.1] we can anticipate that slight difficulties will be encountered in using the recurrence relation

method to obtain the asymptotic value. These minor difficulties may be overcome, as noticed in [5.1], by allowing ν and σ of (6.18) to be commensurable; viz.

$$\left.\begin{aligned} \nu &= k\lambda \\ \text{and}\quad \sigma &= K\lambda \quad , \end{aligned}\right\} \tag{6.19}$$

where k and K are positive integers and,

λ is a positive constant.

We proceed to find the asymptotic value Q_{22} (for $0 < \sigma, \nu < \infty$) of (6.17) with (6.13) and (6.18) in the usual manner, using Laplace transform techniques. The Laplace transform of (6.17) and using (6.13) with (6.18), yields

$$\begin{aligned}[1-\phi^*_{22}(p)]B^*(p) &= C^*_{11}(p+\nu+\sigma) + C^*_{12}(p+\nu) - C^*_{12}(p+\nu+\sigma) + C^*_{21}(p+\sigma) \\ &\quad - C^*_{21}(p+\nu+\sigma) + F^*_{22}(p) - C^*_{22}(p+\nu) - C^*_{22}(p+\sigma) + C^*_{22}(p+\nu+\sigma),\end{aligned} \tag{6.20}$$

where we have isolated $B^*(p)$, and, from $(6.17)_2$,

$$\left.\begin{aligned} C^*_{ij}(p) &= F^*_{ij}(p) + \phi^*_{ij}(p)B^*(p) \quad , \\ F^*_{ij}(p) &= Q_{11}\,\frac{\phi^*_{ij}(r_{11}) - \phi^*_{ij}(p)}{p - r_{11}} \quad , \quad i,j = 1,2 \ . \end{aligned}\right\} \tag{6.21}$$

Hence letting $p \to 0$, the real root of $\phi^*_{22}(p) = 1$, (and the root with the greatest real part) and using the Tauberian result

$$Q_{22} = \lim_{p\to 0} pB^*(p) \ , \quad \text{the asymptotic value,}$$

we obtain from (6.20), the eventual stationary total birth rate as

$$\begin{aligned}\kappa_{22}Q_{22} &= C^*_{11}(\nu+\sigma) + C^*_{12}(\nu) - C^*_{12}(\nu+\sigma) + C^*_{21}(\sigma) - C^*_{21}(\nu+\sigma) \\ &\quad + Q_{11}\,\frac{1 - \phi^*_{22}(r_{11})}{r_{11}} - C^*_{22}(\nu) - C^*_{22}(\sigma) + C^*_{22}(\nu+\sigma) \ ,\end{aligned} \tag{6.22}$$

where

$\kappa_{22} = \int_0^\infty x\phi_{22}(x)dx$, the expected age of childbearing in the subsequent stationary population.

The expression (6.22) with (6.21) for the asymptotic value Q_{22} , contains the unknown quantities $B^*(\nu)$, $B^*(\sigma)$ and $B^*(\nu+\sigma)$ where ν and σ are given by (6.19). Since ν and σ are integer multiples of some constant λ we may recover these unknown quantities by setting up a recurrence relation from (6.20) in the same manner as on previous occasions. Thus putting $p = n\lambda$ in (6.20) and, using (6.19) with (6.21) gives a recurrence relation of the form

$$B^*(n\lambda) = \delta_n + \varepsilon_n^{(1)}B^*[(n+k)\lambda] + \varepsilon_n^{(2)}B^*[(n+K)\lambda] + \varepsilon_n^{(3)}B^*[(n+k+K)\lambda] , \quad n = 1,2,\ldots , \tag{6.23}$$

where k and K are fixed positive integers.

Assuming $B^*(N\lambda) = 0$ for some N , successive approximations to $B^*(k\lambda)$, $B^*(K\lambda)$ and $B^*[(k+K)\lambda]$ may be obtained from (6.23), which in turn, from (6.22), give successive approximations to Q_{22} . As stated earlier on many other occassions, the speed of convergence of the backward recurrence relation method relies on how fast a product $\left\{\varepsilon_n^{(j)}\right\}_{n=1}^{N-1} \to 0$ as N increases. Since, as in all demographically realistic situations $\{\phi_{ij}(x)\}$ have compact support, then $\{\phi_{ij}^*(q\lambda)\}$ decrease exponentially for increasing q and hence $\left\{\varepsilon_n^{(j)}\right\}$ decrease rapidly. The speed of convergence of the method was amply demonstrated in Chapter 4 and we will not do so here.

