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Abstract

2015 Springer Basel Given a system of coverings of k-graphs, we show that the second cohomology of the resulting (k + 1)-graph is isomorphic to that of any one of the k-graphs in the system, and compute the semifinite traces of the resulting twisted (k + 1)-graph C*-algebras. We then consider Bratteli diagrams of 2-graphs whose twisted C*-algebras are matrix algebras over noncommutative tori. For such systems we calculate the ordered K-theory of the resulting twisted 3-graph C*-algebras. We deduce that every such C*-algebra is Morita equivalent to the C*-algebra of a rank-2 Bratteli diagram in the sense of Pask-Raeburn-Rørdam-Sims.

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TWISTED k-GRAPH ALGEBRAS ASSOCIATED TO BRATTELI DIAGRAMS

DAVID PASK, ADAM SIERAKOWSKI, AND AIDAN SIMS

ABSTRACT. Given a system of coverings of k-graphs, we show that the cohomology of the resulting $(k+1)$-graph is isomorphic to that of any one of the k-graphs in the system. We then consider Bratteli diagrams of 2-graphs whose twisted C^*-algebras are matrix algebras over noncommutative tori. For such systems we calculate the ordered K-theory and the gauge-invariant semifinite traces of the resulting 3-graph C^*-algebras. We deduce that every simple C^*-algebra of this form is Morita equivalent to the C^*-algebra of a rank-2 Bratteli diagram in the sense of Pask-Raeburn-Rørdam-Sims.

1. INTRODUCTION

Elliott's classification program, as described in [11], has been a very active field of research in recent years. The program began with Elliott's classification of AF algebras by their K_0-groups in [9] (see also [8]). Elliott subsequently expanded this classification program to encompass all simple A$\mathbb{T}$-algebras of real rank zero [10], and then, in work with Gong, expanded it still further to encompass more general AH algebras [13], leading to the classification of simple AH-algebras of slow dimension growth and of real rank zero [16, 28]. Paralleling these results for stably finite C^*-algebras is the classification by K-theory of Kirchberg algebras in the UCT class by Kirchberg and Phillips in the mid 1990s (see [16, 28]).

Shortly after the introduction of graph C^*-algebras, it was shown in [19] that every simple graph C^*-algebra is either purely infinite or AF, and so is classified by K-theory either by the results of [9] or by those of [16, 28]. As a result of this and results of Szymański [33], the range of Morita-equivalence classes of simple C^*-algebras that can be realised by graph C^*-algebras is completely understood. The introduction of k-graphs and their C^*-algebras in [18] naturally raised the analogous question. But it was shown in [25] that there exist simple k-graph algebras which are direct limits of matrix algebras over $C(\mathbb{T})$ and are neither AF nor purely infinite. The examples constructed there are, nonetheless, classified by their K-theory by [10], and the range of the invariant that they achieve is understood.

The general question of which simple C^*-algebras are Morita equivalent to k-graph C^*-algebras is far from being settled, and the corresponding question for the twisted k-graph C^*-algebras of [22] is even less-well understood. In this paper we consider a class of twisted k-graph C^*-algebras constructed using a procedure akin to that in [25], except that the simple cycles there used to generate copies of $M_n(C(\mathbb{T}))$ are replaced here

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by 2-dimensional simple cycles whose twisted C^*-algebras are matrix algebras over noncommutative tori. When the noncommutative tori all correspond to the same irrational rotation, we compute the ordered K-theory of these examples by adapting Pimsner and Voiculescu's computation of the ordered K-theory of the rotation algebras. Computing the ordered K_0-groups makes heavy use of traces on the approximating subalgebras, and we finish by expanding on this to produce a detailed analysis of traces on the C^*-algebras of rank-3 Bratteli diagrams.

Remarkably, it turns out that our construction does not expand the range of Morita-equivalence classes of C^*-algebras obtained in [25]: For every rank-3 Bratteli diagram E and every irrational θ such that the corresponding twisted C^*-algebra $C^*(\Lambda_E, d_* c^3_\theta)$ of the rank-3 Bratteli diagram Λ_E is simple, there is a rank-2 Bratteli diagram Γ as in [25] whose C^*-algebra is Morita equivalent to $C^*(\Lambda_E, d_* c^3_\theta)$ (see Corollary 5.13). However, this requires both Elliott's classification theorem and the Effros-Handelman-Shen characterisation of Riesz groups as dimension groups. In particular, the rank-2 Bratteli diagram Γ will depend heavily on the value of θ as well as the diagram E.

In Section 3, following a short introduction on twisted $(k+1)$-graph C^*-algebras associated to covering sequences (Λ_n, p_n) of k-graphs, we look at the categorical cohomology of such $(k+1)$-graphs. We prove in Theorem 3.6 that each 2-cocycle on the $(k+1)$-graph Λ associated to a covering sequence (Λ_n, p_n) is — up to cohomology — completely determined by its restriction to the k-graph Λ_1.

In Section 4 we extend the notion of a covering sequence to a Bratteli diagram of covering maps for a singly connected Bratteli diagram E. We construct a $(k+1)$-graph Λ from a Bratteli diagram of covering maps between k-graphs $(\Lambda_v)_{v\in E^0}$ and — upon fixing a 2-cocycle c on Λ — we show how to describe the twisted $(k+1)$-graph C^*-algebra $C^*(\Lambda, c)$, up to Morita equivalence, as an inductive limit of twisted k-graph C^*-algebras, each of which is a direct sum of C^*-algebras of the form $M_{n_v}(C^*(\Lambda_v, c|_{\Lambda_v}))$ (see Theorem 4.4).

In Section 5, we prove our main results. We consider a particular class of 3-graphs associated to Bratteli diagrams of covering maps between rank-2 simple cycles; we call these 3-graphs rank-3 Bratteli diagrams. We show in Theorem 5.4 how to compute the ordered K-theory of twisted C^*-algebras of rank-3 Bratteli diagrams when the twisting cocycle is determined by a fixed irrational angle θ. We investigate when such C*-algebras are simple in Corollary 5.12 and then prove in Corollary 5.13 in the presence of simplicity these C*-algebras can in fact be realised as the C^*-algebras of rank-2 Bratteli diagrams in the sense of [25]. In Section 6 we briefly present some explicit examples of our K-theory calculations. In Section 7, we describe an auxiliary AF algebra $C^*(F)$ associated to each rank-3 Bratteli diagram Λ, and exhibit an injection from semifinite lower-semicontinuous traces on $C^*(F)$ to gauge-invariant semifinite lower-semicontinuous traces of $C^*(\Lambda, c)$. We show that when c is determined by a fixed irrational rotation θ, the map from traces on $C^*(F)$ to traces on $C^*(\Lambda, c)$ is a bijection.

2. Preliminaries and notation

In this section we introduce the notion of k-graphs, covering sequences of k-graphs, and twisted k-graph C^*-algebras which we can associate to covering sequences. These are the main objects of study in this paper.

2.1. **k-graphs.** Following [18, 24, 30] we briefly recall the notion of k-graphs. For $k \geq 0$, a *k-graph* is a nonempty countable small category equipped with a functor $d\colon \Lambda \to \mathbb{N}^k$ that satisfies the *factorisation property*: for all $\lambda \in \Lambda$ and $m, n \in \mathbb{N}^k$ such that $d(\lambda) = m + n$ there exist unique $\mu, \nu \in \Lambda$ such that $d(\mu) = m$, $d(\nu) = n$, and $\lambda = \mu\nu$. When $d(\lambda) = n$ we say λ has *degree* n, and we write $\Lambda^n = d^{-1}(n)$. The standard generators of $\mathbb{N}^k$ are denoted $e_1, \dots, e_k$, and we write n_i for the i^{th} coordinate of $n \in \mathbb{N}^k$. For $m, n \in \mathbb{N}^k$, we write $m \vee n$ for their coordinate-wise maximum, and define a partial order on $\mathbb{N}^k$ by $m \leq n$ if $m_i \leq n_i$ for all i.

If Λ is a k-graph, its *vertices* are the elements of Λ^0. The factorisation property implies that these are precisely the identity morphisms, and so can be identified with the objects. For $\alpha \in \Lambda$ the *source* $s(\alpha)$ is the domain of α, and the *range* $r(\alpha)$ is the codomain of α (strictly speaking, $s(\alpha)$ and $r(\alpha)$ are the identity morphisms associated to the domain and codomain of α).

For $u, v \in \Lambda^0$ and $E \subset \Lambda$, we write $uE := E \cap r^{-1}(u)$ and $Ev := E \cap s^{-1}(v)$. For $n \in \mathbb{N}^k$, we write

$$\Lambda^{\leq n} := \{\lambda \in \Lambda : d(\lambda) \leq n \text{ and } s(\lambda)\Lambda^{e_i} = \emptyset \text{ whenever } d(\lambda) + e_i \leq n\}.$$

We say that Λ is *connected* if the equivalence relation on Λ^0 generated by $\{(v, w) \in \Lambda^0 \times \Lambda^0 : v\Lambda w \neq \emptyset\}$ is the whole of $\Lambda^0 \times \Lambda^0$. We say that Λ is *strongly connected* if $v\Lambda w$ is nonempty for all $v, w \in \Lambda^0$. A *morphism* between k-graphs is a degree-preserving functor. We say that Λ is *row-finite* if $v\Lambda^n$ is finite for all $v \in \Lambda^0$ and $n \in \mathbb{N}^k$, and Λ is *locally convex* if whenever $1 \leq i < j \leq k$, $e \in \Lambda^{e_i}$, $f \in \Lambda^{e_j}$ and $r(e) = r(f)$, we can extend both e and f to paths ee' and ff' in $\Lambda^{e_i + e_j}$. For $\lambda, \mu \in \Lambda$ we write

$$\Lambda^{\min}(\lambda, \mu) := \{(\alpha, \beta) : \lambda\alpha = \mu\beta \text{ and } d(\lambda\alpha) = d(\lambda) \vee d(\mu)\}$$

for the collection of pairs which give minimal common extensions of λ and μ.

A standard example of a k-graph is $T_k := \mathbb{N}^k$ regarded as a k-graph with $d = \mathrm{id}_{\mathbb{N}^k}$.

2.2. Covering sequences.

Following [20] a surjective morphism $p : \Gamma \to \Lambda$ between k-graphs is a *covering* if it restricts to bijections $\Gamma v \mapsto \Lambda p(v)$ and $v\Gamma \mapsto p(v)\Lambda$ for $v \in \Gamma^0$. A covering $p : \Gamma \to \Lambda$ is *finite* if $p^{-1}(v)$ is finite for all $v \in \Lambda^0$.

Definition 2.1. A *covering sequence* (Λ_n, p_n) of k-graphs consists of k-graphs Λ_n and a sequence

$$\Lambda_1 \xleftarrow[p_1]{} \Lambda_2 \xleftarrow[p_2]{} \Lambda_3 \xleftarrow[p_3]{} \cdots .$$

of covering maps $p_n \colon \Lambda_{n+1} \to \Lambda_n$.

Given a covering sequence (Λ_n, p_n) of k-graphs [20, Corollary 2.11] shows that there exist a unique $(k+1)$-graph $\Lambda = \lim_n(\Lambda_n, p_n)$, together with injective functors $\iota_n : \Lambda_n \to \Lambda$ and a bijective map $e : \bigsqcup_{n \geq 2} \Lambda_n^0 \to \Lambda^{e_{k+1}}$, such that, identifying $\mathbb{N}^{k+1}$ with $\mathbb{N}^k \oplus \mathbb{N}$,

(1) $d(\iota_n(\lambda)) = (d(\lambda), 0)$, for $\lambda \in \Lambda_n$,
(2) $\iota_m(\Lambda_m) \cap \iota_n(\Lambda_n) = \emptyset$, for $m \neq n$,
(3) $\bigsqcup_{n \geq 1} \iota_n(\Lambda_n) = \{\lambda \in \Lambda : d(\lambda)_{k+1} = 0\}$,
(4) $s(e(v)) = \iota_{n+1}(v)$, $r(e(v)) = \iota_n(p_n(v))$, for $v \in \Lambda_{n+1}^0$, and
(5) $e(r(\lambda))\iota_{n+1}(\lambda) = \iota_n(p_n(\lambda))e(s(\lambda))$, for $\lambda \in \Lambda_{n+1}$.

We often suppress the inclusion maps ι_n and view the Λ_n as subsets of Λ. For $n > m$ we define $p_{m,n} := p_m \circ p_{m+1} \circ \cdots \circ p_{n-2} \circ p_{n-1} : \Lambda_n \to \Lambda_m$; we define $p_{m,m} = \mathrm{id}_{\Lambda_m}$.

2.3. **Twisted k-graph C^*-algebras.** Let Λ be a row-finite locally convex k-graph and let $c \in Z^2(\Lambda, \mathbb{T})$. A *Cuntz-Krieger (Λ, c)-family* in a C^*-algebra B is a function $s : \lambda \mapsto s_\lambda$ from Λ to B such that

(CK1) $\{s_v : v \in \Lambda^0\}$ is a collection of mutually orthogonal projections;
(CK2) $s_\mu s_\nu = c(\mu, \nu) s_{\mu\nu}$ whenever $s(\mu) = r(\nu)$;
(CK3) $s_\lambda^* s_\lambda = s_{s(\lambda)}$ for all $\lambda \in \Lambda$; and
(CK4) $s_v = \sum_{\lambda \in v\Lambda^{\leq n}} s_\lambda s_\lambda^*$ for all $v \in \Lambda^0$ and $n \in \mathbb{N}^k$.

The following Lemma shows that our definition is consistent with previous literature.

Lemma 2.2. *Let Λ be a row-finite locally convex k-graph and let $c \in Z^2(\Lambda, \mathbb{T})$. A function $s\colon \lambda \mapsto s_\lambda$ from Λ to a C^*-algebra B is a Cuntz-Krieger (Λ, c)-family if and only if it is a Cuntz-Krieger (Λ, c)-family in the sense of* [35].

Proof. Recall that $E \subseteq v\Lambda$ is *exhaustive* if for every $\lambda \in v\Lambda$ there exists $\mu \in E$ such that $\Lambda^{\min}(\lambda, \mu) \neq \emptyset$. According to [32], a Cuntz-Krieger (Λ, c)-family is a function $s : \lambda \mapsto s_\lambda$ such that

(TCK1) $\{s_v : v \in \Lambda^0\}$ is a collection of mutually orthogonal projections;
(TCK2) $s_\mu s_\nu = c(\mu, \nu) s_{\mu\nu}$ whenever $s(\mu) = r(\nu)$;
(TCK3) $s_\lambda^* s_\lambda = s_{s(\lambda)}$ for all $\lambda \in \Lambda$;
(TCK4) $s_\mu s_\mu^* s_\nu s_\nu^* = \sum_{(\alpha,\beta) \in \Lambda^{\min}(\lambda,\mu)} s_{\lambda\alpha} s_{\lambda\alpha}^*$ for all $\mu, \nu \in \Lambda$; and
(CK) $\prod_{\lambda \in E}(s_v - s_\lambda s_\lambda^*) = 0$ for all $v \in \Lambda^0$ and finite *exhaustive* $E \subseteq v\Lambda$.

So we must show that (CK1)–(CK4) are equivalent to (TCK1)–(CK).

First suppose that s satisfies (CK1)–(CK4). Then it clearly satisfies (TCK1)–(TCK3). For (TCK4), fix $\lambda, \mu \in \Lambda$. It suffices to show that $s_\lambda^* s_\mu = \sum_{(\alpha,\beta) \in \Lambda^{\min}(\lambda,\mu)} \overline{c(\lambda, \alpha)} c(\mu, \beta) s_\alpha s_\beta^*$ (see [35, Lemma 3.1.5]). Define $N := d(\lambda) \vee d(\mu)$ and $E' := \{(\alpha, \beta) : \lambda\alpha = \mu\beta, \lambda\alpha \in r(\lambda)\Lambda^{\leq N}\}$. The proof of [30, Proposition 3.5] gives $s_\lambda^* s_\mu = \sum_{(\alpha,\beta) \in E'} \overline{c(\lambda, \alpha)} c(\mu, \beta) s_\alpha s_\beta^*$. Clearly $\Lambda^{\min}(\lambda, \mu) \subseteq E'$ since $r(\lambda)\Lambda^N \subseteq r(\lambda)\Lambda^{\leq N}$. Conversely for any $(\alpha, \beta) \in E'$ we have $\lambda\alpha = \mu\beta$ by definition, and hence $d(\lambda\alpha) \geq N$. Since $\lambda\alpha \in \Lambda^{\leq N}$, we also have $d(\lambda\alpha) \leq N$, and hence we have equality. Thus $E' = \Lambda^{\min}(\lambda, \mu)$. This gives (TCK4).

