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The output of a queue

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THE OUTPUT OF A QUEUE

by

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ABSTRACT

The problem of finding the form of the output of a queue is examined in this thesis, which is divided into two sections.

The first section deals with the distribution of the inter-departure intervals of single-server queueing systems. In Chapter 1 a method of displaying the processes involved in such systems is described and several of the parameters that arise from treatment of such queues are derived in terms of this graphical representation. The assumptions concerning independence of input and equilibrium are also stated. Chapter 2 discusses three methods for determining the inter-departure distribution, giving simple examples to display computational difficulties involved.

The second section examines the serial structure of the output process, basing the work on Daley's generalised auto-correlation formulae for output. The queues $E_k/M/1$ are taken as a case study and the autocorrelation functions of the output of these queues are determined and evaluated by computer. Chapter 5 describes the formulae developed, the computational proceedure undertaken, and the problems inherent in the evaluation. The formulae are thus evaluated and the results given in tabular and graphical form.

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NOTATION:

τ_n	arrival epoch of nth customer
$\delta_n = \tau_{n+1} - \tau_n$	inter-arrival time between nth and (n+1)th arrival
s_n	service time of nth customer
w_n	waiting time of nth customer
χ_n	idle time between (n-1)th departure and nth arrival
d_n	departure epoch of nth customer
$\Delta_n = d_{n+1} - d_n$	inter-departure time between nth and (n+1)th departure
λ	arrival rate
μ	service rate
$\tau = \lambda/\mu$	traffic intensity
γ_n	auto-covariance of lag n
ρ_n	auto-correlation of lag n

Several other variables will appear and be defined later.

Probability density functions will generally be specified by the same letter, while distributions will be specified by the capital. i.e. variable x would have density x(t) and distribution X(t).

Exceptions:

δ	specified by a and A
s	specified by b and B

Laplace transforms will be denoted by an asterisk.

i.e.

$$\mathcal{L}\{x(t)\} = E(e^{-sx}) = x^*(s)$$

SECTION 1

OUTPUT DISTRIBUTION

CHAPTER ONE

Graphical Representation

Assumptions

Preliminaries.

1.1 GRAPHICAL REPRESENTATION OF A QUEUE

Mathematically, a queue may be represented by two sequences of points on the real line.

The two sequences - those of arrival times and departure times - may be described thus.

Let the points of the first sequence occur at $\ldots, \tau_{-1}, \tau_{o}, \tau_{1}, \ldots$, such that the sequence $\{\tau_n\}$ is monotonic non-decreasing. Let the points of the second sequence occur at $\ldots, d_{-1}, d_{o}, d_{1}, \ldots$, such that $\{d_n\}$ is also a monotonic non-decreasing sequence and that

$$d_n \geq \tau_n \qquad \text{for all } n.$$

These two sequences may be regarded as point processes in continuous time, the τ_n being arrival times and the d_n departure times.

Other variables involved in the queue may be defined from these sequences.

The service time s_n is defined as

$$s_n = \begin{cases} d_n - \tau_n & \tau_n \geq d_{n-1} \\ d_n - d_{n-1} & \tau_n \leq d_{n-1}. \end{cases} \tag{1-1}$$

Thus

$$s_n = d_n - \max(\tau_n, d_{n-1}).$$

The number in the system at time t is defined as

$$L(t) = n_a(t) - n_d(t)$$

where $n_a(t)$ is the largest N such that $\tau_N \leq t$ and $n_d(t)$ is the largest M such that $d_M \leq t$.

The number in the waiting line is defined as

$$L_w(t) = \max(L(t)-1, 0).$$

In the literature, L(t) is usually called the queue length and is made up of the number waiting plus the one being served, if any.

In general, when investigating a queue, the sequences $\{\delta_n\}$ and $\{s_n\}$ are known from which the sequence $\{\tau_n\}$ can be found, where $\delta_n = \tau_{n+1} - \tau_n$.

We define the current idle time $\chi(t)$ as the time since the server was last busy. Thus

$$\chi(t) = (t-d_N)H(t-d_N) \qquad (1\text{-}2)$$

where $N = n_a(t)$ now and throughout this section.

Associated with each arrival is a discrete idle time χ_n given by

$$\chi_n = \chi(\tau_n^-) = (\tau_n - d_{n-1})H(\tau_n - d_{n-1}).$$

The total idle time $I(t,t_0)$ between times t_0 and t is given by

$$I(t,t_0) = \sum_{n=N_0+1}^{N} \chi_n + \chi(t) - \chi(t_0)$$

where $N_0 = n_a(t_0)$.

Hence
$$I(t,t_0) = \sum_{n=N_0+1}^{N} (\tau_n - d_{n-1})H(\tau_n - d_{n-1}) + (t-d_N)H(t-d_N)$$
$$- (t_0 - d_{N_0})H(t_0 - d_{N_0}) \quad (1\text{-}3)$$

The waiting time w_n of a customer is defined as the time from arrival to commencement of service

$$\begin{aligned} w_n &= \text{departure time} - \text{arrival time} - \text{service time} \\ &= d_n - \tau_n - s_n \\ &= d_n - \tau_n - (d_n - \max(\tau_n, d_{n-1})) \\ &= \max(\tau_n, d_{n-1}) - \tau_n. \end{aligned}$$

Thus
$$w_n = \begin{cases} d_{n-1} - \tau_n & \tau_n \leq d_{n-1} \\ 0 & \tau_n \geq d_{n-1}. \end{cases}$$

We define virtual waiting time v(t) as

$$\begin{aligned} v(t) &= \text{total service presented} - \text{total busy time} \\ &= (\text{total service presented since } t = t_0) - \\ &\quad (\text{total busy time since } t = t_0) + v(t_0) \\ &= \sum_{n=N_0+1}^{N} s_n - ((t-t_0) - I(t,t_0)) + v(t_0) \end{aligned}$$

$$= \sum_{n=N_0+1}^{N} [s_n + (\tau_n - d_{n-1})H(\tau_n - d_{n-1})] + (t-d_N)H(t-d_N)$$
$$- (t_0 - d_{N_0})H(t_0 - d_{N_0}) - (t-t_0) + v(t_0)$$

which from (1-1) becomes

$$v(t) = \sum_{n=N_0+1}^{N} (d_n - d_{n-1}) + (t-d_N)H(t-d_N)$$
$$- (t_0 - d_{N_0})H(t_0 - d_{N_0}) - (t-t_0) + v(t_0)$$
$$= (d_N - t)H(d_N - t) + v(t_0) - (d_{N_0} - t_0)H(d_{N_0} - t_0).$$

Whenever the system is idle, total service presented equals total busy time and hence $v(t) = 0$.

Also $d_N \leqslant t$.

Thus if t_0 is chosen as a time when the system is idle, then

$$v(t_0) - (d_{N_0} - t_0)\, H(d_{N_0} - t_0) = 0$$

and $$v(t) = (d_N - t)H(d_N - t), \text{ t after any idle period.}$$

Since the above choice of t_0 satisfies this last equation, it is true for all t,

$$\text{i.e. } v(t) = (d_N - t)H(d_N - t) \text{ everywhere,}$$

an equation similar to (1-2) for $\chi(t)$.

Thus this function has a slope of -1 when the server is busy and has a value of zero when the server is idle.

$$v(\tau_n^-) = (d_{n-1} - \tau_n)H(d_{n-1} - \tau_n) = w_n.$$

$v(t)$ can be regarded as the total service remaining for the server at any instant or as the waiting time if at that particular instant a new customer arrived. The value of $v(t)$ jumps up by the service time for each arrival and becomes zero whenever the queue is idle.

A new function, $u(t)$, called the service load can be defined by dropping the Heaviside function from $v(t)$ thus

$$u(t) = d_N - t.$$

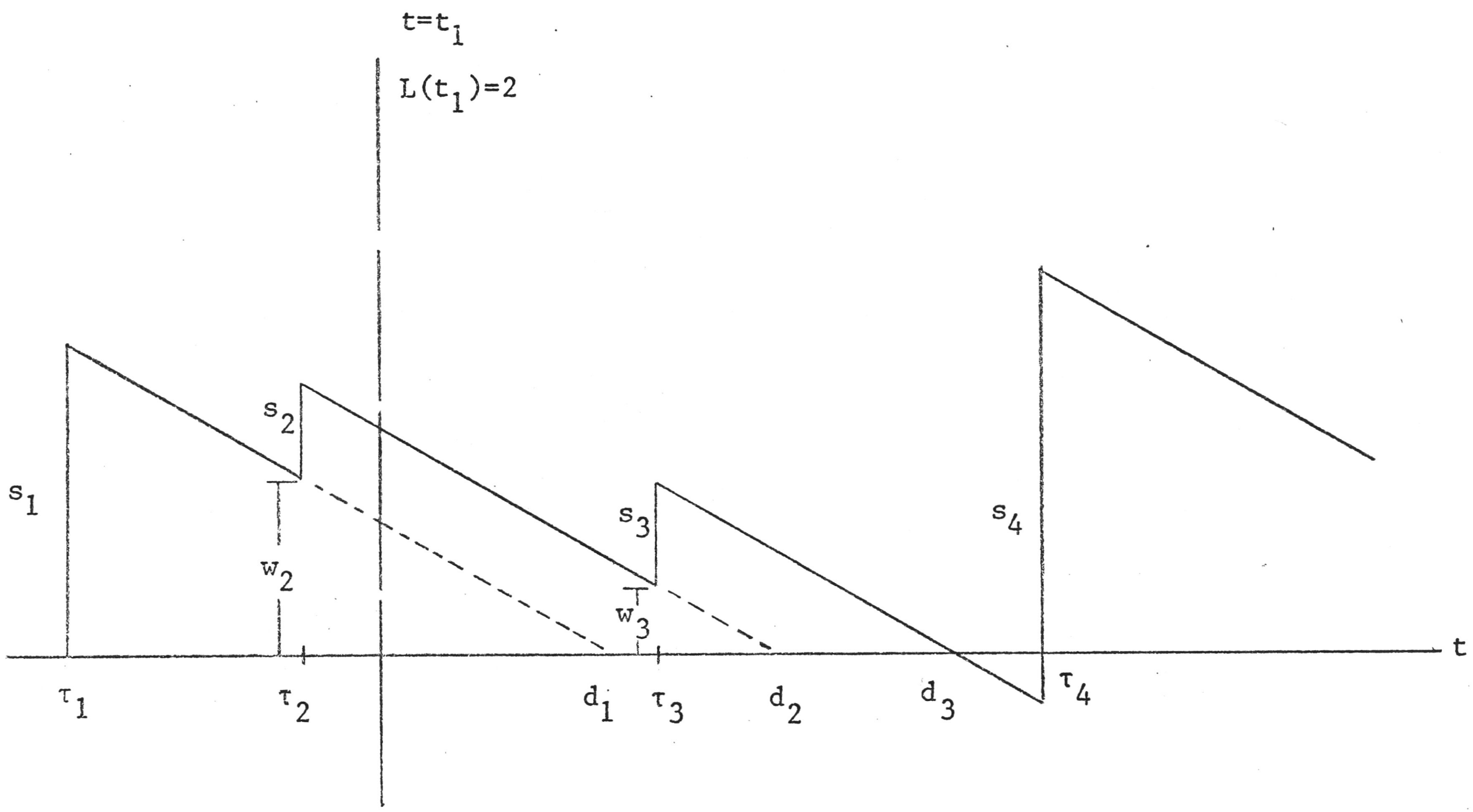


FIG. THE SERVICE LOAD $u(t)$

Thus if $d_N \geq t$, $u(t)$ has the value of $v(t)$ but if $d_n < t$, then $-u(t)$ equals $\chi(t)$. Thus

$$u(t) = v(t) - \chi(t).$$

Hence for $u(t)$, its slope is -1 everywhere except at arrivals at which $u(t)$ changes from

$$u(\tau_n^-) = d_{n-1}-\tau_n$$

to

$$u(\tau_n) = d_n-\tau_n$$

i.e. a jump of $d_n - d_{n-1}$.

Alternatively

$$u(\tau_n) = s_n + \max(u(\tau_n^-),0).$$

A typical $u(t)$ is shown opposite.

At any instant of time t_1, $u(t)$ gives (see figure)

(i) virtual waiting time $v(t_1) = u(t_1)$, $u(t_1) \geq 0$

(ii) current idle time $\chi(t_1) = -u(t_1)$, $u(t_1) < 0$

(iii) number in system $L(t_1)$ by counting number of lines (dotted and undotted) crossed by line $t = t_1$

(iv) how long each customer in system waits before departure time (height of dotted line or undotted).

Thus u(t) is a unique representation of the queue from which all queue parameters may be obtained directly.

Later we will use the discrete service load u_n defined only at an arrival by

$$u_n = u(\tau_n^-).$$

1.2 ASSUMPTIONS

The assumptions made about the queueing systems discussed in this thesis are those made by Lindley (1952):

(I) the δ_n are independent random variables with identical probability density functions with finite mean $1/\lambda$;

(II) the s_n are independent random variables with identical p.d.f.'s and finite mean $1/\mu$;

(III) the two sets $\{\delta_n\}$ and $\{s_n\}$ $n = \ldots,-2,-1,0,1,2,\ldots$ are independent;

(IV) $\lambda < \mu$, i.e. $\tau < 1$ and the system is in equilibrium. (Lindley (1952) showed that if $\lambda < \mu$, the system tends to an equilibrium state.)

Because of assumption (IV), equilibrium states only will be treated and distributions so obtained will be valid only in steady state conditions.

This first section will investigate several techniques for evaluating the distribution of inter-departure times and some examples will illustrate each method's practicability.

It has been shown (Daley (1968), Finch (1959)) that in equilibrium states, excepting the M/M/1 queue, the sequences of waiting times and inter-departure times are auto-correlated. This auto-correlation of the output process will be examined in Section 2.

The examples treated in this thesis are all $M/E_k/1$ and $E_k/M/1$ queues (M/D/1 and D/M/1 queues being treated sometimes on their own or as a limiting case of $M/E_k/1$ and $E_k/M/1$ queues as $k \to \infty$) and serve to illustrate the difficulties which arise in each of the methods used.

1.3 PRELIMINARY

Each δ_n has p.d.f. $a(x)$

and each s_n has p.d.f. $b(x)$.

We define $v_n = s_n - \delta_n$.

Thus the v_n are independent random variables with identical p.d.f.'s given by

$$v(x) = \int_0^\infty b(y)a(y-x)dy = \int_0^\infty b(y+x)a(y)dy.$$

Thus

$$v^*(s) = a^*(-s)b^*(s).$$

Care should be taken to ensure the validity of this equation as it holds, in general, only for s pure imaginary and those points for which the equation can be extended analytically. If in doubt, $v^*(s)$ can be found directly from $v(x)$.

In the examples of the M/G/1 queues, it would be useful to know $w^*(s)$ and $W_0 = w(0)$.

Firstly, W_0 = Prob (server is idle)

$$= 1 - \tau.$$

It can be shown (Cox and Smith (1961)) that

$$w^*(s) = \frac{(1-\tau)s}{s-\lambda+\lambda b^*(s)}.$$

CHAPTER TWO

Transform Method

Moment Generating Function Method

Integral Equation Method

In this chapter we treat three methods for finding the distribution of inter-departure times in the equilibrium state. The three methods each utilise a different technique:-

(1) Transform method (Chang (1963)), in which the basic defining equations for discrete service load u_n are transformed using Laplace transforms and the resulting transform equation solved.

(2) Moment generating function method (Makino (1966)), in which equations relating m.g.f. of output distribution and m.g.f. of input and service distributions are stated without discussion as the method would require identification of any resulting m.g.f. in order to utilise the results for evaluating output distributions.

(3) Integral equation method (based on Saaty (1959)), in which the solution of integral equations for discrete service load distribution are solved by various methods - not a general method such as in (1)

In each case examples show ease of manipulation of each method and problems likely to arise with usage.

2.1 TRANSFORM METHOD

This method is the work of Wei Chang (1963).

It can be shown (Lindley (1952)) that

$$w_n = \begin{cases} w_{n-1} + v_{n-1} & w_{n-1} + v_{n-1} > 0 \\ 0 & w_{n-1} + v_{n-1} \leq 0. \end{cases} \tag{2-1}$$

w_{n-1} is independent of v_{n-1} by virtue of the assumptions. The inter-departure times can be obtained thus

$$\Delta_n = \begin{cases} s_n & w_n > 0 \\ s_n + \chi_n & w_n \leq 0 \end{cases} \tag{2-2}$$

where χ_n, the idle time, is given by

$$\chi_n = (-w_{n-1} - v_{n-1})^+ \tag{2-3}$$

i.e. χ_n can be only positive or zero.

If χ_n^+ is the positive value other than zero then (2-3) becomes

$$\chi_n = \begin{cases} \chi_n^+ = -w_{n-1} - v_{n-1} > 0 & \text{for } w_n = 0 \\ 0 & \text{for } w_n > 0 . \end{cases} \tag{2-4}$$

So (2-2) gives

$$\Delta_n = s_n + \chi_n. \tag{2-5}$$

Thus

$$\text{Prob}(\chi_n=x)=\text{Prob}(\chi_n^+=x)+\text{Prob}(\chi_n=0)\,\delta(x-0)$$

$$=\text{Prob}(\chi_n^+=x)+\text{Prob}(w_n>0)\,\delta(x-0)$$

from (2-4).

