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Application of transfinite numbers and infinitesimals to measure theory

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APPLICATION OF TRANSFINITE NUMBERS AND INFINITESIMALS TO MEASURE THEORY

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ABSTRACT

Measure theory provides one of the most inviting areas in which the transfinite and infinitesimal numbers of non-standard analysis may be applied. This is so because their use becomes not just a convenient tool but an essential requirement for a generalization of the theory.

In Chapter 1 we use a set-theoretic approach to non-standard analysis and establish the basis for our subsequent work. The process involves injective maps (monomorphisms) and allows us to contrast the technique with that using the more concrete but less direct ultrapower method. The chapter provides sufficient framework to allow the self-contained examination of the basic properties of the extended real line carried out in Chapter 2.

Non-standard measure theory is developed in Chapter 3 where we construct a premeasure F and use it to define a non-standard measure μ as an extension of Lebesgue measure to all sets on the real line. The measure is constructed as a point measure such that its standard part agrees with Lebesgue measure where the latter is defined. It is finitely additive in the sense of non-standard analysis and thus provides a natural solution to the "easy problem of measure" solved first by Banach.

In Chapter 4 we show that all sets on the real line are measurable in the sense of μ and apply it to some well known subsets of R to find approximate non-standard measures for them. We also obtain some non-standard cardinality results for our premeasure F by taking standard parts of our measure in those cases where the set under consideration is a standard set which is Lebesgue measurable.

CONTENTS

CHAPTER 1. NON-STANDARD ANALYSIS

INTRODUCTION

Ever since the foundations of calculus were established late in the 17th century by both Newton and Leibniz, its concepts were easily explained if one assumed an enlarged real number system which includes both infinitely small and infinitely large numbers thought of as "ideal" elements. Unfortunately Leibniz and his followers were never able to state with sufficient precision just what rules were supposed to govern their new system. It is thus not surprising that calculus developed more as a descriptive science than as a deductive logical system, and as the axioms of the real number system emerged it became clear that the existence of infinitely small and infinitely large real numbers was inconsistent with these axioms.

To overcome this dilemma, the infinitesimal calculus was reformulated in the nineteenth century and the intuitive insights of Leibniz replaced by the sound but abstruse ε, δ approach of Cauchy and Weierstrass. Since then there have been many mathematicians, more recently Schmieden and Laugwitz [22], who have tried to revive Leibniz's ideas by proposing an extended concept of real numbers on which to base analysis. Based on a generalization of Cantor's construction of the reals, their enlarged number system is however in a certain sense too large in that it contains not only finite, infinitely small and infinitely large numbers but also numbers of an indeterminate size.

In [19] Robinson formulates the properties of R in a first order language to show that there exist proper extensions $^*R$ of the field of real numbers R, which in a certain sense have the same formal properties as R. It is well known that fields which are proper extensions of R are non-Archimedean, so that *R must contain the infinitely small and infinitely large numbers required by Leibniz. This appears paradoxical at first since we stated above that *R has in some sense the same properties as R . There is however no paradox since the statement asserting that *R has the same properties as R refers only to a specified collection of properties of R which are formulated in a certain formal language. Statements of this language have specific interpretations in R as well as in *R , and reinterpretations of higher-order properties like the Archimedean property do not retain their full metamathematical strength. This weak interpretation in the extension gives rise to a class of sets called "internal sets" which the formal language knows about; these have the "same properties", the external ones do not.

In [18] Robinson works within a type-theoretical version of higher order logic. The types he uses are in a certain sense like intuitive set theory; unfortunately their formal description makes them seem obscure. Other authors have independently presented variants of Robinson's theory [16], which however along with certain advantages still do not completely eliminate the complexity of the original theory. Here we choose to develop the subject using the comparatively simple set-theoretic approach in [20] and [26] which is based on the fact that the various branches of mathematics can all be thought of as embedded in set theory. Thus the basic concepts of analysis can be defined in terms of sets and the membership relation within a formal first order language whose variables range over sets or points and whose constants denote certain sets or points. We also employ the "ultrapower" construction due to J. Los, which was developed further by Luxemburg.

1.1 SUPERSTRUCTURE MODELS OF ANALYSIS

In this section we introduce a superstructure $\hat{R}$ as a set constructed on the ground set of real numbers R and large enough to contain standard analysis in the algebraic theory of $(\hat{R}, =, \epsilon)$, where $=$ and ϵ denote respectively the standard predicates of equality and set membership.

We define the following sets inductively:

$$R_o = R \ , \quad R_{n+1} = P\left(\bigcup_{k=0}^{n} R_k\right) \ , \quad n = 0,1,2,\ldots$$

where $P(X)$ denotes the set of all subsets of X.

Definition 1.11 The union $\hat{R} = \bigcup_{n=0}^{\infty} R_n$, together with the notions of equality and membership on the elements of $\hat{R}$ is called the superstructure based on the ground set R_o.

We often refer to the set-elements of R as the entities of R and to those elements contained in $R_o = R$ as the individuals ("Urelemente") of $\hat{R}$. Since individuals are not sets we see that if $a \in R_o$, $x \notin a$ for all x; consequently $R_o \cap R_n = \phi$ (the empty set) for $n > 0$ and $\bigcup_{k=0}^{n} R_k = R_o \cup R_n$.

The properties of $\hat{R}$ follow from the set theoretic properties of its entities and by definition we have:

1.12 (i) $\phi \in \hat{R}$ since $\phi \in P(R_o) = R_1 \subseteq \hat{R}$.

(ii) $R_n \in \hat{R}$ for each n, since $R_n \in P(R_n) \subseteq R_{n+1}$.

(iii) If y is an entity and $x \subseteq y$, then x is an entity. This follows since $y \in \hat{R}$ means that $y \in R_{n+1}$, thus $\bigcup_{k=0}^{n} R_k \supseteq y \supseteq x$, so that $x \in \hat{R}$.

(iv) If x is an entity then $P(x)$ is also an entity. Each entity of $\hat{R}$ is a subset of $\hat{R}$, since $x \in R_{n+1}$ and $x \subseteq R_o \cup R_n \subseteq \hat{R}$. Thus $P(x) \subseteq P(R_o \cup R_n) = R_{n+1}$ and by 1.12 (iii) is therefore an entity.

The above serve as an example of some of the properties of $\hat{R}$ which are a consequence of the entities it contains. In dealing with entities it should be observed that not all arbitrary subsets of $\hat{R}$ are entities, but only those sets which are also elements of $\hat{R}$.

We adopt the definition of ordered pairs as $(x,y) = \left\{\{x\},\{x,y\}\right\}$ and of n- tuples defined inductively as $(x) = x$, $(x_1, x_2, \ldots, x_n) = \left((x_1, x_2, \ldots, x_{n-1}), x_n\right)$.

From this it follows that functions and n-ary relations whose domains and ranges are entities are themselves entities. In particular a binary relation E is a binary relation entity of $\hat{R}$ if and only if its domain $D(E) = \{x : (\exists y)(x,y) \in E\}$ and range $D'(E) = \{y : (\exists x)(x,y) \in E\}$ are elements of $\hat{R}$.

In what follows the notion of equality is assumed to be given and for entities is the set-theoretical notion. Thus entities x,y of $\hat{R}$ are equal if and only if $(\forall z)(z \in x \equiv z \in y)$, that is entities are equal if and only if they have the same elements.

The algebraic operations of R can be defined in terms of the three place relations below:

$A = \{(a,b,c) : a,b,c \in R$ and $a + b = c\}$, for addition, and

$M = \{(a,b,c) : a,b,c \in R$ and $a \cdot b = c\}$, for multiplication,

while the order relation is the binary relation

$E = \{(a,b) : a,b \in R$ and $a \leq b\}$.

This illustrates that all mathematical concepts and objects of standard analysis can be embedded in the entities of $\hat{R}$, so that they become part of the algebraic theory of $(\hat{R}, =, \epsilon)$.

1.2 THE FORMAL LANGUAGE L

The advantage of introducing a formal language L is that it allows us to express statements concerning mathematical objects systematically and with great precision.

Here we adopt a first order language with the basic predicates ϵ (read "member of") and $=$ (read "equal to").

The atomic symbols of L are:

(i) The logical connectives $\wedge, \vee, \supset, \equiv, \sim$ for "and", "or", "implies", "if and only if" and "not" respectively.

(ii) Variables; a countably infinite sequence usually denoted by $x,y,\ldots,$ with or without subscripts.

(iii) Quantifiers; which are the universal quantifier denoted by $(\forall x)$ and the existential quantifier denoted by $(\exists x)$.

(iv) Separating symbols [and] .

(v) Extra logical constants, which form a set of larger cardinality than the cardinality of the set of elements of whatever mathematical system we may subsequently wish to consider.

In considering $\hat{R}$ this ensures that there is a one-to-one correspondence from a subset of the set of all constants of L onto $\hat{R}$. If an object under consideration has already an accepted name, for example "the empty set" or $1,2,3,\ldots$ for the natural numbers or "log" for the logarithmic function, we adopt the convention of using this name also as the corresponding constant symbol of L. We now identify all elements of $\hat{R}$ with the appropriate subset of constants of L, so that elements of $\hat{R}$ are recognizable in L by their usual names. For example, the binary order relation $\leq$ on R defined above is denoted by the constant E of L which stands for the element $\{(a,b) : a,b \in R \wedge a \leq b\}$ of R_3 in $\hat{R}$.

Having established this identification we refer to $\hat{R}$ as an L-superstructure and note that $\hat{R}$ becomes a part of L as a subset of the set of all constants of L.

The atomic formulas of L are obtained by combining $\in$ and $=$ in the usual way with constants or variables; e.g. $x \in y$, $a = b$, $(x_1, x_2, \ldots, x_n) \in y$. From these the well formed formulas (wff.) are now formed through the use of connectives and quantifiers with appropriate placement of the separating symbols [(left hand square bracket) and] (right hand square bracket). A wff. of L is called a sentence provided every variable x contained in it is within the scope of $(\forall x)$ or $(\exists x)$, or in the expression $(\forall x)$ or $(\exists x)$.

<u>Definition 1.21</u> A formula of L is said to be <u>bounded</u> when the quantifiers always appear at the start of subwffs. of the following forms:

(i) $(\forall x)\ [x \in A] \supset W(x)$ and

(ii) $(\exists x)\ [x \in B] \wedge W(x)$, where $W(x)$ is a wff. and A and B are entities of $\hat{R}$, that is constants of L. Set theoretically this corresponds to specifying which entities we are quantifying over.

We now develop interpretations of L, so that we can find a relationship between $\hat{R}$ and L both viewed as mathematical objects in the metalanguage of our text.

Let $L = L(=,\epsilon)$ be the formal language described above.

<u>Definition 1.22</u> A one-to-one mapping I of a subset of the set of all constants of L into a superstructure is called an <u>interpretation map</u> of L in set theory.

Here the basic predicates $=$ and ϵ are always interpreted in the usual set-theoretic way; this is what we mean by an interpretation in set theory in the definition.

For our L-superstructure $\hat{R}$ we have made standard identifications of elements of $\hat{R}$ as constants in L, which enables us to talk about a standard interpretation map ${}^{S}I$ from $\hat{R}$ as a subset of the set of all constants of L onto $(\hat{R}, =, \epsilon)$. Note here that the ability to interpret does not require that the interpretation is true; specifically if y is an individual then $x \in y$ has to have a false interpretation for $x \in D({}^{S}I)$ (i.e. elements of $\hat{R}$), but has no interpretation when $x \notin D({}^{S}I)$. We retain maximum contact with our metalanguage by interpreting the logical connectives as their metamathematical counterparts.

From definition 1.21 we see that since each quantifier in a bounded formula is specified to run over a constant, the interpretation of

$$V = (\forall x)\Big[[x \in A] \supset W(x)\Big] \quad \text{is}$$

${}^{S}I_V$ = "for elements x of ${}^{S}I(A)$, the statement ${}^{S}I_{W(x)}$",

where ${}^{S}I_{W(x)}$ denotes the portion of the formula already interpreted where free occurrences of x are replaced by the elements of ${}^{S}I(A)$. Similarly the interpretation of

$$V = (\exists x)\Big[[x \in A] \wedge W(x)\Big] \quad \text{is}$$

${}^{S}I_V$ = "there is an x in ${}^{S}I(A)$ such that ${}^{S}I_{W(x)}$".

We now show that $(\hat{R}, =, \epsilon)$ is a set-theoretical model of standard analysis based on the definition of a model given below.

Definition 1.23 An interpretation map I provides a model for a set of bounded sentences K in set-theory provided all the constants occurring in sentences of K are in the domain of I and provided the interpretation ${}^{I}V$ is true for each V in K .

Since ${}^{S}I : L \to (\hat{R}, =, \epsilon)$ is the standard interpretation map for L , we see that every bounded sentence V of such a set K , whose constants are in the domain of ${}^{S}I$ has an interpretation in $\hat{R}$. Consider now the set K_0 of all the bounded sentences V of L such that ${}^{S_I}V$ holds in $\hat{R}$. By definition $(\hat{R}, =, \epsilon)$ is a model for K_0 under our standard interpretation ${}^{S}I$ and in particular a standard set-theoretical model of analysis.

Definition 1.24 Suppose ${}^{O}I$ is an interpretation mapping from a subset of constants of L into a superstructure $(\hat{B}, =, \epsilon)$, such that ${}^{O}I$ provides a model for K_0 . If ${}^{O}I/\hat{R} = *$ (that is ${}^{O}I$ with domain restricted to $\hat{R}$) is one-to-one but not onto the set $\bigcup_{n=0}^{\infty} {}^{*}R_n$ (we denote this union by $*(\hat{R})$) we say that the subset $(*(\hat{R}), =, \epsilon)$ of the superstructure $\hat{B}$, is a non-standard set-theoretical model of $((\hat{R}), =, \epsilon)$.

Here we see that ${}^{O}I$ restricted to $\hat{R}$ reinterprets each element $a \in \hat{R}$ as ${}^{O}I(a) = {}^{*}a$ in $*(\hat{R})$. Since ${}^{O}I$ is one-to-one but not onto, $*(\hat{R})$ must by definition contain all elements of R as well as other interpreted constants ${}^{O}I(c)$, $c \in L - \hat{R}$, so that $*(\hat{R})$ is a proper extension of $\hat{R}$.

In section 1.3 we produce a non-standard set-theoretical model of R by giving the axioms of an injective map Φ (monomorphism) from $(\hat{R}, =, \epsilon)$ into $(\hat{B}, =, \epsilon)$ which ensure that Φ produces $*(\hat{R})$ as required by the above definition.

1.3 NON-STANDARD MODELS OF R. MONOMORPHISMS

We now show that given $(\hat{R}, =, \epsilon)$ there is a larger superstructure $(\hat{B}, =, \epsilon)$ and an embedding $\Phi : \hat{R} \to \hat{B}$ which preserves the mathematical structure of $(\hat{R}, =, \epsilon)$. One way to preserve the mathematical structure is to introduce all of the structure as part of the formal language L which may then be reinterpreted in certain ways. We demonstrate this later; here we give the axioms of an algebraic injection Φ which preserves the operations $=$ and ϵ. We modify our notation to comply with common usuage and for A an element of $\hat{R}$, write $*A$ for $\Phi(A)$. Note that this is the same notation as that used in definition 1.24 for ${}^{o}I$-images of elements of $\hat{R}$. Since we ultimately show that Φ is an interpretation mapping in the sense of the definition we adopted the $*$- notation there to minimize notational proliferation.

Let $(\hat{R}, =, \epsilon)$ be the superstructure based on the set of individuals $R_o = R$ and $(B, =, \epsilon)$ be another superstructure based on the ground set $B_o = {}^*R_o$.

<u>Definition 1.31</u> The mapping $\Phi : \hat{R} \to \hat{B}$ is a <u>superstructure monomorphism</u> of $\hat{R}$ if it is a one-to-one map defined on $\hat{R}$ satisfying propositions:

(i) If A is an entity of $\hat{R}$,

$$*\{(x,x) : x \in A\} = \{(y,y) : y \in *A\} .$$

(ii) If A is an entity and $a \in A$, then $*a \in *A$.

(iii) $*\{a_1, a_2, \ldots a_n\} = \{*a_1, *a_2, \ldots *a_n\}$, for $a_1, a_2, \ldots a_n \in \hat{R}$.

(iv) $*\phi = \phi$, $*(A \cup B) = *A \cup *B$, $*(A \cap B) = *A \cap *B$,
$*(A - B) = *A - *B$, $*(A \times B) = *A \times *B$, for entities A and B.

(v) Φ preserves domains and ranges of n-ary relations,
e.g. $D(*\psi) = *D(\psi)$, $D'(*\psi) = *D'(\psi)$,

and commutes with permutations of the variables, that is if $(x,y) \in \psi$ if and only if $(y,x) \in \psi$, then $(z,w) \in {}^*\psi$ if and only if $(w,z) \in {}^*\psi$.

(iv) If A is an entity

$$^*\{(x,y) : x \in y \in A\} = \{(z,w) : z \in w \in {}^*A\} .$$

(vii) ${}^*A \supseteq \Phi[A]$, with equality iff A is a finite set; here $\Phi[A] = \{{}^*a : a \in A\}$.

By definition we see that Φ preserves $=$ and $\in$ as well as finite sets and the basic set operations. Further note that Φ preserves the atomic standard definition of sets (property 1.31(vi)) and produces a proper extension ${}^*A \supset \Phi[A]$ for A infinite.

The set of individuals of $\hat{B}$ is ${}^*R_0$, the Φ- extension of R_0. The properties of the non-standard individuals in ${}^*R_0$ are "the same" as in R_0, but the higher order properties only transfer to a restricted class of sets in $\hat{B}$ called internal sets. The description of those properties could be done ad hoc one at a time as required from the monomorphism axioms above. It is however, easier to utilize the formal language L and develop a systematic method of interpreting formulas in $\hat{R}$ and the image ${}^*(\hat{R}) \subset \hat{B}$.

The existence of *- maps will not be demonstrated until section 1.4, where we show that ultrapower models of R give rise to monomorphisms as non-standard interpretation maps ${}^oI$ in the sense of definition 1.24, and provide a simple method namely the *- transform which allows us to transfer from $\hat{R}$ to ${}^*(\hat{R})$ and requires only a little care with quantifiers.

Given a monomorphism $\Phi : \hat{R} \to \hat{B}$ as in definition 1.31 we define ${}^*(\hat{R}) = \bigcup_{n=0}^{\infty} {}^*R_n$, see [20] and introduce the notions of standard, internal and external elements of $\hat{B}$ as follows:

Definition 1.32 An element $a \in \hat{R}$ is called standard and $*a$ is termed a Φ- standard (briefly, standard) element of $\hat{B}$. Any element of a standard entity of B is called a Φ- internal (briefly, internal) element; other entities of B are called external.

It is in view of definition 1.32 that we refer to $\Phi[A] = \{*a : a \in A\}$ as the embedded standard copy in $\hat{B}$ of the entity $A \in \hat{R}$. The standard subsets of a standard entity $*A$ are the elements of $\Phi[P(A)]$. The internal subsets of $*A$ are the elements of $*P(A)$, while the external subsets of $\hat{B}$ are the elements of $P(*A)$ that are not internal. For general $A \in \hat{R}$ we have $\Phi[P(A)] \subseteq *P(A) \subseteq P(*A)$ where the inclusion is strict if A is infinite. We prove this in Chapter 2, where we deal with external subsets of $\hat{B}$ to show that for infinite $A \in \hat{R}$, $P(*A) - *P(A) \neq \phi$.

