2014

Index theory for locally compact noncommutative geometries

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Abstract

Spectral triples for nonunital algebras model locally compact spaces in noncommutative geometry. In the present text, we prove the local index formula for spectral triples over nonunital algebras, without the assumption of local units in our algebra.

Publication Details

Carey, A. L., Gayral, V., Rennie, A. C. & Sukochev, F. A. (2014). Index theory for locally compact noncommutative geometries. Memoirs of the American Mathematical Society, 231 (1085), 1-131.

Index Theory for Locally Compact Noncommutative Geometries

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Contents

Abstract

Spectral triples for nonunital algebras model locally compact spaces in noncommutative geometry. In the present text, we prove the local index formula for spectral triples over nonunital algebras, without the assumption of local units in our algebra. This formula has been successfully used to calculate index pairings in numerous noncommutative examples. The absence of any other effective method of investigating index problems in geometries that are genuinely noncommutative, particularly in the nonunital situation, was a primary motivation for this study and we illustrate this point with two examples in the text.

In order to understand what is new in our approach in the commutative setting we prove an analogue of the Gromov-Lawson relative index formula (for Dirac type operators) for even dimensional manifolds with bounded geometry, without invoking compact supports. For odd dimensional manifolds our index formula appears to be completely new. As we prove our local index formula in the framework of semifinite noncommutative geometry we are also able to prove, for manifolds of bounded geometry, a version of Atiyah's L^2-index Theorem for covering spaces. We also explain how to interpret the McKean-Singer formula in the nonunital case.

To prove the local index formula, we develop an integration theory compatible with a refinement of the existing pseudodifferential calculus for spectral triples. We also clarify some aspects of index theory for nonunital algebras.

Received by the editor July 5, 2011, and, in revised form, October 29, 2012.

2010 *Mathematics Subject Classification.* Primary 46H30, 46L51, 46L80, 46L87, 19K35, 19K56, 58J05, 58J20, 58J30, 58J32, 58J42.

Key words and phrases. local index formula, nonunital, spectral triple, Fredholm module, Kasparov product.

Introduction

Our objective in writing this memoir is to establish a unified framework to deal with index theory on locally compact spaces, both commutative and noncommutative. In the commutative situation this entails index theory on noncompact manifolds where Dirac-type operators, for example, typically have noncompact resolvent, are not Fredholm, and so do not have a well-defined index. In initiating this study we were also interested in understanding previous approaches to this problem such as those of Gromov-Lawson [**29**] and Roe [**51**] from a new viewpoint: that of noncommutative geometry. In this latter setting the main tool, the Connes-Moscovici local index formula, is not adapted to nonunital examples. Thus our primary objective here is to extend that theorem to this broader context.

Index theory provided one of the main motivations for noncommutative geometry. In [**20**, **21**] it is explained how to express index pairings between the K-theory and K-homology of noncommutative algebras using Connes' Chern character formula. In examples this formula can be difficult to compute. A more tractable analytic formula is established by Connes and Moscovici in [**23**] using a representative of the Chern character that arises from unbounded Kasparov modules or 'spectral triples' as they have come to be known. Their resulting 'local index formula' is an analytic cohomological expression for index pairings that has been exploited by many authors in calculations in fully noncommutative settings.

In previous work [**15**–**17**] some of the present authors found a new proof of the formula that applied for unital spectral triples in semifinite von Neumann algebras. However for some time the understanding of the Connes-Moscovici formula in nonunital situations has remained unsatisfactory.

The main result of this article is a residue formula of Connes-Moscovici type for calculating the index pairing between the K-homology of nonunital algebras and their K-theory. This latter view of index theory, as generalised by Kasparov's bivariant KK functor, is central to our approach and we follow the general philosophy enunciated by Higson and Roe, [**33**]. One of our main advances is to avoid ad hoc assumptions on our algebras (such as the existence of local units, see below).

To illustrate our main result in practice we present two examples in Chapter 5. Elsewhere we will explain how a version of the example of nonunital Toeplitz theory in [**46**] can be derived from our local index formula. To understand what is new about our theorem in the commutative case we apply our residue formula to manifolds of bounded geometry, obtaining a cohomological formula of Atiyah-Singer type for the index pairing. We also prove an L^2-index theorem for coverings of such manifolds.

We now explain in some detail these and our other results.

The noncommutative results. The index theorems we prove rely on a general nonunital noncommutative integration theory and the index theory developed in detail in Chapters 1 and 2.

Chapter 1 presents an integration theory for weights which is compatible with Connes and Moscovici's approach to the pseudodifferential calculus for spectral triples. This integration theory is the key technical innovation, and allows us to treat the unital and nonunital cases on the same footing.

An important feature of our approach is that we can eliminate the need to assume the existence of 'local units' which mimic the notion of compact support, [**27**, **49**, **50**]. The difficulty with the local unit approach is that there are no general results guaranteeing their existence. Instead we identify subalgebras of integrable and square integrable elements of our algebra, without the need to control 'supports'.

In Chapter 2 we introduce a triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ where $\mathcal{H}$ is a Hilbert space, $\mathcal{A}$ is a (nonunital) $*$-algebra of operators represented in a semifinite von Neumann subalgebra of $\mathcal{B}(\mathcal{H})$, and $\mathcal{D}$ is a self-adjoint unbounded operator on $\mathcal{H}$ whose resolvent need not be compact, not even in the sense of semifinite von Neumann algebras. Instead we ask that the product $a(1 + \mathcal{D}^2)^{-1/2}$ is compact, and it is the need to control this product that produces much of the technical difficulty.

We remark that there are good cohomological reasons for taking the effort to prove our results in the setting of semifinite noncommutative geometry, and that these arguments are explained in [**24**]. In particular, [**24**, Théorème 15] identifies a class of cyclic cocycles on a given algebra which have a natural representation as Chern characters, provided one allows semifinite Fredholm modules.

We refer to the case when $\mathcal{D}$ does not have compact resolvent as the 'nonunital case', and justify this terminology in Lemma 2.2. Instead of requiring that $\mathcal{D}$ be Fredholm we show that a spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$, in the sense of Chapter 2, defines an associated semifinite Fredholm module and a KK-class for $\mathcal{A}$.

This is an important point. It is essential in the nonunital version of the theory to have an appropriate definition of the index which we are computing. Since the operator $\mathcal{D}$ of a general spectral triple need not be Fredholm, this is accomplished by following [**35**] to produce a KK-class. Then the index pairing can be defined via the Kasparov product.

The role of the additional smoothness and summability assumptions on the spectral triple is to produce the local index formula for computing the index pairing. Our smoothness and summability conditions are defined using the smooth version of the integration theory in Chapter 1. This approach is justified by Propositions 2.16 and 2.17, which compare our definition with a more standard definition of finite summability.

Having identified workable definitions of smoothness and summability, the main technical obstacle we have to overcome in Chapter 2 is to find a suitable Fréchet completion of $\mathcal{A}$ stable under the holomorphic functional calculus. The integration theory of Chapter 1 provides such an algebra, and in the unital case it reduces to previous solutions of this problem, [**49**, Lemma 16][1].

In Chapter 3 we establish our local index formula in the sense of Connes-Moscovici. The underlying idea here is that Connes' Chern character, which defines

[1]Despite being about nonunital spectral triples, [**49**, Lemma 16] produces a Fréchet completion which only takes smoothness, not integrability, into account.

an element of the cyclic cohomology of $\mathcal{A}$, computes the index pairing defined by a Fredholm module. Any cocycle in the same cohomology class as the Chern character will therefore also compute the index pairing. In this memoir we define several cocycles that represent the Chern character and which are expressed in terms of the unbounded operator $\mathcal{D}$. These cocycles generalise those found in [**15**–**17**] (where semifinite versions of the local index formula were first proved) to the nonunital case. We have to prove that these additional cocycles, including the residue cocycle, are in the class of the Chern character in the (b, B)-complex.

Our main result (stated in Theorem 3.33 of Chapter 3) is then an expression for the index pairing using a nonunital version of the semifinite local index formula of [**15**,**16**], which is in turn a generalisation to the setting of semifinite von Neumann algebras of the original Connes-Moscovici [**25**] formula. Our noncommutative index formula is given by a sum of residues of zeta functions and is easily recognisable as a direct generalisation of the unital formulas of [**15**, **16**, **25**]. We emphasise that even for the standard $\mathcal{B}(\mathcal{H})$ case our local index formula is new.

One of the main difficulties that we have to overcome is that while there is a well understood theory of Fredholm (or Kasparov) modules for nonunital algebras, the 'right framework' for working with unbounded representatives of these K-homology classes has proved elusive. We believe that we have found the appropriate formalism and the resulting residue index formula provides evidence that the approach to spectral triples over nonunital algebras initiated in [**10**] is fundamentally sound and leads to interesting applications. Related ideas on the K-homology point of view for relative index theorems are to be found in [**52**], [**9**] and [**19**], and further references in these texts.

We also discuss some fully noncommutative applications in Chapter 5, including the type I spectral triple of the Moyal plane constructed in [**27**] and semifinite spectral triples arising from torus actions on C^*-algebras, but leave other applications, such as those to the results in [**44**], [**46**] and [**60**], to elsewhere.

To explain how we arrived at the technical framework described here, consider the simplest possible classical case, where $\mathcal{H} = L^2(\mathbb{R})$, $\mathcal{D} = \frac{d}{idx}$ and $\mathcal{A}$ is a certain $*$-subalgebra of the algebra of smooth functions on $\mathbb{R}$. Let $P = \chi_{[0,\infty)}(\mathcal{D})$ be the projection defined using the functional calculus and the characteristic function of the half-line and let u be a unitary in $\mathcal{A}$ such that $u - 1$ converges to zero at $\pm\infty$ 'sufficiently rapidly'. Then the classical Gohberg-Krein theory gives a formula for the index of the Fredholm operator PM_uP where M_u is the operator of multiplication by u on $L^2(\mathbb{R})$. In proving this theorem for general symbols u, one confronts the classical question (studied in depth in [**56**]) of when an operator of the form $(M_u - 1)(1 + \mathcal{D}^2)^{-s/2}$, $s > 0$, is trace class. In the general noncommutative setting of this article, this question and generalisations must still be confronted and this is done in Chapter 1.

The results for manifolds. In the case of closed manifolds, the local index formula in noncommutative geometry (due to Connes-Moscovici [**25**]) can serve as a starting point to derive the Atiyah-Singer index theorem for Dirac type operators. This proceeds by a Getzler type argument enunciated in this setting by Ponge, [**47**], though similar arguments have been used previously with the JLO cocycle as a starting point in [**7**, **23**]. While there is already a version of this Connes-Moscovici formula that applies in the noncompact case [**50**], it relies heavily on the use of compact support assumptions.

For the application to noncompact manifolds M, we find that our noncommutative index theorem dictates that the appropriate algebra $\mathcal{A}$ consists of smooth functions which, together with all their derivatives, lie in $L^1(M)$. We show how to construct K-homology classes for this algebra from the Dirac operator on the spinor bundle over M. This K-homology viewpoint is related to Roe's approach [**52**] and to the relative index theory of [**29**].

Then the results, for Dirac operators coupled to connections on sections of bundles over noncompact manifolds of bounded geometry, essentially follow as corollaries of the work of Ponge [**47**]. The theorems we obtain for even dimensional manifolds are not comparable with those in [**51**], but are closely related to the viewpoint of Gromov-Lawson [**29**]. For odd dimensional manifolds we obtain an index theorem for generalised Toeplitz operators that appears to be new, although one can see an analogy with the results of Hörmander [**34**, section 19.3].

We now digress to give more detail on how, for noncompact even dimensional spin manifolds M, our local index formula implies a result analogous to the Gromov-Lawson relative index theorem [**29**]. What we compute is an index pairing of K-homology classes for the algebra $\mathcal{A}$ of smooth functions which, along with their derivatives, all lie in $L^1(M)$, with differences of classes $[E]-[E']$ in the K-theory of $\mathcal{A}$. We verify that the Dirac operator on a spin manifold of bounded geometry satisfies the hypotheses needed to use our residue cocycle formula so that we obtain a local index formula of the form

$$\langle [E]-[E'],[\mathcal{D}]\rangle = (const)\int \widehat{A}(M)(\mathrm{Ch}(E)-\mathrm{Ch}(E')), \tag{0.1}$$

where $\mathrm{Ch}(E)$ and $\mathrm{Ch}(E')$ are the Chern classes of vector bundles E and E' over M. We emphasise that in our approach, the connections that lead to the curvature terms in $\mathrm{Ch}(E)$ and $\mathrm{Ch}(E')$, do not have to coincide outside a compact set as in [**29**]. Instead they satisfy constraints that make the difference of curvature terms integrable over M.

We reiterate that, for our notion of spectral triple, the operator $\mathcal{D}$ need not be Fredholm and that the choice of the algebra $\mathcal{A}$ is dictated by the noncommutative theory developed in Chapter 2. In that chapter we explain the minimal assumptions on the pair $(\mathcal{D},\mathcal{A})$ such that we can define a Kasparov module and so a KK-class. The further assumptions required for the local index formula are specified, almost uniquely, by the noncommutative integration theory developed in Chapter 1. We verify (in Chapter 4) what these assumptions mean for the commutative algebra $\mathcal{A}$ of functions on a manifold and Dirac-type operator $\mathcal{D}$, in the case of a noncompact manifold of bounded geometry, and prove that in this case we do indeed obtain a spectral triple in the sense of our general definition.

In the odd dimensional case, for manifolds of bounded geometry, we obtain an index formula that is apparently new, although it is of APS-type. The residues in the noncommutative formula are again calculable by the techniques employed by [**47**] in the compact case. This results in a formula for the pairing of the Chern character of a unitary u in a matrix algebra over $\mathcal{A}$, representing an odd K-theory class, with the K-homology class of a Dirac-type operator $\mathcal{D}$ of the form

$$\langle [u],[\mathcal{D}]\rangle = (const)\int \hat{A}(M)\mathrm{Ch}(u). \tag{0.2}$$

We emphasise that the assumptions on the algebra $\mathcal{A}$ of functions on M are such that this integral exists but they do not require compact support conditions.

We were also motivated to consider Atiyah's L^2-index Theorem in this setting. Because we prove our index formula in the general framework of operators affiliated to semifinite von Neumann algebras we are able, with some additional effort, to obtain at the same time a version of the L^2-index Theorem of Atiyah for Dirac type operators on the universal cover of M (whether M is closed or not). We are able to reduce our proof in this L^2-setting to known results about the local asymptotics at small time of heat kernels on covering spaces. The key point here is that our residue cocycle formula gives a uniform approach to all of these 'classical' index theorems.

Summary of the exposition. Chapter 1 begins by introducing the integration theory we employ, which is a refinement of the ideas introduced in [**10**]. Then we examine the interaction of our integration theory with various notions of smoothness for spectral triples. In particular, we follow Higson, [**32**], and [**15**] in extending the Connes-Moscovici pseudodifferential calculus to the nonunital setting. Finally we prove some trace estimates that play a key role in the subsequent technical parts of the discussion. All these generalisations are required for the proof of our main result in Chapter 3.

Chapter 2 explains how our definition of semifinite spectral triple results in an index pairing from Kasparov's point of view. In other words, while our spectral triple does not a priori involve (possibly unbounded) Fredholm operators, there is an associated index problem for bounded Fredholm operators in the setting of Kasparov's KK-theory. We then show that by modifying our original spectral triple we may obtain an index problem for unbounded Fredholm operators without changing the Kasparov class in the bounded picture. This modification of our unbounded spectral triple proves to be essential, in two ways, for us to obtain our residue formula in Chapter 3.

The method we use in Chapter 3 to prove the existence of a formula of Connes-Moscovici type for the index pairing of our K-homology class with the K-theory of the nonunital algebra $\mathcal{A}$ is a modification of the argument in [**17**]. This argument is in turn closely related to the approach of Higson [**32**] to the Connes-Moscovici formula.

The idea is to start with the resolvent cocycle of [**15–17**] and show that it is well defined in the nonunital setting. We then show that there is an extension of the results in [**17**] that gives a homotopy of the resolvent cocycle to the Chern character for the Fredholm module associated to the spectral triple. The residue cocycle can then be derived from the resolvent cocycle in the nonunital case by much the same argument as in [**15, 16**].

In order to avoid cluttering our exposition with proofs of nonunital modifications of the estimates of these earlier papers, we relegate much detail to the Appendix. Modulo these technicalities we are able to show, essentially as in [**17**], that the residue cocycle and the resolvent cocycle are index cocycles in the class of the Chern character. Then Theorem 3.33 in Chapter 3 is the main result of this memoir. It gives a residue formula for the numerical index defined in Chapter 2 for spectral triples.

We conclude Chapter 3 with a nonunital McKean-Singer formula and an example showing that the integrability hypotheses can be weakened still further, though we do not pursue the issue of finding the weakest conditions for our local index formula to hold in this text.

The applications to the index theory for Dirac-type operators on manifolds of bounded geometry are contained in Chapter 4. Also in Chapter 4 is a version of the Atiyah L^2-index Theorem that applies to covering spaces of noncompact manifolds of bounded geometry. In Chapter 5 we make a start on noncommutative examples, looking at torus actions on C^*-algebras and at the Moyal plane. Any further treatment of noncommutative examples would add considerably to the length of this article, and is best left for another place.

Acknowledgements. This research was supported by the Australian Research Council, the Max Planck Institute for Mathematics (Bonn) and the Banff International Research Station. A. Carey also thanks the Alexander von Humboldt Stiftung and colleagues in the University of Münster and V. Gayral also thanks the CNRS and the University of Metz. We would like to thank our colleagues John Phillips and Magda Georgescu for discussions on nonunital spectral flow. Special thanks are given to Steven Lord, Roger Senior, Anna Tomskova and Dmitriy Zanin for careful readings of this manuscript at various stages. We also thank Emmanuel Pedon for discussions on the Kato inequality, Raimar Wulkenhaar for discussions on index computations for the Moyal plane, and Gilles Carron, Thierry Coulhon, Batu Güneysu and Yuri Korduykov for discussions related to heat-kernels on noncompact manifolds. Finally, it is a pleasure to thank the referees: their exceptional efforts have greatly improved this work.

CHAPTER 1

Pseudodifferential Calculus and Summability

In this Chapter we introduce our chief technical innovation on which most of our results rely. It consists of an L^1-type summability theory for weights adapted to both the nonunital and noncommutative settings.

It has become apparent to us while writing, that the integration theory presented here is closely related to Haagerup's noncommutative L^p-spaces for weights, at least for $p = 1, 2$. Despite this, it is sufficiently different to require a self-contained discussion.

It is an essential and important feature in all that follows that our approach comes essentially from an L^2-theory: we are forced to employ weights, and a direct L^1-approach is technically unsatisfactory for weights. This is because given a weight φ on a von Neumann algebra, the map $T \mapsto \varphi(|T|)$ is not subadditive in general.

Throughout this chapter, $\mathcal{H}$ denotes a separable Hilbert space, $\mathcal{N} \subset \mathcal{B}(\mathcal{H})$ is a semifinite von Neumann algebra, $\mathcal{D} : \operatorname{dom} \mathcal{D} \to \mathcal{H}$ is a self-adjoint operator affiliated to $\mathcal{N}$, and τ is a faithful, normal, semifinite trace on $\mathcal{N}$. Our integration theory will also be parameterised by a real number $p \geq 1$, which will play the role of a dimension.

Different parts of the integration and pseudodifferential theory which we introduce rely on different parts of the above data. The pseudodifferential calculus can be formulated for any unbounded self-adjoint operator $\mathcal{D}$ on a Hilbert space $\mathcal{H}$. This point of view is implicit in Higson's abstract differential algebras, [**32**], and was made more explicit in [**15**].

The definition of summability we employ depends on all the data above, namely $\mathcal{D}$, the pair $(\mathcal{N}, \tau)$ and the number $p \geq 1$. We show in Section 1.1 how the pseudodifferential calculus is compatible with our definition of summability for spectral triples, and this will dictate our generalisation of a finitely summable spectral triple to the nonunital case in Chapter 2.

The proof of the local index formula that we use in the nonunital setting requires some estimates on trace norms that are different from those used in the unital case. These are found in Section 1.5. To prepare for these estimates, we also need some refinements of the pseudodifferential calculus introduced by Connes and Moscovici for unital spectral triples in [**22, 25**].

1.1. Square-summability from weight domains

In this Section we show how an unbounded self-adjoint operator affiliated to a semifinite von Neumann algebra provides the foundation of an integration theory suitable for discussing finite summability for spectral triples.

Throughout this Section, we let $\mathcal{D}$ be a self-adjoint operator affiliated to a semifinite von Neumann algebra $\mathcal{N}$ with faithful normal semifinite trace τ, and let $p \geq 1$ be a real number.

