1-1-2015

Von Neumann algebras of strongly connected higher-rank graphs

Marcelo Laca
University of Wollongong, marcelo@uow.edu.au

Nadia S. Larsen
University of Copenhagen

Sergey Neshveyev
University of Oslo

Aidan Sims
University of Wollongong, asims@uow.edu.au

Samuel B. Webster
University of Wollongong, sbgwebster@gmail.com

Abstract
We investigate the factor types of the extremal KMS states for the preferred dynamics on the Toeplitz algebra and the Cuntz-Krieger algebra of a strongly connected finite (Formula presented.)-graph. For inverse temperatures above 1, all of the extremal KMS states are of type (Formula presented.). At inverse temperature 1, there is a dichotomy: if the (Formula presented.)-graph is a simple (Formula presented.)-dimensional cycle, we obtain a finite type (Formula presented.) factor; otherwise we obtain a type III factor, whose Connes invariant we compute in terms of the spectral radii of the coordinate matrices and the degrees of cycles in the graph.

Keywords
von, algebras, strongly, neumann, connected, graphs, higher, rank

Disciplines
Engineering | Science and Technology Studies

Publication Details
Laca, M., Larsen, N. S., Neshveyev, S., Sims, A. D. & Webster, S. B. (2015). Von Neumann algebras of strongly connected higher-rank graphs. Mathematische Annalen, 363 (1), 657-678.

VON NEUMANN ALGEBRAS OF STRONGLY CONNECTED HIGHER-RANK GRAPHS

MARCELO LACA, NADIA S. LARSEN, SERGEY NESHVEYEV, AIDAN SIMS, AND SAMUEL B.G. WEBSTER

Abstract. We investigate the factor types of the extremal KMS states for the preferred dynamics on the Toeplitz algebra and the Cuntz–Krieger algebra of a strongly connected finite k-graph. For inverse temperatures above 1, all of the extremal KMS states are of type I_∞. At inverse temperature 1, there is a dichotomy: if the k-graph is a simple k-dimensional cycle, we obtain a finite type I factor; otherwise we obtain a type III factor, whose Connes invariant we compute in terms of the spectral radii of the coordinate matrices and the degrees of cycles in the graph.

1. Introduction

The C^*-algebras of strongly connected finite higher-rank graphs provide interesting higher-rank analogues of Cuntz–Krieger algebras. In this paper we study von Neumann algebras generated by these C^*-algebras in the representations defined by the extremal KMS states for the preferred dynamics studied in [13]. Results of Enomoto, Fujii and Watatani [6] show that when $k = 1$ and the graph is not a simple cycle, there is a unique KMS state and the associated factor is of type $\mathrm{III}_{\rho(A)^{-p}}$, where $\rho(A)$ is the spectral radius of the adjacency matrix A of the graph, and p is the period of the graph in the sense of Perron-Frobenius theory: the greatest common divisor of the lengths of cycles in the graph.

In the higher-rank case there can be more than one KMS state, and a complete classification of such states has been recently obtained in [13]. Specifically, Theorem 7.1 of [13] shows that the extremal KMS states of the C^*-algebra $C^*(\Lambda)$ of a finite strongly connected k-graph Λ are indexed by the characters of an associated subgroup $\operatorname{Per}\Lambda$ of $\mathbb{Z}^k$, whose group C^*-algebra embeds as a central subalgebra of $C^*(\Lambda)$. The goal of the present paper is to determine the types of these KMS states. Using Feldman–Moore theory [8] and the groupoid description of a k-graph algebra, we obtain a very satisfactory generalisation of Enomoto, Fujii and Watatani's result. Namely, suppose that Λ is not a simple k-dimensional cycle. We define $\mathcal{P}_\Lambda$ to be the subgroup of $\mathbb{Z}^k$ generated by the degrees of cycles in Λ. The k coordinate graphs of Λ determine integer matrices A_i. The vector $\rho(\Lambda) = (\rho(A_1), \dots, \rho(A_k))$ of spectral radii of these matrices determines a homomorphism $n \mapsto \rho(\Lambda)^n$ of $\mathcal{P}_\Lambda$ into the multiplicative group of positive reals. We prove that the closure of its image is the Connes spectrum of the type III factor obtained from any of the extremal KMS states of $C^*(\Lambda)$ described in [13]. We also determine the types of the factors arising from KMS states on the Toeplitz algebra that do not factor through $C^*(\Lambda)$, and from KMS states of $C^*(\Lambda)$ when Λ is a simple k-dimensional cycle. An interesting corollary is that the factors obtained from a k-graph Λ depend only on its skeleton, and are independent of the factorisation property.

In the case when Λ is primitive and aperiodic — or equivalently, when $\mathcal{P}_\Lambda = \mathbb{Z}^k$ and $\operatorname{Per}\Lambda = 0$ — the unique KMS state φ is the most natural state on $C^*(\Lambda)$: it is the unique gauge-invariant

Date: September 22, 2014; minor corrections on January 13, 2015.

2010 *Mathematics Subject Classification.* 46L10 (primary); 46L05, (secondary).

Key words and phrases. von Neumann algebra; KMS state; higher-rank graph; Borel equivalence relation.

This research was supported by the Australian Research Council and the Natural Sciences and Engineering Research Council of Canada. Parts of the work were completed at the workshop *Operator algebras and dynamical systems from number theory* (13w5152) at the Banff International Research Station in November 2013 and at the conference *Classification, Structure, Amenability and Regularity* at the University of Glasgow in September 2014, supported by the EPSRC, GMJT and LMS.

state whose restriction to the AF core of $C^*(\Lambda)$ is tracial. By our result, the Connes spectrum of the associated factor is then the closure of the multiplicative group generated by the spectral radii $\rho(A_i)$ of the connectivity matrices A_i. In some special cases this has been already established by Yang [24, 25]. She studied C^*-algebras and von Neumann algebras of aperiodic k-graphs with a single vertex — the higher-rank analogues of Cuntz algebras. Under the technical condition that the so-called "intrinsic group" of the graph has rank at most 1, she proved that φ is a factor state of type III with Connes spectrum equal to the closure of the multiplicative group generated by the numbers $m_1, \dots, m_k$ of edges of each of the k minimal degrees. This generalises Olesen and Pedersen's result [21] that the unique KMS state for the gauge-action on the Cuntz algebra $\mathcal{O}_n$ is a type $\mathrm{III}_{1/n}$ factor state. Yang's result completely resolved the situation for aperiodic single-vertex 2-graphs. She then asked whether the result remains true for all single-vertex k-graphs, regardless of the intrinsic group. A special case of our main theorem implies that this is indeed the case, under the sole assumption of aperiodicity.

The paper is organised as follows. We introduce necessary background about k-graphs and their C^*-algebras in Section 2. In Section 3, we state our main result, Theorem 3.1, and begin the proof by analysing the factors arising from KMS states at large inverse temperatures. These are all type I_∞ states and the associated von Neumann factors each have a canonical presentation as $\mathcal{B}(\ell^2(\Lambda v))$ for some $v \in \Lambda^0$.

In Section 4 we present a computation of the Connes invariant $S(W^*(\mathcal{Q}))$ of the von Neumann algebra of an ergodic countable equivalence relation $\mathcal{Q}$ with a quasi-invariant measure μ. Corollary 4.2 says that if the sub-relation $\mathcal{Q}^D$ defined by the kernel of the Radon–Nikodym cocycle of μ is ergodic, then $S(W^*(\mathcal{Q}))$ is precisely the essential range of the Radon–Nikodym cocycle. These results are surely known, but we give a self-contained treatment in lieu of an explicit reference.

In Section 5 we apply groupoid methods to study the factors associated to the extremal KMS states of $C^*(\Lambda)$. The groupoid model $\mathcal{G}$ for $C^*(\Lambda)$ [15] determines a Borel equivalence relation $\mathcal{R}$ on the path space Λ^∞. The unique probability measure μ_{eq} on Λ^∞ induced by all KMS states of $C^*(\Lambda)$ (see [13, Proposition 8.1]) is quasi-invariant with respect to $\mathcal{R}$. The key result, Proposition 5.2, says that $W^*(\mathcal{R})$ is isomorphic to the factor determined by any extremal KMS state of $C^*(\Lambda)$; this isomorphism is noncanonical unless Λ is aperiodic. We finish the section by proving that the sub-relation $\mathcal{R}^D$ obtained from $\mathcal{R}$ as in the preceding paragraph contains a still smaller relation $\mathcal{R}^\gamma$ which is an étale topological equivalence relation whose C^*-algebra is the AF core of $C^*(\Lambda)$.

In Section 6, we develop a Frobenius analysis of strongly connected higher-rank graphs. We investigate the group $\mathcal{P}_\Lambda \subseteq \mathbb{Z}^k$ generated by the degrees $d(\lambda)$ of cycles in Λ. We show that there is a map $C : \Lambda^0 \times \Lambda^0 \to \mathbb{Z}^k/\mathcal{P}_\Lambda$ such that $C(r(\lambda), s(\lambda)) = d(\lambda) + \mathcal{P}_\Lambda$ for all λ. The key result of the section, Proposition 6.5, says that there is a strictly positive $p \in \mathcal{P}_\Lambda$ with the following property: $C(v, w) = 0$ if and only if there is a path of degree p connecting v to w. We also show that the relation $\sim$ given by $v \sim w$ if and only if $C(v, w) = 0$ is an equivalence relation on Λ^0. We deduce that there exists a natural free and transitive action of $\mathbb{Z}^k/\mathcal{P}_\Lambda$ on $\Lambda^0/{\sim}$, and that the decomposition of $\mathbb{R}^{\Lambda^0}$ into direct summands indexed by $\Lambda^0/{\sim}$ is a system of imprimitivity for the connectivity matrices A_i. We also deduce that $\mathcal{P}_\Lambda$ always contains the periodicity group $\operatorname{Per}\Lambda$ of [3, 13], with equality if and only if Λ is a simple k-dimensional cycle (see Proposition 6.9).

Finally, in Section 7, we prove our main theorem. We show that the AF core of $C^*(\Lambda)$ decomposes as a direct sum with summands indexed by $\Lambda^0/{\sim}$. Corollary 7.2 shows that the ergodic components of $\mathcal{R}^\gamma$ are the sets X_ω of infinite paths with range in $\omega \in \Lambda^0/{\sim}$. Each characteristic function $\mathbf{1}_{X_\omega}$ is a full projection in $W^*(\mathcal{R})$, so the type of $W^*(\mathcal{R})$ coincides with that of $W^*(\mathcal{R}|_{X_\omega})$; we compute the latter using the results of Section 4. We briefly discuss the relationship between our results and Yang's, and show that the factorisation property in Λ does not affect the factors that arise from it. We conclude by applying our main theorem to a few illustrative examples.

2. Higher-rank graphs

We denote by $\mathbb{N}$ the monoid $\{0, 1, 2, \dots\}$ of nonnegative integers under addition. For an integer $k \geq 1$, we then regard $\mathbb{N}^k$ as a monoid with pointwise addition. The canonical generators of $\mathbb{N}^k$ are denoted e_i, and for $n \in \mathbb{N}^k$ we write n_i for its i^{th} coordinate. We give $\mathbb{N}^k$ its natural partial order $m \leq n$ if and only if each $m_i \leq n_i$ and $m \vee n$ denotes the coordinatewise maximum of $m, n \in \mathbb{N}^k$.

A rank-k graph, or a k-graph, is a small category Λ together with an assignment of a *degree* $d(\lambda) \in \mathbb{N}^k$ to every morphism $\lambda \in \Lambda$ such that

(1) $d(\lambda\mu) = d(\lambda) + d(\mu)$; and
(2) whenever $d(\lambda) = m + n$, there is a unique factorisation $\lambda = \mu\nu$ such that $d(\mu) = m$ and $d(\nu) = n$.

Condition (2) is often called the "factorisation property." It implies in particular that the only morphisms of degree 0 are the identity morphisms.

The set of morphisms of degree $n \in \mathbb{N}^k$ is denoted by Λ^n. Its elements are called paths of degree n in Λ. So Λ^0 is the set of identity morphisms; we regard them interchangeably as paths of degree zero and as vertices. We also identify Λ^0 with the set of objects of Λ in the natural way, so that the codomain and domain maps become functions $r, s : \Lambda \to \Lambda^0$. Throughout the paper we consider only finite k-graphs, meaning that each $|\Lambda^n| < \infty$.

For $\mu \in \Lambda$ and $n \in \mathbb{N}^k$, denote by $\mu\Lambda^n$ the set of morphisms $\mu\lambda$ such that $d(\lambda) = n$ and $s(\mu) = r(\lambda)$. The sets $\Lambda^n\nu$ and $\mu\Lambda^n\nu$ are defined similarly.