The error analysis of [4.1.2] cannot be used here since we do not have a useful upper bound on $B(t)$, and hence, on the unknown quantities $B^*(\nu)$, $B^*(\sigma)$ and $B^*(\nu+\sigma)$ of (6.22) with (6.21). However, successive approximations to Q_{22} can be made, stopping when

the relative error is considered small enough.

For $0 < \nu, \sigma < \infty$ the time dependent net maternity function, (6.12) with (6.13) and (6.18), represents a gradual change in both the age-specific birth rate and the survivor function (and hence in the age-specific death rate). It may be seen that by taking extreme values of ν and σ, not only can we reproduce the models of Chapter 4 and Section [6.1] but, we can also obtain a variety of further possibilities. The quantities ν and σ are known as transition parameters since they characterise a gradual exponential change from an initial to an eventual function. From (6.11), $(6.13)_1$ and (6.18) it can be seen that if we allow a transition parameter to tend to zero, there is no change from the initial data. Letting a transition parameter tend to infinity results in an abrupt change to a new schedule. There are eight various models possible if we allow ν and/or σ to tend to 0 and/or ∞. It is instructive to consider some examples.

If we let $\nu,\sigma \to \infty$ then we have a model in which both the initial age-specific birth rate and survivor, change abruptly to $m_2(x)$ and $\ell_2(x)$ respectively and hence, from (6.22), the asymptotic total birth rate Q_{22} is

$$Q_{22} = Q_{11} \quad \frac{1 - \phi^*_{22}(r_{11})}{r_{11}\kappa_{22}} \quad .$$

We note that the asymptotic, (6.22), only exists if eventually the time dependent net maternity function approaches $\phi_{22}(x)$; otherwise, we would have to go back to (6.20) to determine the asymptotic behaviour. The simplest example of this occurs if we let ν and $\sigma \to 0$; then there is no change, and, the population would continue on its initial trajectory. Letting $\sigma \to 0$ results in the model of Section [4.2] and, from (6.20) and using (6.21), we get

$$[1-\phi_{21}^*(p)]B^*(p) = F_{21}^*(p) + C_{11}^*(p+\nu) - C_{21}^*(p+\nu) ,$$

and hence

$$B(t) \sim Q_{21}e^{r_{21}t} ,$$

where

$$\kappa_{21}Q_{21} = F_{21}^*(r_{21}) + C_{11}^*(r_{21}+\nu) - C_{21}^*(r_{21}+\nu) ,$$

with $\kappa_{21} = \int_0^\infty e^{-r_{21}x}x\phi_{21}(x)dx$ and we have used (6.15).

With the stationary total birth rate Q_{22} given from (6.22), the asymptotic total number P_{22} is given by

$$N(t) \sim P_{22} = \frac{Q_{22}}{b_{22}} ,$$

where

$$b_{22} = \frac{1}{\int_0^\infty \ell_2(x)dx} ,$$ the crude birth rate in the eventual stationary population.

The asymptotic stationary age density is given by

$$a^{(22)}(x) = b_{22}\,\ell_2(x) .$$

The converse problem can be somewhat more difficult than in Section [4.3] since we now have 3 parameters of freedom namely; k, K and λ. If either ν or σ were given then only one transition rate would remain to be determined. For $0 < \nu, \sigma < \infty$ where ν and σ satisfy (6.19) we need to specify two parameters and hence determine the third.

6.3 A Time Dependent Net Maternity Function With Abrupt Changes of the Initial Age Distribution.

The converse problem of determining changes of the initial population parameters which will produce a certain goal is a very important and worthwhile one. There are basically three parameters which are necessary to characterise or determine a particular population, namely, the initial age-specific birth and death rates, and, the initial age distribution, $A(x,0)$. Thus changes of a population are brought about or are due to, changes of these three parameters. The converse problem has already been examined in the present study where, given a particular asymptotic behaviour, the transition rate characterising the variation with time of the initial vital rates, has been sought. If the changes in the vital rates are restricted then, changing the initial age distribution through migration, for example, would have to be implemented if a certain goal is to be realised.

Keyfitz (1971c) discusses the use of migration as a means of controlling a population. Preston (1970) states that the role of age-composition, which is often ignored, is a major factor in the growth of many populations. Both Le Bras (1969) and Tognetti (1976a) discuss the effect of a catastrophe (an abrupt change of the age distribution) on the ensuing population.