DEFINITION 1.1. For any $s > 0$, we define the weight φ_s on $\mathcal{N}$ by

$$T \in \mathcal{N}_+ \mapsto \varphi_s(T) := \tau\big((1+\mathcal{D}^2)^{-s/4}T(1+\mathcal{D}^2)^{-s/4}\big) \in [0,+\infty].$$

As usual, we set

$$\mathrm{dom}(\varphi_s) := \mathrm{span}\{\mathrm{dom}(\varphi_s)_+\} = \mathrm{span}\big\{\big(\mathrm{dom}(\varphi_s)^{1/2}\big)^*\mathrm{dom}(\varphi_s)^{1/2}\big\} \subset \mathcal{N},$$

where

$$\mathrm{dom}(\varphi_s)_+ := \{T \in \mathcal{N}_+ : \varphi_s(T) < \infty\},$$
$$\mathrm{dom}(\varphi_s)^{1/2} := \{T \in \mathcal{N} : T^*T \in \mathrm{dom}(\varphi_s)_+\}.$$

In the following, $\mathrm{dom}(\varphi_s)_+$ is called the positive domain and $\mathrm{dom}(\varphi_s)^{1/2}$ the half domain.

LEMMA 1.2. *The weights φ_s, $s > 0$, are faithful normal and semifinite, with modular group given by*

$$\mathcal{N} \ni T \mapsto (1+\mathcal{D}^2)^{-is/2}T(1+\mathcal{D}^2)^{is/2}.$$

PROOF. Normality of φ_s follows directly from the normality of τ. To prove faithfulness of φ_s, using faithfulness of τ, we also need the fact that the bounded operator $(1+\mathcal{D}^2)^{-s/4}$ is injective. Let $S \in \mathrm{dom}(\varphi_s)^{1/2}$ and $T := S^*S \in \mathrm{dom}(\varphi_s)_+$ with $\varphi_s(T) = 0$. From the trace property, we obtain $\varphi_s(T) = \tau(S(1+\mathcal{D}^2)^{-s/2}S^*)$, so by the faithfulness of τ, we obtain $0 = S(1+\mathcal{D}^2)^{-s/2}S^* = |(1+\mathcal{D}^2)^{-s/4}S^*|^2$, so $(1+\mathcal{D}^2)^{-s/4}S^* = 0$, which by injectivity implies $S^* = 0$ and thus $T = 0$. Regarding semifiniteness of φ_s, one uses semifiniteness of τ to obtain that for any $T \in \mathcal{N}_+$, there exists $S \in \mathcal{N}_+$ of finite trace, with $S \leq (1+\mathcal{D}^2)^{-s/4}T(1+\mathcal{D}^2)^{-s/4}$. Thus $S' := (1+\mathcal{D}^2)^{s/4}S(1+\mathcal{D}^2)^{s/4} \leq T$ is non-negative, bounded and belongs to $\mathrm{dom}(\varphi_s)_+$, as needed. The form of the modular group follows from the definition of the modular group of a weight. □

Domains of weights, and, a fortiori, intersections of domains of weights, are $*$-subalgebras of $\mathcal{N}$. However, $\mathrm{dom}(\varphi_s)^{1/2}$ is not a $*$-algebra but only a left ideal in $\mathcal{N}$. To obtain a $*$-algebra structure from the latter, we need to force the $*$-invariance. Since φ_s is faithful for each $s > 0$, the inclusion of $\mathrm{dom}(\varphi_s)^{1/2}\bigcap(\mathrm{dom}(\varphi_s)^{1/2})^*$ in its Hilbert space completion (for the inner product coming from φ_s) is injective. Hence by [**57**, Theorem 2.6], $\mathrm{dom}(\varphi_s)^{1/2}\bigcap(\mathrm{dom}(\varphi_s)^{1/2})^*$ is a full left Hilbert algebra. Thus we may define a $*$-subalgebra of $\mathcal{N}$ for each $p \geq 1$.

DEFINITION 1.3. Let $\mathcal{D}$ be a self-adjoint operator affiliated to a semifinite von Neumann algebra $\mathcal{N}$ with faithful normal semifinite trace τ. Then for each $p \geq 1$ we define

$$\mathcal{B}_2(\mathcal{D},p) := \bigcap_{s>p}\Big(\mathrm{dom}(\varphi_s)^{1/2}\bigcap(\mathrm{dom}(\varphi_s)^{1/2})^*\Big).$$

The norms

$$\mathcal{Q}_n(T) := \big(\|T\|^2 + \varphi_{p+1/n}(|T|^2) + \varphi_{p+1/n}(|T^*|^2)\big)^{1/2}, \quad n \in \mathbb{N}, \tag{1.1}$$

take finite values on $\mathcal{B}_2(\mathcal{D},p)$ and provide a topology on $\mathcal{B}_2(\mathcal{D},p)$ stronger than the norm topology. Unless mentioned otherwise we will always suppose that $\mathcal{B}_2(\mathcal{D},p)$ has the topology defined by these norms.

Notation. Given a semifinite von Neumann algebra $\mathcal{N}$ with faithful normal semifinite trace τ, we let $\tilde{\mathcal{L}}^p(\mathcal{N},\tau)$, $1 \le p < \infty$, denote the set of τ-measurable operators T affiliated to $\mathcal{N}$ with $\tau(|T|^p) < \infty$. We do not often use this notion of p-integrable elements, preferring to use the bounded analogue, $\mathcal{L}^p(\mathcal{N},\tau) := \tilde{\mathcal{L}}^p(\mathcal{N},\tau) \cap \mathcal{N}$, normed with $T \mapsto \tau(|T|^p)^{1/p} + \|T\|$.

Remarks. (1) If $(1+\mathcal{D}^2)^{-s/2} \in \mathcal{L}^1(\mathcal{N},\tau)$ for all $\Re(s) > p \ge 1$, then $\mathcal{B}_2(\mathcal{D},p) = \mathcal{N}$, since then the weights φ_s, $s > p$, are bounded and the norms $\mathcal{Q}_n$ are all equivalent to the operator norm.

(2) The triangle inequality for $\mathcal{Q}_n$ follows from the Cauchy-Schwarz inequality applied to the inner product $\langle T,S\rangle_n = \varphi_{p+1/n}(T^*S)$, along with the equality $\mathcal{Q}_n(T)^2 = \|T\|^2 + \langle T,T\rangle_n + \langle T^*,T^*\rangle_n$. In concrete terms, an element $T \in \mathcal{N}$ belongs to $\mathcal{B}_2(\mathcal{D},p)$ if and only if for all $s > p$, both $T(1+\mathcal{D}^2)^{-s/4}$ and $T^*(1+\mathcal{D}^2)^{-s/4}$ belong to $\mathcal{L}^2(\mathcal{N},\tau)$, the ideal of τ-Hilbert-Schmidt operators.

(3) The norms $\mathcal{Q}_n$ are increasing, in the sense that for $n \le m$ we have $\mathcal{Q}_n \le \mathcal{Q}_m$. We leave this as an exercise, but observe that this requires the cyclicity of the trace. The following result of Brown and Kosaki gives the strongest statement on this cyclicity. By the preceding Remark (2), we do not need the full power of this result here, but record it for future use.

Proposition 1.4. *[**8**, Theorem 17] Let τ be a faithful normal semifinite trace on a von Neumann algebra $\mathcal{N}$, and let A, B be τ-measurable operators affiliated to $\mathcal{N}$. If AB, $BA \in \tilde{\mathcal{L}}^1(\mathcal{N},\tau)$ then $\tau(AB) = \tau(BA)$.*

Another important result that we will frequently use comes from Bikchentaev's work.

Proposition 1.5. *[**6**, Theorem 3] Let $\mathcal{N}$ be a semifinite von Neumann algebra with faithful normal semfinite trace τ. If A, $B \in \mathcal{N}$ satisfy $A \ge 0$, $B \ge 0$, and are such that AB is trace class, then $B^{1/2}AB^{1/2}$ and $A^{1/2}BA^{1/2}$ are also trace class, with $\tau(AB) = \tau(B^{1/2}AB^{1/2}) = \tau(A^{1/2}BA^{1/2})$.*

Next we show that the topological algebra $\mathcal{B}_2(\mathcal{D},p)$ is complete and thus is a Fréchet algebra. The completeness argument relies on the Fatou property for the trace τ, [**26**].

Proposition 1.6. *The $*$-algebra $\mathcal{B}_2(\mathcal{D},p) \subset \mathcal{N}$ is a Fréchet algebra.*

Proof. Showing that $\mathcal{B}_2(\mathcal{D},p)$ is a $*$-algebra is routine with the aid of the following argument. For T, $S \in \mathcal{B}_2(\mathcal{D},p)$, the operator inequality $S^*T^*TS \le \|T^*T\|\, S^*S$ shows that

$$\varphi_{p+1/n}(|TS|^2) = \varphi_{p+1/n}(S^*T^*TS) \le \|T\|^2 \varphi_{p+1/n}(|S|^2),$$

and, therefore, $\mathcal{Q}_n(TS) \le \mathcal{Q}_n(T)\,\mathcal{Q}_n(S)$.

For the completeness, let $(T_k)_{k\ge1}$ be a Cauchy sequence in $\mathcal{B}_2(\mathcal{D},p)$. Then $(T_k)_{k\ge1}$ converges in norm, and so there exists $T \in \mathcal{N}$ such that $T_k \to T$ in $\mathcal{N}$. For each norm $\mathcal{Q}_n$ we have $|\,\mathcal{Q}_n(T_k) - \mathcal{Q}_n(T_l)\,| \le \mathcal{Q}_n(T_k - T_l)$, so we see that the numerical sequence $(\mathcal{Q}_n(T_k))_{k\ge1}$ possesses a limit. Now since

$$(1+\mathcal{D}^2)^{-p/4-1/4n}T_k^*T_k(1+\mathcal{D}^2)^{-p/4-1/4n} \to (1+\mathcal{D}^2)^{-p/4-1/4n}T^*T(1+\mathcal{D}^2)^{-p/4-1/4n},$$

in norm, it also converges in measure, and so we may apply the Fatou Lemma, [**26**, Theorem 3.5 (i)], to deduce that

$$\tau\big((1+\mathcal{D}^2)^{-p/4-1/4n}T^*T(1+\mathcal{D}^2)^{-p/4-1/4n}\big) \\ \leq \liminf_{k\to\infty}\tau\big((1+\mathcal{D}^2)^{-p/4-1/4n}T_k^*T_k(1+\mathcal{D}^2)^{-p/4-1/4n}\big).$$

Since the same conclusion holds for TT^* in place of T^*T, we see that

$$\mathcal{Q}_n(T) \leq \liminf_{k\to\infty}\mathcal{Q}_n(T_k) = \lim_{k\to\infty}\mathcal{Q}_n(T_k) < \infty,$$

and so $T \in \mathcal{B}_2(\mathcal{D}, p)$. Finally, fix $\varepsilon > 0$ and $n \geq 1$. Now choose N large enough so that $\mathcal{Q}_n(T_k - T_l) \leq \varepsilon$ for all k, $l > N$. Applying the Fatou Lemma to the sequence $(T_k)_{k\geq 1}$, we have $\mathcal{Q}_n(T - T_l) \leq \liminf_{k\to\infty}\mathcal{Q}_n(T_k - T_l) \leq \varepsilon$. Hence $T_k \to T$ in the topology of $\mathcal{B}_2(\mathcal{D}, p)$. □

We now give some easy but useful stability properties of the algebras $\mathcal{B}_2(\mathcal{D}, p)$.

LEMMA 1.7. *Let $T \in \mathcal{B}_2(\mathcal{D}, p)$, $S \in \mathcal{N}$ and let $f \in L^\infty(\mathbb{R})$.*
(1) The operators $Tf(\mathcal{D})$, $f(\mathcal{D})T$ are in $\mathcal{B}_2(\mathcal{D}, p)$. If moreover $T^* = T$, then $Tf(T) \in \mathcal{B}_2(\mathcal{D}, p)$. In all these cases,

$$\mathcal{Q}_n(Tf(\mathcal{D})),\ \mathcal{Q}_n(f(\mathcal{D})T),\ \mathcal{Q}_n(Tf(T)) \leq \|f\|_\infty\mathcal{Q}_n(T).$$

(2) If $S^*S \leq T^*T$ and $SS^* \leq TT^*$, then $S \in \mathcal{B}_2(\mathcal{D}, p)$ with $\mathcal{Q}_n(S) \leq \mathcal{Q}_n(T)$.
(3) We have $S \in \mathcal{B}_2(\mathcal{D}, p)$ if and only if $|S|, |S^*| \in \mathcal{B}_2(\mathcal{D}, p)$.
(4) The real and imaginary parts $\Re(T)$, $\Im(T)$ belong to $\mathcal{B}_2(\mathcal{D}, p)$.
(5) If $T = T^*$, let $T = T_+ - T_-$ be the Jordan decomposition of T into positive and negative parts. Then $T_+, T_- \in \mathcal{B}_2(\mathcal{D}, p)$. Consequently $\mathcal{B}_2(\mathcal{D}, p) = \mathrm{span}\{\mathcal{B}_2(\mathcal{D}, p)_+\}$.

PROOF. (1) Since $T(1+\mathcal{D}^2)^{-s/4}$, $T^*(1+\mathcal{D}^2)^{-s/4} \in \mathcal{L}^2(\mathcal{N}, \tau)$, we immediately see that

$$Tf(\mathcal{D})(1+\mathcal{D}^2)^{-s/4} = T(1+\mathcal{D}^2)^{-s/4}f(\mathcal{D}),\ \bar{f}(\mathcal{D})T^*(1+\mathcal{D}^2)^{-s/4}\ \in \mathcal{L}^2(\mathcal{N}, \tau),$$

and when T is self-adjoint, we also have

$$Tf(T)(1+\mathcal{D}^2)^{-s/4} = f(T)T(1+\mathcal{D}^2)^{-s/4},\ \bar{f}(T)T(1+\mathcal{D}^2)^{-s/4}\ \in \mathcal{L}^2(\mathcal{N}, \tau).$$

To prove the inequality we use the trace property to see that

$$\tau((1+\mathcal{D}^2)^{-s/4}\bar{f}(\mathcal{D})T^*Tf(\mathcal{D})(1+\mathcal{D}^2)^{-s/4}) \\ = \tau(T(1+\mathcal{D}^2)^{-s/4}|f|^2(\mathcal{D})(1+\mathcal{D}^2)^{-s/4}T^*) \\ \leq \|f\|_\infty^2\tau((1+\mathcal{D}^2)^{-s/4}T^*T(1+\mathcal{D}^2)^{-s/4}),$$

and similarly for $Tf(\mathcal{D})$ and $Tf(T)$ when $T^* = T$.
(2) Clearly, $\varphi_s(S^*S) \leq \varphi_s(T^*T)$ and $\varphi_s(SS^*) \leq \varphi_s(TT^*)$. The assertion follows immediately.
(3) This follows from $\mathcal{Q}_n(T) = (\mathcal{Q}_n(|T|) + \mathcal{Q}_n(|T^*|))/2$. Item (4) follows since $\mathcal{B}_2(\mathcal{D}, p)$ is a $*$-algebra, and then item (5) follows from (2), since for a self-adjoint element $T \in \mathcal{B}_2(\mathcal{D}, p)$:

$$T^*T = |T|^2 = (T_+ + T_-)^2 = T_+^2 + T_-^2 \geq\ T_+^2,\ T_-^2.$$

This completes the proof. □

The algebras $\mathcal{B}_2(\mathcal{D},p)$ are stable under the holomorphic functional calculus. We remind the reader that when $\mathcal{B}$ is a nonunital algebra, this means that for all $T\in\mathcal{B}$ and functions f holomorphic in a neighbourhood of the spectrum of T *with* $f(0)=0$ we have $f(T)\in\mathcal{B}$.

LEMMA 1.8. *For any $n\in\mathbb{N}$ the $*$-algebra $M_n(\mathcal{B}_2(\mathcal{D},p))$ is stable under the holomorphic functional calculus in its C^*-completion.*

PROOF. We begin with the $n=1$ case. If $T\in\mathcal{B}_2(\mathcal{D},p)$ is such that $1+T$ is invertible in $\mathcal{N}$, then by (a minor extension of) Lemma 1.7 (1), we see that

$$(1+T)^{-1}-1=-T(1+T)^{-1}\in\mathcal{B}_2(\mathcal{D},p). \tag{1.2}$$

Equation (1.2) and Lemma 1.7 part (1) gives, for z in the resolvent set of T,

$$\mathcal{Q}_n\big((z-T)^{-1}-z^{-1}\big)=\mathcal{Q}_n\big(z^{-1}T(z-T)^{-1}\big)\le\|(1+T)(z-T)^{-1}\|\mathcal{Q}_n\big(z^{-1}T(1+T)^{-1}\big).$$

Set $C_z=\|(1+T)(z-T)^{-1}\|$ and let Γ be a positively oriented contour surrounding the spectrum of T with $0\notin\Gamma$, and f holomorphic in a neighborhood of the spectrum of T containing Γ. Then

$$\mathcal{Q}_n\Big(\frac{1}{2\pi i}\int_\Gamma f(z)\left[(z-T)^{-1}-z^{-1}\right]dz\Big)\le\frac{C}{2\pi}\,\mathcal{Q}_n(T(1+T)^{-1})\int_\Gamma\left|\frac{f(z)dz}{z}\right|<\infty,$$

where $C=\sup_{z\in\Gamma}C_z$. Thus we have (when $\mathcal{B}_2(\mathcal{D},p)\subset\mathcal{N}$ is nonunital)

$$\int_\Gamma f(z)(z-T)^{-1}\,dz\in\ \mathcal{B}_2(\mathcal{D},p)\oplus\mathbb{C}\,\mathrm{Id}_{\mathcal{N}},$$

with the scalar component equal to $f(0)\mathrm{Id}_{\mathcal{N}}$. The general case follows from the $n=1$ case by the main theorem of [**54**]. □

1.2. Summability from weight domains

As in the last Section, we let $\mathcal{D}$ be a self-adjoint operator affiliated to a semifinite von Neumann algebra $\mathcal{N}$ with faithful normal semifinite trace τ and $p\ge1$.

In the previous Section, we have seen that the algebra $\mathcal{B}_2(\mathcal{D},p)$ plays the role of a $*$-invariant L^2-space in the setting of weights. To construct a $*$-invariant L^1-type space associated with the data $(\mathcal{N},\tau,\mathcal{D},p)$, there are two obvious strategies.

One strategy is to define seminorms on $\mathcal{B}_2(\mathcal{D},p)^2$ (the finite span of products) and then to complete this space. The other approach is to take the projective tensor product completion of $\mathcal{B}_2(\mathcal{D},p)\otimes\mathcal{B}_2(\mathcal{D},p)$ and then consider its image in $\mathcal{N}$ under the multiplication map. In fact both approaches yield the same answer, and complementary benefits. We begin by recalling the projective tensor product topology in our setting. It is defined to be the strongest locally convex topology on the algebraic tensor product such that the natural bilinear map

$$\mathcal{B}_2(\mathcal{D},p)\times\mathcal{B}_2(\mathcal{D},p)\mapsto\mathcal{B}_2(\mathcal{D},p)\otimes\mathcal{B}_2(\mathcal{D},p),$$

is continuous, [**58**, Definition 43.2]. The projective tensor product topology can be described in terms of seminorms $\tilde{\mathcal{P}}_{n,m}$ defined for $n,m\in\mathbb{N}$, by

$$\tilde{\mathcal{P}}_{n,m}(T):=\inf\Big\{\sum_{i=1}^k\mathcal{Q}_n(T_{i,1})\,\mathcal{Q}_m(T_{i,2}):\ T=\sum_{i=1}^kT_{i,1}\otimes T_{i,2}\Big\}. \tag{1.3}$$

(In fact, since the $\mathcal{Q}_n$ are norms, so too are the $\tilde{\mathcal{P}}_{n,m}$). Using the fact that the norms $\mathcal{Q}_n$ are increasing and from the arguments of Corollary 1.12, we see that for $k\le n$

and $l \leq m$ we have $\tilde{\mathcal{P}}_{k,l} \leq \tilde{\mathcal{P}}_{n,m}$. This allows us to show that the projective tensor product topology is in fact determined by the subfamily of seminorms $\tilde{\mathcal{P}}_n := \tilde{\mathcal{P}}_{n,n}$, and accordingly we restrict to this family for the rest of this discussion.