For (CK), fix $v \in \Lambda^0$ and a finite exhaustive $E \subseteq v\Lambda$. With $N := \bigvee_{\lambda \in E} d(\lambda)$ and $E' := \{\lambda\nu : \lambda \in E, \nu \in s(\lambda)\Lambda^{\leq N - d(\lambda)}\}$ we have $E' = v\Lambda^{\leq N}$ by an induction using [30, Lemma 3.12]. Relation (TCK4) implies that the $s_\lambda s_\lambda^*$ where $\lambda \in E$ commute, and that the $s_{\lambda\nu} s_{\lambda\nu}^*$ are mutually orthogonal. Also, (TCK2) implies that $s_v - s_\lambda s_\lambda^* \leq s_v - s_{\lambda\nu} s_{\lambda\nu}^*$ for all $\lambda\nu \in E'$, and so

$$\prod_{\lambda \in E}(s_v - s_\lambda s_\lambda^*) \leq \prod_{\mu \in E'}(s_v - s_\mu s_\mu^*) = s_v - \sum_{\mu \in v\Lambda^{\leq N}} s_\mu s_\mu^* = 0.$$

Now suppose that s satisfies (TCK1)–(CK). Then it clearly satisfies (CK1)–(CK3). For (CK4), fix $v \in \Lambda^0$ and $1 \leq i \leq k$ with $E := v\Lambda^{e_i} \neq \emptyset$. Then (TCK4) implies that the $s_\lambda s_\lambda^*$ for $\lambda \in E$ are mutually orthogonal. Since $E \subseteq v\Lambda$ is finite and exhaustive (CK) then gives

$$0 = \prod_{\lambda \in E}(s_v - s_\lambda s_\lambda^*) = s_v - \sum_{\lambda \in E} s_\lambda s_\lambda^*. \qquad \square$$

The *twisted k-graph C^*-algebra* $C^*(\Lambda, c)$ is the universal C^*-algebra generated by a Cuntz-Krieger (Λ, c)-family.

3. Covering sequences and cohomology

We will be interested in twisted C^*-algebras associated to 3-graphs analogous to the rank-2 Bratteli diagrams of [25] (see Section 5). The building blocks for these 3-graphs are covering systems of k-graphs. In this section we investigate their second cohomology groups. Our results substantially simplify the problem of studying the associated C^*-algebras later because it allows us to assume the the twisting 2-cocycles are pulled back from a cocycle of a standard form on $\mathbb{Z}^2$ (see Remark 5.7).

We briefly recap the categorical cohomology of k-graphs (see [22]). Let Λ be a k-graph, and let A be an abelian group. For each integer $r \geq 1$, let $\Lambda^{*r} := \{(\lambda_1, \dots, \lambda_r) \in \prod_{i=1}^r \Lambda : s(\lambda_i) = r(\lambda_{i+1}) \text{ for each } i\}$ be the collection of *composable* r-tuples in Λ, and let $\Lambda^{*0} := \Lambda^0$. For $r \geq 1$, a function $f : \Lambda^{*r} \to A$ is an *r-cochain* if $f(\lambda_1, \dots, \lambda_r) = 0$ whenever $\lambda_i \in \Lambda^0$ for some $i \leq r$. A 0-cochain is any function $f\colon \Lambda^0 \to A$. We write $C^r(\Lambda, A)$ for the group of all r-cochains under pointwise addition. Define maps $\delta^r\colon C^r(\Lambda, A) \to C^{r+1}(\Lambda, A)$ by $\delta^0(f)(\lambda) = f(s(\lambda)) - f(r(\lambda))$ and

$$\begin{aligned}\delta^r(f)(\lambda_0, \dots, \lambda_r) = f(\lambda_1, \dots, \lambda_r) \\ + \sum_{i=1}^{r}(-1)^i f(\lambda_0, \dots, \lambda_{i-2}, \lambda_{i-1}\lambda_i, \lambda_{i+1}, \dots, \lambda_r) \\ + (-1)^{r+1} f(\lambda_0, \dots, \lambda_{r-1}) \quad \text{for } r \geq 1.\end{aligned}$$

Let $B^r(\Lambda, A) := \operatorname{im}(\delta^{r-1})$ and $Z^r(\Lambda, A) = \ker(\delta^r)$. A calculation [22, (3.3)–(3.5)] shows that $B^r(\Lambda, A) \subseteq Z^r(\Lambda, A)$. We define $H^r(\Lambda, A) := Z^r(\Lambda, A)/B^r(\Lambda, A)$. We call the elements of $B^r(\Lambda, A)$ *r-coboundaries*, and the elements of $Z^r(\Lambda, A)$ *r-cocycles*. A 2-cochain $c \in C^2(\Lambda, A)$ is a 2-cocycle if and only if it satisfies the *cocycle identity* $c(\lambda, \mu) + c(\lambda\mu, \nu) = c(\mu, \nu) + c(\lambda, \mu\nu)$.

As a notational convention, if $\Gamma \subseteq \Lambda$ is a subcategory and $c \in Z^r(\Lambda, A)$ then we write $c|_\Gamma$, rather than $c|_{\Gamma^{*r}}$ for the restriction of c to the composable r-tuples of Γ. If Λ and Γ are k-graphs and $\phi\colon \Lambda \to \Gamma$ is a functor, and if $c \in Z^2(\Gamma, A)$, then $\phi_* c\colon \Lambda^{*2} \to A$ is defined by $\phi_* c(\lambda, \mu) = c(\phi(\lambda), \phi(\mu))$.

In [22], the categorical cohomology groups described above were decorated with an underline to distinguish them from the cubical cohomology groups of [21]. In this paper, we deal only with categorical cohomology, so we have chosen to omit the underlines.

Definition 3.1. Let $\Lambda = \lim_n(\Lambda_n, p_n)$ be the $(k+1)$-graph associated to a covering sequence of k-graphs. A sequence (c_n) of cocycles $c_n \in Z^2(\Lambda_n, A)$ is *compatible* if there is a 2-cocycle $c \in Z^2(\Lambda, A)$ such that $c|_{\Lambda_n} = c_n$ for $n \geq 1$; we say that the c_n are compatible *with respect to c*.

Let $\Lambda = \lim_n(\Lambda_n, p_n)$ be the $(k+1)$-graph associated to a covering sequence of k-graphs. For each $v \in \Lambda^0$ there is a unique element ξ_v of $\Lambda_1^0 \Lambda^{\mathbb{N}e_{k+1}} v$; if $v \in \Lambda_n^0$, then $r(\xi_v) = p_{1,n}(v)$. The factorisation property implies that for each $\lambda \in \Lambda$ there is a unique factorisation

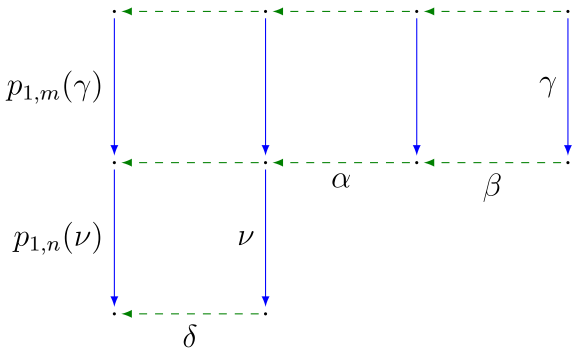
$$\xi_{r(\lambda)}\lambda = \pi(\lambda)\beta \text{ with } \pi(\lambda) \in \Lambda_1 \text{ and } \beta \in \Lambda^{\mathbb{N}e_{k+1}}. \tag{3.1}$$

We call the assignment $\lambda \mapsto \pi(\lambda)$ the *projection* of Λ onto Λ_1.

Observe that if $\lambda \in \Lambda^n$, then $\pi(\lambda) = p_{1,n}(\lambda)$, and if $\lambda \in \Lambda^{\mathbb{N}e_{k+1}}$ with $r(\lambda) \in \Lambda_n^0$, then $\pi(\lambda) = p_{1,n}(r(\lambda))$. In particular, $\pi(\xi_v) = r(\xi_v) = p_{1,n}(v)$ for $v \in \Lambda_n^0$. In general, if $r(\lambda) \in \Lambda_n^0$, then we can factorise $\lambda = \lambda'\lambda''$ with $\lambda' \in \Lambda_n$ and $\lambda'' \in \Lambda^{\mathbb{N}e_{k+1}}$, and then $\pi(\lambda) = p_{1,n}(\lambda')$.

Lemma 3.2. *Let $\Lambda = \lim_n(\Lambda_n, p_n)$ be the $(k+1)$-graph associated to a covering sequence of k-graphs. The projection π of Λ onto Λ_1 of (3.1) is a functor. The formula $\pi_* c(\lambda_1, \lambda_2) = c(\pi(\lambda_1), \pi(\lambda_2))$ for $(\lambda_1, \lambda_2) \in \Lambda^{*2}$ determines a homomorphism $\pi_*\colon Z^2(\Lambda_1, A) \to Z^2(\Lambda, A)$.*

Proof. Fix $(\lambda, \mu) \in \Lambda^{*2}$ and factorise $\lambda = \nu\alpha$ and $\mu = \beta\gamma$ where $\nu \in \Lambda_n$, $\gamma \in \Lambda_m$, and $\alpha, \beta \in \Lambda^{\mathbb{N}e_{k+1}}$. The factorisation property gives $\pi(\lambda\mu) = p_{1,n}(\nu)p_{1,m}(\gamma) = \pi(\lambda)\pi(\mu)$:



Hence π is a functor.

Since functors send identity morphisms to identity morphisms, it follows immediately that $\pi_* c$ is a 2-cocycle. Since the operations in the cohomology groups are pointwise, π_* is a homomorphism. □

Using the covering maps between the k-graphs in a covering sequence we can build a compatible sequence of 2-cocycles from a 2-cocycle on Λ_1.

Theorem 3.3. *Let $\Lambda = \lim_n(\Lambda_n, p_n)$ be the $(k+1)$-graph associated to a covering sequence of k-graphs, and let $c \in Z^2(\Lambda_1, A)$. Let $c_n := (p_{1,n})_* c$ for $n \geq 1$. Then each $c_n \in Z^2(\Lambda_n, A)$, and the c_n are compatible with respect to $\overline{c} := \pi_* c$.*

Proof. For $\lambda \in \Lambda_n$, repeated use of property (5) of Λ shows that $\pi(\lambda) = p_{1,n}(\lambda)$. Hence $\overline{c}|_{\Lambda_n} = (\pi_{|\Lambda_n})_* c = c_n$ for $n \geq 1$. Lemma 3.2 implies that $\overline{c}$ is a 2-cocycle on Λ. Since the restriction of a 2-cocycle is again a 2-cocycle it follows that $c_n = \overline{c}|_{\Lambda_n}$ is a 2-cocycle on Λ_n. □

Theorem 3.3 provides a map $c \mapsto \overline{c}$ from 2-cocycles on Λ_1 to 2-cocycles on Λ. It turns out that this is essentially the only way to construct 2-cocycles on Λ (we will make this more precise in Theorem 3.6).

Lemma 3.4. *Let $\Lambda = \lim_n(\Lambda_n, p_n)$ be the $(k+1)$-graph associated to a covering sequence of k-graphs, and let $c \in Z^2(\Lambda_1, A)$. There exists a unique 2-cocycle c' on Λ extending c such that*

$$c'(\lambda, \mu) = 0 \quad \textit{whenever } \lambda \in \Lambda^{\mathbb{N}e_{k+1}} \textit{ or } \mu \in \Lambda^{\mathbb{N}e_{k+1}}. \tag{3.2}$$

Proof. Theorem 3.3 implies that $\overline{c}$ satisfies $\overline{c}|_{\Lambda_1} = c$, and since $\pi(\lambda) \in \Lambda_1^0$ whenever $\lambda \in \Lambda^{\mathbb{N}e_{k+1}}$,

$$\overline{c}(\lambda, \mu) = c(\pi(\lambda), \pi(\mu)) = 0 \quad \text{whenever } \lambda \in \Lambda^{\mathbb{N}e_{k+1}} \text{ or } \mu \in \Lambda^{\mathbb{N}e_{k+1}}.$$

Now suppose that $c' \in Z^2(\Lambda, A)$ satisfies (3.2).

We claim first that

$$c'(\alpha\lambda, \mu\beta) = c'(\lambda, \mu) \tag{3.3}$$

whenever $\alpha, \beta \in \Lambda^{\mathbb{N}e_{k+1}}$ and $\lambda, \mu \in \Lambda_n$. To see this, observe that (3.2) gives

$$c'(\alpha, \lambda) = 0, \quad c'(\alpha, \lambda\mu\beta) = 0, \quad c'(\mu, \beta) = 0 \quad \text{and} \quad c'(\lambda\mu, \beta) = 0.$$

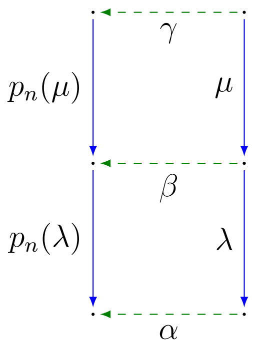
Repeated application of the cocycle identity gives

$$\begin{aligned} c'(\alpha\lambda, \mu\beta) &= c'(\alpha\lambda, \mu\beta) + c'(\alpha, \lambda) + c'(\mu, \beta) \\ &= c'(\alpha, \lambda\mu\beta) + c'(\lambda, \mu\beta) + c'(\mu, \beta) = c'(\lambda, \mu) + c'(\lambda\mu, \beta) = c'(\lambda, \mu). \end{aligned}$$

We now claim that

$$c'(\lambda, \mu) = c'(p_n(\lambda), p_n(\mu)), \tag{3.4}$$

for composable $\lambda, \mu \in \Lambda_{n+1}$. To see this, use property (5) of Λ to find α, β, γ in $\Lambda^{e_{k+1}}$ such that $\alpha\lambda = p_n(\lambda)\beta$ and $\beta\mu = p_n(\mu)\gamma$:



Then

$$c'(p_n(\lambda)\beta, \mu) + c'(p_n(\lambda), \beta) = c'(p_n(\lambda), \beta\mu) + c'(\beta, \mu),$$

and so (3.2) gives $c'(p_n(\lambda)\beta, \mu) = c'(c'(p_n(\lambda), \beta\mu)$. Now (3.3) shows that

$$\begin{aligned} c'(p_n(\lambda), p_n(\mu)) &= c'(p_n(\lambda), p_n(\mu)\gamma) = c'(p_n(\lambda), \beta\mu) \\ &= c'(p_n(\lambda)\beta, \mu) = c'(\alpha\lambda, \mu) = c'(\lambda, \mu). \end{aligned}$$

We now show that $c' = \overline{c}$. We have seen that both c' and $\overline{c}$ satisfy (3.2), and so they both satisfy (3.3). It therefore suffices to show that $c'|_{\Lambda_n} = \overline{c}|_{\Lambda_n}$ for each n. Fix composable $\lambda, \mu \in \Lambda^n$. We have $\overline{c}(\lambda, \mu) = c(p_{1,n}(\lambda), p_{1,n}(\mu))$ by definition. Repeated applications of (3.4) give $c'(\lambda, \mu) = c'(p_{1,n}(\lambda), p_{1,n}(\mu))$. Since c' extends c, we deduce that $c'(\lambda, \mu) = \overline{c}(\lambda, \mu)$. □

Lemma 3.5. *Let $\Lambda = \lim_n(\Lambda_n, p_n)$ be the $(k+1)$-graph associated to a covering sequence of k-graphs, and let $c \in Z^2(\Lambda, A)$. Let $\overline{c|_{\Lambda_1}} = \pi_*(c|_{\Lambda_1})$ as in Theorem 3.3. Then there exists $b \in C^1(\Lambda, A)$ such that*

$$c - \delta^1 b = \overline{c|_{\Lambda_1}}.$$

Proof. For $v \in \Lambda^0$ let ξ_v be the unique element of $\Lambda_1^0 \Lambda^{\mathbb{N}e_{k+1}} v$. For $\lambda \in \Lambda$ define

$$b(\lambda) = c(\xi_{r(\lambda)}, \lambda) - c(\pi(\lambda), \xi_{s(\lambda)}).$$

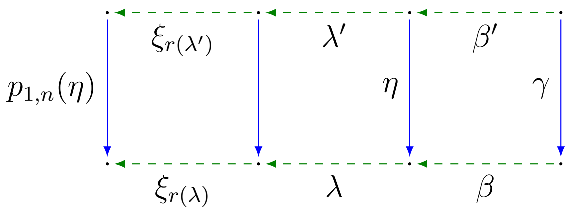
If $\lambda \in \Lambda^0$ then $\pi(\lambda) \in \Lambda^0$ as well, and so $b(\lambda) = 0$. So $b \in C^1(\Lambda, A)$.