Since $\hat{R}$ is a model of K_o, any theorem of $\hat{R}$ is formalizable as a bounded sentence of K_o provided it is written so that its quantifiers are specified to run over specific entities of $\hat{R}$. Since we can transfer "all properties with bounded quantifiers" we would at first expect $*(\hat{R})$ to be isomorphic to $\hat{R}$. That this is not the case follows from the restrictions of the method of transfer which we define below.

Definition 1.33 If $V \in K_o$, form ${}^{s}I_V$ and put a $*$ on each interpreted constant, the result is then the $*$- transform $\Phi\left({}^{s}I_V\right)$ of ${}^{s}I_V$.

To simplify our notation we now talk simply about $V \in K_o$ holding in $\hat{R}$ to mean that ${}^{s}I_V$ is true in $\hat{R}$; we revert to talking about ${}^{s}I$ when discussing the actual mapping.

The meta-theorem 3.2 of [20] proves that the axioms of Φ imply that the image $*(\hat{R})$ is a non-standard model of K_o by showing that any bounded sentence V of K_o holds in $\hat{R}$ if and only if its *- transform $*V$ holds in $*(\hat{R})$.

From the transfer constraints and our remarks about the entities of R we see that a statement like "every subset of N has a first element" is not permissible but must be replaced by "every element of $P(N)$ has a first element". We will return to this case in our discussion of internal and external subsets of $*R$ in Chapter 2.

We now state a general transfer principle which follows from our discussion above and as a consequence of our explicit ultrapower construction in section 1.4, which we prefer in the interest of concreteness.

<u>1.34</u> <u>Transfer Principle</u> A sentence in $\hat{R}$ that has a bounded formalization in L is true if and only if its *- transform is true.

In [26] Zakon uses a version of 1.34 as part of his monomorphism definition. As indicated earlier we will now demonstrate the use of our transfer method from $\hat{R}$ to $*(\hat{R})$ via the *- transforms of definition 1.33. The transfer principle guarantees that the transfer really works, provided we take the necessary care with quantifiers.

<u>1.35</u> Let $a, b, a_1, a_2, \ldots a_n$ be elements of $\hat{R}$, then:

(i) $a \in b \equiv *a \in *b$

(ii) $(a_1, a_2, \ldots a_n) \in b \equiv (*a_1, *a_2, \ldots *a_n) \in *b$

(iii) $a \subseteq b \equiv *a \subseteq *b$

(iv) $a = b \equiv *a = *b$

(v) $a \in R_o \equiv *a \in B_o$.

Properties 1.35 (i), (ii) and (iv) are immediate from the definition of Φ and (v) is a special case of (i) with $B_o = {}^*R_o$. To prove (iii) note that $a \subseteq b$ stands for the bounded sentence $(\forall x)\Big[[x \in a] \supset x \in b\Big]$ which by our transfer principle is equivalent to its *- transform $(\forall x)\Big[[x \in {}^*a] \supset x \in {}^*b\Big]$, that is $a \subseteq b \equiv {}^*a \subseteq {}^*b$.

We do not give other specific examples at this stage since the extensive use of *- transforms in later chapters is necessarily required to make true statements in non-standard analysis and will serve as further examples. We now use the ultrapower construction to show that the monomorphism as required by definition 1.31 exists and give concrete examples of the *- images of entities of the superstructure $\hat{R}$.

1.4 ULTRAPOWER MODELS

In this section we show that non-standard set-theoretical models of $(\hat{R}, =, \in)$ can be constructed using the ultrapower method. The method has the advantage of being a concrete approach (see [1]) and provides a general technique for generating set-theoretical monomorphisms as non-standard ultrafilter-dependent interpretation maps for $\hat{R}$. Diagrammatically this can be depicted as:

$$\begin{tikzcd} & L \arrow[dl, "{}^{s}I"'] \arrow[dr, "{}^{o}I"] & \\ \Phi : (\hat{R}, =, \in) \arrow[rr, "*"] & & (\hat{B}, =, \in) \end{tikzcd} ,$$

where ${}^{S}I$ is the standard interpretation map discussed in section 1.2 and ${}^{O}I$ is a non-standard interpretation map (meaning here as distinct from ${}^{S}I$) from $\hat{R}$ as a subset of the set of all constants of L into the superstructure $\hat{B}$.

As we will see below, things are not quite as straight forward as they may appear, in particular we need to use the ultrapower method in constructing ${}^{O}I$. The ultrapower technique serves only as a tool. Once we have used it we dispense with ultrapowers by reverting back to dealing with the superstructure $(\hat{B}, =, \epsilon)$ based on the set of individuals ${}^{O}I(R) = B_O$. This allows us to talk about ${}^{O}I$ as a non-standard interpretation map in set theory and produce a non-standard set-theoretical model of $(\hat{R}, =, \epsilon)$. We now employ the ultrapower technique to constructively exhibit non-standard internal elements of $\hat{B}$ which are needed to ensure that in $\hat{B}$ we obtain proper extensions of infinite entities A of $\hat{R}$, as required by our monomorphism axioms. How we actually achieve this is outlined in the following sections.

<u>Definition 1.41</u> Let J be a countable set. A non-empty set U of subsets of $J[\phi \subset U \subset P(J)]$ is called a <u>free ultrafilter</u> provided

(i) $\phi \in U$ (PROPER FILTER).

(ii) If A and $B \in U$, then $A \cap B \in U$ (FINITE INTERSECTION PROPERTY).

(iii) If $A \in U$ and $B \in P(J)$ and if $A \subseteq B$ then $B \in U$ (SUPERSET PROPERTY).

(iv) If $B \in P(J)$, then either $B \in U$ or $J - B = \{j \in J : j \notin B\} \in U$ (MAXIMALITY).

(v) No finite subset of J is an element of U (FREENESS).

The ultrafilter U is said to be δ- incomplete whenever there exists a sequence $F_n \in U$, $n = 1,2,\ldots$ such that $\bigcap_{n=1}^{\infty} F_n \notin U$. If a given ultrafilter U over J is δ- incomplete, then there exists a countable partition $\{J_n : n = 1,2,\ldots\}$ of the set J such that $J_n \notin U$ for all $n = 1,2,\ldots$.

To show this consider the decreasing sequence $F_n \in U$, $n = 1,2,3,\ldots$ such that $\bigcap_{n=1}^{\infty} F_n = \phi \notin U$. Letting $J_n = J - F_n$, we have $J_n \notin U$, for all $n = 1,2,3,\ldots$ while

$$\begin{aligned} \bigcup_{n=1}^{\infty} J_n &= \bigcup_{n=1}^{\infty} J - F_n \\ &= J - \bigcap_{n=1}^{\infty} F_n \\ &= J \in U , \end{aligned}$$

so that the subsets J_n as defined constitute a countable partition of J with $J_n \notin U$, $n = 1,2,\ldots$ as required.

Let U be a δ- incomplete ultrafilter of subsets of J and consider $\hat{R}^J$, the set of all mappings of J into $\hat{R}$.

The reason we deal with $\hat{R}^J$ is that there is a self-evident interpretation mapping from a subset of constants of L into $\hat{R}^J$ which interprets each constant of L which is an element of $\hat{R}$ as the respective constant sequence in $\hat{R}^J$. Thus any $a \in \hat{R}$ is interpreted in $\hat{R}^J$ as the constant sequence $a' : J \to \hat{R}$ of $\hat{R}^J$ defined by $a'(j) = a$ for all $j \in J$.

For example the positive integer 5 is interpreted in $\hat{R}^J$ as the constant sequence $5' = (5,5,5,\ldots)$; similarly R_1 is given by $R_1' = (R_1,R_1,R_1,\ldots)$ in $\hat{R}^J$.

Till now we have produced nothing new except for reinterpreting elements of $\hat{R}$ as constant sequences in $\hat{R}^J$. That we do obtain something other than sequence extensions of elements of $\hat{R}$ in $\hat{R}^J$ is a consequence of the following ultrafilter-dependent extension of the undefined basic predicates $=$ and ϵ of L to $\hat{R}^J$.

<u>Definition 1.42</u> If $a,b \in \hat{R}^J$, then

$a =_u b \equiv \{j : a(j) = b(j)\} \in U$ and

$a \in_u b \equiv \{j : a(j) \in b(j)\} \in U$.

Since $J \in U$ it follows that for $a,b \in \hat{R}$ we have:

<u>1.43</u> (i) $a = b \equiv a' =_u b'$ and

(ii) $a \in b \equiv a' \in_u b'$,

so that the relations $=_u$ and $\in_u$ are indeed U-extensions of $=$ and $\in$ of $\hat{R}$ as suggested.

In view of definition 1.42 we denote our interpretation map for $\hat{R}^J$ by ${}^uI$ to indicate its U-dependence.

As stated earlier the U-dependent interpretation of $=$ and $\in$ in $\hat{R}^J$ is to allow us to produce proper extensions of $\hat{R}$ in $\hat{R}^J$. In particular we show in Chapter 2 that as a consequence of the U-dependent interpretation of the binary relation $\leq$ of L in $\hat{R}^J$ we are able to construct an individual in ${}^uI(R) = R'$, which is larger than any standard real number in R. Thus R' is a proper extension of R since $R' - R \neq \phi$. The infinitesimals, which are the foundation of non-standard analysis, are then obtained as the non-standard elements in R' which are the inverses of the "infinite" elements of $R' - R$.

We now examine a consequence of the ultrafilter-dependence of definition 1.42. With $a \neq_u b \equiv \{j : a(j) \neq b(j)\} \in U$ and $a \notin_u b \equiv \{j : a(j) \notin b(j)\} \in U$ we see that either $a =_u b$ or $a \neq_u b$, and either $a \in_u b$ or $a \notin_u b$. The proof follows from the ultrafilter properties and we verify it for $\in_u$ to illustrate the technique. Let $a,b \in \hat{R}^J$, and set $T_1 = \{j : a(j) \in b(j)\}$ and $T_2 = \{j : a(j) \notin b(j)\}$. Since $T_1 \cup T_2 = J \in U$, it follows from 1.41 (iv) that either T_1 or $J - T_1 = T_2$ belongs to U so that either $a \in_u b$ or $a \notin_u b$.

Note also that $=_u$ as defined is an equivalence relation in R^J . That $=_u$ is reflexive and symmetric is clear; that it is transitive follows from definition 1.41 since:

Let $a,b,c \in \hat{R}^J$, with $a =_u b$ and $b =_u c$;

then $T_1 = \{j : a(j) = b(j)\} \in U$ and

$T_2 = \{j : b(j) = c(j)\} \in U$,

thus $T_3 = \{j : a(j) = c(j)\}$

$\supseteq T_1 \cap T_2 \in U$ by 1.41 (ii) .

Thus $T_3 \in U$ by 1.41 (iii) and $a =_u c$ as required.

As another concrete example we now show that $5' \in_u A'$, where $\hat{R} \ni A = \{2,3,5\}$.

Proceeding from first principles we see that $5'$ is the constant sequence $\in \hat{R}^J$ given by $5'(j) = 5$ for all $j \in J$. Similarly since A' is the constant map $A' : J \to R_1$, such that $A'(j) = \{2,3,5\}$ for all $j \in J$ we have $5'(j) \in A'(j)$ for all $j \in J$, so that $\{j : 5'(j) \in A'(j)\} = J \in U$ and $5' \in_u A'$ as required.

The algebraic operations in R^J are introduced pointwise and U-dependent, that is for $a,b \in R^J$:

(1) $a + b =_u c$ if and only if $\{j : a(j) + b(j) = c(j)\} \in U$.

(2) $a \,.\, b =_u c$ if and only if $\{j : a(j) \,.\, b(j) = c(j)\} \in U$.

(3) $a \leq_u b$ if and only if $\{j : a(j) \leq b(j)\} \in U$.

Thus $2' \cdot 3' =_u 6'$ in R^J, since $2' = (2,2,2,\ldots.)$ and $3' = (3,3,3,\ldots.)$. Multiplying pointwise, $2' \cdot 3' = (6,6,6,\ldots.) = 6'$ and since $\{j : 6'(j) = 6\} = J \in U$, our result follows by (2) above.

Note that the δ- incompleteness of U assures that whenever A is an infinite entity of $\hat{R}$, there are sequences in A^J which are inequivalent mod U to all constant sequences in A^J, so that $\hat{R}^J$ is a proper extension.

For example since R is an infinite entity of $\hat{R}$ there exists a sequence $\{a_n : n = 1,2,\ldots\}$ of elements of R such that $a_n \neq a_m$ for all $n,m = 1,2,\ldots$ and $n \neq m$. Consider the sequence a of J into R such that $a(j) = a_n$ for all $j \in J_n$ (elements of our countable partition of J discussed earlier) and $n = 1,2,3,\ldots$. Then $a \in \hat{R}^J$, in fact $a \in R'$ but a is not equal to any element b' of $\bigcup_{n=0}^{\infty} R'_n$ (denoted in what follows by $(\hat{R})'$). Thus not all sequences in R^J are generated by our interpretation map ${}^uI$; this makes the construction useful and is the origin of the internal-external terminology. However it is clear that our interpretation map ${}^uI$ embeds $\hat{R}$ into the subset $(\hat{R})'$ of $\hat{R}^J$ via the constant sequences.

It can be shown by using properties 1.41 (i)-(v) of our δ- incomplete ultrafilter U that one by one all K_o sentences of L hold in $(\hat{R})'$ under the extended U-dependent interpretation of our basic predicates. We do not intend doing this here but refer to [7] where it is shown that $(\hat{R})'$ is a model of K_o under the interpretation ${}^uI$ defined above.

However since we want a non-standard set-theoretical model of K_o we now embed part of $\hat{R}^J$ in set theory by making $=_u$ into ordinary equality and $\in_u$ sets into real sets. This process is part of the construction of ${}^{o}I$ as an interpretation in set theory and will give us the full non-standard set-theoretical model in the sense of definition 1.24. Here we follow the approach in [20] where the process is called "collapsing"; in [16] Machover and Hirschfeld use an alternative approach and introduce "pseudosets": the internal map sets of $\hat{R}^J$ and "real" sets.

The reason we embed part of $\hat{R}^J$ in an ordinary superstructure is that we want to be able to compare sets and functions which arise as sequences in $\hat{R}^J$ with arbitrary ones. For example in Chapter 2 we see that the set of infinitesimals in $\Phi(R) = {}^*R$ is a collection of all the real-valued sequences $x(j) \in R^J$ which satisfy $-\frac{1}{n} \leq x(j) \leq \frac{1}{n}$ for each constant sequence $\frac{1}{n}$, $n \in N$. This is a real set and its elements are described by the metamathematical $\in$ rather than by $\in_u$. As we see later the infinitesimals cannot be described by $\in_u$ since they are an external set, yet we certainly want to be able to discuss them.

Above we showed that $=_u$ is an equivalence relation in $\hat{R}^J$. That $=_u$ has substitutivity properties with respect to $\in_u$ follows from the K_o sentence :

$$(\forall x)(\forall y)(\forall z)(\forall u)\Big[[x \in R_n \wedge y \in R_n \wedge z \in R_n \wedge u \in R_n]$$
$$\supset \Big[[[x \in y] \wedge [x=z] \wedge [y=u]] \supset [z \in u]\Big]\Big] \text{ , which holds in } \hat{R} \text{ for all n,}$$

and hence in $(\hat{R})'$ under our interpretation ${}^{u}I$. We now modify $\hat{R}^J$ by replacing each element $a' \in (\hat{R})'$ by its equivalence class

$${}^*a = [a'] = \{b \in R^J : b =_u a'\} .$$

(Here our use of the *- notation is deliberate and justifiable by what follows.) For $d \in \hat{R}^J$ not constant we associate the sequence with the equivalence class $[d\ ] = d$ by abuse of notation.

Once this replacement has been completed it allows us to use ordinary equality in $\hat{R}^J$ as modified and we tacitly assume from now on that this process has always been carried out and continue to talk simply about $\hat{R}^J$.

Since $\hat{R}^J$ is a model of K_o we have an interpretation map ${}^oI : \hat{R} \to \hat{R}^J$ given by setting for each $a \in \hat{R}$, ${}^oI(a) = {}^*a$, as defined above. That is K_o sentences hold in $^*(\hat{R})$ when each constant c of K_o (element of $\hat{R}$) is replaced by ${}^oI(c) = {}^*c$.

<u>Definition 1.44</u> Setting ${}^oI(R_n) = {}^*R_n$, $n = 0,1,2,\ldots$ we say that an element $a \in \hat{R}^J$ is <u>internal</u> if $a \in_u {}^*R_n$ for some n . An internal element $a \in R^J$ is called <u>standard</u> whenever there exists an element $b \in \hat{R}$ such that $a = {}^*b$. All entities which are not internal are called <u>external</u>. (Although the definition is simply a special case of 1.32, we restate it here in the context of interpretation mappings.)

By definition we see that no internal elements $x \in_u {}^*(\hat{R})$ can belong to any $y \in_u {}^*R_o$. This follows from the K_o sentence: $(\forall y \in R_o)(\forall x \in R_n)\ x \notin y$ which holds in $\hat{R}$, hence in $\hat{R}^J$ under our interpretation ${}^oI$, that is: $(\forall y \in_u {}^*R_o)(\forall x \in_u {}^*R_n)\ x \notin_u y$, so $x \notin_u y$ for every $x \in_u {}^*(\hat{R})$ as asserted.

We say that x is an "element" of X in the metamathematical sense if $x \in_u X$. Now $X \in_u {}^*R_1$ if and only if $X \subset_u {}^*R_o$, that is $(\forall x)\Big[[x \in_u X] \supset [x \in_u {}^*R_o]\Big]$, so that we can replace each $X \in_u {}^*R_1$ by the genuine set $\{x : x \in_u X\}$. Proceeding inductively we do the same for each $X \in_u {}^*R_n$, $n = 2,3,\ldots$ so that for internal members of $\hat{R}^J$, $\in_u$ becomes the ordinary set theoretical membership relation. We refer to $\hat{R}^J$ modified as above as a collapsed model of K_o and from now on use ordinary set theoretical notation in such models.

<u>Definition 1.45</u> The set $\bigcup_{n=0}^{\infty} {}^*R_n$ of all internal members of the collapsed model $\hat{R}^J$ is called an <u>ultrapower model of $\hat{R}$</u> with respect to the δ- incomplete ultrafilter U, and will be denoted by $*(\hat{R})$.

Notice that by definition $*(\hat{R})$ is a non-standard set-theoretical model of $(\hat{R}, =, \epsilon)$. From the above definition it follows that for every collapsed ultrapower model $\hat{R}^J$ of $\hat{R}$ there is a superstructure monomorphism Φ on $\hat{R}$ into $\hat{B}$ such that the Φ- internal members of $\hat{B}$ are exactly the elements of the ultrapower $*(\hat{R})$.

To see this let $\Phi : \hat{R} \to \hat{R}^J$ be the interpretation map ${}^oI$ of the collapsed model $\hat{R}^J$ of $\hat{R}$. We showed above that members of $\Phi(R_o) = {}^*R_o$ have no elements in $*(\hat{R})$ so that we may treat them as the set of individuals ${}^*R_o = B_o$ of the superstructure $\hat{B}$. Notice that it was working with $\hat{R}^J$ which allowed us to identify non-standard individuals and obtain ${}^*R_o$ as the proper extension of the ground set R_o of individuals of $\hat{R}$. Apart from the concrete nature of the constant sequence extensions of members of $\hat{R}$ this was the main purpose of our involvement with $\hat{R}^J$. Since we no longer require it we now replace $\hat{R}^J$ by embedding it in the superstructure $\hat{B} = \bigcup_{n=0}^{\infty} B_n$, where $B_o = {}^*R_o$ and deal with it. This does not affect the map Φ and the internal members of R^J (that is elements of $*(R)$) since they have internal elements only. Since the interpretation map Φ above satisfies properties 1.31 (i)-(vii), Φ defines a superstructure monomorphism as well as the interpretation map of $\hat{R}^J$ so that Φ- internal elements are by definition 1.44 the internal elements of $\hat{R}^J$, namely elements of $*(R)$.