The *connectivity matrices* $A_1, \dots, A_k \in \mathrm{Mat}_{\Lambda^0}(\mathbb{N})$ of Λ are given by

$$A_i(v, w) = |v\Lambda^{e_i}w|.$$

The factorisation property implies that the matrices A_i pairwise commute. For $n \in \mathbb{N}^k$, we define

$$A^n := \textstyle\prod_{i=1}^k A_i^{n_i}. \tag{2.1}$$

We then have $A^n(v, w) = |v\Lambda^n w|$ for all v, w, and $n \mapsto A^n$ is a semigroup homomorphism. We write $\rho(B)$ for the spectral radius of a square matrix B. Define

$$\rho(\Lambda) := (\rho(A_1), \rho(A_2), \dots, \rho(A_k)) \in [0, \infty)^k.$$

For $g \in \mathbb{Z}^k$ we write $\rho(\Lambda)^g$ for the product $\prod_{i=1}^k \rho(A_i)^{g_i}$.

A finite k-graph Λ is *strongly connected* if $v\Lambda w \neq \emptyset$ for all $v, w \in \Lambda^0$. If there exists p such that $v\Lambda^p w \neq \emptyset$ for all v and w, then Λ is called *primitive*.

When working with strongly connected k-graphs, there is no loss of generality in assuming that Λ^n is nonempty for every $n \in \mathbb{N}^k$, and it is then not difficult to check (see [13, Lemma 2.1] and the paragraph before it) that

$$v\Lambda^n \neq \emptyset \text{ and } \Lambda^n v \neq \emptyset \quad \text{ for all } v \in \Lambda^0 \text{ and } n \in \mathbb{N}^k. \tag{2.2}$$

So each column and each row of each A^n is nonzero. It then follows from [13, Corollary 4.2] and [11, Lemma A.1] that each $\rho(A^n) \geq 1$ and that $n \mapsto \rho(A^n)$ is a homomorphism of $\mathbb{N}^k$ into the multiplicative semigroup $[1, \infty)$. Hence $\rho(A^n) = \rho(\Lambda)^n$ for all n.

The Toeplitz algebra $\mathcal{T}C^*(\Lambda)$ of the k-graph Λ is the universal C^*-algebra generated by elements $\{t_\lambda \mid \lambda \in \Lambda\}$ such that

(TCK1) $\{t_v \mid v \in \Lambda^0\}$ is a family of mutually orthogonal projections;
(TCK2) $t_\mu t_\nu = t_{\mu\nu}$ whenever $s(\mu) = r(\nu)$;
(TCK3) $t_\mu^* t_\mu = t_{s(\mu)}$ for all μ;
(TCK4) $t_v \geq \sum_{\mu \in v\Lambda^n} t_\mu t_\mu^*$ for all $v \in \Lambda^0$ and $n \in \mathbb{N}^k$; and
(TCK5) $t_\mu^* t_\nu = \sum_{\mu\alpha = \nu\beta \in \Lambda^{d(\mu) \vee d(\nu)}} t_\alpha t_\beta^*$ for all μ, ν.

The C^*-algebra $C^*(\Lambda)$ of Λ is the quotient of $\mathcal{T}C^*(\Lambda)$ by the ideal generated by $\{t_v - \sum_{\mu \in v\Lambda^n} t_\mu t_\mu^* \mid v \in \Lambda^0, n \in \mathbb{N}^k\}$. It is universal for families $\{s_\lambda \mid \lambda \in \Lambda\}$ satisfying (TCK1)–(TCK3) and

(CK) $s_v = \sum_{\mu \in v\Lambda^n} s_\mu s_\mu^*$ for all v, n.

3. The main result, and the factor types of KMS states on the Toeplitz algebra

Let Λ be a strongly connected finite k-graph. Our main result is a characterisation of the factor types of the extremal KMS states of $\mathcal{T}C^*(\Lambda)$ and $C^*(\Lambda)$ studied in [13]. Each $r \in [0,\infty)^k$ determines an action $\alpha^r\colon \mathbb{R} \to \operatorname{Aut} \mathcal{T}C^*(\Lambda)$ by $\alpha^r_t(t_\lambda) = e^{itr\cdot d(\lambda)}t_\lambda$, and this descends to an action, also denoted α^r, on $C^*(\Lambda)$. Corollary 4.6 of [13] shows that, up to rescaling, the only value of r for which α^r admits a KMS state that factors through $C^*(\Lambda)$ is $r = \log \rho(\Lambda)$. We write an unadorned α for this dynamics, and call it the "preferred dynamics". Corollary 4.6 of [13] also shows that there are α-KMS$_\beta$ states for all $\beta \geq 1$, and the only ones that factor through $C^*(\Lambda)$ occur at $\beta = 1$. Given a state ϕ, we write π_ϕ for the associated GNS representation.

Theorem 3.1. *Let Λ be a strongly connected finite k-graph. Let α be the preferred dynamics and suppose that ϕ is an extremal α-KMS$_\beta$ state of $\mathcal{T}C^*(\Lambda)$.*

(1) *Suppose that $\beta > 1$. Then $\pi_\phi(\mathcal{T}C^*(\Lambda))''$ is the type I_∞ factor.*

(2) *Suppose that $\beta = 1$, and write $\overline{\phi}$ for the corresponding KMS state of $C^*(\Lambda)$.*

 (a) *If $\rho(\Lambda) = (1,\dots,1)$, then $\pi_\phi(\mathcal{T}C^*(\Lambda))'' = \pi_{\overline{\phi}}(C^*(\Lambda))$ is the type $\mathrm{I}_{|\Lambda^0|}$ factor.*

 (b) *Otherwise, let*

$$S := \{\rho(\Lambda)^{d(\mu)-d(\nu)} \mid \mu,\nu \in \Lambda \text{ are cycles}\},$$

 and let $\lambda := \sup\{s \in S \mid s < 1\}$. Then $\lambda \in (0,1]$ and $\pi_\phi(\mathcal{T}C^*(\Lambda))'' = \pi_{\overline{\phi}}(C^*(\Lambda))''$ is the injective type III_λ factor.

The rest of the paper mainly consists of the proof of Theorem 3.1, which will be completed in Section 7. Most of the work lies in statement (2b); in particular, (1) is fairly straightforward, and the proof works for any α^r and any β with $\beta r > \log \rho(\Lambda)$ coordinatewise.

Theorem 6.1 of [12] describes KMS states of $\mathcal{T}C^*(\Lambda)$ as follows. Take $r \in [0,\infty)^k$ and $\beta \in \mathbb{R}$ such that $\beta r > \log \rho(\Lambda)$ coordinatewise. Then for each v, the series $\sum_{\mu \in \Lambda v} e^{-\beta r\cdot d(\mu)}$ converges to some $y_v \geq 1$. Set $y = (y_v)_{v\in\Lambda^0}$. For each $\epsilon \in [0,\infty)^{\Lambda^0}$ such that $\epsilon \cdot y = 1$, define $\Delta\colon \Lambda \to \mathbb{R}^+$ by $\Delta_\lambda = e^{-\beta r\cdot d(\lambda)}\epsilon_{s(\lambda)}$. Let $\{h_\lambda \mid \lambda \in \Lambda\}$ be the orthonormal basis for $\ell^2(\Lambda)$, and $\pi_S : \mathcal{T}C^*(\Lambda) \to \mathcal{B}(\ell^2(\Lambda))$ the path-space representation $\pi_S(t_\lambda)h_\mu = \delta_{s(\lambda),r(\mu)}h_{\lambda\mu}$ [22, Example 7.7]. Then the formula $\varphi_\epsilon(a) = \sum_{\lambda\in\Lambda} \Delta_\lambda(\pi_S(a)h_\lambda|h_\lambda)$ defines an α-KMS$_\beta$ state φ_ϵ of $\mathcal{T}C^*(\Lambda)$. Moreover, putting $m^\epsilon := \prod_{i=1}^k (1 - e^{-\beta r_i}A_i)^{-1}\epsilon$, we have $\varphi_\epsilon(t_\mu t_\nu^*) = \delta_{\mu,\nu}e^{-\beta r\cdot d(\mu)}m^\epsilon_{s(\mu)}$ for all μ,ν. The map $\epsilon \mapsto \varphi_\epsilon$ from $\Sigma_\beta = \{\epsilon \in [0,\infty)^{\Lambda^0} \mid \epsilon\cdot y = 1\}$ to the simplex of α-KMS$_\beta$ states of $\mathcal{T}C^*(\Lambda)$ is an affine isomorphism. The simplex Σ_β is the closed convex hull of $\{y_v^{-1}\delta_v \mid v \in \Lambda^0\}$.

Proposition 3.2. *Let Λ be a strongly connected finite k-graph. Suppose that $r \in [0,\infty)^k$ and $\beta > 0$ satisfy $\beta r > \log\rho(\Lambda)$ coordinatewise. Fix $v \in \Lambda^0$ and let $\epsilon := y_v^{-1}\delta_v \in [0,\infty)^{\Lambda^0}$. Let $\pi_\epsilon := \pi_{\varphi_\epsilon}$ be the GNS representation associated to the KMS$_\beta$ state φ_ϵ. Then $\pi_\epsilon(\mathcal{T}C^*(\Lambda))''$ is a factor of type I_∞, and $\pi_\epsilon(t_\mu) \mapsto \pi_S(t_\mu)|_{\ell^2(\Lambda v)}$ determines a von Neumann algebra isomorphism $\pi_\epsilon(\mathcal{T}C^*(\Lambda))'' \cong \mathcal{B}(\ell^2(\Lambda v))$.*

Proof. By [12, Theorem 6.1], the state φ_ϵ is an extremal point in the simplex of KMS$_\beta$ states. Hence $\pi_\epsilon(\mathcal{T}C^*(\Lambda))''$ is a factor (see, for example [2, Theorem 5.3.30(3)]).

Let $q_v := \prod_{i=1}^k (t_v - \sum_{\gamma\in v\Lambda^{e_i}} t_\gamma t_\gamma^*)$. We have $\pi_S(q_v)h_v = h_v$, and for $\mu \in \Lambda \setminus v\Lambda$ we have $q_v \leq t_v \perp t_{r(\mu)}$ giving $\pi_S(q_v)h_\mu = 0$. For $\mu \in v\Lambda \setminus \{v\}$, we have $\mu = \mu_1\mu'$ for some $\mu_1 \in \bigcup_i v\Lambda^{e_i}$, and then $\pi_S(t_v - t_{\mu_1}t_{\mu_1}^*)h_\mu = 0$, forcing $\pi_S(q_v)h_\mu = 0$. Hence $\pi_S(q_v)$ is the projection onto $\mathbb{C}h_v$, and $\pi_S(q_v\mathcal{T}C^*(\Lambda)q_v) = \mathbb{C}\pi_S(q_v)$. Since π_S is faithful by [22, Corollary 7.7], we then have $q_v\mathcal{T}C^*(\Lambda)q_v = \mathbb{C}q_v$, and hence $\pi_\epsilon(q_v)\pi_\epsilon(\mathcal{T}C^*(\Lambda))''\pi_\epsilon(q_v) = \mathbb{C}\pi_\epsilon(q_v)$. Thus $\pi_\epsilon(q_v)$ is either a minimal projection or zero. Since $\epsilon = y_v^{-1}\delta_v$, we have

$$\varphi_\epsilon(q_v) = \textstyle\sum_{\lambda\in\Lambda} \Delta_\lambda(\pi_S(q_v)h_\lambda \mid h_\lambda) = \Delta_v = e^{-\beta r\cdot d(v)}\epsilon_v = y_v^{-1} > 0. \tag{3.1}$$

Hence $\pi_\epsilon(q_v) \neq 0$, and so $\pi_\epsilon(\mathcal{T}C^*(\Lambda))''$ has a minimal projection $\pi_\epsilon(q_v)$, and is therefore of type I.