Since the restriction of the maps b and $\delta^1 b(\lambda, \mu) = b(\lambda) + b(\mu) - b(\lambda\mu)$ to Λ_1 are identically zero, the cocycle $c' := c - \delta^1 b$ extends c. To conclude that $c' = \overline{c|_{\Lambda_1}}$ it now suffices, by Lemma 3.4, to verify that c' satisfies (3.2).

We first prove that

$$c'(\lambda,\mu)=0 \text{ whenever } \lambda\in\Lambda^{\mathbb{N}e_{k+1}}. \tag{3.5}$$

Fix such a composable pair $\lambda,\mu\in\Lambda$, and factorise $\mu=\eta\beta'$ with $\eta\in\Lambda_n$ and $\beta'\in\Lambda^{\mathbb{N}e_{k+1}}$. Let l be the integer such that $r(\lambda)\in\Lambda^0_l$. By property (5) of Λ there exist λ',β in $\Lambda^{\mathbb{N}e_{k+1}}$ with $r(\lambda')\in\Lambda^0_l$ and $r(\beta)\in\Lambda^0_1$, and $\gamma\in\Lambda^{d(\eta)}_m$ such that $\xi_{r(\lambda)}\lambda\beta\gamma=p_{1,n}(\eta)\xi_{r(\lambda')}\lambda'\beta'$:



We have $\pi(\mu)=\pi(\eta)=p_{1,n}(\eta)=\pi(\gamma)=p_{1,m}(\gamma)$. We prove (3.5) in three steps.

(1) First we show that $c'(\beta,\gamma)=0$ and $c'(\lambda\beta,\gamma)=0$. The cocycle identity gives $c(\xi_{r(\beta)},\beta\gamma)+c(\beta,\gamma)=c(\xi_{r(\beta)}\beta,\gamma)+c(\xi_{r(\beta)},\beta)$. Since $\xi_{r(\beta)}\beta=\xi_{r(\gamma)}$, the definition of b gives

$$\begin{aligned}
c'(\beta,\gamma)&=c(\beta,\gamma)-b(\beta)-b(\gamma)+b(\beta\gamma)\\
&=c(\beta,\gamma)-\Big(c(\xi_{r(\beta)},\beta)-0\Big)-\Big(c(\xi_{r(\gamma)},\gamma)-c(p_{1,n}(\gamma),\xi_{s(\gamma)})\Big)\\
&\qquad+\Big(c(\xi_{r(\beta)},\beta\gamma)-c(p_{1,n}(\gamma),\xi_{s(\gamma)})\Big)\\
&=\Big(c(\xi_{r(\beta)},\beta\gamma)+c(\beta,\gamma)\Big)-\Big(c(\xi_{r(\beta)}\beta,\gamma)+c(\xi_{r(\beta)},\beta)\Big)=0.
\end{aligned}$$

Applying this calculation to $\lambda\beta$ rather than β gives $c'(\lambda\beta,\gamma)=0$ as well.

(2) Next we show that $c'(\lambda,\beta)=0$. We have $c'(\xi_{r(\lambda)},\lambda\beta)+c'(\lambda,\beta)=c'(\xi_{r(\lambda)}\lambda,\beta)+c'(\xi_{r(\lambda)},\lambda)$. Since $\xi_{r(\lambda)}\lambda=\xi_{r(\beta)}$ it follows that

$$\begin{aligned}
c'(\lambda,\beta)&=c(\lambda,\beta)-b(\lambda)-b(\beta)+b(\lambda\beta)\\
&=c(\lambda,\beta)-\Big(c(\xi_{r(\lambda)},\lambda)-0\Big)-\Big(c(\xi_{r(\beta)},\beta)-0\Big)+\Big(c(\xi_{r(\lambda)},\lambda\beta)-0\Big)\\
&=\Big(c(\xi_{r(\lambda)},\lambda\beta)+c(\lambda,\beta)\Big)-\Big(c(\xi_{r(\lambda)}\lambda,\beta)+c(\xi_{r(\lambda)},\lambda)\Big)=0.
\end{aligned}$$

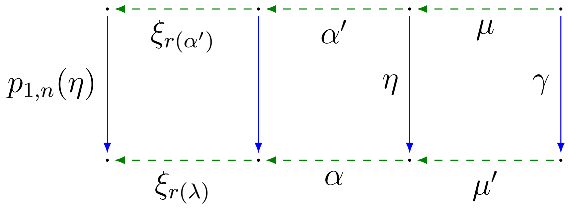
(3) Finally, to establish (3.5), we apply the cocycle identity $c'(\lambda\beta,\gamma)+c'(\lambda,\beta)=c'(\lambda,\beta\gamma)+c'(\beta,\gamma)$ and steps (1) and (2) to see that

$$c'(\lambda,\mu)=c'(\lambda,\eta\beta')=c'(\lambda,\beta\gamma)=0.$$

It remains to show that

$$c'(\lambda,\mu)=0 \quad \text{whenever } \mu\in\Lambda^{\mathbb{N}e_{k+1}}. \tag{3.6}$$

Fix such a composable pair $\lambda,\mu\in\Lambda$, and factorise $\lambda=\alpha\eta$ with $\alpha\in\Lambda^{\mathbb{N}e_{k+1}}$ and $\eta\in\Lambda_n$. Using the factorisation property, we obtain $\alpha'\in\Lambda^{d(\alpha)}$, $\mu'\in\Lambda^{d(\mu)}$ and $\gamma\in\Lambda^{d(\eta)}_m$ that make the following diagram commute.



Equation (3.5) gives $c'(\alpha, \eta) = 0$ and $c'(\alpha, \eta\mu) = 0$. By the cocycle identity $c'(\alpha\eta, \mu) + c'(\alpha, \eta) = c'(\alpha, \eta\mu) + c'(\eta, \mu)$. So we need only check that $c'(\eta, \mu) = 0$. We consider the cocycle identity

$$c(p_{1,m}(\gamma), \xi_{r(\mu)}\mu) + c(\xi_{r(\mu)}, \mu) = c(p_{1,m}(\gamma)\xi_{r(\mu)}, \mu) + c(p_{1,m}(\gamma), \xi_{r(\mu)}).$$

Since $\xi_{s(\gamma)} = \xi_{r(\mu)}\mu$, we have $p_{1,n}(\eta) = p_{1,m}(\gamma)$. Since $\xi_{s(\eta)} = \xi_{r(\mu)}$ and $p_{1,m}(\gamma)\xi_{r(\mu)} = \xi_{r(\mu')}\eta$, we obtain

$$c(p_{1,n}(\eta), \xi_{s(\eta)}) - c(\xi_{r(\mu)}, \mu) - c(p_{1,m}(\gamma), \xi_{s(\gamma)}) = -c(\xi_{r(\mu')}\eta, \mu). \tag{3.7}$$

Hence

$$\begin{aligned}
c'(\eta, \mu) &= c(\eta, \mu) - b(\eta) - b(\mu) + b(\eta\mu) \\
&= c(\eta, \mu) - b(\eta) - b(\mu) + b(\mu'\gamma) \\
&= c(\eta, \mu) - \Big(c(\xi_{r(\eta)}, \eta) - c(p_{1,n}(\eta), \xi_{s(\eta)})\Big) \\
&\qquad - \Big(c(\xi_{r(\mu)}, \mu) - 0\Big) + \Big(c(\xi_{r(\mu')}, \mu'\gamma) - c(p_{1,m}(\gamma), \xi_{s(\gamma)})\Big) \\
&= c(\eta, \mu) - c(\xi_{r(\eta)}, \eta) + c(\xi_{r(\mu')}, \mu'\gamma) - c(\xi_{r(\mu')}\eta, \mu) \quad \text{by (3.7)} \\
&= c(\eta, \mu) - c(\xi_{r(\mu')}, \eta) + c(\xi_{r(\mu')}, \eta\mu) - c(\xi_{r(\mu')}\eta, \mu) = 0,
\end{aligned}$$

establishing (3.6). □

Theorem 3.6. *Let $\Lambda = \lim_n(\Lambda_n, p_n)$ be the $(k+1)$-graph associated to a covering sequence of k-graphs. The restriction map $c \mapsto c|_{\Lambda_1}$ from $Z^2(\Lambda, A)$ to $Z^2(\Lambda_1, A)$ induces an isomorphism $H^2(\Lambda, A) \cong H^2(\Lambda_1, A)$.*

Proof. Surjectivity follows from Lemma 3.4. To verify injectivity fix 2-cocycles c_1, c_2 on Λ such that $c_1|_{\Lambda_1} - c_2|_{\Lambda_1} = \delta^1 b$ for some $b \in C^1(\Lambda_1, A)$. Lemma 3.5 gives $b_1, b_2 \in C^1(\Lambda, A)$ such that

$$c_1 - \delta^1 b_1 = \pi_*(c_1|_{\Lambda_1}), \qquad \text{and} \qquad c_2 - \delta^1 b_2 = \pi_*(c_2|_{\Lambda_1}).$$

Hence

$$\begin{aligned}
c_1 - c_2 &= \pi_*(c_1|_{\Lambda_1}) + \delta^1 b_1 - \pi_*(c_2|_{\Lambda_1}) - \delta^1 b_2 \\
&= \delta^1 b_1 - \delta^1 b_2 + \pi_*(c_1|_{\Lambda_1} - c_2|_{\Lambda_1}) \\
&= \delta^1 b_1 - \delta^1 b_2 + \pi_*(\delta^1 b).
\end{aligned}$$

We have $\pi_*(b) \in C^1(\Lambda, A)$ and $\pi_*(\delta^1 b) = \delta^1(\pi_* b)$, so we deduce that c_1 and c_2 are cohomologous. □

Remark 3.7. Since twisted C^*-algebras do not "see" a perturbation of a 2-cocycle by a coboundary [22, Proposition 5.6], Theorem 3.6 says that a twisted k-graph C^*-algebra $C^*(\Lambda, c)$ associated to a cocycle c on the $(k+1)$-graph associated to covering sequence

(Λ_n, p_n) is determined by the covering sequence and $c|_{\Lambda_1}$. One might expect that a similar statement applies to traces; we addresses this in the next section.

4. Bratteli diagrams of covering maps

We now consider a more complicated situation than in the preceding two sections: Rather than a single covering system, we consider Bratteli diagrams of covering maps between k-graphs. Roughly speaking this consists of a Bratteli diagram to which we associate a k-graph at each vertex and a covering map at each edge. In particular, each infinite path in the diagram corresponds to a covering sequence of k-graphs to which we can apply the results of the preceding two sections. We show how to construct a $(k+1)$-graph Λ from a Bratteli diagram of covering maps, and how to view a full corner of a twisted C^*-algebra $C^*(\Lambda, c)$ as a direct limit of direct sums of matrix algebras over the algebras $C^*(\Lambda_v, c|_{\Lambda_v})$.

We take the convention that a Bratteli diagram is a 1-graph E with a partition of E^0 into finite subsets $E^0 = \bigsqcup_{n=1}^{\infty} E^0_n$ such that

$$E^1 = \bigsqcup_{n=1}^{\infty} E^0_n E^1 E^0_{n+1}.$$

We let E^* denote the set of finite paths in E. Following [6] we insist that

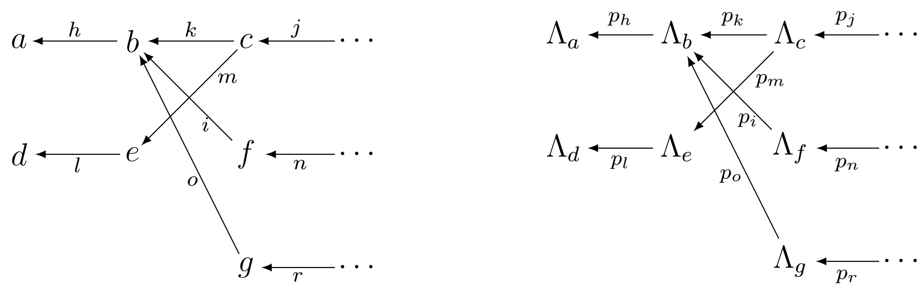
$$vE^1 \neq \emptyset \text{ for all } v, \quad \text{and} \quad E^1 v \neq \emptyset \text{ for all } v \in E^0 \setminus E^0_1.$$

This implies that each E^0_n is non-empty. In the usual convention for drawing directed graphs the vertex set E^0 has levels E^0_n arranged horizontally, and edges point from right to left: each edge in E points from some level E^0_{n+1} to the level E^0_n immediately to its left.

We say that a Bratteli diagram E is *singly connected* if $|vE^1w| \leq 1$ for all $v, w \in E^0$.

Definition 4.1. A *Bratteli diagram of covering maps between k-graphs* consists of a singly connected Bratteli diagram E, together with a collection $(\Lambda_v)_{v\in E^0}$ of k-graphs and a collection $(p_e)_{e\in E^1}$ of covering maps $p_e\colon \Lambda_{s(e)} \to \Lambda_{r(e)}$.

A Bratteli diagram of covering maps is sketched below.



Given a Bratteli diagram of covering maps between k-graphs there exist a unique $(k+1)$-graph, denoted Λ_E (or simply Λ), together with injective functors $\iota_v : \Lambda_v \to \Lambda$, $v \in E^0$, and a bijective map $e : \bigsqcup_{v\in E^0\setminus E^0_1} \Lambda^0_v \times E^1 v \to \Lambda^{e_{k+1}}$, such that

1. (1) $d(\iota_v(\lambda)) = (d(\lambda), 0)$ for $\lambda \in \Lambda_v$,
2. (2) $\iota_v(\Lambda_v) \cap \iota_w(\Lambda_w) = \emptyset$ for $v \neq w \in E^0$,
3. (3) $\bigsqcup_{v\in E^0} \iota_v(\Lambda_v) = \{\lambda \in \Lambda : d(\lambda)_{k+1} = 0\}$,
4. (4) $s(e(w, f)) = \iota_v(w)$ and $r(e(w, f)) = \iota_{r(f)}(p_f(w))$ for $v \in E^0\setminus E^0_1$ and $(w, f) \in \Lambda^0_v \times E^1 v$, and

(5) $e(r(\lambda), f)\iota_v(\lambda) = \iota_{r(f)}(p_f(\lambda))e(s(\lambda), f)$ for $v \in E^0 \backslash E^0_1$ and $(\lambda, f) \in \Lambda^0_v \times E^1 v$.

The construction of the $(k+1)$-graph Λ is taken from [20], with the exception that we describe the underlying data as a Bratteli diagram rather than a sequence of $\{0,1\}$-valued matrices. We will view each Λ_v as a subset of Λ.

Definition 4.2 (c.f. Definition 3.1)**.** Let $\Lambda = \Lambda_E$ be the $(k+1)$-graph associated to a Bratteli diagram of covering maps between k-graphs, and let $c_v \in Z^2(\Lambda_v, A)$ for each $v \in E^0$. The collection of 2-cocycles (c_v) is called *compatible* if there exists a 2-cocycle $c \in Z^2(\Lambda, A)$ such that $c|_{\Lambda_v} = c_v$ for all $v \in E^0$.

Definition 4.2 shows how to build a twisted $(k+1)$-graph C^*-algebra from a Bratteli diagram of covering maps. We will exhibit this C^*-algebra as an inductive limit. This involves considering homomorphisms between twisted k-graph C^*-algebras associated to subgraphs of the ambient $(k+1)$-graph Λ_E associated to the Bratteli diagram of covering maps. In keeping with this, we use the same symbol s to denote the generating twisted Cuntz-Krieger families $s\colon \lambda \mapsto s_\lambda$ of the C^*-algebras of the different subgraphs. It will be clear from context which k-graph we are working in at any given time.

