2014

Products and powers, and exponentiations

Martin W. Bunder
University of Wollongong, mbunder@uow.edu.au

Publication Details

Bunder, M. W. (2014). Products and powers, and exponentiations. The Fibonacci Quarterly: a journal devoted to the study of integers with special properties, 52 (2), 172-174.

Products and powers, and exponentiations

Keywords
powers, products, exponentiations

Disciplines
Engineering | Science and Technology Studies

Publication Details
Bunder, M. W. (2014). Products and powers, and exponentiations. The Fibonacci Quarterly: a journal devoted to the study of integers with special properties, 52 (2), 172-174.

PRODUCTS AND POWERS, POWERS AND EXPONENTIATIONS, ...

MARTIN W. BUNDER

Abstract. The Horadam recurrence relation $w_{n+1}(a,b;p,q) = pw_n(a,b;p,q) - qw_{n-1}(a,b;p,q)$ (with $w_0 = a$ and $w_1 = b$) has inspired consideration of the recurrence $z_n(a,b;p,q) = z_n^p(a,b;p,q).z_{n-1}^q$ (with $z_0 = a$ and $z_1 = b$). This paper defines a natural sequence of such recurrence relations of which w_n and z_n are the first and second.

1. The functions $w_n(a,b;p,q)$ and $z_n(a,b;p,q)$

The Horadam functions (Horadam [6] p 161) and the functions $z_n(a,b;p,q)$ (Bunder [2] p 279 and Larcombe, Bagdasar and Fennesey [8]) are given by:

Definition 1.1. $w_0(a,b;p,q) = a, \quad w_1(a,b;p,q) = b,$
$w_{n+1}(a,b;p,q) = pw_n(a,b;p,q) - qw_{n-1}(a,b;p,q).$

Definition 1.2. $z_0(a,b;p,q) = a, \quad z_1(a,b;p,q) = b,$
$z_{n+1}(a,b;p,q) = (z_n(a,b;p,q))^p.(z_{n-1}(a,b;p,q))^q.$

$w_n(a,b;p,q)$ will usually be written as w_n and $z_n(a,b;p,q)$ as z_n.

2. A sequence of functions starting with w_n and z_n

The Horadam recurrence of Definition 1.1 involves the sum of two products (i.e. repeated additions) pw_n and $(-q)w_{n-1}$. The recurrence in Definition 1.2 involves the product of two powers (i.e. repeated multiplications) z_n^p and z_{n-1}^q. Taking this to the next level, the recurrence would involve the exponentiation of repeated exponentiations $\left(t_n^{\cdot^{\cdot^{\cdot^{t_n}}}}\right)$ and $\left(t_{n-1}^{\cdot^{\cdot^{\cdot^{t_{n-1}}}}}\right)$, where there are p t_ns and q t_{n-1}s. There are of course two different exponentiations, but we will consider only one.

The first aim of this paper is to generate a natural infinite sequence of such functions $< w_n, z_n, t_n, ... >$ and the second to see whether t_n and later functions can be defined in simple terms or in terms of functions coming earlier in the sequence, just as z_n can be defined in terms of w_n. Bunder [2] and Larcombe, Bagdasar and Fennesey [5] show that:

$$z_n = a^{w_n(1,0;p,-q)} b^{w_n(0,1;p,-q)}.$$

The first aim can be achieved by using the following function due to Ackermann [1]:

Definition 2.1. $\phi(m,n,0) = m+n, \quad \phi(m,0,1) = 0, \quad \phi_(m,0,2) = 1,$
$\phi(m,0,r) = m, \quad$ *for* $r > 2$
$\phi(m,n,r) = \phi(m,\phi(m,n-1,r),r-1), \quad$ *for* $n > 0, r > 0.$

This gives $\phi(m,n,1) = mn$, $\phi(m,n,2) = m^n$, $\phi(m,n,3) = m^{\cdot^{\cdot^{\cdot^{m}}}}$ (n ms).

Ackermann considered such functions to clarify Hilbert's proposed proof of the continuum hypothesis. It is also one of the earliest and simplest examples of a total function that is computable but not primitive recursive (see van Heijenoort [9]). The function $\phi(m,n,3)$, often written as ${}^n m$ was already known to Euler. $\phi(m,n,r)$ is sometimes written as $ack(m,n,r)$, see for example Giesler [5]. Knuth [7] and Conway and Guy [4] have other notations for the ϕ or ack function.

Note that Ackermann's $\phi(m,n,r)$ is related to, but not the same as, what is these days usually called the Ackermann function.

3. A general Horadam-style recurrence

A general Horadam recurrence, motivated by the discussion in Section 1, is given by:

Definition 3.1. $s_{i,0}(a,b;p,q)=a, \quad s_{i,1}(a,b;p,q)=b,$
$s_{i,n+1}(a,b;p,q)=\phi(\phi(s_{i,n}(a,b;p,q),p,i+1),\phi(s_{i,n-1}(a,b;p,q),q,i+1),i).$

We will usually write $s_{i,n}(a,b;p,q)$ as $s_{i,n}$.

Clearly $s_{1,n}=w_n(a,b;p,-q)$, $s_{2,n}=z_n$ and $s_{3,n+1}=\left(s_{3,n}{}^{\cdot^{\cdot^{\cdot^{s_{3,n}}}}}\right)^{\left(s_{3,n-1}{}^{\cdot^{\cdot^{\cdot^{s_{3,n-1}}}}}\right)}$, where there are p $s_{3,n}$s and q $s_{3,n-1}$s.