For $\lambda \in \Lambda v$, let ξ_λ denote the class of $\sqrt{y_v} t_\lambda q_v$ in the GNS space $\mathcal{H}_\epsilon$ of φ_ϵ. For $\lambda, \mu \in \Lambda v$, we have

$$\begin{aligned}\big(\xi_\lambda \mid \xi_\mu\big)_{\mathcal{H}_\epsilon} = y_v \varphi_\epsilon(q_v t_\mu^* t_\lambda q_v) &= y_v \sum_{(\gamma,\gamma') \in \Lambda^{\min}(\mu,\lambda)} \varphi_\epsilon(q_v t_\gamma t_{\gamma'}^* q_v) \\ &= y_v \sum_{(\gamma,\gamma') \in \Lambda^{\min}(\mu,\lambda)} \varphi_\epsilon(\delta_{v,\gamma}\delta_{v,\gamma'} q_v) \\ &= y_v \delta_{\lambda,\mu}\delta_{s(\lambda),v}\, \varphi_\epsilon(q_v) = \delta_{\lambda,\mu}\delta_{s(\lambda),v}\end{aligned}$$

by (3.1). So $\{\xi_\lambda \mid \lambda \in \Lambda v\}$ is an infinite orthonormal set in $\mathcal{H}_\epsilon$. Define $\mathcal{H} := \overline{\operatorname{span}}\{\xi_\lambda \mid \lambda \in \Lambda v\}$. For $\lambda \in \Lambda v$ and $\mu, \nu \in \Lambda$, we have

$$\pi_\epsilon(t_\mu t_\nu^*)\xi_\lambda = \sqrt{y_v}[t_\mu t_\nu^* t_\lambda q_v] = \sqrt{y_v}\Big[\sum_{(\gamma,\gamma') \in \Lambda^{\min}(\nu,\lambda)} t_{\mu\gamma}(q_v t_{\gamma'})^*\Big].$$

We have $\pi_S(t_{\gamma'})\ell^2(\Lambda) = \overline{\operatorname{span}}\{h_{\gamma'\lambda} \mid \lambda \in s(\gamma')\Lambda\}$, which is orthogonal to $\pi_S(q_v)\ell^2(\Lambda) = \mathbb{C}\delta_v$ unless $\gamma' = v$. Since π_S is injective, we deduce that $\pi_\epsilon(t_\mu t_\nu^*)\xi_\lambda$ is zero unless $\lambda = \nu\lambda'$ and $\Lambda^{\min}(\nu, \lambda) = \{(\lambda', v)\}$, giving $\pi_\epsilon(t_\mu t_\nu^*)\xi_\lambda = \xi_{\mu\lambda'} \in \mathcal{H}$. So $\mathcal{H}$ is invariant for $\pi_\epsilon(\mathcal{T}C^*(\Lambda))$, and hence also for its double commutant. Since $\pi_\epsilon(\mathcal{T}C^*(\Lambda))''$ is a factor, it follows that the restriction map $T \to T|_{\mathcal{H}}$ is a von Neumann algebra isomorphism.

For $\mu, \nu \in \Lambda v$, we have $\pi_\epsilon(t_\mu q_v t_\nu^*)\xi_\lambda = \delta_{\nu,\lambda}\xi_\mu$, and so $\pi_\epsilon(t_\mu q_v t_\nu^*)$ is the rank-one operator θ_{ξ_μ,ξ_ν}. So $\mathcal{K}(\mathcal{H}) \subseteq \pi_\epsilon(\mathcal{T}C^*(\Lambda))$, giving $\pi_\epsilon(\mathcal{T}C^*(\Lambda))''|_{\mathcal{H}} = \mathcal{B}(\mathcal{H})$. The formula $U\colon \xi_\lambda \mapsto h_\lambda$ defines a unitary isomorphism $\mathcal{H} \cong \ell^2(\Lambda v)$, and we have $U\pi_\epsilon(t_\mu)|_{\mathcal{H}}U^* = \pi_S(t_\mu)$. □

4. The Connes invariant of the von Neumann algebra of an equivalence relation

Our analysis of the types of extremal KMS states on the C^*-algebra of a k-graph will rely on identifying the associated factor with the von Neumann algebra of the equivalence relation determined by the k-graph groupoid. Computing the Connes invariant S of a factor is in general a difficult problem; but it simplifies drastically in the presence of a faithful normal state φ with factorial centraliser. In this instance, Connes' result [4, Théorème 3.2.1] implies that S is equal to the spectrum of the modular operator for φ. In this section we describe what this result says for von Neumann algebras of equivalence relations; this result is surely known, but we provide a proof as we were unable to find a reference.

Let us review Feldman and Moore's construction of the von Neumann algebra $W^*(\mathcal{Q})$ of a countable Borel equivalence relation $\mathcal{Q}$ on a space X and a quasi-invariant measure μ on X (see [7, Section 2] and [8, Section 2]). Recall that a Borel equivalence relation $\mathcal{Q}$ on a standard Borel space X is said to be *countable* if $\{y \mid (x, y) \in \mathcal{Q}\}$ is countable for each $x \in X$. A Borel measure μ on X is *quasi-invariant* for $\mathcal{Q}$ if, whenever $\mu(A) = 0$, the $\mathcal{Q}$-saturation $\mathcal{Q}(A) := \bigcup_{x\in A}\{y \mid (x, y) \in \mathcal{Q}\}$ of A is also μ-null. Equip $\mathcal{Q}$ with the left counting measure ν:

$$\nu(C) = \int_X \big|\{y \mid (x, y) \in C\}\big|\, d\mu(x) \quad \text{for Borel } C \subseteq \mathcal{Q}.$$

(Unlike Feldman and Moore, we work with left counting measure, not right, as this is consistent with Renault's representation theory of groupoids.) Identifying the diagonal of $\mathcal{Q}$ with X via $(x, x) \mapsto x$, the restriction of ν to the diagonal coincides with μ. A Borel subset A of $\mathcal{Q}$ is called a *bisection* if the projection maps $(x, y) \mapsto x$ and $(x, y) \mapsto y$ are injective on A. Consider the $*$-algebra $\mathbb{C}[\mathcal{Q}]$ of functions $f \in L^\infty(\mathcal{Q}, \nu)$ supported on finitely many Borel bisections of $\mathcal{Q}$, under the convolution

$$(f_1 * f_2)(x, z) = \sum_{(x,y)\in\mathcal{Q}} f_1(x, y) f_2(y, z)$$

and involution $f^*(x,y) = \overline{f(y,x)}$. Write D for the Radon–Nikodym cocycle on $\mathcal{Q}$ determined by μ. Then $\mathbb{C}[\mathcal{Q}]$ has a representation π on $L^2(\mathcal{Q}, d\nu)$ given by

$$(\pi(f)\xi)(x,z) = \sum_{(x,y)\in\mathcal{Q}} D(x,y)^{-1/2} f(x,y)\xi(y,z), \tag{4.1}$$

and by definition, $W^*(\mathcal{Q}) = \pi(\mathbb{C}[\mathcal{Q}])''$. The characteristic function $\mathbf{1}_{\{(x,x)\mid x\in X\}}$ is a cyclic separating vector for $W^*(\mathcal{Q})$, so the formula

$$\varphi(f) = \int_X f(x,x)d\mu(x) \quad \text{for } f \in \mathbb{C}[\mathcal{Q}] \tag{4.2}$$

defines a faithful normal state φ of $W^*(\mathcal{Q})$.

Proposition 4.1. *Let $\mathcal{Q}$ be a countable measurable equivalence relation on a standard Borel space (X,μ) with Radon–Nikodym cocycle D. Let $\mathcal{Q}^D$ denote the finer equivalence relation*

$$x \sim_{\mathcal{Q}^D} y \text{ if and only if } x \sim_{\mathcal{Q}} y \text{ and } D(x,y) = 1.$$

Identify $W^*(\mathcal{Q}^D)$ with the strong-operator closure of the subalgebra

$$\{\pi(f) \mid f \in \mathbb{C}[\mathcal{Q}],\ \operatorname{supp}(f) \subseteq \mathcal{Q}^D\} \subset W^*(\mathcal{Q}).$$

Then $W^*(\mathcal{Q}^D)$ is equal to the centraliser of the state φ on $W^*(\mathcal{Q})$ defined by (4.2)*.*

Proof. Let $M := W^*(\mathcal{Q})$ and $N := W^*(\mathcal{Q}^D) \subseteq M$. Let ν denote left counting measure on $\mathcal{Q}$, and let $\xi := \mathbf{1}_{\{(x,x)\mid x\in X\}} \in L^2(\mathcal{Q}, d\nu)$; so φ is the vector state associated to ξ. Let Δ be the modular operator for φ. Then [8, Proposition 2.8] implies that Δ is given by multiplication by D on $L^2(\mathcal{Q}, d\nu)$. Hence the eigenspace of Δ corresponding to the eigenvalue 1 is $L^2(\mathcal{Q}^D, d\nu)$, which is precisely $\overline{N\xi}$.

Hence N is contained in the centraliser M_φ of φ and $\overline{N\xi} = \overline{M_\varphi\xi}$. In particular, N is invariant under the modular group. So there is a unique φ-preserving conditional expectation $E\colon M \to N$. Let p be the projection onto $\overline{N\xi}$. For $a \in M$ we have $E(a)\xi = pa\xi$. Hence, if $a \in M_\varphi$, then $E(a)\xi = pa\xi = a\xi$, so $a = E(a) \in N$. Thus $N = M_\varphi$. □

Corollary 4.2. *Let $\mathcal{Q}$, $D : \mathcal{Q} \to (0,\infty)$ and $\mathcal{Q}^D$ be as in Proposition 4.1. If $\mathcal{Q}^D$ is ergodic on (X,μ), then the Connes invariant $S(W^*(\mathcal{Q}))$ is the essential range of D.*

Proof. Since $\mathcal{Q}^D$ is ergodic, Proposition 4.1 implies that the centraliser of φ is a factor. So [4, Théorème 3.2.1] shows that $S(W^*(\mathcal{Q}))$ is equal to the spectrum of the operator of multiplication by D on $L^2(\mathcal{Q}, d\nu)$. Since this spectrum is exactly the essential range of D, the result follows. □

Remark 4.3. An alternative proof of Corollary 4.2 is as follows. First apply [8, Proposition 2.11], to see that $S(W^*(\mathcal{Q}))$ is equal to the ratio set of D, and then argue directly from its definition that this coincides with the essential range when $\mathcal{Q}^D$ is ergodic.

In our application the equivalence relation will arise as the orbit equivalence relation defined by the action of a groupoid on its unit space. For the convenience of the reader, we record the following useful relation between the two Radon–Nikodym cocycles arising in this situation.

Lemma 4.4. *Let $\mathcal{G}$ be a second countable étale groupoid and let μ be a quasi-invariant measure on $\mathcal{G}^0$ with Radon–Nikodym cocycle $c\colon \mathcal{G} \to (0,\infty)$. Consider the orbit equivalence relation $\mathcal{Q}$ defined by $\mathcal{G}$ on $\mathcal{G}^0$. Then the measure μ is quasi-invariant with respect to $\mathcal{Q}$, and if $D\colon \mathcal{Q} \to (0,\infty)$ is the corresponding Radon–Nikodym cocycle, then there exists a $\mathcal{Q}$-invariant μ-conull Borel subset $X \subseteq \mathcal{G}^0$ such that $D(x,y) = c(g)$ for all $(x,y) \in \mathcal{Q} \cap (X \times X)$ and $g \in \mathcal{G}^x_y := \{g \in \mathcal{G} \mid x = r(g),\ y = s(g)\}$.*

Proof. Choose a countable cover $\{V_n\}_{n=1}^\infty$ of $\mathcal{G}$ by open bisections; that is, open sets V_n on which r and s are injective. For each n define $T_n : r(V_n) \to s(V_n)$ by $T_n(r(g)) = s(g)$ for $g \in V_n$. By quasi-invariance of μ for $\mathcal{G}$, the T_n preserve the measure class of μ and

$$\tfrac{d(T_n)_*\mu}{d\mu}(s(g)) = c(g) \quad \text{for } g \in V_n.$$

So if $\mu(A) = 0$, then $\mu\big(\bigcup_n T_n^{-1}(A)\big) = 0$ too. Since $\bigcup_n T_n^{-1}(A)$ is precisely the $\mathcal{Q}$-saturation of A, the measure μ is quasi-invariant for $\mathcal{Q}$.

By Proposition 2.2 of [7] applied to the Borel isomorphism $T_n : r(V_n) \to s(V_n)$ with graph in $\mathcal{Q}$, there is a μ-null $Y_n \subseteq r(V_n)$ such that for $g \in V_n \setminus r^{-1}(Y_n)$, we have

$$D(r(g), s(g)) = D(T_n^{-1}(s(g)), s(g)) = \frac{d(T_n)_*\mu}{d\mu}(s(g)) = c(g).$$

Now $Y := \bigcup_n Y_n$ is μ-null, so its $\mathcal{Q}$-saturation $\mathcal{Q}(Y)$ is μ-null and $\mathcal{Q}$-invariant. Thus $X := \mathcal{G}^0 \setminus \mathcal{Q}(Y)$ suffices. □

Remark 4.5. As a byproduct we see that for μ-a.e. $x \in \mathcal{G}^0$ we have $c(g) = 1$ for all $g \in \mathcal{G}_x^x$, since $D(x, x) = 1$ for μ-a.e. x.

5. Equivalence relations and KMS states of $C^*(\Lambda)$

We will now use the groupoid picture for the C^*-algebra of a strongly connected finite k-graph. We briefly recount the construction of the groupoid $\mathcal{G}$ associated to a k-graph Λ and refer to [15] or Section 12 of [13] for more details.