The age-specific birth and death rates alone determine the intrinsic rate of change of a population. Whereas, the stable equivalent births and numbers are a function of both vital rates and also the initial age distribution. Given $\phi_{22}(x)$ of $(6.13)_2$ such that $\phi^*_{22}(0) = 1$ then, the population resulting from the model (6.17)-(6.18) will eventually become stationary. Specifying a particular goal for the total number, then we may determine transition rates ν and σ that will achieve the

set target. Such transition rates may not exist however, and to realise the goal, changes to the initial age distribution through migration would have to be imposed. The changes of the initial age distribution would have to be assumed to be abrupt since it is not known by the author how these changes can be implemented otherwise, unless, the change only related to those of the initial population. Allowing for changes of the age distribution with time for the continuous model is an area for further research.

7. Conclusion

In the present work, stable population theory has been extended to allow exponential-based time variation in the age-specific, vital rates of birth and death. Thus the analysis allows for the determination of the effect of a time dependent net maternity function on the ensuing population.

Using Laplace transform techniques and residue theory, (reviewed in Chapter 2) a general method for finding the asymptotic total birth rate has been obtained where the time dependent net maternity function has been scaled exponentially to replacement level. The method, which involves setting up a recurrence relation, converges very rapidly and is dependent on how fast the replacement level rates are achieved. With such a model, a change at a rate of ten times the initial rate of increase of a population, gives of the order of 10% increase in the eventual total number, over the abrupt change of Keyfitz. A slower transition towards replacement level fertility rates produces a greater amount of impetus for further growth. The momentum or potential of a population is evaluated by comparing the asymptotic total number, obtained as a consequence of the recurrence relation method, with the initial total number.

The transient behaviour of the population is obtained by three methods which are thought by the author to be novel to this thesis.

Firstly, a strictly numerical quadrature method (the modified block-by-block method of Campbell and Day (1971)) is used to solve the Volterra integral equation of the second kind, which governs the total birth rate of the Sharpe-Lotka population model. To do this, spline interpolation of the discrete data was used to obtain continuous

integrands.

Secondly, with the net maternity function defined in a piecewise manner, a stepping procedure has been developed which may suffice to obtain a solution for simple time dependent changes in the net maternity function. However, this procedure does become less efficient the further away from the origin the solution is sought.

Thirdly, the recurrence relation method used to obtain the stable equivalent, has been extended to obtain the contribution from some of the complex roots of the characteristic equation with greatest real part.

A number of generalisations and extensions have been presented which allow the determination of the effect of various time paths of change of the initial net maternity function. A model which is of substantial importance is that which allows for various age-groups to change differently with time, thus depicting variations with age of the influence of certain policies which produce the change. Differential fertility models between the parent and subsequent populations have also been examined, and the effect on the momentum of population growth analysed.

The problem of paramount importance in the management of populations is that of determining the time path of change in the net maternity function, given the desired asymptotic behaviour. This converse problem was analysed in Chapter 4 where the transition rate λ (which characterises the assumed exponential change) was determined, given a desired asymptotic value.

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APPENDIX A. Special Solutions of the System (3.29)-(3.31).

(A) $\mu_1 = \mu_2$ $(\theta = 0)$.

The solution to (3.29)-(3.31), with $\theta = 0$ (that is, when only the birth rate changes with time), is given by (3.20) and (3.19); with (3.24) and $b(t) = e^{-r_2 t}B_o(t)$ (from (3.25)). Thus,

$$b(t) = \frac{Q_1}{\gamma_1}[\gamma_2 + \omega\xi(t)]\exp\left\{\omega\int_o^t \xi(s)ds\right\} , \quad \omega = \gamma_1 - \gamma_2 . \tag{A.1}$$

If $\lim_{t\to\infty} \int_o^t \xi(s)ds = \xi_\infty < \infty$ then $b(t)$, as defined in (A.1), admits an asymptotic value $\left(Q_1 \frac{\gamma_2}{\gamma_1} e^{\omega\xi_\infty}\right)$ and, we have used the condition on $\xi(t)$, $(3.17)_1$.

(B) $r_1 = r_2$ $(\theta = \omega)$.