Using Cauchy's integral theorem, the transform $\chi_n^*(s)$ is given by

$$\chi_n^*(s) = E(e^{-s\chi_n}) = \frac{-1}{2\pi i}\oint_C \frac{E[e^{z(w_{n-1}+v_{n-1})}]}{z-s}\,dz+\text{Pr}(w_n>0) \qquad (2\text{-}6)$$

with C the contour from $-i\infty$ to $i\infty$ in the right half plane. (2-6) is valid, even if the variables are not independent. However, with w_{n-1} and v_{n-1} independent, then

$$E(e^{z(w_{n-1}+v_{n-1})}) = E(e^{zw_{n-1}})E(e^{zv_{n-1}}) \,. \qquad (2\text{-}7)$$

In the steady state, dropping subscripts and using (2-7), (2-6) becomes, using transform notation

$$\chi^*(s) = \frac{-1}{2\pi i}\oint_C \frac{w^*(-z)v^*(-z)}{z-s}\,dz + (1-W_0) \qquad (2\text{-}8)$$

where $W_0 = W(0)$.

Also (2-5) becomes

$$\Delta^*(s) = b^*(s)\chi^*(s). \qquad (2\text{-}9)$$

Summarising, the technique is

(i) calculate $\chi^*(s)$ from (2-8), and

(ii) use (2-9) to find $\Delta^*(s)$ and then invert the transform

Examples of Transform Method

(a) M/M/1 queue

$$a^*(s) = \frac{\lambda}{\lambda+s}$$

$$b^*(s) = \frac{\mu}{\mu+s}$$

$$v^*(s) = b^*(s)a^*(-s) = \frac{\lambda\mu}{(\mu+s)(\lambda-s)}$$

$$w^*(s) = \frac{(1-\tau)s}{s-\lambda+\lambda b^*(s)} = \frac{(1-\tau)(s+\mu)}{s+(\mu-\lambda)}$$

$$W_0 = 1-\tau$$

$$\chi^*(s) = \frac{-1}{2\pi i}\oint_C \frac{w^*(-z)v^*(-z)}{z-s}\,dz + (1-W_0)$$

$$= \frac{\lambda\mu(1-\tau)}{2\pi i}\oint_C \frac{dz}{(z+\lambda)(z-s)(z-(\mu-\lambda))} + \tau.$$

Using residue theorem at poles $z = s$ and $z = \mu-\lambda$

$$\chi^*(s) = \frac{\lambda(1-\tau)}{s+\lambda} + \tau = \frac{\lambda+\tau s}{\lambda+s}.$$

Thus

$$\Delta^*(s) = b^*(s)\chi^*(s) = \frac{\lambda+\tau s}{\lambda+s}\cdot\frac{\mu}{\mu+s}$$

$$= \frac{\lambda}{\lambda+s}.$$

Thus

$$\Delta(t) = \lambda e^{-\lambda t}$$

i.e. the output of an M/M/1 queue (in the equilibrium state) is Poisson, the same as its input - a well-known result (Burke (1956)).

(b) D/M/1 queue

$$a^*(s) = e^{-s/\lambda}$$

$$b^*(s) = \frac{\mu}{\mu+s}$$

$$v^*(s) = \frac{\mu e^{s/\lambda}}{\mu+s}$$

$$w^*(s) = \frac{U_0(s+\mu)}{s+\mu U_0} \qquad W_0 = U_0$$

where U_0 is defined and the derivation of $w^*(s)$ above can be found in 2.3 Example (b) Eqn.(2-28).

$$\chi^*(s) = \frac{-1}{2\pi i}\oint_C \frac{U_0(\mu-z)\mu e^{-z/\lambda}dz}{(\mu U_0-z)(\mu-z)(z-s)} + (1-U_0)$$

$$= \frac{1}{2\pi i}\oint_C \frac{\mu U_0 e^{-z/\lambda}}{(z-s)(z-\mu U_0)}dz + (1-U_0).$$

Using residue theorem at poles $z = s$ and $z = \mu U_o$

$$\chi^*(s) = \frac{\mu U_0}{s-\mu U_0}(e^{-\mu U_0/\lambda} - e^{-s/\lambda}) + (1-U_0).$$

Thus

$$\Delta^*(s) = \frac{\mu^2 U_0(e^{-\mu U_0/\lambda} - e^{-s/\lambda})}{(s+\mu)(s-\mu U_0)} + \frac{\mu(1-U_0)}{s+\mu}$$

$$= \frac{\mu U_0}{1+U_0}\left(\frac{1}{s-\mu U_0} - \frac{1}{s+\mu}\right)\left(e^{-\mu U_0/\lambda} - e^{-s/\lambda}\right) + \frac{\mu(1-U_0)}{s+\mu}.$$

Inverting we get

$$\Delta(x) = \frac{\mu U_0}{1+U_0}\left\{\left(e^{\mu U_0 x} - e^{-\mu x}\right)e^{-\mu U_0/\lambda} - \left(e^{\mu U_0(x-1/\lambda)} - e^{-\mu(x-1/\lambda)}\right)H\left(x-\frac{1}{\lambda}\right)\right\}$$

$$+ \mu(1-U_0)e^{-\mu x}.$$

Splitting the range we get

$$\Delta(x) = \begin{cases} \left(\frac{\mu(1-U_0)}{1+U_0} + \frac{\mu U_0}{1+U_0} e^{\mu/\lambda}\right)e^{-\mu x} & x > \frac{1}{\lambda} \\ \frac{\mu(1-U_0)}{1+U_0} e^{-\mu x} + \frac{\mu U_0(1-U_0)}{1+U_0} e^{\mu U_0 x} & x \leq \frac{1}{\lambda}. \end{cases}$$

(c) <u>M/D/1 queue</u>

$$a^*(s) = \frac{\lambda}{\lambda+s}$$

$$b^*(s) = e^{-s/\mu}$$

$$v^*(s) = \frac{\lambda e^{-s/\mu}}{\lambda-s}$$

$$w^*(s) = \frac{(1-\tau)s}{s-\lambda+\lambda e^{-s/\mu}} \qquad W_0 = 1-\tau.$$

Equation (2-8) gives

$$\chi^*(s) = \frac{-\lambda(1-\tau)}{2\pi i}\oint_C \frac{ze^{z/\mu}\,dz}{(z-s)(z+\lambda)(z+\lambda-\lambda e^{z/\mu})} + \tau.$$

The expression $z + \lambda - \lambda e^{z/\mu}$ is an exponential polynomial with no principal term and hence has an unbounded number of zeros with arbitrary large positive part (Bellman and Cooke (1963)). Consequently the evaluation of the contour integral is difficult.

However, it may be shown from later work (2.3 Example (a)(ii)) that

$$\chi(x) = \begin{cases} \lambda(1-\tau)e^{-\lambda x} & x > 0 \\ \tau & x < 0. \end{cases}$$

Consequently,

$$\chi^*(s) = \frac{\lambda+\tau s}{\lambda+s}$$

so (2-9) gives

$$\Delta^*(s) = \frac{\lambda+\tau s}{\lambda+s}\, e^{-s/\mu} \quad .$$

Inverting this, we get

$$\Delta(x) = \begin{cases} \lambda(1-\tau)e^{\lambda(1/\mu-x)} & x > 1/\mu \\ \tau & x = 1/\mu \\ 0 & x < 1/\mu. \end{cases}$$

CONCLUSIONS

Although the transform method can be used to evaluate the inter-departure distribution of any GI/G/1 queue, there are difficulties inherent in the evaluation of the contour integrals and transform inversions involved which limit the method's practical usage.

2.2 MOMENT GENERATING FUNCTION METHOD

This is the work of Makino (1966).

This method gives as a result the moment generating function of the output process and as such cannot be directly used in the work of Section 2. For this reason it is included here without discussion.

If $M_A(\theta)$ is the moment generating function of the inter-arrival distribution;

$M_S(\theta)$ is the m.g.f. of the service time distribution;

and $M_U(\theta)$ is the m.g.f. of the output distribution, it has been shown that

(i) for the M/G/1 queue

$$M_U(\theta) = \frac{\mu-\theta}{\mu} \cdot \frac{\lambda}{\lambda-\theta} M_S(\theta)$$

giving

$$M/M/1 \text{ as } M_U(\theta) = \frac{\lambda}{\lambda-\theta}$$

$$M/E_K/1 \text{ as } M_U(\theta) = \left(\frac{\mu-\theta}{\mu}\right)\left(\frac{\lambda}{\lambda-\theta}\right)\left(\frac{k\mu}{k\mu-\theta}\right)^k$$

and

$$M/D/1 \text{ as } M_U(\theta) = \left(\frac{\mu-\theta}{\mu}\right)\left(\frac{\lambda}{\lambda-\theta}\right)e^{\theta/\mu}$$

(ii) for the $E_K/M/1$ queue

$$M_U(\theta) = \left(\frac{\mu}{\mu-\theta}\right)\left[\left(\frac{k\lambda}{k\lambda-\theta}\right)^k \cdot \left\{\frac{1-((1-\frac{\theta}{k\lambda})v)^k}{1-(1-\frac{\theta}{k\lambda})v}\right\} \cdot (1-v)+v^k\right]$$

where v is a positive root less than unity of

$$v^k + v^{k-1} + \ldots + v - k\tau = 0.$$

Makino also treated the $E_2/E_2/1$ and several tandem queueing systems of two and three stages.

2.3 INTEGRAL EQUATION METHOD

The main work is from Saaty (1959).

Using the variable v_n defined in 1.3, the discrete service load u_n can be defined by the recurrence relation

$$u_n = \begin{cases} u_{n-1} + v_{n-1} & u_{n-1} > 0 \\ v_{n-1} & u_{n-1} \leq 0. \end{cases} \qquad (2\text{-}10)$$

As stated before for u(t), u_n contains the information thus:-

if $u_n \geq 0$, $w_n = u_n$ $\quad \chi_n = 0$

if $u_n \leq 0$, $w_n = 0$ $\quad \chi_n = -u_n$.

Define $u_n(x)$ as

$$\begin{aligned} u_n(x) &= \text{Prob}(u_n = x) \\ &= \text{Prob}(u_n=x, u_{n-1}>0) + \text{Prob}(u_n=x, u_{n-1}\leq 0) \\ &= \text{Prob}(u_{n-1}+v_{n-1}=x, u_{n-1}>0) + \text{prob}(v_{n-1}=x, u_{n-1}\leq 0). \end{aligned}$$

Therefore

$$u_n(x) = \int_0^\infty u_{n-1}(y)v(x-y)\,dy + v(x)\int_{-\infty}^0 u_{n-1}(y)\,dy. \qquad (2\text{-}11)$$

In a steady state, that is as $n \to \infty$, $u_{n-1}(x)$ and $u_n(x)$ both tend to the equilibrium p.d.f. $u(x)$ which from (2-11) is given by

$$u(x) = \int_0^\infty u(y)v(x-y)\,dy + v(x)\int_{-\infty}^0 u(y)\,dy. \qquad (2-12)$$

Integrating (2-12) from $-\infty$ to x and introducing distribution notation

$$U(x) = \int_{-\infty}^x \int_0^\infty u(y)v(x-y)\,dy\,dx + \int_{-\infty}^x v(x)\int_{-\infty}^0 u(y)\,dy\,dx.$$

Reversing the order of integration we get

$$U(x) = \int_0^\infty \left\{\int_{-\infty}^x v(x-y)dx\right\}u(y)\,dy + V(x)U(0).$$

Integrating the first part by parts with respect to y gives

$$U(x) = \left[\left\{\int_{-\infty}^x v(x-y)dy\right\}\left\{\int u(y)\,dy\right\}\right]_0^\infty$$

$$- \int_0^\infty \left\{\frac{d}{dy}\int_{-\infty}^x v(x-y)\,dx\right\}\left\{\int u(y)\,dy\right\}dy + V(x)U(0).$$

Simplifying this, the equation finally becomes

$$U(x) = \int_0^\infty U(y)v(x-y)\,dy \qquad (2-13)$$

where $-\infty < x < \infty$.

This is a homogeneous integral equation over the full plane, but the integral involved is only over the half plane. This variation from the standard Weiner-Hopf type results in solutions being more readily obtained by methods devised for each individual equation encountered.

The above work is the essence of Saaty's technique.

From the resulting $U(x)$, the waiting time distribution is given by

$$W(x) = \begin{cases} U(x) & x > 0 \\ U_0 & x = 0 \end{cases} \qquad (2\text{-}14)$$

where $U_0 = U(0)$ henceforth.

From the definitions of w_n and d_n we find

$$\begin{aligned} d_n &= \tau_n + s_n + w_n \\ &= \begin{cases} \tau_n + s_n + u_n & u_n > 0 \\ \tau_n + s_n & u_n \leq 0. \end{cases} \end{aligned} \qquad (2\text{-}15)$$

Hence

$$\begin{aligned} \Delta_n &= d_{n+1} - d_n \\ &= \delta_n + s_{n+1} - s_n + \begin{cases} u_{n+1} - u_n & u_{n+1}, \quad u_n > 0 \\ u_{n+1} & u_{n+1} > 0, \; u_n \leq 0 \\ -u_n & u_{n+1} \leq 0, \; u_n > 0 \\ 0 & u_{n+1}, \quad u_n \leq 0. \end{cases} \end{aligned}$$

Thus

$$\Delta_n = s_{n+1} - v_{n+1} + \begin{cases} v_{n+1} & u_{n+1} > 0 \\ v_{n+1} - u_{n+1} & u_{n+1} \leq 0. \end{cases}$$

Finally we have

$$\Delta_n = \begin{cases} s_{n+1} & u_{n+1} > 0 \\ s_{n+1} - u_{n+1} & u_{n+1} \leq 0 \end{cases} \qquad (2\text{-}16)$$

s_{n+1} and u_{n+1} are independent and immediately assuming steady state conditions, all the Δ_n have density $\Delta(x)$ given by

$$\Delta(x) = \int_0^x b(y)u(y-x)dy + (1-U_0)b(x). \qquad (2\text{-}17)$$

Note that only $u(x)$ $x \leq 0$ need be known to evaluate $\Delta(x)$. However, for completeness, in the following examples the full solution for $u(x)$ will be evaluated.

<u>Examples of the Integral Equation Method</u>

(a) <u>M/G/1 queue</u>

$$a(x) = \begin{cases} \lambda e^{-\lambda\tau} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

$$v(x) = \int_0^\infty b(y)a(y-x)dy$$

$$= \begin{cases} \int_x^\infty b(y)\lambda e^{-\lambda(y-x)}dy, & x > 0 \\ \int_0^\infty b(y)\lambda e^{-\lambda(y-x)}dy, & x \leq 0 \end{cases}$$

$$= \lambda e^{\lambda x} \times \begin{cases} \int_x^\infty b(y)e^{-\lambda y}dy, & x>0 \\ \int_0^\infty b(y)e^{-\lambda x}dy, & x\leq 0. \end{cases}$$

Using (2-13)

$$U(x) = \int_0^\infty U(y)v(x-y)dy$$

$$= \begin{cases} \lambda\int_0^x U(y)e^{\lambda(x-y)}\int_{x-y}^\infty b(z)e^{-\lambda z}dzdy \\ \quad +\lambda\int_x^\infty U(y)e^{\lambda(x-y)}dy\int_0^\infty b(y)e^{-\lambda y}dy, & x>0 \\ \lambda\int_0^\infty b(y)e^{-\lambda y}dy\int_0^\infty U(y)e^{\lambda(x-y)}dy & x\leq 0 \end{cases}$$

$$U(x) = \lambda e^{\lambda x} \times \begin{cases} \int_0^x U(y)e^{-\lambda y}\int_{x-y}^\infty b(z)e^{-\lambda z}dzdy \\ \quad +b^*(\lambda)\int_x^\infty U(y)e^{-\lambda y}dy & x>0 \\ b^*(\lambda)\int_0^\infty U(y)e^{-\lambda y}dy & x\leq 0 \end{cases}$$

The above integral equation is solved for $x>0$ in a different way for each $b(x)$.

However for $x\leq 0$, $U(x)$ is given by

$$U(x) = \lambda e^{\lambda x} b^*(\lambda)\int_0^\infty U(y)e^{-\lambda y}dy.$$

Putting $\lambda b^*(\lambda)\int_0^\infty U(y)e^{-\lambda y}dy = A$ independent of x

$$U(x) = Ae^{\lambda x} \quad , \qquad x\leq 0$$

At $x = 0$ $\quad U(0) = A = U_0$

Thus $\quad U(x) = U_0 e^{\lambda x} \quad , \qquad x\leq 0$

From 1.3 $\quad U_0 = 1-\tau$

Thus $\quad U(x) = (1-\tau)e^{\lambda x}, \qquad x\leq 0$

and $\quad u(x) = \lambda(1-\tau)e^{\lambda x}, \qquad x\leq 0$

Thus (2-17) gives

$$\Delta(x) = \int_0^x b(y)u(y-x)dy + (1-U_0)b(x)$$

$$= \int_0^x b(y)\lambda(1-\tau)e^{\lambda(y-x)}dy + \tau b(x)$$

i.e. $$\Delta(x) = \lambda(1-\tau)e^{-\lambda x}\int_0^x b(y)e^{\lambda y}dy + \tau b(x) \; . \qquad (2\text{-}18)$$

For the first two M/G/1 queues U(x), x>0 will be evaluated for completeness.