The unit space of $\mathcal{G}$ is the space Λ^∞ of infinite paths in Λ, which is defined as follows. Let Ω_k denote the k-graph with objects $\mathbb{N}^k$, morphisms $\{(m, n) \in \mathbb{N}^k \times \mathbb{N}^k \mid m \leq n\}$, structure maps $r(m, n) = m$, $s(m, n) = n$ and $d(m, n) = n - m$, and composition $(m, n)(n, p) = (m, p)$. Then an infinite path $x \in \Lambda^\infty$ is a degree-preserving functor $\Omega_k \to \Lambda$.

For $n \in \mathbb{N}^k$ denote by σ^n the shift on Λ^∞ corresponding to n, so $\sigma^n(x)(p, q) = x(p + n, q + n)$. Then, as a set,

$$\mathcal{G} = \{(x, g, y) \in \Lambda^\infty \times \mathbb{Z}^k \times \Lambda^\infty \mid \sigma^{g+n}(x) = \sigma^n(y) \text{ for some } n \in \mathbb{N}^k\}.$$

The source and range maps are $s(x, g, y) = y$ and $r(x, g, y) = x$, and composition is

$$(x, g, y)(y, h, z) = (x, g + h, z).$$

For $x \in \Lambda^\infty$ and $\mu \in \Lambda$ with $r(x) := x(0, 0) = s(\mu)$, there is a unique $\mu x \in \Lambda^\infty$ such that $(\mu x)(0, d(\mu)) = \mu$ and $\sigma^{d(\mu)}(\mu x) = x$. The sets $Z(\mu, \nu) = \{(\mu x, d(\mu) - d(\nu), \nu x) \mid x \in \Lambda^\infty,\ s(\mu) = s(\nu) = r(x)\}$ indexed by pairs $\mu, \nu \in \Lambda$ form a basis of compact open sets for the topology.

There is an isomorphism $C^*(\Lambda) \cong C^*(\mathcal{G})$ that carries each s_λ to $\mathbf{1}_{Z(\lambda, s(\lambda))}$. The preferred dynamics α on $C^*(\Lambda) = C^*(\mathcal{G})$ corresponds to the cocycle $c\colon \mathcal{G} \to \mathbb{R}$ given by

$$c(x, g, y) = g \cdot \log \rho(\Lambda).$$

That is, for $f \in C_c(\mathcal{G}) \subset C^*(\mathcal{G})$ we have

$$\alpha_t(f)(x, g, y) = e^{itc(x,g,y)} f(x, g, y) = \rho(\Lambda)^{itg} f(x, g, y)$$

for all $t \in \mathbb{R}$ and $(x, g, y) \in \mathcal{G}$.

We now briefly recap the description of the α-KMS states of $C^*(\Lambda)$ from [13]. By [13, Proposition 8.1], there is a unique Borel probability measure μ_{eq} on Λ^∞ with Radon–Nikodym cocycle e^{-c}. Let

$$\operatorname{Per}\Lambda := \{m - n \mid m, n \in \mathbb{N}^k, \sigma^m(x) = \sigma^n(x) \text{ for all } x \in \Lambda^\infty\}.$$

Then $\operatorname{Per}\Lambda$ is a subgroup of $\mathbb{Z}^k$, and [13, Theorem 7.1] describes a one-to-one correspondence between the extremal α-KMS$_1$-states on $C^*(\mathcal{G})$ and the characters of $\operatorname{Per}\Lambda$: the state φ_χ corresponding to a character $\chi \in (\operatorname{Per}\Lambda)\widehat{}$ is given by

$$\varphi_\chi(f) = \int_{\Lambda^\infty} \sum_{g \in \operatorname{Per}\Lambda} \chi(g) f(x, g, x)\, d\mu_{\mathrm{eq}}(x) \quad \text{for } f \in C_c(\mathcal{G}). \tag{5.1}$$

Let π_{φ_χ} denote the GNS representation associated to φ_χ. Our goal is to understand the corresponding von Neumann algebra $\pi_{\varphi_\chi}(C^*(\mathcal{G}))''$. The following technical result from [13] will play a key role.

Proposition 5.1. [13, Lemma 12.1] *Let Λ be a strongly connected finite k-graph, and $\mathcal{G}$ be the associated groupoid. For μ_{eq}-a.e. $x \in \Lambda^\infty$, we have $\mathcal{G}_x^x = \{x\} \times \operatorname{Per}\Lambda \times \{x\}$.*

Note that by Remark 4.5, quasi-invariance of μ_{eq} immediately gives $\mathcal{G}_x^x \subseteq \{(x,g,x) \mid \rho(\Lambda)^g = 1\}$ for μ_{eq}-a.e. x. But Proposition 5.1 says much more for $k \geq 2$, since the inclusion

$$\operatorname{Per}\Lambda \subseteq \{m-n \mid m,n \in \mathbb{N}^k, A^m = A^n\}, \tag{5.2}$$

proved in [13, Remark 7.2], shows that $\operatorname{Per}\Lambda$ is generally an infinite index subgroup of $\{g \mid \rho(\Lambda)^g = 1\}$.

Consider the orbit equivalence relation $\mathcal{R}$ on Λ^∞ defined by $\mathcal{G}$, so

$$x \sim_{\mathcal{R}} y \text{ if and only if } \sigma^m(x) = \sigma^n(y) \text{ for some } m,n \in \mathbb{N}^k.$$

By Lemma 4.4, the measure μ_{eq} is quasi-invariant with respect to $\mathcal{R}$, and the corresponding Radon–Nikodym cocycle D can be described as follows. There is an $\mathcal{R}$-invariant μ_{eq}-conull set $X \subseteq \Lambda^\infty$ such that for all $x,y \in X$ and $m,n \in \mathbb{N}^k$ satisfying $\sigma^m(x) = \sigma^n(y)$, we have

$$D(x,y) = e^{-c(x,m-n,y)} = \rho(\Lambda)^{n-m}. \tag{5.3}$$

From now on we view $\mathcal{R}$ as a measurable equivalence relation on $(\Lambda^\infty, \mu_{\mathrm{eq}})$. We write ν_{eq} for the left counting measure on $\mathcal{R}$ obtained from μ_{eq} as in Section 4, and write $\pi : \mathbb{C}[\mathcal{R}] \to \mathcal{B}(L^2(\mathcal{R}, d\nu_{\mathrm{eq}}))$ for the representation (4.1), so that $W^*(\mathcal{R}) = \pi(\mathbb{C}[\mathcal{R}])''$.

The following result underpins our computation of the factor type of the extremal α-KMS_1 states described in (5.1). The idea for the isomorphism in part (2) comes from [18, Remark 2.5].

Proposition 5.2. *Let Λ be a strongly connected finite k-graph and $\mathcal{G}$ be the associated groupoid. Let χ be a character of $\operatorname{Per}\Lambda$, let φ_χ be the extremal KMS state of $C^*(\Lambda)$ defined in (5.1), and let $\tilde{\chi}$ be a character of $\mathbb{Z}^k$ extending χ. Then*

1. *there is a $*$-homomorphism $\Phi_{\tilde{\chi}} \colon C_c(\mathcal{G}) \to \mathbb{C}[\mathcal{R}]$ such that*
$$\Phi_{\tilde{\chi}}(f)(x,y) = \sum_{(x,g,y)\in\mathcal{G}} \tilde{\chi}(g) f(x,g,y) \quad \textit{for } f \in C_c(\mathcal{G});$$
2. *the map $\pi_{\varphi_\chi}(f) \mapsto \pi(\Phi_{\tilde{\chi}}(f))$ for $f \in C_c(\mathcal{G})$ extends uniquely to a von Neumann algebra isomorphism of $\pi_{\varphi_\chi}(C^*(\mathcal{G}))''$ onto $W^*(\mathcal{R})$;*
3. *$W^*(\mathcal{R})$ is a factor and the equivalence relation $\mathcal{R}$ on $(\Lambda^\infty, \mu_{\mathrm{eq}})$ is ergodic.*

Proof. Part (1): Since $f \in C_c(\mathcal{G})$, the formula for $\Phi_{\tilde{\chi}}(f)$ has only finitely many nonzero terms, so the series converges, and the function $\Phi_{\tilde{\chi}}(f)$ is supported on finitely many Borel bisections of $\mathcal{R}$. It is clear that $\Phi_{\tilde{\chi}}(f)$ is Borel. Thus $\Phi_{\tilde{\chi}}(f)$ belongs to $\mathbb{C}[\mathcal{R}]$. Direct calculation shows that $\Phi_{\tilde{\chi}}$ is a $*$-homomorphism.

Part (2): Proposition 5.1 implies that for $f \in C_c(\mathcal{G})$, we have

$$\Phi_{\tilde{\chi}}(f)(x,x) = \sum_{g\in\operatorname{Per}\Lambda} \chi(g) f(x,g,x) \quad \text{for } \mu_{\mathrm{eq}}\text{-a.e. } x \in \Lambda^\infty.$$

Hence the state φ on $W^*(\mathcal{R})$ defined by (4.2) satisfies $\varphi\circ\Phi_{\tilde{\chi}} = \varphi_\chi$. Since $\Phi_{\tilde{\chi}}(C_c(\mathcal{G}))$ is strong-operator dense in $W^*(\mathcal{R})$, we can identify $\pi \circ \Phi_{\tilde{\chi}}$ with the GNS-representation of φ_χ. Hence $\pi_{\varphi_\chi}(C^*(\mathcal{G}))'' \cong W^*(\mathcal{R})$.

Part (3): Since φ_χ is an extremal KMS state, [2, Theorem 5.3.30(3)] implies that $W^*(\mathcal{R}) \cong \pi_{\varphi_\chi}(C^*(\Lambda))''$ is a factor. Proposition 2.9(2) of [8] implies that $\mathcal{R}$ is ergodic. ☐

Equation (5.3) shows that the finer equivalence relation $\mathcal{R}^D$ of Proposition 4.1 for $\mathcal{Q} = \mathcal{R}$ and $\mu = \mu_{\mathrm{eq}}$ is given, up to a set of measure 0, by

$$x \sim_{\mathcal{R}^D} y \text{ if and only if } \sigma^m(x) = \sigma^n(y) \text{ for some } m,n \in \mathbb{N}^k \text{ with } \rho(\Lambda)^{m-n} = 1. \tag{5.4}$$

Corollary 4.2 leads us to analyse this equivalence relation $\mathcal{R}^D$. To do this, it will help to consider an even finer equivalence relation.

Recall that $C^*(\Lambda)$ carries a canonical action γ of $\mathbb{T}^k$, and that $C^*(\Lambda)^\gamma$ denotes the fixed-point algebra for γ (see [15]).

Lemma 5.3. *Let Λ be a strongly-connected finite k-graph and $\mathcal{G}$ be the associated groupoid. Let $\mathcal{R}^\gamma$ denote the relation*

$$x \sim_{\mathcal{R}^\gamma} y \text{ if and only if } \sigma^n(x) = \sigma^n(y) \text{ for some } n \in \mathbb{N}^k,$$

regarded as a subgroupoid of the measurable equivalence relation $\mathcal{R}^D$. Then, using the isomorphism $(x, y) \mapsto (x, 0, y)$ of $\mathcal{R}^\gamma$ onto the kernel $\mathcal{G}_0$ of the canonical $\mathbb{Z}^k$-valued cocycle $(x, m, y) \mapsto m$ on $\mathcal{G}$, we can consider $\mathcal{R}^\gamma$ as an étale topological equivalence relation, and $C^*(\mathcal{R}^\gamma) \cong C^*(\Lambda)^\gamma$.

Proof. The isomorphism $(x, y) \mapsto (x, 0, y)$ of $\mathcal{R}^\gamma$ onto $\mathcal{G}_0$ is Borel, so it can be used to define on $\mathcal{R}^\gamma$ the structure of an étale groupoid. The isomorphism $C^*(\Lambda) \cong C^*(\mathcal{G})$ carries the gauge action to the action given by $\beta_z(f)(x, g, y) = z^g f(x, g, y)$ for $f \in C_c(\mathcal{G})$ (see the proof of [15, Corollary 3.5(i)]). So it carries the fixed-point algebra $C^*(\Lambda)^\gamma$ to the completion of the functions on $\mathcal{G}$ supported on $\mathcal{G}_0$. Hence $C^*(\mathcal{R}^\gamma) \cong C^*(\Lambda)^\gamma$. $\square$

Let χ be a character of $\operatorname{Per}\Lambda$, and let $\Phi_{\tilde{\chi}}$ be as in Proposition 5.2(1). Lemma 5.3 implies that $W^*(\mathcal{R}^\gamma) \subseteq W^*(\mathcal{R})$ is the von Neumann algebra generated by the image under (the extension of) $\pi \circ \Phi_{\tilde{\chi}}$ of $C^*(\Lambda)^\gamma$. Note in passing that the gauge action need not pass from $C^*(\Lambda)$ to $W^*(\mathcal{R})$; it extends precisely when $\operatorname{Per}\Lambda = 0$.