Then we let $\mathcal{B}_2(\mathcal{D},p) \otimes_\pi \mathcal{B}_2(\mathcal{D},p)$ denote the completion of $\mathcal{B}_2(\mathcal{D},p) \otimes \mathcal{B}_2(\mathcal{D},p)$ in the projective tensor product topology. The projective tensor product topology is the unique topology on $\mathcal{B}_2(\mathcal{D},p) \otimes \mathcal{B}_2(\mathcal{D},p)$ such that, [**58**, Proposition 43.4], for any locally convex topological vector space G, the canonical isomorphism

$$\big\{\text{bilinear maps } \mathcal{B}_2(\mathcal{D},p) \times \mathcal{B}_2(\mathcal{D},p) \to G\big\} \longrightarrow \big\{\text{linear maps } \mathcal{B}_2(\mathcal{D},p) \otimes \mathcal{B}_2(\mathcal{D},p) \to G\big\},$$

gives an (algebraic) isomorphism

$$\big\{\text{continuous bilinear maps } \mathcal{B}_2(\mathcal{D},p) \times \mathcal{B}_2(\mathcal{D},p) \to G\big\} \longrightarrow \big\{\text{continuous linear maps } \mathcal{B}_2(\mathcal{D},p) \otimes \mathcal{B}_2(\mathcal{D},p) \to G\big\}.$$

As multiplication is a continuous bilinear map $m : \mathcal{B}_2(\mathcal{D},p) \times \mathcal{B}_2(\mathcal{D},p) \to \mathcal{B}_2(\mathcal{D},p)$, we obtain a continuous (with respect to the projective tensor product topology) linear map $\tilde{m} : \mathcal{B}_2(\mathcal{D},p) \otimes \mathcal{B}_2(\mathcal{D},p) \to \mathcal{B}_2(\mathcal{D},p)$. We extend $\tilde{m}$ to the completion $\mathcal{B}_2(\mathcal{D},p) \otimes_\pi \mathcal{B}_2(\mathcal{D},p)$ and denote by $\tilde{\mathcal{B}}_1(\mathcal{D},p) \subset \mathcal{B}_2(\mathcal{D},p)$ the image of $\tilde{m}$. Since $\tilde{m}$ is continuous, $\tilde{m}$ has closed kernel, and there is an isomorphism of topological vector spaces between $\tilde{\mathcal{B}}_1(\mathcal{D},p)$ with the quotient topology (defined below) and $\mathcal{B}_2(\mathcal{D},p) \otimes_\pi \mathcal{B}_2(\mathcal{D},p)/\ker \tilde{m}$. Now by [**58**, Theorem 45.1], any $\Theta \in \mathcal{B}_2(\mathcal{D},p) \otimes_\pi \mathcal{B}_2(\mathcal{D},p)$ admits a representation as an absolutely convergent sum (i.e. convergent for all $\tilde{\mathcal{P}}_n$)

$$\Theta = \sum_{i=0}^{\infty} \lambda_i R_i \otimes S_i, \qquad R_i,\, S_i \in \mathcal{B}_2(\mathcal{D},p), \quad \lambda_i \geq 0,$$

such that

$$\sum_{i=0}^{\infty} \lambda_i < \infty \quad \text{and} \quad \mathcal{Q}_n(R_i),\, \mathcal{Q}_n(S_i) \to 0,\ i \to \infty \quad \text{for all } n \in \mathbb{N}. \tag{1.4}$$

By defining $\tilde{R}_i = \lambda_i^{1/2} R_i$ and $\tilde{S}_i = \lambda_i^{1/2} S_i$, we see that we can represent Θ as an absolutely convergent sum in each of the norms $\tilde{\mathcal{P}}_n$

$$\Theta = \sum_{i=0}^{\infty} \tilde{R}_i \otimes \tilde{S}_i, \tag{1.5}$$

such that for all $n \in \mathbb{N}$, the numerical sequences $\big(\mathcal{Q}_n(\tilde{R}_i)\big)_{i\geq 0}$ and $\big(\mathcal{Q}_n(\tilde{S}_i)\big)_{i\geq 0}$ belong to $\ell^2(\mathbb{N}_0)$.

Having considered the basic features of the projective tensor product approach, we now consider the approach based on products of elements of $\mathcal{B}_2(\mathcal{D},p)$. So we let $\mathcal{B}_2(\mathcal{D},p)^2$ be the finite linear span of products from $\mathcal{B}_2(\mathcal{D},p)$, and define a family of norms, $\{\mathcal{P}_{n,m} : n, m \in \mathbb{N}\}$, on $\mathcal{B}_2(\mathcal{D},p)^2$, by setting

$$\mathcal{P}_{n,m}(T) := \inf \Big\{ \sum_{i=1}^{k} \mathcal{Q}_n(T_{1,i})\, \mathcal{Q}_m(T_{2,i}) \Big\}, \tag{1.6}$$

where the infimum is taken over all possible such representations of T of the form

$$T = \sum_{i=1}^{k} T_{1,i} T_{2,i} \quad \text{with} \quad T_{1,i},\, T_{2,i} \in \mathcal{B}_2(\mathcal{D},p).$$

Just as we did for the norms $\tilde{\mathcal{P}}$ after Equation (1.3), we may use the fact that the $\mathcal{Q}_n$ are increasing to show that the topology determined by the norms $\mathcal{P}_{n,m}$ is the same as that determined by the smaller set of norms $\mathcal{P}_n := \mathcal{P}_{n,n}$. Thus we may restrict attention to the norms $\mathcal{P}_n$.

Now $\mathcal{B}_2(\mathcal{D},p)^2 \subset \tilde{\mathcal{B}}_1(\mathcal{D},p)$ and, regarding $\tilde{\mathcal{B}}_1(\mathcal{D},p)$ as a quotient as above, we claim that the norms $\mathcal{P}_n$ are the natural seminorms (restricted to $\mathcal{B}_2(\mathcal{D},p)^2$) defining the Fréchet topology on the quotient, [**58**, Proposition 7.9]. To see this, recall that the quotient seminorms $\tilde{\mathcal{P}}_{n,q}$ on $\tilde{\mathcal{B}}_1(\mathcal{D},P)$ are defined, for $T \in \tilde{\mathcal{B}}_1(\mathcal{D},p) \cong \mathcal{B}_2(\mathcal{D},p) \otimes_\pi \mathcal{B}_2(\mathcal{D},p)/\ker \tilde{m}$, by

$$\tilde{\mathcal{P}}_{n,q}(T) := \inf_{T=\tilde{m}(\Theta)} \tilde{\mathcal{P}}_n(\Theta).$$

Then for $T \in \mathcal{B}(\mathcal{D},p)^2$ we have the elementary equalities

$$\begin{aligned}
\mathcal{P}_n(T) &= \inf\Big\{\sum_{i=1}^k \mathcal{Q}_n(T_{i,1})\mathcal{Q}_n(T_{i,2}) :\ T = \sum_{i=1}^k T_{i,1}T_{i,2}\Big\}\\
&= \inf\Big\{\sum_{i=1}^k \mathcal{Q}_n(T_{i,1})\mathcal{Q}_n(T_{i,2}) :\ \Theta = \sum_{i=1}^k T_{i,1}\otimes T_{i,2}\ \&\ \ \tilde{m}(\Theta) = T\Big\}\\
&= \inf_{\tilde{m}(\Theta)=T} \tilde{\mathcal{P}}_n(\Theta).
\end{aligned}$$

Thus the $\mathcal{P}_n$ are norms on $\mathcal{B}_2(\mathcal{D},p)^2$.

DEFINITION 1.9. Let $\mathcal{B}_1(\mathcal{D},p)$ be the completion of $\mathcal{B}_2(\mathcal{D},p)^2$ with respect to the topology determined by the family of norms $\{\mathcal{P}_n : n \in \mathbb{N}\}$.

THEOREM 1.10. *We have an equality of Fréchet spaces* $\mathcal{B}_1(\mathcal{D},p) = \tilde{\mathcal{B}}_1(\mathcal{D},p)$.

PROOF. For $T \in \tilde{\mathcal{B}}_1(\mathcal{D},p)$, there exists $\Theta = \sum_{i=0}^\infty R_i \otimes S_i \in \mathcal{B}_2(\mathcal{D},p)\otimes_\pi \mathcal{B}_2(\mathcal{D},p)$ with $\tilde{m}(\Theta) = T$ and such that the sequences $(\mathcal{Q}_n(R_i))_{i\geq 0}$, $(\mathcal{Q}_n(S_i))_{i\geq 0}$ are in $\ell^2(\mathbb{N}_0)$ for each n. Now

$$\Theta = \lim_{N\to\infty} \sum_{i=0}^N R_i \otimes S_i \quad \text{and} \quad \tilde{m}\Big(\sum_{i=0}^N R_i\otimes S_i\Big) = \sum_{i=0}^N R_iS_i,$$

so by the continuity of $\tilde{m}$

$$T = \tilde{m}(\Theta) = \lim_{N\to\infty}\sum_{i=0}^N R_iS_i.$$

Here the limit defining T is with respect to the family of norms $\tilde{\mathcal{P}}_{n,q} = \mathcal{P}_n$ on $\mathcal{B}_2(\mathcal{D},p)^2$. Hence, by definition, $T \in \mathcal{B}_1(\mathcal{D},p)$, and so $\tilde{\mathcal{B}}_1(\mathcal{D},p) \subset \mathcal{B}_1(\mathcal{D},p)$. Now observe that we have the containments

$$\mathcal{B}_2(\mathcal{D},p)^2 \subset \tilde{\mathcal{B}}_1(\mathcal{D},p) \subset \mathcal{B}_1(\mathcal{D},p),$$

and as $\mathcal{B}_2(\mathcal{D},p)^2$ is dense in $\mathcal{B}_1(\mathcal{D},p)$ by definition, $\mathcal{B}_2(\mathcal{D},p)^2$ is dense in $\tilde{\mathcal{B}}_1(\mathcal{D},p)$. As $\tilde{\mathcal{P}}_{n,q} = \mathcal{P}_n$ on $\mathcal{B}_2(\mathcal{D},p)^2$, we see that $\tilde{\mathcal{B}}_1(\mathcal{D},p)$ is a dense and closed subset of $\mathcal{B}_1(\mathcal{D},p)$. Hence $\tilde{\mathcal{B}}_1(\mathcal{D},p) = \mathcal{B}_1(\mathcal{D},p)$. □

On the basis of this result, we will employ the single notation $\mathcal{B}_1(\mathcal{D},p)$ from now on.

Remark. For R, $S \in \mathcal{B}_2(\mathcal{D}, p)$, the product $RS \in \mathcal{B}_1(\mathcal{D}, p)$ with the estimate $\mathcal{P}_n(RS) \leq \mathcal{Q}_n(R)\mathcal{Q}_n(S)$. By applying $\tilde{m}$ to a representation of $\Theta \in \mathcal{B}_2(\mathcal{D}, p) \otimes_\pi \mathcal{B}_2(\mathcal{D}, p)$ as in Equation (1.5), this allows us to see that every $T \in \mathcal{B}_1(\mathcal{D}, p)$ can be represented as a sum, convergent for every $\mathcal{P}_n$,

$$T = \sum_{i=0}^{\infty} R_i S_i, \quad \text{such that for all } n \in \mathbb{N}, \quad (\mathcal{Q}_n(R_i))_{i\geq 0}\,, \, (\mathcal{Q}_n(S_i))_{i\geq 0} \in \ell^2(\mathbb{N}_0).$$

We now show that $\mathcal{B}_1(\mathcal{D}, p)$ is a $*$-algebra, and that the norms $\mathcal{P}_n$ are submultiplicative. The first step is to show that $\mathcal{B}_1(\mathcal{D}, p)$ is naturally included in $\mathcal{B}_2(\mathcal{D}, p)$.

Lemma 1.11. *The algebra $\mathcal{B}_1(\mathcal{D}, p)$ is continuously embedded in $\mathcal{B}_2(\mathcal{D}, p)$. In particular, for all $T \in \mathcal{B}_1(\mathcal{D}, p)$ and all $n \in \mathbb{N}$, $\mathcal{Q}_n(T) \leq \mathcal{P}_n(T)$.*

Proof. Let $T \in \mathcal{B}_1(\mathcal{D}, p)$. That T belongs to $\mathcal{B}_2(\mathcal{D}, p)$ follows from the submultiplicativity of the norms $\mathcal{Q}_n$. To see this, fix $n \in \mathbb{N}$. Then, for any representation $T = \sum_{i=0}^{\infty} R_i S_i$, the submultiplicativity of the norms $\mathcal{Q}_n$ gives us

$$\mathcal{Q}_n(T) = \mathcal{Q}_n\Big(\sum_{i=0}^{\infty} R_i S_i\Big) \leq \sum_{i=0}^{\infty} \mathcal{Q}_n(R_i S_i) \leq \sum_{i=0}^{\infty} \mathcal{Q}_n(R_i)\, \mathcal{Q}_n(S_i).$$

Since this is true for any representation $T = \sum_{i=0}^{\infty} R_i S_i$, we find $\mathcal{Q}_n(T) \leq \mathcal{P}_n(T)$, proving that $\mathcal{B}_1(\mathcal{D}, p)$ embeds continuously in $\mathcal{B}_2(\mathcal{D}, p)$. □

Corollary 1.12. *The Fréchet space $\mathcal{B}_1(\mathcal{D}, p)$ is a $*$-subalgebra of $\mathcal{N}$. Moreover, the norms $\mathcal{P}_n$ are $*$-invariant, submultiplicative, and for $n \leq m$ satisfy $\mathcal{P}_n \leq \mathcal{P}_m$.*

Proof. We begin by showing that each $\mathcal{P}_n$ is a $*$-invariant norm. Using the $*$-invariance of $\mathcal{Q}_n(\cdot)$, we have for any $T \in \mathcal{B}_2(\mathcal{D}, p)^2$

$$\begin{aligned}
\mathcal{P}_n(T^*) &= \inf\Big\{ \sum_i \mathcal{Q}_n(S_{1,i})\, \mathcal{Q}_n(S_{2,i}) \;:\; T^* = \sum_i S_{1,i} S_{2,i} \Big\} \\
&\leq \inf\Big\{ \sum_i \mathcal{Q}_n(T_{2,i}^*)\, \mathcal{Q}_n(T_{1,i}^*) \;:\; T = \sum_i T_{1,i} T_{2,i} \Big\} \\
&= \inf\Big\{ \sum_i \mathcal{Q}_n(T_{2,i})\, \mathcal{Q}_n(T_{1,i}) \;:\; T = \sum_i T_{1,i} T_{2,i} \Big\} = \mathcal{P}_n(T).
\end{aligned}$$

Hence $\mathcal{P}_n(T^*) \leq \mathcal{P}_n(T)$, and by replacing T^* with T we find that $\mathcal{P}_n(T^*) = \mathcal{P}_n(T)$. It now follows that each $\mathcal{P}_n$ is $*$-invariant on all of $\mathcal{B}_1(\mathcal{D}, p)$.

That $\mathcal{B}_1(\mathcal{D}, p)$ is an algebra, follows from the embedding $\mathcal{B}_1(\mathcal{D}, p) \subset \mathcal{B}_2(\mathcal{D}, p)$ proven in Lemma 1.11:

$$\mathcal{B}_1(\mathcal{D}, p) \cdot \mathcal{B}_1(\mathcal{D}, p) \subset \mathcal{B}_2(\mathcal{D}, p) \cdot \mathcal{B}_2(\mathcal{D}, p) \subset \mathcal{B}_1(\mathcal{D}, p).$$

For the submultiplicativity of the norms $\mathcal{P}_n$, we observe for T, $S \in \mathcal{B}_1(\mathcal{D}, p)$

$$\mathcal{P}_n(TS) \leq \mathcal{Q}_n(T)\mathcal{Q}_n(S) \leq \mathcal{P}_n(T)\mathcal{P}_n(S),$$

where the first inequality follows from the definition of $\mathcal{P}_n$ and the second from the norm estimate of Lemma 1.11.

To prove that $\mathcal{P}_n(\cdot) \leq \mathcal{P}_m(\cdot)$ for $n \leq m$, take $T \in \mathcal{B}_2(\mathcal{D}, p)^2$ and consider any representation $T = \sum_{i=1}^{k} T_{i,1}\, T_{i,2}$. Then, since $\mathcal{Q}_n(\cdot) \leq \mathcal{Q}_m(\cdot)$ for $n \leq m$, we have

$$\sum_{i=1}^{k} \mathcal{Q}_n(T_{i,1})\, \mathcal{Q}_n(T_{i,2}) \leq \sum_{i=1}^{k} \mathcal{Q}_m(T_{i,1})\, \mathcal{Q}_m(T_{i,2}),$$

and thus

$$\mathcal{P}_n(T) \leq \sum_{i=1}^{k} \mathcal{Q}_m(T_{i,1})\, \mathcal{Q}_m(T_{i,2}). \tag{1.7}$$

Since (1.7) is true for any such representation, we have $\mathcal{P}_n(T) \leq \mathcal{P}_m(T)$. Now let $T \in \mathcal{B}_1(\mathcal{D}, p)$ be the limit of the sequence $(T_N)_{N\geq 1} \subset \mathcal{B}_2(\mathcal{D}, p)^2$. Then $\mathcal{P}_n(T) = \lim_{N\to\infty} \mathcal{P}_n(T_N) \leq \lim_{N\to\infty} \mathcal{P}_m(T_N) = \mathcal{P}_m(T)$. □

Next we show the compatibility of the norms $\mathcal{P}_n$ with positivity.

LEMMA 1.13. *Let $0 \leq A \in \mathcal{N}$. Then $A \in \mathcal{B}_1(\mathcal{D}, p)$ if and only if $A^{1/2} \in \mathcal{B}_2(\mathcal{D}, p)$ with*

$$\mathcal{P}_n(A) = \mathcal{Q}_n(A^{1/2})^2, \quad \textit{for all } n \in \mathbb{N}.$$

Moreover if $0 \leq A \leq B \in \mathcal{N}$ and $B \in \mathcal{B}_1(\mathcal{D}, p)$, then $A \in \mathcal{B}_1(\mathcal{D}, p)$, and we have $\mathcal{P}_n(A) \leq \mathcal{P}_n(B)$ for all $n \in \mathbb{N}$.

PROOF. Given $0 \leq A \in \mathcal{N}$ with $A^{1/2} \in \mathcal{B}_2(\mathcal{D}, p)$, it follows from the definitions that $A \in \mathcal{B}_1(\mathcal{D}, p)$ and $\mathcal{P}_n(A) \leq \mathcal{Q}_n(A^{1/2})^2$. So suppose $0 \leq A \in \mathcal{B}_1(\mathcal{D}, p)$ and choose any representation

$$A = \sum_{i=0}^{\infty} R_i S_i, \qquad \sum_{i=0}^{\infty} \mathcal{Q}_n(R_i)\mathcal{Q}_n(S_i) < \infty, \text{ for all } n \in \mathbb{N}.$$

Then using the self-adjointness of A, the definitions, and the Cauchy-Schwarz inequality yields

$$\begin{aligned}
\mathcal{Q}_n(A^{1/2})^2 &= \mathcal{Q}_n\Big(\big(\sum_{i=0}^{\infty} R_i S_i\big)^{1/2}\Big)^2 \\
&= \big\|\sum_{i=0}^{\infty} R_i S_i\big\| + \varphi_{p+1/n}\big(\sum_{i=0}^{\infty} R_i S_i\big) + \varphi_{p+1/n}\big(\sum_{i=0}^{\infty} S_i R_i\big) \\
&\leq \sum_{i=0}^{\infty} \|R_i\|\,\|S_i\| + \big|\varphi_{p+1/n}\big(R_i S_i\big)\big| + \big|\varphi_{p+1/n}\big(S_i R_i\big)\big| \\
&\leq \sum_{i=0}^{\infty} \|R_i\|\,\|S_i\| + \varphi_{p+1/n}\big(R_i R_i^*\big)^{1/2} \varphi_{p+1/n}\big(S_i^* S_i\big)^{1/2} \\
&\qquad\qquad + \varphi_{p+1/n}\big(S_i S_i^*\big)^{1/2} \varphi_{p+1/n}\big(R_i^* R_i\big)^{1/2} \\
&\leq \sum_{i=0}^{\infty} \mathcal{Q}_n(R_i)\mathcal{Q}_n(S_i).
\end{aligned}$$

The last inequality follows from applying the Cauchy-Schwarz inequality,

$$(r_1 s_1 + r_2 s_2 + r_3 s_3)^2 \leq (r_1^2 + r_2^2 + r_3^2)(s_1^2 + s_2^2 + s_3^2),$$

to each term in the sum. Thus for any representation of A we have $\mathcal{Q}_n(A^{1/2})^2 \leq \sum_{i=0}^{\infty} \mathcal{Q}_n(R_i)\mathcal{Q}_n(S_i)$, which entails $\mathcal{Q}_n(A^{1/2})^2 \leq \mathcal{P}_n(A)$ as needed. For the last statement, suppose that $0 \leq B \in \mathcal{B}_1(\mathcal{D}, p)$ and that $0 \leq A \in \mathcal{N}$ satisfies $B \geq A$. Then $B^{1/2} \geq A^{1/2}$ and $B^{1/2} \in \mathcal{B}_2(\mathcal{D}, p)$, so Lemma 1.7 (2) completes the proof. □

Since $\mathcal{B}_1(\mathcal{D},p)$ is a $*$-algebra, we have $T \in \mathcal{B}_1(\mathcal{D},p)$ if and only if $T^* \in \mathcal{B}_1(\mathcal{D},p)$. Thus given $T = T^* \in \mathcal{B}_1(\mathcal{D},p)$, it is natural to ask whether the positive and negative parts T_+, T_- of the Jordan decomposition of T are in $\mathcal{B}_1(\mathcal{D},p)$. We can not answer this question, but can nevertheless prove that $\mathcal{B}_1(\mathcal{D},p)$ is the (finite) span of its positive cone.