We have thus achieved our main aim, that of justifying the validity of $*$- transforms in $\hat{B}$, by exhibiting a concrete example of a superstructure monomorphism $\Phi : \hat{R} \to \hat{B}$ as the interpretation map ${}^{o}I$ of our ultrapower construction. In summary we have

$$\Phi(a) = {}^{*}a = {}^{o}I(a) = [a'] , \quad a \in R_o \quad \text{and}$$

$$\Phi(R_n) = {}^{*}R_n = {}^{o}I(R_n) = [R_n'] , \quad n \geq 1 .$$

This is best represented diagrammatically as below:

$$\begin{tikzcd}
& L \arrow[ddl, "{}^{s}I"'] \arrow[d] \arrow[ddr, "{}^{o}I"] & \\
& \hat{R}^J \arrow[dr, "\text{COLLAPSING}"] & \\
\Phi : \hat{R} \arrow[ur, "\text{CONSTANT SEQUENCES}"] \arrow[rr, "*"] & & \hat{B}
\end{tikzcd}$$

1.5 STRICT MONOMORPHISMS

From the point of view of applications, the nicest way to proceed in non-standard analysis is to simply use *- transforms inside $\hat{B}$ and distinguish when necessary between internal and external sets (that is sets which arise as mappings $J \to R_n$ and arbitrary subsets of ${}^{*}R_n$). This distinction is not necessary if we work within ${}^{*}(\hat{R}) \subset \hat{B}$ generated by a strict superstructure monomorphism Φ , since then all members of ${}^{*}(\hat{R})$ have internal elements only (if any).

<u>Definition 1.51</u> The monomorphism $\Phi : \hat{R} \to \hat{B}$ is said to be <u>strict</u> if and only if every member of ${}^{*}(\hat{R})$ has internal elements only (if any), that is: $(\forall y)\Big[\big[y \in ({}^{*}(R) - {}^{*}R_o)\big] \supset \big[y \subseteq {}^{*}(R)\big]\Big]$.

With Φ strict, we see that any internal element $y \in {}^*R_{n+1}$ is a subset of ${}^*R_n \cup {}^*R_o$ and that within $\hat{B}$ we are working only with internal entities. It should be noted however that Φ does not provide a mechanism for identifying the external subsets of the standard sets ${}^*R_n$, $n \geq 1$. Thus we are guaranteed for example that all entities of ${}^*R_1$ are internal by definition, but our monomorphism does not identify the external entities, that is those belonging to $P({}^*R_o) - {}^*R_1$. As a consequence strict monomorphisms provide a convenient aid but provide no additional structural insight. Note that every monomorphism Φ can be changed into a strict one by replacing each $y \in {}^*R_n$ $(n \geq 1)$ by $y \cap {}^*(\hat{R})$ since this removes from y all its external elements, if any.

1.6 ENLARGEMENTS

Definition 1.61 A binary relation $S \in \hat{R}$ is said to be concurrent if, for any finite number of elements $a_1, a_2, \ldots, a_n$ of its domain $D(S)$, there is some $b \in \hat{R}$ such that $(a_k,b) \in S$, $k = 1,2,\ldots n$.

For example the relation $\leq$ between real numbers is concurrent since for any $a_1, a_2, \ldots, a_n \in R$ and $b = \max\{a_1,a_2,\ldots,a_n\}$ we have $a_k \leq b$, $k = 1,2,\ldots,n$.

Definition 1.62 A monomorphism Φ is said to be enlarging if and only if for every concurrent relation $S \in \hat{R}$, there is some $b \in {}^*(\hat{R})$ such that $({}^*x,b) \in {}^*S$ for all $x \in D(S)$ simultaneously. If this is the case we say that Φ bounds all concurrent relations.

We refer to ${}^*(\hat{R})$ generated by an enlarging monomorphism Φ as an enlargement of $\hat{R}$; a strict enlargement if Φ is enlarging and strict.

The existence of enlargements and their significance was first discovered by A. Robinson [18]. In [15] Luxemburg uses the fact that many results in mathematics can be reformulated to read that a certain binary relation is concurrent.

In set theory one of the basic concurrent binary relations is the binary relation of membership between the elements of an infinite set and its family of finite subsets. This means that in any enlargement of a mathematical theory every infinite set is contained externally in the enlargement in a $*$- finite set (see definition 2.33 of chapter 2) of the enlargement.

In [20] theorem 4.2 states that for every superstructure $\hat{A}$ (thus specifically for our $\hat{R}$) there is a superstructure $\hat{B}$ and an enlarging superstructure monomorphism $\Phi : \hat{A} \to \hat{B}$. In our case this implies that $\hat{R}$ has an enlargement $*(\hat{R})$ generated by the enlarging monomorphism Φ ; that this is the case rests on the fact that a specific ultrafilter U can be chosen in such a way that all concurrent relations S in $*(\hat{R})$ are bounded by Φ .

CHAPTER 2. THE NON-STANDARD REAL LINE

2.1 BASIC PROPERTIES OF *R

In Chapter 1 we established the basis for non-standard analysis. We now make extensive use of the transfer method to establish certain properties of the set of "hyperreal" numbers $*R \in *(\hat{R})$ in the form of *- transforms. Here we adopt a formal approach and deal with $*R$ as the set of individuals generated by an enlarging monomorphism. For an informal descriptive approach see [8] and [9].

That we deal with $\Phi(R) = *R$ is appropriate since in Chapter 3 we develop non-standard measure theory on the non-standard unit interval $*[0,1) = \{x \in *R : 0 \leq x < 1\}$ of $*R$ and then apply it in Chapter 4.

By definition $*R$ is a totally ordered field, which contains the embedded standard copy $\Phi[R] = \{*a : a \in R\}$ as a proper subfield. This follows since each totally ordered field axiom of R can be written as a K_o- sentence whose ${}^{S}I$- interpretation is that axiom for R. Each *- transform is then that axiom for $*R$ and our assertion follows. Specifically, that $\leq$ totally orders R involves trichotomy, which can be expressed as the K_o- sentence:

$$\underline{2.11} \qquad (\forall x)(\forall y)\Big[[x \in R \wedge y \in R] \supset [x < y] \vee [x = y] \vee [x > y]\Big] .$$

Transforming this we obtain

$$\underline{2.12} \qquad (\forall x)(\forall y)\Big[[x \in *R \wedge y \in *R] \supset [x* < y] \vee [x = y] \vee [x* > y]\Big] ,$$

note that we are writing $=$ for $*=$, so that for every $x,y \in *R$, either $x *\leq y$ or $y *\leq x$, which implies that $*R$ is totally ordered by $*\leq$; here $*\leq$ is the extension of $\leq$ from R to $*R$. The algebraic operations, absolute value relation, integral part operation etc., of R extend to *R in a similar fashion and for notational convenience we continue to use the ordinary symbols to denote these unless confusion arises, or we wish to emphasize the non-standard nature of particular entities.

Since $\Phi[R]$ is an isomorphic copy of R in *R we further simplify matters by not using the *- notation for standard individuals of *R , denoting *3 for example simply by 3. Thus from now on we will identify R with the proper subfield $\Phi[R]$ of standard individuals of *R and feel free to write $R \subset {}^*R$. This convention does not apply to set entities however since in general for sets $E \in \hat{R}$, $\Phi[E]$ is quite different from E .

Since $N = \{1,2,\ldots\}$ is a subset of R , *N is a subset of *R and is a standard entity having the same properties as N as far as these can be expressed as K_o- sentences. *N is called the extended natural number system and is totally ordered by $\leq$ as above. Note also that $\leq$ is concurrent on N , since considering $\leq$ on N we have

$$\leq \; = \{(x,y) : x,y \in N \wedge x \leq y\} \text{ , with } D(\leq) = N \text{ .}$$

Thus for $a_1,a_2,\ldots,a_n \in N$ and $b = \max\{a_1,a_2,\ldots,a_n\}$, $(a_n,b) \in \leq$ so that as Φ is enlarging there is a $\gamma \in {}^*N$ such that $(\forall a \in N)a = {}^*a \leq \gamma$, an "infinite" natural number. This prompts us to formalize the following facts about *R :

<u>Definition 2.13</u> (i) A real number $a \in {}^*R$ is said to be <u>finite</u> if $|a| \leq n_o$, for some $n_o \in N$; a is said to be <u>infinite</u> if $|a| \geq n$, for all $n \in N$.

(ii) A real number $a \in {}^*R$ is said to be <u>infinitesimal</u> if $|a| \leq \frac{1}{n}$ for all $0 \neq n \in N$.

The set of all finite real numbers of $*R$ will be denoted by M_o and the set of all infinitesimals by M_1. Since $*R$ is a field, each non zero $a \in *R$ has a multiplicative inverse, in particular with τ infinite, $\frac{1}{\tau}$ is infinitesimal. Above we showed the existence of infinite natural numbers using the fact that $\leq$ is concurrent and Φ is enlarging. We now construct a particular infinite positive integer $\tau \in *N$ and an infinitesimal $\frac{1}{\tau} \in *R$ as its inverse. As well as exhibiting $\tau \in *N - N$ our construction serves as a good illustration of the concrete nature of the ultrapower method. Thus consider the mapping $\tau \in \hat{R}^J$ which is one to one and onto N, that is $\tau : J \to N$ satisfies $\{j : \tau(j) = n\} = \{j_n\}$, a singleton for each $n \in N$, so $\tau \neq n$ for any $n \in N$ (recall here that n is a constant sequence). τ as defined is clearly positive, and infinite by definition 2.13, since the set where it is positive is J and the set where any $n \in N$ exceeds it is at most finite and thus not in U. Since $\tau \neq 0$, $\frac{1}{\tau} \in *R$ is infinitesimal by definition as $0 < \frac{1}{\tau} < \frac{1}{n}$ for all $n \in N$. The illustration above highlights the value of the ultrapower construction, since clearly for any infinite τ as above we have $\tau \neq *a$, for all $a \in R$, as $*a$ is standard and hence finite by definition. Having completed our construction above we now continue to learn more about $*R$ by using *- transforms of well known properties of R.

<u>Theorem 2.14</u> Any $n \in *N$ is finite if and only if it is standard, that is $*N \cap M_o = N$.

<u>Proof</u>: Clearly $N \subseteq M_o$ and $N \subseteq *N$.

If $n \in *N$ is finite then $n \leq n_o$, for some $n_o \in N$. K_o contains the sentence

$$(\forall x)(x \in N \supset x \leq n_o \equiv x = 1 \vee x = 2 \vee \ldots \vee x = n_o) .$$

The transformation of this says that n is one of the standard numbers $1, 2, \ldots n_o$.

Thus the finite elements of $*N$ are the standard ones and the infinite elements are the non-standard ones, that is the set of infinitely large positive integers given by $*N - N = \{n \in *N : n \text{ infinite}\}$. Till now we have identified only one infinite integer τ , as constructed above. However this is sufficient to generate "blocks" of infinite positive integers $\in *N - N$ by using the *- transforms of a K_o- property of N .

Thus since $(\forall x)\big[[x \in N] \equiv [x+1 \in N]\big]$ we see that by transforming we obtain $(\forall x)\big[[x \in *N] \equiv [x+1 \in *N]\big]$, so that the "block" of infinite positive integers $\ldots \tau-2, \tau-1, \tau, \tau+1, \tau+2, \ldots$ belongs to $*N$. Since $2\tau, \tau\cdot\tau, \tau^\tau, \tau^{\tau^\tau}$, and so forth also belong to $*N$, so do blocks of positive integers with respect to them, e.g. $\ldots, (\tau\cdot\tau)-1, (\tau\cdot\tau), (\tau\cdot\tau)+1, \ldots$. It is clear that these blocks are densely ordered with no first or last element and that each block is itself order isomorphic to the integers $\ldots, -2, -1, 0, 1, 2, \ldots$. Thus we can think of $*N$ as consisting of N as an initial segment, followed by an ordered set of blocks as above, see [4].

We now look more closely at some of the algebraic properties of $*R$ and its subsets. First note that M_o is an integral domain since it is a subring of $*R$ without zero divisors. The set of infinitesimals constitutes a subring of M_o with the property that $(\forall \ell)(\forall a)\big[[\ell \in M_1 \wedge a \in M_o] \supset [\ell a \in M_1]\big]$, that is M_1 is an ideal of M_o ; it is in fact a maximal ideal, see theorem 4.4.3 of [23].

Following Robinson [18] we introduce the relations $\sim$ and $\simeq$ below.

<u>Definition 2.15</u> Let $a,b \in {}^*R$. We write:

(i) $a \sim b$, if and only if there is some positive $r \in R$ such that $|a - b| < r$, and

(ii) $a \simeq b$, if and only if $|a - b| < r$ for all such r .

The above defines what is referred to as the infinitesimal relation $\simeq$, that is for $a,b \in {}^*R$, 2.15 (ii) holds if and only if a and b are infinitesimally close.

Both $\sim$ and $\simeq$ define equivalence classes in ${}^*R$ and we use Zakon's notation [27] in calling them respectively the galaxy and the monad of a , denoted $G(a)$ and $M(a)$. For a detailed study of monads see [14] and [25]. In [18] Robinson uses $\mu(a)$ to denote the monad of a , however we want to retain this notation for our later non-standard measure theory. From the beforegoing we see that $G(0) = M_o$, consists of the union of all monads of standard points and $M(0) = M_1$.

At this stage it is important to realize that the equivalence relation $\simeq$ allows us to express ideas in calculus in a very intuitive and natural way. For instance consider an internal function f defined on the infinitesimal neighbourhood of $a \in {}^*R$ such that $f(x)$ is infinitesimally close to $f(a)$ whenever x is infinitesimally close to a . This expresses the "intuitive notion" of continuity of a function at $x = a$, namely that a small change in the independent variable produces a small change in the answer. This intuitive formulation is equivalent to the ε - δ definition of continuity [23], although only for $a \in M_o$ that is standard points of ${}^*R$.

Thus for $f(x) = x^2$ and $\varepsilon \in M_1$, $f(x+\varepsilon) = x^2 + \varepsilon(2x+\varepsilon)$ so that

$f(x)$ is continuous at finite x since in the infinitesimal neighbourhood of x, $f(x+\varepsilon) \simeq x^2$. This follows as for $\varepsilon \in M_1$, $\varepsilon(2x+\varepsilon) \in M_1$ since M_1 is an ideal in M_o. Note that when x is not finite, $f(x)$ need not satisfy the infinitesimal perturbation condition. To see this let $x = \tau \in {}^*N - N$ and consider the infinitesimal change $\varepsilon = \frac{1}{\tau}$. Then

$$f(x+\varepsilon) = \left(\tau + \frac{1}{\tau}\right)^2 = \tau^2 + 2 + \frac{1}{\tau^2} \simeq \tau^2 + 2 .$$

2.2 THE STANDARD PART HOMOMORPHISM

The order homomorphism of M_o with kernel M_1 onto R is called the standard part homomorphism and we denote it by st. That the unique map st of M_o onto R exists follows from the fact that the quotient ring M_o/M_1 is order isomorphic to R. To prove this first note that M_o/M_1 is a totally ordered field by theorem A.1.2 and A.2.5 of [23]. Further note that M_o/M_1 is Archimedean since M_o is Archimedean. To show this let $k,\ell \in N$ and $k \neq \ell$; then $k + M_1 \neq \ell + M_1$ so that the natural copy of N in M_o/M_1 is $\{n + M_1 : n \in N\}$. Now for any $a \in M_o$, $|a| \leq m_o$ for some $m_o \in N$, so that $|a + M_1| \leq |m_o + M_1|$ in M_o/M_1, since the canonical order homomorphism preserves order. Since M_o/M_1 is a totally ordered Archimedean field, it is isomorphic to a subfield of R (Theorem A.3.2 of [23]). To show it is actually R requires us to show that it contains a natural copy of R.

Let $a,b \in R \subset M_o$, $a \neq b$. Since $b - a \in M_o$, $a + M_1 \neq b + M_1$. Under these circumstances the canonical homomorphism restricted to R

is one-to-one so the image M_0/M_1 is a natural copy of R. This completes the proof of the existence of st ; we now summarize some of its properties for later reference.

2.21 Let $a, b \in M_0$, then

(i) $st(a+b) = st(a) + st(b)$

(ii) $st(a \cdot b) = st(a) \cdot st(b)$

(iii) $a \leqslant b \supset st(a) \leqslant st(b)$

(iv) $st(a) = 0 \equiv a \in M_1$

(v) $(\forall r)\Big[[r \in R] \supset [st(r) = r]\Big]$

(vi) $a \simeq b \equiv st(a) = st(b)$.

As a consequence of the properties of st we see that by 2.21 (iv) all infinitesimals belong to the monad of zero, while 2.21 (v) shows that for $r \in R$, st is the identity map. Property 2.21 (vi) says, as expected, that two finite numbers are infinitesimally close if they belong to the same monad.

2.3 EXTERNAL ENTITIES

So far we have considered properties of R which involve quantification over numbers only, and now examine some higher order properties. Above we introduced certain specific sets of individuals, e.g. $*N - N$, M_1, M_0 and the monads $M(a)$, $a \in R$.

It is now natural to ask whether any of these sets are internal or not, that is whether or not they belong to $*R_1$. To resolve these specific questions, as well as more general cases we proceed as follows.

We assume the set in question is internal and examine whether or not it violates any of all the *- transform properties it should posess on that premise, since a set is internal only if all *- transforms of all standard properties hold. Because the fact that a particular *- transform property holds does not necessarily imply that the set is internal, this requires a "judicious" choice on our part in finding a K_o- property whose *- transform does not hold for the set under consideration. Although these K_o- properties are fairly obvious for the sets under investigation, it emphasizes the fact that explicit external knowledge about our non-standard model is relatively hard to obtain.

For example since N is well ordered, every nonempty subset of N has a first element. This can be expressed as a K_o- sentence expressing a higher-order property of N having a universal quantifier ranging over all subsets of N . The limitations of transferring properties of set entities from R to *(R) insofar as they can be expressed in L now require this to be interpreted as

<u>2.31</u> every nonempty <u>internal</u> subset of *N (that is every element of *P(N)) has a first element.

Assuming *N - N is internal contradicts 2.31 since there is no smallest infinite natural number. Thus we conclude that *N - N is external, that is $*N - N \in P(*N) - *P(N)$. Similarly, assume that M_1 is internal. Now $M_1 \neq \phi$, and $\ell \in M_1$ implies $|\ell| < 1$, that is M_1 is bounded above. From the Dedekind completeness property of R it follows that every nonempty internal subset of *R which is bounded above has a least upper bound. Applying this to M_1 , let ℓ_o be the least upper bound. Now $0 \in M_1$ so that $\ell_o > 0$ (since $R \cap M_1 = \{0\}$ we regard zero as a special individual, namely the only standard infinitesimal). Furthermore, $\ell_o \notin M_1$ since if it were, $\ell_o < 2\ell_o \in M_1$, a contradiction. On the other hand if $\ell_0 \notin M_1$ then $\ell_0/2$ is an upper bound for M_1 and $\ell_0/2 < \ell_0$ so ℓ_0 is not the least upper bound. Since we have a contradiction in both cases we conclude that M_1 is external.