6. Frobenius analysis of strongly connected higher-rank graphs

In this section we analyse strongly connected finite k-graphs that are not necessarily primitive to obtain a further refinement of the Perron–Frobenius theory for them developed in [16, Lemma 4.1] and [13, Corollary 4.2]. Generalisations of Perron–Frobenius theory have been studied by a number of authors (see for example [9] and the references therein), so part of the content of this section may be known to experts. We give a self-contained presentation since we could not find exactly what we need in the literature.

Lemma 6.1. *Let Λ be a strongly connected k-graph. For each vertex $v \in \Lambda^0$ let $\mathcal{P}_v^+ := d(v\Lambda v) \subseteq \mathbb{N}^k$. Then $\mathcal{P}_v^+$ is a subsemigroup of $\mathbb{N}^k$ and $\mathcal{P}_v^+ - \mathcal{P}_v^+$ a subgroup of $\mathbb{Z}^k$. Let*

$$\mathcal{P}_\Lambda := \{d(\mu) - d(\nu) \mid \mu, \nu \text{ are cycles in } \Lambda\}.$$

Then $\mathcal{P}_\Lambda = \mathcal{P}_v^+ - \mathcal{P}_v^+$ for every $v \in \Lambda^0$. We call $\mathcal{P}_\Lambda$ the group of periods *of Λ.*

Proof. Fix $v \in \Lambda^0$. The set $\mathcal{P}_v^+$ is a semigroup because d carries composition to addition, and then $\mathcal{P}_v^+ - \mathcal{P}_v^+$ is obviously a subgroup of $\mathbb{Z}^k$. We clearly have $\mathcal{P}_v^+ - \mathcal{P}_v^+ \subseteq \mathcal{P}_\Lambda$ for each v. For the reverse inclusion, fix cycles $\mu, \nu \in \Lambda$ and $v \in \Lambda^0$. Since Λ is strongly connected, there exist $\lambda \in r(\nu)\Lambda r(\mu)$, $\lambda' \in r(\mu)\Lambda r(\nu)$, $\eta \in v\Lambda r(\nu)$ and $\eta' \in r(\nu)\Lambda v$. Hence

$$d(\mu) - d(\nu) = d(\eta\lambda\mu\lambda'\eta') - d(\eta\nu\lambda\lambda'\eta') \in \mathcal{P}_v^+ - \mathcal{P}_v^+. \qquad \square$$

When $k = 1$, the group $\mathcal{P}_\Lambda$ is the subgroup of $\mathbb{Z}$ generated by the classical period of the directed graph $(\Lambda^0, \Lambda^1, r, s)$ (see, for example, [23, 14]).

Remark 6.2. Since $|v\Lambda^n w| = A^n(v, w) = \big(\prod_{i=1}^k A_i^{n_i}\big)(v, w)$ for all v, w, n, the group $\mathcal{P}_\Lambda$ depends only on the connectivity matrices of Λ and is independent of the factorisation property. By contrast, for $k \geq 2$ the group $\operatorname{Per}\Lambda$ is not determined by the A_i alone (see [5]).

We now establish a number of properties of $\mathcal{P}_\Lambda$ that we will need in order to compute the types of the KMS states of $C^*(\Lambda)$.

Lemma 6.3. *Let Λ be a strongly connected k-graph. Then $d(\lambda) - d(\mu) \in \mathcal{P}_\Lambda$ whenever $\lambda, \mu \in \Lambda$ satisfy $r(\lambda) = r(\mu)$ and $s(\lambda) = s(\mu)$. In particular, there is a function $C : \Lambda^0 \times \Lambda^0 \to \mathbb{Z}^k/\mathcal{P}_\Lambda$ such that $C(r(\lambda), s(\lambda)) = d(\lambda) + \mathcal{P}_\Lambda$ for all $\lambda \in \Lambda$. For $u, v, w \in \Lambda^0$, we have*

$$C(u, v) + C(v, w) = C(u, w), \quad C(u, u) = 0, \quad \text{and} \quad C(u, v) = -C(v, u).$$

In particular, there is an equivalence relation $\sim$ on Λ^0 such that $v \sim w$ if and only if $C(v,w) = 0$.

Proof. Fix λ, μ with $r(\lambda) = r(\mu)$ and $s(\lambda) = s(\mu)$. Choose $\nu \in s(\lambda)\Lambda r(\lambda)$, so $\mu\nu$ and $\lambda\nu$ are cycles. Then $d(\lambda) - d(\mu) = d(\lambda\nu) - d(\mu\nu) \in \mathcal{P}_\Lambda$. Since $v\Lambda w$ is nonempty for all $v, w \in \Lambda^0$, it follows that there is a well-defined function $C : \Lambda^0 \times \Lambda^0 \to \mathbb{Z}^k/\mathcal{P}_\Lambda$ satisfying $C(r(\lambda), s(\lambda)) = d(\lambda) + \mathcal{P}_\Lambda$ for all $\lambda \in \Lambda$.

Choose $\mu \in u\Lambda v$ and $\nu \in v\Lambda w$, and note that

$$C(u,w) = d(\mu\nu) + \mathcal{P}_\Lambda = d(\mu) + d(\nu) + \mathcal{P}_\Lambda = C(u,v) + C(v,w).$$

Now $C(u,u) = C(u,u) + C(u,u)$ forces $C(u,u) = 0$, and then $0 = C(u,u) = C(u,v) + C(v,u)$ forces $C(u,v) = -C(v,u)$ for all u and v. The last statement follows. $\square$

Corollary 6.4. *Let Λ be a strongly connected finite k-graph, and $\mathcal{G}$ be the associated groupoid. For $\lambda \in \Lambda$ and $w \in \Lambda^0$, we have $C(r(\lambda), w) = d(\lambda) + C(s(\lambda), w)$, and for $(x, g, y) \in \mathcal{G}$ we have $C(r(x), r(y)) = g + \mathcal{P}_\Lambda$.*

Proof. The first assertion follows from $C(r(\lambda), w) - C(s(\lambda), w) = C(r(\lambda), s(\lambda))$ and the definition of C. For the second, take $(x, g, y) \in \mathcal{G}$, and pick $m, n \in \mathbb{N}^k$ with $m - n = g$ and $\sigma^m(x) = \sigma^n(y)$. Put $\mu = x(0, m)$ and $\nu = x(0, n)$. Then $s(\mu) = r(\sigma^m(x)) = r(\sigma^n(x)) = s(\nu)$. Hence $C(r(x), r(y)) = C(r(\mu), s(\mu)) - C(r(\nu), s(\nu)) = m + \mathcal{P}_\Lambda - n + \mathcal{P}_\Lambda = g + \mathcal{P}_\Lambda$. $\square$

The following is the main technical result we will need later.

Proposition 6.5. *Let Λ be a strongly connected finite k-graph. There exists $p \in \mathcal{P}_\Lambda \cap (\mathbb{N} \setminus \{0\})^k$ such that for all $v, w \in \Lambda^0$, we have $v \sim w$ if and only if $v\Lambda^p w \neq \emptyset$.*

Proof. Fix a vertex $u_0 \in \Lambda^0$. For each $v \in \Lambda^0$ fix paths $\lambda_v \in v\Lambda u_0$ and $\mu_v \in u_0\Lambda v$. For each $v, w \in \Lambda^0$, define $g_{v,w} := d(\lambda_v\mu_w) \in \mathbb{N}^k$.

We define $p \in (\mathbb{N} \setminus \{0\})^k$ as follows. For each $v, w \in \Lambda$ with $v \sim w$, we have $d(\lambda_v\mu_w) + \mathcal{P}_\Lambda = C(v,w) = 0 + \mathcal{P}_\Lambda$, and so $g_{v,w} \in \mathcal{P}_\Lambda$. By Lemma 6.1 there are cycles α, β in $u_0\Lambda u_0$ with $d(\alpha) - d(\beta) = g_{v,w}$. Since Λ has no sources, there exists $\tau \in u_0\Lambda^{(1,\dots,1)}$ and since Λ is strongly connected, there exists $\tau' \in s(\tau)\Lambda u_0$. Now $\alpha\tau\tau'$ and $\beta\tau\tau'$ are cycles, and $m_{v,w} := d(\alpha\tau\tau')$ and $n_{v,w} := d(\beta\tau\tau')$ belong to $\mathcal{P}^+_{u_0}$. In particular, $m_{v,w}, n_{v,w} \in \mathcal{P}_\Lambda \cap (\mathbb{N} \setminus \{0\})^k$ and $g_{v,w} = m_{v,w} - n_{v,w}$. Let

$$p := \sum_{v \sim w} m_{v,w}.$$

Since $\mathcal{P}^+_{u_0}$ is a semigroup, we have $p \in \mathcal{P}_\Lambda \cap (\mathbb{N} \setminus \{0\})^k$. We show that $v \sim w$ if and only if $v\Lambda^p w \neq \emptyset$.

First suppose that $v \sim w$. We have

$$p = \sum_{v' \sim w'} m_{v',w'} = g_{v,w} + \Big(n_{v,w} + \sum_{v' \sim w', (v',w') \neq (v,w)} m_{v',w'}\Big).$$

Let $n := \Big(n_{v,w} + \sum_{v' \sim w', (v',w') \neq (v,w)} m_{v',w'}\Big)$. Then $n \in \mathcal{P}^+_{u_0}$; say $\nu \in u_0\Lambda^n u_0$. So $d(\lambda_v\nu\mu_w) = g_{v,w} + n = p$, giving $\lambda_v\nu\mu_w \in v\Lambda^p w$.

Now suppose that $v\Lambda^p w \neq \emptyset$, say $\lambda \in v\Lambda^p w$. Then $C(v,w) = d(\lambda) + \mathcal{P}_\Lambda = p + \mathcal{P}_\Lambda = 0 + \mathcal{P}_\Lambda$, giving $v \sim w$. $\square$

Proposition 6.6. *Let Λ be a strongly connected finite k-graph. For $v \in \Lambda^0$, the map $C(\cdot, v) : \Lambda^0 \to \mathbb{Z}^k/\mathcal{P}_\Lambda$ induces a bijection $\widetilde{C}_v : \Lambda^0/\sim \to \mathbb{Z}^k/\mathcal{P}_\Lambda$. There is a free and transitive action of $\mathbb{Z}^k/\mathcal{P}_\Lambda$ on $\Lambda^0/\sim$ such that $\widetilde{C}_v\big((g + \mathcal{P}_\Lambda) \cdot [w]\big) = g + \widetilde{C}_v([w])$ for all $g \in \mathbb{Z}^k$, $v \in \Lambda^0$ and $[w] \in \Lambda^0/\sim$.*

Proof. If $u \sim w$ then $C(u,w) = 0$, and so Lemma 6.3 gives $C(u,v) = C(u,w) + C(w,v) = C(w,v)$. So $C(\cdot, v)$ descends to a function $\widetilde{C}_v : \Lambda^0/\sim \to \mathbb{Z}^k/\mathcal{P}_\Lambda$. If $\widetilde{C}_v([u]) = \widetilde{C}_v([w])$, then Lemma 6.3 gives $C(u,w) = C(u,v) + C(v,w) = C(u,v) - C(w,v) = \widetilde{C}_v([u]) - \widetilde{C}_v([w]) = 0$ and so $u \sim w$. So $\widetilde{C}_v$ is injective. For surjectivity, fix $g + \mathcal{P}_\Lambda \in \mathbb{Z}^k/\mathcal{P}_\Lambda$, and express $g = m - n$ with $m, n \in \mathbb{N}^k$. By (2.2), there exist $\lambda \in \Lambda^m v$, and then $\mu \in r(\lambda)\Lambda^n$. Another application of Lemma 6.3 gives

$$\widetilde{C}_v([s(\mu)]) = C(s(\mu), v) = C(r(\lambda), v) - C(r(\lambda), s(\mu)) = m - n + \mathcal{P}_\Lambda = g + \mathcal{P}_\Lambda,$$

so $\widetilde{C}_v$ is surjective.