PROPOSITION 1.14. *For every $T \in \mathcal{B}_1(\mathcal{D},p)$, there exist four positive operators $T_0,\dots,T_3 \in \mathcal{B}_1(\mathcal{D},p)$ such that*

$$T = \big(T_0 - T_2\big) + i\big(T_1 - T_3\big).$$

Here $\Re(T) = T_0 - T_2$ and $\Im(T) = T_1 - T_3$, but this need not be the Jordan decomposition since it may not be that $T_0T_2 = T_1T_3 = 0$. Nevertheless, the space $\mathcal{B}_1(\mathcal{D},p)$ is the linear span of its positive cone.

PROOF. Let $T \in \mathcal{B}_1(\mathcal{D},p)$ have the representation $T = \sum_j R_jS_j$. By Equation (1.5), this means that for each n the sequences $(\mathcal{Q}_n(R_j))_{j=0}^\infty$ and $(\mathcal{Q}_n(S_j))_{j=0}^\infty$ belong to $\ell^2(\mathbb{N}_0)$. Now, from the polarization identity

$$4R^*S = \sum_{k=0}^{3} i^k(S + i^kR)^*(S + i^kR),$$

we can decompose $T = \sum_{k=0}^3 i^kT_k$, with

$$4T_k = \sum_{j=0}^{\infty}(S_j + i^kR_j^*)^*(S_j + i^kR_j^*) \geq 0.$$

Since $(\mathcal{Q}_n(R_j))_{j=0}^\infty$ and $(\mathcal{Q}_n(S_j))_{j=0}^\infty$ belong to $\ell^2(\mathbb{N}_0)$, and using the $*$-invariance of the norms $\mathcal{Q}_n$, we see that the four elements T_k, $k = 0,1,2,3$, all belong to $\mathcal{B}_1(\mathcal{D},p)$. Now it is straightforward to check that $\Re(T) = T_0 - T_2$ and $\Im(T) = T_1 - T_3$, however, these need not give the canonical decomposition into positive and negative parts since we may not have $T_0T_2 = 0$ and $T_1T_3 = 0$. □

Remark. The previous proposition shows that we can represent elements of $\mathcal{B}_1(\mathcal{D},p)$ as finite sums of products of elements of $\mathcal{B}_2(\mathcal{D},p)$, and so have a correspondingly simpler description of the norms. We will not pursue this further here.

The next lemma is analogous to Lemma 1.7 (1). It shows that $\mathcal{B}_1(\mathcal{D},p)$ is a bimodule for the natural actions of the commutative von Neumann algebra generated by the spectral family of the operator $\mathcal{D}$.

LEMMA 1.15. *Let $T \in \mathcal{B}_1(\mathcal{D},p)$ and $f \in L^\infty(\mathbb{R})$. Then $Tf(\mathcal{D})$ and $f(\mathcal{D})T$ belong to $\mathcal{B}_1(\mathcal{D},p)$ with $\mathcal{P}_n\big(Tf(\mathcal{D})\big), \mathcal{P}_n\big(f(\mathcal{D})T\big) \leq \|f\|_\infty \mathcal{P}_n(T)$ for all $n \in \mathbb{N}$.*

PROOF. Fix $T \in \mathcal{B}_1(\mathcal{D},p)$, $f \in L^\infty(\mathbb{R})$ and $n \in \mathbb{N}$. Consider an arbitrary representation $T = \sum_{i=0}^\infty R_i\, S_i$. Then we claim that $\sum_{i=0}^\infty R_i\,\big(S_if(\mathcal{D})\big)$ is a representation of $Tf(D)$. Indeed, it follows by Lemma 1.7 (1) that

$$\sum_{i=0}^{\infty} \mathcal{Q}_n(R_i)\, \mathcal{Q}_n\big(S_if(\mathcal{D})\big) \leq \|f\|_\infty \sum_{i=0}^{\infty} \mathcal{Q}_n(R_i)\, \mathcal{Q}_n(S_i) < \infty,$$

showing that $Tf(\mathcal{D}) \in \mathcal{B}_1(\mathcal{D}, p)$. Moreover, the preceding inequality entails that

$$\begin{aligned}\mathcal{P}_n\big(Tf(\mathcal{D})\big) &\leq \inf\Big\{\sum_{i=0}^{\infty} \mathcal{Q}_n(R_i)\,\mathcal{Q}_n\big(S_i f(\mathcal{D})\big) \;:\; T = \sum_{i=0}^{\infty} R_i\, S_i\Big\}\\ &\leq \|f\|_\infty \inf\Big\{\sum_{i=0}^{\infty} \mathcal{Q}_n(R_i)\,\mathcal{Q}_n(S_i) \;:\; T = \sum_{i=0}^{\infty} R_i\, S_i\Big\} = \|f\|_\infty\, \mathcal{P}_n(T).\end{aligned}$$

The case of $f(\mathcal{D})T$ is similar. □

Our next aim is to prove that $\mathcal{B}_1(\mathcal{D}, p)$ is stable under the holomorphic functional calculus in its C^*-completion. This will be a corollary of the following two lemmas.

Lemma 1.16. *Let T, R be elements of $\mathcal{B}_2(\mathcal{D}, p)$ with $1 + R$ invertible in $\mathcal{N}$. Then $T(1 + R)^{-1} \in \mathcal{B}_2(\mathcal{D}, p)$, and for all $n \in \mathbb{N}$ we have*

$$\mathcal{Q}_n\big(T(1 + R)^{-1}\big) \leq C_n(R)\,\mathcal{Q}_n(T),$$

where the constant $C_n(R)$ is given by

$$C_n(R) := 4\sqrt{2}\max\{1, \|(1 + R)^{-1}\|\}\max\{1, \mathcal{Q}_n(R)\}.$$

Proof. For any $n \in \mathbb{N}$ we have

$$\begin{aligned}\mathcal{Q}_n(T(1 + R)^{-1})^2 &= \|T(1 + R)^{-1}\|^2 + \varphi_{p+1/n}((1 + R^*)^{-1}|T|^2(1 + R)^{-1})\\ &\qquad\qquad + \varphi_{p+1/n}(T|1 + R|^{-2}T^*)\\ &\leq \|(1 + R)^{-1}\|^2\left(\|T\|^2 + \varphi_{p+1/n}(TT^*)\right)\\ &\qquad\qquad + \varphi_{p+1/n}((1 + R^*)^{-1}|T|^2(1 + R)^{-1})\\ &\leq \|(1 + R)^{-1}\|^2\,\mathcal{Q}_n(T)^2 + \varphi_{p+1/n}((1 + R^*)^{-1}|T|^2(1 + R)^{-1}),\end{aligned} \tag{1.8}$$

where the first inequality follows by an application of the operator inequality $A^*B^*BA \leq \|B\|^2A^*A$, while the second follows from the definition of the norm $\mathcal{Q}_n$. Writing

$$\begin{aligned}&(1 + R^*)^{-1}|T|^2(1 + R)^{-1} =\\ &\quad |T|^2 - R^*(1 + R^*)^{-1}|T|^2 - |T|^2R(1 + R)^{-1} + R^*(1 + R^*)^{-1}|T|^2R(1 + R)^{-1},\end{aligned}$$

the Cauchy-Schwarz inequality for the weight $\varphi_{p+1/n}$ gives

$$\begin{aligned}&\varphi_{p+1/n}((1 + R^*)^{-1}|T|^2(1 + R)^{-1}) \leq\\ &\quad \varphi_{p+1/n}(|T|^2) + \varphi_{p+1/n}(R^*(1 + R^*)^{-1}|T|^2R(1 + R)^{-1})\\ &\quad + \varphi_{p+1/n}(|T|^4)^{1/2}\Big(\varphi_{p+1/n}(|R|^2|1 + R|^{-2})^{1/2} + \varphi_{p+1/n}(|R^*|^2|1 + R^*|^{-2})^{1/2}\Big).\end{aligned}$$

Using the operator inequality $A^*B^*BA \leq \|B\|^2A^*A$ as above, we deduce that

$$\begin{aligned}&\varphi_{p+1/n}((1 + R^*)^{-1}|T|^2(1 + R)^{-1}) \leq\\ &\qquad \varphi_{p+1/n}(|T|^2) + \|T\|^2\,\|(1 + R)^{-1}\|^2\,\varphi_{p+1/n}(|R|^2)\\ &\qquad + \|T\|\,\|(1 + R)^{-1}\|\,\varphi_{p+1/n}(|T|^2)^{1/2}\Big(\varphi_{p+1/n}(|R|^2)^{1/2} + \varphi_{p+1/n}(|R^*|^2)^{1/2}\Big).\end{aligned}$$

Simplifying this last expression, using $\|T\|$, $\varphi(|T|^2)^{1/2} \leq \mathcal{Q}_n(T)$ and similarly for R, we find

$$\varphi_{p+1/n}((1 + R^*)^{-1}|T|^2(1 + R)^{-1}) \leq \mathcal{Q}_n(T)^2\,\left(1 + \|(1 + R)^{-1}\|\,\mathcal{Q}_n(R)\right)^2.$$

This yields

$$\mathcal{Q}_n(T(1+R)^{-1}) \le \sqrt{\|(1+R)^{-1}\|^2 + (1+\|(1+R)^{-1}\|\,\mathcal{Q}_n(R))^2}\,\mathcal{Q}_n(T).$$

Finally we employ, for $a, b > 0$, the numerical inequalities

$$\begin{aligned}\sqrt{a^2+(1+ab)^2} &\le \sqrt{(ac)^2+(1+ac)^2}, \quad c := \max\{1,b\}\\ &\le \sqrt{2}(1+ac) \le \sqrt{2}(1+a)(1+c)\\ &\le 4\sqrt{2}\max\{1,a\}\max\{1,c\} \le 4\sqrt{2}\max\{1,a\}\max\{1,b\},\end{aligned}$$

to arrive at the inequality of the statement of the Lemma. □

LEMMA 1.17. *Let $T \in \mathcal{B}_1(\mathcal{D},p)$ and $R \in \mathcal{B}_2(\mathcal{D},p)$, with $1+R$ invertible in $\mathcal{N}$. Then the operator $T(1+R)^{-1}$ belongs to $\mathcal{B}_1(\mathcal{D},p)$, with*

$$\mathcal{P}_n\big(T(1+R)^{-1}\big) \le C_n(R)\mathcal{P}_n(T), \quad \text{for all } n \in \mathbb{N},$$

for the finite constant $C_n(R)$ of Lemma 1.16.

PROOF. To see this, fix $n \in \mathbb{N}$ and consider any representation of T

$$T = \sum_{i=0}^{\infty} T_{1,i}T_{2,i} \quad \text{with} \quad T_{1,i}, T_{2,i} \in \mathcal{B}_2(\mathcal{D},p) \quad \text{and} \quad \sum_{i=0}^{\infty} \mathcal{Q}_n(T_{1,i})\mathcal{Q}_n(T_{2,i}) < \infty.$$

Then

$$\mathcal{P}_n(T(1+R)^{-1}) \le \sum_{i=0}^{\infty} \mathcal{Q}_n(T_{1,i})\,\mathcal{Q}_n(T_{2,i}(1+R)^{-1}) \le C_n(R)\sum_{i=0}^{\infty} \mathcal{Q}_n(T_{1,i})\,\mathcal{Q}_n(T_{1,i}),$$

where we used Lemma 1.16 to obtain the second estimate. Since the constant does not depend on the representation chosen, we have the inequality

$$\mathcal{P}_n\big(T(1+R)^{-1}\big) \le C_n(R)\,\mathcal{P}_n(T),$$

which completes the proof. □

PROPOSITION 1.18. *For any $n \in \mathbb{N}$ and $p \ge 1$, the $*$-algebra $M_n(\mathcal{B}_1(\mathcal{D},p))$ is stable under the holomorphic functional calculus.*

PROOF. We begin with the case $n = 1$. Let $T \in \mathcal{B}_1(\mathcal{D},p)$ and let f be a function holomorphic in a neighborhood of the spectrum of T. Let Γ be a positively oriented contour surrounding the spectrum of T, taking care that 0 does not lie on Γ. We want to show that (when $\mathcal{B}_1(\mathcal{D},p)$ is a nonunital subalgebra of $\mathcal{N}$)

$$\int_\Gamma f(z)(z-T)^{-1}\,dz \in\, \mathcal{B}_1(\mathcal{D},p) \oplus \mathbb{C}\,\mathrm{Id}_{\mathcal{N}},$$

with the scalar component equal to $f(0)\mathrm{Id}_{\mathcal{N}}$. Since

$$\int_\Gamma f(z)(z-T)^{-1}\,dz - f(0)\,\mathrm{Id}_{\mathcal{N}} = \int_\Gamma f(z)Tz^{-1}(z-T)^{-1}\,dz,$$

we get for all $n \in \mathbb{N}$

$$\mathcal{P}_n\Big(\int_\Gamma f(z)(z-T)^{-1}\,dz - f(0)\,\mathrm{Id}_{\mathcal{N}}\Big) \le \int_\Gamma \left|\frac{f(z)}{z^2}\right| \mathcal{P}_n(T)C_n(-T/z)\,dz,$$

where C_n is the constant from Lemmas 1.16 and 1.17, and we have used Lemma 1.11 to see that $T/z \in \mathcal{B}_2(\mathcal{D},p)$. Then the inequality

$$C_n(-T/z) \le 4\sqrt{2}\max\{1, \|(1-T/z)^{-1}\|\}\max\{1, \mathcal{Q}_n(T)/|z|\},$$

allows us to conclude that the integral above is finite. Again, the general case follows from [**54**]. □

We conclude this Section by showing that when the weights φ_s, $s>0$, are tracial, then our space of integrable element $\mathcal{B}_1(\mathcal{D},p)$, coincides with an intersection of trace-ideals. This fact will be of relevance in two of our applications (Chapter 4 and Section 5.2), where the restriction of the faithful normal semifinite weights φ_s to an appropriate sub-von Neumann algebra are faithful normal semifinite traces.

Proposition 1.19. *Assume that there exists a von Neumann subalgebra $\mathcal{M}\subset\mathcal{N}$ such that for all $n\in\mathbb{N}$, the restriction of the faithful normal semifinite weight $\tau_n:=\varphi_{p+1/n}|_{\mathcal{M}}$ is a faithful normal semifinite trace. Then*

$$\mathcal{B}_1(\mathcal{N},\tau)\bigcap\mathcal{M}=\bigcap_{n\geq1}\mathcal{L}^1(\mathcal{M},\tau_n).$$

Here $\mathcal{L}^1(\mathcal{M},\tau_n)$ denotes the trace ideal of $\mathcal{M}$ associated with the faithful normal semifinite trace τ_n. Moreover, for any $n\in\mathbb{N}$, $\mathcal{P}_n(\cdot)=\|\cdot\|+2\|\cdot\|_{\tau_n}$, where $\|\cdot\|_{\tau_n}$ is the trace-norm on $\mathcal{L}^1(\mathcal{M},\tau_n)$.

Proof. Note first that the tracial property of the faithful normal semifinite trace $\tau_n:=\varphi_{p+1/n}|_{\mathcal{M}}$, immediately implies that

$$\mathcal{B}_2(\mathcal{N},\tau)\bigcap\mathcal{M}=\bigcap_{n\geq1}\mathcal{L}^2(\mathcal{M},\tau_n),$$

that is, the half-domain of τ_n on $\mathcal{M}$ is already $*$-invariant and moreover

$$\mathcal{Q}_n(T)=\big(\|T\|^2+2\||T|^2\|_{\tau_n}\big)^{1/2}.$$

Now, take $T\in\mathcal{B}_1(\mathcal{D},p)\bigcap\mathcal{M}$, and any representation $T=\sum_{i=1}^\infty R_iS_i$. Observe then that the Hölder inequality gives

$$\|T\|+2\|T\|_{\tau_n}\leq\sum_{i=1}^\infty\Big(\|R_iS_i\|+2\|R_iS_i\|_{\tau_n}\Big)\leq\sum_{i=1}^\infty\mathcal{Q}_n(R_i)\,\mathcal{Q}_n(S_i).$$

Since this inequality is valid for any such representation, it gives

$$\|T\|+2\|T\|_{\tau_n}\leq\mathcal{P}_n(T),$$

and hence $\mathcal{B}_1(\mathcal{N},\tau)\bigcap\mathcal{M}\subset\bigcap_{n\geq1}\mathcal{L}^1(\mathcal{M},\tau_n)$. Conversely, let $T\in\bigcap_n\mathcal{L}^1(\mathcal{M},\tau_n)$. If $T\geq0$ then $T=\sqrt{T}\sqrt{T}$ and $\sqrt{T}\in\mathcal{B}_2(\mathcal{D},p)\cap\mathcal{M}$, by the first part of the proof and the fact that $\sqrt{T}\in\bigcap_n\mathcal{L}^2(\mathcal{M},\tau_n)$. Thus $T\in\mathcal{B}_1(\mathcal{D},p)\cap\mathcal{M}$ and, by Lemma 1.13, $\mathcal{P}_n(T)=\mathcal{Q}_n(\sqrt{T})^2=\|T\|+2\|T\|_{\tau_n}$. If T is now arbitrary in $\bigcap_n\mathcal{L}^1(\mathcal{M},\tau_n)$, we may write it as a linear combination of four positive elements, $T=c_1T_1+c_2T_2+c_3T_3+c_4T_4$, with: $|c_j|=1$ for each $j=1,2,3,4$; $0\leq T_j\in\mathcal{L}^1(\mathcal{M},\tau_n)$ for each n; and $\|T_j\|+2\|T_j\|_{\tau_n}\leq\|T\|+2\|T\|_{\tau_n}$. Hence $\bigcap_{n\geq1}\mathcal{L}^1(\mathcal{M},\tau_n)\subset\mathcal{B}_1(\mathcal{N},\tau)\bigcap\mathcal{M}$. Regarding the equality of norms, for $T\in\bigcap_{n\geq1}\mathcal{L}^1(\mathcal{M},\tau_n)=\mathcal{B}_1(\mathcal{N},\tau)\bigcap\mathcal{M}$, write $T=S|T|$ for the polar decomposition. Then by construction of the norms $\mathcal{P}_n$ and the value of the norms $\mathcal{Q}_n$ we see that

$$\mathcal{P}_n(T)\leq\mathcal{Q}_n(S|T|^{1/2})\mathcal{Q}_n(|T|^{1/2})\leq\||T|^{1/2}\|^2+2\||T|\|_{\tau_n}=\|T\|+2\|T\|_{\tau_n},$$

and we conclude using the converse inequality already proven. □

1.3. Smoothness and summability

Anticipating the pseudodifferential calculus, we introduce dense subalgebras of $\mathcal{B}_1(\mathcal{D}, p)$ which 'see' smoothness as well as summability. There are several operators naturally associated to our notions of smoothness.

We recall that $\mathcal{D}$ is a self-adjoint operator affiliated to a semifinite von Neumann algebra $\mathcal{N}$ with faithful normal semifinite trace τ, and $p \geq 1$. For a few definitions, like the next, we do not require all of this information.