With $\theta = \omega$, namely when the initial and final rates of change are equal, the differential equation (3.29)-(3.30) is reduced to

$$b''(t) + [\gamma_2 + u(t) - u'(t)/u(t)]b'(t) = 0 , \tag{A.2}$$

where the initial conditions (3.31) become

$$\left.\begin{aligned} b(0) &= Q_1 , \\ \text{and} \quad b'(0) &= 0 . \end{aligned}\right\} \tag{A.3}$$

From (A.2), by reducing the order of the differential equation, we obtain

$$b'(t) = Ae^{-\gamma_1 t}\xi(t)u(t)\exp\left\{\theta\int_o^t \xi(s)ds\right\} ,$$

where A is an arbitrary constant of integration. Upon using the initial condition $(A.3)_2$ we obtain the rather surprising result that

$$b(t) = Q_1 , \tag{A.4}$$

for all $t \in \mathbb{R}_+$ and for arbitrary $\xi(t)$ satisfying conditions $(3.17)_1$.

Thus from (3.25) $B(t)$ continues with its initial trajectory of $B(t) = Q_1 e^{r_1 t}$, $t > 0$, since $r_1 = r_2$.

(C) $\underline{|\theta-\omega| = \varepsilon << 1}$.

The solution of (3.29)-(3.31) with $\theta = \omega$, namely (A.4), suggests that we may formally seek a perturbation type solution of the form

$$b(t) = Q_1 + \varepsilon b_1(t) , \tag{A.5}$$

where $|\theta-\omega| = \varepsilon << 1$.

Substitution of (A.5) into the system (3.29)-(3.31) and neglecting $0(\varepsilon^2)$ terms results in the differential equation for $b_1(t)$ given by

$$b_1''(t) + [\gamma_2 + u(t) - u'(t)/u(t)]\, b_1'(t) = \xi(t)\left[\frac{u'(t)}{u(t)} - \frac{\xi'(t)}{\xi(t)} - \gamma_2\right]Q_1, \tag{A.6}$$

subject to the initial conditions

$$\left.\begin{aligned} & b_1(0) = 0 \\ \text{and} \quad & b_1'(0) \pm \left(1 + \frac{\xi'(0)}{r_1+\mu_2}\right)Q_1 = 0 , \quad \theta \begin{Bmatrix} > \\ < \end{Bmatrix} \omega , \end{aligned}\right\} \tag{A.7}$$

obtained from (3.31) and (A.5). Hence from (A.6), reduction of the order of integration results in

$$I(t)b_1'(t) = Q_1\int_0^t I(s)\xi(s)\left[\frac{u'(s)}{u(s)} - \frac{\xi'(s)}{\xi(s)} - \gamma_2\right]ds + A \tag{A.8}$$

where,

$$I(t) = \exp\left\{\gamma_1 t - \theta\int_0^t \xi(s)ds\right\}\Big/\xi(t)u(t) , \tag{A.9}$$

and upon using $(A.7)_2$ and assuming $\theta > \omega$,

$$A = \left(\frac{1}{\xi'(0)} + \frac{1}{r_1+\mu_2}\right)Q_1 . \tag{A.10}$$

Hence, noting that $\frac{u'(s)}{(u(s))^2} = -\frac{d}{ds}\left(\frac{1}{u(s)}\right)$ and integrating a portion of (A.8) by parts we obtain, after some algebra,

$$I(t)[b_1'(t) + Q_1\xi(t)] = Q_1V(t) + A \ , \tag{A.11}$$

where,

$$V(t) = \int_o^t I(s)\xi(s)[\omega - \theta\xi(s) - \xi'(s)/\xi(s)]ds \ . \tag{A.12}$$

Rearranging and then integrating (A.11) and using (A.12) yields

$$\frac{b_1(t)}{Q_1} = \int_o^t [V(v) - \xi(v)]dv + \left(\frac{1}{\xi'(0)} + \frac{1}{r_1+\mu_2}\right)\int_o^t [I(v)]^{-1}dv \ , \tag{A.13}$$

where $I(v)$ and $V(v)$ are defined by (A.9) and (A.12) respectively, and, use has been made of the initial condition $(A.7)_1$ and of (A.10).

In the above special cases (A), (B) and (C), conditions have been imposed on the vital parameters (birth and death rates) of the Malthusian extreme net maternity functions in order to obtain a solution in a closed form for (2.32), with $\Phi(x,t)$ given by (3.10) and in particular (3.10) with (3.24).

Consider now, functions $\xi(t)$ satisfying conditions $(3.17)_1$ for which the differential equation (3.29)-(3.30) can be reduced to an equation which has a solution expressible in a closed form.