(i) M/M/1 queue

$$a(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

$$b(x) = \begin{cases} \mu e^{-\mu x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

$$v(x) = \frac{\lambda\mu}{\lambda+\mu} \times \begin{cases} e^{-\mu x} & x > 0 \\ e^{\lambda x} & x \leq 0 \end{cases}$$

$$U(x) = \frac{\lambda\mu}{\lambda+\mu} \times \begin{cases} e^{-\mu x} \int_0^x U(y) e^{\mu y} dy + e^{\lambda x} \int_x^\infty U(y) e^{-\lambda y} dy & x > 0 \\ e^{\lambda x} \int_0^\infty U(y) e^{-\lambda y} dy & x \leq 0 \end{cases}$$

For $x \leq 0$

$$U(x) = (1-\tau) e^{\lambda x}$$

Thus
$$\int_0^\infty U(y) e^{-\lambda y} dy = \frac{\mu^2 - \lambda^2}{\lambda \mu^2} .$$

For x>0 we have

$$U(x) = \frac{\lambda\mu}{\lambda+\mu}\left(e^{-\mu x}\int_0^x U(y) e^{\mu y} dy + e^{\lambda x}\int_x^\infty U(y) e^{-\lambda y} dy\right). \qquad (2\text{-}19)$$

Taking Laplace transforms of both sides, putting

$$L(s) = \int_0^\infty U(y)e^{-sy}dy.$$

Then

$$\mathcal{L}\left\{e^{-\mu x}\int_0^x U(y)e^{\mu y}dy\right\} = \frac{L(s)}{s+\mu}$$

and
$$\mathcal{L}\left\{e^{\lambda x}\int_x^\infty U(y)e^{-\lambda y}dy\right\} = \frac{L(s)-L(\lambda)}{\lambda-s}.$$

Then (2-19) becomes

$$L(s) = \frac{\lambda\mu}{\lambda+\mu}\left\{\frac{L(s)}{s+\mu} + \frac{L(s)}{\lambda-s} - \frac{L(\lambda)}{\lambda-s}\right\}.$$

Collecting terms in L(s) this becomes

$$L(s) = \frac{\lambda\mu}{\mu^2-\lambda^2}L(\lambda)\left\{\frac{\mu}{s} - \frac{\lambda}{s+\mu-\lambda}\right\}.$$

But
$$L(\lambda) = \frac{\mu^2-\lambda^2}{\lambda\mu^2}$$

so
$$L(s) = \frac{1}{s} - \frac{\tau}{s+\mu-\lambda}.$$

Thus for $x>0$

$$U(x) = 1 - \tau e^{-(\mu-\lambda)x}.$$

Hence the full solution is

$$U(x) = \begin{cases} 1-\tau e^{-(\mu-\lambda)x} & x>0 \\ (1-\tau)e^{\lambda x} & x\leq 0 \end{cases}$$

Therefore

$$u(x) = \begin{cases} \lambda(1-\tau)e^{-(\mu-\lambda)x} & x>0 \\ \lambda(1-\tau)e^{\lambda x} & x\leq 0 \end{cases}$$

Using (2-14), the waiting time distribution is given by

$$W(x) = \begin{cases} 1-\tau & x=0 \\ 1-\tau e^{-(\mu-\lambda)x} & x>0 \end{cases}$$

From (2-17) or (2-18)

$$\Delta(x) = \lambda e^{-\lambda x} .$$

Again the well known result.

(ii) <u>M/D/1 queue</u>

$$a(t) = \begin{cases} \lambda e^{-\lambda t} & t\geq 0 \\ 0 & t<0 \end{cases}$$

$$b(t) = \delta(t - \frac{1}{\mu})$$

$$v(x) = \int_0^\infty b(y)a(y-x)dy$$

$$= \int_0^\infty \delta(y - \frac{1}{\mu})a(y-x)dy$$

$$v(x) = \begin{cases} 0 & x>\frac{1}{\mu} \\ \lambda e^{-\lambda(\frac{1}{\mu} - x)} & x\leq\frac{1}{\mu} \end{cases}$$

$$U(x) = \lambda e^{\lambda(x - \frac{1}{\mu})} \times \begin{cases} \int_{x-1/\mu}^{\infty} U(y)e^{-\lambda y}dy & x>\frac{1}{\mu} \\ \int_{0}^{\infty} U(y)e^{-\lambda y}dy & x\leq\frac{1}{\mu} \end{cases}$$

Thus in this case for $x\leq\frac{1}{\mu}$,

$$U(x) = U_0e^{\lambda x}.$$

Put
$$A = \int_0^{\infty} U(y)e^{-\lambda y}dy = \frac{U_0}{\lambda e^{-\lambda/\mu}}$$

Consider now the region $x>\frac{1}{\mu}$ where

$$U(x) = \lambda e^{\lambda(x - \frac{1}{\mu})}\int_{x-1/\mu}^{\infty} U(y)e^{-\lambda y}dy.$$

Intorducing A yields

$$U(x) = \lambda e^{\lambda(x - \frac{1}{\mu})}\left(A - \int_0^{x-1/\mu} U(y)e^{-\lambda y}dy\right) \quad (2\text{-}20)$$

Define

$$F(x) = e^{-\lambda x}U(x)$$

Then (2-10) becomes

$$F(x) = \lambda e^{-\lambda/\mu}\left\{A - \int_0^{x-1/\mu} F(y)\,dy\right\} \qquad (2\text{-}21)$$

and

$$A = \int_0^{\infty} F(y)\,dy.$$

Substituting $x = x' + \frac{n}{\mu}$ and then dropping the dashes (2-21) becomes

$$F(x + \tfrac{n}{\mu}) = \lambda e^{-\lambda/\mu}\left\{A - \int_0^{x+\frac{n-1}{\mu}} F(y)\,dy\right\} \quad x > -\frac{n-1}{\mu} \qquad (2\text{-}22)$$

Now introduce subscripts on F thus

$$F_0(x) = A\,\lambda e^{-\lambda/\mu} \qquad x \leq \frac{1}{\mu}$$

and

$$F_n(x + \tfrac{n}{\mu}) = F(x + \tfrac{n}{\mu}) \qquad 0 < x \leq \frac{1}{\mu}$$

Then (2-22) becomes

$$F_n(x + \tfrac{n}{\mu}) = \lambda e^{-\lambda/\mu}\left\{A - \int_0^{x+\frac{n-1}{\mu}} F(y)\,dy\right\}$$

Therefore

$$F_n(x + \tfrac{n}{\mu}) = \lambda e^{-\lambda/\mu}\left\{A - \sum_{i=0}^{n-2} \int_{i/\mu}^{\frac{i+1}{\mu}} F_i(y)\,dy - \int_{\frac{n-1}{\mu}}^{x+\frac{n-1}{\mu}} F_{n-1}(y)\,dy\right\} \qquad (2\text{-}23)$$

for $n \geq 2$ and $0 < x \leq \frac{1}{\mu}$.

Redefining integral parameters (2-23) gives

$$F_n(x + \frac{n}{\mu}) = \lambda e^{-\lambda/\mu}\left(A - \sum_{i=0}^{n-2} \int_0^{1/\mu} F_i(y + \frac{i}{\mu})dy - \int_0^x F_{n-1}(y + \frac{n-1}{\mu})dy\right)$$

for $n \geq 2$. (2-24)

and

$$F_1(x + \frac{1}{\mu}) = \lambda e^{-\lambda/\mu}\left(A - \int_0^x F_0(y)dy\right)$$

in the region $0 < x \leq \frac{1}{\mu}$.

The terms in x^j in $F_n(x + \frac{n}{\mu})$ are obtained in (2-24) by integrating $F_{n-1}(x + \frac{n-1}{\mu})$ over the region $(0,x)$.

Since $F_0(x)$ is a constant, $F_1(x + \frac{1}{\mu})$ must be linear in x, and hence $F_n(x + \frac{n}{\mu})$ must be an nth order polynomial.

Put

$$F_n(x + \frac{n}{\mu}) = A\lambda e^{-\lambda/\mu}\sum_{j=0}^{n} f_{nj}x^j$$

so that $f_{00} = 1.$

Substituting into (2-24)

$$F_n(x + \frac{n}{\mu}) = \lambda e^{-\lambda/\mu}\left(A - \sum_{i=0}^{n-2} \int_0^{1/\mu} A\lambda e^{-\lambda/\mu}\sum_{j=0}^{i} f_{ij}y^j dy - \int_0^x A\lambda e^{-\lambda/\mu}\sum_{j=0}^{n-1} f_{n-1,j}y^j dy\right)$$

Thus

$$F_n(x + \frac{n}{\mu}) = A\lambda e^{-\lambda/\mu}\left\{1 - \lambda e^{-\lambda/\mu}\sum_{i=0}^{n-2}\sum_{j=0}^{i} \frac{f_{ij}}{j+1}(\frac{1}{\mu})^{j+1} - \lambda e^{-\lambda/\mu}\sum_{j=0}^{n-1} \frac{f_{n-1,j}}{j+1}x^{j+1}\right\}$$

$$= A\lambda e^{-\lambda/\mu}\sum_{j=0}^{n} f_{nj}x^j.$$

Comparing terms

$$f_{nj} = -\lambda e^{-\lambda/\mu}\frac{f_{n-1,j-1}}{j} \qquad j>0$$

$$f_{n0} = 1-\lambda e^{-\lambda/\mu}\sum_{i=0}^{n-2}\sum_{j=0}^{i} \frac{f_{ij}}{j+1}(\frac{1}{\mu})^{j+1}$$

$$= 1 + \sum_{i=0}^{n-2}\sum_{j=0}^{i} \frac{f_{i+1,j+1}}{\mu^{j+1}} \qquad (2\text{-}25)$$

$$f_{n0} = 1 + \sum_{i=1}^{n-1}\sum_{j=1}^{i} \frac{f_{ij}}{\mu^j}$$

$$= 1 + \sum_{i=1}^{n-2}\sum_{j=1}^{i} \frac{f_{ij}}{\mu^j} + \sum_{j=1}^{n-1} \frac{f_{n-1,j}}{\mu^j}$$

$$= f_{n-1,0} + \sum_{j=1}^{n-1} \frac{f_{n-1,j}}{\mu^j} \quad \text{using (3-24).}$$

Thus $$f_{n0} = \sum_{j=0}^{n-1} \frac{f_{n-1,j}}{\mu^j}.$$

Summarising

$$F_n(x + \frac{n}{\mu}) = A\lambda e^{-\lambda/\mu} \sum_{j=0}^{n} f_{nj} x^j \qquad 0<x\leq\frac{1}{\mu} \qquad n = 0,1,2,\ldots$$

where $$f_{ij} = -\lambda e^{-\lambda/\mu} \frac{f_{i-1,j-1}}{j} \qquad i\geq j>0$$

and $$f_{i0} = \sum_{j=0}^{i-1} \frac{f_{i-1,j}}{\mu^j} \qquad (2\text{-}26)$$

with $f_{00} = 1$

and taking $F_0(x) = A\lambda e^{-\lambda/\mu}$ for $x\leq\frac{1}{\mu}$

Hence

$$U_n(x) = A\lambda e^{\lambda(x - \frac{1}{\mu})} \sum_{j=0}^{n} f_{nj} (x - \frac{n}{\mu})^j \qquad \frac{n}{\mu} < x \leq \frac{n+1}{\mu}$$

$$n = 1,2,3,\ldots$$

so that

$$U(x) = \begin{cases} U_n(x) & \frac{n}{\mu} < x \leq \frac{n+1}{\mu} \quad n>0 \\ A\lambda e^{\lambda(x-\frac{1}{\mu})} & x\leq\frac{1}{\mu} \end{cases}$$

So $U_0 = 1 - \tau = A\lambda e^{-\lambda/\mu}$

or $$A = \frac{1 - \tau}{\lambda e^{-\lambda/\mu}}$$

and

$$U(x) = \begin{cases} (1-\tau)e^{\lambda x} \sum_{j=0}^{n} f_{nj}(x - \frac{n}{\mu})^j, \frac{n}{\mu} < x \leq \frac{n+1}{\mu}, & n>0 \\ (1-\tau)e^{\lambda x} & x \leq \frac{1}{\mu} \end{cases}$$

with f_{nj} as defined in (2-26).

For the purposes of finding $\Delta(x)$ only $u(x)$ for $x \leq 0$ is required which for all M/G/1 queues is

$$u(x) = \lambda(1-\tau)e^{\lambda x}.$$

Hence from (2-17) or (2-18)

$$\Delta(x) = \begin{cases} \lambda(1-\tau)e^{\lambda(\frac{1}{\mu} - x)} & x > \frac{1}{\mu} \\ \tau & x = \frac{1}{\mu} \\ 0 & x < \frac{1}{\mu} \end{cases}$$

(iii) M/E_K/1 queue

$$a(t) = \begin{cases} \lambda e^{-\lambda t} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$b(t) = \begin{cases} \frac{(\mu k)^k}{\Gamma(k)} e^{-k\mu t} t^{k-1} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$v(x) = \left(\frac{k\mu}{\lambda+k\mu}\right)^k \lambda \times \begin{cases} e^{\lambda x} & x\leq 0 \\ e^{\lambda x}\left(1-\frac{\gamma(k,(\lambda+k\mu)x)}{\Gamma(k)}\right), & x>0 \end{cases}$$

where $\gamma(a,x)$ is the incomplete Gamma function given by

$$\gamma(a,x) = \int_0^x e^{-t}t^{a-1}dt$$

Hence

$$U(x) = \int_0^\infty U(y)v(x-y)\,dy$$

$$= \lambda\left(\frac{k\mu}{\lambda+k\mu}\right)^k \times \begin{cases} \frac{1}{\Gamma(k)}\int_0^x U(y)\left(\Gamma(k)-\gamma(k,(\lambda+k\mu)(x-y))\right)dy + \int_x^\infty U(y)e^{\lambda(x-y)}dy, & x>0 \\ \int_0^\infty U(y)e^{\lambda(x-y)}dy & x\leq 0. \end{cases}$$

For this case, only $U(x)$ for $x\leq 0$ will be determined.

For $x\leq 0$,

$$U(x) = (1-\tau)e^{\lambda x}$$

and

$$u(x) = \lambda(1-\tau)e^{\lambda x}.$$

Thus

$$\Delta(x) = \frac{\lambda(\mu k)^k}{\Gamma(k)}\left\{\frac{(1-\tau)e^{-\lambda x}}{(k\mu-\lambda)^k}\gamma(k,(k\mu-\lambda)x) + \frac{1}{\mu}x^{k-1}e^{-k\mu x}\right\}$$

(b) D/M/1 queue

$$a(t) = \delta(t - \frac{1}{\lambda})$$

$$b(t) = \begin{cases} \mu e^{-\mu t} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$v(x) = \begin{cases} \mu e^{-\mu(x + \frac{1}{\lambda})} & x \geq -\frac{1}{\lambda} \\ 0 & x < -\frac{1}{\lambda} \end{cases}$$

$$U(x) = \begin{cases} \mu e^{-\mu(x + \frac{1}{\lambda})} \int_0^{x+1/\lambda} U(y) e^{\mu y} dy & x > -\frac{1}{\lambda} \\ 0 & x \leq -\frac{1}{\lambda} \end{cases}$$

Replacing x by $x' - \frac{1}{\lambda}$ and putting $y' = x'-y$ and finally dropping the dashes

$$U(x - \frac{1}{\lambda}) = \begin{cases} \mu \int_0^x U(x-y) e^{-\mu y} dy & x > 0 \\ 0 & x \leq 0. \end{cases}$$

Try as solution

$$U(x) = 1 - (1 - U_0)e^{cx} \qquad x > -\frac{1}{\lambda}$$

Substituting we have

$$1-(1-U_0)e^{-c/\lambda}e^{cx} = \mu \int_0^x \left(e^{-\mu y} - (1-U_0)e^{cx}e^{-(c+\mu)y}\right) dy$$

$$= 1-e^{-\mu x} - \frac{\mu}{c+\mu}(1-U_0)(e^{cx}-e^{-\mu x}).$$

Equating coefficients

$$(1-U_0)e^{-c/\lambda} = \frac{\mu(1-U_0)}{c+\mu}$$

or

$$e^{-c/\lambda} = \frac{\mu}{c+\mu}$$

and
$$1 - \frac{\mu(1-U_0)}{c+\mu} = 0$$

or
$$c = -\mu U_0.$$

Thus

$$U(x) = \begin{cases} 1-(1-U_0)e^{-\mu U_0 x} & x > -\frac{1}{\lambda} \\ 0 & x \leq -\frac{1}{\lambda} \end{cases}$$

where U_0 is the non-zero root of

$$e^{\mu U_0/\lambda} = \frac{1}{1-U_0}$$

(compare this with expression for θ in (4-7))

and

$$u(x) = \begin{cases} \mu U_0(1-U_0)e^{-\mu U_0 x} & x > -\frac{1}{\lambda} \\ 0 & x \leq -\frac{1}{\lambda} \end{cases} \qquad (2\text{-}27)$$

Thus

$$w(x) = \begin{cases} \mu U_0(1-U_0)e^{-\mu U_0 x} & x > 0 \\ U_0 & x = 0 \end{cases} \qquad (2\text{-}28)$$

Hence $w^*(s) = U_0 + \frac{\mu U_0(1-U_0)}{s+\mu U_0} = \frac{(s+\mu)U_0}{s+\mu U_0}$

Substituting (2-27) into (2-17) gives

$$\Delta(x) = \frac{\mu}{1+U_0} \times \begin{cases} U_0 e^{-\mu(x-1/\lambda)} + (1-U_0)e^{-\mu x} & x > \frac{1}{\lambda} \\ U_0 e^{\mu U_0(x-1/\lambda)} + (1-U_0)e^{-\mu x} & x \leq \frac{1}{\lambda} \end{cases}$$

2.4 SUMMARY

In this section, the inter-departure distributions of several simple queueing systems have been investigated using a variety of techniques so as to illustrate the structure associated with the system.

The same techniques may be used for more complex systems and will be of theoretical value in the investigation of any GI/G/1 system. However in most of these complex systems, the techniques are unsuitable for computational purposes.

The output of the M/G/∞ queue has been shown to be Poisson (Mirasol (1963)).

Joint distributions of output parameters with other parameters of the queue have been obtained such as that of the number of departures in a time interval (0,t] and the queue length at t for the M/G/1 queueing system (Shanbhag (1966)) and for the M/M/1 queue in particular (Greenberg and Greenberg (1966)).

It should be emphasised that even if a complete description of the inter-departure distribution were available this would be inadequate as, except for the M/M/1 queue, the inter-departure process is auto-correlated and this is the subject of the next section.