Definition 1.20. Let $\mathcal{D}$ be a self-adjoint operator affiliated to a semifinite von Neumann algebra $\mathcal{N} \subset \mathcal{B}(\mathcal{H})$, where $\mathcal{H}$ is a Hilbert space. Set $\mathcal{H}_\infty = \bigcap_{k\geq 0} \operatorname{dom} \mathcal{D}^k$. For an operator $T \in \mathcal{N}$ such that $T : \mathcal{H}_\infty \to \mathcal{H}_\infty$ we set

$$\delta(T) := [|\mathcal{D}|, T], \quad \delta'(T) := [(1 + \mathcal{D}^2)^{1/2}, T], \quad T \in \mathcal{N}. \tag{1.9}$$

In addition, we recursively set

$$T^{(n)} := [\mathcal{D}^2, T^{(n-1)}],\ n \in \mathbb{N} \quad \text{and} \quad T^{(0)} := T. \tag{1.10}$$

Finally, let

$$L(T) := (1 + \mathcal{D}^2)^{-1/2}[\mathcal{D}^2, T], \quad R(T) := [\mathcal{D}^2, T](1 + \mathcal{D}^2)^{-1/2}. \tag{1.11}$$

We have defined δ, δ', L, R for operators in $\mathcal{N}$ preserving $\mathcal{H}_\infty$, and so consider the domains of δ, δ', L, R to be subsets of $\mathcal{N}$. If $T \in \operatorname{dom}\delta$, say, so that $\delta(T)$ is bounded, then it is straightforward to check that $\delta(T)$ commutes with every operator in the commutant of $\mathcal{N}$, and hence $\delta(T) \in \mathcal{N}$. Similar comments apply to δ', L, R. It follows from the proof of [**15**, Proposition 6.5] and $R(T)^* = -L(T^*)$ that

$$\bigcap_{n\geq 0} \operatorname{dom} L^n = \bigcap_{n\geq 0} \operatorname{dom} R^n = \bigcap_{k,\, l\geq 0} \operatorname{dom} L^k \circ R^l. \tag{1.12}$$

Similarly, using the fact that $|x| - (1 + x^2)^{1/2}$ is a bounded function, it is proved after the Definition 2.2 of [**15**] that

$$\bigcap_{n\in\mathbb{N}} \operatorname{dom} \delta^n = \bigcap_{n\in\mathbb{N}} \operatorname{dom} \delta'^n. \tag{1.13}$$

Finally, it is proven in [**22, 25**] and [**15**, Proposition 6.5] that we have equalities of all the smooth domains in Equations (1.12), (1.13).

Definition 1.21. Let $\mathcal{D}$ be a self-adjoint operator affiliated to a semifinite von Neumann algebra $\mathcal{N}$ with faithful normal semifinite trace τ, and $p \geq 1$. Then define for $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$

$$\mathcal{B}_1^k(\mathcal{D}, p) := \big\{T \in \mathcal{N} \,:\, \text{for all } l = 0, \ldots, k,\ \delta^l(T) \in \mathcal{B}_1(\mathcal{D}, p)\big\},$$

where $\delta = [|\mathcal{D}|, \cdot]$ as in Equation (1.9). Also set

$$\mathcal{B}_1^\infty(\mathcal{D}, p) := \bigcap_{k=0}^{\infty} \mathcal{B}_1^k(\mathcal{D}, p).$$

We equip $\mathcal{B}_1^k(\mathcal{D}, p)$, $k \in \mathbb{N}_0 \cup \{\infty\}$, with the topology determined by the seminorms

$$\mathcal{P}_{n,l}(T) := \sum_{j=0}^{l} \mathcal{P}_n(\delta^j(T)), \quad n \in \mathbb{N}, \quad l \in \mathbb{N}_0. \tag{1.14}$$

The triangle inequality for the seminorms $\mathcal{P}_{n,l}$ follows from the linearity of δ^l and the triangle inequality for the norm $\mathcal{P}_n$. Submultiplicativity then follows from the Leibniz rule as well as the triangle inequality and submultiplicativity for $\mathcal{P}_n$. For k finite, it is sufficient to consider the subfamily of norms $\{\mathcal{P}_{n,k}\}_{n\in\mathbb{N}}$.

Remarks. (1) Defining $\mathcal{B}_2^k(\mathcal{D},p) := \big\{T \in \mathcal{N} \,:\, \text{for all } l = 0,\dots,k,\ \delta^l(T) \in \mathcal{B}_2(\mathcal{D},p)\big\}$, an application of the Leibniz rule shows that $\mathcal{B}_2^k(\mathcal{D},p)^2 \subset \mathcal{B}_1^k(\mathcal{D},p)$.

We observe that $\mathcal{B}_2^\infty(\mathcal{D},p)$ is non-empty, and so $\mathcal{B}_1^\infty(\mathcal{D},p)$ is non-empty. Note first that $\mathcal{B}_2(\mathcal{D},p)$ is non empty as it contains $\mathcal{L}^2(\mathcal{N},\tau)$. Then, for $T \in \mathcal{B}_2(\mathcal{D},p)$, and $f \in C_c(\mathbb{R})$ and $k,l \in \mathbb{N}_0$ arbitrary, $|\mathcal{D}|^k f(\mathcal{D}) T f(\mathcal{D}) |\mathcal{D}|^l$ is well defined and is in $\mathcal{B}_2(\mathcal{D},p)$ by Lemma 1.7. This implies that $\delta^k\big(f(\mathcal{D})Tf(\mathcal{D})\big) \in \mathcal{B}_2(\mathcal{D},p)$ for any $k \in \mathbb{N}_0$ and thus $f(\mathcal{D})Tf(\mathcal{D})$ is in $\mathcal{B}_2^\infty(\mathcal{D},p)$.
(2) Using Lemma 1.15, we see that the topology on the algebras $\mathcal{B}_1^k(\mathcal{D},p)$ could have been equivalently defined with $\delta' = [(1+\mathcal{D}^2)^{1/2},\cdot]$ instead of δ. This follows since $f(\mathcal{D}) = |\mathcal{D}| - (1+\mathcal{D}^2)^{1/2}$ is bounded. Indeed, Lemma 1.15 shows that

$$\mathcal{P}_n(\delta(T)) = \mathcal{P}_n(\delta'(T) + [f(\mathcal{D}),T]) \le \mathcal{P}_n(\delta'(T)) + 2\|f\|_\infty\, \mathcal{P}_n(T),$$

and similarly that $\mathcal{P}_n(\delta'(T)) \le \mathcal{P}_n(\delta(T)) + 2\|f\|_\infty\, \mathcal{P}_n(T)$. Hence convergence in the topology defined using δ implies convergence in the topology defined by δ', and conversely. Similar comments apply for $\mathcal{B}_2^k(\mathcal{D},p)$.
(3) In Lemma 1.29, we will show that we can also use the seminorms $\mathcal{P}_n \circ L^k$ (and similarly for R^k and $L^k \circ R^j$) to define the topologies of $\mathcal{B}_1^\infty(\mathcal{D},p)$ and $\mathcal{B}_2^\infty(\mathcal{D},p)$.

We begin by proving that the algebra $\mathcal{B}_1^k(\mathcal{D},p)$ is a Fréchet $*$-subalgebra of $\mathcal{N}$.

Proposition 1.22. *For any $n \in \mathbb{N}$, $l = \mathbb{N}_0 \cup \{\infty\}$ and $p \ge 1$, the $*$-algebra $M_n(\mathcal{B}_1^l(\mathcal{D},p))$ is Fréchet and stable under the holomorphic functional calculus.*

Proof. We first regard the question of completeness and treat the case $l = 1$ and $n = 1$ only, since the general case is similar. Let $(T_k)_{k\ge 0}$ be a Cauchy sequence in $\mathcal{B}_1^1(\mathcal{D},p)$. Since

$$\mathcal{P}_{n,1}(T_k - T_l) \ge \mathcal{P}_n\big(\delta(T_k) - \delta(T_l)\big),\ \mathcal{P}_n(T_k - T_l),$$

we see that both $(S_k)_{k\ge0} := (\delta(T_k))_{k\ge0}$ and $(T_k)_{k\ge0}$ are Cauchy sequences in $\mathcal{B}_1(\mathcal{D},p)$. Since $\mathcal{B}_1(\mathcal{D},p)$ is complete, both $(S_k)_{k\ge0}$ and $(T_k)_{k\ge0}$ converge, say to $S \in \mathcal{B}_1(\mathcal{D},p)$ and $T \in \mathcal{B}_1(\mathcal{D},p)$ respectively. Next observe that $\delta : \operatorname{dom}\delta \subset \mathcal{N} \to \mathcal{N}$ is bounded, where we give on $\operatorname{dom}\delta$ the topology determined by the norm $\|\cdot\| + \|\delta(\cdot)\|$. Hence δ has closed graph, and since $T_k \to T$ in norm and $\delta(T_k)$ converges in norm also, we have $S = \delta(T)$. Finally, since $(\delta(T_k))_{k\ge0}$ is Cauchy in $\mathcal{B}_1(\mathcal{D},p)$, we have $S = \delta(T) \in \mathcal{B}_1(\mathcal{D},p)$.

We now pass to the question of stability under holomorphic functional calculus. As before, the proof for $M_n(\mathcal{B}_1^k(\mathcal{D},p))$, will follow from the proof for $\mathcal{B}_1^k(\mathcal{D},p)$. By completeness of $\mathcal{B}_1^k(\mathcal{D},p)$, it is enough to show that for $T \in \mathcal{B}_1^k(\mathcal{D},p)$, $T(1+T)^{-1} \in \mathcal{B}_1^k(\mathcal{D},p)$ (see the proof of Proposition 1.18). But this follows from an iterative use of the relation

$$\delta\big(T(1+T)^{-1}\big) = \delta(T)(1+T)^{-1} - T(1+T)^{-1}\delta(T)(1+T)^{-1},$$

together with Lemma 1.17 and the fact that $\mathcal{B}_1(\mathcal{D},p)$ is an algebra. $\square$

1.4. The pseudodifferential calculus

The pseudodifferential calculus of Connes-Moscovici, [**22, 25**], depends only on an unbounded self-adjoint operator $\mathcal{D}$. In its original form, this calculus characterises those operators which are smooth 'as far as $\mathcal{D}$ is concerned'. In Section 1.2 we saw that we could also talk about operators which are 'integrable as far as $\mathcal{D}$ is concerned'. This latter notion also requires the trace τ and the dimension p. We combine all these ideas in the following definition, to obtain a notion of pseudodifferential operator adapted to the nonunital setting.

Once again, throughout this Section we let $\mathcal{D}$ be a self-adjoint operator affiliated to a semifinite von Neumann algebra $\mathcal{N}$ with faithful normal semifinite trace τ and $p \geq 1$.

Definition 1.23. The set of **order-r tame pseudodifferential operators** associated with $(\mathcal{H}, \mathcal{D})$, $(\mathcal{N}, \tau)$ and $p \geq 1$ is given by

$$\mathrm{OP}_0^r := (1+\mathcal{D}^2)^{r/2}\mathcal{B}_1^\infty(\mathcal{D}, p), \quad r \in \mathbb{R}, \qquad \mathrm{OP}_0^* := \bigcup_{r\in\mathbb{R}} \mathrm{OP}_0^r.$$

We topologise OP_0^r with the family of norms

$$\mathcal{P}^r_{n,l}(T) := \mathcal{P}_{n,l}\big((1+\mathcal{D}^2)^{-r/2}T\big), \quad n \in \mathbb{N}, \quad l \in \mathbb{N}_0. \tag{1.15}$$

Remark. To lighten the notation, we do not make explicit the important dependence on the real number $p \geq 1$ and the operator $\mathcal{D}$ in the definition of the tame pseudodifferential operators.

With this definition, OP_0^r is a Fréchet space and OP_0^0 is a Fréchet $*$-algebra. In Corollary 1.30 we will see that $\bigcup_{r<-p} \mathrm{OP}_0^r \subset \mathcal{L}^1(\mathcal{N}, \tau)$, which is the basic justification for the introduction of tame pseudodifferential operators. However, since $\mathcal{B}_1^\infty(\mathcal{D}, p)$ is a priori a nonunital algebra, functions of $\mathcal{D}$ alone do not belong to OP_0^*. In particular, not all 'differential operators', such as powers of $\mathcal{D}$, are tame pseudodifferential operators.

Definition 1.24. The set of **regular order-r pseudodifferential operators** is

$$\mathrm{OP}^r := (1+\mathcal{D}^2)^{r/2}\Big(\bigcap_{n\in\mathbb{N}} \mathrm{dom}\,\delta^n\Big), \quad r \in \mathbb{R}, \qquad \mathrm{OP}^* := \bigcup_{r\in\mathbb{R}} \mathrm{OP}^r.$$

The natural topology of OP^r is associated with the family of norms

$$\sum_{k=0}^{l} \|\delta^k((1+\mathcal{D}^2)^{-r/2}T)\|, \quad l \in \mathbb{N}_0.$$

By a slight adaptation of Lemma 1.11, we see that $\mathcal{B}_1^\infty(\mathcal{D}, p) \subset \mathcal{B}_2^\infty(\mathcal{D}, p)$ with $\mathcal{Q}_{n,k}(\cdot) \leq \mathcal{P}_{n,k}(\cdot)$ for all $n \geq 1$ and $k \geq 0$. Moreover, we have from the definition that $\mathcal{B}_2^\infty(\mathcal{D}, p) \subset \bigcap_{n\in\mathbb{N}} \mathrm{dom}\,\delta^n$, with $\|\delta^k(\cdot)\| \leq \mathcal{Q}_{n,k}(\cdot)$. Thus $\mathcal{B}_1^\infty(\mathcal{D}, p) \subset \bigcap_{n\in\mathbb{N}} \mathrm{dom}\,\delta^n$, with $\|\delta^k(\cdot)\| \leq \mathcal{P}_{n,k}(\cdot)$. Hence, we have a continuous inclusion $\mathrm{OP}_0^r \subset \mathrm{OP}^r$. For $r > 0$, OP^r contains all polynomials in $\mathcal{D}$ of order smaller than r. In particular, $\mathrm{Id}_\mathcal{N} \in \mathrm{OP}^0$.

To prove that our definition of tame pseudodifferential operators is symmetric, namely that

$$\mathrm{OP}_0^r = (1+\mathcal{D}^2)^{r/2-\theta}\mathcal{B}_1^\infty(\mathcal{D}, p)(1+\mathcal{D}^2)^\theta, \quad \text{for all } \theta \in [0, r/2], \tag{1.16}$$

we introduce σ, the complex one-parameter group of automorphisms of OP^* defined by

$$\sigma^z(T) := (1+\mathcal{D}^2)^{z/2}\, T\, (1+\mathcal{D}^2)^{-z/2}, \quad z \in \mathbb{C},\ T \in \mathrm{OP}^*. \tag{1.17}$$

It is then clear that if we know that σ preserves each OP_0^r, then Equation (1.16) will follow immediately. The next few results show that σ restricts to a group of automorphisms of each OP^r and each OP_0^r, $r \in \mathbb{R}$.

LEMMA 1.25. *There exists $C > 0$ such that for every $T \in \mathcal{B}_1^\infty(\mathcal{D}, p)$ and $\varepsilon \in [0, 1/3]$, we have $\mathcal{P}_n\big([(1+\mathcal{D}^2)^{\varepsilon/2}, T]\big) \le C\, \mathcal{P}_n\big(\delta(T)\big)$.*

PROOF. Let g be a function on $\mathbb{R}$ such that the Fourier transform of g' is integrable. The elementary equality

$$[g(|\mathcal{D}|), T] = -2i\pi \int_{\mathbb{R}} \widehat{g}(\xi)\xi \int_0^1 e^{-2i\pi\xi s|\mathcal{D}|}\, [|\mathcal{D}|, T]\, e^{-2i\pi\xi(1-s)|\mathcal{D}|}\, ds\, d\xi,$$

implies by Lemma 1.15 that

$$\mathcal{P}_n\big([g(|\mathcal{D}|), T]\big) \le \|\widehat{g'}\|_1\, \mathcal{P}_n\big(\delta(T)\big).$$

The estimate $\|\widehat{g'}\|_1 \le \sqrt{2}(\|g'\|_2 + \|g''\|_2)$ is well known. Setting $g_\varepsilon(t) = (1+t^2)^{\varepsilon/2}$, an explicit computation of the associated 2-norms proves that for $\varepsilon \in [0, \frac{1}{2})$ we have

$$\|\widehat{g_\varepsilon'}\|_1 \le \varepsilon\, \pi^{1/4}\Big(\frac{\Gamma(\frac{1}{2}-\varepsilon)^{1/2}}{\Gamma(2-\varepsilon)^{1/2}} + \frac{\sqrt{6}(2-\varepsilon)\Gamma(\frac{3}{2}-\varepsilon)^{1/2}}{2\Gamma(4-\varepsilon)^{1/2}}\Big). \tag{1.18}$$

Since this estimate is uniform in ε on compact subintervals of $[0, \frac{1}{2})$, in particular on $[0, \frac{1}{3}]$ and is independent of $T \in \mathcal{B}_1^\infty(\mathcal{D}, p)$, the assertion follows immediately. $\square$

LEMMA 1.26. *There is a constant $C \ge 1$ such that for all $T \in \mathcal{B}_1^\infty(\mathcal{D}, p)$ and $z \in \mathbb{C}$*

$$\mathcal{P}_{n,l}\big(\sigma^z(T)\big) \le \sum_{k=l}^{\lfloor 3\Re(z)\rfloor + l + 1} C^k \mathcal{P}_{n,k}(T).$$

Thus σ_z preserves $\mathcal{B}_1^\infty(\mathcal{D}, p)$.

PROOF. It is clear that

$$\begin{aligned} \sigma^z(T) &= T + [(1+\mathcal{D}^2)^{z/2}, T](1+\mathcal{D}^2)^{-z/2} \\ &= T + (1+\mathcal{D}^2)^{z/2}[(1+\mathcal{D}^2)^{-z/2}, T]. \end{aligned} \tag{1.19}$$

It follows from Lemma 1.15 and Lemma 1.25 that for $z \in [-1/3, 1/3]$ we have

$$\mathcal{P}_n\big(\sigma^z(T)\big) \le \mathcal{P}_n(T) + C\, \mathcal{P}_n\big(\delta(T)\big) \le C\, \mathcal{P}_{n,1}(T),$$

with the same constant as in Lemma 1.25 (which is thus independent of $T \in \mathcal{B}_1^\infty(\mathcal{D}, p)$ and $z \in \mathbb{C}$). By the group property, we have

$$\mathcal{P}_n\big(\sigma^z(T)\big) \le \sum_{k=0}^{\lfloor 3\Re(z)\rfloor + 1} C^k \mathcal{P}_{n,k}(T),$$

for $z \in \mathbb{R}$, and as σ^z commutes with δ, we have the inequality $\mathcal{P}_{n,l}\big(\sigma^z(T)\big) \le \sum_{k=l}^{\lfloor 3\Re(z)\rfloor + l + 1} C^k \mathcal{P}_{n,k}(T)$ for every $z \in \mathbb{R}$. Finally, as $\sigma^z = \sigma^{i\Im(z)}\sigma^{\Re(z)}$ and $\sigma^{i\Im(z)}$ is isometric for each $\mathcal{P}_{n,l}$ (by Lemma 1.15 again), the assertion follows. $\square$

Proposition 1.27. *The maps $\sigma^z : \mathcal{B}_1^\infty(\mathcal{D},p) \to \mathcal{B}_1^\infty(\mathcal{D},p)$, $z \in \mathbb{C}$, form a strongly continuous group of automorphisms which is uniformly continuous on vertical strips.*

Proof. Fix $T \in \mathcal{B}_1^\infty(\mathcal{D},p)$. We need to prove that the map $z \mapsto \sigma^z(T)$ is continuous from $\mathbb{C}$ to $\mathcal{B}_1^\infty(\mathcal{D},p)$, for the topology determined by the norms $\mathcal{P}_{n,l}$. By Lemma 1.26 we know that σ^z preserves $\mathcal{B}_1^\infty(\mathcal{D},p)$ and since $\{\sigma^z\}_{z\in\mathbb{C}}$ is a group of automorphisms, continuity everywhere will follow from continuity at $z=0$. So, let $z \in \mathbb{C}$ with $|z| \le \frac{1}{3}$. From Equation (1.19), it is enough to treat the case $\Re(z) \ge 0$. Moreover, Lemma 1.15 gives us

$$\mathcal{P}_{n,l}\big(\sigma^z(T) - T\big) \le \mathcal{P}_{n,l}\big([(1+\mathcal{D}^2)^{z/2}, T]\big),$$

and from the same reasoning as that leading to the estimate (1.18), we obtain

$$\begin{aligned}\mathcal{P}_{n,l}\big([(1+\mathcal{D}^2)^{z/2}, T]\big) \le& \\ &|z|\,\pi^{1/4}\Big(\frac{\Gamma(\frac{1}{2}-|\Re(z)|)^{1/2}}{\Gamma(2-|\Re(z)|)^{1/2}} + \frac{\sqrt{6}(2-|\Re(z)|)\Gamma(\frac{3}{2}-|\Re(z)|)^{1/2}}{2\Gamma(4-|\Re(z)|)^{1/2}}\Big)\ \mathcal{P}_{n,l+1}(T)\\ &=: |z|\,C(z).\end{aligned}$$

Since $C(z)$ is uniformly bounded on the vertical strip $0 \le \Re(z) \le \frac{1}{3}$, we obtain the result. □

Remark. Using Lemma 1.7 in place of Lemma 1.15, we see that Lemmas 1.25, 1.26 and Proposition 1.27 hold also with $\mathcal{B}_2^\infty(\mathcal{D},p)$ instead of $\mathcal{B}_1^\infty(\mathcal{D},p)$.

We now deduce that these continuity results also hold for both tame and regular pseudodifferential operators.