(D) $\underline{\xi(t) = \left[1 + Ce^{-\theta t}\right]^{-1}}$, $\underline{\theta < 0 \quad (\mu_1 < \mu_2)}$.

Consider the differential equation (3.29)-(3.30) with initial conditions (3.31) and let us seek $\xi(t)$ which makes $u(t)$, the coefficient of the second derivative term, zero and also satisfies conditions $(3.17)_1$. That is, we want $\xi(t)$ such that $u(t) = 0$, or

$$\theta\big(1-\xi(t)\big) - \xi'(t)/\xi(t) = 0 \ . \tag{A.14}$$

Upon dividing by $\xi(t)$, equation (A.14) can be written as

$$\frac{d}{dt}\left[\frac{1}{\xi(t)}\right] + \theta\left[\frac{1}{\xi(t)}\right] = \theta \ ,$$

which has solution,

$$\frac{1}{\xi(t)} = 1 + Ce^{-\theta t} \quad ,$$

and hence,

$$\xi(t) = \left[1 + Ce^{-\theta t}\right]^{-1} \quad , \tag{A.15}$$

with C an arbitrary constant, and on applying the conditions $(3.17)_1$, we have the restriction $\theta < 0$ $(\mu_1 < \mu_2)$.

With $\xi(t)$ as given by (A.15), then $u(t) = 0$ so that (3.29)-(3.30) become,

$$b'(t) + (\theta-\omega)\xi(t)b(t) = 0 \quad ,$$

whose solution is given by,

$$b(t) = Q_1 \exp\left\{(\omega-\theta)\int_0^t \xi(s)ds\right\} \quad , \tag{A.16}$$

where we have used the initial conditions (3.31). Thus, upon using (A.15) and integrating, (A.16) becomes

$$\begin{aligned} b(t) &= Q_1\left[(1+C)e^{-\theta t}\xi(t)\right]^{1 - \frac{\omega}{\theta}} \\ &= Q_1\left[(1+C)/(e^{\theta t} + C)\right]^{1 - \frac{\omega}{\theta}} \quad , \end{aligned} \tag{A.17}$$

where $\theta < 0$ and C is arbitrary.

The solution for $b(t)$, given by (A.17), has an asymptotic value,

$$Q_2 = \lim_{t\to\infty} b(t) = Q_1\left[1 + \frac{1}{C}\right]^{1 - \frac{\omega}{\theta}} \quad , \tag{A.18}$$

where, from (3.25) $b(t) = e^{-r_2 t}B(t)$ and hence the total birth rate $B(t)$ behaves asymptotically like $Q_2 e^{r_2 t}$.

We note that with $\theta = \omega$, (A.17) correctly agrees with (A.4).

(E) $\underline{\xi(t) = \left[1 + D\left(e^{\gamma_2 t} - e^{-\theta t}\right)\right]^{-1}}$.

Let us now seek a function $\xi(t)$ which will make $q(t)$ (the coefficient of the $b(t)$ term in (3.29)-(3.30)) vanish, and also such

that $\xi(t)$ satisfies the conditions $(3.17)_1$. If such a $\xi(t)$ exists then, the differential equation (3.29) will be reducible and we will be able to obtain the solution in a closed form. That is, we require a $\xi(t)$ that satisfies

$$u(t)[\xi'(t) + \gamma_2\xi(t)] - \xi(t)u'(t) = 0 ,$$

which becomes, upon substitution for $u(t)$ from $(3.30)_3$,

$$\xi''(t) - 2\frac{(\xi'(t))^2}{\xi(t)} + (\theta-\gamma_2)\xi'(t) + \gamma_2\theta\xi(t)[1-\xi(t)] = 0 , \qquad (A.19)$$

with conditions $(3.17)_1$ on $\xi(t)$.