Lemma 4.3. *Let $\Lambda = \Lambda_E$ be the $(k+1)$-graph associated to a Bratteli diagram of covering maps between row finite locally convex k-graphs, together with a compatible collection (c_v) of 2-cocycles. For each $e \in E^1$ there exists an embedding $\iota_e\colon C^*(\Lambda_{r(e)}, c_{r(e)}) \to C^*(\Lambda_{s(e)}, c_{s(e)})$ such that*

$$\iota_e(s_\lambda) = \sum_{p_e(\mu)=\lambda} s_\mu \quad \textit{for all } \lambda \in \Lambda_{r(e)}.$$

Proof. We follow the argument of [20, Remark 3.5.(2)] (which applies in the situation where $c_{r(e)} = c_{s(e)} \equiv 1$). The argument goes through mutatis mutandis when the cocycles are nontrivial. ☐

With slight abuse of notation, in the situation of Lemma 4.3, given $n \geq 1$, we also write ι_e for the induced map $\iota_e^{(n)}\colon M_n(C^*(\Lambda_{r(e)}, c_{r(e)})) \to M_n(C^*(\Lambda_{s(e)}, c_{s(e)}))$ given by

$$\begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix} \mapsto \begin{pmatrix} \iota_e(a_{11}) & \dots & \iota_e(a_{1n}) \\ \vdots & \ddots & \vdots \\ \iota_e(a_{n1}) & \dots & \iota_e(a_{nn}) \end{pmatrix}.$$

Theorem 4.4. *Let $\Lambda = \Lambda_E$ be the $(k+1)$-graph associated to a Bratteli diagram of covering maps between row finite locally convex k-graphs, together with a compatible collection (c_v) of 2-cocycles. Let $c \in Z^2(\Lambda, \mathbb{T})$ be a 2-cocycle such that $c|_{\Lambda_v} = c_v$. The projection $P_0 := \sum_{v \in E^0_1, w \in \Lambda^0_v} s_w$ is full in $C^*(\Lambda, c)$. For $n \in \mathbb{N}$, let $A_n := \overline{\operatorname{span}}\{s_\mu s_\nu^* : \mu, \nu \in \Lambda, d(\mu)_{k+1} = d(\nu)_{k+1} = n\}$. Then each $A_n \subseteq A_{n+1}$, and*

$$P_0 C^*(\Lambda, c) P_0 = \overline{\textstyle\bigcup_n A_n}.$$

For each $\alpha = \alpha_1 \dots \alpha_n \in E^0_1 E^* E^0_n$, let $F(\alpha) := \{\mu = \mu_1 \dots \mu_n \in \Lambda^{n e_{k+1}} : r(\mu_i) \in \Lambda^0_{r(\alpha_i)}$ for $i \leq n$, and $s(\mu_n) \in \Lambda^0_{s(\alpha_n)}\}$. Then each $T_\alpha := \sum_{\mu \in F(\alpha)} s_\mu$ is a partial isometry, and there is an isomorphism $\omega_n\colon \bigoplus_{v \in E^0_n} M_{E^0_1 E^* v}(C^*(\Lambda_v, c_v)) \to A_n$ such that

$$\omega_n\Big(\big((a_{\alpha,\beta})_{\alpha,\beta \in E^0_1 E^* v}\big)_v\Big) = \sum_v \sum_{\alpha,\beta \in E^0_1 E^* v} T_\alpha a_{\alpha,\beta} T_\beta^*.$$

We have $\omega_{n+1} \circ (\mathrm{diag}_{e\in E^1 w}\, \iota_e) = \omega_n$*, and so*

$$P_0 C^*(\Lambda, c) P_0 \cong \varinjlim \left(\bigoplus_{v\in E^0_n} M_{E^0_1 E^* v}(C^*(\Lambda_v, c_v)), \sum_{v\in E^0_n} a_v \mapsto \sum_{w\in E^0_{n+1}} \mathrm{diag}_{e\in E^1 w} \big(\iota_e(a_{r(e)})\big) \right).$$

Proof. We follow the proof of [20, Theorem 3.8], using Lemma 4.3 in place of [20, Remark 3.5.2]. □

5. Rank-3 Bratteli diagrams and their C^*-algebras

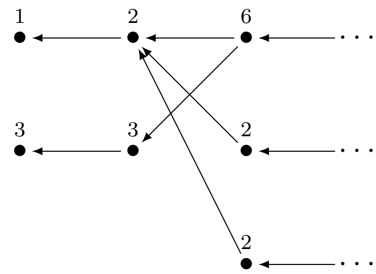
We are now ready to prove our main results. We consider a special class of 3-graphs, which we call "rank-3 Bratteli diagrams" (see Definition 5.2). We will compute the K-theory of the twisted C^*-algebras of these 3-graphs associated to twists by irrational angles using the inductive-limit decomposition just described and the well-known formula for the ordered K-theory of the irrational rotation algebras. We will deduce from Elliott's theorem that these C^*-algebras are classified by K-theory whenever they are simple, and deduce that such C^*-algebras, when simple, can all be realised by rank-2 Bratteli diagrams as in [25].

Recall that a Bratteli diagram is singly connected if $|vE^1w| \leq 1$ for all $v, w \in E^0$. Our rank-3 Bratteli diagrams will be constructed from singly connected Bratteli diagrams of coverings of 2-graphs where the individual 2-graphs are rank-2 simple cycles. The lengths of the cycles are encoded by an additional piece of information: a weight map on the underlying Bratteli diagram.

Definition 5.1. Let E be a Bratteli diagram. A *weight map* on E is a function $w\colon E^0 \to \mathbb{N}\setminus\{0\}$ such that $w(r(e))$ divides $w(s(e))$ for all $e \in E^1$. A *weighted Bratteli diagram* is a Bratteli diagram E together with a weight map.

An example of the first few levels of a singly connected weighted Bratteli diagram is sketched below (the weight map w is identified by labelling the vertices):

(5.1)



To construct our rank-3 Bratteli diagrams, we need to recall the skew-product construction for k-graphs. Following [18], fix a k-graph Λ and a functor $\eta : \Lambda \to G$ into a countable group G. The *skew product graph*, denoted $\Lambda \times_\eta G$, is the k-graph with morphisms $\Lambda \times G$, source, range and degree maps given by

$$r(\lambda, g) = (r(\lambda), g), \quad s(\lambda, g) = (s(\lambda), g\eta(\lambda)), \quad d(\lambda, g) = d(\lambda),$$

and composition given by $(\lambda, g)(\mu, h) = (\lambda\mu, g)$ whenever $s(\lambda, g) = r(\mu, h)$.

Definition 5.2. Let E be a singly connected weighted Bratteli diagram. For $v \in E^0$, let a_v, b_v be the blue and red (respectively) edges in a copy T_2^v of T_2. For each v, let $1 : T_2^v \to \mathbb{Z}/w(v)\mathbb{Z}$ be the functor such that $1(a_v) = 1(b_v) = 1$, the generator of $\mathbb{Z}/w(v)\mathbb{Z}$.

The *rank-3 Bratteli diagram* Λ_E (or simply Λ) associated to E is the unique 3-graph arising from the Bratteli diagram of covering maps given by

$$\Lambda_v = T_2^v \times_1 \mathbb{Z}/w(v)\mathbb{Z}, \qquad v \in E^0,$$
$$p_f(a^s_{s(f)}b^t_{s(f)}, m) = (a^s_{r(f)}b^t_{r(f)}, m \bmod w(r(f)), \qquad s,t \in \mathbb{N}, f \in E^1, m \in \mathbb{Z}/w(s(f))\mathbb{Z}.$$

To keep notation compact we write $\{(v,m) : m = 0,\dots,w(v)-1\}$ for the vertices of Λ_v.

Figure 1 illustrates the portion of the skeleton of a rank-3 Bratteli diagram corresponding to the portion of a weighted Bratteli in (5.1).

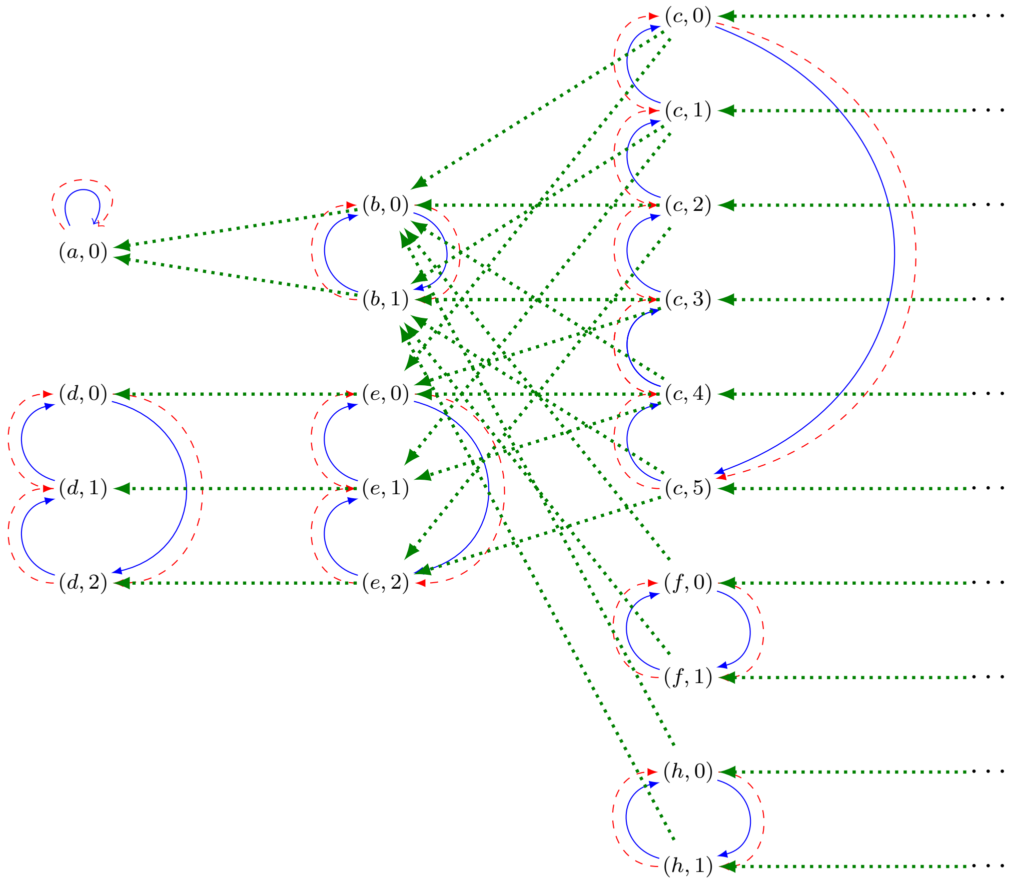


FIGURE 1.

Remark 5.3. Our definition of rank-3 Bratteli diagram relates to the rank-2 Bratteli diagrams introduced by Pask, Raeburn, Rørdam and Sims. Both constructions are based on Bratteli diagrams as initial data, with the difference that here we construct 3-graphs rather than 2-graphs. See [25] for the details.

Given a $\mathbb{T}$-valued 2-cocycle c on $\mathbb{Z}^k$ and a k-graph Λ, we obtain a 2-cocycle d_*c on Λ by $(d_*c)(\mu,\nu) = c(d(\mu), d(\nu))$. An example of a $\mathbb{T}$-valued 2-cocycle on $\mathbb{Z}^k$ is the map

$$c^k_\theta(m,n) = e^{2\pi\theta m_2 n_1}. \tag{5.2}$$

The 2-cocycle (5.2) on $\mathbb{Z}^2$ will be denoted c_θ. We let A_θ denote the *rotation C^*-algebra* corresponding to the angle $\theta \in \mathbb{R}$ (see [21, Example 7.7]).

Our main theorem describes the ordered K-theory of the twisted C^*-algebras of rank-3 Bratteli diagrams corresponding to irrational θ. We state the result now, but the proof will require some more preliminary work.

To state the theorem, we take the convention that given a direct sum $G = \bigoplus_i G_i$ of groups and $g \in G$, we write $g\delta_i$ for the image of g in the i^{th} direct summand of G.

Theorem 5.4. *Let $\Lambda = \Lambda_E$ be the rank-3 Bratteli diagram associated to a singly connected weighted Bratteli diagram E, and take $\theta \in \mathbb{R}\backslash\mathbb{Q}$. Let $c^3_\theta \in Z^2(\mathbb{Z}^3, \mathbb{T})$ be as in (5.2), and let $c = d_* c^3_\theta \in Z^2(\Lambda, \mathbb{T})$. For $n \in \mathbb{N}$, define $A_n : \oplus_{v\in E^0_n}(\frac{1}{w(v)}\mathbb{Z} + \theta\mathbb{Z}) \to \oplus_{u\in E^0_{n+1}}(\frac{1}{w(u)}\mathbb{Z} + \theta\mathbb{Z})$ and $B_n \colon \oplus_{v\in E^0_n} \mathbb{Z}^2 \to \oplus_{u\in E^0_{n+1}}\mathbb{Z}^2$ by*

$$A_n\Big(\frac{1}{w(v)}p + \theta q\Big)\delta_v = \sum_{e\in vE^1} \Big(\frac{1}{w(v)}p + \theta q\Big)\delta_{s(e)} \qquad \text{and}$$

$$B_n(p, q)\delta_v = \sum_{e\in vE^1} \Big(p + \Big(1 - \frac{w(s(e))}{w(v)}\Big)q,\ \frac{w(s(e))}{w(v)}q\Big)\delta_{s(e)}.$$

Endow $\varinjlim \Big(\bigoplus_{v\in E^0_n}(\frac{1}{w(v)}\mathbb{Z} + \theta\mathbb{Z}), A_n\Big)$ with the positive cone and order inherited from the approximating subgroups. Then there are an order isomorphism

$$h_0 \colon K_0(C^*(\Lambda, c)) \to \varinjlim \Big(\bigoplus_{v\in E^0_n} \Big(\frac{1}{w(v)}\mathbb{Z} + \theta\mathbb{Z}\Big), A_n\Big)$$

and an isomorphism

$$h_1 \colon K_1(C^*(\Lambda, c)) \to \varinjlim \Big(\bigoplus_{v\in E^0_n} \mathbb{Z}^2, B_n\Big)$$

with the following properties: Let $v \in E^0_n$, let $i < w(v)$, let μ_i be the unique element of $(v, i)\Lambda_v^{(w(v),0)}$ and let ν_i be the unique element of $(v, i)\Lambda_v^{(w(v)-1,1)}$. Then

$$h_0\big(\big[s_{(v,i)}\big]\big) = \tfrac{1}{w(v)}\delta_v \in \textstyle\bigoplus_{u\in E^0_n}(\frac{1}{w(u)}\mathbb{Z} + \theta\mathbb{Z});$$

$$h_1\big(\big[s_{\mu_i} + \textstyle\sum_{j\neq i} s_{(v,j)}\big]\big) = (1, 0)\delta_v \in \bigoplus_{u\in E^0_n} \mathbb{Z}^2; \text{ and}$$

$$h_1\big(\big[s_{\nu_i} + \textstyle\sum_{j\neq i} s_{(v,j)}\big]\big) = (0, 1)\delta_v \in \bigoplus_{u\in E^0_n} \mathbb{Z}^2.$$

There is an isomorphism $\theta \colon K_1(C^*(\Lambda, c)) \to K_0(C^*(\Lambda), c)$ such that $h_0 \circ \theta \circ h_1^{-1}\big((a, b)\delta_v\big) = \frac{b}{w(v)} + (a + b)\theta$ for all $v \in E^0$ and $(a, b) \in \mathbb{Z}^2$.

Remark 5.5. If we regard the maps $A_n \colon \oplus_{v\in E^0_n} (\frac{1}{w(v)}\mathbb{Z} + \theta\mathbb{Z}) \to \oplus_{u\in E^0_{n+1}}(\frac{1}{w(u)}\mathbb{Z} + \theta\mathbb{Z})$ as $E^0_n \times E^0_{n+1}$ matrices of homomorphisms $A_n(v, u) \colon \frac{1}{w(v)}\mathbb{Z} + \theta\mathbb{Z} \to \frac{1}{w(u)}\mathbb{Z} + \theta\mathbb{Z}$, then each $A_n(v, w)$ is the inclusion map $\frac{1}{w(v)}\mathbb{Z} + \theta\mathbb{Z} \subseteq \frac{1}{w(u)}\mathbb{Z} + \theta\mathbb{Z}$. Likewise, if we think of each B_n as an $E^0_n \times E^0_{n+1}$ matrix of homomorphisms $B_n(v, u) : \mathbb{Z}^2 \to \mathbb{Z}^2$, then writing $l(v, u) := w(u)/w(v)$, each $B_n(v, u)$ is implemented by multiplication by the matrix $\left(\begin{smallmatrix} 1 & 1-l(v,u) \\ 0 & l(v,u) \end{smallmatrix}\right)$.