Proposition 1.28. *The group σ is strongly continuous on OP_0^r for its natural topology, and similarly for OP^r.*

Proof. Since $T \in \mathrm{OP}_0^r$ if and only if $(1+\mathcal{D}^2)^{-r/2}T \in \mathcal{B}_1^\infty(\mathcal{D},p)$ and since σ^z commutes with the left multiplication by $(1+\mathcal{D}^2)^{-r/2}$, the proof is a direct corollary of Proposition 1.27. The proof for OP^r is simpler since it uses only the operator norm and not the norms $\mathcal{P}_n^r$; we refer to [**15, 22, 25**] for a proof. □

We can now show that $\mathcal{B}_1^\infty(\mathcal{D},p)$ has an equivalent definition in terms of the L and/or R operators, defined in Equation (1.11). Unlike the equivalent definition in terms of δ' mentioned in the remark after Definition 1.21, this does not work for $\mathcal{B}_1^k(\mathcal{D},p)$, $k \neq \infty$.

Lemma 1.29. *We have the equality*

$$\mathcal{B}_1^\infty(\mathcal{D},p) = \big\{T \in \mathcal{N} : \forall l \in \mathbb{N}_0,\ \ L^l(T) \in \mathcal{B}_1(\mathcal{D},p)\big\},$$

where $L(\cdot) = (1+\mathcal{D}^2)^{-1/2}[\mathcal{D}^2,\cdot]$ is as in Definition 1.20. The analogous statement with R replacing L is also true.

Proof. We have the simple identity $L = (1+\sigma^{-1})\circ\delta'$, which with Proposition 1.27 yields one of the inclusions. For the other direction, it suffices to show that for every $m, n \in \mathbb{N}$ we have

$$\mathcal{P}_m(\delta'^n(A)) \le \max_{n\le k\le 2n} \mathcal{P}_m(L^k(A)).$$

Using the integral formula for fractional powers we have

$$\delta'(T) = [(1+\mathcal{D}^2)(1+\mathcal{D}^2)^{-1/2}, T] = \frac{1}{\pi}\int_0^\infty \lambda^{-1/2}[(1+\mathcal{D}^2)(1+\lambda+\mathcal{D}^2)^{-1}, T]d\lambda.$$

However, a little algebra gives

$$\Big[\frac{1+\mathcal{D}^2}{1+\lambda+\mathcal{D}^2}, T\Big] = \Big(\frac{(1+\mathcal{D}^2)^{1/2}}{1+\lambda+\mathcal{D}^2} - \frac{(1+\mathcal{D}^2)^{3/2}}{(1+\lambda+\mathcal{D}^2)^2}\Big)L(T) + \lambda\frac{1+\mathcal{D}^2}{(1+\lambda+\mathcal{D}^2)^2}L^2(T)\frac{1}{1+\lambda+\mathcal{D}^2}.$$

The following formula can be proved in the scalar case, and by an appeal to the spectral representation proved in general:

$$\int_0^\infty \lambda^{-1/2}\Big(\frac{(1+\mathcal{D}^2)^{1/2}}{1+\lambda+\mathcal{D}^2} - \frac{(1+\mathcal{D}^2)^{3/2}}{(1+\lambda+\mathcal{D}^2)^2}\Big)d\lambda = \frac{\pi}{2}.$$

Therefore,

$$\delta'(T) = \tfrac{1}{2}L(T) + \frac{1}{\pi}\int_0^\infty \lambda^{1/2}\frac{1+\mathcal{D}^2}{(1+\lambda+\mathcal{D}^2)^2}L^2(T)\frac{1}{1+\lambda+\mathcal{D}^2}d\lambda.$$

An induction now shows that

$$\delta'^n(T) = 2^{-n}\sum_{k=0}^n \binom{n}{k}\Big(\frac{2}{\pi}\Big)^k \int_{\mathbb{R}_+^k}\prod_{j=1}^k \frac{\lambda_j^{1/2}(1+\mathcal{D}^2)}{(1+\lambda_j+\mathcal{D}^2)^2}L^{n+k}(T)\prod_{j=1}^k\frac{d\lambda_j}{1+\lambda_j+\mathcal{D}^2}.$$

The functional calculus then gives

$$(1+\lambda+\mathcal{D}^2)^{-1} \le (1+\lambda)^{-1}, \qquad \lambda^{1/2}(1+\mathcal{D}^2)(1+\lambda+\mathcal{D}^2)^{-2} \le \lambda^{-1/2}/4,$$

and so by Lemma 1.15 we have

$$\mathcal{P}_m\big(\delta'^n(T)\big) \le 2^{-n}\Big(1+\sum_{k=1}^n\binom{n}{k}\Big(\frac{2}{\pi}\Big)^k\prod_{j=1}^k\int_0^\infty\frac{d\lambda_j}{4\lambda_j^{1/2}(1+\lambda_j)}\Big)\max_{n\le k\le 2n}\mathcal{P}_m\big(L^k(T)\big).$$

The assertion now follows by the second remark following Definition 1.21 that we may equivalently use δ' to define $\mathcal{B}_1^k(\mathcal{D},p)$ for $k\in\mathbb{N}\cup\{\infty\}$. □

We now begin to prove the important properties of this pseudodifferential calculus, such as trace-class properties and the pseudodifferential expansion. First, by combining Proposition 1.28 with the Definition 1.23, we obtain our first trace class property.

Corollary 1.30. *For $r>p$, we have $\mathrm{OP}_0^{-r}\subset\mathcal{L}^1(\mathcal{N},\tau)$.*

Proof. Let $T_r\in\mathrm{OP}_0^{-r}$. By Definition 1.23 and Proposition 1.28, we see that the symmetric definition of OP_0^r in Equation (1.16) is equivalent to the original definition. Thus, there exists $A\in\mathcal{B}_1^\infty(\mathcal{D},p)\subset\mathcal{B}_1(\mathcal{D},p)$ such that

$$T_r = (1+\mathcal{D}^2)^{-r/4}A(1+\mathcal{D}^2)^{-r/4}.$$

Define $n:=\lfloor (r-p)^{-1}\rfloor$ and write $A=\sum_{k=0}^3 i^kA_k$ with $A_k\in\mathcal{B}_1(\mathcal{D},p)$ positive, as in Proposition 1.14. The Hölder inequality then entails that

$$\|T_r\|_1 = \|(1+\mathcal{D}^2)^{-r/4}A(1+\mathcal{D}^2)^{-r/4}\|_1 \le \|(1+\mathcal{D}^2)^{-p/4-1/4n}A(1+\mathcal{D}^2)^{-p/4-1/4n}\|_1$$
$$\le \sum_{k=0}^3\big\|(1+\mathcal{D}^2)^{-p/4-1/4n}\sqrt{A_k}\big\|_2^2 \le \sum_{k=0}^3\mathcal{Q}_n\big(\sqrt{A_k}\big)\mathcal{Q}_n\big(\sqrt{A_k}\big) = \sum_{k=0}^3\mathcal{P}_n(A_k),$$

which is finite, and so allows us to conclude. □

As expected, the product of a tame pseudodifferential operator by a regular pseudodifferential operator is a tame pseudodifferential operator.

LEMMA 1.31. *For all* $r,\, t \in \mathbb{R}$ *we have* $\big(\mathrm{OP}_0^r\,\mathrm{OP}^t \cup \mathrm{OP}^t\,\mathrm{OP}_0^r\big) \subset \mathrm{OP}_0^{r+t}$.

PROOF. Since σ preserves both OP_0^r and OP^r, it suffices to prove the claim for $r = t = 0$. Indeed, for $T_r \in \mathrm{OP}_0^r$ and $T_s \in \mathrm{OP}^s$, there exist $A \in \mathrm{OP}_0^0$ and $B \in \mathrm{OP}^0$ such that $T_r = (1+\mathcal{D}^2)^{r/2}A$ and $T_s = (1+\mathcal{D}^2)^{s/2}B$. Thus, the general case will follow from the case $t = s = 0$ by writing

$$T_rT_s = (1+\mathcal{D}^2)^{(r+s)/2}\sigma^{-s}(A)B.$$

So let $T \in \mathrm{OP}_0^0$ and $S \in \mathrm{OP}^0$. We need to show that $TS \in \mathrm{OP}_0^0 = \mathcal{B}_1^\infty(\mathcal{D},p)$. For this, let $T = \sum_{i=0}^\infty T_{1,i}T_{2,i}$ any representation. We will prove that

$$\sum_{i=0}^\infty T_{1,i}\,(T_{2,i}S),$$

is a representation of the product TS. Indeed, we have

$$\begin{aligned}
\mathcal{Q}_n(T_{2,i}S)^2 &= \|T_{2,i}S\|^2 + \|T_{2,i}S(1+\mathcal{D}^2)^{-p/4-1/4n}\|_2^2 + \|S^*T_{2,i}^*(1+\mathcal{D}^2)^{-p/4-1/4n}\|_2^2\\
&\le \|S\|^2\|T_{2,i}\|^2 + \|\sigma^{p/4+1/4n}(S)\|^2\|T_{2,i}(1+\mathcal{D}^2)^{-p/4-1/4n}\|_2^2\\
&\qquad\qquad + \|S\|^2\|T_{2,i}^*(1+\mathcal{D}^2)^{-p/4-1/4n}\|_2^2\\
&\le \big(\|S\| + \|\sigma^{p/4+1/4n}(S)\|\big)^2\mathcal{Q}_n(T_{2,i})^2,
\end{aligned}$$

which is finite because $\mathrm{OP}^0 = \bigcap_{n\in\mathbb{N}}\operatorname{dom}\delta^n$ is invariant under σ by Proposition 1.28. This immediately shows that $TS \in \mathcal{B}_1(\mathcal{D},p)$ since

$$\begin{aligned}
\mathcal{P}_n(TS) &\le \sum_{i=0}^\infty \mathcal{Q}_n(T_{1,i})\,\mathcal{Q}_n(T_{2,i}S)\\
&\le \big(\|S\| + \|\sigma^{p/4+1/4n}(S)\|\big)\sum_{i=0}^\infty \mathcal{Q}_n(T_{1,i})\,\mathcal{Q}_n(T_{2,i}) < \infty.
\end{aligned}$$

In particular, one finds $\mathcal{P}_n(TS) \le \big(\|S\| + \|\sigma^{p/4+1/4n}(S)\|\big)\mathcal{P}_n(T)$. Now the formula $\delta^k(TS) = \sum_{j=0}^k\binom{k}{j}\delta^j(T)\delta^{k-j}(S)$ and the last estimate shows that $\mathcal{P}_{n,k}(TS) = \mathcal{P}_n(\delta^k(TS))$ is finite and so $TS \in \mathcal{B}_1^\infty(\mathcal{D},p)$. That $\mathrm{OP}^t\,\mathrm{OP}_0^r \subset \mathrm{OP}_0^{r+t}$ can be proven in the same way. □

Remark. Lemma 1.31 shows that $\mathcal{B}_1^\infty(\mathcal{D},p)$ is a two-sided ideal in $\bigcap\operatorname{dom}\delta^k$.

The following is a Taylor-expansion type theorem for OP_0^r just as in [**22**, **25**], and adapted to our setting.

PROPOSITION 1.32. *Let* $T \in \mathrm{OP}_0^r$ *and* $z = n+1-\alpha$ *with* $n \in \mathbb{N}_0$ *and* $\Re(\alpha) \in (0,1)$. *Then we have*

$$\sigma^{2z}(T) - \sum_{k=0}^n C_k(z)\,(\sigma^2 - \mathrm{Id})^k(T) \in \mathrm{OP}_0^{r-n-1},$$

with

$$C_k(z) := \frac{z(z-1)\dots(z-k+1)}{k!}.$$

PROOF. The proof is exactly the same as that in [**22**, **25**] once we realise that if $T \in \mathrm{OP}_0^r$ then $(\sigma^2 - \mathrm{Id})^k(T) \in \mathrm{OP}_0^{r-k}$. This follows from

$$(\sigma^2 - \mathrm{Id})^k(T) = (1+\mathcal{D}^2)^{-k/2}\sigma^k\big(\delta'^k(T)\big),$$

and the invariance of each OP_0^r under $\delta' = [(1+\mathcal{D}^2)^{1/2},\cdot]$ and σ. For δ' this follows from the second remark following Definition 1.21. □

LEMMA 1.33. *If $A \in \mathrm{OP}_0^r$ and $n \in \mathbb{N}_0$, then $A^{(n)} \in \mathrm{OP}_0^{r+n}$, where $A^{(n)}$ is as in Definition 1.20.*

PROOF. For $n = 1$, by the assumption there is an operator $T \in \mathrm{OP}_0^0$ such that $A = (1+\mathcal{D}^2)^{r/2}T$. Then $A^{(1)} = (1+\mathcal{D}^2)^{r/2}T^{(1)} = (1+\mathcal{D}^2)^{(r+1)/2}L(T)$. So the proof follows from the relation $L = (1+\sigma^{-1})\circ\delta'$ and the fact that both σ^{-1} and δ' preserve OP_0^0, by Lemma 1.26. The general case follows by induction. □

PROPOSITION 1.34. *The derivation LD defined by $LD(T) := [\log(1+\mathcal{D}^2), T]$, preserves OP_0^r, for all $r \in \mathbb{R}$.*

PROOF. Set $g(t) = \log(1+t^2)$. We have $\|\widehat{g'}\|_1 < \infty$ and

$$LD(T) = [g(|\mathcal{D}|), T] = -2i\pi\int_{\mathbb{R}}\widehat{g}(\xi)\xi\int_0^1 e^{-2i\pi\xi s|\mathcal{D}|}\,\delta(T)\,e^{-2i\pi\xi(1-s)|\mathcal{D}|}\,ds\,d\xi.$$

The assertion follows as in Lemma 1.25. □

We next improve Proposition 1.28.

PROPOSITION 1.35. *The map $\sigma : \mathbb{C}\times\mathrm{OP}_0^r \to \mathrm{OP}_0^r$, is strongly holomorphic (entire), with*

$$\frac{d}{dz}\sigma^z = \tfrac{1}{2}\sigma^z\circ LD.$$

PROOF. If $z - z_0 = u$, then we have

$$\Big(\frac{\sigma^z - \sigma^{z_0}}{z - z_0} - \tfrac{1}{2}\sigma^{z_0}\circ LD\Big) = \sigma^{z_0}\circ\Big(\frac{\sigma^u - 1}{u} - \tfrac{1}{2}LD\Big).$$

Since σ^{z_0} is strongly continuous, it is sufficient to prove holomorphy at $z_0 = 0$. Then for $T \in \mathrm{OP}_0^r$ we see that

$$\begin{aligned}\frac{\sigma^z(T) - T}{z} &- \tfrac{1}{2}LD(T)\\ &= [g_z(\mathcal{D}), T] + z^{-1}[(1+\mathcal{D}^2)^{z/2}, T]\big((1+\mathcal{D}^2)^{-z/2} - 1\big),\end{aligned} \tag{1.20}$$

with $g_z(s) = z^{-1}\big((1+s^2)^{z/2}-1\big) - \frac{1}{2}\log(1+s^2)$. An explicit computation shows that $\|g_z'\|_2 + \|g_z''\|_2 = O(|z|)$. Since $\sqrt{2}(\|g_z'\|_2 + \|g_z''\|_2) \geq \|\widehat{g_z'}\|_1$, we see that $\|\widehat{g_z'}\|_1 \to 0$ as $z \to 0$. It follows, as in Lemma 1.25, that the first term tends to 0 in the $\mathcal{P}_{n,l}^r$-norms, as $z \to 0$. It remains to treat the second commutator in Equation (1.20). We let $z \in \mathbb{C}$ with $0 < \Re(z) < 1$. Employing the integral formula for complex powers of a positive operator $A \in \mathcal{N}$

$$A^z = \pi^{-1}\sin(\pi z)\int_0^\infty \lambda^{-z}A(1+\lambda A)^{-1}d\lambda,\quad 0 < \Re(z) < 1, \tag{1.21}$$

gives

$$\begin{aligned}(1+\mathcal{D}^2)^{-z/2} &= \Big((1+\mathcal{D}^2)^{-1/2}\Big)^z \\ &= \pi^{-1}\sin(\pi z)\int_0^\infty \lambda^{-z}(1+\mathcal{D}^2)^{-1/2}(1+\lambda(1+\mathcal{D}^2)^{-1/2})^{-1}d\lambda \\ &= \pi^{-1}\sin(\pi z)\int_0^\infty \lambda^{-z}((1+\mathcal{D}^2)^{1/2}+\lambda)^{-1}d\lambda.\end{aligned}$$

We apply this formula by choosing $0<\varepsilon<(1-\Re(z))$ and writing

$$\begin{aligned}&\frac{1}{z}[(1+\mathcal{D}^2)^{z/2},T]\big((1+\mathcal{D}^2)^{-z/2}-1\big) \\ &= -\frac{1}{z}(1+\mathcal{D}^2)^{z/2}[(1+\mathcal{D}^2)^{-z/2},T](1+\mathcal{D}^2)^{z/2}\big((1+\mathcal{D}^2)^{-z/2}-1\big) \\ &= \frac{\sin(\pi z)}{\pi z}\int_0^\infty \lambda^{-z}(1+\mathcal{D}^2)^{z/2}((1+\mathcal{D}^2)^{1/2}+\lambda)^{-1}\delta'(T)((1+\mathcal{D}^2)^{1/2}+\lambda)^{-1} \\ &\qquad\qquad \times(1+\mathcal{D}^2)^{(z+\varepsilon)/2}(1+\mathcal{D}^2)^{-\varepsilon/2}\big((1+\mathcal{D}^2)^{-z/2}-1\big)d\lambda.\end{aligned}$$

Using the elementary estimate

$$\|((1+\mathcal{D}^2)^{1/2}+\lambda)^{-1}(1+\mathcal{D}^2)^{z/2}\|_\infty \le (1+\lambda)^{\Re(z)-1},$$

we have

$$\begin{aligned}&\mathcal{P}^r_{n,l}\Big(\frac{1}{z}[(1+\mathcal{D}^2)^{z/2},T]\big((1+\mathcal{D}^2)^{-z/2}-1\big)\Big) \le \\ &\qquad \frac{|\sin(\pi z)|}{\pi}\,\mathcal{P}^r_{n,l}(\delta'(T))\,\Big\|\frac{1}{z}(1+\mathcal{D}^2)^{-\varepsilon/2}\big((1+\mathcal{D}^2)^{-z/2}-1\big)\Big\|_\infty \\ &\qquad\qquad \times\int_0^\infty \lambda^{-\Re(z)}(1+\lambda)^{2\Re(z)-2+\varepsilon}d\lambda.\end{aligned}$$

This concludes the proof since, as $0<\Re(z)<1-\varepsilon$, the last norm is bounded in a neighborhood of $z=0$, while the integral over λ is bounded (provided ε is small enough) and $|\sin(\pi z)|$ goes to zero with z. $\square$

Last, we prove that the derivation $LD(\cdot)=[\log(1+\mathcal{D}^2),\cdot]$ 'almost' lowers the order of a tame pseudodifferential operator by one.

PROPOSITION 1.36. *For all $r\in\mathbb{R}$ and for any $\varepsilon\in(0,1)$, LD continuously maps OP^r_0 to $\mathrm{OP}^{r-1+\varepsilon}_0$.*

PROOF. Since the proof for a generic $r\in\mathbb{R}$ will follows from those of a fixed $r_0\in\mathbb{R}$, we may assume that $r=0$. Let $T\in\mathrm{OP}^0_0$. We need to show that $LD(T)\in\mathrm{OP}^{-1+\varepsilon}_0$ for any $\varepsilon>0$, or equivalently, that $LD(T)(1+\mathcal{D}^2)^{1/2-\varepsilon/2}\in\mathrm{OP}^0_0$ for any $\varepsilon>0$. We use the integral representation

$$\log(1+\mathcal{D}^2)=\mathcal{D}^2\int_0^1(1+w\mathcal{D}^2)^{-1}\,dw,$$

which follows from $\log(1+x) = \int_0^x \frac{1}{1+\lambda}\,d\lambda$ via the change of variables $\lambda = xw$. Then

$$[\log(1+\mathcal{D}^2), T](1+\mathcal{D}^2)^{1/2-\varepsilon/2} = [\mathcal{D}^2, T](1+\mathcal{D}^2)^{-1/2}\int_0^1 \frac{(1+\mathcal{D}^2)^{1-\varepsilon/2}}{1+w\mathcal{D}^2}\,dw$$
$$- \mathcal{D}^2\int_0^1 \frac{w}{1+w\mathcal{D}^2}[\mathcal{D}^2, T](1+\mathcal{D}^2)^{-1/2}\frac{(1+\mathcal{D}^2)^{1-\varepsilon/2}}{1+w\mathcal{D}^2}\,dw.$$

Now elementary calculus shows that for $0<\alpha<1$ and $0\le x\le 1$ we have

$$\frac{(1+x)^\alpha}{(1+xw)} \le \left(\frac{\alpha}{w}\right)^\alpha\left(\frac{1-\alpha}{1-w}\right)^{1-\alpha} \quad\text{and}\quad \int_0^1 w^{-\alpha}(1-w)^{\alpha-1}dw = \Gamma(1-\alpha)\,\Gamma(\alpha),$$

and so we obtain the integral estimate

$$\int_0^1 \frac{(1+x)^\alpha}{(1+xw)}dw \le \alpha^\alpha\,(1-\alpha)^{1-\alpha}\,\Gamma(1-\alpha)\,\Gamma(\alpha).$$

Then using $R(T) = [\mathcal{D}^2, T](1+\mathcal{D}^2)^{-1/2}$ and elementary spectral theory gives

$$\mathcal{P}_{n,k}\Big([\log(1+\mathcal{D}^2), T](1+\mathcal{D}^2)^{1/2-\varepsilon/2}\Big) \le$$
$$2\mathcal{P}_{n,k}(R(T))\,(1-\varepsilon/2)^{1-\varepsilon/2}\,(\varepsilon/2)^{\varepsilon/2}\,\Gamma(\varepsilon/2)\,\Gamma((1-\varepsilon)/2),$$

which gives the bound for all $0<\varepsilon<1$. □

1.5. Schatten norm estimates for tame pseudodifferential operators

In this Section we prove the Schatten norm estimates we will require in our proof of the local index formula. As before, we let $\mathcal{D}$ be a self-adjoint operator affiliated to a semifinite von Neumann algebra $\mathcal{N}$ with faithful normal semifinite trace τ and $p\ge 1$.