Dividing (A.19) by $\xi^2(t)$ and letting

$$h(t) = \frac{-1}{\xi(t)} , \qquad (A.20)$$

we obtain a non-homogeneous second order linear differential equation with constant coefficients in $h(t)$,

$$h''(t) + (\theta-\gamma_2)h'(t) - \theta\gamma_2 h(t) = \theta\gamma_2 , \qquad (A.21)$$

with conditions from $(3.17)_1$ and (A.20)

$$h(0) = -1 \quad \text{and} \quad \lim_{t\to\infty} h(t) = -\infty ,$$

which has solution

$$h(t) = -\left[1 + D\left(e^{\gamma 2t} - e^{-\theta t}\right)\right] , \qquad (A.22)$$

with restrictions on D , so that $h(t)$ and hence $\xi(t)$ will satisfy the second of the above conditions,

$$\left.\begin{aligned} &D < 0 \quad \text{if} \quad 0 < \gamma_2 < -\theta , \\ \text{and} \quad &D > 0 \quad \text{if} \quad 0 < -\theta < \gamma_2 \quad \text{or} \quad -\theta < 0 \quad (\mu_1 > \mu_2) . \end{aligned}\right\} \qquad (A.23)$$

Hence the equation (A.19) with $(3.17)_1$ has solution, on using (A.20) and (A.22),

$$\xi(t) = \left[1 + D\left(e^{\gamma 2t} - e^{-\theta t}\right)\right]^{-1} , \qquad (A.24)$$

where (A.23) gives the restrictions on D .

Since $\xi(t)$, as given by (A.24), was chosen so that $q(t) = 0$ in (3.29), we have; from (3.29), upon dividing by $u(t)$

$$b''(t) + [\gamma_2 + u(t) + (\theta-\omega)\xi(t) - u'(t)/u(t)]b'(t) = 0 , \qquad (A.25)$$

and the associated initial conditions from (3.31), with $\xi(t)$ given by (A.24) and using $(3.30)_3$, are

$$\left.\begin{aligned} & b(0) = Q_1 , \\ \text{and} \quad & b'(0) + (\theta-\omega)\left[1 - D\left(\frac{r_2+\mu_1}{r_1+\mu_2}\right)\right]b(0) = 0 . \end{aligned}\right\} \qquad (A.26)$$

Reducing the order of the differential equation (A.25), integrating twice and using the conditions (A.26) we obtain

$$b(t) = E\int_0^t u(s)\xi(s)\exp\left\{\int_0^s \left(\omega\xi(v)-r_2-\mu_1\right)dv\right\}ds + Q_1 , \qquad (A.27)$$

where,

$u(t)$ and $\xi(t)$ are given by $(3.30)_3$ and (A.24) respectively,

$$E = \frac{b'(0)}{u(0)} = (\theta-\omega)\left[\frac{1}{r_1+\mu_2} - \frac{1}{D(r_2+\mu_1)}\right] ,$$

and D is given by (A.23). Thus, (A.27) becomes

$$b(t) = E.D.(\theta+\gamma_2)\int_0^t \xi^2(s)\exp\left\{\int_0^s \left(\omega\xi(v)-\theta\right)dv\right\}ds + Q_1 , \qquad (A.28)$$

where $\xi(t)$ is given by (A.24).

We note that if $\theta = \omega$, with $\xi(t)$ as given by (A.24), then the solution (A.28) (or (A.27)) reduces to (A.4).

(F) $\underline{\xi(t) = e^{-\lambda t}}$, $\underline{\lambda > 0}$.

The substitution;

$$b(t) = \eta(\xi) , \quad \xi(t) = e^{-\lambda t} , \quad \lambda > 0 , \qquad (A.29)$$

transforms equation (3.29)-(3.30), after some algebra, into

$$\xi(\sigma-\theta\xi)\eta''(\xi) + \left[\tau_1 + \tau_2\xi + \tau_3\xi^2\right]\eta'(\xi) + \left[\rho_1 + \rho_2\xi\right]\eta(\xi) = 0, \quad \xi \in (0,1), \qquad (A.30)$$

a second order homogeneous linear differential equation with polynomial coefficients; where,

$$\left.\begin{aligned} \sigma &= \lambda + \theta \quad , & \lambda\tau_1 &= -\sigma(\mu_1+r_2) \\ \theta &= \mu_1 - \mu_2 \quad , & \lambda\tau_2 &= \theta(\sigma+\gamma_1) + \lambda\omega \\ \omega &= \gamma_1 - \gamma_2 \quad , & \lambda\tau_3 &= -\theta\omega \\ r_i &= \gamma_i - \mu_i \ , \ i = 1,2 \quad , & \lambda^2\rho_1 &= \sigma(\gamma_2-\lambda)(r_2-r_1) \\ & & \text{and } \lambda^2\rho_2 &= -\theta\gamma_2(r_2-r_1) \quad ; \end{aligned}\right\} \tag{A.31}$$

with the initial conditions (3.31) now becoming, under the transformation (A.29),

$$\left.\begin{aligned} &\eta(1) = Q_1 \\ &\text{and } \eta'(1) + \frac{(\theta-\omega)}{\lambda(r_1+\mu_2)} (\sigma-\gamma_1)\eta(1) = 0 \ . \end{aligned}\right\} \tag{A.32}$$