SECTION 2

SERIAL STRUCTURE

CHAPTER THREE

Auto-covariance and Auto-correlation

The Correlation Structure of the Output Process.

3.1 AUTO-COVARIANCE AND AUTO-CORRELATION

Consider a general sequence

$$x_1, x_2, x_3, \ldots , x_n, \ldots$$

which has been generated by some stochastic process. Thus the elements constitute a time series. Generally the process can be described by the joint probability density function

$$f(x_1, x_2, x_3, \ldots, x_n, \ldots).$$

For any fixed k we can find simple moments

$$E(x_k^\ell) = \int_{-\infty}^{\infty} x_k^\ell \, f\,(x_k) dx_k$$

Therefore there exist a mean function μ_k and a variance function σ_k^2 which generally vary with k.

Similarly bivariate moments can be evaluated

$$E(x_i^a \, x_j^b) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_i^a \, x_j^b \, f(x_i, x_j) dx_i dx_j$$

to describe the dependence of the values of the time series at two neighbouring points i and j.

The prime bivariate moment (which will be treated later) is the auto-covariance function (a.c.v.f.)

$$\gamma(i,j) = E\big((x_i - \mu_i)(x_j - \mu_j)\big)$$

with $\gamma(i,i) = \sigma_i^2$.

This can be normalised by dividing by the product of the standard deviation σ_i and σ_j to give the auto-correlation function (a.c.f.)

$$\rho(i,j) = \frac{\gamma(i,j)}{\sigma_i \sigma_j}$$

which lies between -1 and +1.

A stochastic process is said to be stationary if for all k, n, all sets of i's that

$$f(x_{i_1}, x_{i_2}, \ldots, x_{i_n}) = f(x_{i_1+k}, x_{i_2+k}, x_{i_3+k}, \ldots, x_{i_n+k}).$$

That is, that the joint p.d.f. of any n elements of the time series is the same as the joint p.d.f. of a set of n elements k elements distant respectively of the first set for any n and any k.

Consequently,

$$\mu_i = \mu \text{ for all } i$$

$$\gamma(i,j) = E\big((x_i-\mu_i)(x_j-\mu_j)\big)$$

$$= E\big((x_i-\mu)(x_j-\mu)\big)$$

and since $E(x_i, x_j) = E(x_{i+k}, x_{j+k})$ only the difference j-i is required.

$$\gamma(i,j) = \text{Covariance } \big(x_i,\ x_{i+u}\big)$$

where u = j-i is called the lag. Thus the auto-covariance function is a function of the lag u only, stated as

$$\gamma_u = \gamma(i,j) \qquad u = j-i.$$

Thus the auto-correlation function is also a function of u only and is defined

$$\rho_u = \frac{\gamma_u}{\gamma_0} = \frac{\gamma_u}{\sigma^2} \; .$$

In this section, only stationary processes are considered.

In the case of queueing systems, the correlation structure of waiting times (Daley (1968)) and of inter-departure intervals (next chapter) have been examined.

3.2 THE CORRELATION STRUCTURE OF THE OUTPUT PROCESS

In the case of output, it has been shown (Daley (1968), Burke (1956)) that the inter-departure intervals are not independent of each other, except in the case of the M/M/1 queue. Thus work on tandem queues must take into account any correlation in the input to the second (and any subsequent) service points.

It is assumed in the following work that the system is already in operation at the time 0 i.e. there exists a sequence

$$\ldots, \delta_{-5}, \delta_{-4}, \ldots, \delta_0, \delta_1, \delta_2, \ldots$$

and similarly for the other sequences.

Jenkins (1966) calculated the auto-correlations ρ_1 and ρ_2 of the output of the $M/E_k/1$ queue by finding the joint p.d.f. of Δ_{n-1}, Δ_n and Δ_{n-1}, Δ_n, Δ_{n+1}. This work yielded the formulae.

$$\rho_1 = \frac{(1-\tau)k}{\tau^2(1-k)+k}\left\{\frac{(k-1)\tau-k}{\tau+k} + \left(\frac{k}{\tau+k}\right)^k\right\}$$

$$\rho_2 = \frac{(1-\tau)k}{\tau^2(1-k)+k}\left\{\frac{(k-1)\tau-k}{\tau+k} + \frac{\tau^2 k}{(\tau+k)^2}\left(\frac{k}{\tau+k}\right)^k + \frac{(k+1)\tau+k}{\tau+k}\left(\frac{k}{\tau+k}\right)^{2k}\right\}.$$

However, because of the complexity of the evaluation of the joint distributions, this technique cannot be practically applied to higher lags.

Daley (1968) developed a quite general approach and revealed equations for the evaluation of auto-covariances of GI/M/1 and M/G/1 queue outputs.

For the GI/M/1 queue

$$\gamma_k = (\tau^{-1}-\theta^{-1})\frac{1}{\mu}\left(E(\Delta_k/w_0=0) - \frac{1}{\lambda}\right) \qquad (3\text{-}1)$$

where θ is the unique root in $0 < \theta < 1$ of

$$\theta = \int_0^\infty e^{-\mu(1-\theta)x}dA(x) . \qquad (3\text{-}2)$$

For the M/G/1 queue, the equation linked the γ_k in the expression

$$\lambda^2(1-\tau)^{-1}\sum_{n=1}^{\infty}\gamma_n z^n = (w-z)(1-z)^{-1}(1-\omega)^{-1}+(2\omega'-\omega)((1-z)\omega\omega)^{-1} \qquad (|z| < 1) \qquad (3\text{-}3)$$

where $\omega' = \frac{dw}{dz}$, $\omega = \omega(z)$ is the root of smallest modulus of

$$\omega = z\int_0^\infty e^{-\lambda(1-\omega)x}dB(x). \qquad (3\text{-}4)$$

As a consequence, Daley showed that in a stationary M/G/1 queueing system, the maximum auto-correlation of lag 1 occurred in the queue M/D/1 and that its value is given by

$$\rho_1 = \frac{e^{-\tau}+\tau-1}{\tau+1} \quad .$$

The use of eqns. (3-1) and (3-2) to evaluate the auto-correlation is described in the next chapter for the example of the $E_k/M/1$ queue.

The use of equations (3-3) and (3-4) requires a little more work. We require ω either in closed form or as a series. Equation (3-4) yields an expression for ω not directly expressable as required. For example, the $M/E_k/1$ queue yields (3-4) as

$$\omega = z\left\{\frac{k/\tau}{(1-1/\tau)-\omega}\right\}^k$$

and the M/D/1 queue gives

$$\omega = ze^{-\tau(1-\omega)}$$

putting $\mu = 1$ and $\tau = \lambda$.

One possible way of expressing ω as a series is by using Lagrange's theorem. (Whittaker and Watson (1902)). Then evaluate the right-hand side of (3-3) as a series in z and finally compare terms with the left-hand side.

CHAPTER FOUR

Preliminary

$E_k/M/1$

D/M/1

Summary

4.1 PRELIMINARY

Equation (3-1) states that for the GI/M/1 queue

$$\gamma_k = (\tau^{-1}-\theta^{-1})\frac{1}{\mu}\{E(\Delta_n/w_0=0) - \frac{1}{\lambda}\}$$

with θ the unique root in $0 < \theta < 1$ of (3-2) which is

$$\theta = \int_0^\infty e^{-\mu(1-\theta)x}dA(x) \ .$$

Another result (Daley (1968)) states that

$$\mathrm{Var}(\Delta) = \mathrm{Var}(\delta) - (\tau^{-1}-\theta^{-1})2\theta(E(s))^2(1-\theta)^{-1} \ . \qquad (4\text{-}1)$$

Equation (2-17) states that

$$\Delta(x) = \int_0^x b(y)u(y-x)dy + (1-U_0)b(x) \ .$$

Thus

$$E(\Delta) = \int_0^\infty x\Delta(x)dx$$

can be expressed as

$$E(\Delta) = \frac{1}{\mu} - \int_{-\infty}^0 xu(x)dx \ . \qquad (4\text{-}2)$$

In the following work all densities will be conditional upon $w_0 = 0$ if they are dashed

ie. $u'_k(x)$ = p.d.f. of u_k conditional on $w_0 = 0$.

The requirement $w_0 = 0$ is obviously equivalent to the condition $u_0 \leq 0$.

From

$$u_n = \begin{cases} u_{n-1}+v_n & u_{n-1} > 0 \\ v_n & u_{n-1} \leq 0 \end{cases}$$

If $u_0 \leq 0$

$$u_1 = v_1$$

and further u_n can be evaluated as before

$$\therefore \quad u_1'(x) = v_1(x) = v(x)$$

and $$u_n'(x) = \int_0^{\infty} u_{n-1}'(y)v(x-y)dy + v(x)\int_{-\infty}^{0} u_{n-1}'(y)dy \qquad (4\text{-}3)$$

$$\Delta_n'(x) = \int_0^{x} b(y)u_n'(y-x)dy + (1-U_0)b(x) \quad . \qquad (4\text{-}4)$$

The rest of this chapter will describe the evaluation of the required expressions for the determination of γ_k and ρ_k for the $E_k/M/1$ queue.

In Chapter 5, numerical results for this queue will be given and interpreted.

4.2

For the $E_k/M/1$ queue

$$a(t) = \frac{(k\lambda)^k}{\Gamma(k)} t^{k-1}e^{-k\lambda t} \qquad t \geq 0$$

$$b(t) = \mu e^{-\mu t} \qquad t \geq 0$$

Equation for θ is

$$\theta = \int_0^\infty e^{-\mu(1-\theta)x} dA(x)$$

$$= \int_0^\infty e^{-\mu(1-\theta)x} \frac{(k\lambda)^k}{\Gamma(k)} x^{k-1} e^{-k\lambda x} dx$$

$$= \left(\frac{k\lambda+\mu-\mu\theta}{k\lambda}\right)^{-k}$$

or

$$\theta\left(1 + \frac{\mu}{k\lambda}(1-\theta)\right)^k = 1 \qquad (4\text{-}5)$$

$$\mathrm{Var}(\delta) = E((\delta - \tfrac{1}{\lambda})^2) = \frac{1}{k\lambda^2} \qquad (4\text{-}6)$$

$$v(x) = \mu\left(\frac{k\lambda}{k\lambda+\mu}\right)^k \times \begin{cases} e^{-\mu x} & x \geq 0 \\ \dfrac{e^{-\mu x}}{\Gamma(k)} \displaystyle\int_{-(k\lambda+\mu)x}^{\infty} y^{k-1} e^{-y} dy & x < 0 \end{cases}$$

$$= \mu\left(\frac{k\lambda}{k\lambda+\mu}\right)^k \times \begin{cases} e^{-\mu x} & x \geq 0 \\ \dfrac{e^{-\mu x}}{\Gamma(k)} \Gamma(k,-(k\lambda+\mu)x) & x \leq 0 \end{cases}$$

Now $\Gamma(k,y) = \Gamma(k)(1-P(k,y))$ (incomplete Γ-function)

$$= \Gamma(k) e_{k-1}(y) e^{-ky}$$

where

$$e_{k-1}(y) = 1+y+\frac{y^2}{2!} + \ldots + \frac{y^{k-1}}{(k-1)!} = \sum_{i=0}^{k-1} \frac{y^i}{i!}$$

Thus

$$v(x) = \mu\left(\frac{k\lambda}{k\lambda+\mu}\right)^k \begin{cases} e^{-\mu x} & x \geq 0 \\ e^{k\lambda x} \sum_{i=0}^{k-1} \frac{(-(k\lambda+\mu)x)^i}{i!}, & x < 0 \end{cases}$$

Putting $\mu\left(\frac{k\lambda}{k\lambda+\mu}\right)^k = A$

Then

$$v(x) = \begin{cases} Ae^{-\mu x} & x \geq 0 \\ Ae^{k\lambda x} \sum_{i=1}^{k} \frac{(-(k\lambda+\mu)x)^{i-1}}{\Gamma(i)} & x < 0 \end{cases}$$

Firstly, consider $x \geq 0$.

By inspection of the equation for $u_n'(x)$, set

$$u_n'(x) = Ae^{-\mu x} \sum_{j=1}^{n} a_{nj} x^{j-1}$$

with $\qquad u_1'(x) = v(x) = Ae^{-\mu x}$

$\therefore \qquad a_{11} = 1.$

Equation (4-3) states

$$u_{n+1}'(x) = \int_0^\infty u_n'(y)v(x-y)dy + U_n'(0)v(x)$$

where

$$U_n'(0) = \int_{-\infty}^{0} u_n'(y)dy.$$

$$u_{n+1}'(x) = \int_0^x (Ae^{-\mu y} \sum_{j=1}^{n} a_{nj} y^{j-1})(Ae^{-\mu(x-y)})dy$$

$$+ \int_x^\infty (Ae^{-\mu y} \sum_{j=1}^{n} a_{nj} y^{j-1})(Ae^{k\lambda(x-y)} \sum_{i=1}^{k} \frac{(-(k\lambda+\mu)(x-y))^{i-1}}{\Gamma(i)})dy$$

$$+ U_n'(0)Ae^{-\mu x}$$

$$= A^2 e^{-\mu x}\sum_{j=1}^{n} a_{nj}\int_0^x y^{j-1}dy + A^2 e^{-\mu x}\sum_{j=1}^{n}\sum_{i=1}^{k}\frac{a_{nj}}{\Gamma(i)} \times$$

$$\int_x^\infty e^{(k\lambda+\mu)(x-y)} y^{j-1}(-(k\lambda+\mu)(x-y))^{i-1}dy + AU_n'(0)e^{-\mu x}.$$

Putting

$$-(k\lambda+\mu)(x-y) = t$$

then $$y = \frac{t}{k\lambda+\mu} + x$$

and $$y^{j-1} = \left(\frac{t}{k\lambda+\mu} + x\right)^{j-1} = \sum_{\ell=0}^{j-1}\frac{\Gamma(j)}{\Gamma(\ell+1)\Gamma(j-\ell)}\left(\frac{t}{k\lambda+\mu}\right)^{\ell} x^{j-\ell-1}$$

Then

$$u_{n+1}'(x)=Ae^{-\mu x}\left\{U_n'(0)+A\sum_{j=1}^{n}\frac{a_{nj}x^j}{j}+A\sum_{j=1}^{n}\sum_{i=1}^{k}\sum_{\ell=0}^{j-1}\frac{a_{nj}\Gamma(j)x^{j-\ell-1}}{\Gamma(\ell+1)\Gamma(j-\ell)\Gamma(i)} \times \frac{1}{(k\lambda+\mu)^{\ell+1}}\int_0^\infty e^{-t}t^{i+\ell-1}dt\right\}$$

Putting $m = j - \ell$ this becomes

$$u_{n+1}'(x) = Ae^{-\mu x}\{U_n'(0)+A\sum_{j=2}^{n+1}\frac{a_{n,j-1}x^{j-1}}{j-1} + A\sum_{j=1}^{n}\sum_{i=1}^{k}\sum_{m=1}^{j}\frac{a_{nj}\Gamma(j)\Gamma(i+j-m)x^{m-1}}{\Gamma(j-m+1)\Gamma(m)\Gamma(i)(k\lambda+\mu)^{j-m+1}}\}$$

Changing the order of summation we get

$$u_{n+1}'(x)=Ae^{-\mu x}\left(U_n'(0)+A\sum_{j=2}^{n+1}\frac{a_{n,j-1}x^{j-1}}{j-1}+A\sum_{j=1}^{n}x^{j-1}\left(\frac{1}{\Gamma(j)}\sum_{\ell=j}^{n}\frac{a_{n\ell}\Gamma(\ell)}{\Gamma(\ell-j+1)(k\lambda+\mu)^{\ell-j+1}} \times \sum_{i=1}^{k}\frac{\Gamma(i+\ell-j)}{\Gamma(i)}\right)\right).$$

Thus

$$a_{n+1,1} = U_n'(0) + AF_n(1)$$

$$a_{n+1,j} = \frac{Aa_{n\,j-1}}{j-1} + AF_n(j) \qquad j = 2,\dots,n$$

$$a_{n+1,n+1} = \frac{Aa_{nn}}{n}$$

with

$$F_n(j) = \frac{1}{\Gamma(j)}\sum_{\ell=j}^{n} \frac{a_{n\ell}\Gamma(\ell)}{\Gamma(\ell-j+1)(k\lambda+\mu)^{\ell-j+1}} \sum_{i=1}^{k} \frac{\Gamma(i+\ell-j)}{\Gamma(i)} .$$

Now, consider $x < 0$.