LEMMA 1.37. *Suppose that $A\in \mathrm{OP}_0^0$ and $\alpha,\beta\ge 0$ with $\alpha+\beta>0$. Then $(1+\mathcal{D}^2)^{-\beta/2}A(1+\mathcal{D}^2)^{-\alpha/2}$ belongs to $\mathcal{L}^q(\mathcal{N},\tau)$ for all $q>p/(\alpha+\beta)$, provided $q\ge 1$.*

PROOF. Since $(1+\mathcal{D}^2)^{-\beta/2}A(1+\mathcal{D}^2)^{-\alpha/2} = \sigma^{-\beta}(A)(1+\mathcal{D}^2)^{-\alpha/2-\beta/2}$ and because σ is continuous, Proposition 1.27, on $\mathrm{OP}_0^0 = \mathcal{B}_1^\infty(\mathcal{D},p)$ we can assume $\beta=0$.

So let $A\in\mathrm{OP}_0^0$. Note first that for $y\in\mathbb{R}$ we have $A(1+\mathcal{D}^2)^{iy/2}\in\mathcal{N}$ and by Corollary 1.30 $A(1+\mathcal{D}^2)^{-\alpha q/2+iy/2}\in\mathcal{L}^1(\mathcal{N},\tau)$, since $\alpha q>p$. Consider then, on the strip $0\le\Re(z)\le 1$ the holomorphic operator-valued function given by $F(z) := A(1+\mathcal{D}^2)^{-\alpha q z/2}$. The previous observation gives $F(iy)\in\mathcal{N}$ and $F(1+iy)\in\mathcal{L}^1(\mathcal{N},\tau)$. Then, a standard complex interpolation argument gives $F(1/q+iy)\in\mathcal{L}^q(\mathcal{N},\tau)$, for $q\ge 1$, which was all we needed. □

LEMMA 1.38. *For $\alpha\in[0,1]$, $\beta,\gamma\in\mathbb{R}$ with $\alpha+\beta+\gamma>0$ and $A\in\mathrm{OP}_0^0$ we let*

$$B_{\alpha,\beta,\gamma} := (1+\mathcal{D}^2)^{-\beta/2}\big[(1+\mathcal{D}^2)^{(1-\alpha)/2}, A\big](1+\mathcal{D}^2)^{-\gamma/2},$$
$$C_{\alpha,\beta,\gamma} := (1+\mathcal{D}^2)^{-\beta/2}\big[(1+\mathcal{D}^2)^{(1-\alpha)/2}, A\big](1+\mathcal{D}^2)^{-\gamma/2}\log(1+\mathcal{D}^2),$$
$$D_{\alpha,\beta,\gamma} := (1+\mathcal{D}^2)^{-\beta/2}\big[(1+\mathcal{D}^2)^{(1-\alpha)/2}\log(1+\mathcal{D}^2), A\big](1+\mathcal{D}^2)^{-\gamma/2}.$$

Then, provided $q\ge 1$, we find that $B_{\alpha,\beta,\gamma}, C_{\alpha,\beta,\gamma}, D_{\alpha,\beta,\gamma}\in\mathcal{L}^q(\mathcal{N},\tau)$ for all $q>p/(\alpha+\beta+\gamma)$. Moreover, the same conclusion holds with $|\mathcal{D}|$ instead of $(1+\mathcal{D}^2)^{1/2}$ in the commutator.

PROOF. There exists $\varepsilon > 0$ such $\alpha+\beta+\gamma-\varepsilon > 0$. Since $(1+\mathcal{D}^2)^{-\varepsilon/2}\log(1+\mathcal{D}^2)$ is bounded for all $\varepsilon > 0$, we see that the assertion for $B_{\alpha,\beta,\gamma-\varepsilon/2}$ implies the assertion for $C_{\alpha,\beta,\gamma}$. Note also that the Leibniz rule implies

$$D_{\alpha,\beta,\gamma} = C_{\alpha,\beta,\gamma} + (1+\mathcal{D}^2)^{1/2-(\alpha+\beta)/2} LD(A)(1+\mathcal{D}^2)^{-\gamma/2},$$

so the third case follows from the second case using Proposition 1.36 and Lemma 1.37. Thus it suffices to treat the case of $B_{\alpha,\beta,\gamma}$. Moreover, we can further assume that $\alpha \in (0,1)$ (for $\alpha = 1$ there is nothing to prove and for $\alpha = 0$, the statement follows from Lemma 1.37) and, as in the proof of the preceding lemma, we can assume $\beta = 0$. Using the integral formula for fractional powers, Equation (1.21), for $0 < \alpha < 1$, we see that

$$\begin{aligned} B_{\alpha,0,\gamma} &= -(1+\mathcal{D}^2)^{(1-\alpha)/2}[(1+\mathcal{D}^2)^{(\alpha-1)/2}, A](1+\mathcal{D}^2)^{(1-\alpha)/2}(1+\mathcal{D}^2)^{-\gamma/2} \\ &= \pi^{-1}\sin\pi(1-\alpha)/2 \int_0^\infty \lambda^{(1-\alpha)/2}(1+\mathcal{D}^2)^{(1-\alpha)/2}(1+\mathcal{D}^2+\lambda)^{-1} \\ &\qquad\qquad \times [\mathcal{D}^2, A](1+\mathcal{D}^2+\lambda)^{-1}(1+\mathcal{D}^2)^{(1-\alpha-\gamma)/2} d\lambda \\ &= \pi^{-1}\sin\pi(1-\alpha)/2 \int_0^\infty \lambda^{(1-\alpha)/2}(1+\mathcal{D}^2)^{1-\alpha/2}(1+\mathcal{D}^2+\lambda)^{-1} \\ &\qquad\qquad \times L(A)(1+\mathcal{D}^2)^{(\varepsilon-\alpha-\gamma)/2}(1+\mathcal{D}^2+\lambda)^{-1}(1+\mathcal{D}^2)^{(1-\varepsilon)/2} d\lambda. \end{aligned}$$

By Lemma 1.37 we see that for $\varepsilon > 0$ sufficiently small, $L(A)(1+\mathcal{D}^2)^{(\varepsilon-\alpha-\gamma)/2} \in \mathcal{L}^q(\mathcal{N},\tau)$ for all $q > p/(\alpha+\gamma-\varepsilon)$ provided $q \geq 1$. So estimating in the q norm with $q := p/(\alpha+\gamma-2\varepsilon) > p/(\alpha+\gamma-\varepsilon)$ gives

$$\|B_{\alpha,0,\gamma}\|_q \leq \|L(A)(1+\mathcal{D}^2)^{(\varepsilon-\alpha-\gamma)/2}\|_q \int_0^\infty \lambda^{-(1-\alpha)/2}(1+\lambda)^{-\alpha/2}(1+\lambda)^{-1/2-\varepsilon/2}\, d\lambda,$$

which is finite. Finally, the same conclusion holds with $|\mathcal{D}|$ instead of $(1+\mathcal{D}^2)^{1/2}$ in the commutator, and this follows from the same estimates and the fact that $|\mathcal{D}|^{1-\alpha} - (1+\mathcal{D}^2)^{(1-\alpha)/2}$ extends to a bounded operator for $\alpha \in [0,1]$. □

In the course of our proof of the local index formula, we will require additional parameters. In the following lemma we use the same notation as later in the paper for ease of reference.

LEMMA 1.39. *Assume that there exists $\mu > 0$ such that $\mathcal{D}^2 \geq \mu^2$. Let $A \in \mathrm{OP}_0^0$, $\lambda = a + iv$, $0 < a < \mu^2/2$, $v \in \mathbb{R}$, $s \in \mathbb{R}$ and $t \in [0,1]$, and set*

$$R_{s,t}(\lambda) = (\lambda - (t + s^2 + \mathcal{D}^2))^{-1}.$$

Let also $q \in [1,\infty)$ and $N_1, N_2 \in \frac{1}{2}\mathbb{N} \cup \{0\}$, with $N_1 + N_2 > p/2q$. Then for each $\varepsilon > 0$, there exists a finite constant C such that

$$\|R_{s,t}(\lambda)^{N_1} A R_{s,t}(\lambda)^{N_2}\|_q \leq C((t+\mu^2/2+s^2-a)^2+v^2)^{-(N_1+N_2)/2+p/4q+\varepsilon}.$$

(For half integers, we use the principal branch of the square root function).

Remark. In Section 3.2, we will be integrating operator valued functions along the contour $\ell = \{a + iv : v \in \mathbb{R}\}$. The trace estimates above are where we require $0 < a < \mu^2/2$ in the definition of our contour of integration ℓ. It is clear from the proof below, where this condition is used, that there is some flexibility to reformulate this condition.

Proof. By the functional calculus (see the proof of [**15**, Lemmas 5.2 & 5.3] for more details) and the fact that $a < \mu^2/2$, we have the operator inequalities for any $N \in \frac{1}{2}\mathbb{N} \cup \{0\}$ and $Q < N$

$$|R_{s,t}(\lambda)^N| \leq (\mathcal{D}^2 - \mu^2/2)^{-Q} \left((t + \mu^2/2 + s^2 - a)^2 + v^2\right)^{-N/2+Q/2},$$

which gives the following estimate

$$\begin{aligned}\|R_{s,t}(\lambda)^{N_1} A R_{s,t}(\lambda)^{N_2}\|_q &\leq \|R_{s,t}(\lambda)^{N_1}(\mathcal{D}^2 - \mu^2/2)^{Q_1}\| \|R_{s,t}(\lambda)^{N_2}(\mathcal{D}^2 - \mu/2)^{Q_2}\| \\ &\qquad \times \|(\mathcal{D}^2 - \mu^2/2)^{-Q_1} A (\mathcal{D}^2 - \mu^2/2)^{-Q_2}\|_q \\ &\leq ((t + \mu^2/2 + s^2 - a)^2 + v^2)^{-(N_1+N_2)/2+(Q_1+Q_2)/2} \\ &\qquad \times \|(\mathcal{D}^2 - \mu^2/2)^{-Q_1} A (\mathcal{D}^2 - \mu^2/2)^{-Q_2}\|_q.\end{aligned}$$

One concludes the proof using Lemma 1.37 by choosing $Q_1 \leq N_1$, $Q_2 \leq N_2$ such that $Q_1 + Q_2 = p/2q + \varepsilon$. □

Remark. For $\lambda = 0$ and with the same constraints on q and N as above, the same operator inequalities as those of [**17**, Lemma 5.10], gives

$$\begin{aligned}&\|A(t + s^2 + \mathcal{D}^2)^{-N}\|_q \\ &\qquad \leq \|A(\mathcal{D}^2 - \mu^2/2)^{-(p/q+\varepsilon)/2}\|_q (\mu^2/2 + s^2)^{-N+(p/2q+\varepsilon)}.\end{aligned} \tag{1.22}$$

CHAPTER 2

Index Pairings for Semifinite Spectral Triples

In this Chapter we define the notion of a *smoothly summable semifinite spectral triple* $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ relative to a semifinite von Neumann algebra with faithful normal semifinite trace $(\mathcal{N}, \tau)$, and show that such a spectral triple produces, via Kasparov theory, a well-defined numerical index pairing with $K_*(\mathcal{A})$, the K-theory of $\mathcal{A}$.

The 'standard case' of spectral triples with $(\mathcal{N}, \tau) = (\mathcal{B}(\mathcal{H}), \mathrm{Tr})$ for some separable Hilbert space $\mathcal{H}$, is presented in [**20**]. In this case there is an associated Fredholm module, and hence K-homology class. Then there is a pairing between K-theory and K-homology, integer valued in this case, that is well-defined and explained in detail in [**33**, Chapter 8]. The discussion in [**33**] applies to both the unital and nonunital situations. The extension of [**33**, Chapter 8] to deal with both the semifinite situation and nonunitality require some refinements that are not difficult, but are worth making explicit to the reader for the purpose of explaining the basis of our approach.

When the spectral triple is semifinite *and* has $(1+\mathcal{D}^2)^{-s/2} \in \mathcal{L}^1(\mathcal{N}, \tau)$ for all $s > p \geq 1$, for some p, then there is an analytic formula for the index pairing, given in terms of the $\mathbb{R}$-valued index of suitable τ-Fredholm operators, [**4**, **12**, **13**, **16**].

However, for a semifinite spectral triple with $(1+\mathcal{D}^2)^{-1/2}$ not τ-compact, we need a different approach, and so we follow the route indicated in [**35**]. There it is shown that we can associate a Kasparov module, and so a KK-class, to a semifinite spectral triple. This gives us a well-defined pairing with $K_*(\mathcal{A})$ via the Kasparov product, with and modulo some technicalities, this pairing takes values in $K_0(\mathcal{K}_\mathcal{N})$, the K-theory of the τ-compact operators $\mathcal{K}_\mathcal{N}$ in $\mathcal{N}$. Composing this pairing with the map on $K_0(\mathcal{K}_\mathcal{N})$ induced by the trace τ gives us a numerical index which computes the usual index when the triple is 'unital'. When we specialise to particular representatives of our Kasparov class, we will see that we are also computing the $\mathbb{R}$-valued indices of suitable τ-Fredholm operators.

2.1. Basic definitions for spectral triples

In this Section, we give the minimal definition for a semifinite spectral triple, in order to have a Kasparov (and also Fredholm) module. Recall that we denote by $\mathcal{K}(\mathcal{N}, \tau)$, or $\mathcal{K}_\mathcal{N}$ when τ is understood, the ideal of τ-compact operators in $\mathcal{N}$. This is the norm closed ideal in $\mathcal{N}$ generated by projections with finite trace.

Definition 2.1. A semifinite spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$, relative to $(\mathcal{N}, \tau)$, is given by a Hilbert space $\mathcal{H}$, a $*$-subalgebra $\mathcal{A} \subset \mathcal{N}$ acting on $\mathcal{H}$, and a densely defined unbounded self-adjoint operator $\mathcal{D}$ affiliated to $\mathcal{N}$ such that:

1. $a \cdot \mathrm{dom}\,\mathcal{D} \subset \mathrm{dom}\,\mathcal{D}$ for all $a \in \mathcal{A}$, so that $da := [\mathcal{D}, a]$ is densely defined. Moreover, da extends to a bounded operator in $\mathcal{N}$ for all $a \in \mathcal{A}$;
2. $a(1+\mathcal{D}^2)^{-1/2} \in \mathcal{K}(\mathcal{N}, \tau)$ for all $a \in \mathcal{A}$.

We say that $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is even if in addition there is a $\mathbb{Z}_2$-grading such that $\mathcal{A}$ is even and $\mathcal{D}$ is odd. This means there is an operator γ such that $\gamma = \gamma^*$, $\gamma^2 = \mathrm{Id}_{\mathcal{N}}$, $\gamma a = a\gamma$ for all $a \in \mathcal{A}$ and $\mathcal{D}\gamma + \gamma\mathcal{D} = 0$. Otherwise we say that $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is odd.

Remark. (1) We will write γ in all our formulae, with the understanding that, if $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is odd, $\gamma = \mathrm{Id}_{\mathcal{N}}$ and of course, we drop the assumption that $\mathcal{D}\gamma + \gamma\mathcal{D} = 0$.
(2) By density, we immediately see that the second condition in the definition of a semifinite spectral triple, also holds for all elements in the C^*-completion of $\mathcal{A}$.
(3) The condition $a(1 + \mathcal{D}^2)^{-1/2} \in \mathcal{K}(\mathcal{N}, \tau)$ is equivalent to $a(i + \mathcal{D})^{-1} \in \mathcal{K}(\mathcal{N}, \tau)$. This follows since $(1 + \mathcal{D}^2)^{1/2}(i + \mathcal{D})^{-1}$ is unitary.

Our first task is to justify the terminology 'nonunital' for the situation where $\mathcal{D}$ does not have τ-compact resolvent. What we show is that if $\mathcal{A}$ is unital, then we obtain a spectral triple on the Hilbert space $\overline{1_{\mathcal{A}}\mathcal{H}}$ for which $1_{\mathcal{A}}\mathcal{D}\,1_{\mathcal{A}}$ has compact resolvent. On the other hand, one can have a spectral triple with nonunital algebra whose 'Dirac' operator has compact resolvent, as in [**28**, **29**, **61**].

Lemma 2.2. *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a semifinite spectral triple relative to $(\mathcal{N}, \tau)$, and suppose $\mathcal{A}$ possesses a unit $P \neq \mathrm{Id}_{\mathcal{N}}$. Then $(P + (P\mathcal{D}P)^2)^{-1/2} \in \mathcal{K}(P\mathcal{N}P, \tau|_{P\mathcal{N}P})$. Hence, $(\mathcal{A}, P\mathcal{H}, P\mathcal{D}P)$ is a unital spectral triple relative to $(P\mathcal{N}P, \tau|_{P\mathcal{N}P})$.*

Proof. It is a short exercise to show that $\tau|_{P\mathcal{N}P}$ is a faithful normal semifinite trace on $P\mathcal{N}P$. We just need to show that $(Pi + P\mathcal{D}P)^{-1}$ is compact in $P\mathcal{N}P$. To do this we show that we can approximate $(Pi + P\mathcal{D}P)^{-1}$ by $P(i + \mathcal{D})^{-1}P$ up to compacts. This follows from

$$(Pi + P\mathcal{D}P)P(i + \mathcal{D})^{-1}P = P(i + \mathcal{D})P(i + \mathcal{D})^{-1}P = P[\mathcal{D}, P](i + \mathcal{D})^{-1}P + P,$$

the compactness of $(i+\mathcal{D})^{-1}P$ and the boundness of $P[\mathcal{D}, P]$ and of $(Pi+P\mathcal{D}P)^{-1}$. $\square$

Thus, we may without loss of generality assume that a spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ whose operator $\mathcal{D}$ does not have compact resolvent, must have a nonunital algebra $\mathcal{A}$. Adapting this proof shows that similar results hold for spectral triples with additional hypotheses such as summability or smoothness, introduced below.

2.2. The Kasparov class and Fredholm module of a spectral triple

In this Section, we use Kasparov modules for trivially graded C^*-algebras, [**36**]. *Nonunital algebras are assumed to be separable*, with the exception of $\mathcal{K}(\mathcal{N}, \tau)$ which typically is not separable nor even σ-unital. By separable, we always mean separable for the norm topology and not necessarily for other topologies like the δ-φ-topology introduced in Definition 2.19. Information about C^*-modules and their endomorphisms can be found in [**48**]. Given a C^*-algebra B and a right B-C^*-module X, we let $\mathrm{End}_B(X)$ denote the C^*-algebra of B-linear adjointable endomorphisms of X, and let $\mathrm{End}^0_B(X)$ be the ideal of B-compact adjointable endomorphisms.

We briefly recall the definition of Kasparov modules, and the equivalence relation on them used to construct the KK-groups.

Definition 2.3. Let A and B be C^*-algebras, with A separable. An odd Kasparov A-B-module consists of a countably generated ungraded right B-C^*-module X, with $\pi : A \to \mathrm{End}_B(X)$ a $*$-homomorphism, together with $F \in \mathrm{End}_B(X)$ such that $\pi(a)(F - F^*)$, $\pi(a)(F^2 - 1)$, $[F, \pi(a)]$ are compact adjointable endomorphisms of X, for each $a \in A$. An even Kasparov A-B-module is an odd Kasparov A-B-module, together with a self-adjoint adjointable endomorphism γ satisfying $\gamma^2 = 1$, $\pi(a)\gamma = \gamma\pi(a)$, and $F\gamma + \gamma F = 0$.