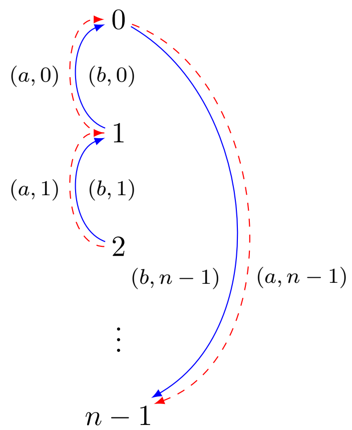
Remark 5.6. In the preceding theorem, given $v \in E^0_n$, it is not so easy to specify explicitly the projection in $C^*(\Lambda, c)$ which maps to $\theta\delta_v \in \bigoplus_{u\in E^0_n}(\frac{1}{w(u)}\mathbb{Z} + \theta\mathbb{Z})$. However, the description of the connecting maps in K_0 in Lemma 5.10 yields the following description: Lemma 5.8 describes an isomorphism $\phi \colon A_{w(v)\theta} \otimes M_{w(v)}(\mathbb{C}) \cong C^*(\Lambda_v, c)$, and if p_θ denotes the Rieffel projection in $A_{w(v)\theta}$, then h_0 carries the K_0-class of $\phi(p_\theta \oplus 0_{w(v)-1})$ to $\theta\delta_v$.

Remark 5.7. Theorem 5.4 applies only when $c = d_* c_\theta^3 \in Z^2(\Lambda, \mathbb{T})$ for some θ. But Theorem 3.6 suggests that this is a fairly mild hypothesis. To make this precise, first suppose that $v, w \in E_1^0$ have the property that there exist $\alpha \in vE^*$ and $\beta \in wE^*$ with $s(\alpha) = s(\beta)$. By choosing any infinite path x in E with range $s(\alpha)$ we can pick out covering systems corresponding to αx and to βx, and then Theorem 3.6 implies that c_v and c_w are cohomologous. Now consider any connected component C of E. An induction using what we have just showed proves that the c_v corresponding to vertices v in C are all cohomologous. So, decomposing, E into connected components, we see that $C^*(\Lambda_E, c)$ is a direct sum of subalgebras $C^*(\Lambda_C, c)$ in which $v \mapsto c_v$ is constant up to cohomology.

Each Λ_v is the quotient of the 2-graph Δ_2 (see [21, Examples 2.2(5)]) by the canonical action of $\{(m, n) : m+n \in w(v)\mathbb{Z}\} \leq \mathbb{Z}^2$, and so [21, Theorem 4.9] shows that $H_2(\Lambda_v, \mathbb{T}) \cong \mathbb{T}$; in particular $\{[d_* c_\theta] : \theta \in [0, 2\pi)\}$ is all of $H_2(\Lambda_v, \mathbb{T})$. So for each connected component C of E, there is some θ such that $c_v \sim d_* c_\theta$ for each vertex v in C.

The first step to proving Theorem 5.4 is to describe the building blocks in the direct-limit decomposition of Theorem 4.4 for a rank-3 Bratteli diagram. Recall that if E is a singly connected weighted Bratteli diagram and $n = w(v)$, then $\Lambda_v = T_2^v \times_1 \mathbb{Z}/n\mathbb{Z}$ as illustrated below:

(5.3)



When $n = 1$ the C^*-algebra $C^*(\Lambda_v, d_* c_\theta^3)$ is isomorphic to the irrational rotation algebra A_θ (see [21]). We prove that, in general, $C^*(\Lambda_v, d_* c_\theta^3)$ is isomorphic to $M_n(A_{n\theta})$.

Lemma 5.8. *Let Λ be the 2-graph $T_2 \times_1 \mathbb{Z}/n\mathbb{Z}$ of (5.3). Let $c = d_* c_\theta$ for $\theta \in \mathbb{R}$. Let u, v denote the generators for $A_{n\theta}$, and $(\zeta_{i,j})$ the standard matrix units for $M_n(\mathbb{C})$. Let μ_i (resp. ν_i) denote the unique element in Λ of degree $(n, 0)$ (resp. $(n-1, 1)$) with source and range (v, i), and let $\alpha_0, \ldots, \alpha_{n-1}$ be any elements in $\mathbb{T}$ such that $e^{2\pi i\theta}\alpha_{i-1} = \alpha_i$. Then there is an isomorphism*

$$\phi \colon A_{n\theta} \otimes M_n(\mathbb{C}) \cong C^*(\Lambda, c)$$

such that

$$\phi(u \otimes 1_n) = \sum_{i=0}^{n-1} s_{\mu_i}, \quad \phi(v \otimes 1_n) = \sum_{i=0}^{n-1} \alpha_i s_{\nu_i}, \quad \text{and} \quad \phi(1 \otimes \zeta_{j,j+1}) = s_{(a,j-1)}.$$

Proof. Define elements U, V and $e_{j,j+1}$, $j < n$ of $C^*(\Lambda, c)$ by

$$U = \sum_{i=0}^{n-1} s_{\mu_i}, \quad V = \sum_{i=0}^{n-1} \alpha_i s_{\nu_i} \quad \text{and } e_{j,j+1} = s_{(a,j-1)}$$

The set $\{e_{j,j+1} : j = 1, \dots, n-1\}$ generates a system of matrix units $(e_{i,j})_{i,j=1,\dots,n}$. Each $e_{i,j}$ is non-zero by [30, Theorem 3.15]. Straightforward calculations show that U and V are both unitaries, and that these unitaries commute with the $e_{j,j+1}$ and their adjoints.

We claim that $VU = e^{2\pi i n\theta} UV$. Since $s(\mu_i) = r(\nu_j)$ only for $i = j$, we have

$$UV = \Big(\sum_{i=0}^{n-1} s_{\mu_i}\Big)\Big(\sum_{i=0}^{n-1} \alpha_i s_{\nu_i}\Big) = \sum_{i=0}^{n-1} \alpha_i c(\mu_i, \nu_i) s_{\mu_i \nu_i} \quad \text{and similarly}$$
$$VU = \Big(\sum_{i=0}^{n-1} \alpha_i s_{\nu_i}\Big)\Big(\sum_{i=0}^{n-1} s_{\mu_i}\Big) = \sum_{i=0}^{n-1} \alpha_i c(\nu_i, \mu_i) s_{\nu_i \mu_i}.$$

Since $c(\mu_i, \nu_i) = c_\theta\big((n,0),(n-1,1)\big) = 1$ and $c(\nu_i, \mu_i) = c_\theta\big((n-1,1),(n,0)\big) = e^{2\pi i n\theta}$, we obtain $VU = e^{2\pi i n\theta} UV$ as claimed.

By definition the elements $U, V, e_{i,j}$ (and $\{s_\lambda : \lambda \in \Lambda\}$) belong to the algebra generated by the elements $\{s_{(a,0)}, \dots, s_{(b,n-1)}\}$. Conversely, for $j < n$ we have

(5.4)
$$s_{(a,j)} = \begin{cases} U e_{n,1} & \text{if } j = n-1 \\ e_{j+1,j+2}, & \text{otherwise,} \end{cases} \quad \text{and} \quad s_{(b,j)} = \begin{cases} V e_{n,1} & \text{if } j = n-1 \\ e^{-2\pi i\theta} U^* V e_{j+1,j+2} & \text{otherwise.} \end{cases}$$

Hence $C^*(\Lambda_v)$ is generated by U, V and the $e_{i,j}$.

The universal property of $A_{n\theta} \otimes M_n(\mathbb{C})$ gives a surjective homomorphism $\phi\colon A_{n\theta} \otimes M_n(\mathbb{C}) \to C^*(\Lambda, c)$ such that

$$\phi(u \otimes 1_n) = U, \quad \phi(v \otimes 1_n) = V, \quad \text{and} \quad \phi(1 \otimes \zeta_{i,j}) = e_{i,j}.$$

The formulas (5.4) describe a Cuntz-Krieger ϕ-representation in the sense of [21, Definition 7.4], so the universal property of $C^*_\phi(\Lambda)$ gives an inverse for ϕ, and so ϕ is an isomorphism[1]. Hence $A_{n\theta} \otimes M_n(\mathbb{C}) \cong C^*_\phi(\Lambda) \cong C^*(\Lambda, c)$ by [22, Corollary 5.7]. □

Lemma 5.9. *Let $\Lambda = \Lambda_E$ be the rank-3 Bratteli diagram associated to a singly connected weighted Bratteli diagram E together with a compatible collection (c_v) of 2-cocycles. Take $e \in E^1$, and suppose that $c_{r(e)} = d_* c_\theta$ and $c_{s(e)} = d_* c_\theta$ where $\theta \in \mathbb{R} \backslash \mathbb{Q}$. Let $n = w(r(e))$, $m = w(s(e))$ and $l = m/n$. Let $\iota_e\colon C^*(\Lambda_{r(e)}, c_{r(e)}) \to C^*(\Lambda_{s(e)}, c_{s(e)})$ be as in Lemma 4.3, and let $\phi_r\colon A_{n\theta} \otimes M_n(\mathbb{C}) \to C^*(\Lambda_{r(e)}, c_{r(e)})$ and $\phi_s\colon A_{m\theta} \otimes M_m(\mathbb{C}) \to C^*(\Lambda_{s(e)}, c_{s(e)})$ be the isomorphisms obtained from Lemma 5.8. Let u_r and v_r be the generators of $A_{n\theta}$, and let $\rho_r\colon K_1(A_{n\theta} \otimes M_n(\mathbb{C})) \to \mathbb{Z}^2$ be the isomorphism such that $\rho_r([u_r \oplus 1_{n-1}]) = \binom{1}{0}$ and $\rho_r([v_r \oplus 1_{n-1}]) = \binom{0}{1}$; define u_s, v_s and $\rho_s\colon K_1(A_{m\theta} \otimes M_m(\mathbb{C})) \to \mathbb{Z}^2$ similarly. Then the diagram*

(5.5)
$$\begin{array}{ccc}
K_1(C^*(\Lambda_{r(e)}, c_{r(e)})) & \xrightarrow{K_1(\iota_e)} & K_1(C^*(\Lambda_{s(e)}, c_{s(e)})) \\
\uparrow{\scriptstyle K_1(\phi_r)} & & \uparrow{\scriptstyle K_1(\phi_s)} \\
K_1(A_{n\theta} \otimes M_n(\mathbb{C})) & & K_1(A_{m\theta} \otimes M_m(\mathbb{C})) \\
\downarrow{\scriptstyle \rho_r} & & \downarrow{\scriptstyle \rho_s} \\
\mathbb{Z}^2 & \xrightarrow{\left(\begin{smallmatrix} 1 & 1-l \\ 0 & l \end{smallmatrix}\right)} & \mathbb{Z}^2
\end{array}$$

commutes.

[1] Of course, if θ is irrational then A_θ and hence $A_\theta \otimes M_n(\mathbb{C})$ is simple, and so ϕ is automatically injective.

Proof. Recall $n = w(r(e))$, $m = w(s(e))$, and $l = m/n$. Since the maps $a \mapsto a \oplus 1_{n-1}$ and $b \mapsto b \oplus 1_{m-1}$ induce isomorphisms in K_1, the classes $[u_r \oplus 1_{n-1}]$ and $[v_r \oplus 1_{n-1}]$ generate $K_1(A_{n\theta} \otimes M_n(\mathbb{C}))$ and $[u_s \oplus 1_{m-1}]$ and $[v_s \oplus 1_{m-1}]$ generate $K_1(A_{m\theta} \otimes M_m(\mathbb{C}))$.

We claim that $K_1(\iota_e)$ maps $[\phi_r(u_r \oplus 1_{n-1})]$ to $[\phi_s(u_s \oplus 1_{m-1})]$. To see this, for $i < n$, let μ_i^r denote the unique element of $(r(e), i)\Lambda_{r(e)}^{(n,0)}$, and let

$$U_r := \sum_{i=0}^{n-1} s_{\mu_i^r} \in C^*(\Lambda_{r(e)}, c_{r(e)}).$$

By Lemma 5.8, $U_r = \phi_r(u_r \otimes 1_n)$, and so

$$U_r = \phi_r(u_r \otimes 1) = \prod_{i=0}^{n-1} \phi_r(1_i \oplus u_r \oplus 1_{n-i-1}).$$

Hence $[U_r] = \sum_{i=0}^{n-1} [\phi_r(1_i \oplus u_r \oplus 1_{n-i-1})] = n[\phi_r(u_r \oplus 1_{n-1})]$. An identical argument shows that if μ_i^s denotes the unique element of $(s(e), i)\Lambda_{s(e)}^{(m,0)}$ for $i < m$, then $U_s := \sum_{i=0}^{m-1} s_{\mu_i^s}$ satisfies $[U_s] = m[\phi_s(u_s \oplus 1_{m-1})]$. Direct computation using that $ln = m$ shows that ι_e maps $(\sum_{i=0}^{n-1} s_{\mu_i^r})^l$ to $\sum_{i=0}^{m-1} s_{\mu_i^s}$. We have

$$m[\phi_r(u_r \oplus 1_{n-1})] = ln[\phi_r(u_r \oplus 1_{n-1})] = l[U_r] = \Big[\Big(\sum_{i=0}^{n-1} s_{\mu_i^r}\Big)^l\Big].$$

Hence

$$K_1(\iota_e)(m[\phi_r(u_r \oplus 1_{n-1})]) = \Big[\sum_{i=0}^{m-1} s_{\mu_i^s}\Big] = [U_s] = m[\phi_s(u_s \oplus 1_{m-1})].$$

Since $\theta \in \mathbb{R}\backslash\mathbb{Q}$, we have $K_1(A_{n\theta} \otimes M_n(\mathbb{C})) = \langle [u_r \oplus 1_{n-1}], [v_r \oplus 1_{n-1}] \rangle \cong \mathbb{Z}^2$, and we deduce that $K_1(\iota_e)$ maps $[\phi_r(u_r \oplus 1_{n-1})]$ to $[\phi_s(u_s \oplus 1_{m-1})]$.

We now show that $K_1(\iota_e)$ maps $[\phi_r(v_r \oplus 1_{n-1})]$ to $l[\phi_s(v_s \oplus 1_{m-1})] - (l-1)[\phi_s(u_s \oplus 1_{m-1})]$. Let ν_i^r, ν_j^s be the unique elements of $(r(e), i)\Lambda_{r(e)}^{(n-1,1)}$ and $(s(e), j)\Lambda_{s(e)}^{(m-1,1)}$ for each i, j. Let $W_r = \sum_{i=0}^{n-1} s_{\nu_i^r}$. Fix $\alpha_0, \dots, \alpha_{n-1}$ in $\mathbb{T}$ such that $e^{2\pi i\theta}\alpha_{i-1} = \alpha_i$, and set $V_r = \sum_{i=0}^{n-1} \alpha_i s_{\nu_i^r}$. Lemma 5.8 gives $V_r = \phi_r(v_r \otimes 1_n)$. Since $V_r = W_r \cdot \sum_{i=0}^{n-1} \alpha_i s_{(r(e),i)}$, and since $[1] = [\sum_{i=0}^{n-1} \alpha_i s_{(r(e),i)}]$ it follows that $n[\phi_r(v_r \oplus 1_{n-1})] = [V_r] = [W_r]$. Similarly $m[\phi_s(v_s \oplus 1_{m-1})] = [\sum_{i=0}^{m-1} s_{\nu_i^s}]$, and

$$\begin{aligned} n[\phi_r(v_r \oplus 1_{n-1})] &+ (l-1)n[\phi_r(u_r \oplus 1_{n-1})] \\ &= [W_r] + (l-1)[U_r] = \Big[\Big(\sum_{i=0}^{n-1} s_{\nu_i^r}\Big)\Big(\sum_{i=0}^{n-1} s_{\mu_i^r}\Big)^{l-1}\Big] = n\Big[\sum_{i=0}^{n-1} s_{\nu_i^r(\mu_i^r)^{l-1}}\Big]. \end{aligned}$$

For $j < n$, we have $p_e^{-1}(\nu_i^r(\mu_i^r)^{l-1}) = \{\nu_i^s, \nu_{i+n}^s, \dots, \nu_{i+(l-1)n}^s\}$. Hence

$$\begin{aligned} K_1(\iota_e)\big(n[\phi_r(v_r \oplus 1_{n-1})] &+ (l-1)n[\phi_r(u_r \oplus 1_{n-1})]\big) \\ &= n\big[\iota_e(s_{\nu_i^r(\mu_i^r)^{l-1}})\big] = \Big[\sum_{i=0}^{m-1} s_{\nu_i^s}\Big] = m[\phi_s(v_s \oplus 1_{m-1})]. \end{aligned}$$

Since $m = nl$, we deduce that $K_1(\iota_e)$ sends $[\phi_r(v_r \oplus 1_{n-1})] + (l-1)[\phi_r(u_r \oplus 1_{n-1})]$ to $l[\phi_s(v_s \oplus 1_{m-1})]$. We saw above that $K_1(\iota_e)\big((l-1)[\phi_r(u_r \oplus 1_{n-1})]\big) = (l-1)[\phi_s(u_s \oplus 1_{m-1})]$, so subtracting gives $K_1(\iota_e)\big([\phi_r(v_r \oplus 1_{n-1})]\big) = l[\phi_s(v_s \oplus 1_{m-1})] - (l-1)[\phi_s(u_s \oplus 1_{m-1})]$. So the diagram (5.5) commutes as claimed. □

Let $T(A)$ denote the set of *tracial states*, i.e., positive linear functionals with the trace property and norm one, on a C^*-algebra A. For any $\tau \in T(A)$ there is a map $K_0(\tau)\colon K_0(A) \to \mathbb{R}$ such that $K_0(\tau)([p]-[q]) = \sum_i \tau(p_{ii} - q_{ii})$ for any projections $p, q \in M_n(A)$. When $\tau = \mathrm{Tr}$, the unique tracial state on $A_\theta \otimes M_k(\mathbb{C})$ for $\theta \in \mathbb{R}\backslash\mathbb{Q}$, the map $K_0(\tau)$ is an order isomorphism of $K_0(A_\theta \otimes M_k(\mathbb{C}))$ onto $\frac{1}{k}\mathbb{Z} + \frac{\theta}{k}\mathbb{Z}$ [29].