Substituting in (4-3) the form of $u_n'(x)$, $x \geq 0$ yields

$$u_{n+1}'(x) = \int_0^\infty \Big(Ae^{-\mu y}\sum_{j=1}^{n} a_{nj}y^{j-1}\Big)\Big(Ae^{k\lambda(x-y)}\sum_{i=1}^{k}\frac{(-(k\lambda+\mu)(x-y))^{i-1}}{\Gamma(i)}\Big)dy$$

$$+\; U_n'(0)Ae^{k\lambda x}\sum_{i=1}^{k}\frac{(-(k\lambda+\mu)x)^{i-1}}{\Gamma(i)} .$$

Putting $(y-x)^{i-1} = \sum_{\ell=0}^{i-1}\frac{\Gamma(i)y^{\ell}(-x)^{i-\ell-1}}{\Gamma(\ell+1)\Gamma(i-\ell)}$ and integrating

the integral term becomes

$$A^2e^{k\lambda x}\sum_{j=1}^{n}\sum_{i=1}^{k}\sum_{\ell=0}^{i-1}\frac{a_{nj}(k\lambda+\mu)^{i-1}(-x)^{i-\ell-1}}{\Gamma(\ell+1)\Gamma(i-\ell)}\,\frac{\Gamma(j+\ell)}{(k\lambda+\mu)^{j+\ell}} .$$

Putting $m = i - \ell$ and changing order of summation this becomes

$$A^2e^{k\lambda x}\sum_{j=1}^{k}\frac{(-(k\lambda+\mu)x)^{j-1}}{\Gamma(j)}\left(\sum_{i=j}^{k}\frac{1}{\Gamma(i-j+1)}\sum_{\ell=1}^{n}\frac{a_{n\ell}\Gamma(\ell+i-j)}{(k\lambda+\mu)^{\ell}}\right) .$$

Thus in full, for $x < 0$

$$u'_{n+1}(x) = Ae^{k\lambda x}\left\{\sum_{j=1}^{k}\left[U'_n(0)+A\sum_{i=j}^{k}\frac{1}{\Gamma(i-j+1)}\sum_{\ell=1}^{n}\frac{a_{n\ell}\Gamma(\ell+i-j)}{(k\lambda+\mu)^{\ell}}\right]\frac{(-(k\lambda+\mu)x)^{j-1}}{\Gamma(j)}\right\}$$

i.e.

$$u'_n(x) = Ae^{k\lambda x}\sum_{j=1}^{k}\frac{b_{nj}(-(k\lambda+\mu)x)^{j-1}}{\Gamma(j)}$$

with

$$b_{n+1,j} = U'_n(0) + A\sum_{i=j}^{k}\frac{1}{\Gamma(i-j+1)}\sum_{\ell=1}^{n}\frac{a_{n\ell}\Gamma(\ell+i-j)}{(k\lambda+\mu)^{\ell}} .$$

To use the above as a recurrence relation the value of $U'_n(0)$ is required and can be expressed as

$$U'_n(0) = \int_{-\infty}^{0} u'_n(x)dx$$

$$= A\sum_{j=1}^{k}\frac{b_{nj}(k\lambda+\mu)^{j-1}}{(k\lambda)^{j}} .$$

From (4-4)

$$E(\Delta_n/w_0=0) = \frac{1}{\mu} - \int_{-\infty}^{0} xu'_n(x)dx$$

$$= \frac{1}{\mu} + A\sum_{j=1}^{k} b_{nj}\frac{(k\lambda+\mu)^{j-1}}{(k\lambda)^{j+1}}j .$$

As $k \to \infty$ the $E_k/M/1$ queue tends to the D/M/1 queue. Results for the $E_k/M/1$ queue could be used to find the results for the D/M/1 but a special treatment of the D/M/1 is somewhat simpler.

4.3 D/M/1

For the D/M/1 queue

$$a(t) = \delta(t - \frac{1}{\lambda})$$

$$b(t) = \mu e^{-\mu t} \qquad t \geq 0$$

Equation for θ is

$$\theta = \int_0^\infty e^{-\mu(1-\theta)x} dA(x)$$

$$= \int_0^\infty e^{-\mu(1-\theta)x} \delta(x - \frac{1}{\lambda}) dx$$

$$= e^{-\mu(1-\theta)/\lambda}$$

or $$\theta e^{(1-\theta)\mu/\lambda} = 1 \qquad (4\text{-}7)$$

$$\mathrm{Var}(\delta) = 0 \qquad (4\text{-}8)$$

$$v(x) = \begin{cases} \mu e^{-\mu(x + \frac{1}{\lambda})} & x \geq -\frac{1}{\lambda} \\ 0 & x < -\frac{1}{\lambda} \end{cases}$$

From (4-3) and the form of $v(x)$ it can be shown that

$$u_n'(x) = 0 \qquad x < -\frac{1}{\lambda} \text{ for all } n.$$

For $x \geq -\frac{1}{\lambda}$, try the form

$$u_n'(x) = e^{-\mu(x + \frac{1}{\lambda})} \sum_{i=1}^{n} a_{ni} (x + \frac{n}{\lambda})^{i-1}$$

with $a_{11} = \mu$ (from $u_1'(x) = v(x)$).

Equation (4-3) then becomes

$$u'_{n+1}(x) = \int_0^{x+1/\lambda} \{\mu e^{-\mu(x-y+\frac{1}{\lambda})}\}\{e^{-\mu(y+\frac{1}{\lambda})}\sum_{i=1}^{n} a_{ni}(y+\tfrac{n}{\lambda})^{i-1}\}dy + U'_n(0)\mu e^{-\mu(x+\frac{1}{\lambda})}$$

which on integration becomes

$$u'_{n+1}(x) = e^{-\mu(x+\frac{1}{\lambda})}\{\mu U'_n(0) - \mu e^{-\mu/\lambda}\sum_{i=1}^{n} \frac{a_{ni}}{i}(\tfrac{n}{\lambda})^{i} + \mu e^{-\mu/\lambda}\sum_{i=1}^{n} \frac{a_{ni}}{i}(x+\frac{n+1}{\lambda})^{i}\}$$

Thus

$$a_{n+1,1} = \mu U'_n(0) - \sum_{i=1}^{n} \frac{\mu e^{-\mu/\lambda} a_{ni}}{i (\frac{n}{\lambda})^{i}}$$

$$a_{n+1,i} = \mu e^{-\mu/\lambda}\frac{a_{n,i-1}}{i-1} \qquad i = 2,\ldots,n+1$$

and hence $a_{n+1,1}$ can be re-written as

$$a_{n+1,1} = \mu U'_n(0) - \sum_{i=1}^{n} a_{n+1,i+1}(\tfrac{n}{\lambda})^{i}$$

Thus

$$u'_n(x) = e^{-(x+1/\lambda)}\sum_{i=1}^{n} a_{ni}(x+n/\lambda)^{i-1}$$

is the solution for D/M/1 with

$$a_{11} = \mu$$

$$a_{ni} = \mu e^{-\mu/\lambda}\frac{a_{n-1,i-1}}{i-1} \qquad i = 2,\ldots,n$$

$$a_{n1} = \mu U'_{n-1}(0) - \sum_{i=2}^{n} a_{ni}(\tfrac{n-1}{\lambda})^{i-1}$$

Now define

$$I_k^n = \int_{-1/\lambda}^{0} (x + \tfrac{n}{\lambda})^{k-1} e^{-\mu(x+1/\lambda)}dx$$

Then

$$I_1^n = \frac{1}{\mu}(1 - e^{-\mu/\lambda})$$

$$I_k^n = \frac{e^{-\mu/\lambda}}{\mu}(\frac{1}{\lambda})^{k-1}\{(n-1)^{k-1}e^{\mu/\lambda} - n^{k-1}\} + \frac{k-1}{\mu} I_{k-1}^n \ .$$

Using the above expressions

$$U_n'(0) = \int_{-\frac{1}{\lambda}}^{0} u_n'(x)dx$$

$$= \sum_{i=1}^{n} a_{ni} I_i^n$$

and

$$E(\Delta_n/w_0=0) = \frac{1}{\mu} - \int_{-\frac{1}{\lambda}}^{0} xu_n'(x)dx$$

$$= \frac{1}{\mu} - \sum_{i=1}^{n} a_{ni}(I_{i+1}^n - \frac{n}{\lambda} I_i^n)$$

4.4 SUMMARY

4.4.1

For the $E_k/M/1$ queue we have

$$\theta(1 + \frac{\mu}{k\lambda}(1-\theta)^k) = 1$$

$$Var(\delta) = \frac{1}{k\lambda^2}$$

$$u_n'(x) = \begin{cases} Ae^{-\mu x}\sum_{j=1}^{n} a_{nj}x^{j-1} & x \geq 0 \\ \\ Ae^{k\lambda x}\sum_{j=1}^{k} b_{nj}\frac{(-(k\lambda+\mu)x)^{j-1}}{\Gamma(j)} & x < 0 \end{cases}$$

with

$$A = \mu\left(\frac{k\lambda}{k\lambda+\mu}\right)^k$$

$$F_n(j) = \frac{1}{\Gamma(j)} \sum_{\ell=j}^{n} \frac{a_{n\ell}\,\Gamma(\ell)}{\Gamma(\ell-j+1)(k\lambda+\mu)^{\ell-j+1}} \sum_{i=1}^{k} \frac{\Gamma(i+\ell-j)}{\Gamma(i)}$$

and

$$a_{11} = 1$$

$$a_{n+1,1} = U_n'(0) + AF_n(1)$$

$$a_{n+1,j} = \frac{Aa_{n,j-1}}{j-1} + AF_n(j) \qquad j = 2,\ldots,n$$

$$a_{n+1,n+1} = \frac{Aa_{nn}}{n}$$

$$b_{n+1,j} = U_n'(0) + A\sum_{i=j}^{k} \frac{1}{\Gamma(i-j+1)} \sum_{\ell=1}^{n} \frac{a_{n\ell}\Gamma(\ell+i-j)}{(k\lambda+\mu)^{\ell}}$$

$$U_n'(0) = A\sum_{j=1}^{k} \frac{b_{nj}(k\lambda+\mu)^{j-1}}{(k\lambda)^j}$$

$$U_1'(0) = V(0) = 1 - \frac{A}{\mu}$$

$$E(\Delta_n/w_0=0) = \frac{1}{\mu} + A\sum_{j=1}^{k} b_{nj}\frac{(k\lambda+\mu)^{j-1}}{(k\lambda)^{j+1}}j\ .$$

From the above, along with (3-1) and (4-1) the values of

$$\gamma_k \text{ and } \rho_k$$

can be found for the $E_k/M/1$ queue.

4.4.2

For the D/M/1 queue we have

$$\theta e^{\mu/\lambda(1-\theta)} = 1$$

$$\mathrm{Var}(\delta) = 0$$

$$u_n'(x) = \begin{cases} 0 & x < -\frac{1}{\lambda} \\ e^{-\mu(x+\frac{1}{\lambda})} \sum_{i=1}^{n} a_{ni}(x + \frac{n}{\lambda})^{i-1}, & x \geq -\frac{1}{\lambda} \end{cases}$$

with

$$a_{11} = \mu$$

$$a_{ni} = \mu e^{-\mu/\lambda} \frac{a_{n-1,i-1}}{i-1} \qquad i = 2,\ldots,n$$

$$a_{n1} = \mu U_{n-1}'(0) - \sum_{i=2}^{n} a_{ni} (\frac{n-1}{\lambda})^{i-1}.$$

If

$$I_k^n = \int_{-\frac{1}{\lambda}}^{0} (x + \frac{n}{\lambda})^{k-1} e^{-\mu(x+\frac{1}{\lambda})} dx$$

then $\quad I_1^n = \frac{1}{\mu}(1 - e^{-\mu/\lambda})$

$$I_k^n = \frac{e^{-\mu/\lambda}}{\mu}(\frac{1}{\lambda})^{k-1}\left\{(n-1)^{k-1}e^{\mu/\lambda} - n^{k-1}\right\} + \frac{k-1}{\mu} I_{k-1}^n .$$

Thus $\quad U_n'(0) = \sum_{i=1}^{n} a_{ni} I_i^n$

and $\quad E(\Delta_n/w_0=0) = \frac{1}{\mu} - \sum_{i=1}^{n} a_{ni}(I_{i+1}^n - \frac{n}{\lambda}I_i^n).$

From the above, along with (3-1) and (4-1) the values of

γ_k and ρ_k

can be found for the D/M/1 queue.

4.4.3 Appendix

An alternative expression for $u_n'(x)$ for the D/M/1 can be found by trying

$$u_n'(x) = e^{-\mu(x+\frac{n}{\lambda})}\sum_{i=1}^{n}\alpha_{ni}(x+\frac{n}{\lambda})^{i-1} \qquad x \geq -\frac{1}{\lambda}$$

i.e. $\alpha_{ni} = a_{ni}e^{n\mu/\lambda}$.

Then

$$\alpha_{11} = \mu$$
$$\alpha_{ni} = \mu\frac{\alpha_{n-1,i-1}}{i-1}$$

$$\alpha_{n1} = \mu U_{n-1}'(0)e^{(n-1)\mu/\lambda} - \sum_{i=2}^{n}\alpha_{ni}(\frac{n-1}{\lambda})^{i-1} .$$

Define
$$J_k^n = \int_{-\frac{1}{\lambda}}^{0}(x+\frac{n}{\lambda})^{k-1}e^{-\mu(x+\frac{n}{\lambda})x}dx.$$

Then
$$J_1^n = \frac{e^{-n\mu/\lambda}}{\mu}(e^{\mu/\lambda}-1)$$

$$J_k^n = \frac{e^{-n\mu/\lambda}}{\mu}(\frac{1}{\lambda})^{k-1}\left\{(n-1)^{k-1}e^{\mu/\lambda}-n^{k-1}\right\}+\frac{k-1}{\mu}J_{k-1}^n$$

$$U_n'(0) = \sum_{i=1}^{n}\alpha_{ni}J_i^n$$

$$E(\Delta_n/w_0=0) = \frac{1}{\mu} - \sum_{i=1}^{n}\alpha_{ni}(J_{i+1}^n - \frac{n}{\lambda}J_i^n).$$

CHAPTER FIVE

Method

Programming

Results

5.1 METHOD

The work in this chapter centres around the evaluation of the auto-correlation function ρ_k for the $E_k/M/1$ and D/M/1 queues and the interpretation of the values obtained.

In this investigation $\mu = 1$ and hence $\tau = \lambda$. Thus the expressions obtained in Chapter 4 may be rewritten more simply.

5.1.1 The $E_k/M/1$ queue

The equation for θ becomes

$$\theta(1 + \frac{1}{k\tau}(1-\theta))^k = 1 \qquad (5\text{-}1)$$

$$\text{Var}(\delta) = \frac{1}{k\tau^2} \qquad (5\text{-}2)$$

and hence

$$\text{Var}(\Delta) = \frac{1}{k\tau^2} - (\tau^{-1}-\theta^{-1})\frac{2\theta}{1-\theta} \qquad (5\text{-}3)$$

$$\gamma_n = (\tau^{-1}-\theta^{-1})\left(1 - \frac{1}{\tau} - \int_{-\infty}^{0} xu_n'(x)dx\right) \qquad (5\text{-}4)$$

and thus

$$\rho_n = \frac{(\tau^{-1}-\theta^{-1})}{\frac{2\theta}{1-\theta}(\tau^{-1}-\theta^{-1}) - \frac{1}{k\tau^2}}\left(\int_{-\infty}^{0} xu_n'(x)dx + \frac{1}{\tau} - 1\right) \qquad (5\text{-}5)$$

$$A = (\frac{k\tau}{k\tau+1})^k \qquad (5\text{-}6)$$

$$a_{11} = 1$$

$$a_{n+1,1} = U_n'(0) + AF_n(1)$$

$$a_{n+1,j} = \frac{Aa_{n,j-1}}{j-1} + AF_n(j) \qquad j = 2,\dots,n \quad (5\text{-}7)$$

$$a_{n+1,n+1} = \frac{Aa_{nn}}{n}$$

with $$F_n(j) = \frac{1}{\Gamma(j)} \sum_{\ell=j}^{n} \frac{a_{n\ell}\Gamma(\ell)}{\Gamma(\ell-j+1)(k\tau+1)^{\ell-j+1}} \sum_{i=1}^{k} \frac{\Gamma(i+\ell-j)}{\Gamma(i)} \qquad (5\text{-}8)$$

and

$$b_{n+1,j} = U_n'(0) + A \sum_{i=j}^{k} \frac{1}{\Gamma(i-j+1)} \sum_{\ell=1}^{n} \frac{a_{n\ell}\Gamma(\ell+i-j)}{(k\tau+1)^{\ell}} \qquad (5\text{-}9)$$

$$U_n'(0) = A \sum_{j=1}^{k} b_{nj} \frac{(k\tau+1)^{j-1}}{(k\tau)^{j}} \qquad (5\text{-}10)$$

with

$$U_1'(0) = 1 - A$$

and

$$\int_{-\infty}^{0} x u_n'(x)dx = -A \sum_{j=1}^{k} b_{nj} \frac{(k\tau+1)^{j-1}}{(k\tau)^{j+1}} \; j \; . \qquad (5\text{-}11)$$

5.1.2 The D/M/1 Queue (First Method)

The equation for θ becomes

$$\theta e^{(1-\theta)/\tau} = 1 \qquad (5\text{-}12)$$

$$\mathrm{Var}(\delta) = 0 \qquad (5\text{-}13)$$

and hence

$$\mathrm{Var}(\Delta) = -(\tau^{-1}-\theta^{-1})\frac{2\theta}{1-\theta} \qquad (5\text{-}14)$$

$$\gamma_n = (\tau^{-1}-\theta^{-1})\left(1 - \frac{1}{\tau} - \int_{-\infty}^{0} x u_n'(x)dx\right) \qquad (5\text{-}15)$$

and thus

$$\rho_n = \frac{1-\theta}{2\theta}\left(\int_{-\infty}^{0} x u_n'(x)dx + \frac{1}{\tau} - 1 \right) \qquad (5\text{-}16)$$

$$a_{11} = 1$$

$$a_{ni} = e^{-1/\tau} \frac{a_{n-1,i-1}}{i-1} \qquad i = 2,\ldots,n \qquad (5\text{-}17)$$

$$a_{n1} = U_{n-1}'(0) - \sum_{i=2}^{n} a_{ni} \left(\frac{n-1}{\tau}\right)^{i-1}$$

$$I_1^n = 1 - e^{-1/\tau}$$

$$I_k^n = e^{-1/\tau}(\frac{1}{\tau})^{k-1}\{(n-1)^{k-1}e^{1/\tau}-n^{k-1}\}+(k-1)I_{k-1}^n \quad (5\text{-}18)$$

and $$U_n'(0) = \sum_{i=1}^{n} a_{ni} I_i^n \quad (5\text{-}19)$$

$$\int_{-\infty}^{0} x u_n'(x)dx = \sum_{i=1}^{n} a_{ni}(I_{i+1}^n - \frac{n}{\tau}I_i^n). \quad (5\text{-}20)$$

For the case of auto-correlation of lag 1, (5-16) gives

$$\rho_1 = \frac{1-\theta}{2\theta}\left(\int_{-\infty}^{0} x u_1'(x)dx + \frac{1}{\tau} - 1\right).$$

Using the fact that $u_1'(x) = v(x)$ we finally get

$$\rho_1 = -\frac{1}{2}(\frac{1-\theta}{\theta})e^{-1/\tau} .$$

Because of the programming problems described in the next section, the limit of ρ_1 as τ tended to zero was required. That is

$$\lim_{\tau\to 0} -\frac{1}{2}\frac{(1-\theta)}{\theta}e^{-1/\tau}$$

which can be shown to be equal to -1/2. Thus

$$\lim_{\tau\to 0} \rho_1 = -\frac{1}{2} .$$

5.2 PROGRAMMING

The equations given in 5.1 were the basis of several computer programs used to evaluate the ρ_n for the $E_k/M/1$ and D/M/1 queue.