Unless the meaning of repeated exponentiation can somehow be generalised, this, of course, requires p and q to be positive integers.

(Note that our notation would have been neater, giving $w_n=s_{1,n}$, if we had q for $-q$ on the right hand side of the recurrence in Definition 1.1, as this gives $s_{1,n}=w_n$!)

4. $s_{m,n}$ in simple terms or in terms of $s_{j,n}$ where $j<m$?

$s_{1,n}$ can be expressed as:

If $n\geq 0$, $p^2\neq -4q$, $C=(p+\sqrt{(p^2+4q)})/2$ and $D=(p+\sqrt{(p^2+4q)})/2$,
$s_{1,n}=\left(\dfrac{b-aC}{C-D}\right)C^n+\left(\dfrac{b-aD}{D-C}\right)D^n.$
If $n\geq 0$, $\quad s_{1,n}(a,b,p,-p^2/4)=nb(p/2)^{n-1}-(n-1)a(p/2)^n.$

(See Horadam [6] pp 161,175 and Bunder [3]).

In Section 2, $s_{2,n}(=z_n)$ was given in terms of $w_n(0,1;p,-q)$ and $w_n(1,0;p,-q)$, we also have:

$$s_{1,n}=w_n(a,b;p,-q)=aw_n(1,0;p,-q)+bw_n(0,1;p,-q),$$

so we might expect $s_{3,n}=\left(b^{\cdot^{\cdot^{\cdot^{b}}}}\right)^{\left(a^{\cdot^{\cdot^{\cdot^{a}}}}\right)}$, where there are $w_n(0,1;p,-q)$ b s and $w_n(1,0;p,-q)$ a s. However the examples below show that this is not generally the case. Even in simple cases such as $i=3$ and $p,q<5$, there seems to be no simple expressions for $s_{i,n}$, nor one in terms of $s_{j,n}$ where $j<i$.

Example 4.1. *Let $p=-q=1$ then*
$< s_{1,n} >=< a,b,a+b,a+2b,2a+3b,... >$
$< z_n >=< s_{2,n} >=< a,b,ab,ab^2,a^2b^3,... >$
$s_{3,n+1}=s_{3,n}^{s_{3,n-1}}$ *and* $< s_{3,n} >=< a,b,b^a,b^{ab},b^{ab^{1+a}},b^{ab^{1+a+ab+ab^{1+a}}},... >.$

Example 4.2. *Let* $p = 3, q = -2$ *then*

$< s_{1,n} >=< a, b, 2a + 3b, 6a + 11b, 22a + 39b, ... >$

$< z_n >=< s_{2,n} >=< a, b, a^2b^3, a^6b^{11}, a^{22}b^{39}, ... >$

$$s_{3,n+1} = \left(s_{3,n}^{s_{3,n}^{s_{3,n}}}\right)^{\left({s_{3,n-1}}^{s_{3,n-1}}\right)} \quad \textit{and} < s_{3,n} >=< a, b, b^{b^b.a^a}, \left(b^{b^b.a^a}\right)^{\left(\left(\left(b^{b^b.a^a}\right)^{\left(b^{b^b.a^a}\right)}\right)\right).b^b}, ... >$$

.

5. Summary

A sequence of functions $< s_{1,n}, s_{2,n}, ... >$ has been defined, (with $s_{1,n}$ the Horadam function $w_n(a, b; p, -q)$ and $s_{2,n} = z_n$), each element of which is generated by a Horadam like recurrence relation, with higher order operations than the previous one. The first two of these can be represented in terms of elementary arithmetical functions, z_n can also be written in terms of w_n. Later functions in the sequence, it seems, cannot be represented in terms of such elementary functions except for specific values of n. Perhaps later work, maybe with new notation, can change this situation.

References

[1] Wilhelm Ackermann, *Zum Hilbertischen Aufbau der reellen Zahlen*, Math Annalen, **99** (1928), 118-133.
[2] M.W.Bunder, *Products and powers*, The Fibonacci Quarterly **13**, (1975), 279.
[3] M.W.Bunder, *Horadam functions and powers of irrationals*, The Fibonacci Quarterly **13**, to appear.
[4] J.H.Conway and R.K.Guy *The Book of Numbers* Springer-Verlag, New York 1995.
[5] Daniel Giesler *What lies beyond exponentiation* www.Tetration.org
[6] A.F. Horadam, *Basic Properties of a certain generalised sequence of numbers*, The Fibonacci Quarterly **Oct**, (1965), 161–175.
[7] Donald, E Knuth *Mathematics and Computer Science: Coping with Finiteness* Science 194 (4271) 1235-1242.
[8] Peter L. Larcombe, Ovidiu D. Bagdasar and Eric J. Fennesey, *On a result of Bunder involving Horadam sequences: a proof and generalization*, The Fibonacci Quarterly, to appear.
[9] J. van Heijenoort, *From Frege to Godel*, Harvard, (1967).

MSC2010: 11B39

School of Mathematics and Applied Statistics, University of Wollongong, New South Wales, 2522, Australia

E-mail address: mbunder@uow.edu.au