That M_1 is external demonstrates that $^*R$ is not complete in the external sense, that is there are bounded subsets of $^*R$ with no least upper bound. The *- transform of the formal sentence describing completeness of R does of course hold. We could call this property *- completeness or internal completeness, that is bounded internal sets do have least upper bounds as required. Thus $^*N$ is *- well ordered but not well ordered and $^*R$ is *- complete but not complete. This highlights the fact that interplay between internal and external notions is at the crux of Robinson's infinitesimal foundations.

Using a similar procedure to the above it can be shown that the subsets M_0, $M(a)(a \in R)$ and $^*R - M_0$ of $^*R$ are all external. More generally Theorem 5.2 of [13] shows that:

<u>2.32</u> If $A \in \hat{R}$ then the set $^*A - \Phi[A]$ of all the non-standard elements of $^*A$ is either empty or external.

In the latter case the set $\Phi[A] = \{^*a : a \in A\}$ is also external.

Having shown various sets to be external we now show that the standard part homomorphism is not an internal map. The set-theoretic properties of $\hat{R}$ show that if b is a binary relation entity of $\hat{R}$, then the domain and range of b are entities of $\hat{R}$. Transforming this implies that "the domain and range of an internal binary relation is internal". Since st is a mapping of M_0 onto R and both M_0 and R are external subsets of $^*R$, we conclude on the basis of the remarks in inverted commas that st is an external operation.

Although 2.32 shows that the set of non-standard elements of the extension of an infinite set of $\hat{R}$ is external, any finite set of entities which are not standard is internal. Thus the set $\{\tau, \tau+1, \tau+2\}$ is internal even though its elements are not standard (here τ is the infinite individual of $*N$ constructed earlier). That $\{\tau, \tau+1, \tau+2\}$ is internal holds since as a set it has all the properties of a finite set of standard individuals as far as they can be expressed as K_o- sentences. We now transfer the notion of finiteness to $*(\hat{R})$ by transforming the statement expressing the property of being finite in $\hat{R}$. Thus since:

"An entity A is finite if there is a bijection entity from an initial segment $\{1,2,\ldots,n\}$ of N onto A",
we have by transforming that;

<u>Definition 2.33</u> An internal entity A of $*(\hat{R})$ is <u>*- finite</u> if there is an internal one-to-one mapping from an initial segment of $*N$ onto A.

Here of course, an initial segment can be externally infinite, for example $\{n \in *N : 0 < n \leq \tau\}$, $\tau \in *N - N$.

<u>Theorem 2.34</u> Every *- finite set of internal entities is internal.

<u>Proof</u>: Since the range of an internal function is internal (see example 3.9 (iv) of [13]) it follows from definition 2.33 that a *- finite set is internal.

In particular we see that since every finite set of real numbers $\in R$ has a largest and smallest element, the *- transform of the K_o- sentence expressing this property tells us that every *- finite set of real numbers in $*R_1$ has both a largest and smallest element. Thus if A is a *- finite subset of $*R$ there is a unique smallest integer γ, such that $\{1,2,3,\ldots,\gamma\}$, $\gamma \in *N$ is one-to-one and onto A. In this case we say that the internal cardinal of A is γ or shortly that A has γ- elements and write $||A|| = \gamma$ to denote this. Note that any externally infinite *- finite set A has an external cardinal at least as big as $\aleph_0$ since $N \subset \{1,2,\ldots,\gamma\}$, for any $\gamma \in *N - N$. Further A as just described contains at least one internal element which is not standard and so externally infinitely many of these. For example, if $*N \supseteq A \in *R_1$ and $A = \{1,2,3,\ldots,\tau\}$, $\tau \in *N - N$ then A contains the externally infinite block $\ldots\ldots \tau-1, \tau.$, discussed earlier.

Given a finite sequence $a_1, a_2, \ldots, a_n \in R$ we can form the finite sum $\sum_{k=1}^{n} a_k$. The ability to form sums extends to $*R$ and when $n \in *N$ we say that the *- finite sequence has a *- finite sum. For example the *- finite sequence $(1,1,1,\ldots,1_\lambda)$ having λ- elements, $\lambda \in *N - N$ has the *- finite sum

$$\sum_{k=1}^{\lambda} 1 = \lambda .$$ This allows us to say that

$$\sum_{k=1}^{\lambda} 2 = 2\lambda$$, that is the *- finite sum

of $(2,2,\ldots,2_\lambda)$ is twice that of $(1,1,1,\ldots,1_\lambda)$.

To conclude this chapter we look briefly at the non-standard interval $*[0,1]$ of $*R$ to obtain a cardinality result for *- finite sets.

Fixing an infinite positive integer $\gamma \in *N - N$ we have

$$*[0,1] = \bigcup_{k<\gamma} \left[\frac{k}{\gamma}, \frac{k+1}{\gamma}\right] ,$$ where

$$0, \frac{1}{\gamma}, \frac{2}{\gamma}, \ldots, \frac{k+1}{\gamma}, \ldots, 1, \qquad {}^*N \ni k < \gamma$$

constitutes an infinitesimal partition of $^*[0,1]$ into subintervals $\left[\frac{k}{\gamma}, \frac{k+1}{\gamma}\right]$ each of length $\frac{1}{\gamma}$. Since this length is an infinitesimal, each interval is contained in exactly one of the disjoint monads $M(x)$ which cover $^*[0,1]$ and each such $M(x)$ with $x \in [0,1]$ contains at least one such interval. Hence the number of intervals cannot be less than that of the $M(x)$, $x \in [0,1]$, that is it is $\geq 2^{\aleph_0}$, since there are as many $M(x)$ as there are standard points in $^*[0,1]$. This allows us to be more specific about the cardinality of $*$- finite sets, in fact we see that any $*$- finite set is of power $\geq 2^{\aleph_0}$; that is $\gamma \geq 2^{\aleph_0}$ in some sense.

CHAPTER 3. NON-STANDARD MEASURE THEORY

In this chapter we develop the non-standard measure theory applied in Chapter 4.

Non-standard measure theory has been examined by several authors using various approaches. Robinson in [18] gives the first brief outline of the extension of Lebesgue integration to non-standard models. In [10] - [12] Loeb uses set partitions to develop a non-standard representation for measures.

Here we use Bernstein and Wattenberg's approach in [3] to construct a non-standard measure which is an extension of Lebesgue measure to all sets in the unit interval and thus provide a natural solution to the "easy problem of measure" solved first by Banach [17].

3.1 STANDARD MEASURE THEORY

The concept of measure of a set of real numbers is a generalization of the notion of length to arbitrary sets A on R .

Ideally the measure $m(A)$ of a set A should be defined for all $A \subseteq R$ and should satisfy the following requirements:

3.11 (i) $m(A) \geq 0$

(ii) Finite Additivity: If $A = \bigcup_{k=1}^{n} A_k$, where the components A_k are mutually disjoint, then

$$m(A) = \sum_{k=1}^{n} m(A_k) .$$

(iii) Countable Additivity: If $A = \bigcup_{k=1}^{\infty} A_k$, with A_k's mutually disjoint, then

$$m(A) = \sum_{k=1}^{\infty} m(A_k) \ .$$

(iv) Monotonicity: If $A_1 \subseteq A_2$, then

$m(A_1) \leq m(A_2)$.

(v) Translation Invariance:

$m(A) = m(A+r)$, where $r \in R$ is the distance by which each point of A is translated.

(vi) If A is an interval then

$m(A)$ = length of the interval A .

It is well known that for arbitrary $A \subseteq R$ not all of the properties 3.11(i)-(vi) can be satisfied so that we must sacrifice some of them to ensure that all subsets of R do have a measure.

Definition 3.12 Let A be an arbitrary set and F a finite subset of $S = [0,1)$. Denote by $||A \cap F||$ the number of elements (in the finite set) $A \cap F$. For every $A \subseteq S$ we define a measure m_F on A relative to F as:

$$m_F(A) = \frac{||A \cap F||}{||F||} \ .$$

As defined, m_F satisfies properties 3.11(i), (ii), (iii) and (iv) and is a normalized $\left(\text{i.e. } m_F(F) = 1\right)$ non-negative, finitely additive measure on all subsets of S . However note that m_F is not particularly useful as it fails to distinguish on a measure basis between subsets $S \supseteq A \supseteq F$, since for all such A we have $m_F(A) = 1$. In the remaining sections of Chapter 3 we show that a non-standard measure can be constructed on S which is in a sense an extension of m_F to $*(\hat{R})$.

Specifically we obtain a non-standard measure μ_F , defined on all subsets of the unit interval, such that its standard part agrees with Lebesgue measure where the latter is defined.

The measure μ_F is constructed as a point measure finite in the sense of non-standard analysis. That is we choose F to be a well-distributed *- finite set of points in $\Phi(S) = {}^*S$ such that $S \subseteq F$. Then given any set $A \subseteq S$, the measure $\mu_F(A)$ is defined to be the number of points of the set F which lie in A divided by the total number of points in F. How we go about selecting a suitable F such that all our desired requirements are met will be detailed in the sections to follow.

3.2 NON-STANDARD MEASURE THEORY

We continue the approach in Chapter 2 and work within a fixed enlargement ${}^*(\hat{R})$ generated by our superstructure monomorphism Φ. We tacitly assume that all sets and relations subsequently discussed in ${}^*(\hat{R})$ are internal. In particular we deal only with internal subsets of ${}^*[0,1)$; however we underline this fact by occasionally restating it in the work to follow.

Let $X \in \hat{R}$ be the set which consists of all finite sets of real numbers. X extends to ${}^*X$ in ${}^*(\hat{R})$, where any element of ${}^*X$ is *- finite, so that ${}^*X$ contains all finite sets of real numbers as well as sets such as $\{1,2,\ldots,\lambda\}$, where λ is any infinite positive integer.

The function c which assigns to each $A \in X$ a positive integer $c(A)$ which is its standard cardinality extends to $\Phi(c) = {}^*c$ in ${}^*(\hat{R})$ and assigns to each $A \in {}^*X$ a positive integer in ${}^*N$ which is its non-standard cardinality, see [5]. We retain our earlier notation and for $A \in {}^*X$ simply write $||A||$ to denote this integer, so that ${}^*c(A) = ||A||$, that is $A \in {}^*X$ has $||A||$ elements.

Following Bernstein and Wattenberg we let $S = [0,1)$ in $\hat{R}$ and write points and addition on S modulo 1. For $x,y \in S$ we write $x \dotplus y$ to denote this, so that $x \dotplus y = x + y$ for $x + y < 1$ and $x \dotplus y = x + y - 1$ for $x + y \geq 1$.

In $*(\hat{R})$, S extends to $\Phi(S) = *S = *[0,1) = \{x \in *R : 0 \leq x < 1\}$.

<u>Definition 3.21</u> (Bernstein and Wattenberg)

A non-empty *- finite subset of *S will be called a <u>sample</u>. Any sample F has an associated sample measure which assigns to every subset A of $*S$ a non-standard real number $\mu_F(A)$ defined by

$$\mu_F(A) = \frac{\|F \cap A\|}{\|F\|} \quad ; \quad \text{(compare with 3.12).}$$

For any sample F we have:

<u>3.22</u> (i) $\mu_F(*S) = 1$, $\mu_F(\phi) = 0$,

$\mu_F(A) \geq 0$, for any subset $A \subseteq *S$.

(ii) If $A \subseteq B$, then $\mu_F(A) \leq \mu_F(B)$.

(iii) If $\{A_i\}_{i \in *N}$ is any sequence of disjoint subsets of $*S$, then there is a non-standard integer L such that if $i > L$, then $\mu_F(A_i) = 0$ and

$$\mu_F\left(\bigcup_{i \in *N} A_i\right) = \sum_{i=1}^{L} \mu_F(A_i)$$

<u>Proof:</u> Since F is a sample, $\|F\| \neq 0$ we have

(i) $$\mu_F(*S) = \frac{\|F \cap *S\|}{\|F\|} = \frac{\|F\|}{\|F\|} = 1 \; ;$$

$$\mu_F(\phi) = \frac{\|F \cap \phi\|}{\|F\|} = \frac{\|\phi\|}{\|F\|} = 0 \; ;$$

for any $A \subseteq *S$, $\mu_F(A) = \dfrac{\|F \cap A\|}{\|F\|}$. Now $\|F \cap A\| \geq 0$ and so $\mu_F(A) \geq 0$.

(ii) If $A \subseteq B$ then $\|F \cap A\| \leq \|F \cap B\|$ and $\mu_F(A) \leq \mu_F(B)$.

(iii) In $\hat{R}$ we have the following true statement:

"If F is finite and $\{A_i\}_{i \in N}$ any sequence of disjoint subsets of S , then there is an integer $L \in N$ such that if $i > L$, $A_i \cap F = \phi$ (i.e. $\mu_F(A_i) = 0$) and

$$\mu_F\left(\bigcup_{i \in N} A_i\right) = \sum_{i=1}^{L} \mu_F(A_i) \text{ . "}$$

3.22 (iii) is now obtained in $*(\hat{R})$ as the Φ- transform of the above statement, where it is important to note that i ranges over the set $*N$.

Notice that we can bound L , namely $L \leq \| F \|$ if we adopt the convention of rewriting each sequence $\{A_i\}_{i \in *N}$ as $\{B_i\}_{i \in *N}$, where B_j is the j-th set A_i for which $A_i \cap F \neq \phi$. In Chapter 4 we develop this further and show that as expected L depends on how the sequence $\{A_i\}_{i \in *N}$ has been defined.

Since F is *- finite we can write $F = \{x_1, x_2, \ldots, x_\nu\}$, for some $\nu \in *N\text{-}N$, that is $\| F \| = \nu$ and we tacitly assume this value for $\| F \|$ in our subsequent work unless otherwise stated. We now show that $*S - F \neq \phi$, so that we can obtain zero for the measure of certain subsets of $*S$. Since F is *- finite and bounded it has a least element x_k . Let g be the standard inverse function on $R - \{0\}$. Then $\Phi(g) = *g$ and for positive $r \in *R - *S$, $*g(r) \in *S$. Thus for any $s \in *R$ such that $s > (*g)^{-1}(x_k)$ we have $*g(s) \in *S - F$.

Checking with 3.11 μ_F defines a non-negative measure, *- finitely additive in $*(\hat{R})$. Note that μ_F cannot be countably additive in the old sense, since standard countability is an external notion in $*(\hat{R})$.

The Φ- transform of standard countability is of course internal so that sets such as $*N$ itself as well as $*Q$ for example, are both $*$- countable.

Property 3.11 (v) requires a measure to be translation invariant. This is the case for Lebesgue measure on S (mod 1) and we show the following:

Theorem 3.23 Let $A \subseteq S$ be Lebesgue measurable. Then for each $x \in S$, $A \dotplus x$ is Lebesgue measurable and $L(A \dotplus x) = L(A)$.

Proof: Let
$$A_1 = A \cap [0, 1-x) \text{ and}$$
$$A_2 = A \cap [1-x, 1) .$$
Then $A_1 \cap A_2 = \phi$ and $A = A_1 \cup A_2$, so that
$$L(A) = L(A_1) + L(A_2) .$$
Now $A_1 \dotplus x = A_1 + x$ so $L(A_1 \dotplus x) = L(A_1)$ since Lebesgue measure is translation invariant. Similarly, $L(A_2 \dotplus x) = L\big(A_2 + (x-1)\big) = L(A_2)$. But $A \dotplus x = (A_1 \dotplus x) \cup (A_2 \dotplus x)$, where $(A_1 \dotplus x) \cap (A_2 \dotplus x) = \phi$. Hence $A \dotplus x$ is Lebesgue measurable and
$$\begin{aligned} L(A \dotplus x) &= L(A_1) + L(A_2) \quad \text{from above} \\ &= L(A) . \end{aligned}$$

We shall see later that μ_F is not strictly translation invariant on $*S$ in the above sense, but at this stage we investigate invariance of subsets $A \subseteq *S$ through distances $\frac{1}{n}$, $n \in *N$.

Definition 3.24 (Bernstein and Wattenberg)

If n $*N$ and F is a sample, F is said to be n-invariant if $F = F \dotplus \frac{1}{n}$; that is, if whenever $x \in F$ so does $x \dotplus \frac{1}{n}$ and vice versa.

Theorem 3.25 If F is n-invariant then for any internal subset $A \subseteq *S$,

(i) $\mu_F(A) = \mu_F\left(A \mp \frac{1}{n}\right)$

(ii) $\mu_F\left(\left[a,\ a + \frac{t}{n}\right)\right) = \frac{t}{n}$, $t \in {}^*N$, $0 \leqslant t \leqslant n$

(iii) $\left|\mu_F([a,b)) - (b-a)\right| \leqslant \frac{1}{n}$.

Proof: (i) Clearly

$$\|(A \cap F) \mp \frac{1}{n}\| = \|\left(A \mp \frac{1}{n}\right) \cap \left(F \mp \frac{1}{n}\right)\|$$

$$= \|\left(A \mp \frac{1}{n}\right) \cap F\| ,$$

since F is n-invariant.

Using this result we have the following true statement in $\hat{R}$:

" $(\forall F)(\forall A) A \in P(S) \wedge F \in X \wedge F$ is n-invariant

$$\supset \|A \cap F\| = \|(A \cap F) \mp \frac{1}{n}\| = \|\left(A \mp \frac{1}{n}\right) \cap F\| \text{ "} .$$

Transforming this statement now gives us the required result, namely for all $A \in {}^*P(S)$, that is all internal subsets A of ${}^*S$, we have

$\|A \cap F\| = \|A \mp \frac{1}{n} \cap F\|$ so that

$\mu_F(A) = \mu_F\left(A \mp \frac{1}{n}\right)$ as required.

We obtain (ii) by partioning any interval $[a, a+1)$ as follows:

$$1 = \mu_F\left(\left[a,\ a + \frac{1}{n}\right)\right) + \mu_F\left(\left[a + \frac{1}{n},\ a + \frac{2}{n}\right)\right) + \dots$$

$$\dots + \mu_F\left(\left[a + \frac{n-1}{n},\ a + 1\right)\right) .$$

Setting $A = \left[a,\ a + \frac{1}{n}\right)$ in the above and using 3.25 (i) and 3.22 (iii) with $n = L$ gives

$$1 = n\,\mu_F\left(\left[a + \frac{1}{n}\right)\right) .$$

Hence $\mu_F\left(\left[a,\ a + \frac{1}{n}\right)\right) = \frac{1}{n}$ and

$$\mu_F\left(\left[a,\ a + \frac{t}{n}\right)\right) = \frac{t}{n} \quad \text{by additivity.}$$

To obtain (iii) note that if t is the greatest integer such that $\frac{t}{n} < (b-a)$, then

$$\frac{t}{n} = \mu_F\left(\left[a,\ a + \frac{t}{n}\right)\right) \text{ , from 3.25 (ii) .}$$

Also $[a,b) = [a,\ a+(b-a))$

$$\supseteq \left[a,\ a + \frac{t}{n}\right) \quad , \quad \text{since } \frac{t}{n} < (b-a) \ .$$

Now $\frac{t+1}{n} \geqslant (b-a)$ from our requirement on t , and so

$$\mu_F([a,b)) \leqslant \mu_F\left(\left[a,\ a + \frac{t+1}{n}\right)\right) = \frac{t+1}{n} \ .$$

That is $\mu_F([a,b)) - \frac{t}{n} \leqslant \frac{1}{n}$ and since $\frac{t}{n} < (b-a)$ we have $\mu_F[a,b) - (b-a) \leqslant \frac{1}{n}$. Thus $-\frac{1}{n} \leqslant \mu_F[a,b) - (b-a)$ and we have the required result namely, $|\mu_F([a,b)) - (b-a)| \leqslant \frac{1}{n}$.