Lemma 5.10. *Let $\Lambda = \Lambda_E$ be the rank-3 Bratteli diagram associated to a singly connected weighted Bratteli diagram E together with a compatible collection (c_v) of 2-cocycles. Take $e \in E^1$ and suppose that $c_{r(e)} = d_* c_\theta$ and $c_{s(e)} = d_* c_\theta$ where $\theta \in \mathbb{R}\backslash\mathbb{Q}$. Let $n = w(r(e))$ and $m = w(s(e))$. Let $\iota_e\colon C^*(\Lambda_{r(e)}, c_{r(e)}) \to C^*(\Lambda_{s(e)}, c_{s(e)})$ be as in Lemma 4.3, and let $\phi_r\colon A_{n\theta} \otimes M_n(\mathbb{C}) \to C^*(\Lambda_{r(e)}, c_{r(e)})$ and $\phi_s\colon A_{m\theta} \otimes M_m(\mathbb{C}) \to C^*(\Lambda_{s(e)}, c_{s(e)})$ be the isomorphisms obtained from Lemma 5.8. Let τ_r and τ_s be the unique tracial states on $A_{n\theta} \otimes M_n(\mathbb{C})$ and $A_{m\theta} \otimes M_m(\mathbb{C})$. Then the diagram*

$$\begin{array}{ccc} K_0(C^*(\Lambda_{r(e)}, c_{r(e)})) & \xrightarrow{K_0(\iota_e)} & K_0(C^*(\Lambda_{s(e)}, c_{s(e)})) \\ \big\uparrow{\scriptstyle K_0(\phi_r)} & & \big\uparrow{\scriptstyle K_0(\phi_s)} \\ K_0(A_{n\theta} \otimes M_n(\mathbb{C})) & & K_0(A_{m\theta} \otimes M_m(\mathbb{C})) \\ \big\downarrow{\scriptstyle K_0(\tau_r)} & & \big\downarrow{\scriptstyle K_0(\tau_s)} \\ \frac{1}{n}\mathbb{Z} + \theta\mathbb{Z} & \xrightarrow{\subseteq} & \frac{1}{m}\mathbb{Z} + \theta\mathbb{Z} \end{array} \tag{5.6}$$

commutes.

Proof. Define $l = m/n$. Let p_r and p_s denote the Powers-Rieffel projections in $A_{n\theta}$ and $A_{m\theta}$ respectively. Since the maps $a \mapsto a \oplus 0_{n-1}$ and $b \mapsto b \oplus 0_{m-1}$ induce an isomorphisms in K_0, the elements $[1 \oplus 0_{n-1}]$ and $[p_r \oplus 0_{n-1}]$ generate $K_0(A_{n\theta} \otimes M_n(\mathbb{C}))$, and $[1 \oplus 0_{m-1}]$ and $[p_s \oplus 1_{m-1}]$ generate $K_0(A_{m\theta} \otimes M_m(\mathbb{C}))$. Then $K_0(\iota_e)$ maps $[\phi_r(1 \oplus 0_{n-1})]$ to $l[\phi_s(1 \oplus 0_{m-1})]$ because

$$K_0(\iota_e)\big(n[\phi_r(1 \oplus 0_{n-1})]\big) = K_0(\iota_e)\big([1_{C^*(\Lambda_{r(e)})}]\big) = [1_{C^*(\Lambda_{s(e)})}] = m[\phi_r(1 \oplus 0_{m-1})],$$

and $m = nl$.

We show that $K_0(\iota_e)$ maps $[\phi_r(p_r \oplus 0_{n-1})]$ to $[\phi_s(p_s \oplus 0_{m-1})]$. Let $\tilde{\tau}_r := \tau_r \circ \phi_r^{-1}$ and $\tilde{\tau}_s := \tau_s \circ \phi_s^{-1}$ be the unique tracial states on $C^*(\Lambda_{r(e)}, c_{r(e)})$ and $C^*(\Lambda_{s(e)}, c_{s(e)})$. Since $\iota_e(C^*(\Lambda_{r(e)}, c_{r(e)})) \subseteq C^*(\Lambda_{s(e)}, c_{s(e)})$ uniqueness of $\tilde{\tau}_r$ implies that $\tilde{\tau}_s \circ \iota_e = \tilde{\tau}_r$, and since the unique tracial state Tr on $A_{n\theta}$ satisfies $\mathrm{Tr}(p_r) = n\theta$, we have $\tau_r(p_r \oplus 0_{n-1}) = \theta$. Hence

$$\tilde{\tau}_s \circ \iota_e(\phi_r(p_r \oplus 0_{n-1})) = \tilde{\tau}_r(\phi_r(p_r \oplus 0_{n-1})) = \tau_r(p_r \oplus 0_{n-1}) = \theta = \tilde{\tau}_s(\phi_s(p_s \oplus 0_{m-1})).$$

In particular

$$K_0(\tilde{\tau}_s \circ \iota_e)([\phi_r(p_r \oplus 0_{n-1}]) = K_0(\tilde{\tau}_s)([\phi_s(p_s \oplus 0_{m-1})]).$$

Since θ is irrational, $K_0(\tilde{\tau}_s)$ is an isomorphism, and so we deduce that $K_0(\iota_e)\big([\phi_r(p_r \oplus 0_{n-1})]\big) = [\phi_s(p_s \oplus 0_{m-1})]$, and that the diagram (5.6) commutes. □

Corollary 5.11. *Let $\Lambda = \Lambda_E$ be the rank-3 Bratteli diagram associated to a singly connected weighted Bratteli diagram E together with a compatible collection (c_v) of 2-cocycles.*

Let $v \in E^0$ and suppose that $c_v = d_* c_\theta$ for some $\theta \in \mathbb{R}\backslash\mathbb{Q}$. Then the ordered K-theory of $C^*(\Lambda_v, c_v)$ is given by

$$(K_0, K_0^+, K_1) = \Big(\frac{1}{w(v)}\mathbb{Z} + \theta\mathbb{Z}, \Big(\frac{1}{w(v)}\mathbb{Z} + \theta\mathbb{Z}\Big) \cap [0,\infty), \mathbb{Z}^2\Big).$$

Proof. The irrational rotation algebra A_θ is a stably finite unital exact C^*-algebra with a simple, weakly unperforated K_0-group [29, 5] (see also [2, p. 36]). Hence [2, p. 42] gives

$$K_0(C^*(\Lambda_v, c_v))^+ = \{0\} \cup \{x : K_0(\tau)(x) > 0 \text{ for all } \tau \in T(C^*(\Lambda_v, c_v))\}.$$

Since $C^*(\Lambda_v, c_v) \cong A_\theta \otimes M_k(\mathbb{C})$ admits a unique tracial state τ, the map $K_0(\tau)\colon K_0(A_\theta \otimes M_k(\mathbb{C})) \to \mathbb{R}^+$ is an order isomorphism onto its range, and so $K_0(A_\theta \otimes M_k(\mathbb{C}))^+ = (\frac{1}{k}\mathbb{Z} + \frac{\theta}{k}\mathbb{Z}) \cap [0,\infty)$. The result follows. □

Proof of Theorem 5.4. Theorem 4.4 shows that

$$\begin{aligned} K_*(C^*(\Lambda, c)) &\cong K_*(P_0 C^*(\Lambda) P_0) \\ &\cong \varinjlim \Big(\bigoplus_{v \in E_n^0} K_*(M_{E_1^0 E^* v}(C^*(\Lambda_v, c_v))), \\ &\qquad\qquad K_*\Big(\sum_{v \in E_n^0} a_v \mapsto \sum_{w \in E_{n+1}^0} \operatorname{diag}_{e \in E^1 w}(\iota_e(a_{r(e)}))\Big)\Big). \end{aligned}$$

Each $K_*(M_{E_1^0 E^* v}(C^*(\Lambda_v, c_v))) \cong K_*(C^*(\Lambda_v, c_v))$ and these isomorphisms are compatible with the connecting maps. Lemma 5.8 shows that each $C^*(\Lambda_v, c_v) \cong A_{w(v)\theta} \otimes M_{w(v)}(\mathbb{C})$ and hence has K-theory $(\frac{1}{w(v)}\mathbb{Z} + \theta\mathbb{Z}, \mathbb{Z}^2)$, and Lemmas 5.9 and 5.10 show that the connecting maps are as claimed. The order on K_0 follows from Corollary 5.11.

For the final statement, observe that under the canonical isomorphisms $\frac{1}{w(v)}\mathbb{Z} + \theta\mathbb{Z} \cong \mathbb{Z}^2$, the inclusion maps $A_n(v,u)$ of Remark 5.5 are implemented by the matrices $\left(\begin{smallmatrix} w(u)/w(v) & 0 \\ 0 & 1 \end{smallmatrix}\right)$ (see also the proof of Lemma 5.10). The corresponding maps $B_n(v,u)$ are implemented by the matrices $\left(\begin{smallmatrix} 1 & 1-w(u)/w(v) \\ 0 & 1 \end{smallmatrix}\right)$. Fix $e \in E^1$, let $v = r(e)$ and $u = s(e)$ and $l = w(u)/w(v)$, and calculate:

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1-l \\ 0 & l \end{pmatrix} = \begin{pmatrix} 0 & l \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} l & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

So the automorphisms $T_n := \bigoplus_{v \in E_0^n} \left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right) : \bigoplus_{v \in E_n^0} \mathbb{Z}^2 \to \bigoplus_{v \in E_n^0} \mathbb{Z}^2$ satisfy $T_n B_n = A_n T_n$ and so there is a group isomorphism $\varinjlim(\bigoplus_{v \in E_n^0} \mathbb{Z}^2, B_n) \cong \varinjlim(\bigoplus_{v \in E_n^0} \mathbb{Z}^2, A_n)$ that carries each $(a,b)\delta_v$ to $(b, a+b)\delta_v$ according to the communing diagram

$$\begin{array}{ccccccc} \mathbb{Z}^2\delta_v & \xrightarrow{\cong} & \frac{1}{w(v)}\mathbb{Z} + \theta\mathbb{Z} & \xrightarrow{A_n(v,u)} & \frac{1}{w(v)}\mathbb{Z} + \theta\mathbb{Z} & \xrightarrow{\cong} & \mathbb{Z}^2\delta_u \, . \\ \uparrow{\scriptstyle T_n(v,v)} & & & & & & \uparrow{\scriptstyle T_n(u,u)} \\ \mathbb{Z}^2\delta_v & & & \xrightarrow{\quad B_n(v,u) \quad} & & & \mathbb{Z}^2\delta_u \end{array}$$

After identifying each $\mathbb{Z}^2\delta_v$ with $\frac{1}{w(v)}\mathbb{Z} + \theta\mathbb{Z}$ as above, we obtain the desired isomorphism $K_1(C^*(\Lambda), c) \cong K_0(C^*(\Lambda), c)$. □

Having computed the K-theory of the $C^*(\Lambda, c)$, we conclude by observing that they are all classifiable by their K-theory. We say that a weighted Bratteli diagram is cofinal if the underlying Bratteli diagram is cofinal.

Corollary 5.12. *Let $\Lambda = \Lambda_E$ be the rank-3 Bratteli diagram associated to a singly connected weighted Bratteli diagram E. Take $\theta \in \mathbb{R}\backslash\mathbb{Q}$, let $c^3_\theta \in Z^2(\mathbb{Z}^3, \mathbb{T})$ be as in* (5.2)*, and let $c = d_* c^3_\theta \in Z^2(\Lambda, \mathbb{T})$. Then $C^*(\Lambda, c)$ is an $A\mathbb{T}$-algebra of real rank zero, and is simple if and only if E is cofinal, in which case it is classified up to isomorphism by ordered K-theory and scale.*

Proof. Each $A_{w(v)\theta}$ is an $A\mathbb{T}$ algebra [12], and has real rank zero since it has a unique trace (see, for example, [4, Theorem 1.3]). Since direct limits of $A\mathbb{T}$ algebras are $A\mathbb{T}$ and since direct limits of C^*-algebras of real rank zero also have real rank zero, $C^*(\Lambda, c)$ is also an $A\mathbb{T}$-algebra of real rank zero.

It is straightforward to verify that E is cofinal if and only if Λ is cofinal. Hence [32, Lemma 7.2] implies that if E is not cofinal then $C^*(\Lambda, c)$ is not simple. Now suppose that E is cofinal. Following the argument of [1, Proposition 5.1] shows that any ideal of $C^*(\Lambda, c)$ which contains some s_v is all of $C^*(\Lambda, c)$. So if ψ is a nonzero homomorphism of $C^*(\Lambda, c)$, then $\psi(s_v) \neq 0$ for all v. That is $\psi|_{C^*(\Lambda_v, c)}$ is nonzero for each $v \in E^0$. But each $C^*(\Lambda_v, c) \cong A_{w(v)\theta} \otimes M_{w(v)}(\mathbb{C})$ is simple, and it follows that ψ is injective on each $C^*(\Lambda_v, c)$ and hence isometric on each $\bigoplus_{v \in E^0_n} C^*(\Lambda_v, c)$. So ψ is isometric on a dense subspace of $C^*(\Lambda, c)$ and hence on all of $C^*(\Lambda, c)$. Thus $C^*(\Lambda, c)$ is simple.

The final assertion follows from Elliott's classification theorem [10]. □

Corollary 5.13. *Let $\Lambda = \Lambda_E$ be the rank-3 Bratteli diagram associated to a singly connected cofinal weighted Bratteli diagram E. Take $\theta \in \mathbb{R}\backslash\mathbb{Q}$, let $c^3_\theta \in Z^2(\mathbb{Z}^3, \mathbb{T})$ be as in* (5.2)*, and let $c = d_* c^3_\theta \in Z^2(\Lambda, \mathbb{T})$. Then there is a rank-2 Bratteli diagram Γ (as described in* [25, Definition 4.1]*) such that $C^*(\Gamma)$ is Morita equivalent to $C^*(\Lambda, c)$.*

Proof. We have seen above that $C^*(\Lambda, c)$ is a simple $A\mathbb{T}$ algebra of real rank zero. We claim that $K_0(C^*(\Lambda, c))$ is a Riesz group in the sense of [7, Section 1]. To see this, observe that it is clearly a countably group satisfying $na \geq 0$ implies $a \geq 0$ for all $a \in K_0(C^*(\Lambda, c))$, Fix finite sets $\{a_i : i \in I\}$ and $\{b_j : j \in J\}$ of elements of G such that $a_i \leq b_j$ for all i, j; we must find c such that $a_i \leq c \leq b_j$ for all i, j. We may assume that the a_i, b_j all belong to some fixed $\oplus_{v \in E^0_n} \frac{1}{w(v)}\mathbb{Z} + \theta\mathbb{Z}$, and since the order on this group is the coordinatewise partial order, it suffices to suppose that they all belong to some fixed $(\frac{1}{w(v)}\mathbb{Z} + \theta\mathbb{Z})\delta_v$; but this is a totally ordered subgroup of $\mathbb{R}$, so we can take $c = \max_i a_i$.