In both cases, there were programming problems inherent in the equations.

(i) As $\tau \to 0$ the value of $\theta \to 0$ even faster thus underflowing the computer for $\tau < 0.05$ in the case of the D/M/1.

(ii) $E(\Delta_n / w_0 = 0) \to \frac{1}{\tau}$ as $n \to \infty$

and hence

$$E(\Delta_n / w_0 = 0) - \frac{1}{\tau} \to 0$$

(iii) Terms in the calculation of $E(\Delta_n / w_0 = 0) \to 0$ as the series subscript $\to$ n.

That is, earlier terms required more significant figures so that significance to later terms was not lost

(iv) $U'_n(0) \to 1$ as $\tau \to 0$

and hence small variations in τ failed to change

$U'_n(0)$ (to the accuracy used).

In an effort to eliminate, or at least to ease some of these difficulties all programs were run on an IBM 1620 with 16 significant figure accuracy. The results of these programs appear in this chapter.

5.3 RESULTS

The auto-correlation function (a.c.f.) for the $E_k/M/1$ queue (with certain values of k) was evaluated and graphed to determine the basic shape and relative magnitude. These graphs are shown on the following pages.

As can be seen from the graphs, all the a.c.f's. have a

MINIMA OF AUTO-CORRELATION FUNCTION

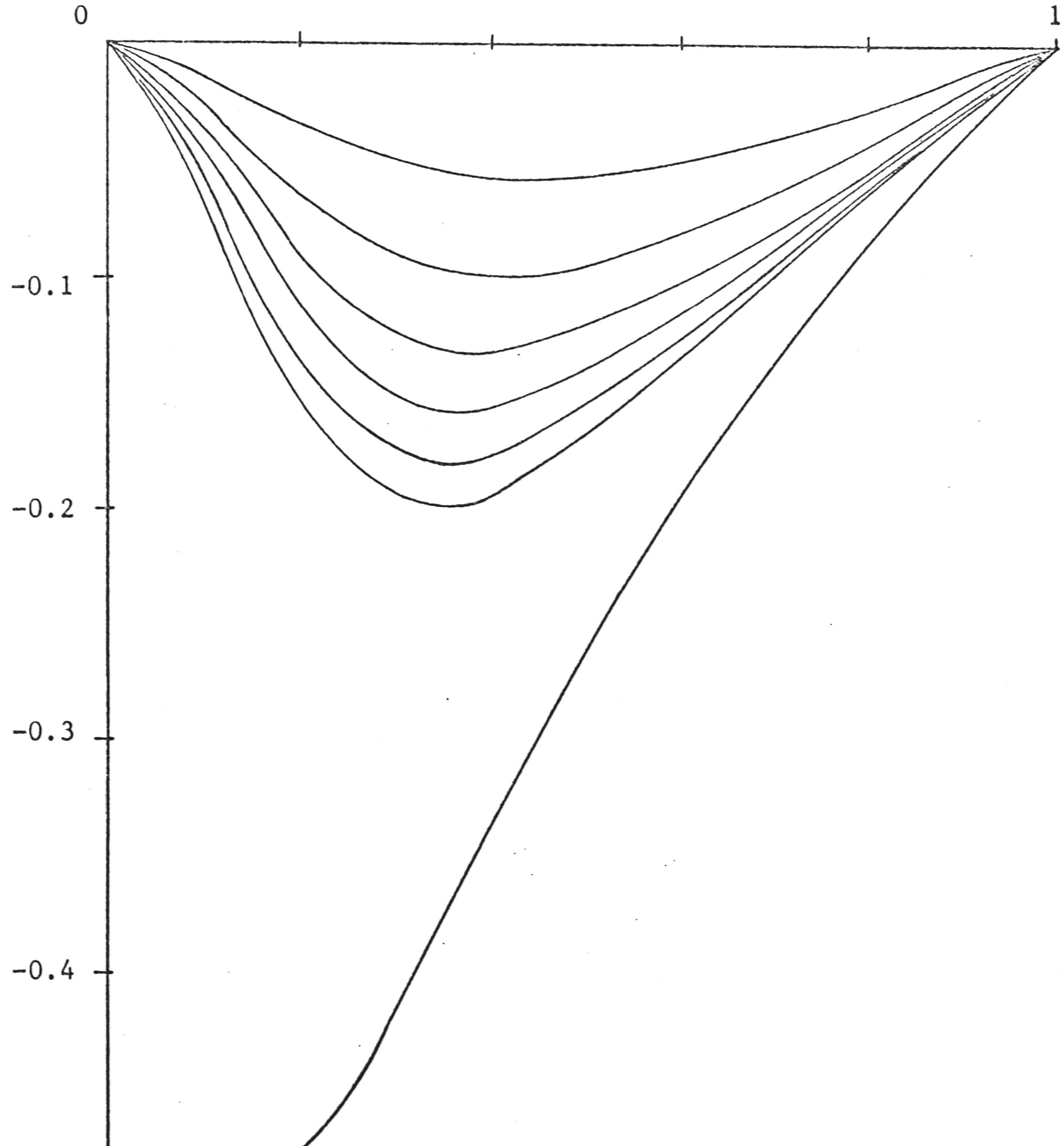
k	=	2	3	4	5	6	7	∞
Lag 1	τ	0.45	0.41	0.38	0.37	0.35	0.34	0.0
	ρ	-0.0568	-0.0992	-0.1317	-0.1576	-0.1789	-0.1968	-0.5
Lag 2	τ	0.56	0.55	0.54	0.53	0.53	0.52	0.50
	ρ	-0.0288	-0.0456	-0.0565	-0.0642	-0.0700	-0.0742	-0.1077
Lag 3	τ	0.63	0.62	0.61	0.61	0.61	0.61	0.60
	ρ	-0.0190	-0.0291	-0.0352	-0.0392	-0.0422	-0.0444	-0.0597
Lag 4	τ	0.67	0.66	0.66	0.66	0.66	0.66	0.66
	ρ	-0.0142	-0.0212	-0.0253	-0.0280	-0.0300	-0.0313	-0.0407
Lag 5	τ	0.70	0.70	0.70	0.70	0.70	0.70	0.70
	ρ	-0.0113	-0.0166	-0.0196	-0.0216	-0.0230	-0.0240	-0.0307

minimum point occurring at varying positions in the region $0<\tau<1$. (In the case of the D/M/1 queue, the a.c.f. tends to a minimum as $\tau\to0$, this minimum having a value of -0.5).

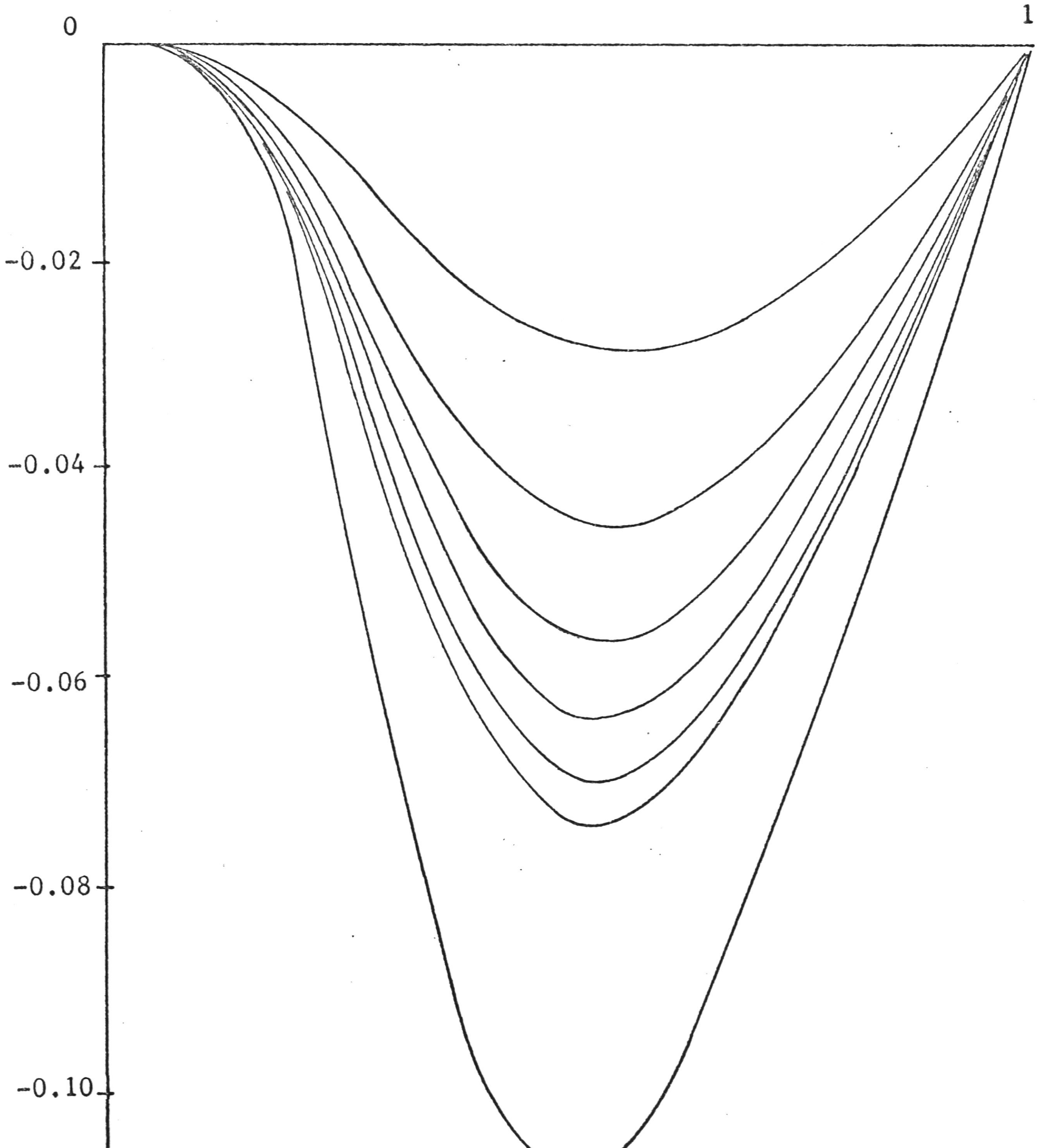
From the table (opposite) of positions of this minimum it can be seen that, for fixed lag, the value of τ at which the minimum auto-correlation occurs decreases with k, tending to the value of τ at which the D/M/1 queue a.c.f. of that lag has its minimum.

From the graphs, it can be seen that all the a.c.f's (except that of lag 1 for the D/M/1) tend to zero as $\tau\to0$ and as $\tau\to1$, and all are negative in the region $0<\tau<1$.

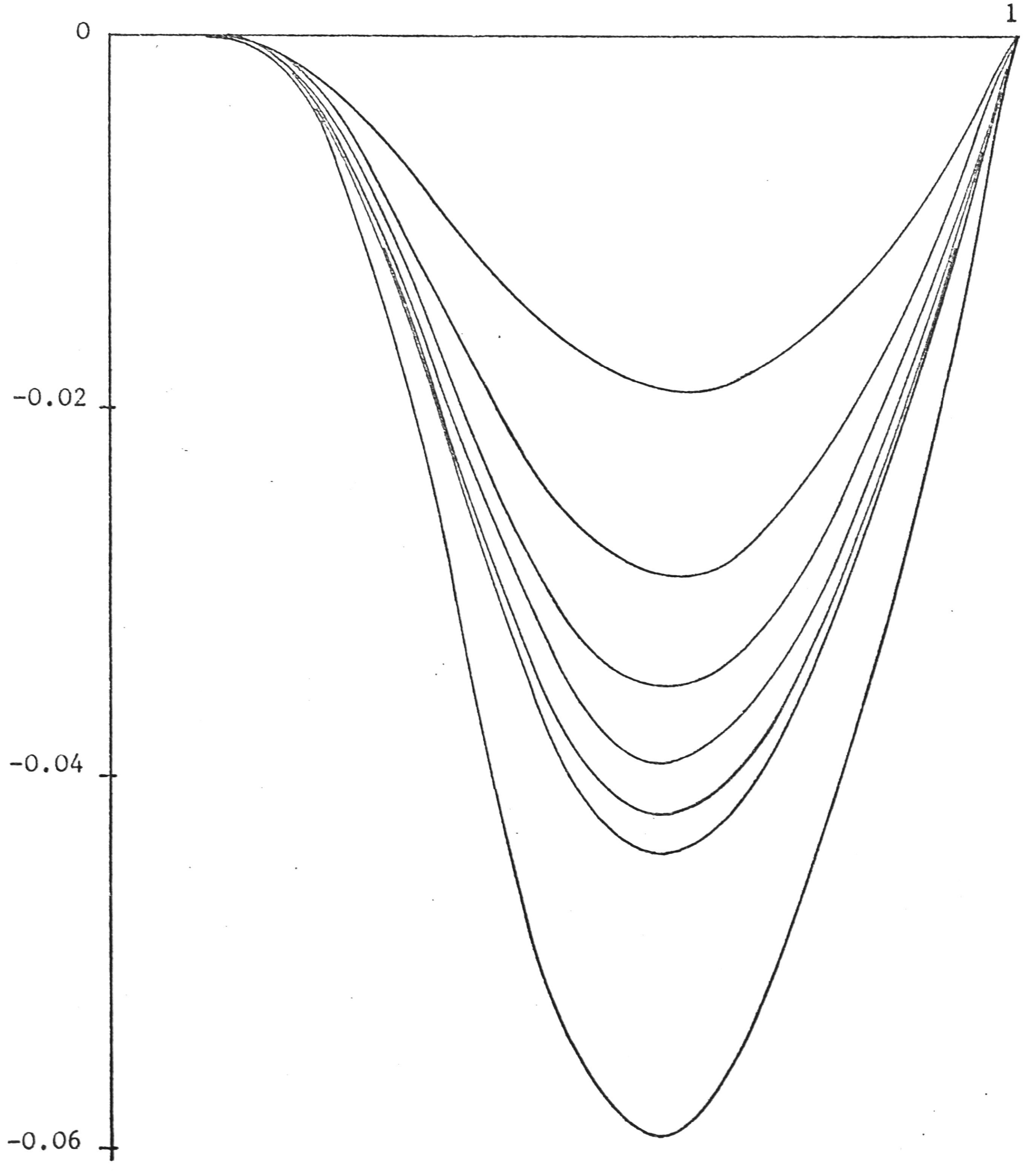
Autocorrelation function of lag 1 versus traffic intensity for the queues $E_k/M/1$, $k=2,3,4,5,6,7,\infty$ ($k=\infty$ being the D/M/1 queue) proceeding from upper to lower curve.


0
1
-0.1
-0.2
-0.3
-0.4
-0.5

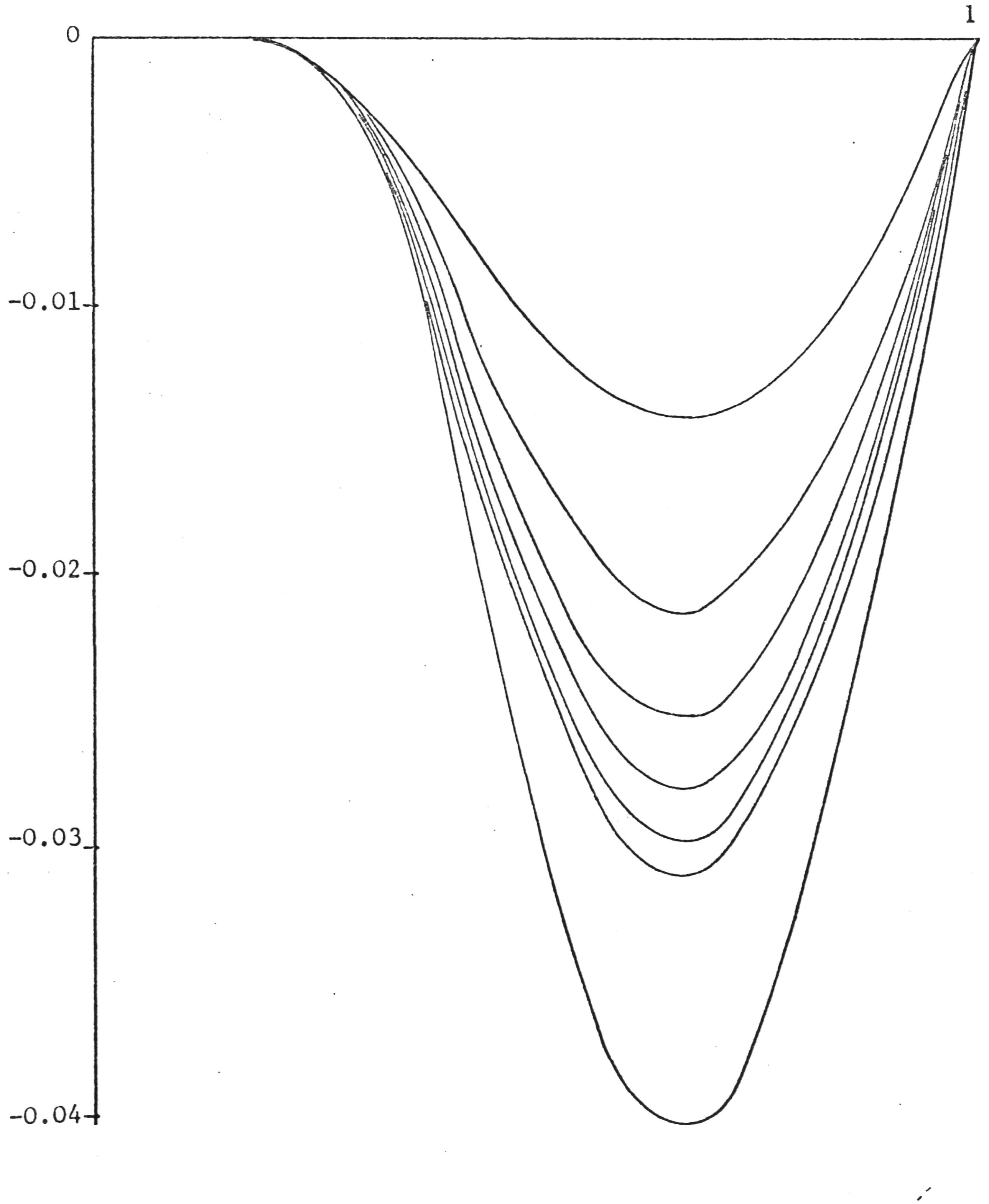
Autocorrelation function of lag 2 versus traffic intensity for the queues $E_k/M/1$, $k=2,3,4,5,6,7,\infty$ ($k=\infty$ being the D/M/1 queue) proceeding from upper to lower curve.