Pulling back the action of $\mathbb{Z}^k/\mathcal{P}_\Lambda$ on itself by translation along the bijection $\widetilde{C}_v$ gives the desired free and transitive action on $\Lambda^0/\!\sim$. Choose another $v' \in \Lambda^0$ and let $\diamond$ denote the action of $\mathbb{Z}^k/\mathcal{P}_\Lambda$ such that $\widetilde{C}_{v'}\big((g+\mathcal{P}_\Lambda)\diamond[w]\big) = g + \widetilde{C}_{v'}([w])$. Choose $u \in (g+\mathcal{P}_\Lambda)\cdot[w]$ and $u' \in (g+\mathcal{P}_\Lambda)\diamond[w]$. Using Lemma 6.3 repeatedly, we check that

$$\begin{aligned}C(u,u') &= C(u,v) + C(v,v') - C(u',v') = C(v,v') + \widetilde{C}_v\big((g+\mathcal{P}_\Lambda)\cdot[w]\big) - \widetilde{C}_{v'}\big((g+\mathcal{P}_\Lambda)\diamond[w]\big)\\ &= C(v,v') + (g + \widetilde{C}_v([w])) - (g + \widetilde{C}_{v'}([w])) = C(v,v') + C(w,v) - C(w,v') = 0.\end{aligned}$$

So $(g+\mathcal{P}_\Lambda)\cdot[w] = (g+\mathcal{P}_\Lambda)\diamond[w]$ for all w. □

Corollary 6.7. *Let Λ be a strongly connected finite k-graph. Then $\mathcal{P}_\Lambda$ has index at most $|\Lambda^0|$ in $\mathbb{Z}^k$, and $\operatorname{Per}\Lambda \subseteq \mathcal{P}_\Lambda$. We have $\mathcal{P}_\Lambda = \mathbb{Z}^k$ if and only if Λ is primitive.*

Proof. The bijection $\mathbb{Z}^k/\mathcal{P}_\Lambda \to \Lambda^0/\!\sim$ of Proposition 6.6 shows that $|\mathbb{Z}^k/\mathcal{P}_\Lambda| \le |\Lambda^0|$. We have $\operatorname{Per}\Lambda \subseteq \{d(\mu) - d(\nu) \mid r(\mu) = r(\nu) \text{ and } s(\mu) = s(\nu)\}$ by (5.2) and so Lemma 6.3 gives $\operatorname{Per}\Lambda \subseteq \mathcal{P}_\Lambda$.

Suppose that Λ is primitive; say $v\Lambda^p w \neq \emptyset$ for all v, w. Then each $v\Lambda^p v \neq \emptyset$, so $p \in \mathcal{P}_\Lambda$. For $v, w \in \Lambda^0$ and $\lambda \in v\Lambda^p w$, we have $C(v,w) = d(\lambda) + \mathcal{P}_\Lambda = p + \mathcal{P}_\Lambda = 0 + \mathcal{P}_\Lambda$. So $\Lambda^0/\!\sim$, and hence also $\mathbb{Z}^k/\mathcal{P}_\Lambda$, is a singleton, giving $\mathcal{P}_\Lambda = \mathbb{Z}^k$. Now suppose that $\mathcal{P}_\Lambda = \mathbb{Z}^k$. Then $C(v,w) = 0$ for all v, w, and then Λ is primitive by Proposition 6.5. □

With $\sim$ as in Lemma 6.3, for an equivalence class $\omega \in \Lambda^0/\!\sim$ we identify $\mathbb{R}^\omega$ with $\operatorname{span}\{\delta_v \mid v \in \omega\} \subseteq \mathbb{R}^{\Lambda^0}$. So $\mathbb{R}^{\Lambda^0}$ decomposes as the internal direct sum

$$\mathbb{R}^{\Lambda^0} = \bigoplus\nolimits_{\omega \in \Lambda^0/\sim} \mathbb{R}^\omega. \tag{6.1}$$

Observe that the action of Proposition 6.6 determines a transitive action of $\mathbb{N}^k$ on $\Lambda^0/\!\sim$.

Proposition 6.8. *Let Λ be a strongly connected finite k-graph. The decomposition* (6.1) *defines a system of imprimitivity for the semigroup of matrices $(A^n)_{n\in\mathbb{N}^k}$; that is, $A^n\mathbb{R}^\omega \subseteq \mathbb{R}^{n\cdot\omega}$ for all $n \in \mathbb{N}^k$ and $\omega \in \Lambda^0/\!\sim$.*

Proof. Take $\omega \in \Lambda^0/\!\sim$ and $v \in \omega$. If $|w\Lambda^n v| = A^n(w,v) > 0$, then $C(w,v) = n + \mathcal{P}_\Lambda$ and hence the class of w in $\Lambda^0/\!\sim$ is $n \cdot \omega$. □

For the next result, let Δ_k denote the k-graph with objects $\mathbb{Z}^k$, morphisms $\{(g,h) \in \mathbb{Z}^k \times \mathbb{Z}^k \mid g \le h\}$, structure maps $r(g,h) = g$, $s(g,h) = h$ and $d(g,h) = h - g$, and composition $(g,h)(h,l) = (g,l)$. There is a free and transitive action of $\mathbb{Z}^k$ on Δ_k by translation, and for any subgroup $G \le \mathbb{Z}^k$, the quotient Δ_k/G is a k-graph under the inherited operations. If G has finite index, then Δ_k/G is finite and strongly connected and we think of it as a simple k-dimensional cycle. We write $[g,h]$ for the image of $(g,h) \in \Delta_k$ in Δ_k/G, and we identify $(\Delta_k/G)^0$ with $\mathbb{Z}^k/G$ via $[g,g] \mapsto [g]$.

Proposition 6.9. *Let Λ be a strongly connected finite k-graph. The following are equivalent.*

1. $\Lambda \cong \Delta_k/\mathcal{P}_\Lambda$*;*
2. $|v\Lambda^{e_i}| = 1$ *for all* $v \in \Lambda^0$ *and* $1 \le i \le k$*;*
3. $\mathcal{P}_\Lambda = \operatorname{Per}\Lambda$*; and*
4. $\rho(\Lambda) = (1, \dots, 1)$.

Proof. (1) $\Longrightarrow$ (2). Take an isomorphism $\phi : \Lambda \to \Delta_k/\mathcal{P}_\Lambda$. For $v \in \Lambda^0$, pick $g \in \mathbb{Z}^k$ with $\phi(v) = [g]$. Then $|v\Lambda^{e_i}| = |[g](\Delta_k/\mathcal{P}_\Lambda)^{e_i}| = |\{[g, g+e_i]\}| = 1$.

(2) $\Longrightarrow$ (3). We have $\operatorname{Per}\Lambda \subseteq \mathcal{P}_\Lambda$ by Corollary 6.7. Condition (2) implies that each $|v\Lambda^\infty| = 1$. So if μ is a cycle and $x \in s(\mu)\Lambda^\infty$, then $\mu x \in s(\mu)\Lambda^\infty$ forces $\mu x = x$. Hence $\sigma^{d(\mu)}(x) = x$ for all

$x \in s(\mu)\Lambda^\infty$, and [13, Lemma 5.1] gives $\sigma^{d(\mu)}(y) = y$ for all y. Since the degrees of cycles generate $\mathcal{P}_\Lambda$ we obtain $\mathcal{P}_\Lambda \subseteq \operatorname{Per}\Lambda$.

(3) $\Longrightarrow$ (4). We prove the contrapositive. Suppose that $\rho(\Lambda) \neq (1, \dots, 1)$. By (5.2), $\operatorname{Per}\Lambda$ is contained in the group of elements $g \in \mathbb{Z}^k$ such that $g \cdot \log \rho(\Lambda) = 0$. Hence $\operatorname{Per}\Lambda$ has infinite index in $\mathbb{Z}^k$, while $\mathcal{P}_\Lambda$ has finite index by Corollary 6.7.

(4) $\Longrightarrow$ (1). Proposition 6.5 gives $p \in \mathbb{N}^k$ such that $v \sim w$ if and only if $v\Lambda^p w \neq \emptyset$. So A^p is block-diagonal with strictly positive diagonal blocks indexed by $\omega \in \Lambda^0/\!\sim$. Since $\rho(A^p) = \rho(\Lambda)^p = 1$, each diagonal block of A^p has spectral radius 1. The only strictly positive square integer matrix with spectral radius 1 is the 1×1 identity matrix, so A^p is the identity matrix, and $\sim$ is the trivial relation. Now Propositions 6.6 and 6.8 give a bijection $\phi : \Lambda^0 \to \mathbb{Z}^k/\mathcal{P}_\Lambda$ such that $u\Lambda^n w \neq \emptyset$ implies $\phi(u) = n + \phi(w)$. This shows that every column and row of the matrix A^n has at most one nonzero entry. Since A^n has no zero columns and rows, and $\rho(A^n) = 1$, this implies that A^n is a permutation matrix. Specifically, we have $|u\Lambda^n w| = \delta_{n+\mathcal{P}_\Lambda, \phi(u)-\phi(w)}$. So $\lambda \mapsto [\phi(r(\lambda)), \phi(s(\lambda))]$ is a bijection of Λ onto $\Delta^k/\mathcal{P}_\Lambda$ that preserves range, source and degree. Since each $|v\Lambda^n| = 1$ this bijection automatically preserves composition, and hence is an isomorphism of k-graphs. □

7. Type classification of the KMS$_1$ states of $C^*(\Lambda)$

Let Λ be a strongly connected finite k-graph, and let $C^*(\Lambda)^\gamma$ be the fixed-point algebra for the gauge action γ of $\mathbb{T}^k$ on $C^*(\Lambda)$. By Lemma 5.3, we have $C^*(\Lambda)^\gamma \cong C^*(\mathcal{R}^\gamma)$, and so $C^*(\mathcal{R}^\gamma)$ is AF (see [15]). Specifically, for each $n \in \mathbb{N}^k$ and $v \in \Lambda^0$, put $\mathcal{F}_\Lambda(n, v) := \overline{\operatorname{span}}\{s_\mu s_\nu^* \mid \mu, \nu \in \Lambda^n v\}$, and then put $\mathcal{F}_\Lambda(n) := \overline{\operatorname{span}}\{s_\mu s_\nu^* \mid \mu, \nu \in \Lambda^n\}$ for each n. The $s_\mu s_\nu^*$ in any given $\mathcal{F}_\Lambda(n, v)$ are matrix units, so $\mathcal{F}_\Lambda(n, v) \cong \operatorname{Mat}_{\Lambda^n v}(\mathbb{C})$, and we have $\mathcal{F}_\Lambda(n) = \bigoplus_v \mathcal{F}_\Lambda(n, v)$. Relation (CK) shows that if $\mu, \nu \in \Lambda^n v$ then $s_\mu s_\nu^* = \sum_{\lambda \in v\Lambda^m} s_{\mu\lambda} s_{\nu\lambda}^* \in \mathcal{F}_\Lambda(m + n)$ for all m. So each $\mathcal{F}_\lambda(n) \subset \mathcal{F}_\Lambda(m + n)$, with inclusion matrix A^m, and the inductive limit of the algebras $\mathcal{F}_\Lambda(n)$ is $C^*(\Lambda)^\gamma$.

Given a matrix $B \in \operatorname{Mat}_N(\mathbb{N})$, let $\mathcal{F}_B$ denote the unital AF algebra whose Bratteli diagram is stationary with N vertices $\{v_{n,i} \mid 1 \leq i \leq N\}$ at level n, and $B(i, j)$ edges connecting $v_{n,i}$ to $v_{n+1,j}$ for all i, j. That is, $\mathcal{F}_B = \varinjlim C_n$, where $C_n = \bigoplus_{i=1}^N \operatorname{Mat}_{\sum_k B^n(k,i)}(\mathbb{C})$, and the partial inclusions $C_{n,i} \hookrightarrow C_{n+1,j}$ have multiplicity $B(i, j)$.

Proposition 7.1. *Let Λ be a strongly connected finite k-graph, and let $\sim$ be the equivalence relation on Λ^0 described in Lemma 6.3. Take $p \in (\mathbb{N} \setminus \{0\})^k$ as in Proposition 6.5, so $v \sim w$ if and only if $v\Lambda^p w \neq \emptyset$. For $\omega \in \Lambda^0/\!\sim$, define $A^p_\omega \in \operatorname{Mat}_\omega(\mathbb{N})$ by $A^p_\omega(v, w) = |v\Lambda^p w|$ for $v, w \in \omega$, and define $q_\omega := \sum_{v \in \omega} s_v \in C^*(\Lambda)$. Then each A^p_ω is primitive, the projections q_ω are central in $C^*(\Lambda)^\gamma$, each $q_\omega C^*(\Lambda)^\gamma \cong \mathcal{F}_{A^p_\omega}$, and $C^*(\Lambda)^\gamma = \bigoplus_{\omega \in \Lambda^0/\sim} q_\omega C^*(\Lambda)^\gamma$.*

Proof. Each A^p_ω is primitive — indeed its entries are all strictly positive — by choice of p. Since $p_i \neq 0$ for all i, the sequence $(np)_{n \in \mathbb{N}}$ is cofinal in $\mathbb{N}^k$. Hence $C^*(\Lambda)^\gamma = \varinjlim_{n \in \mathbb{N}^k} \mathcal{F}_\Lambda(n) = \varinjlim_{n \in \mathbb{N}} \mathcal{F}_\Lambda(np)$. As explained above, the inclusion $\mathcal{F}_\Lambda(np) \hookrightarrow \mathcal{F}_\Lambda((n + 1)p)$ has matrix A^p, which is block-diagonal with blocks A^p_ω by choice of p. So $C^*(\Lambda)^\gamma \cong \bigoplus_\omega \mathcal{F}_{A^p_\omega}$ as claimed. Each q_ω is the identity projection in A^p_ω, so is central. □

Since the A^p_ω are primitive, the stationary Bratteli diagrams they determine are cofinal, so each $q_\omega C^*(\Lambda)^\gamma \cong \mathcal{F}_{A^p_\omega}$ is simple by [1, Corollary 3.5]. It follows also that each $q_\omega C^*(\Lambda)^\gamma$ has a unique tracial state, see for example [19, Proposition 10.4.9]. The trace vector of the approximating subalgebra $\bigoplus_{v \in \omega} \mathcal{F}_\Lambda(np, v)$ of $q_\omega C^*(\Lambda)^\gamma$ is given by $\rho(A^p_\omega)^{-n} \xi^\omega$, where ξ^ω is the Perron-Frobenius eigenvector of A^p_ω with unit 1-norm. As a special case, we deduce that the following are equivalent:

(1) Λ is primitive;
(2) $C^*(\Lambda)^\gamma$ is simple; and
(3) $C^*(\Lambda)^\gamma$ carries a unique tracial state.