Till now we have only looked at the effect of n-invariance on F , without specifying which values of n we are interested in. Since we want to use μ_F for all sets $*A = \Phi(A)$, $A \subseteq S$, we require F to be n-invariant for at least every positive integer n , and below we see the consequence of such a requirement.

Theorem 3.26 If F is a sample which is n-invariant for every standard positive integer n , then there is a *- finite positive integer σ ($\sigma \in *N-N$) such that F is k-invariant for any $k \in *N$, $k \leqslant \sigma$.

Proof: Consider the set T of all standard positive integers n such that F is k-invariant for $k \leq n$. Now for all finite positive integers $n \in T$, $\leq$ is a concurrent binary relation on T, so that by extending n-invariance to $*(\hat{R})$ under our enlarging monomorphism we are guaranteed the existence of a *- finite positive integer $\sigma \in {}^*N - N$ for which F is k-invariant for all $k \leq \sigma$. That is F is k-invariant for integers $k \in {}^*N$ ranging over the values $1,2,3,\ldots,\sigma-3,\sigma-2,\sigma-1,\sigma$. We refer to the largest such σ as the mesh of F.

If F has mesh σ then for positive integers $q,r \in {}^*N$, $q \leq r \leq \sigma$ we have as an immediate consequence of Theorem 3.25 that:

3.27 (i) $\mu_F\left(\left[a + \frac{q}{r}\right)\right) = \frac{q}{r}$

(ii) $\mu_F\big([a,b) - (b-a)\big) \leq \frac{1}{r} \simeq 0$, for $r \in {}^*N - N$.

We can construct a sample F which is n-invariant for every standard positive integer n and which includes all standard points of ${}^*S$. Although $F \ni y = x \mp \frac{k}{n}$, $x \in S$ and $k \leq n \leq \sigma$ (the mesh of F), we cannot write $F = \bigcup_{k \leq n \leq \sigma} S \mp \frac{k}{n}$, since for $a \in F$ points like $a \mp \frac{1}{\sigma-1} \mp \left(1 - \frac{1}{\sigma}\right) = a \mp \frac{1}{\sigma(\sigma-1)}$ also belong to F.

However for $x \in S$, any $x \mp \frac{p}{r} \in F$, where $p,r \in {}^*N$, $(p,r) = 1$ and $r = \ell_1^{i_1}\ell_2^{i_2}\ldots\ell_n^{i_n}$, $i_1, i_2, \ldots, i_n$ non-negative integers $\in N$ and $\ell_1^{i_1}, \ell_2^{i_2}, \ldots, \ell_n^{i_n}$, the prime powers $\leq \sigma$.

This kind of sample gives the appropriate measure to intervals and gives each standard point in ${}^*S$ an infinitesimal measure $\frac{1}{\|F\|}$. However the sample may produce a sample measure which gives inaccurate measures to some Lebesgue measurable sets and which behaves poorly with respect to translation of subsets. To illustrate this let $Q_S = Q \cap S$, where Q is the set of rationals, then ${}^*Q_S = {}^*(Q \cap S) = {}^*Q \cap {}^*S$. Let $\|{}^*Q_S \cap F\| = \gamma$, then

$$\mu_F({}^*Q_S) = \frac{\|{}^*Q_S \cap F\|}{\|F\|} = \frac{\gamma}{\nu}$$, where clearly $\gamma < \nu$ since $S \subset F$. Now form another sample $F' \supset F$ by adding to F some non-standard rationals $q \in {}^*Q_S - F$ and their translates $q \mp \frac{k}{n}$, $k \lesssim n \in N$, to retain the n-invariance of F' (here we add only a *- finite number of new points to F, thus assuring that F' as formed remains internal). For F' formed as above write $F' = F \cup H$. Since $F \cap H = \phi$, $\|F'\| = \|F\| + \|H\|$; further as $H \subset {}^*Q_S$, $\|H\| = \|{}^*Q_S \cap H\|$ and

$$\mu_{F'}({}^*Q_S) = \frac{\|{}^*Q_S \cap F'\|}{\|F'\|} = \frac{\|{}^*Q_S \cap F\| + \|H\|}{\|F'\|}$$

$$= \frac{\gamma + \|H\|}{\nu + \|H\|} .$$

Thus by adding enough non-standard rationals $q \in {}^*Q_S - F$ to form F', we can make $\|H\|$ very large in comparison with γ and ν and give ${}^*Q_S$ a large sample measure $\mu_{F'}({}^*Q_S)$. On the other hand we can assign a very small measure to ${}^*Q_S$ by adding to F many points $x \in ({}^*S - {}^*Q_S) - F$ and their translates to form F'. Then since ${}^*Q_S \cap (F' - F) = \phi$

$$\mu_{F'}({}^*Q_S) = \frac{\|{}^*Q_S \cap F'\|}{\|F'\|} = \frac{\|{}^*Q_S \cap F\|}{\|F'\|}$$

$$= \frac{\gamma}{\|F'\|}$$, which can be made small by making F' large enough. For a particular Lebesgue measurable set $V \subset S$ the above possibilities show that to keep $\mu_F(V)$ close to $L(V)$ in constructing F, by adding points x and their translates $x \mp \frac{k}{n}$ to the sample,

it is vital to be able to choose points x such that just enough of the points $x \dotplus \frac{k}{n}$ belong to V. Requirement 3.28 (iii) below specifies what we need to aim for and in section 3.3 we state the technical lemma which allows us to achieve this.

We now have a clear conception of the overall properties we wish a sample F to have if μ_F is to be defined for all subsets of $*S$. Following Bernstein and Wattenberg we say that a sample F is called a premeasure and its associated sample measure called a measure, provided the conditions below are satisfied.

<u>3.28</u> (i) $S \subset F$

(ii) $F = F \dotplus \frac{1}{n}$ for all $n \in N$

(iii) If A is any Lebesgue measurable set, then $\mathrm{st}\left(\mu_F(*A)\right)$ is its Lebesgue measure, denoted $L(A)$.

(iv) If α is any standard point of $*S$ and A is any internal subset of $*S$ then

$$\mu_F(A) \approx \mu_F(A \dotplus \alpha) .$$

Condition (i) above gives each standard point in $*S$ an infinitesimal measure $\frac{1}{\|F\|}$. Note that the third requirement does not restrict us to Lebesgue measurable standard sets since $\mathrm{st}\left(\mu_F(*A)\right)$ is not *Lebesgue measure in $*(\hat{R})$ but an extension of Lebesgue measure on R. To see this note that F is *- finite and so has *Lebesgue measure zero in $*(R)$, but $\mu_F(F) = 1$. In Theorem 3.23 we proved that Lebesgue measure is translation invariant on S (mod 1). Now we show that in the sense of μ_F the translate of a set may be smaller than the original set, so that 3.28 (iv) is the best we can hope for. However, even though we do not have strict translation invariance, what we do have is as good as what we have for standard Lebesgue measure since

$$\mathrm{st}\left(\mu_F(A)\right) = \mathrm{st}\left(\mu_F(A \dotplus \alpha)\right) .$$

Let $T = \{kx : k \in N , \ x$ a fixed irrational$\}$.
Translating T through x gives the set $T + x$, and since we are working in the unit interval (modulo 1) we have $T = (T \dotplus x) \cup \{x\}$, where $x \notin T \dotplus x$.

Since $\mu_F(\{x\}) = \frac{1}{\|F\|}$ and our measure is finitely additive the measure of T is infinitesimally larger than that of $(T \dotplus x)$. We list below some conditions on F which are equivalent to requirement 3.28 (iii) and illustrate the general relationship between μ_F and standard Lebesgue measure. In particular if F is a sample then the following conditions on F are equivalent.

<u>3.29</u> (i) For every standard open set 0 , $st(\mu_F(*0)) \leq L(0)$.

(ii) For every standard set A , if $\overline{m}(A)$ is its outer measure and $\underline{m}(A)$ is its inner measure then

$$\underline{m}(A) \leq st(\mu_F(*A)) \leq \overline{m}(A) .$$

(iii) For every Lebesgue measurable set A ,

$$st(\mu_F(*A)) = L(A) .$$

To show (i) implies (ii) we use the fact that

$$\overline{m}(A) = \inf \{L(0) : 0 \text{ open}, \ A \subseteq 0\} .$$

Now $A \subseteq 0 \supset *A \subseteq *0$ so that using 3.22 (ii) we have by 3.29 (i) that

$$st(\mu_F(*A)) \leq st(\mu_F(*0)) \leq L(0) ,$$

that is $st(\mu_F(*A)) \leq \overline{m}(A)$.

Now $A \subseteq S$ and since $*(S-A) = *S - *A$ we obtain
$\underline{m}(A) = 1 - \overline{m}(S-A) \leq 1 - st(\mu_F(*S - *A)) = st(\mu_F(*A))$.
Further (ii) $\supset$ (iii) since if A is Lebesgue measurable $L(A) = \underline{m}(A) = \overline{m}(A)$.
To show (iii) $\supset$ (i) let 0 be an open set. Then 0 is Lebesgue measurable and by (iii) $st(\mu_F(*0)) = L(0)$ so that

$$st(\mu_F(*0)) \leq L(0) .$$

3.3 THE EXISTENCE OF PREMEASURES IN $*(\hat{R})$

To show the existence of premeasures in $*(\hat{R})$, we construct a sample F which satisfies the premeasure requirements. In general terms we achieve this by showing that a relation Q which holds between F and a certain quintuple, if and only if F satisfies 3.28 (i) - (iv), is finitely satisfiable (i.e. concurrent) in $\hat{R}$. Our enlarging monomorphism Φ then guarantees the existence of a premeasure F in $*(\hat{R})$ which simultaneously satisfies Q for all standard objects in its domain.

As part of our construction we need several technical lemmas from [3] which we state here without proof, except in those cases where a proof clarifies the concepts involved.

In section 3.2 we highlighted the need to exercise control over our sample during its construction. In particular for any Lebesgue measurable set V we have to ensure that not too many of the points $x \dotplus \frac{k}{n}$, $k \in Z$, $n \in N$, lie in V when we add new points x and their translates $x \dotplus \frac{k}{n}$ to build up F. The following lemma in $\hat{R}$ allows us to do just that.

<ins>Lemma 3.31</ins> Let V be any Lebesgue measurable set with Lebesgue measure L(V) and suppose p, q are integers such that $\frac{p}{q} \leq L(V) < \frac{p+1}{q}$. Then there is a point x in S such that at most p of the points in the set

$$T(x) = \left\{x \dotplus \frac{t}{q} : 0 \leq t \leq q\ ,\quad t \in N\right\}$$

are in V.

The proof of the lemma rests on the contradiction $L(V) \geq \frac{p+1}{q}$ obtained by assuming that for each $x \in S$ at least p + 1 points of T(x) are in V.

The corollary of lemma 3.31 in [3] states that if in addition to V, p, q as above we are given a set W of Lebesgue measure zero, then a point x can be chosen satisfying the lemma for V but such that no point of $T(x)$ lies in W.

Applying the corollary to Q_S we see that since $L(Q_S) = 0$ we can choose any irrational point $y \in S$ and have $T(y) \cap Q_S = \phi$ for any $q \in N$.

For a given integer $n \in N$ we obtain n-invariance in building up a sample by adding both new points x and their translates $x \mp \frac{k}{n}$, $k \in Z$. Similarly for any real number y we can obtain a sample which assigns to a set A a measure which is close to the measure it assigns to $A \mp y$, by adding to the sample a lot of the translates $x \mp ky$ of points x in the sample. Again the definition and lemmas below allow us to do this without losing control of the sample we obtain.

<u>Definition 3.32</u> The standard points $z_1, z_2, \ldots, z_t \in S$ are said to be <u>independent</u> if whenever

$$\sum_{i=1}^{t} k_i z_i = 0 \quad \text{for} \quad k_i \in Z ,$$

then all of the k_i's must be zero. Since we are working modulo 1, if any z_i is rational and $t > 1$ then the points $z_1, z_2, \ldots, z_t$ are not independent.

<u>Lemma 3.33</u> Let $z_1, z_2, \ldots, z_t$ be independent, $x \in S$, $q \in N$ and $T = \{x \mp \frac{t}{q} : t \in N\}$.

Then all of the sets of the form $T \mp \sum_{i=1}^{t} k_i z_i$ for $k_i \in Z$ are pairwise disjoint.

Proof: Suppose they are not pairwise disjoint. Then there is a point y which can be expressed as

$$x \dotplus \frac{p}{q} \dotplus k_1 z_1 \dotplus \dots \dotplus k_t z_t \quad \text{and as}$$

$$x \dotplus \frac{p'}{q} \dotplus k_1' z_1 \dotplus \dots \dotplus k_t' z_t \quad \text{with some } k_i \neq k_i' .$$

Then $\frac{(p-p')}{q} \dotplus \left(k_1 - k_1'\right) z_1 \dotplus \dots \dotplus \left(k_t - k_t'\right) z_t = 0$

and since we are working modulo 1, this requires $\sum_{i=1}^{t} \left(k_i - k_i'\right) z_i = 0$ which contradicts the independence of the z_i's since for some i, $k_i \neq k_i'$.

Lemma 3.34 Suppose $y_1, y_2, \dots, y_s$ are irrational points of S. Then there is a set of independent points $z_1, z_2, \dots, z_t$ and an integer T such that each y_i is of the form $k_1 z_1 \dotplus k_2 z_2 \dotplus \dots + k_t z_t$ for some integers k_i, $|k_i| \leq T$.

The proof is by induction on s. Here we only note the for $s = 1$ we can write

$$y_1 = \sum_{i=1}^{t} k_i z_i \quad , \quad \text{where } t = 1 ,$$

$k_1 = 1$ and $y_1 = z_1$.

When building up a sample there are always some points over which we have no control and which adversely affect our measure. This problem is overcome by adding enough points to the sample to suppress their effect and the following lemma allows us to do this.

Lemma 3.35 Suppose F and H are samples with $\frac{||H||}{||F||} < \varepsilon$ for some standard positive real number ε. Let $F' = F \cup H$. Then for any set A, $|\mu_F(A) - \mu_{F'}(A)| < \varepsilon$.

Proof: Let $\|F \cap A\| = p$. Then $\mu_F(A) = \frac{p}{\|F\|}$ and the smallest $\mu_{F'}(A)$ can be, occurs when $\|F' \cap A\|$ has its least value. Since $F' = F \cup H$, this occurs for $A \subseteq F$, in which case

$$\begin{aligned} \|F' \cap A\| &= \|(F \cup H) \cap A\| \\ &= \|(F \cap A)\| + \|H \cap A\| \\ &= p \, . \end{aligned}$$

Thus the smallest $\mu_{F'}(A)$ can possibly be, is

$$\frac{p}{\|F'\|} = \frac{p}{\|F\| + \varepsilon\|F\|} \, .$$

Now

$$\begin{aligned} \frac{p}{\|F\|} - \frac{p}{\|F\| + \varepsilon\|F\|} &= \frac{p}{\|F\|}\left\{1 - \frac{1}{1+\varepsilon}\right\} \\ &= \frac{p}{\|F\|}\left\{\frac{\epsilon}{\epsilon+1}\right\} < \epsilon \quad \text{since} \quad \frac{p}{\|F\|} \leq 1 \, . \end{aligned}$$

Thus $\mu_{F'}(A) \geq \mu_F(A) - \epsilon$. Similarly the largest $\mu_{F'}(A)$ can be, occurs when $H \subseteq A$ and $H \cap F = \phi$. Then

$$\mu_{F'}(A) = \frac{p + \|H\|}{\|F\| + \varepsilon\|F\|}$$

and so $\mu_{F'}(A)$ cannot be larger than $\frac{p + \|H\|}{\|F\|}$.

But $\frac{p + \|H\|}{\|F\|} - \frac{p}{\|F\|} = \frac{\|H\|}{\|F\|} < \epsilon$.

Thus $\mu_{F'}(A) \leq \mu_F(A) + \epsilon$, and combining the two results we have

$$|\mu_F(A) - \mu_{F'}(A)| < \epsilon \, .$$

It is important to note that by making $F' - F$ sufficiently small, we can make μ_F and $\mu_{F'}$ arbitrarily close in the standard sense. As a consequence of Lemma 3.35 we are now in a position to obtain a sample which keeps the measure of any set A close to the measure of $A \mp y$ for a standard real y .

<u>Corollary 3.36</u> Let $y \in R$ and suppose F' is a sample such that $F' \cap (F' \mp ky) = \phi$ for each integer k. Let F be the sample given by

$$F = \{x \mp ky : x \in F', \quad |k| \leq L\} .$$

Then for any set A, $|\mu_F(A) - \mu_F(A \mp y)| < \frac{1}{L}$.

<u>Proof</u>: Since F' is a sample we can write its elements as

$$F' = \{x_1, x_2, \ldots, x_{||F'||}\} .$$

Since $F' \mp ky$ are pairwise disjoint for integers $|k| \leq L$ we have $||F|| = (2L+1)||F'||$ and we can write

$$F = \left\{\begin{array}{ccccc} x_1 - Ly, \ldots\ldots, & x_1, & \ldots\ldots\ldots, & x_1 + Ly \\ \vdots & , \; x_2, & & \vdots \\ \vdots & & & \vdots \\ x_{||F'||} - Ly, \ldots\ldots, & x_{||F'||}, & \ldots\ldots, & x_{||F'||} + Ly \end{array}\right\}$$

Now let $H = F \cap (F-y)$. Then both H and $H \mp y$ are subsets of F with

$$||H|| = 2L||F'|| \quad \text{and} \quad ||H \mp y|| = 2L||F'|| \text{ , from which}$$

(i) $$\frac{||F - H||}{||F||} = \frac{||F'||}{(2L + 1)||F'||} = \frac{1}{2L + 1} < \frac{1}{2L} \quad \text{and}$$

(ii) $$\frac{||F - (H \mp y)||}{||F||} < \frac{1}{2L} .$$

Let $H' = H \mp y$, then $\mu_{H'}(A \mp y) = \mu_H(A)$. We now write $F = (F-H) \cup H$ which is in a form to which we can immediately apply Lemma 3.35.

Using the lemma, with $\epsilon = \frac{1}{2L}$ we obtain from inequality (i) above that

(iii) $|\mu_H(A) - \mu_F(A)| < \frac{1}{2L}$ and similarly

$|\mu_{H'}(A \mp y) - \mu_F(A \mp y)| < \frac{1}{2L}$, that is

(iv) $|\mu_H(A) - \mu_F(A \mp y)| < \frac{1}{2L}$.

By combining inequalities (iii) and (iv) we now obtain

$$|\mu_F(A) - \mu_F(A \dotplus y)| < \frac{1}{L}, \quad \text{as required.}$$

Having established the necessary construction aids above, we now define the relation Q mentioned at the beginning of section 3.3 and point out that the four parts of the definition correspond exactly to those of 3.28 (i) - (iv) as required.

Definition 3.37 Let $Q(\langle 0,n,\epsilon,x,y\rangle, F)$ be the relation which holds between a sample F and a quintuple consisting of an open set 0, an integer $n \in N$, a positive real number ϵ, a point $x \in S$ and an irrational real number y, if and only if

(i) $x \in F$

(ii) F is an n-invariant sample

(iii) $|\mu_F(0) - L(0)| < \epsilon$, where $L(0)$ denotes the Lebesgue measure of 0

(iv) $|\mu_F(A) - \mu_F(A \dotplus y)| < \epsilon$ for every $A \subseteq S$.