The problem (A.30)-(A.31) has already been solved, indirectly, for $\theta = 0$, in the form of (A.1) with $\xi(t) = e^{-\lambda t}$, $\lambda > 0$. That is,

$$\eta(\xi) = \frac{Q_1}{\gamma_1} \left[\gamma_2 + \omega\xi\right] \exp\left[\frac{\omega}{\lambda} (1-\xi)\right] \quad , \tag{A.33}$$

is the solution of

$$\xi\eta''(\xi) + [\alpha+\beta\xi]\eta'(\xi) + \beta(1+\alpha)\eta(\xi) = 0 \ , \tag{A.34}$$

where,

$$\alpha = -\frac{\gamma_2}{\lambda} \ , \quad \beta = \frac{\omega}{\lambda} \ , \quad \omega = \gamma_1 - \gamma_2 \ ,$$

with initial condtions

$$\eta(1) = Q_1$$

$$\text{and} \quad \eta'(1) + \frac{\omega}{\lambda\gamma_1} (\gamma_1-\lambda)\eta(1) = 0 \ .$$

We note that if $\gamma_1 = \gamma_2$ in (A.34), so that $\omega = 0$ ($\beta = 0$), then $\omega = \theta = 0$ and the solution (A.33) is in agreement with (A.4).

The integral equation (3.12); with $\xi(t) = e^{-\lambda t}$, $\lambda > 0$ and

$\zeta(t) = 1 - \xi(t)$, defined for all positive t , has been transformed by (A.29) (with (3.25)) into the differential equation (A.30) defined for $\xi \in (0,1)$. Numerical procedures for solving (3.12) are dependent on advancing in small steps of time t , whereas, there are numerous methods such as the Runge-Kutta method, (see for example Fröberg (1969)) for solving the differential equation (A.30) in which the step size can be varied. Thus we can take smaller steps near $\xi = 1$, corresponding to $t = 0$, where the solution varies the most.

We may, however, proceed more directly, in solving the differential equation (A.30), by formally assuming a power series solution. We note that regular singular points exist at $\xi = 0$ and $\xi = \frac{\sigma}{\theta}$ $(\theta \neq 0)$, the latter being outside the interval (0,1) when $\theta > 0$. With $\theta < 0$ we may assume a formal Frobenius type solution (see for example Coddington (1961) or Boyce and Di Prima (1969)), about the regular singular point $\xi_1 = \frac{\sigma}{\theta}$. That is there exists a solution of the form,

$$\eta(\xi) = \sum_{n=0}^{\infty} c_n(\xi_1 - \xi)^{n+k} \quad , \tag{A.35}$$

where k and $\{c_n\}$ are determined by substitution into (A.30).

If however $\theta > 0$, and hence ξ_1 is outside the interval (0,1), then a Taylor series expansion may be assumed about $\xi = 1$ and hence facilitate the use of the boundary conditions (A.32), which are given at $\xi = 1$. Thus we have a formal solution of the form

$$\eta(\xi) = \sum_{n=0}^{\infty} a_n(1-\xi)^n \quad . \tag{A.36}$$

For convenience we make the transformation

$$\eta(\xi) = y(x) \quad , \quad \xi = \varepsilon - x \quad , \tag{A.37}$$

in (A.30) where ε is arbitrary for the time being, and we obtain

$$(\varepsilon-x)(\nu+\theta x)y''(x) + \left[\alpha_1 + \alpha_2 x + \alpha_3 x^2\right]y'(x)$$

$$+ \left[\beta_1 + \beta_2 x\right]y(x) = 0 , \quad x \in (\varepsilon-1,\varepsilon) , \tag{A.38}$$

where,

$$\left.\begin{aligned} \nu &= \sigma - \theta\varepsilon \\ \alpha_1 &= -\left(\tau_1 + \tau_2\varepsilon + \tau_3\varepsilon^2\right) \\ \alpha_2 &= \tau_2 + 2\tau_3\varepsilon \quad , \qquad \beta_1 = \rho_1 + \rho_2\varepsilon \quad , \\ \alpha_3 &= -\tau_3 \qquad \text{and} \quad \beta_2 = -\rho_2 \quad . \end{aligned}\right\} \tag{A.39}$$