Equivalence of the first two assertions has been already proved in [17, Theorem 7.2].

Corollary 7.2. *Let Λ be a strongly connected finite k-graph, and let $\sim$ be the equivalence relation on Λ^0 described in Lemma 6.3. Let $\mathcal{R}^\gamma$ be the equivalence relation of Lemma 5.3. Then the ergodic components of $\mathcal{R}^\gamma$ with respect to μ_{eq} are the sets*

$$X_\omega := \{x \in \Lambda^\infty \mid r(x) \in \omega\} \quad \textit{for} \quad \omega \in \Lambda^0/\!\sim. \tag{7.1}$$

Proof. The isomorphism $C^*(\Lambda)^\gamma \cong C^*(\mathcal{R}^\gamma)$ carries each projection q_ω described in Proposition 7.1 to the characteristic function $\mathbf{1}_{X_\omega}$ of X_ω regarded as a subset of the diagonal in $\mathcal{R}^\gamma$. Hence the projections $\mathbf{1}_{X_\omega}$ are central in $C^*(\mathcal{R}^\gamma)$, so the sets X_ω are $\mathcal{R}^\gamma$-invariant and $\Lambda^\infty = \bigsqcup_\omega X_\omega$. For $\omega \in \Lambda^0/\!\sim$, let $\mathcal{R}^\gamma_\omega := \mathcal{R}^\gamma \cap (X_\omega \times X_\omega)$, regarded as an equivalence relation on $(X_\omega, \mu_{\mathrm{eq}}|_{X_\omega})$. Then $\mathcal{R}^\gamma_\omega$ is clopen in $\mathcal{R}^\gamma$, and the canonical inclusion $C_c(\mathcal{R}^\gamma_\omega) \subseteq C_c(\mathcal{R}^\gamma)$ extends to an isomorphism $W^*(\mathcal{R}^\gamma_\omega) \cong \mathbf{1}_{X_\omega} W^*(\mathcal{R}^\gamma)$. The formula $\varphi(f) = \frac{1}{\mu_{\mathrm{eq}}(X_\omega)} \int_{\Lambda^\infty} f(x,x)\,d\mu_{\mathrm{eq}}(x)$ for f supported on $\mathcal{R}^\gamma_\omega$ gives a normal tracial state on $W^*(\mathcal{R}^\gamma_\omega)$. Uniqueness of the trace on the dense subalgebra $C^*(\mathcal{R}^\gamma_\omega) \cong q_\omega C^*(\Lambda)^\gamma$ implies that φ is a unique normal tracial state. Thus $W^*(\mathcal{R}^\gamma_\omega)$ is a factor, and so $\mathcal{R}^\gamma_\omega$ is ergodic by [8, Proposition 2.9(2)]. □

We can now compute the factor types of the extremal KMS states of $C^*(\Lambda)$.

Theorem 7.3. *Let Λ be a strongly connected finite k-graph, let α denote the preferred dynamics on $C^*(\Lambda)$, and let φ_χ be the extremal α-KMS$_1$-state on $C^*(\Lambda)$ corresponding to a character χ of* Per Λ *as in* [13, Theorem 7.1]. *Let $\mathcal{P}_\Lambda \subseteq \mathbb{Z}^k$ be the group of periods of Λ from Lemma 6.1. Then the Connes invariant S of $\pi_{\varphi_\chi}(C^*(\Lambda))''$ is*

$$S = \overline{\{\rho(\Lambda)^g \mid g \in \mathcal{P}_\Lambda\}} \subseteq [0,\infty).$$

Proof. By Proposition 5.2 it suffices to show that $S(W^*(\mathcal{R})) = \overline{\{\rho(\Lambda)^g \mid g \in \mathcal{P}_\Lambda\}}$. Let $\sim$ be as in Lemma 6.3, and fix $\omega \in \Lambda^0/\!\sim$. Let X_ω be as in (7.1). Since $W^*(\mathcal{R})$ is a factor, $S(W^*(\mathcal{R})) = S(\mathbf{1}_{X_\omega} W^*(\mathcal{R}) \mathbf{1}_{X_\omega})$. The corner $\mathbf{1}_{X_\omega} W^*(\mathcal{R}) \mathbf{1}_{X_\omega}$ is also a factor, and is the von Neumann algebra of the relativised equivalence relation $\mathcal{R}_\omega := \mathcal{R} \cap (X_\omega \times X_\omega)$ on $(X_\omega, \mu_{\mathrm{eq}}|_{X_\omega})$.

Consider the relation $\mathcal{R}^D_\omega \subseteq \mathcal{R}_\omega$ as in (5.4), so

$$x \sim_{\mathcal{R}^D_\omega} y \text{ if and only if } x \sim_{\mathcal{R}_\omega} y \text{ and } D(x,y) = 1.$$

Corollary 7.2 implies that $\mathcal{R}^\gamma_\omega = \mathcal{R}^\gamma \cap (X_\omega \times X_\omega) \subseteq \mathcal{R}^D_\omega$ is ergodic, and so $\mathcal{R}^D_\omega$ is ergodic too. So Corollary 4.2 applied to $\mathcal{Q} = \mathcal{R}_\omega$ implies that $S(W^*(\mathcal{R})) = S(W^*(\mathcal{R}_\omega))$ is the set of essential values of $D|_{\mathcal{R}_\omega}$ with respect to the left counting measure ν_{eq} induced by μ_{eq}. We must show that this set is precisely $\overline{\{\rho(\Lambda)^g \mid g \in \mathcal{P}_\Lambda\}}$.

In order to prove that $\overline{\{\rho(\Lambda)^g \mid g \in \mathcal{P}_\Lambda\}}$ contains the set of essential values, it suffices to show that $D(x,y) \in \{\rho(\Lambda)^g \mid g \in \mathcal{P}_\Lambda\}$ for ν_{eq}-a.e. $(x,y) \in \mathcal{R}_\omega$. Take $(x,y) \in \mathcal{R}_\omega$ and choose $m,n \in \mathbb{N}^k$ with $(x, m-n, y) \in \mathcal{G}$. Since $r(x) \sim r(y)$, Corollary 6.4 gives $m - n \in \mathcal{P}_\Lambda$. Hence, by (5.3), we ν_{eq}-almost surely have

$$D(x,y) = \rho(\Lambda)^{n-m} \in \{\rho(\Lambda)^g \mid g \in \mathcal{P}_\Lambda\}.$$

Now fix $s \in \{\rho(\Lambda)^g \mid g \in \mathcal{P}_\Lambda\}$; say $m,n \in \mathbb{N}^k$ satisfy $m - n \in \mathcal{P}_\Lambda$ and $s = \rho(\Lambda)^{n-m}$. For every $x \in X_\omega$ we can find $y \in \Lambda^\infty$ such that $\sigma^m(x) = \sigma^n(y)$, so $(x, m-n, y) \in \mathcal{G}$. Then by Corollary 6.4 we have $r(x) \sim r(y)$, hence $y \in X_\omega$. In particular, the projection of the closed set

$$Z := \{(x,y) \in X_\omega \times X_\omega \mid \sigma^m(x) = \sigma^n(y)\}$$

onto the first coordinate is the entire set X_ω. It follows that $\nu_{\mathrm{eq}}(Z) > 0$, and since $D(x,y) = \rho(\Lambda)^{n-m} = s$ for ν_{eq}-a.e. $(x,y) \in Z$, we see that s is an essential value of D on $\mathcal{R}_\omega$. Since the set of essential values of D is closed, the result follows. □

Remark 7.4. We computed the type of φ_χ without describing the centre of the centraliser of φ_χ in $\pi_{\varphi_\chi}(C^*(\Lambda))''$. But such a description falls easily out of our arguments: We want to understand the ergodic components of the equivalence relation $\mathcal{R}^D$ on $(\Lambda^\infty, \mu_{\mathrm{eq}})$ defined as in (5.4). Since the ergodic components of $\mathcal{R}^\gamma \subseteq \mathcal{R}^D$ are the sets X_ω, the ergodic components of $\mathcal{R}^D$ are unions of these sets. In other words, the ergodic components are defined by a coarser equivalence relation $\approx$ than the relation $\sim$ on Λ^0. A moment's reflection shows that this equivalence relation must be given by $v \approx w$ if and only if there are $\lambda, \mu \in \Lambda$ with

$$v = r(\lambda), \quad w = r(\mu), \quad s(\lambda) = s(\mu) \quad \text{and} \quad \rho(\Lambda)^{d(\lambda)-d(\mu)} = 1.$$

So the minimal central projections of the centraliser are the images of the projections $\sum_{v\in\omega} s_v$ for $\omega \in \Lambda^0/_{\approx}$ (cf. [20]).

Proof of Theorem 3.1. Part (1) follows immediately from Proposition 3.2, so suppose that $\beta = 1$. Then [13, Corollary 4.6] shows that ϕ factors through a state $\overline{\phi}$ of $C^*(\Lambda)$, and [13, Theorem 7.1] implies that $\overline{\phi} = \varphi_\chi$ for some character χ of $\operatorname{Per}(\Lambda)$.

If $\rho(\Lambda) = (1, \dots, 1)$, then Proposition 6.9 shows that $\Lambda \cong \Delta_k/\mathcal{P}_\Lambda$. So each $v\Lambda^n$ and hence each $v\Lambda^\infty$ is a singleton set. Choose $u_0 \in \Lambda^0$ and fix paths $\lambda_v \in v\Lambda u_0$ indexed by $v \in \Lambda^0$. Let x be the unique infinite path with range u_0. Then $(\lambda_v x, d(\lambda_v) - d(\lambda_w), \lambda_w x) \in \mathcal{G}$ for all v, w, and it follows that the equivalence relation $\mathcal{R}$ is the full equivalence relation $\Lambda^0 \times \Lambda^0$. Hence $W^*(\mathcal{R}) \cong \operatorname{Mat}_{\Lambda^0}(\mathbb{C})$. This proves statement (2a).

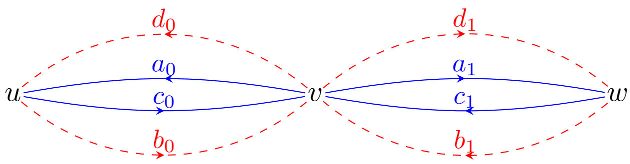
Now suppose that $\rho(\Lambda) \neq (1, \dots, 1)$. Then Theorem 7.3 shows that the Connes invariant of $\pi_{\overline{\phi}}(C^*(\Lambda))''$ is the closure of the set S described in statement (2b). Since $\mathcal{P}_\Lambda$ is a finite index subgroup of $\mathbb{Z}^k$, this set has nontrivial intersection with $(0, 1)$, and the result follows. $\square$

Remark 7.5. Theorem 3.1 generalises Yang's result [25, Theorem 5.3] to finite k-graphs with more than one vertex, and removes the technical hypothesis that the intrinsic group of the k-graph has rank at most one. To see this, observe that if Λ has just one vertex, then $\mathcal{P}_\Lambda = \mathbb{Z}^k$, and $\rho(\Lambda) = (|\Lambda^{e_1}|, \dots, |\Lambda^{e_k}|)$. So Theorem 3.1 shows that the Connes invariant of the associated factor is the closure of the subgroup of $(0, \infty)^\times$ generated by the numbers $|\Lambda^{e_1}|, \dots, |\Lambda^{e_k}|$. Yang considers only the situation where each $|\Lambda^{e_i}| \geq 2$. In this case, just as Yang says, if some $\log(|\Lambda^{e_i}|)/\log(|\Lambda^{e_j}|)$ is irrational, the factor is of type III_1. Otherwise we can uniquely write each $|\Lambda^{e_1}|^{a_i} = |\Lambda^{e_i}|^{b_i}$ with $\gcd(a_i, b_i) = 1$, and then the factor is of type III_λ, where $\lambda = |\Lambda^{e_1}|^{-1/\operatorname{lcm}(b_2,\cdots,b_k)}$.