Theorem 3.38 The relation Q as defined above is concurrent and hence both premeasures and measures exist in $*(\hat{R})$.

To show that Q as defined is concurrent we must exhibit a sample F which satisfies Q for the finite collection of quintuples $(0_1, n_1, \varepsilon_1, x_1, y_1)$ .... $(0_s, n_s, \varepsilon_s, x_s, y_s)$ in its domain.

Using Lemma 3.34 we obtain a set of independent points $z_1, z_2, \ldots, z_t$ and an integer T_1, so that each $y_1, \ldots, y_s$ can be written in the form

$$y_i = \sum_{j=1}^{t} k_j z_j, \quad \text{for integers } |k_j| \leq T_1.$$

With $\varepsilon = \min(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_s)/4$ choose $T_2 \in N$ such that $\frac{1}{T_2} < \varepsilon$ and let $T = tT_1T_2$.

Let W be the finite collection of open sets consisting of all sets of the form $0_i \mp \sum_{j=1}^{t} k_j z_j$, where now $|k_j| \leq T$. Enumerating the elements of W we can write $W = \{W_1, W_2, \ldots, W_r\}$, where each W_i is an open set and as such can be represented as a countable union of disjoint open intervals $W_i = \bigcup_j I_{ij}$. Fix an integer J such that for each i,

$$L\left(\bigcup_{j>J} I_{ij}\right) < \frac{\varepsilon}{2^i}$$ and note that as a consequence, letting

$$0 = \bigcup_{\substack{j>J \\ 1 \leq i \leq r}} I_{ij} \text{, yields } L(0) \leq \varepsilon \sum_{i=1}^{r} \frac{1}{2^i} \leq \varepsilon .$$

Let J' be any integer $> \frac{J}{\varepsilon}$ and let n be the product $J'n_1 \; n_2 \; \ldots \; n_s$. Now any sample which is n-invariant is n_i-invariant for each i, $1 \leq i \leq s$. To see this note that if x belongs to a particular sample then so does $x \mp \frac{k}{n}$, $k \in N$ and we have n_i invariance by choosing an integer $k = \left(J'n_1 \; n_2 \; \ldots \; n_s\right) / n_i$ since then $x \mp \frac{1}{n_i}$ beongs to the sample. We now construct an n-invariant sample H containing all the x_i's by putting

$$H = \left\{x_i \mp \frac{k}{n} : \; 1 \leq i \leq s \; , \; k \in N\right\} .$$

However, since we have no control over the measure it assigns to 0 we construct another n-invariant sample K satisfying conditions (ii) - (iv) of definition 3.37 and large enough to overwhelm H. The final sample will then be $H \cup K$ and we proceed inductively with our construction.

Let p be the integer such that $\frac{p}{n} \leq L(0) < \frac{p+1}{n}$. Using 3.31 we can choose a point $v \in S$ such that at most p of the points of

$$T(v) = \left\{v \mp \frac{k}{n} : \; k \in N\right\} \text{, lie in } 0 .$$

Let $F_1 = T(v)$, then

$$\mu_{F_1}(0) = \frac{\|0 \cap F_1\|}{\|F_1\|} \leq \frac{p}{\|F_1\|} \ .$$

Since $\|F_1\| = n$ and $\frac{p}{n} \leq L(0) \leq \epsilon$ by our choice of p, we have $\mu_{F_1}(0) \leq \epsilon$.

By Lemma 3.33 the samples $F_1 \mp \sum_{i=1}^{t} k_i z_i$ are all pairwise disjoint.

Let $Z_1 = \left\{x \mp \sum_{i=1}^{t} k_i z_i \ : \ x \in F_1 \ , \quad k_i \ \text{integers}\right\}$.

Since F_1 is finite (it has n distinct elements), Z_1 is countable and $L(Z_1) = 0$. Hence we can apply the corollary to Lemma 3.31 to Z_1 and obtain a point v_1 such that at most p of the points in

$$T(v_1) = \left\{v_1 \mp \frac{k}{n} : \ k \in N\right\}$$

lie in 0 and none lie in Z_1.

Suppose now inductively that F_u has been chosen so that $\mu_{F_u}(0) \leq \epsilon$ and the samples $F_u \mp \sum_{i=1}^{t} k_i z_i$ are pairwise disjoint. Let

$$Z_u = \left\{x \mp \sum_{i=1}^{t} k_i z_i \ : \ x \in F_u \ , \quad k_i \ \text{integers}\right\} \ .$$

As Z_u is countable $L(Z_u) = 0$ and we can obtain a point v_u such that at most p of the points in $T(v_u)$ lie in 0 and none in Z_u, that is

$$T(v_u) \cap \left(F_u \mp \sum_{i=1}^{t} k_i z_i\right) = \phi \ .$$

Let $F_{u+1} = F_u \cup T(v_u)$ and k_i', k_i'' be integers, $1 \leq i \leq t$. We are required to prove that the samples $F_{u+1} \mp \sum k_i z_i$ are pairwise disjoint, and do so by showing that $\left\|\left(F_{u+1} \mp \sum k_i' z_i\right) \cap \left(F_{u+1} \mp \sum k_i'' z_i\right)\right\|$ is zero. Since we have

$$\left\|\left(F_{u+1} \dot{+} \sum k_i' z_i\right) \cap \left(F_{u+1} \dot{+} \sum k_i'' z_i\right)\right\| = \left\|\left(F_{u+1} \dot{+} \sum k_i z_i\right) \cap F_{u+1}\right\| ,$$

where $k_i = k_i' - k_i''$ and $F_{u+1} = F_u \cup T(v_u)$ by definition, we substitute for F_{u+1} to obtain:

$$\left(F_{u+1} \dot{+} \sum k_i z_i\right) \cap F_{u+1} = \left[\left(F_u \dot{+} \sum k_i z_i\right) \cup \left(T(v_u) \dot{+} \sum k_i z_i\right)\right]$$

$$\cap \left[F_u \cap T(v_u)\right]$$

$$= \left[\left(F_u \dot{+} \sum k_i z_i\right) \cap F_u\right] \cup \left[\left(T(v_u) \dot{+} \sum k_i z_i\right) \cap T(v_u)\right]$$

$$\cup \left[\left(F_u \dot{+} \sum k_i z_i\right) \cap T(v_u)\right] \cup \left[F_u \cap \left(T(v_u) \dot{+} \sum k_i z_i\right)\right] .$$

Now the first term is ϕ by our inductive hypothesis. Term 2 is ϕ by Lemma 3.33, and v_u was chosen in such a way that term 3 is ϕ. Finally note that if $f_1, f_2, \ldots, f_r \in F_u \cap \left(T(v_u) \dot{+} \sum k_i z_i\right)$ then each $f_\ell = v_u \dot{+} \sum k_i z_i$, $1 \leq \ell \leq r$, so that

$v_u = f_\ell - \sum k_i z_i$ and the number of elements in

$F_u \cap \left(T(v_u) \dot{+} \sum k_i z_i\right)$ and $\left(F_u - \sum k_i z_i\right) \cap T(v_u)$ are the same and we have,

$$\left\|F_u \cap \left(T(v_u) \dot{+} \sum k_i z_i\right)\right\| = \left\|\left(F_u - \sum k_i z_i\right) \cap T(v_u)\right\| .$$

Since we can write $F_u - \sum k_i z_i = F_u \dot{+} \sum(-k_i) z_i$, we see that the fourth term reduces to $\left(F_u \dot{+} \sum(-k_i) z_i\right) \cap T(v_u)$ which is ϕ by construction so that $F_{u+1} \dot{+} \sum k_i z_i$ are pairwise disjoint as required.

Since F_u has $u\,n$ elements, we have $\frac{\|H\|}{\|F_u\|} = \frac{s\,n}{u\,n} = \frac{s}{u}$, so by choosing $u > \frac{s}{\epsilon}$ we have $\frac{\|H\|}{\|F_u\|} < \epsilon$. Thus putting $G = F_\eta$, $\eta > \frac{s}{\epsilon}$ we obtain a sample G which is n-invariant and for which $\frac{\|H\|}{\|G\|} < \epsilon$.

Further we have $G \bar{+} \sum k_i z_i$ disjoint and $\mu_G(0) \leq \epsilon$.

Let
$$K = \bigcup_{|k_i| \leq T} \left(G \bar{+} \sum_{i=1}^{t} k_i z_i\right)$$
$$= \left\{g \bar{+} \sum_{i=1}^{t} k_i z_i \ : \ g \in G \ , \ |k| \leq T\right\}$$

and put $F = H \cup K$. Since G is finite, so is K and so is F .

We now have to show that F as constructed satisfies all of the requirements of Definition 3.37. Condition (i) is satisfied by F since it is already satisfied by H . Also note that H is n-invariant by construction and K is n-invariant since G is. Thus F is n-invariant which implies, as shown earlier, that F is n_i-invariant, $1 \leq i \leq s$. It now remains to show that $|\mu_F(0) - L(0_i)| < \varepsilon_j$ and $|\mu_F(A) - \mu_F(A \bar{+} y_j)| < \varepsilon_i$, $1 \leq i,j \leq s$.

Returning to our earlier set W , if W_i is a set in W , then
$$W_i = \bigcup_j I_{ij} = \left(\bigcup_{j \leq J} I_{ij}\right) \cup \left(\bigcup_{j > J} I_{ij}\right) \quad \text{and}$$
$$\mu_G(W_i) = \mu_G\left(\bigcup_{j \leq J} I_{ij}\right) + \mu_G\left(\bigcup_{j > J} I_{ij}\right) .$$

Now since $\bigcup_{j>J} I_{ij} \subset 0$ and $\mu_G(0) \leq \varepsilon$ we obtain $\mu_G\left(\bigcup_{j>J} I_{ij}\right) \leq \varepsilon$, by property 3.2.3(ii). Using this above yields
$$\left|\mu_G(W_i) - \mu_G\left(\bigcup_{j \leq J} I_{ij}\right)\right| \leq \varepsilon .$$

Similarly since $L(W_i) = L\left(\bigcup_{j \leq J} I_{ij}\right) + L\left(\bigcup_{j>J} I_{ij}\right)$ and
$$L\left(\bigcup_{j>J} I_{ij}\right) < \frac{\varepsilon}{2^i} < \varepsilon \quad \text{for} \quad 1 \leq i \leq r ,$$

we have $\left|L(W_i) - L\left(\bigcup_{j \leq J} I_{ij}\right)\right| \leq \varepsilon$. Combining these results and noting that G is n-invariant we have by 3.25 (iii) that

$$\left|\mu_G(I_{ij}) - L(I_{ij})\right| < \frac{1}{n}, \quad \text{and hence}$$

$$\left|\mu_G\left(\bigcup_{j\le J} I_{ij}\right) - L\left(\bigcup_{j\le J} I_{ij}\right)\right| \le \frac{J}{n} < \varepsilon ,$$

by countable additivity and our choice of n. Combining the above results we get $|\mu_G(W_i) - L(W_i)| < 3\varepsilon$.

Now let X be the set of all points of the form

$$y = \sum_{i=1}^{t} k_i z_i \quad \text{where} \quad |k_i| \le T .$$

Note that for each of our original 0_i's and each $y \in X$ we have $\mu_{G\mp y}(0_i) = \mu_G(0_i - y)$ and since $(0_i - y) \in W$, (i.e. $(0_i - y)$ is one of the W_i above) we obtain $|\mu_{G\mp y}(0_i) - L(0_i)| < 3\varepsilon$. Now since G was chosen so that the sets of the form $G \mp y$, $y \in X$ are pairwise disjoint, and $K = \bigcup_{y\in X} (G \mp y)$ it follows that

$$\mu_K(0_i) - L(0_i) = \frac{\sum_{y\in X} \|G \mp y \cap 0_i\|}{\sum_{y\in X} \|G \mp y\|} - L(0_i)$$

$$= \frac{\sum_{y\in X}\left(\|G \mp y \cap 0_i\| - \|G \mp y\| L(0_i)\right)}{\sum_{y\in X} \|G \mp y\|}$$

$$= \frac{1}{\sum_{y\in X} \|G \mp y\|} \sum_{y\in X} \|G \mp y\| \left(\frac{\|G + y \cap 0_i\|}{\|G \mp y\|} - L(0_i)\right)$$

but $|\mu_{G\mp y}(0_i) - L(0_i)| < 3\epsilon$ from above so that

$$|\mu_K(0_i) - L(0_i)| < 3\epsilon .$$

By Lemma 3.35 we obtain $|\mu_F(0_i) - \mu_K(0_i)| < \varepsilon$, so that

$$|\mu_F(0_i) - L(0_i)| < 4\varepsilon < \varepsilon_j \quad \text{for} \quad 1 \le i,j \le s .$$

Thus F satisfies condition (iii) of Definition 3.37. Turning now to our last requirement on F, for each z_i let

$$K_i = \left\{ v \mp \sum_{j=1}^{t} k_j z_j \ : \ v \in G\ , \ \ |k_j| \leq T\ , \ \ k_i = 0 \right\} .$$

Applying Corollary 3.36 to K_i above and the set $K = \{x \mp kz_i \ : \ x \in K_i\ , \ |k_j| \leq T\}$ we obtain $|\mu_K(A) - \mu_K(A \mp z_i)| < \frac{1}{T}$ for any set A. Now since each $y_i = \sum_{i=1}^{t} k_i z_i$, with $|k_i| \leq T_1$ we get by repeated application of the above inequality, with the triangle inequality that

$$|\mu_K(A) - \mu_K(A \mp y)| < \frac{tT_1}{T} \leq \frac{1}{T_2} < \varepsilon ,$$

by our earlier choice of the constants $T, T_1, T_2 \in N$. Since $|\mu_K(A) - \mu_F(A)| \leq \varepsilon$ and hence $|\mu_K(A \mp y_i) - \mu_F(A \mp y_i)| \leq \varepsilon$ we finally obtain

$$|\mu_F(A) - \mu_F(A + y_j)| < 3\varepsilon < \varepsilon_i , \quad 1 \leq i \leq s .$$

Thus F satisfies all the requirements of Definition 3.37 and our proof is complete. That is, F as constructed is a sample which finitely satisfies the relation Q for the set of quintuples

$$<0_1, n_1, \varepsilon_1, x_1, y_1>, \ \ldots\ldots, <0_s, n_s, \varepsilon_s, x_s, y_s> ,$$

in the domain of Q.

This shows that Q is concurrent as defined and our enlarging monomorphism Φ guarantees the existence of a *- finite premeasure F' and its associated measure in $*(\hat{R})$ such that F' simultaneously satisfies the relation $*Q$ for all standard objects in its domain. Note that $F' \neq \Phi(F) = *F$ for any sample as constructed above. Indeed since $F \in \hat{R}$ is finite in any of these cases we would simply obtain the same sample after transformation by Φ. Informally we can think of F'

as a specific limiting sample F_λ , based on the above construction with $\lambda \in {}^*N - N$, since this serves as an aid in visualizing the construction and *- finiteness of our premeasure in $^*(R)$. At this stage we modify our notation, as a matter of convenience, and in all subsequent work deal simply with a premeasure F on $^*S$.

In this section we have been successful in producing a measure for a finite interval, and it is evident from our construction that a similar approach will produce a measure on the entire real line. In [2] Bernstein constructs such a measure. Here we only outline his approach since in Chapter 4 we prefer to work with measures defined for sets $A \subseteq {}^*R$ reduced mod 1.

Denoting the general interval $[-n,n)$ by S_n and writing $\bar{+}$ to denote summation modulo $2n$, we see that the sequence $\{S_n\}$ of intervals extends to $\Phi(\{S_n\}) = {}^*\{S_n\}$ in $^*(\hat{R})$ and for any $d \in {}^*N$,

$$^*S_d = \{x \in {}^*R : -d \leq x < d\} .$$

With the requisite modifications of our earlier technical lemmas Bernstein shows that there exists a $\eta \in {}^*N - N$ and a *- finite $G \subseteq {}^*S_\eta$ such that

(i) $r \in G$ for all $r \in R$

(ii) G is n-invariant on $^*S_\eta$ for all $n \in N$

(iii) $\mu_G(^*A) \simeq L(A)$ for Lebesgue measurable $A \subseteq R$

(iv) $\mu_G(A) \simeq \mu_G(A \bar{+} y)$ for every $y \in R$ and $A \subseteq {}^*S_\eta$.

He then shows that G as above represents a premeasure on the extended real line $^*S_\eta \supset R$ so that for our earlier premeasure F on $^*S$ we have the relationship

$$||G|| = 2\eta||F|| .$$

CHAPTER 4. NON-STANDARD MEASURES OF SETS ON THE REAL LINE

Standard Lebesgue measure does not distinguish between various denumerable sets, nor even between a denumerable set and sets such as Cantor's ternary set.

Here we will show that in the sense of F we can assign non-standard cardinals as upper and lower bounds for the number of points in such sets, so that in each case we are able to find an approximate infinitesimal measure for the set in question. As the standard part of these measures must agree with the Lebesgue measure where the latter is defined, we also find a more accurate relationship between $||F|| = \nu$ and the non-standard cardinals of the various sets under investigation.

4.1 A NON-STANDARD MEASURE

For all subsets B of S we define a non-standard measure μ as:

Definition 4.11 $\mu(B) = \mu_F(*B) = \dfrac{||*B \cap F||}{||F||}$, where F is as defined at the end of Chapter 3.

Note that in considering any $B \subseteq S$, we are guaranteed that $*B$ is internal since it is a Φ- standard set. From the above definition we see that our measure depends entirely on $||F \cap *B||$ which we shall refer to as the non-standard cardinal of B corresponding to F . Since $||F|| = \nu$ we have in general that $||F \cap *B|| = \nu\mu(B)$ and note here that $*B$ is not necessarily a subset of F , (e.g. put $B = S$).

Before proceeding to find actual non-standard measures of particular sets we must show that all $B \in P(S)$ are measurable in the sense of 4.11. To show this we must prove that $||*B \cap F||$ is always defined for all $B \in P(S)$, that is that $*B \cap F$ is $*$- finite. This follows directly from the fact that in $\hat{R}$ the intersection of two sets, one of which is finite, is itself finite. Writing this more formally we have that in $\hat{R}$,

<u>4.12</u> $(\forall u) : u \in P(S) \supset : (\forall v) . v \in X \supset u \cap v \in X$, where $X \in \hat{R}$ is the set of all finite subsets of R . Under our monomorphism Φ , 4.12 is transformed into the true statement

$$(\forall u) : u \in {}^*P(S) \supset : (\forall v) . v \in {}^*X \supset u \cap v \in {}^*X ,$$

in ${}^*(\hat{R})$. Since $B \in P(S)$ we have ${}^*B \in {}^*P(S)$ so that with $F \in {}^*X$ we obtain ${}^*B \cap F \in {}^*X$, that is ${}^*B \cap F$ is *- finite.

4.2 *- FINITE CARDINALS

In this section we use the fact that $*(\hat{R})$ is an enlargement to obtain some results for non-standard cardinals.

For a given set E, consider the relation

$$K = \{(x,y) : x \in y \text{ and } y \text{ is a finite subset of } E\} .$$

Clearly K is concurrent on E and since $*(\hat{R})$ is an enlargement there is a *- finite subset I of $*E$ satisfying $(*e,I) \in K$, that is $*e \in I$ for each $e \in E$.