With $\theta > 0$ and $\varepsilon = 1$ we thus try, from (A.36) and (A.37), a solution for (A.38) of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n \quad , \tag{A.40}$$

from which, after substitution into (A.38), we obtain a recurrence relation for the coefficients $\{a_n\}$ of the power series (A.40) by equating coefficients of successive powers of x. Namely we obtain

$$\left.\begin{aligned} &2\lambda a_2 + \alpha_1 a_1 + \beta_1 a_o = 0 \ , \\ &6\lambda a_3 + 2(\alpha_1-\sigma)a_2 + (\beta_1+\alpha_2)a_1 + \beta_2 a_o = 0 \ , \\ \text{and} \quad &(n+2)(n+1)\lambda a_{n+2} + (n+1)(\alpha_1-n\sigma)a_{n+1} \\ &\qquad + \left\{n[\alpha_2 + \theta(n-1)] + \beta_1\right\}a_n + [(n-1)\alpha_3 + \beta_2]a_{n-1} = 0, \\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{for} \quad n = 2,3,\ldots \ , \end{aligned}\right\} \tag{A.41}$$

where,

$\{\alpha_n\}$ and $\{\beta_n\}$ are given by (A.39) with $\varepsilon = 1$,

and we have used $\sigma - \theta = \lambda$ from $(A.31)_1$.

The solution to (A.38) may then be written as a sum of two linearly independent solutions y_1 and y_2 , that is

$$y(x) = F.y_1(x) + G.y_2(x) \ , \tag{A.42}$$

where y_1 and y_2 can be obtained by taking, for example, $a_o = 0$,

$a_1 = 1$ and $a_o = 1$, $a_1 = 0$ respectively in (A.41). We note, from (A.41), that a_n for $n \geq 2$ is a function of arbitrary a_o and a_1, that is

$$a_n = a_n(a_o, a_1) \quad \text{for} \quad n = 2,3,\ldots . \tag{A.43}$$

So that if we let

$$\left.\begin{aligned} A_n &= a_n(0,1) \quad , \\ \text{and} \quad C_n &= a_n(1,0) \quad , \quad n = 2,3,\ldots \quad , \end{aligned}\right\} \tag{A.44}$$

then,

$$\left.\begin{aligned} y_1(x) &= x + \sum_{n=2}^{\infty} A_n x^n \quad , \\ \text{and} \quad y_2(x) &= 1 + \sum_{n=2}^{\infty} C_n x^n \quad . \end{aligned}\right\} \tag{A.45}$$

Hence from; (A.37), with $\varepsilon = 1$, (A.42) and (A.45),

$$\eta(\xi) = F.\left[1 - \xi + \sum_{n=2}^{\infty} A_n (1-\xi)^n\right] + G.\left[1 + \sum_{n=2}^{\infty} C_n (1-\xi)^n\right], \tag{A.46}$$

where

$\{A_n\}$ and $\{C_n\}$ are given by (A.41) on noting (A.43) and (A.44). Also, using the boundary conditions (A.32), F and G are given by

$$F = \frac{(\theta-\omega)}{\lambda(r_1+\mu_2)}(\sigma-\gamma_1)Q_1 \quad \text{and} \quad G = Q_1 \ .$$

We note that for $\theta < 0$, from (A.35) and (A.37) with $\varepsilon = \xi_1$, a formal Frobenius solution for (A.38) of the form

$$y(x) = \sum_{n=0}^{\infty} c_n \, x^{n+k} \quad ,$$

where $\varepsilon = \xi_1 = \frac{\sigma}{\theta}$, may be readily obtained but will not be pursued further here.

APPENDIX B. Publications During the Author's Candidature.

Cerone, P. and Keane, A.

--1977 Series of Roots of a Transcendental Equation. Journal and Proceedings, Royal Society of New South Wales, 110:29-31.

--1978a The Momentum of Population Growth With Time Dependent Net Maternity Function. Demography, 15:131-134.

--1978b The Stable Births Resulting From a Time Dependent Change Between Two Net Maternity Functions. Demography, 15:135-137.

Hill, J.M., Laird P.G. and Cerone, P.

--1979 Mellin Type Integral Equations for Solutions of Differential-Difference Equations. Accepted for publication in Utilitas Mathematica.

APPENDIX B