It now follows from [7, Theorem 2.2] that $K_0(C^*(\Lambda, c))$ is a dimension group. We claim that it is simple. Indeed, suppose that J is a nontrivial ideal of $K_0(C^*(\Lambda, c))$. Then each $J_v := J \cap (\frac{1}{w(v)}\mathbb{Z} + \theta\mathbb{Z})\delta_v$ is an ideal in this subgroup, and therefore the whole subgroup since each $\frac{1}{w(v)}\mathbb{Z} + \theta\mathbb{Z}$ is a simple dimension group. Since J is nontrivial, we may fix $v \in E^0$, say $v \in E^0_p$ such that $J_v \neq \emptyset$, and therefore $J_v = (\frac{1}{w(v)}\mathbb{Z} + \theta\mathbb{Z})\delta_v$. Choose $v' \in E^0$, say $v' \in E^0_m$; we just have to show that $J_{v'}$ is nontrivial. Since E is cofinal, there exists n sufficiently large so that $s(v'E^n) \subseteq s(vE^*)$ (see, for example, [23, Proposition A.2]). It follows that the element $1\delta_{v'}$ of $(\frac{1}{w(v')}\mathbb{Z} + \theta\mathbb{Z})\delta_{v'}$ satisfies $1\delta_{v'} = \sum_{\mu \in v'E^n} 1\delta_{s(\mu)}$. Let $N := |v'E^n|$. Then

$$N\delta_v = N\Big(\sum_{\nu \in vE^{m+n-p}} 1\delta_{s(\nu)}\Big) \geq \sum_{u \in E^0_{m+n}, vE^* u \neq \emptyset} N\delta_u \geq \delta_{v'}.$$

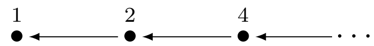
Since $N\delta_v \in J$ and J is an ideal of the Riesz group $K_0(C^*(\Lambda, c))$, it follows that $\delta_{v'} \in J$. Hence $K_0(C^*(\Lambda, c))$ is a simple dimension group.

For any v, we have $(\frac{1}{w(v)}\mathbb{Z} + \theta\mathbb{Z})\delta_v \cong \mathbb{Z}^2$ as a group, so $K_0(C^*(\Lambda, c))$ is not $\mathbb{Z}$. Now the argument of the proof of [25, Theorem 6.2(2)] shows that there is a sequence of proper nonnegative matrices $A'_n \in M_{q_n,q_{n+1}}(\mathbb{N})$ such that $K_0(C^*(\Lambda, c)) = \varinjlim(\mathbb{Z}^{q_n}, A'_n)$. We may now apply [25, Theorem 6.2(2)] with $B_n = A_n = A'_n$ and $T_n = \mathrm{id}_{q_n}$ for all n to see that there is a rank-2 Bratteli diagram Γ such that $C^*(\Gamma)$ is simple and has real rank zero and ordered K-theory is identical to that of $C^*(\Lambda, c)$. So the two are Morita equivalent by Corollary 5.12. □

6. Examples

In this section we present a few illustrative examples of our K-theory calculations from the preceding section.

Example 6.1. Consider the rank-3 Bratteli diagram Λ associated to the singly connected weighted Bratteli diagram E pictured below.

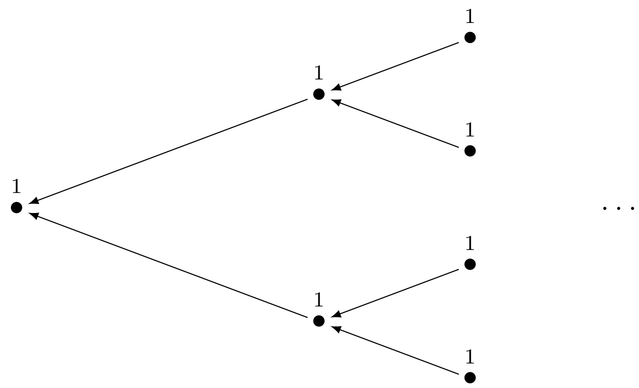


Let v_n be the vertex at level n, and let e_n denote the unique edge with range v_n. Let $\theta \in \mathbb{R} \setminus \mathbb{Q}$, let $c^3_\theta \in Z^2(\mathbb{Z}^3, \mathbb{T})$ be as in (5.2), and let $c = d_* c^3_\theta \in Z^2(\Lambda, \mathbb{T})$; this is (up to cohomology) the unique 2-cocycle extending $c_1 = d_* c_\theta \in Z^2(\Lambda_1, \mathbb{T})$. By Theorem 4.4 the twisted 3-graph C^*-algebra $C^*(\Lambda, c)$ is Morita equivalent to $\varinjlim\,(C^*(\Lambda_{v_n}, c_{v_n}),\ \iota_{e_n})$. Hence

$$K_0(C^*(\Lambda, c)) \cong \bigcup_n \big(\frac{1}{2^n}\mathbb{Z} + \theta\mathbb{Z}\big),$$

with positive cone $(\mathbb{Z}[\frac{1}{2}] + \theta\mathbb{Z}) \cap [0, \infty)$, and $K_1(C^*(\Lambda, c)) \cong K_0(C^{(}\Lambda, c))$ as groups.

Example 6.2. Let $\Lambda = \Lambda_E$ be the rank-3 Bratteli diagram associated to the singly connected weighted Bratteli diagram E given by

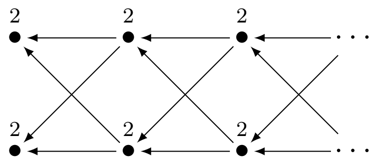


For each $n \geq 1$ let $v_{n,j}$ denote the j'th vertex of E at level n counting from top to bottom, and for each $n \geq 2$ let $e_{n,j}$ denote the unique edge of source $v_{n,j}$. Let $\theta \in \mathbb{R} \setminus \mathbb{Q}$, let $c^3_\theta \in Z^2(\mathbb{Z}^3, \mathbb{T})$ be as in (5.2), and let $c = d_* c^3_\theta \in Z^2(\Lambda, \mathbb{T})$. Then

$$K_1(C^*(\Lambda, c)) \cong K_0(C^*(\Lambda, c)) \cong \left(\bigoplus_{i=1}^{2^{n-1}} (\mathbb{Z} + \theta\mathbb{Z}), a \mapsto (a, a)\right),$$

which is isomorphic to $(\mathbb{Z}+\theta\mathbb{Z})^\infty \subseteq \mathbb{R}^\infty$, with positive cone carried to $(\mathbb{Z}+\theta\mathbb{Z})^\infty \cap [0, \infty)^\infty$.

Example 6.3. Consider the rank-3 Bratteli diagram Λ associated to the singly connected weighted Bratteli diagram E pictured below.



Let $\theta \in \mathbb{R} \setminus \mathbb{Q}$, let $c_\theta^3 \in Z^2(\mathbb{Z}^3, \mathbb{T})$ be as in (5.2), and let $c = d_* c_\theta^3 \in Z^2(\Lambda, \mathbb{T})$. Then

$$K_1(C^*(\Lambda, c)) \cong K_0(C^*(\Lambda, c)) \cong \varinjlim \left((\mathbb{Z} + \theta\mathbb{Z}) \oplus (\mathbb{Z} + \theta\mathbb{Z}), \left(\begin{smallmatrix} \mathrm{id} & \mathrm{id} \\ \mathrm{id} & \mathrm{id} \end{smallmatrix}\right)\right),$$

which is isomorphic to $\mathbb{Z}[\frac{1}{2}] + \mathbb{Z}[\frac{\theta}{2}]$ with positive cone $(\mathbb{Z}[\frac{1}{2}] + \mathbb{Z}[\frac{\theta}{2}]) \cap [0, \infty)$.

7. Rank-3 Bratteli diagrams and traces

In this section we show how to identify traces on twisted C^*-algebras associated to rank-3 Bratteli diagrams.

First we briefly introduce densely defined traces on C^*-algebras, following [27]. (Note that there are other definitions of a trace; see for example [17].) We let A^+ denote the positive cone in a C^*-algebra A, and we extend arithmetic on $[0, \infty]$ so that $0 \times \infty = 0$. A *trace* on a C^*-algebra A is an additive map $\tau : A^+ \to [0, \infty]$ which respects scalar multiplication by non-negative reals and satisfies the *trace property* $\tau(a^*a) = \tau(aa^*)$, $a \in A$. A trace τ is *faithful* if $\tau(a) = 0$ implies $a = 0$. It is *semifinite* if it is finite on a norm dense subset of A^+ i.e, $\overline{\{a \in A^+ : 0 \le \tau(a) < \infty\}} = A^+$. A trace τ is *lower semicontinuous* if $\tau(a) \le \liminf_n \tau(a_n)$ whenever $a_n \to a$ in A^+. We may extend a semifinite trace τ by linearity to a linear functional on a dense subset of A. The domain of definition of a densely defined trace is a two-sided ideal $I_\tau \subset A$.

Following [27, 34] a *graph trace* on a k-graph Λ is a function $g \colon \Lambda^0 \to \mathbb{R}^+$ satisfying the *graph trace property*

$$g(v) = \sum_{\lambda \in v\Lambda^{\le n}} g(s(\lambda)) \quad \text{for all } v \in \Lambda^0 \text{ and } n \in \mathbb{N}^k.$$

A graph trace is *faithful* if it is non-zero on every vertex in Λ.

Lemma 7.1 ([27])**.** *Let Λ be a row-finite locally convex k-graph, and let $c \in Z^2(\Lambda, \mathbb{T})$. For each semifinite trace τ on $C^*(\Lambda, c)$ there is a graph trace g on Λ such that $g(v) = \tau(s_v)$ for all $v \in \Lambda^0$.*

Proof. Fix $v \in \Lambda^0$. Since τ is semifinite we may extend it to the two-sided ideal $I_\tau = \{a : \tau(a) \le \infty\}$. Choose $a \in (I_\tau)_+$ such that $\|s_v - a\| < 1$. Then $\|s_v - s_v a s_v\| < 1$, and so $s_v = b s_v a s_v \in I_\tau$, where b is the inverse of $s_v a s_v$ in $s_v C^*(\Lambda, c) s_v$. In particular $g(v) = \tau(s_v) < \infty$. The graph trace property follows from applying τ to (CK4). □

It turns out, conversely, that each graph trace corresponds to a trace. This however requires a bit more machinery which we now introduce. Recall that each twisted k-graph C^*-algebra $C^*(\Lambda, c)$ carries a *gauge action* γ of $\mathbb{T}^k$ such that $\gamma_z(s_\lambda) = z^{d(\lambda)} s_\lambda$. Averaging against Haar measure over this action gives a faithful conditional expectation $\Phi^\gamma \colon a \mapsto \int_{\mathbb{T}} \gamma_z(a)\, dz$ onto the fixed-point algebra $C^*(\Lambda, c)^\gamma$, which is called the *core*. We have $\Phi^\gamma(s_\mu s_\nu^*) = \delta_{d(\mu), d(\nu)} s_\mu s_\nu^*$, and so $C^*(\Lambda, c)^\gamma = \overline{\operatorname{span}}\{s_\mu s_\nu^* : d(\mu) = d(\nu)\}$. For every finite set $F \subseteq \Lambda$ there is a smallest finite set $F' \subseteq \Lambda$ such that $F \subseteq F'$ and $A_{F'} :=$

$\operatorname{span}\{s_\mu s_\nu^* \in C^*(\Lambda, c)^\gamma : \mu, \nu \in F'\}$ is a finite dimensional C^*-algebra [31, Lemma 3.2]. For two finite sets $F \subseteq G \subseteq \Lambda$, we have $F' \subseteq G'$ so $A_{F'} \subseteq A_{G'}$. So any increasing sequence of finite subsets F_n such that $\bigcup_n F_n = \Lambda$ gives an AF decomposition of $C^*(\Lambda, c)^\gamma$.

The following lemma was proved for $c = 1$ by Pask, Rennie and Sims using the augmented boundary path representation on $\ell^2(\partial\Lambda) \otimes \ell^2(\mathbb{Z}^k)$ (see the first arXiv version of [27]). When $c \neq 1$ it is not clear how to represent $C^*(\Lambda, c)$ on $\ell^2(\partial\Lambda)$, so we proceed in a different way:

Lemma 7.2. *Let Λ be a row-finite locally convex k-graph and let $c \in Z^2(\Lambda, \mathbb{T})$. There is a faithful conditional expectation E of $C^*(\Lambda, c)$ onto $\overline{\operatorname{span}}\{s_\mu s_\mu^*\}$ which satisfies*

$$E(s_\mu s_\nu^*) = \begin{cases} s_\mu s_\mu^* & \text{if } \mu = \nu \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Recall that the linear map $E : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ such that $E(\theta_{i,j}) = \delta_{i,j}\theta_{i,i}$ is a faithful conditional expectation.

Fix a finite set $F \subseteq \Lambda$. Select the smallest finite set $F' \subseteq \Lambda$ such that $F \subseteq F'$ and $A_{F'} = \operatorname{span}\{s_\mu s_\nu^* : \mu, \nu \in F', d(\mu) = d(\nu)\}$ is a finite-dimensional C^*-algebra. There exist integers k_i and an isomorphism $A_{F'} \cong \bigoplus_{i=1}^n M_{k_i}(\mathbb{C})$ which carries $\operatorname{span}\{s_\mu s_\mu^* : \mu \in F'\}$ to $\operatorname{span}\{\theta_{ii}\}$, and carries each $s_\mu s_\nu^*$ with $\mu \neq \nu$ into $\operatorname{span}\{\theta_{ij} : i \neq j\}$ [32, Equation (3.2)]. Hence the map $s_\mu s_\nu^* \mapsto \delta_{\mu,\nu} s_\mu s_\mu^*$, from $A_{F'}$ into its canonical diagonal subalgebra $\operatorname{span}\{s_\mu s_\mu^* : \mu \in F'\}$ is a faithful conditional expectation. Extending this map by continuity to $C^*(\Lambda, c)^\gamma = \overline{\bigcup_{F'} A_{F'}}$ gives a norm-decreasing linear map $\Psi : C^*(\Lambda, c)^\gamma \to \overline{\operatorname{span}}\{s_\mu s_\mu^*\}$ satisfying $\Psi(s_\mu s_\nu^*) = \delta_{\mu,\nu} s_\mu s_\mu^*$. This Ψ is an idempotent of norm one, and is therefore a conditional expectation by [3, Theorem II.6.10.2]. Since Ψ agrees with the usual expectation of the AF-algebra $C^*(\Lambda, c)^\gamma$ onto its canonical diagonal subalgebra, it is faithful. Hence the composition $E := \Psi \circ \Phi^\gamma$ is the desired faithful conditional expectation from $C^*(\Lambda, c)$ onto $\overline{\operatorname{span}}\{s_\mu s_\mu^*\}$. □

Lemma 7.3 (c.f. [27, Proposition 3.10])**.** *Let Λ be a row-finite locally convex k-graph and let $c \in Z^2(\Lambda, \mathbb{T})$. For each faithful graph trace g on Λ there is a faithful, semifinite, lower semicontinuous, gauge invariant trace τ_g on $C^*(\Lambda, c)$ such that $\tau_g(s_\mu s_\nu^*) = \delta_{\mu,\nu} g(s(\mu))$ for all $\mu, \nu \in \Lambda$.*

Proof. Take a finite $F \subseteq \Lambda$ and scalars $\{a_\mu : \mu \in F\}$. Suppose that $\sum_{\mu \in F} a_\mu s_\mu s_\mu^* = 0$. Let $N := \bigvee_{\mu \in F} d(\mu)$. Relation (CK) implies that $\sum_{\mu \in F} \sum_{\alpha \in s(\mu)\Lambda^{\leq N - d(\mu)}} a_\mu s_{\mu\alpha} s_{\mu\alpha}^* = 0$. Let $G := \{\mu\alpha : \mu \in F, \alpha \in s(\mu)\Lambda^{\leq N - d(\mu)}\}$, and for $\lambda \in G$, let $b_\lambda := \sum_{\mu \in F, \lambda = \mu\mu'} a_\mu$. Then

$$0 = \sum_{\mu \in F} \sum_{\alpha \in s(\mu)\Lambda^{\leq N - d(\mu)}} a_\mu s_{\mu\alpha} s_{\mu\alpha}^* = \sum_{\lambda \in G} b_\lambda s_\lambda s_\lambda^*.$$

Since (CK) implies that the $s_\lambda s_\lambda^*$ where $\lambda \in G$ are mutually orthogonal, we deduce that each $b_\lambda = 0$. Now the graph-trace property gives

$$\sum_{\mu \in F} a_\mu g(s(\mu)) = \sum_{\mu \in F} \sum_{\alpha \in s(\mu)\Lambda^{\leq N - d(\mu)}} a_\mu g(s(\alpha)) = \sum_{\lambda \in G} b_\lambda g(s(\lambda)) = 0.$$

So there is a well-defined linear map $\tau_g^0 \colon \operatorname{span}\{s_\mu s_\mu^* : \mu \in \Lambda\} \to \mathbb{R}^+$ such that $\tau_g^0(s_\mu s_\mu^*) = g(s(\mu))$ for all μ. Let $E : C^*(\Lambda, c) \to \overline{\operatorname{span}}\{s_\mu s_\mu^* : \mu \in \Lambda\}$ be the map of Lemma 7.2. Then E restricts to a map from $A_c := \operatorname{span}\{s_\mu s_\nu^* : \mu, \nu \in \Lambda\}$ to $\operatorname{span}\{s_\mu s_\mu^* : \mu \in \Lambda\}$. Define $\tau_g := \tau_g^0 \circ E \colon A_c \to \operatorname{span}\{s_\mu s_\mu^* : \mu \in \Lambda\}$.