Let $\Omega \in *N - N$, and put $E = N = \{1,2,\ldots\}$ in the above. Then we can embed N into the internal *- finite subset $N_\Omega = \{1,2,3,\ldots,\Omega\}$ of $*N$. Since $N \notin *P(R)$ the embedding is external, in fact any embedding as above is external whenever E is an infinite standard set. That N_Ω is internal follows since it is a *- finite set of individuals $n \in *N$, $1 \leq n \leq \Omega$. Note that N is externally infinite, but since $N \subset N_\Omega$ for any $\Omega \in *N - N$, we can informally write $\|N\| < \Omega$.

Let $g(x) = \frac{1}{x}$, for $x \in R - \{0\}$. In $\hat{R}$ we know that for a specific finite set of positive integers N_a, g maps N_a onto the finite set $g(N_a)$, where $\|N_a\| = \|g(N_a)\|$.

Since Φ preserves these properties, the extension $\Phi(g) = *g$ maps *- finite sets onto *- finite sets having the same number of elements. With X the set of all finite subsets of R we can write this formally as:

$$(\forall Y)\big(Y \in X \supset g(Y) \in X \wedge \|Y\| = \|g(Y)\|\big) ,$$

where $g(Y) = \left\{\frac{1}{y} : y \in Y\right\}$. Transforming this we obtain

4.21 $$(\forall Y)\big(Y \in *X \supset *g(Y) \in *X \wedge \|Y\| = \|*g(Y)\|\big) ,$$

that is $*g(Y)$ is *- finite.

Let $A = \left\{\frac{1}{n} : n \in N\right\}$, then $A \subset S$ so $*A \subset *S$. With mesh $F = \sigma$ we can write $*A = \left\{\frac{1}{n} : n \in *N\right\}$ as

$$*A = \left\{\frac{1}{n} : *N \ni n \lesssim \sigma\right\} \cup \left\{\frac{1}{n} : \sigma < n \in *N\right\} .$$

Since F is n-invariant for $*N \ni n \lesssim \sigma$ we have $\left\{\frac{1}{n} : *N \ni n \lesssim \sigma\right\} \subset F$ and so

$$\|*A \cap F\| = \sigma + \left\| \left\{\frac{1}{n} : \sigma < n \in *N\right\} \cap F\right\| .$$

Clearly $\left\{\frac{1}{n} : \sigma < n \in *N\right\} \cap F \neq \phi$ since for example $\frac{1}{\sigma(\sigma-1)} \in F$. Further, with X as above, we have in $\hat{R}$ the true statement:

$(\forall y) . y \in X \supset (\exists n) n \in N \wedge \frac{1}{n} \notin y$, so that by transforming we obtain

$$(\forall y) . y \in *X \supset (\exists n) n \in *N \wedge \frac{1}{n} \notin y .$$

This shows that there is a $\tau \in *N - N$ with $*A \ni \frac{1}{\tau} \notin F$, so that $\left\{\frac{1}{n} : \sigma < n \in *N\right\} \nsubseteq F$. However since F is *- finite there exists a least element $q = \inf. *A \cap F$, where $q = \frac{1}{\rho}$, $\rho \in *N - N$. Setting

$$\rho' = \sup\{x : x \in *N \wedge (\forall y) . y \lesssim x \wedge y \in *N \supset \frac{1}{y} \in *A \cap F\}$$

we thus have the inequality.

<u>4.22</u> $\quad \sigma \lesssim \rho' \lesssim \rho$.

Note here that if $\sigma = \rho'$, we obtain $\rho' < \rho$ by our remarks above; however observe that we have explicit information only about those points of the set $\left\{\frac{1}{n} : n \in *N \wedge \sigma < n \lesssim \rho\right\}$ which can be generated as m-invariant translates $(\sigma \gtrsim m \in *N)$ of points in the set $\left\{\frac{1}{n} : *N \ni n \lesssim \sigma\right\}$. Allowing as well for points $\frac{1}{\lambda}$, $\lambda \in *N$ for which we may have $*A \cap F \not\ni \frac{1}{\lambda} \in *A$, $\rho' < \lambda < \rho$ we thus find ourselves restricted to the general inequality 4.22.

4.23 If we let $\|{}^*A \cap F\| = \omega$, we then have

$$\rho' \lesssim \omega = \sigma + \left\| \left\{\frac{1}{n} : \sigma < n \lesssim \rho\right\} \cap F \right\| \lesssim \rho ,$$

that is $\sigma < \|{}^*A \cap F\| \lesssim \rho$.

Note here that although $\left\{\frac{1}{n} : n \lesssim \sigma\right\}$ is not the Φ-transform of any standard set, it is still an internal, *- finite subset of *S and can be assigned a measure using definition 3.21. Within P(*S) we can regard it informally as the σ-extension of the standard set A as above.

We now establish bounds for non-standard cardinals of other subsets of S in terms of both σ and ρ . Specifically we consider the following subsets B,C,D of S .

Let

$$B = \left\{ 1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \ldots \right\}$$

$$C = \left\{ \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots \right\} \quad \text{and}$$

$$D = \left\{ 1, \frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}, \frac{1}{16}, \ldots \right\} .$$

Transforming we see that

$${}^*B = \left\{\frac{1}{n} : n \in {}^*N \wedge 2|n-1\right\} \quad \text{so that}$$

$${}^*B \cap F = \left\{ \left\{\frac{1}{n} : {}^*N \ni n \lesssim \sigma \wedge 2|n-1\right\} \cup \left\{\frac{1}{n} : \sigma < n \lesssim \rho \wedge 2|n-1\right\} \right\} \cap F$$

and $\|{}^*B \cap F\| = \left[\frac{\sigma}{2}\right] + \left\| \left\{\frac{1}{n} : \sigma < n \lesssim \rho \wedge 2|n-1\right\} \cap F \right\|$, that is

$$\left[\frac{\sigma}{2}\right] \lesssim \|{}^*B \cap F\| \lesssim \left[\frac{\rho+1}{2}\right] , \quad \text{where}$$

[...] stands for the "integral part of" operation.

As it is conceivable that all odd points $\frac{1}{n}$ belong to F ,$^*N \ni n \leqslant \rho$, but only few of the even ones do for $n > \sigma$ we could have at worst that

$$\|{}^*A \cap F\| \approx \sigma + \left[\frac{\rho-\sigma}{2}\right] = \left[\frac{\rho+\sigma}{2}\right] \quad \text{and}$$

$$\|{}^*B \cap F\| = \left[\frac{\sigma}{2}\right] + \left[\frac{\rho-\sigma}{2}\right] \approx \left[\frac{\rho}{2}\right] .$$ We certainly have

$\rho \geqslant \sigma(\sigma-1)$ so that $\|{}^*A \cap F\|$ may be approximately equal to $\|{}^*B \cap F\|$ rather than double.

Transforming the set C we obtain ${}^*C = \left\{ \frac{1}{2^n} : n \in {}^*N \right\}$ so that

$$[\log_2\sigma] \lesssim \|{}^*C \cap F\| \lesssim [\log_2\rho] .$$

For the set D above, representing the set of all finite decimals in base 2 belongong to S , we see that since for k , $n \in {}^*N$ and $k \lesssim n \lesssim \sigma$ all points $\frac{k}{n} \in F$ we obtain

$$2^{[\log_2\sigma]} \lesssim \|{}^*D \cap F\| \lesssim 2^{[\log_2\rho]} \quad \text{, so that}$$

$$1 + \left[\frac{\sigma}{2}\right] \lesssim \|{}^*D \cap F\| \lesssim \rho .$$

Although one might expect $\|{}^*D \cap F\|$ to be larger than $\|{}^*A \cap F\|$, we observe that on the basis of the bounds it could well be smaller.

Note that A,B,C and D are all Lebesque measurable subsets of S so that by 3.28(iii) we require the standard part of each of their non-standard measures to be zero so that $\nu > n\omega$ for all $n \in N$.

4.3 NON-STANDARD MEASURES ON R

Our non-standard measure μ as defined by 4.11 can be immediately applied to all subsets B of S. However, we want to assign non-standard measures to arbitrary subsets of the real line and indicated in Chapter 3 that there are several ways of achieving this. In [2] Bernstein constructs a premeasure for a non-standard interval $*S_\lambda = \{x \in *R : -\lambda \leqslant x < \lambda\}$, where $\lambda \in *N - N$ so that $R \subset *S_\lambda$, to obtain the required result. Here we prefer to modify each subset $B \subseteq R$ by reducing it mod 1 and then applying Definition 4.11.

For bounded sets $B \subseteq R$ let $a = \inf. B$ and $b = \sup. B$. Then with $m = [a]$ (the integral part of a) and $n = [b] + 1$ we reduce B mod 1 and have

4.31 $$\mu(B) = \sum_{i=m+1}^{n} \mu\big((B - i + 1) \cap S\big) \quad ,$$

where we simplify our notation by writing $B_i = (B - i + 1) \cap S$ in the above expression.

In line with 4.22 we now fix our extended real line $*R$ to be of length 2ω, so that for unbounded sets $B \subseteq R$, 4.31 reduces to

4.32 $$\mu(B) = \sum_{i=-\omega+1}^{\omega} \mu(B_i) \quad .$$

Applying this to $N = \{1,2,...\}$ we have

$$\mu(N) = \mu_F(*N) = \sum_{i=-\omega+1}^{\omega} \mu_F(N_i) \quad ,$$

where $N_i \cap F = 1 \quad , \quad 1 \leqslant i \leqslant \omega$

$N_i \cap F = \phi \quad , \quad -\omega+1 \leqslant i < 1 \ .$

Thus $\mu(N) = \frac{1}{\nu} \sum_{i=1}^{\omega} 1 = \frac{\omega}{\nu}$, which shows that in the F sense, as defined in 4.32 the number of elements of $A = \left\{1, \frac{1}{2}, \frac{1}{3}, \ldots\right\}$ is the same as that of $\{1,2,3,\ldots\}$.

For arbitrary closed intervals $A = \left[a, a + \frac{p}{q}\right]$ we have A bounded by integers $m = [a]$ and $n = \left[a + \frac{p}{q}\right] + 1$ so that in these cases we are really only concerned with the measure of intervals $\left[0, \frac{r}{s}\right]$, $r \leq s$ obtained by applying Definition 4.11; the result in a specific case being obtained by using 4.31 first. Let $A = \left[0, \frac{r}{s}\right) \subset S$, then $*A \subset *S$ and

$$\begin{aligned} \mu\left(\left[0, \frac{r}{s}\right)\right) &= \mu_F\left(*\left[0, \frac{r}{s}\right)\right) \\ &= \frac{\left\|*\left[0, \frac{r}{s}\right) \cap F\right\|}{\|F\|} \\ &= \frac{r}{s}, \quad \text{by 3.25 (ii)} . \end{aligned}$$

Now $st\left(\frac{r}{s}\right) = \frac{r}{s} = L(A)$, as required since A is a Lebesgue measurable standard set. In general we thus have $\mu_F\left(*\left[a, a + \frac{p}{q}\right)\right) = \frac{p}{q}$ and

<u>4.33</u> (i) $\left\|*\left[a, a + \frac{p}{q}\right) \cap F\right\| = \frac{p\nu}{q}$; thus

(ii) $\left\|*\left[a, a + \frac{p}{q}\right] \cap F\right\| = \frac{p\nu}{q} + 1$ and

(iii) $\left\|*\left(a, a + \frac{p}{q}\right) \cap F\right\| = \frac{p\nu}{q} - 1$.

<u>4.35</u> <u>EXAMPLE</u>

The preceding results have an immediate application to non-standard probability. To see this consider the interval $[0,\frac{1}{2}) = [0,\frac{1}{4}) \cup [\frac{1}{4},\frac{1}{2})$. Since μ is translation invariant through standard rational distances,

$$\mu([0,\tfrac{1}{2})) = 2\mu([0,\tfrac{1}{4})) \quad \text{and}$$

$$\|{}^*[0,\tfrac{1}{2}) \cap F\| = 2\|{}^*[0,\tfrac{1}{4}) \cap F\| \quad .$$

We are thus in a position to state that corresponding to F there are twice as many points in $[0,\frac{1}{2})$ as there are in $[0,\frac{1}{4})$. This allows us to compare the probabilities of picking a particular real y belonging to both intervals, by using a non-standard probability function β . Here the non-standard probability β of picking a given real in the standard interval $[0,\frac{1}{4})$ can be defined as

$$\frac{1}{\|{}^*[0,\tfrac{1}{4}) \cap F\|} \quad .$$

We are thus able to say that the chance of picking a particular real in the interval $[0,\frac{1}{4})$ is twice that of picking it in the interval $[0,\frac{1}{2})$, where the "chance of picking" has been defined in terms of the non-standard probability β as given above. With respect to standard probabilities we know that our chance of picking a given real in any interval is zero, because there are infinitely many points to choose from. This violates the intuitive feeling that after all there is some chance of picking the point and the above example provides a solution for this dilemma.

4.4 THE SET OF RATIONALS IN THE UNIT INTERVAL

Since the B_i's in 4.33 are all standard subsets of S, we now deal with subsets of S in greater detail before finding bounds for the non-standard measure of the set of rationals in S.

In Chapter 3 we had property 3.22 (iii) which states that if $\{A_i\}_{i\in *N}$ is any sequence of internal disjoint subsets of $*S$, then there is a non-standard integer L such that if $i > L$ then

$$\mu_F(A_i) = 0 \quad \text{and} \quad \mu_F\left(\bigcup_{i\in *N} A_i\right) = \sum_{i=1}^{L} \mu_F(A_i) \quad .$$

For any $A \subseteq {}^*S$, in particular $A = {}^*B$, $B \subseteq S$, we can write $A = \bigcup_{i\in *N} A_i$, where the A_i's are nonempty disjoint internal subsets of $*S$. Clearly only intersections for which $A_i \cap F \neq \phi$ contribute to the measure and if the A_i can be selected to contain at most one element of F each, we can write:

4.41 $$\mu_F(A) = \mu_F\left(\bigcup_{i\in *N} A_i\right) = \sum_{i=1}^{L'} \mu_F(C_i) \quad ,$$

for $$C_i = \begin{cases} \text{the } A_j \text{ that contains } x_i \\ \phi \text{ if no } A_j \text{ contains } x_i \end{cases} ,$$

where $\{x_1, x_2, \ldots, x_\nu\} = F$, $\quad 1 \leqslant i \leqslant \nu$.

Clearly $C_i \cap F = \phi$ for $i > \nu$ which allows us to write:

4.42 $$A \cap F = \bigcup_{i=1}^{\nu} C_i \ ,$$ where $L' = \nu$ in 4.41, that is

4.43 $$\mu_F(A) = \sum_{i=1}^{\nu} \mu_F(C_i) \quad .$$

In the special case where $C_i = \{x_i\}$ for all $x_i \in F$ we have

$$\mu_F(F) = \sum_{i=1}^{\nu} \mu_F(C_i)$$

$$= \sum_{i=1}^{\nu} \mu_F(\{x_i\})$$

$$= \nu \cdot \frac{1}{\nu} = 1 \quad , \text{ as required.}$$

For general $A \subseteq {}^*S$, we can set the $L \leqslant \nu$ in 3.22 (iii) if our decomposition of A is in terms of its disjoint components C_i as in 4.41. To illustrate this consider $A = \bigcup_{i \in {}^*N} A_i$, where

$$A_i = \phi \ , \quad 1 < i < \nu \quad \text{and} \quad F \supset A_i = \{x_{i-\nu+1}\} \ , \quad \nu \leqslant i < \nu+\delta \ .$$

Then $\|F \cap \bigcup_{i \in {}^*N} A_i\| = \delta$ and $\mu_F\left(\bigcup_{i \in {}^*N} A_i\right) = \frac{\delta}{\nu}$, but

$\sum_{i=1}^{\nu} \mu_F(A_i) = \frac{1}{\nu}$. This is rectified if as required, we set

$C_i = A_{\nu+(i-1)}$, since then $C_i \cap F = \phi$ for $i > \delta$ and

$$\mu_F\,(A) = \sum_{i=1}^{\nu} \mu_F(C_i) \quad \text{reduces to}$$

$$\mu_F\,(A) = \sum_{i=1}^{\delta} \frac{1}{\nu} = \frac{\dot{\delta}}{\nu} \ .$$

To fix a value for L for transformed sets we examine the situation for finite sets in $\hat{R}$ and transform the appropriate statement. For $\bigcup_{i \in N} A_i \subseteq S$, we have in $\hat{R}$ the true statement:

<u>4.44</u> $$(\forall y).y \in X \supset (\exists L)L \in N \wedge \|\bigcup_{i \in N} A_i \cap y\| = \sum_{i=1}^{L} \|A_i \cap y\|$$

$$\wedge\ L = \|\{i : (\exists j)j \in N \wedge i \in N \wedge j \geqslant i \wedge A_j \cap y \neq \phi\}\| \ .$$

Transforming 4.44 we obtain the corresponding statement in ${}^*(\hat{R})$, and using F for the *- finite set $y \in {}^*X$ we have:

<u>4.45</u> $$(\exists L) L \in {}^*N \wedge \left\| \bigcup_{i \in {}^*N} {}^*A_i \cap F \right\| = \sum_{i=1}^{L} \| {}^*A_i \cap F \|$$

$$\wedge\ L = \| \{ i : (\exists j) j \in {}^*N \wedge i \in {}^*N \wedge j \geq i \wedge {}^*A_j \cap F \neq \phi \} \| .$$

In the special case where

<u>4.46</u> $${}^*A_j \cap F \neq \phi \supset (\forall i)(i < j \supset {}^*A_i \cap F \neq 0) ,$$

we have $L = \| \{ i : i \in {}^*N \wedge {}^*A_i \cap F \neq \phi \} \|$

$\leq \nu$, as we saw earlier.

So $L = \nu$ for the case

$F = \bigcup_{i \in {}^*N} A_i$, $A_i = \{x_i\}$, since then $A_i = \phi$ for $i > \nu$; on the other hand $L < \nu$ if B_i is a subset of F for values $i < \nu$ and $B_i \cap F = \phi$, $i \geq \nu$. For example consider the sequence $\{B_i\}_{i \in {}^*N}$, where $B_i = \{x_{3i-2}, x_{3i-1}, x_{3i}\}$, $x_i \in F$, $i \leq \nu$. By definition $B_i \cap F = \phi$ for $i > \left[\frac{\nu}{3}\right] + 1$. Even though ν is fixed, we do not know to which residue class (mod 3) it belongs. However, we can set $L = \left[\frac{\nu}{3}\right] + 1$ and consider the following cases:

(i) $\nu = 3L - 3$ or

(ii) $\nu = 3L - 2$ or

(iii) $\nu = 3L - 1$.

We then have $$\mu_F\left(\bigcup_{i \in {}^*N} B_i \right) = \sum_{i=1}^{L} \mu_F(B_i)$$

$$= (L-1)\left[\frac{3}{\nu}\right] = 1 \quad \text{if} \quad \nu = 3L - 3$$

$$= L\left[\frac{3}{\nu}\right] - \frac{2}{\nu} = 1 \quad \text{if} \quad \nu = 3L - 2$$

$$= L\left[\frac{3}{\nu}\right] - \frac{1}{\nu} = 1 \quad \text{if} \quad \nu = 3L - 1 .$$

So in each case we have $\mu_F\left(\bigcup_{i \in {}^*N} B_i \right) = 1 = \mu_F(F)$, as required. Thus we see that the value of L depends in general on the definition of the sequence of disjoint subsets under consideration; however, if in each case we modify our sequence in an way which allowed us to write 4.42, then we always have $L \leq \nu$.