0
1
-0.02
-0.04
-0.06
-0.08
-0.10

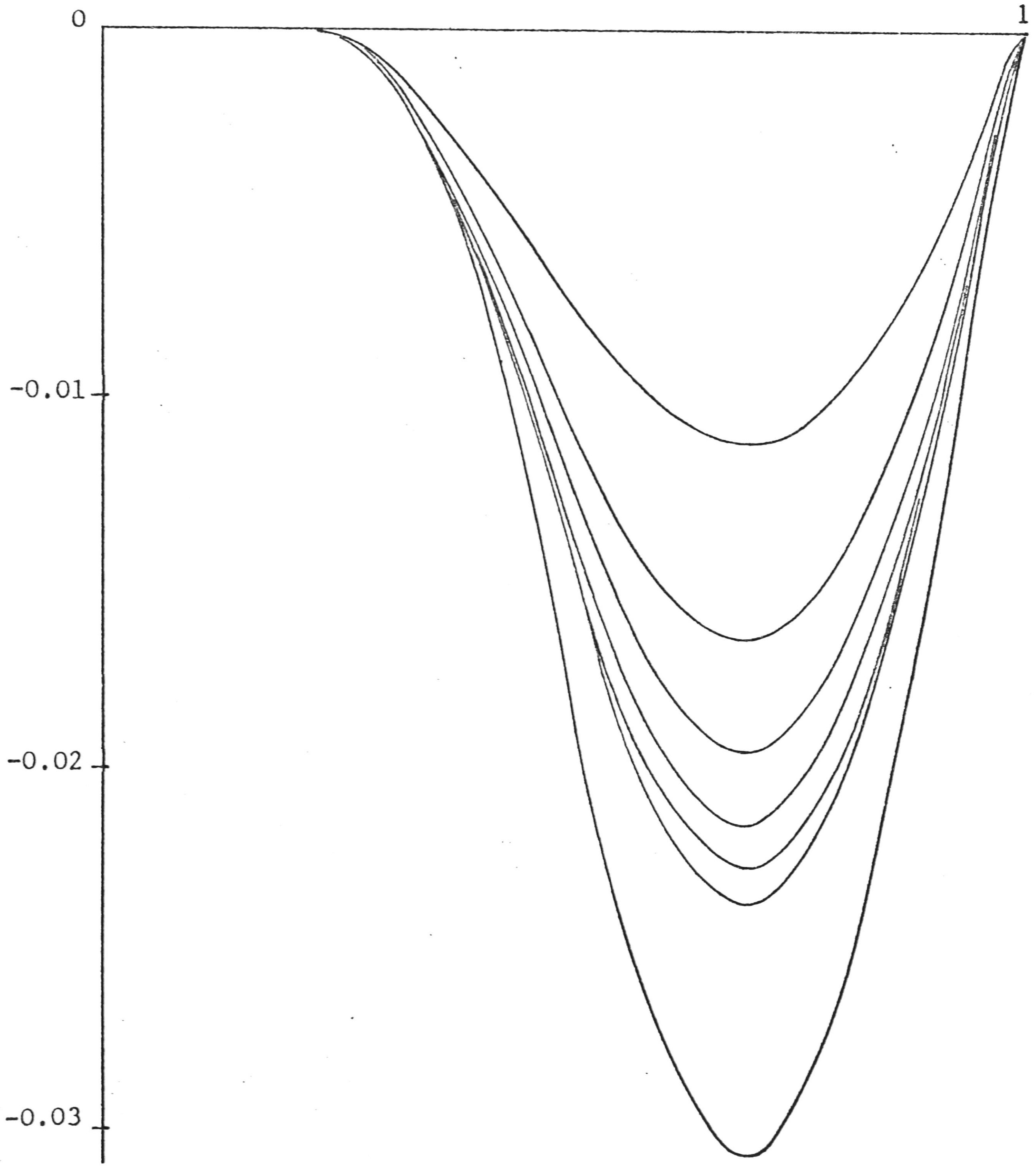
Autocorrelation function of lag 3 versus traffic intensity for the queues $E_k/M/1$, $k=2,3,4,5,6,7,\infty$ ($k=\infty$ being the D/M/1 queue) proceeding from upper to lower curve.


0
1
-0.02
-0.04
-0.06

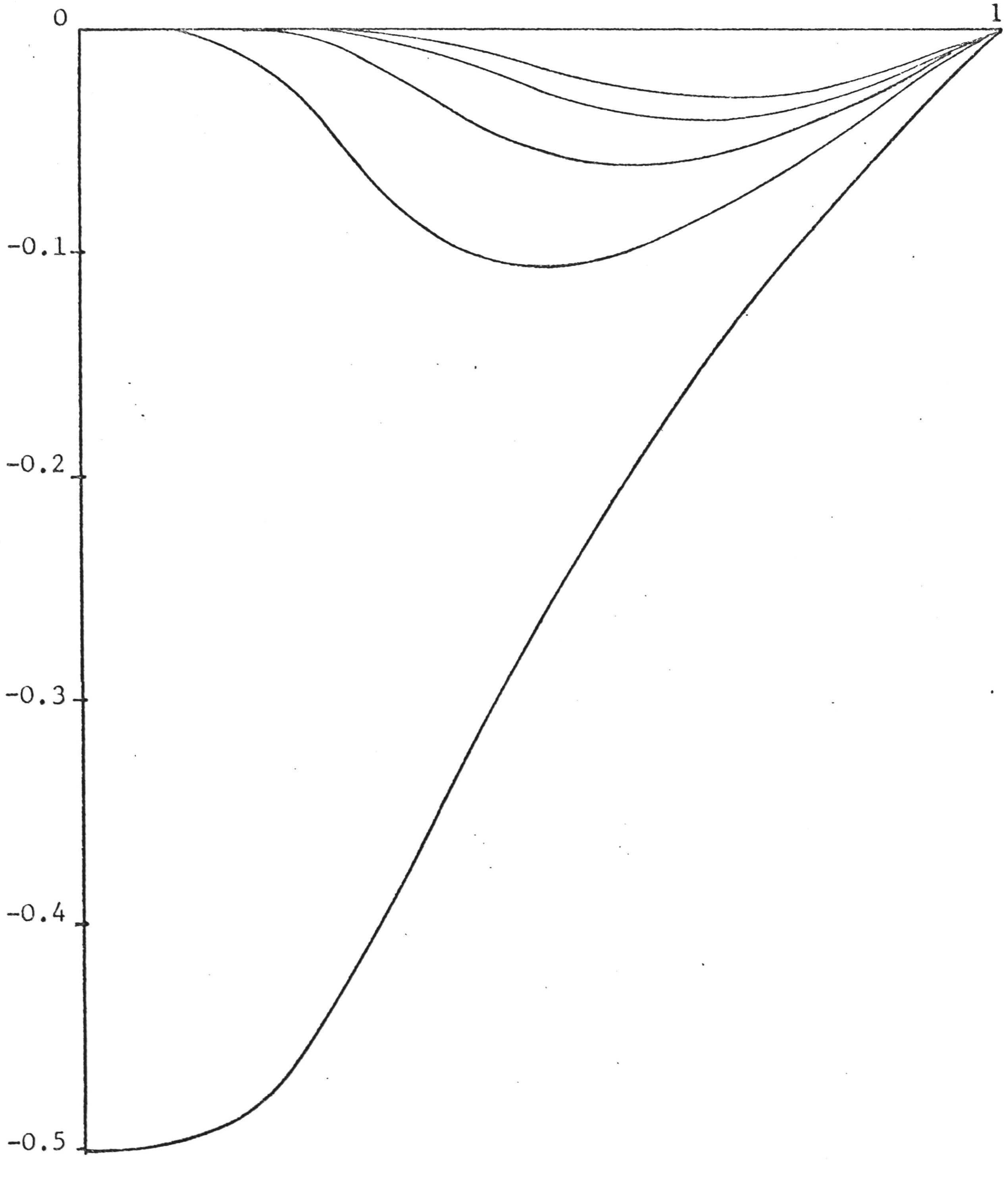
Autocorrelation function of lag 4 versus traffic intensity for the queues $E_k/M/1$, $k=2,3,4,5,6,7,\infty$ ($k=\infty$ being the D/M/1 queue) proceeding from upper to lower curve.


0
1
-0.01
-0.02
-0.03
-0.04

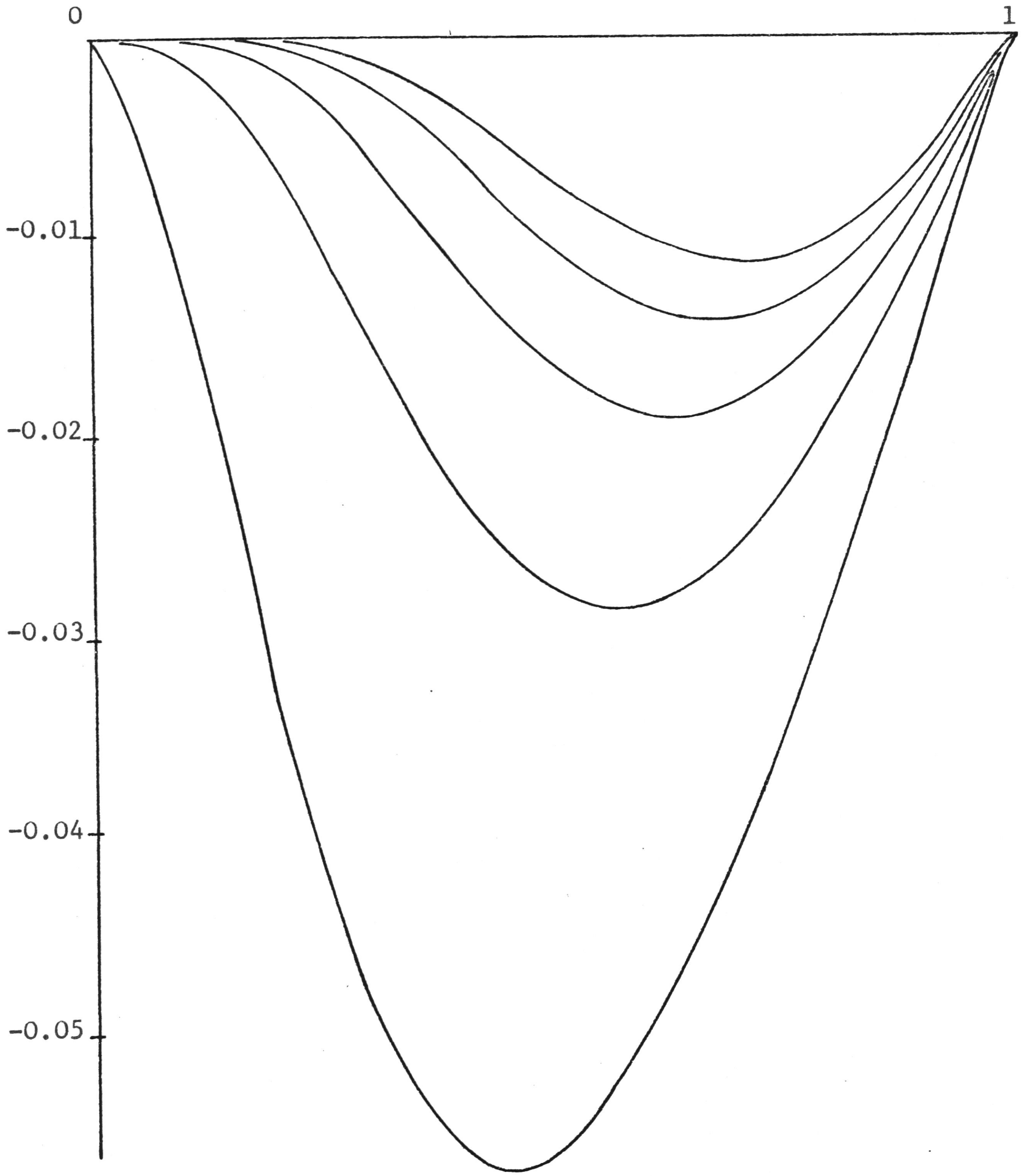
Autocorrelation function of lag 5 versus traffic intensity for the queues $E_k/M/1$, $k=2,3,4,5,6,7,\infty$ ($k=\infty$ being the D/M/1 queue) proceeding from upper to lower curve.


0
1
-0.01
-0.02
-0.03

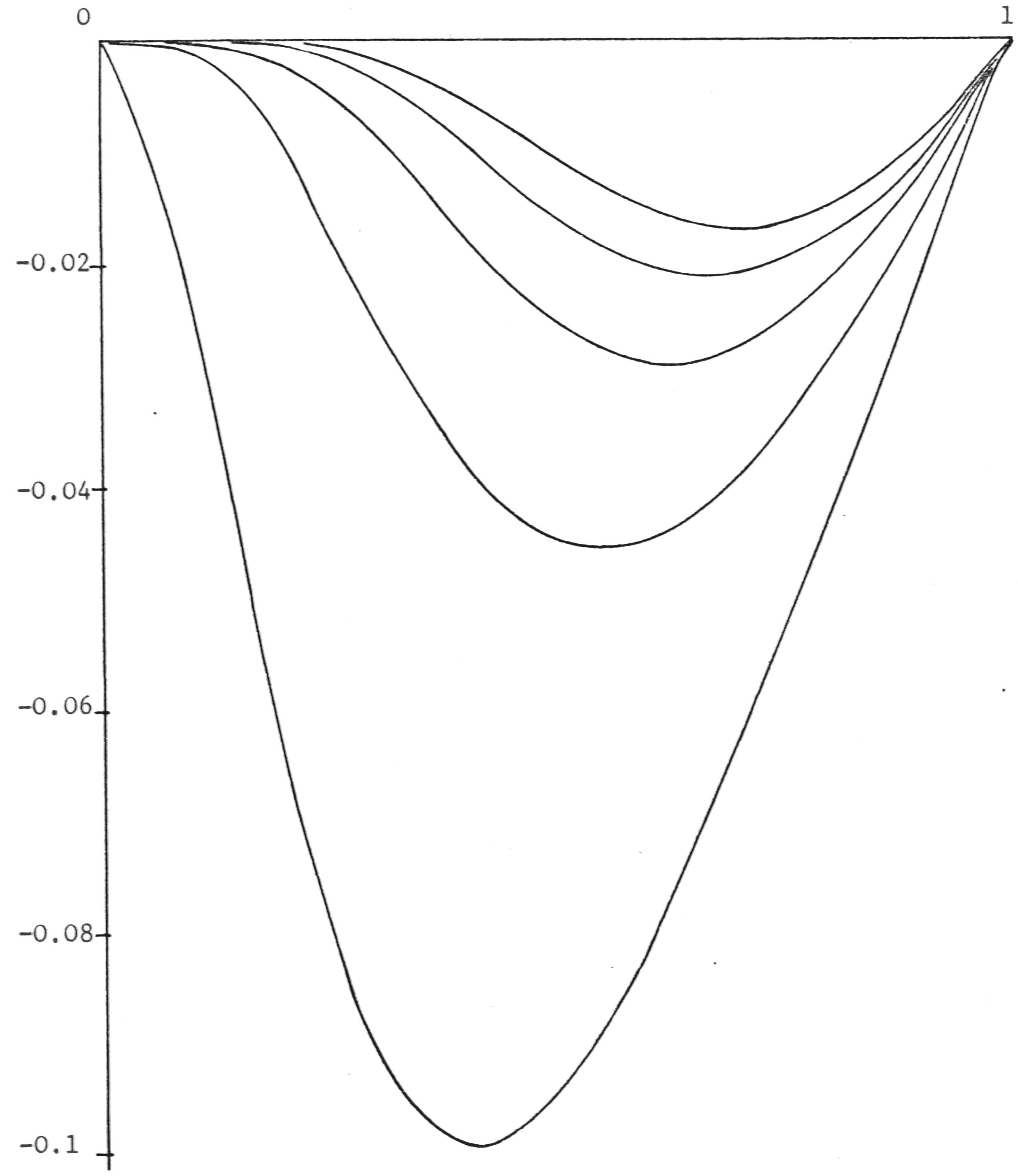
Autocorrelation function of the D/M/1 queue versus traffic intensity for lags 1,2,3,4,5, proceeding from lower to upper curve.