We claim that τ_g satisfies the trace condition. For this, it suffices to show that

$$\tau_g(s_\lambda s_\mu^* s_\eta s_\zeta^*) = \tau_g(s_\eta s_\zeta^* s_\lambda s_\mu^*) \quad \text{for all } \lambda, \mu, \eta, \zeta. \tag{7.1}$$

Since $E(s_\mu s_\nu^*) = 0$ unless $d(\mu) = d(\nu)$, both sides of (7.1) are zero unless $d(\lambda) - d(\mu) = d(\zeta) - d(\eta)$. We have

$$\begin{aligned}\tau_g(s_\lambda s_\mu^* s_\eta s_\zeta^*) &= \sum_{(\alpha,\beta)\in\Lambda^{\min}(\mu,\eta)} c(\lambda,\alpha)\overline{c(\mu,\alpha)}c(\eta,\beta)\overline{c(\zeta,\beta)}\tau_g(s_{\lambda\alpha}s_{\zeta\beta}^*)\\ &= \sum_{\substack{(\alpha,\beta)\in\Lambda^{\min}(\mu,\eta)\\ \lambda\alpha=\zeta\beta}} c(\lambda,\alpha)\overline{c(\mu,\alpha)}c(\eta,\beta)\overline{c(\zeta,\beta)}g(s(\alpha)). \qquad (7.2)\end{aligned}$$

Similarly,

$$\tau_g(s_\eta s_\zeta^* s_\lambda s_\mu^*) = \sum_{\substack{(\beta,\alpha)\in\Lambda^{\min}(\zeta,\lambda)\\ \eta\beta=\mu\alpha}} c(\lambda,\alpha)\overline{c(\mu,\alpha)}c(\eta,\beta)\overline{c(\zeta,\beta)}g(s(\beta)) \tag{7.3}$$

The argument of the paragraph following Equation (3.6) of [14] shows that $(\alpha,\beta) \mapsto (\beta,\alpha)$ is a bijection from the indexing set on the right-hand side of (7.2) to that on the right-hand side of (7.3), giving (7.1).

We now follow the proof of Proposition 3.10 of [26], beginning from the second sentence, except that in the final line of the proof, we apply the gauge-invariant uniqueness theorem [32, Theorem 3.15] with $\mathcal{E} = \mathrm{FE}(\Lambda)$ rather than [30, Theorem 4.1]. □

Theorem 7.4. *Let Λ be a row-finite locally convex k-graph and let $c \in Z^2(\Lambda, \mathbb{T})$. The map $g \mapsto \tau_g$ of Lemma 7.3 is a bijection between faithful graph traces on Λ and faithful, semifinite, lower semicontinuous, gauge invariant traces on $C^*(\Lambda, c)$.*

Proof. Combine Lemma 7.1 and Lemma 7.3. □

Remark 7.5. If g is a (not necessarily faithful) graph trace, then the graph-trace condition ensures that $H_g := \{v \in \Lambda^0 : g(v) = 0\}$ is saturated and hereditary in the sense of [30, Setion 5], and so $\Lambda \setminus \Lambda H_g$ is also a locally convex row-finite k-graph [30, Theorem 5.2(b)]. If I_{H_g} is the ideal of $C^*(\Lambda, c)$ generated by $\{s_v : v \in H_g\}$, then [32, Corollary 4.5] shows that $C^*(\Lambda, c)/I_{H_g}$ is canonically isomorphic to $C^*(\Lambda \setminus \Lambda H_g, c|_{\Lambda\setminus\Lambda H_g})$. It is easy to see that g restricts to a faithful graph trace on $\Lambda \setminus \Lambda H_g$, so Lemma 7.3 gives a faithful semifinite lower-semicontinuous gauge-invariant trace on $C^*(\Lambda \setminus \Lambda H_g, c|_{\Lambda\setminus\Lambda H_g})$. Composing this with the canonical homomorphism $\pi_{H_g} \colon C^*(\Lambda, c) \to C^*(\Lambda \setminus \Lambda H_g, c|_{\Lambda\setminus\Lambda H_g})$ gives a semifinite lower-semicontinuous gauge-invariant trace on $C^*(\Lambda, c)$. So Theorem 7.4 remains valid if the word "faithful" is removed throughout.

Definition 7.6. Let (E, Λ, p) be a Bratteli diagram of covering maps between k-graphs. For each $v \in E^0$, let $g_v : \Lambda_v^0 \to \mathbb{R}^+$ be a graph trace. We say that the collection (g_v) of graph traces is *compatible* if

$$g_v(u) = \sum_{e\in vE^1} \sum_{p_e(w)=u} g_{s(e)}(w) \quad \text{for all } v \in E^0 \text{ and } u \in \Lambda_v^0.$$

We will show in Lemma 7.8 that the compatibility requirement is necessary and sufficient to combine the g_v into a graph trace on the $(k+1)$-graph Λ_E associated to the Bratteli diagram E of covering maps.

Lemma 7.7. *A function $g\colon \Lambda^0 \to \mathbb{R}^+$ on the vertices of a locally convex k-graph Λ is a graph trace if and only if for all $v \in \Lambda^0$ and $i \in \{1, \dots, k\}$ with $v\Lambda^{e_i} \neq \emptyset$ we have*

$$g(v) = \sum_{\lambda \in v\Lambda^{e_i}} g(s(\lambda)). \tag{7.4}$$

Proof. It is clear that every graph trace satisfies (7.4). The reverse implication is a straightforward induction along the lines of, for example, the proof of [30, Proposition 3.11]. □

Lemma 7.8. *Let $\Lambda = \Lambda_E$ be the $(k+1)$-graph associated to a Bratteli diagram of covering maps between row finite locally convex k-graphs. Let $g_v\colon \Lambda_v^0 \to \mathbb{R}^+$ be a graph trace for each $v \in E^0$. Define $g\colon \Lambda^0 \to \mathbb{R}^+$ by $g \circ \iota_v = g_v$ for all $v \in E^0$. Then g is a graph trace if and only if (g_v) is compatible.*

Proof. If g is a graph trace, then (7.4) with $i = k+1$ shows that the g_v are compatible. Conversely, if the g_v are compatible, then (7.4) holds for $i \le k$ because each g_v is a graph trace, and for $i = k+1$ by compatibility. So the result follows from Lemma 7.7. □

Lemma 7.9. *Let $\Lambda = \Lambda_E$ be the rank-3 Bratteli diagram associated to a singly connected weighted Bratteli diagram E. Consider the Bratteli diagram F such that $F^0 = E^0$ and $F^1 = \bigsqcup_{e \in E^1} \{e\} \times \mathbb{Z}/(w(s(e))/w(r(e)))\mathbb{Z}$, with $r(e,i) = r(e)$ and $s(e,i) = s(e)$. For each graph trace h on F, there is a graph trace g_h on Λ such that $g_h((v,j)) = h(v)$ for all $v \in F^0$ and $j \in \mathbb{Z}/w(v)\mathbb{Z}$, and the map $h \mapsto g_h$ is bijection between graph traces on F and graph traces on Λ.*

Proof. Given a graph trace h on F, define functions $g_v : \Lambda_v^0 \to [0,\infty)$ by $g_v(v,i) = h(v)$ for all i. Since each $(v,i)\Lambda_v^{e_1} = (v,i)\Lambda_v^{e_1}(v,i+1) = \{(a_v,i)\}$ and $(v,i)\Lambda_v^{e_2} = (v,i)\Lambda_v^{e_2}(v,i+1) = \{(b_v,i)\}$, the g_v are all graph traces by Lemma 7.7. Since h is a graph trace, each $h(v) = \sum_{e \in vF^1} h(s(e))$. So each

$$\begin{aligned} g_v(v,i) = \sum_{e \in vF^1} h(s(e)) &= \sum_{e \in vE^1, j < w(s(e))/w(r(e))} g_{s(e)}\big(s(e), i + jw(r(e))\big) \\ &= \sum_{e \in vE^1, p_e(s(e),j) = (v,i)} g_{s(e)}(s(e),j). \end{aligned}$$

So the g_v are compatible, and there is a graph trace g as claimed.

Conversely, given a graph trace g on Λ, define $h : F^0 \to [0,\infty)$ by $h(v) = g(v,0)$. Since each $(v,i)\Lambda_v^{e_1} = \{(a_v,i)\}$ and $s(a_v,i) = (v,i+1)$, we have $g(v,i) = g(v,i+1)$ for all i, and so $g(v,i) = g(v,j)$ for all $v \in E^0$ and $i,j \in \mathbb{Z}/w(v)\mathbb{Z}$. So each

$$h(v) = g(v,0) = \sum_{\alpha \in (v,0)\Lambda^{e_3}} g(s(\alpha)) = \sum_{e \in vE^1} w(s(e))/w(r(e)) g(s(e),0) = \sum_{f \in vF^1} h(s(f)).$$

So h is a graph trace, and $g = g_h$. □

Corollary 7.10. *Let $\Lambda = \Lambda_E$ be the rank-3 Bratteli diagram associated to a singly connected weighted Bratteli diagram E, and take $\theta \in \mathbb{R}$. Let $c_\theta^3 \in Z^2(\mathbb{Z}^3, \mathbb{T})$ be as in (5.2), and let $c = d_* c_\theta^3 \in Z^2(\Lambda, \mathbb{T})$. Consider the Bratteli diagram F of Lemma 7.9. Let τ be a semifinite lower-semicontinuous trace on $C^*(F)$. There is a gauge-invariant semifinite lower-semicontinuous trace $\tilde{\tau}$ on $C^*(\Lambda, c)$ such that $\tilde{\tau}(p_{(v,i)}) = \tau(p_v)$ for all $v \in F^0$ and $i \in \mathbb{Z}/w(v)\mathbb{Z}$. The map $\tau \mapsto \tilde{\tau}$ is a bijection between semifinite lower-semicontinuous* *traces on $C^*(F)$ and gauge-invariant semifinite lower-semicontinuous traces on $C^*(\Lambda, c)$. If θ is irrational then every semifinite lower-semicontinuous trace on $C^*(\Lambda, c)$ is gauge-invariant.*

Proof. Lemma 7.1 show that each semifinite trace τ on $C^*(F)$ determines a graph trace $h = h_\tau$ on F such that $h_\tau(v) = \tau(p_v)$. Lemma 7.9 shows that there is then a graph trace $g = g_h$ on Λ such that $g(v, i) = h(v) = \tau(p_v)$ for all $v \in F^0$. Now Lemma 7.3 and Remark 7.5 yield a gauge-invariant semifinite lower-semicontinuous trace $\tilde{\tau} = \tau_g$ such that $\tilde{\tau}(p_{(v,i)}) = g(v, i) = h(v) = \tau(p_v)$ as claimed. We have

$$C^*(F) = \overline{\bigcup_n \operatorname{span}\{s_\mu s_\nu^* : s(\mu) = s(\nu) \in E_n^0\}}.$$

Each $\operatorname{span}\{s_\mu s_\nu^* : s(\mu) = s(\nu) \in E_n^0\} \cong \bigoplus_{v \in E_n^0} M_{F^*v}(\mathbb{C})$ via $s_\mu s_\nu^* \mapsto \theta_{\mu,\nu}$; in particular this isomorphism carries each p_v to a minimal projection in the summand $M_{F^*v}(\mathbb{C})$. So each trace on $C^*(F)$ is completely determined by its values on the p_v, and so $\tau \mapsto \tilde{\tau}$ is injective.

If ρ is a gauge-invariant semifinite lower-semicontinuous trace on $C^*(\Lambda, c)$ then Theorem 7.4 and Remark 7.5 shows that $\rho = \tau_g$ where g is the graph trace on Λ such that $g(v, i) = \rho(p_{(v,i)})$. Now Lemma 7.9 shows that $g = g_h$ and $g_h(v, i) = h(v)$ for some graph trace h on $C^*(F)$, and then Theorem 7.4 and Remark 7.5 give a gauge-invariant semifinite lower-semicontinuous trace $\tau = \tau_h$ on $C^*(F)$ such that $\tau(p_v) = h(v)$. Hence $\rho(p_{(v,i)}) = \tilde{\tau}(p_{(v,i)})$. Now $\rho = \tilde{\tau}$ because they are both gauge-invariant traces, and so Theorem 7.4 (and Lemma 7.3) shows that gauge-invariant traces are completely determined by their values on vertex projections.

Suppose that θ is irrational and that τ is a semifinite lower-semicontinuous trace on $C^*(\Lambda, c)$. For $v \in E^0$, let $c_v := c|_{\Lambda_v}$. Then τ restricts to a trace on each $C^*(\Lambda_v, c_v)$. Lemma 5.8 shows that each $C^*(\Lambda_v, c_v)$ is isomorphic to $M_{w(v)}(A_\theta)$. The gauge-invariant trace τ_g on $C^*(\Lambda, c)$ determined by the graph trace $g(v, i) = \tau(p_{(v,i)})$ restricts to a trace on each $C^*(\Lambda_v, c_v)$ such that $\tau_g(s_\mu s_\nu^*) = \delta_{\mu,\nu} g(s(\mu)) = \delta_{\mu,\nu}\tau(p_{s(\mu)})$ for all $\mu, \nu \in \Lambda_v$, and in particular $\|\tau_g|_{C^*(\Lambda_v,c_v)}\| = \sum_{i \in \mathbb{Z}/w(v)\mathbb{Z}} g(v, i) = \|\tau|_{C^*(\Lambda_v,c_v)}\|$. Since there is only one trace on $M_{w(v)}(A_\theta)$ with this norm, it follows that $\tau|_{C^*(\Lambda_v,c_v)} = \tau_g|_{C^*(\Lambda_v,c_v)}$ and in particular

$$\tau(s_\mu s_\nu^*) = \delta_{\mu,\nu}\tau(p_{s(\mu)}) \quad \text{for } \mu, \nu \in \Lambda_v. \tag{7.5}$$

Now suppose that $\alpha, \beta \in \Lambda$ and $s(\alpha) = s(\beta)$. Write $\alpha = \eta\mu$ and $\beta = \zeta\nu$ where $\eta, \zeta \in \Lambda^{\mathbb{N}e_3}$ and $\mu, \nu \in \iota_v(\Lambda_v)$ for some $v \in E^0$. Since $d(\eta)_2 = d(\zeta)_2 = 0$, we have $c(\eta, \mu) = 1 = c(\zeta, \nu)$. This and the trace condition give

$$\tau(s_\alpha s_\beta^*) = \tau(s_\eta s_\mu s_\nu^* s_\zeta^*) = \tau(s_\mu s_\nu^* s_\zeta^* s_\eta).$$

This is zero unless $r(\zeta) = r(\eta)$, so suppose that $r(\zeta) = r(\eta) \in \iota_w(\Lambda_w^0)$. Let $m, n \in \mathbb{N}$ be the elements such that $v \in E_n^0$ and $w \in E_m^0$. Since $s(\zeta)$ and $s(\eta)$ both belong to $\iota_v(\Lambda_v^0)$, we have $d(\zeta) = d(\eta) = (n - m)e_3$, and then $s_\zeta^* s_\eta = \delta_{\zeta,\eta} p_{s(\zeta)}$. So (7.5) gives

$$\tau(s_\alpha s_\beta^*) = \delta_{\zeta,\eta}\tau(s_\mu s_\nu^*) = \delta_{\zeta,\eta}\delta_{\mu,\nu}\tau(p_{s(\mu)}) = \delta_{\alpha,\beta}\tau(p_{s(\alpha)}) = \tau_g(s_\alpha s_\beta^*).$$

So $\tau = \tau_g$, and is gauge invariant as claimed. □

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