Remark 7.6. As discussed in Remark 6.2, the group $\mathcal{P}_\Lambda$ depends only on the skeleton of Λ and is independent of the factorisation property. The same is true of the spectral-radius vector $\rho(\Lambda)$. So Theorem 3.1 shows that the type of the factors obtained from extremal KMS states depends only on the skeleton of Λ and not on the factorisation property.

We finish with an explicit example of the phenomenon described in the preceding remark: a pair of 2-graphs Λ_1, Λ_2 with the same adjacency matrices, and hence determining the same factors, in which $\operatorname{Per}\Lambda_1$ and $\operatorname{Per}\Lambda_2$ are distinct proper subgroups of the common group of periods, which is itself a strict subgroup of $\mathbb{Z}^2$.

Example 7.7. Consider the 2-coloured graph below.



Any 2-graph with this skeleton must satisfy $a_i b_i = d_i c_i$ for $i = 0, 1$ and $a_i b_{1-i} = d_i c_{1-i}$, and so there are two possible 2-graphs with this skeleton: the 2-graph Λ_1 in which $c_i d_i = b_i a_i$ for $i = 0, 1$; and the 2-graph Λ_2 in which $c_i d_i = b_{1-i} a_{1-i}$ for $i = 0, 1$ (see [10]). Every cycle μ in either Λ_1 or Λ_2 satisfies $d(\mu)_1 + d(\mu)_2 \in 2\mathbb{Z}$. Since $d(a_1 c_1) = (2, 0)$ and $d(a_1 b_1) = (1, 1)$ generate $\{m \mid m_1 + m_2 \in 2\mathbb{Z}\}$, we see that $\mathcal{P}_{\Lambda_i} = \{m \mid m_1 + m_2 \in 2\mathbb{Z}\}$ for $i = 1, 2$.

It is easy to see that Λ_1 is the pullback of the 1-graph E consisting of blue (solid) paths in Λ over $f : \mathbb{N}^2 \to \mathbb{N}$, $(m, n) \mapsto m + n$, as in [15, Definition 1.9]. Since every cycle in E has an entrance, the periodicity group of E is trivial, and one can use this to check that $\operatorname{Per}\Lambda_1 = \mathbb{Z}(-1, 1) \subsetneq \mathcal{P}_{\Lambda_1} \subsetneq \mathbb{Z}^2$.

We claim that $\operatorname{Per}\Lambda_2 = 2\mathbb{Z}(-1,1)$. To see this, first note that every infinite path $x \in u\Lambda_2^\infty$ satisfies $\sigma^{(1,0)}(c_1d_1c_1x) = d_1c_1x = a_1b_1x \in Z(a_1)$ whereas $\sigma^{(0,1)}(c_1d_1c_1x) = \sigma^{(0,1)}(b_2a_2c_1x) = a_2c_1x \in Z(a_2)$. Since $Z(a_1)$ and $Z(a_2)$ are disjoint, we deduce that $(1,0)-(0,1) \notin \operatorname{Per}\Lambda_2$. Since $\operatorname{Per}\Lambda_2 \subseteq \mathcal{P}_{\Lambda_2}$, we have $(m,0) \notin \operatorname{Per}\Lambda_2$ for m odd, and if m is even and nonzero then any path of the form $(c_0a_0)^{m/2}c_1x$ satisfies $(c_0a_0)^{m/2}c_1x \in Z(c_0)$ and $\sigma^{(m,0)}((c_0a_0)^{m/2}c_1x) \in Z(c_1)$ giving $(m,0) \notin \operatorname{Per}\Lambda_2$. The same argument gives $(0,m) \notin \operatorname{Per}\Lambda_2$. So $\operatorname{Per}\Lambda_2 = \mathbb{Z}n$ for some $n \in (\mathbb{Z}\setminus\{0\})^2$. We have $v\Lambda_2^\infty = \{b_{i_0}a_{i_0}b_{i_1}a_{i_1}\cdots \mid (i_n)_{n=1}^\infty \in \{0,1\}^{\mathbb{N}}\}$, and repeated application of the factorisation rules using this description shows that $c_ia_ix = b_{1-i}d_{1-i}x$ for $i = 1,2$ and $x \in v\Lambda_2^\infty$. Since $Z(v) = Z(c_1a_1) \cup Z(c_2a_2)$, it follows that $\sigma^{(2,0)}(x) = \sigma^{(0,2)}(x)$ for all $x \in v\Lambda_2^\infty$, and so $(2,-2) \in \operatorname{Per}\Lambda_2$ by [13, Lemma 5.1], and therefore $\operatorname{Per}\Lambda_2 = 2\mathbb{Z}(1,-1)$ as claimed.

For either of Λ_1 or Λ_2, we have

$$A_1 = A_2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

So $A_1(1,\sqrt{2},1)^t = \sqrt{2}(1,\sqrt{2},1)^t$, and since A_1 is irreducible, the Perron-Frobenius theorem gives $\rho(\Lambda_i) = (\sqrt{2},\sqrt{2})$ for $i = 1,2$. By inspection, for $n \in \mathbb{N}^2$, there is a cycle in Λ_i of degree n if and only if $n_1 + n_2 \in 2\mathbb{Z}$, so that $\rho(\Lambda_i)^n = 2^{(n_1+n_2)/2}$ is a power of 2. So Theorem 3.1(2b) says that for $i = 1,2$ and any extremal α-KMS$_1$ state ϕ of $C^*(\Lambda_i)$, the factor $\pi_\phi(C^*(\Lambda_i))''$ is of type III$_{1/2}$.

Remark 7.8. We could modify the 2-coloured graph of the preceding example by replacing every red (dashed) edge f with a pair of parallel red edges $(f,0),(f,1)$. For either of $i = 0,1$, we could then specify factorisation rules on this amplified 2-coloured graph by $e(f,j) = (f',j)e'$ $(j = 0,1)$ whenever $ef = f'e'$ in Λ_i above, obtaining a new 2-graph $\Lambda_{i,2}$. We then have $\rho(\Lambda_{i,2}) = (\sqrt{2}, 2\sqrt{2})$. Again the cycles in $\Lambda_{i,2}$ all satisfy $d(\lambda)_1 + d(\lambda_2) \in 2\mathbb{Z}$, so each $\rho(\Lambda_{i,2})^{d(\lambda)} = 2^{(d(\lambda_1)+d(\lambda_2))/2+d(\lambda)_2}$ is a power of 2. So we deduce from Theorem 3.1(2b) that each $\pi_\phi(C^*(\Lambda_{i,2}))''$ is still of type III$_{1/2}$.

If, instead of replacing each red edge with two edges $(f,0)$ and $(f,1)$ we insert three parallel red edges $(f,0),(f,1),(f,2)$ to obtain 2-graphs $\Lambda_{i,3}$, we obtain $\rho(\Lambda_{i,3}) = (\sqrt{2}, 3\sqrt{2})$. Since $\log 2$ and $\log 3$ are rationally independent, so are $\log\sqrt{2} = \frac{1}{2}\log 2$ and $\log(3\sqrt{2}) = \log 3 + \frac{1}{2}\log 2$. So Theorem 3.1(2b) says that $\pi_\phi(C^*(\Lambda_{i,3}))''$ is the injective III$_1$ factor for $i = 1,2$.

References

[1] O. Bratteli, *Inductive limits of finite dimensional C^*-algebras*, Trans. Amer. Math. Soc. **171** (1972), 195–234.

[2] O. Bratteli and D.W. Robinson, Operator algebras and quantum statistical mechanics. 2, Equilibrium states. Models in quantum statistical mechanics, Springer-Verlag, Berlin, 1997, xiv+519.

[3] T.M. Carlsen, S. Kang, J. Shotwell and A. Sims, *The primitive ideals of the Cuntz-Krieger algebra of a row-finite higher-rank graph with no sources*, J. Funct. Anal. **266** (2014), no. 4, 2570–2589.

[4] A. Connes, *Une classification des facteurs de type* III, Ann. Sci. École Norm. Sup. (4) **6** (1973), 133–252.

[5] K.R. Davidson and D. Yang, *Periodicity in rank 2 graph algebras*, Canad. J. Math. **61** (2009), no. 6, 1239–1261.

[6] M. Enomoto, M. Fujii and Y. Watatani, *KMS states for gauge action on $\mathcal{O}_A$*, Math. Japon. **29** (1984), no. 4, 607–619.

[7] J. Feldman and C.C. Moore, *Ergodic equivalence relations, cohomology, and von Neumann algebras. I*, Trans. Amer. Math. Soc. **234** (1977), no. 2, 289–324.

[8] J. Feldman and C.C. Moore, *Ergodic equivalence relations, cohomology, and von Neumann algebras. II*, Trans. Amer. Math. Soc. **234** (1977), no. 2, 325–359.

[9] N. Gao and V.G. Troitsky, *Irreducible semigroups of positive operators on Banach lattices*, Linear Multilinear Algebra **62** (2014), no. 1, 74–95.

[10] R. Hazlewood, I. Raeburn, A. Sims and S.B.G. Webster *Remarks on some fundamental results about higher-rank graphs and their C^*-algebras*, Proc. Edinb. Math. Soc. (2) **56** (2013), no. 2, 575–597.

[11] A. an Huef, M. Laca, I. Raeburn, and A. Sims, *KMS states on the C^*-algebras of finite graphs*, J. Math. Anal. Appl. **405** (2013), no. 2, 388–399.

[12] A. an Huef, M. Laca, I. Raeburn and A. Sims, *KMS states on C^*-algebras associated to higher-rank graphs* , J. Funct. Anal. **266** (2014), no. 1, 265–283.

[13] A. an Huef, M. Laca, I. Raeburn and A. Sims, *KMS states on the C^*-algebra of a higher-rank graph and periodicity in the path space*, J. Funct. Anal., to appear.
[14] B.P. Kitchens, Symbolic dynamics, One-sided, two-sided and countable state Markov shifts, Springer-Verlag, Berlin, 1998, x+252.
[15] A. Kumjian and D. Pask, Higher rank graph C^*-algebras, *New York J. Math.* **6** (2000), 1–20.
[16] A. Kumjian and D. Pask, *Actions of $\mathbb{Z}^k$ associated to higher rank graphs*, Ergodic Theory Dynam. Systems **23** (2003), no. 4, 1153–1172.
[17] B. Maloney and D. Pask, *Simplicity of the C^*-algebras of skew product k-graphs*, preprint 2013 (arXiv:1306.6107 [math.OA]).
[18] S. Neshveyev, *KMS states on the C^*-algebras of non-principal groupoids*, J. Operator Theory **70** (2013), no. 2, 513–530.
[19] S. Neshveyev and E. Størmer, *Dynamical Entropy in Operator Algebras*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics. Springer Verlag, Berlin, 2006.
[20] R. Okayasu, *Type III factors arising from Cuntz-Krieger algebras*, Proc. Amer. Math. Soc. **131** (2003), no. 7, 2145–2153.
[21] D. Olesen and G.K. Pedersen, *Some C^*-dynamical systems with a single KMS state*, Math. Scand. **42** (1978), no. 1, 111–118.
[22] I. Raeburn and A. Sims, *Product systems of graphs and the Toeplitz algebras of higher-rank graphs*, J. Operator Theory **53** (2005), no. 2, 399–429.
[23] E. Seneta, *Non-Negative Matrices and Markov Chains*, second edition, Springer Series in Statistics. Springer Verlag, New York, 1981.
[24] D. Yang, *Type III von Neumann algebras associated with 2-graphs*, Bull. Lond. Math. Soc. **44** (2012), no. 4, 675–686.
[25] D. Yang, *Factoriality and type classification of k-graph von Neumann algebras*, preprint 2013 arXiv:1311.4638 [math.OA].

(Marcelo Laca) Department of Mathematics and Statistics, University of Victoria, PO Box 3060, Victoria BC, Canada V8W 3R4
E-mail address: `laca@math.uvic.ca`

(Nadia Larsen) Department of Mathematics, University of Oslo, PO BOX 1053 Blindern, N-0316 Oslo, Norway
E-mail address: `nadiasl@math.uio.no`

(Sergey Neshveyev) Department of Mathematics, University of Oslo, PO BOX 1053 Blindern, N-0316 Oslo, Norway
E-mail address: `sergeyn@math.uio.no`

(Aidan Sims) School of Mathematics and Applied Statistics, University of Wollongong, NSW 2522, Australia
E-mail address: `asims@uow.edu.au`

(Sam Webster) School of Mathematics and Applied Statistics, University of Wollongong, NSW 2522, Australia
E-mail address: `sbgwebster@gmail.com`