0
1
-0.1
-0.2
-0.3
-0.4
-0.5

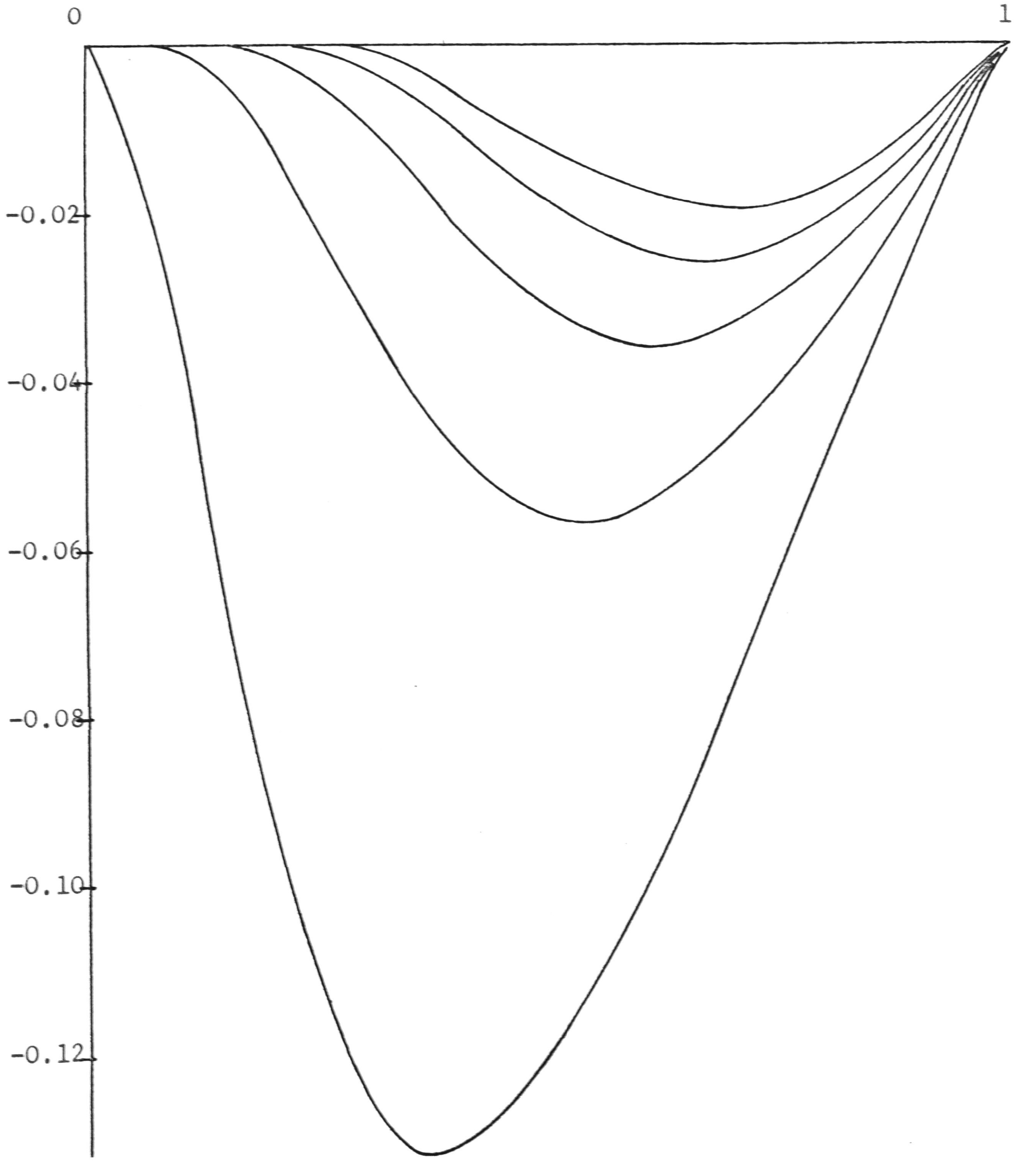
Autocorrelation function of the $E_2/M/1$ queue versus traffic intensity for lags 1,2,3,4,5, proceeding from lower to upper curve.


0
1
-0.01
-0.02
-0.03
-0.04
-0.05

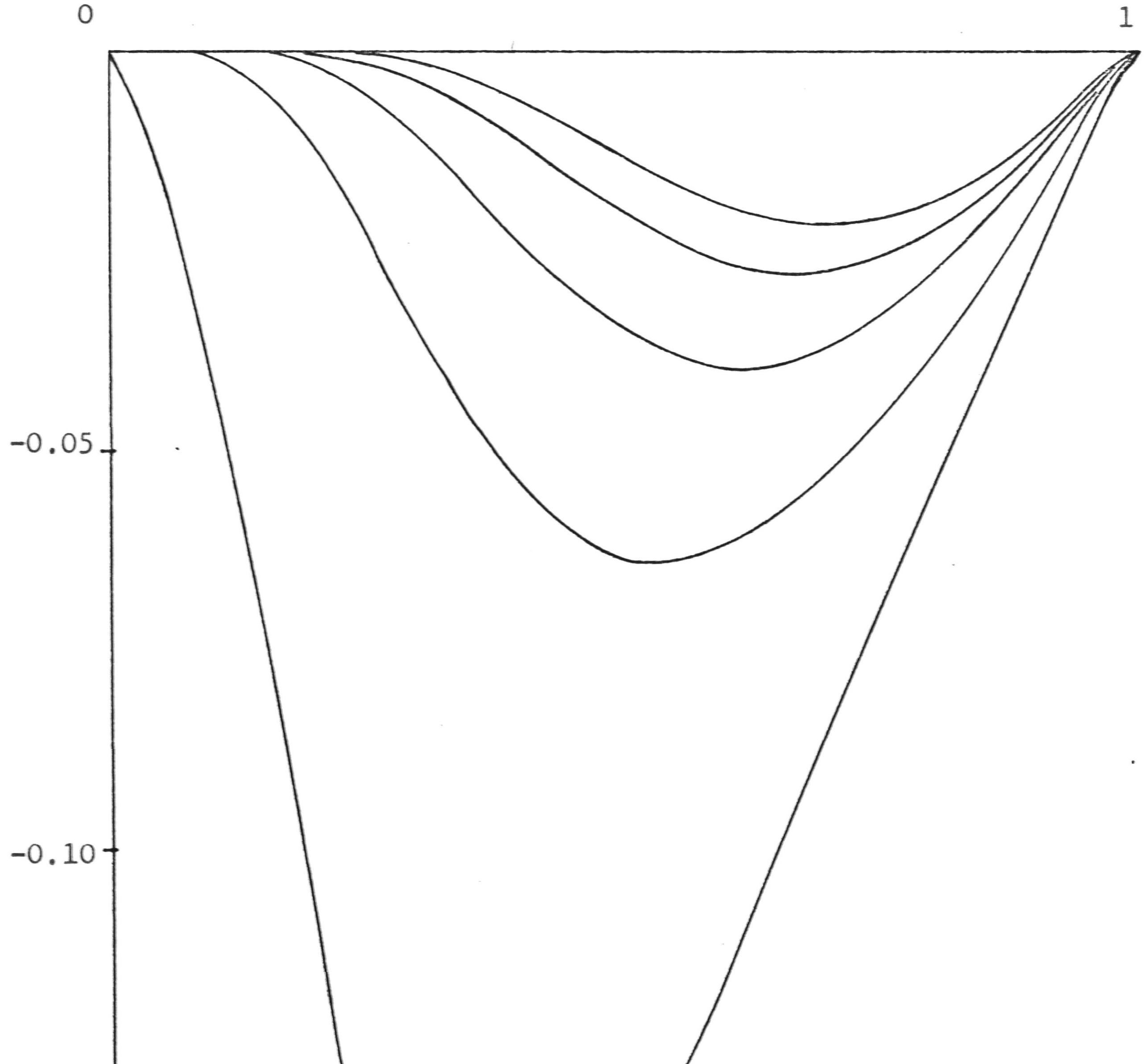
Autocorrelation function of the $E_3/M/1$ queue versus traffic intensity for lags 1,2,3,4,5, proceeding from lower to upper curve.


0
1
-0.02
-0.04
-0.06
-0.08
-0.1

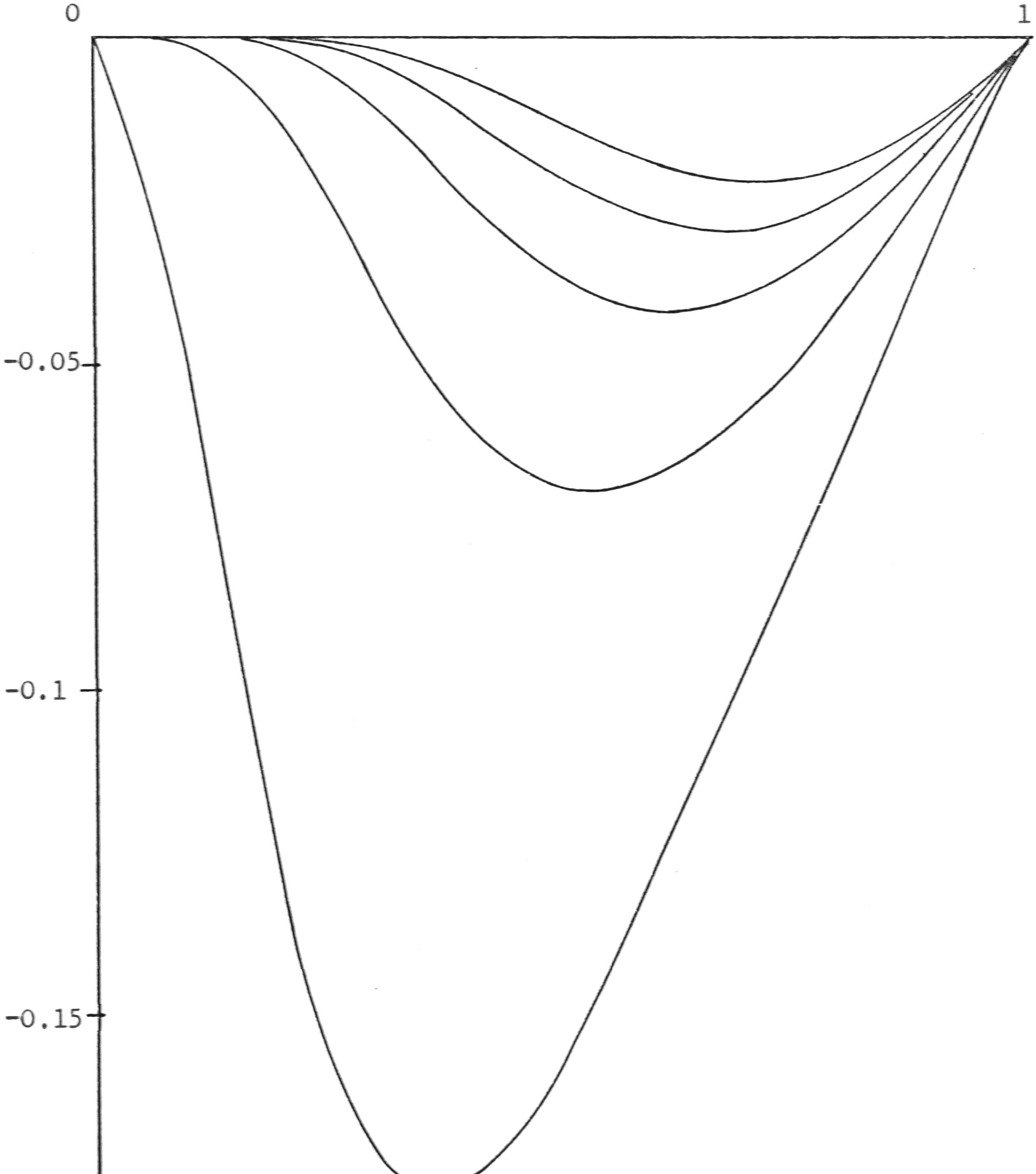
Autocorrelation function of the $E_4/M/1$ queue versus traffic intensity for lags 1,2,3,4,5, proceeding from lower to upper curve.


0
1
-0.02
-0.04
-0.06
-0.08
-0.10
-0.12

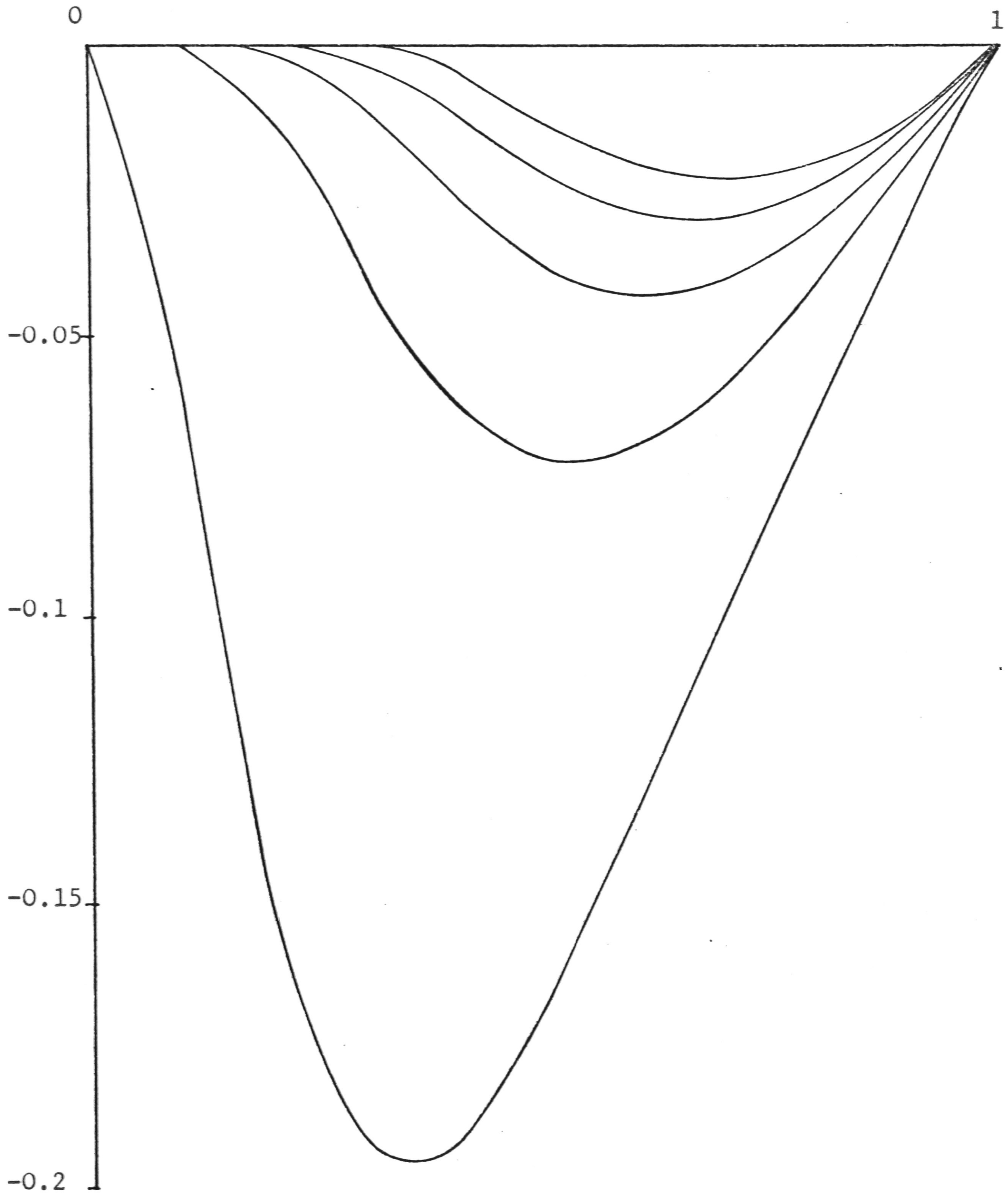
Autocorrelation function of the $E_5/M/1$ queue versus traffic intensity for lags 1,2,3,4,5, proceeding from lower to upper curve.


0
1
-0.05
-0.10
-0.15

Autocorrelation function of the $E_6/M/1$ queue versus traffic intensity for lags 1,2,3,4,5, proceeding from lower to upper curve.


0
1
-0.05
-0.1
-0.15

Autocorrelation function of the $E_7/M/1$ queue versus traffic intensity for lags 1,2,3,4,5, proceeding from lower to upper curve.


0
1
-0.05
-0.1
-0.15
-0.2

CONCLUSIONS

In this thesis, we have investigated methods of determining the output process - distribution and autocorrelation. We have demonstrated the methods through various examples. These examples were relatively simple compared to the complexity of the calculations involved in more general queueing systems.

The problems involved in GI/G/1 queues are compounded in the systems in which arrivals are autocorrelated. Some work has been done (Chaudhry (1965), Tuteja (1966)) on M/M/s queues with autocorrelated arrival patterns but this is a rather simple case of the more general problem. The work of Lloyd (1963) and Odoom and Lloyd (1965) on reservoirs may be useful in this area. Simulation perhaps might yield information about G/G/1 queues. However the generation of sample correlated input directly is generally an impossible task.

The investigation of tandem queueing systems has been generally in the field of systems with finite waiting room. If the work mentioned in the last paragraph were available, a more direct solution of tandem systems with infinite waiting area could possibly be found.

The work of this thesis was originally designed as preliminary to an investigation of traffic control systems where the general assumption that arrival patterns are random is not made. The work herein may be used to find better traffic synchronisation techniques, perhaps utilising some adaptive forecasting. Reduction of idle time in general auto-correlated input queues may result from using forecasting.

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