Let Q_s be the set of rationals in S , that is

$Q_s = Q \cap S = \bigcup_{i \in N} A_i$, $N \ni i \geq 1$ where

$$A_i = \left\{ \frac{j}{i} : i,j \in N \wedge (i,j) = 1 \wedge 1 \leq j \leq i \right\} .$$

Now $Q_s \subset S \supset {}^*Q_s \subset {}^*S$, so ${}^*Q_s = {}^*\left(\bigcup_{i \in N} A_i \right) = \bigcup_{i \in {}^*N} {}^*A_i$,

can be written as a sequence of disjoint subsets of ${}^*S$. Let Φ be the Euler function. Then

$$\Phi(i) = \|A_i\| = \|\{j : j \in N \wedge 1 \leq j \leq i \wedge (i,j) = 1\}\|$$

so that

${}^*\Phi(i) = {}^*\|A_i\| = \|{}^*A_i\|$, since ${}^*\Phi$ is the extension of Φ , which associates with each ${}^*A_i \subset {}^*Q_s$, $i \in {}^*N$, the number of points in ${}^*A_i$ namely $\|{}^*A_i\|$.

Now ${}^*A_i \subset F$ $1 \leq i \leq \sigma$ (mesh of F), so

$$\|{}^*A_i \cap F\| = \|{}^*A_i\| = {}^*\Phi(i) \qquad 1 \leq i \leq \sigma .$$

Thus writing

$$\|{}^*Q_s \cap F\| = \sum_{i=1}^{L} \|{}^*A_i \cap F\|$$

$$= \sum_{i=1}^{\sigma} \|{}^*A_i \cap F\| + \sum_{i=\sigma+1}^{\rho} \|{}^*A_i \cap F\|$$

from section 4.2. So

$$\|{}^*Q_s \cap F\| = \sum_{i=1}^{\sigma} {}^*\Phi(i) + \sum_{i=\sigma+1}^{\rho} \|{}^*A_i \cap F\|$$

$$\leq \sum_{i=1}^{\rho} {}^*\Phi(i)$$

by the n-invariance properties of F. In [6] it is shown that the average order of $\Phi(n)$ is $\frac{6n}{\pi^2}$ and that the inequality

4.47 $$\left| \sum_{i=1}^{n} \Phi(i) - \frac{3n^2}{\pi^2} \right| < n \log n \quad ,$$

due to R. Tambs-Lyche, gives us an asymptotic estimate for the sum of the first n terms of the Euler function.

Applying this in our case we can fix both an upper and lower bound for $\|{}^*Q_s \cap F\|$ by considering the largest possible error in the value of $\sum_{i=1}^{n} \Phi(i)$ from 4.47. Thus we see that

$$\frac{3\sigma^2}{\pi^2} - \sigma \log \sigma \leq \|{}^*Q_s \cap F\| \leq \frac{3\rho^2}{\pi^2} + \rho \log \rho \ , \quad \text{so that}$$

4.48 $$\frac{1}{\nu}\left\{\frac{3\sigma^2}{\pi^2} - \sigma \log \sigma\right\} \leq \mu(Q_s) \leq \frac{1}{\nu}\left\{\frac{3\rho^2}{\pi^2} + \rho \log \rho\right\} .$$

As $L(Q_s) = st\big(\mu(Q_s)\big) = 0$ we have $n\mu(Q_s) < 1$, for all $n \in N$ so that applying this to 4.48 yields

4.49 $$n\left\{\frac{3\sigma^2}{\pi^2} - \sigma \log \sigma\right\} < \nu \ , \quad \text{for all} \quad n \in N \ .$$

4.5 CANTOR'S TERNARY SET

Cantor's ternary set is interesting since it is a non denumerable set of Lebesgue measure zero.

Consider the unit interval and remove in succession the following open intervals

(i) U_1 , the middle third $\left(\frac{1}{3} , \frac{2}{3}\right)$

(ii) U_2 , the middle thirds of the remaining two intervals, viz. $\left(\frac{1}{9} , \frac{2}{9}\right) \cup \left(\frac{7}{9} , \frac{8}{9}\right)$ and continue this process to form the sequence U_i , $i \in N$, of disjoint subsets of S . We see that in general the Lebesgue measure of the open intervals removed at stage i is $L(U_i) = \frac{1}{3}\left(\frac{2}{3}\right)^{i-1}$ and that at the n-th step we have removed intervals of total length

4.51 $$\sum_{i=1}^{n} L(U_i) = 1 - \left(\frac{2}{3}\right)^n .$$

Consider the points remaining after $U_1, U_2, \ldots$ have been removed. These form a set called Cantor's ternary set, which we shall denote by T . Clearly T has Lebesgue measure zero since in 4.51 n can be chosen arbitrarily large. As T is also non denumerable [24] it is an interesting set, as stated earlier, and we now proceed to find upper and lower bounds for its non-standard measure $\mu(T)$.

For $i \in N$, $U_i \subset S$ and we have by transformation that $*U_i$, $i \in *N$ forms a sequence of disjoint internal subsets of $*S$ to which 3.22 (iii) applies. Now at step one we are removing one interval U_1 of measure $\mu(U_1) = \mu_F(*U_1) = \mu_F\left(\left\{x \in *R : \frac{1}{3} < x < \frac{2}{3}\right\}\right)$

$$= \mu_F\left(*\left(\frac{1}{3} , \frac{2}{3}\right)\right)$$

$$= \frac{1}{3} - \frac{1}{\nu} \quad \text{by 4.34 (iii)} .$$

So the measure of intervals $U_1, U_2, \ldots$ removed are

$$\mu(U_1) = \frac{1}{3} - \frac{1}{\nu}$$

$$\mu(U_2) = \frac{2}{3^2} - \frac{2}{\nu} \quad \text{and in general}$$

$$\mu(U_n) = \frac{1}{3}\left(\frac{2}{3}\right)^{n-1} - \frac{2^{n-1}}{\nu} .$$

At the nth step we have thus removed intervals of total measure

$$\sum_{i=1}^{n} \mu(U_i) = \sum_{i=1}^{n} \mu_F({}^*U_i) = 1 - \left(\frac{2}{3}\right)^n - \left(\frac{2^n - 1}{\nu}\right) ,$$

from above and by 4.51.

Now $T = S - \bigcup_{i \in N} U_i$ and since $T \cap \bigcup_{i \in N} U_i = \phi$, we have

<u>4.52</u> $\quad {}^*T = {}^*S - \bigcup_{i \in {}^*N} {}^*U_i$. Since the ${}^*U_i$'s are disjoint, we have by 3.22 (iii) that for some $L \in {}^*N - N$:

$$\mu_F\left(\bigcup_{i \in {}^*N} {}^*U_i\right) = \sum_{i=1}^{L} \mu_F({}^*U_i) .$$

By 4.52

$$\mu(T) = \mu_F({}^*T) = \mu_F({}^*S) - \mu_F\left(\bigcup_{i \in {}^*N} {}^*U_i\right)$$

$$= 1 - \sum_{i=1}^{L} \mu_F({}^*U_i) \quad \text{from above.}$$

Now $\mu_F({}^*U_i) = \frac{1}{\nu} \|{}^*U_i \cap F\|$ and with $\lambda \in {}^*N$, for $3^\lambda < \sigma$ (mesh of F) $\wedge$ $0 \leq i,j \leq 3^\lambda$ we have

$$\left\|\left(\frac{i}{3^\lambda}, \frac{i+1}{3^\lambda}\right) \cap F\right\| = \left\|\left(\frac{j}{3^\lambda}, \frac{j+1}{3^\lambda}\right) \cap F\right\| = \frac{\nu}{3^\lambda} - \frac{1}{\nu} .$$

Now the largest λ can be above, such that $3^\lambda < \sigma$, is $\lambda = [\log_3 \sigma] \in {}^*N - N$, so that

$$\sum_{i=1}^{L} \mu_F(*U_i) \geq \sum_{i=1}^{\lambda} \mu_F(*U_i)$$

$$= 1 - \left(\frac{2}{3}\right)^{\lambda} - \frac{(2^{\lambda} - 1)}{\nu} \quad .$$

Thus

<u>4.53</u> $$\mu(T) = \mu_F(*S) - \mu_F\left(\bigcup_{i\in *N} *U_i\right) \leq \left(\frac{2}{3}\right)^{\lambda} + \frac{2^{\lambda}}{\nu} - \frac{1}{\nu} \; .$$

Note here that each step i in our construction involves the interval $\left(\frac{1}{3^i}, \frac{2}{3^i}\right)$ and that for $*N \ni i \geq [\log_3 \rho] + 1$ none of the end-points $\frac{1}{3^i} \in *T$ belong to F, although there may be points of F in some of the components of U_i even for $i \geq [\log_3 \rho] + 1$.

To fix a lower bound for $\mu(T)$ we use the fact that at each step i of our construction we guarantee the presence of 2^i more (end)-points in T. We also know that by the n-invariance of F, all end-points of the components of each U_i , $i \leq \lambda$ belong to F, so that $\|*T \cap F\| \geq \sum_{i=1}^{\lambda} 2^i$. As we also have $0, \frac{1}{4}$ and $\frac{3}{4} \in *T$ we thus have

$2^{\lambda+1} < \|*T \cap F\|$, and combining this with 4.53 we now obtain

<u>4.54</u> $$\frac{2^{\lambda+1}}{\nu} < \mu(T) \leq \left(\frac{2}{3}\right)^{\lambda} + \frac{2^{\lambda}}{\nu} - \frac{1}{\nu} \quad .$$

Since T is a standard set of Lebesgue measure zero we have

$$L(T) = 1 - \lim_{n\to\infty} \sum_{i=1}^{n} L(U_i)$$

$$= st\big(\mu(T)\big) \quad .$$

Chapter 3 guarantees that st $\mu(T) = 0$ so that applying the standard part operation to 4.54 and noting that $\left(\frac{2}{3}\right)^{\lambda} - \frac{1}{\nu}$ is infinitesimal we have

4.55 $$n\ 2^{\lambda} < \nu\ , \quad \text{for all} \quad n \in N\ , \text{ and } \quad \lambda = [\log_3 \sigma]\ .$$

Notice here that taking standard parts has produced nothing new in 4.55 since clearly

$$n\ 2^{[\log_3 \sigma]} < 2^{[\log_2 \sigma]} < \sigma < \nu\ ,$$

so that 4.55 is a much weaker inequality than 4.49. There we have $n\sigma(\frac{3}{\pi^2}\sigma - \log\sigma) < \nu$ for all $n \in N$ so that certainly

4.56 $$n\ \sigma < \nu\ , \quad \text{for all} \quad n \in N\ .$$

It is interesting to note that 4.55 can be obtained from the construction of T without resorting to non-standard measures. Indeed if we choose to write each U_i above in component form, e.g. U_3 as:

$$U_3^1 = \left(\frac{1}{27}, \frac{2}{27}\right),\quad U_3^2 = \left(\frac{7}{27}, \frac{8}{27}\right),\quad U_3^3 = \left(\frac{19}{27}, \frac{20}{27}\right),\quad U_3^4 = \left(\frac{25}{27}, \frac{26}{27}\right)\ ,$$

and similarly for all other U_i , $i \in N$, we are removing at step i , 2^{i-1} intervals U_i^k each of Lebesgue measure $\frac{1}{3^i}$. Now we can write

$$T = S - \bigcup_{i \in N} \bigcup_{k=1}^{2^{i-1}} U_i^k\ ,$$

where each U_i^k , $i \in N$, $1 \leq k \leq 2^{i-1}$, is a subset of S . By the n-invariance of F we see that ${}^*U_i^k \cap F \neq \phi$ for $i \leq \lambda$ defined as above. But there are $2^{\lambda} - 1$ of these intervals ${}^*U_i^k$ and since for $1 \leq i \leq \lambda$, $\|{}^*U_i^k \cap F\| > n$ for all $n \in N$ we have $\nu > n(2^{\lambda} - 1)$, which agrees with 4.55 as required.

4.6 A STANDARD SUBSET OF S THAT IS LEBESGUE NON MEASURABLE

We define the equivalence relation $\sim_r$ on S as follows:

4.61 Definition For $x,y \in S$, x and y are said to be equivalent points if $|x-y| \in Q \cap S$, and in such case we write $x \sim_r y$.

By definition we see that $\sim_r$ partitions S into equivalence classes A_α, where $x,y \in A_\alpha$ if $x \sim_r y$. It is clear that the A's are countable subsets of S and that there are a non-denumerable number of them; this follows since distinct A_α are mutually disjoint and their union is S.

Since each $a \in S$ belongs to A_α for some α, we denote by A_a the equivalence class generated by a and write:

4.62 $$A_a = \{x : x \sim_r a\} .$$

From above we see that the sets A_a form disjoint subsets of S, the union of which is S.

Using the axiom of choice we form the set W by taking one element x_a from each set A_a. The set W is then Lebesgue non-measurable [21], but we show below that we can still find an approximate non-zero value for $\mu(W)$. Note at this stage that since 3.28 (iii) applies only to subsets $A \subseteq S$ which are Lebesgue measurable we are not forced to apply the standard part homomorphism here, and no contradiction results.

Consider the sets $(W \dotplus r)$, $r \in Q \cap S$. These are mutually disjoint subsets of S and for any $x \in S$ we have $x \in A_a$ for some $a \in S$. But $x_a \in A_a$ and it follows that $x \in (W \dotplus r)$ for some

$r \in Q \cap S$. Thus $(\forall x)\ x \in S . \supset (\exists r)\ x \in (W \dot{+} r) \wedge r \in Q \cap S$, from which we conclude that

<u>4.63</u> $$S = \bigcup_{r \in Q \cap S} (W \dot{+} r) \ .$$

Now $(W \dot{+} r_i) \cap (W \dot{+} r_j) = \phi$ for $r_i, r_j \in Q \cap S$, $r_i \neq r_j$. Thus transforming we have in $*(\hat{R})$ that for $r_i, r_j \in *(Q \cap S)$:

$$*\left[(W \dot{+} r_i) \cap (W \dot{+} r_j)\right] = *(W \dot{+} r_i) \cap *(W \dot{+} r_j)$$
$$= *\phi = \phi \ ,$$

and the sets $*(W \dot{+} r)$, $r \in *(Q \cap S)$ form a sequence of disjoint subsets of $*S$. Transforming 4.63 we thus have

<u>4.64</u> $$*S = \bigcup_{r \in *(Q \cap S)} *(W \dot{+} r) \ .$$

By transforming 4.63, we see that each $x \in *(W \dot{+} r)$ is of the form $a \dot{+} r$, $a \in *S$, $r \in *(Q \cap S)$.

By Theorem 3.25 we see that for $r = \frac{q}{s}$, $q \lesssim s \lesssim \sigma$ we have

<u>4.65</u> $$\|*(W \dot{+} r) \cap F\| = \|(*W \dot{+} r) \cap F\|$$
$$= \|*W \cap F\| \ .$$

We now examine this case in detail in the standard, finite case and transform our result to obtain an approximation for $\mu(W)$ as outlined below. In $\hat{R}$ we have the following true statement:

$$(\forall y) . y \in X \wedge \left[(\forall r) r \in (Q \cap S) \cap y \supset \|(W \dot{+} r) \cap y\| = \|W \cap y\|\right]$$
$$\supset \Big\| \bigcup_{r \in (Q \cap S) \cap y} (W \dot{+} r) \cap y \Big\| = \|(Q \cap S) \cap y\| \bullet \|W \cap y\| \ .$$

Transforming this we have in $*(R)$ that:

$$(\forall y) . y \in *X \wedge \left[(\forall r) r \in *(Q \cap S) \cap y \supset \|*(W \dot{+} r) \cap y\| = \|*W \cap y\|\right]$$
$$\supset \Big\| \bigcup_{r \in *(Q \cap S) \cap y} *(W + r) \cap y \Big\| = \|*(Q \cap S) \cap y\| \bullet \|*W \cap y\| \ .$$

With $F \in {}^*X$ we now write

4.66
$$\Big\| \bigcup_{r \in {}^*(Q \cap S) \cap F} {}^*(W \mp r) \cap F \Big\| = \Big\| \bigcup_{\substack{\frac{q}{s} \in F;\, q,s \in {}^*N \\ q \leq s \leq \sigma}} {}^*(W \mp \tfrac{q}{s}) \cap F \Big\|$$

$$+ \Big\| \bigcup_{\substack{\frac{t}{u} \in F;\, t,u \in {}^*N \\ \sigma < u,\, t \leq u}} {}^*(W \mp \tfrac{t}{u}) \cap F \Big\|$$

$$= \sum_{i=1}^{\sigma} \Phi(i) \bullet \| {}^*W \cap F \| + \Big\| \bigcup_{\substack{\frac{t}{u} \in F;\, t,u \in {}^*N \\ \sigma < u,\, t \leq u}} {}^*(W \mp \tfrac{t}{u}) \cap F \Big\|$$

by 4.65 and section 4.4.

But $F = {}^*S \cap F$

$$= \bigcup_{r \in {}^*(Q \cap S) \cap F} {}^*(W \mp r) \cap F \text{ , by 4.64}$$

Since we want to apply 4.66 we write

$$r \in {}^*(Q \cap S) \equiv r \in {}^*(Q \cap S) \cap F \;\cup\; {}^*(Q \cap S) - F$$

4.67 Thus

$$F = \bigcup_{r \in {}^*(Q \cap S) \cap F} {}^*(W \mp r) \cap F \;\cup \bigcup_{r \in ({}^*Q \cap S) - F} {}^*(W \mp r) \cap F \text{ .}$$

From earlier work we know that ${}^*(Q \cap S) - F \neq \phi$, however in general we cannot be sure which rationals are in ${}^*(Q \cap S) - F$, nor for which of these ${}^*(W \mp r) \cap F \neq \phi$. Using 4.67 we can however make the approximation

$$\|F\| \geq \Big\| \bigcup_{r \in {}^*(Q \cap S) \cap F} {}^*(W \mp r) \cap F \Big\| \geq \Big\| \bigcup_{\substack{\frac{q}{s} \in F;\, q,s \in {}^*N \\ q \leq s \leq \sigma}} {}^*(W \mp \tfrac{q}{s}) \cap F \Big\| \text{ ,}$$

so that $\|F\| \geq \sum_{i=1}^{\sigma} \Phi(i) \bullet \| {}^*W \cap F \|$, by 4.66 .

Thus $$\|{}^*W \cap F\| \leq \frac{\nu}{\sum_{i=1}^{\sigma} \Phi(i)}$$

and $$\mu(W) \leq \frac{1}{\sum_{i=1}^{\sigma} \Phi(i)} .$$

To find a lower bound for $\mu(W)$ we note that the number of A_a's having an element of F in it is $\|{}^*W \cap F\|$. Each element of $A_a \cap F$ will have at most $\sum_{i=1}^{\rho} \Phi(i)$ rational translates also in F and $A_a \cap F$ will have no other points. Thus

$$\|{}^*W \cap F\| \bullet \sum_{i=1}^{\rho} \Phi(i) \geq \|F\| \qquad \text{and}$$

$$\mu(W) \geq \frac{1}{\sum_{i=1}^{\rho} \Phi(i)} .$$

Combining our inequality above with 4.47 we now obtain

$$\frac{1}{\frac{3\rho^2}{\pi^2} + \rho \log \rho} < \mu(W) < \frac{1}{\frac{3\sigma^2}{\pi^2} - \sigma \log \sigma} .$$

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