We will use the notation $({}_AX_B, F)$ or $({}_AX_B, F, \gamma)$ for Kasparov modules, generally omitting the representation π. A Kasparov module $({}_AX_B, F)$ with

$$\pi(a)(F - F^*) = \pi(a)(F^2 - 1) = [F, \pi(a)] = 0,$$

for all $a \in A$, is called *degenerate*.

We recall the equivalence relation on Kasparov A-B-modules which defines classes in the abelian group $KK(A, B) = KK^0(A, B)$ (even case) or $KK^1(A, B)$ (odd case). The relation consists of three separate equivalence relations: unitary equivalence, stable equivalence and operator homotopy. More details can be found in [**36**].

Two Kasparov A-B-modules $({}_A(X_1)_B, F_1)$ and $({}_A(X_2)_B, F_2)$ are *unitarily equivalent* if there is an adjointable unitary B-module map $U : X_1 \to X_2$ such that $\pi_2(a) = U\pi_1(a)U^*$, for all $a \in A$ and $F_2 = U\, F_1\, U^*$.

Two Kasparov A-B-modules $({}_A(X_1)_B, F_1)$ and $({}_A(X_2)_B, F_2)$ are *stably equivalent* if there are degenerate Kasparov A-B-modules $({}_A(X_3)_B, F_3)$ and $({}_A(X_4)_B, F_4)$ with

$$({}_A(X_1 \oplus X_3)_B, F_1 \oplus F_3) = ({}_A(X_2 \oplus X_4)_B, F_2 \oplus F_4),$$

and $\pi_1 \oplus \pi_3 = \pi_2 \oplus \pi_4$.

Two Kasparov A-B-modules $({}_A(X)_B, G)$ and $({}_A(X)_B, H)$ (with the same representation π of A) are called *operator homotopic* if there is a norm continuous family $(F_t)_{t\in[0,1]} \subset \mathrm{End}_B(X)$ such that for each $t \in [0, 1]$ $({}_A(X)_B, F_t)$ is a Kasparov module and $F_0 = G$, $F_1 = H$.

Two Kasparov modules $({}_A(X)_B, G)$ and $({}_A(X)_B, H)$ are *equivalent* if after the addition of degenerate modules, they are operator homotopic to unitarily equivalent Kasparov modules. The equivalence classes of even (resp. odd) Kasparov A-B modules form an abelian group denoted $KK^0(A, B)$ (resp. $KK^1(A, B)$). The zero element is represented by any degenerate Kasparov module, and the inverse of a class $[({}_A(X)_B, F)]$ is the class of $({}_A(X)_B, -F)$, with grading $-\gamma$ in the even case.

This equivalence relation, in conjunction with the Kasparov product, implies further equivalences between Kasparov modules, such as Morita equivalence. This is discussed in [**5**, **36**], where more information on the Kasparov product can also be found. With these definitions in hand, we can state our first result linking semifinite spectral triples and Kasparov theory.

LEMMA 2.4 (see [**35**]). *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a semifinite spectral triple relative to $(\mathcal{N}, \tau)$ with $\mathcal{A}$ separable. For $\varepsilon > 0$ (resp $\varepsilon \geq 0$ when $\mathcal{D}$ is invertible), set $F_\varepsilon := \mathcal{D}(\varepsilon + \mathcal{D}^2)^{-1/2}$ and let A be the C^*-completion of $\mathcal{A}$. Then, $[F_\varepsilon, a] \in \mathcal{K}_\mathcal{N}$ for all $a \in A$. In particular, provided that $\mathcal{K}_\mathcal{N}$ is σ-unital, and letting $X := \mathcal{K}_\mathcal{N}$ as a right $\mathcal{K}_\mathcal{N}$-C^*-module, the data $({}_AX_{\mathcal{K}_\mathcal{N}}, F_\varepsilon)$ defines a Kasparov module with class $[({}_AX_{\mathcal{K}_\mathcal{N}}, F_\varepsilon)] \in KK^\bullet(A, \mathcal{K}_\mathcal{N})$, where $\bullet = 0$ if the spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is $\mathbb{Z}_2$-graded and $\bullet = 1$ otherwise. The class $[({}_AX_{\mathcal{K}_\mathcal{N}}, F_\varepsilon)]$ is independent of $\varepsilon > 0$ (or even $\varepsilon \geq 0$ if $\mathcal{D}$ is invertible).*

PROOF. Regarding $X = \mathcal{K}_\mathcal{N}$ as a right $\mathcal{K}_\mathcal{N}$-C^*-module via $(T_1|T_2) := T_1^*T_2$, we see immediately that left multiplication by F_ε on $\mathcal{K}_\mathcal{N}$ gives $F_\varepsilon \in \mathrm{End}_{\mathcal{K}_\mathcal{N}}(\mathcal{K}_\mathcal{N})$, the adjointable endomorphisms, see [**48**], and left multiplication by $a \in A$, the C^*-completion of $\mathcal{A}$, gives a representation of A as adjointable endomorphisms of X also. Since the algebra of compact endomorphisms of X is just $\mathcal{K}_{\mathcal{N}}$, and we have assumed $\mathcal{K}_{\mathcal{N}}$ is σ-unital, we see that X is countably generated, by [**48**, Proposition 5.50]. That $F_\varepsilon^* = F_\varepsilon$ as an endomorphism follows from the functional calculus. Now let $a, b \in \mathcal{A}$. The integral formula for fractional powers gives

$$(\varepsilon + \mathcal{D}^2)^{-1/2} = \pi^{-1} \int_0^\infty \lambda^{-1/2}(\varepsilon + \lambda + \mathcal{D}^2)^{-1} d\lambda,$$

and with a nod to [**12**, Lemma 3.3] we obtain

$$\mathcal{D}\big[(\varepsilon + \mathcal{D}^2)^{-1/2}, a\big]b = \pi^{-1} \int_0^\infty \lambda^{-1/2}\Big(\mathcal{D}^2(\varepsilon + \lambda + \mathcal{D}^2)^{-1}[\mathcal{D}, a](\varepsilon + \lambda + \mathcal{D}^2)^{-1}b + \mathcal{D}(\varepsilon + \lambda + \mathcal{D}^2)^{-1}[\mathcal{D}, a]\mathcal{D}(\varepsilon + \lambda + \mathcal{D}^2)^{-1}b\Big)d\lambda.$$

By the definition of a spectral triple, the integrand is τ-compact, and so is in the compact endomorphisms of our module. The functional calculus yields the norm estimates

$$\|\mathcal{D}^2(\varepsilon + \lambda + \mathcal{D}^2)^{-1}[\mathcal{D}, a](\varepsilon + \lambda + \mathcal{D}^2)^{-1}b\| \leq \|[\mathcal{D}, a]\|\|b\|(\varepsilon + \lambda)^{-1},$$

and

$$\|\mathcal{D}(\varepsilon + \lambda + \mathcal{D}^2)^{-1}[\mathcal{D}, a]\mathcal{D}(\varepsilon + \lambda + \mathcal{D}^2)^{-1}b\| \leq \|[\mathcal{D}, a]\|\|b\|(\varepsilon + \lambda)^{-1}.$$

Therefore, the integral above is norm-convergent. Thus, $\mathcal{D}[(\varepsilon + \mathcal{D}^2)^{-1/2}, a]b$ is τ-compact and

$$[F_\varepsilon, a]b = \mathcal{D}[(\varepsilon + \mathcal{D}^2)^{-1/2}, a]b + [\mathcal{D}, a](\varepsilon + \mathcal{D}^2)^{-1/2}b,$$

is τ-compact too. Similarly, $a[F_\varepsilon, b]$ is τ-compact. Finally, $[F_\varepsilon, ab] = a[F_\varepsilon, b] + [F_\varepsilon, a]b$ is τ-compact, and so a compact endomorphism. Taking norm limits now shows that $[F_\varepsilon, ab]$ is τ-compact for all $a, b \in A$. By the norm density of products in A, one concludes that $[F_\varepsilon, a]$ is compact for all $a \in A$. Finally for $a \in A$ we have $a(1 - F_\varepsilon^2) = a\varepsilon(\varepsilon + \mathcal{D}^2)^{-1}$, and this is τ-compact since $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is a spectral triple. Thus $({}_A X_{\mathcal{K}_{\mathcal{N}}}, F_\varepsilon)$ is a Kasparov module.

To show that the associated KK-class is independent of ε, it suffices to show that $\varepsilon \mapsto F_\varepsilon$ is continuous in operator norm, [**36**]. This follows from the integral formula for fractional powers which shows that

$$F_{\varepsilon_1} - F_{\varepsilon_2} = \frac{\varepsilon_2 - \varepsilon_1}{\pi} \int_0^\infty \lambda^{-1/2}\mathcal{D}(\varepsilon_1 + \lambda + \mathcal{D}^2)^{-1}(\varepsilon_2 + \lambda + \mathcal{D}^2)^{-1}\, d\lambda,$$

since the integral converges in norm independent of ε_1, $\varepsilon_2 > 0$. If $\mathcal{D}$ is invertible we can also take $\varepsilon_i = 0$. This completes the proof. □

The assumption that $\mathcal{K}_{\mathcal{N}}$ is σ-unital is never satisfied in the type II setting, and so we do not obtain a countably generated C^*-module. In order to go beyond this assumption, we adopt the method of [**35**].

DEFINITION 2.5. Given $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ relative to $(\mathcal{N}, \tau)$, we let $\mathcal{C} \subset \mathcal{K}_{\mathcal{N}}$ be the algebra generated by the operators

$$F_\varepsilon[F_\varepsilon, a], \quad b[F_\varepsilon, a], \quad [F_\varepsilon, a], \quad F_\varepsilon b[F_\varepsilon, a], \quad a\varphi(\mathcal{D}), \quad a,\, b \in \mathcal{A}, \quad \varphi \in C_0(\mathbb{R}).$$

If $\mathcal{A}$ is separable, so too is $\mathcal{C}$. This allows us to repeat the construction of Lemma 2.4 using $\mathcal{C}$ instead of $\mathcal{K}_\mathcal{N}$. The result is a Kasparov module $({}_AX_C, F_\varepsilon)$ with class in $KK^\bullet(A, C)$, where C is the norm closure of $\mathcal{C}$.

COROLLARY 2.6. *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a semifinite spectral triple relative to $(\mathcal{N}, \tau)$ with $\mathcal{A}$ separable. For $\varepsilon > 0$ (respectively $\varepsilon \geq 0$ when $\mathcal{D}$ is invertible), define $F_\varepsilon := \mathcal{D}(\varepsilon + \mathcal{D}^2)^{-1/2}$ and let A be the C^*-completion of $\mathcal{A}$. Then, $[F_\varepsilon, a] \in C \subset \mathcal{K}_\mathcal{N}$ for all $a \in A$. In particular, letting $X := C$ as a right C-C^*-module, the data $({}_AX_C, F_\varepsilon)$ defines a Kasparov module with class $[({}_AX_C, F_\varepsilon)] \in KK^\bullet(A, C)$, where $\bullet = 0$ if the spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is $\mathbb{Z}_2$-graded and $\bullet = 1$ otherwise. The class $[({}_AX_C, F_\varepsilon)]$ is independent of $\varepsilon > 0$ (or even $\varepsilon \geq 0$ if $\mathcal{D}$ is invertible).*

Using the Kasparov product we now have a well-defined map

$$\cdot \otimes_A [({}_AX_C, F_\varepsilon)] \, : \, K_\bullet(A) = KK^\bullet(\mathbb{C}, A) \to \; K_0(C). \tag{2.1}$$

For this pairing to make sense it is required that A be separable, [**5**, Theorem 18.4.4], and we remind the reader that we always suppose this to be the case. We refer to the map given in Equation (2.1) as the *K-theoretical index pairing.*

Let $\mathcal{F}_\mathcal{N}$ denote the ideal of 'finite rank' operators in $\mathcal{K}_\mathcal{N}$; that is, $\mathcal{F}_\mathcal{N}$ is the ideal of $\mathcal{N}$ generated by projections of finite trace, without taking the norm completion. In [**35**, Section 6], it shown that for all $n \geq 1$, $M_n(\mathcal{F}_\mathcal{N})$ is stable under the holomorphic functional calculus inside $M_n(\mathcal{K}_\mathcal{N})$, and so $K_0(\mathcal{F}_\mathcal{N}) \cong K_0(\mathcal{K}_\mathcal{N})$. One may now deduce that $M_n(C \cap \mathcal{F}_\mathcal{N})$ is stable under the holomorphic functional calculus inside $M_n(C \cap \mathcal{K}_\mathcal{N}) = M_n(C)$. Thus every class in $K_0(C)$ may be represented as $[e] - [f]$ where e, f are projections in a matrix algebra over the unitisation of $C \cap \mathcal{F}_\mathcal{N}$. As in [**35**], the map $\tau_* : K_0(C) \to \mathbb{R}$ is then well-defined.

DEFINITION 2.7. Let $\mathcal{A}$ be a $*$-algebra (continuously) represented in $\mathcal{N}$, a semifinite von Neumann algebra with faithful semifinite normal trace τ. A semifinite pre-Fredholm module for $\mathcal{A}$ relative to $(\mathcal{N}, \tau)$, is a pair $(\mathcal{H}, F)$, where $\mathcal{H}$ is a separable Hilbert space carrying a faithful representation of $\mathcal{N}$ and F is an operator in $\mathcal{N}$ satisfying:

1. $a(1 - F^2),\ a(F - F^*) \in \mathcal{K}_\mathcal{N}$, *and*
2. $[F, a] \in \mathcal{K}_\mathcal{N}$ *for* $a \in \mathcal{A}$.

If $1 - F^2 = 0 = F - F^*$ we drop the prefix "pre-". If our (pre-)Fredholm module satisfies $[F, a] \in \mathcal{L}^{p+1}(\mathcal{N}, \tau)$ and $a(1 - F^2) \in \mathcal{L}^{(p+1)/2}(\mathcal{N}, \tau)$ for all $a \in \mathcal{A}$, we say that $(\mathcal{H}, F)$ is $(p + 1)$-summable. We say that $(\mathcal{H}, F)$ is even if in addition there is a $\mathbb{Z}_2$-grading such that $\mathcal{A}$ is even and F is odd. This means there is an operator γ such that $\gamma = \gamma^*$, $\gamma^2 = \mathrm{Id}_\mathcal{N}$, $\gamma a = a\gamma$ for all $a \in \mathcal{A}$ and $F\gamma + \gamma F = 0$. Otherwise we say that $(\mathcal{H}, F)$ is odd.

A semifinite pre-Fredholm module for a $*$-algebra $\mathcal{A}$ extends to a semifinite pre-Fredholm module for the norm completion of $\mathcal{A}$ in $\mathcal{N}$, by essentially the same proof as Lemma 2.4. For completeness we state this as a lemma.

LEMMA 2.8. *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a semifinite spectral triple relative to $(\mathcal{N}, \tau)$. Let A be the C^*-completion of $\mathcal{A}$. If $F_\varepsilon := \mathcal{D}(\varepsilon + \mathcal{D}^2)^{-1/2}$, $\varepsilon > 0$, then the operators $[F_\varepsilon, a]$ and $a(1 - F_\varepsilon^2)$ are τ-compact for every $a \in A$. Hence $(\mathcal{H}, F_\varepsilon)$ is a pre-Fredholm module for A.*

2.3. The numerical index pairing

We will now make particular Kasparov products explicit by choosing specific representatives of the classes. We will focus on the condition $F^2 = 1$ for Kasparov modules. Imposing this condition simplifies the description of the Kasparov product with K-theory. In the context of Lemma 2.4, this will be the case if and only if $\varepsilon = 0$, that is, if and only if $\mathcal{D}$ is invertible. We will shortly show how to modify the pair $(\mathcal{H}, \mathcal{D})$ in the data given by a semifinite spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$, in order that $\mathcal{D}$ is always invertible. Before doing that, we need some more Kasparov theory for nonunital C^*-algebras.

Suppose that we have two C^*-algebras A, B and a graded Kasparov module $(X = {}_AX_B, F, \gamma)$. Assume also that A is nonunital. Let e and f be projections in a (matrix algebra over a) unitization of A, which we can take to be the minimal unitization $A^\sim = A \oplus \mathbb{C}$ (see [**48**]), by excision in K-theory, and suppose also that we have a class $[e] - [f] \in K_0(A)$. That is, $[e] - [f] \in \ker(\pi_* : K_0(A^\sim) \to K_0(\mathbb{C}))$ where $\pi : A^\sim \to \mathbb{C}$ is the quotient map. Then the Kasparov product over A of $[e] - [f]$ with $[(X, F, \gamma)]$ gives us a class in $K_0(B)$. We now show that if $F^2 = \mathrm{Id}_X$, we can represent this Kasparov product as a difference of projections over B (in the unital case) or $B^\sim$ (in the nonunital case).

Here and in the following, we always represent elements $a + \lambda\,\mathrm{Id}_{A^\sim} \in A^\sim$ on X as $a + \lambda\,\mathrm{Id}_X$, $\lambda \in \mathbb{C}$. Set $X_\pm := \frac{1\pm\gamma}{2}X$ and, ignoring the matrices to simplify the discussion, let $e \in A^\sim$. To show that $eF_\pm e : eX_\pm \to eX_\mp$ is Fredholm (which in this context means invertible modulo $\mathrm{End}^0_B(X_\pm, X_\mp)$), we must display a parametrix. Taking $eF_\mp e$ yields

$$eF_\mp eF_\pm e = eF_\mp[e, F_\pm]e + e(F_\mp F_\pm - \mathrm{Id}_{X_\pm})e + \mathrm{Id}_{eX_\pm}.$$

We are left with showing that $e(F_\mp F_\pm - \mathrm{Id}_{X_\pm})e$ and $eF_\mp[e, F_\pm]e$ are (B-linear) compact endomorphisms of the C^*-module $X_\pm$. The compactness of $eF_\mp[e, F_\pm]e$ follows since e is represented as $a + \lambda\mathrm{Id}_X$ for some $a \in A$ and $\lambda \in \mathbb{C}$, and thus $[e, F_\pm] = [a, F_\pm]$ which is compact by definition of a Kasparov module.

However $e(F_\mp F_\pm - \mathrm{Id}_{X_\pm})e$ is generally not compact, because we are only guaranteed that $a(F_\mp F_\pm - \mathrm{Id}_{X_\pm})$ is compact for $a \in A$, not $a \in A^\sim$! Nevertheless, if the Kasparov module is normalized, i.e. if $F^2 = \mathrm{Id}_X$, we have $F_\mp F_\pm - \mathrm{Id}_{X_\pm} = 0$, and so we have a parametrix, showing that $eF_\pm e$ is Fredholm. In this case, the Kasparov product $([e] - [f]) \otimes_A [(X, F)]$ is given by

$$\big[\mathrm{Index}(eF_\pm e)\big] - \big[\mathrm{Index}(fF_\pm f)\big] \in K_0(B).$$

Here the index is defined as the difference $[\ker \widetilde{eF_\pm e}] - [\mathrm{coker}\, \widetilde{eF_\pm e}]$, where $\widetilde{eF_\pm e}$ is any regular amplification of $eF_\pm e$, see [**30**, Lemma 4.10]. This index is independent of the amplification chosen, the kernel and cokernel projections can be chosen finite rank over B, or $B^\sim$ if B is nonunital, and the index lies in $K_0(B)$ by [**30**, Proposition 4.11].

Similarly, in the odd case we would like to have (see [**35**, Appendix] and [**42**, Appendix]),

$$[u] \otimes_A \big[(X, F)\big] = \Big[\mathrm{Index}\left(\tfrac{1}{4}(1 + F)u(1 + F) - \tfrac{1}{2}(1 - F)\right)\Big] \in K_0(B),$$

where $[u] \in K_1(A)$. As in the even case, to see that $\frac{1}{4}(1 + F)u(1 + F) - \frac{1}{2}(1 - F)$ is Fredholm in the nonunital case, it is easier to assume that $F^2 = 1$, and in this case, writing $(1+F)/2 = P$ for the positive spectral projection of F, we have

$$[u] \otimes_A \big[(X,F)\big] = \big[\operatorname{Index}(PuP)\big] = [\ker \widetilde{PuP}] - [\operatorname{coker} \widetilde{PuP}] \in K_0(B),$$

there being no contribution to the index from $P^\perp = (1-F)/2$. As in the even case above, $\widetilde{PuP}$ is a regular amplification of PuP, and the projections onto $\ker \widetilde{PuP}$ and $\operatorname{coker} \widetilde{PuP}$ are finite rank over B or $B^\sim$. We show in Section 2.7 an alternative method to avoid the simplifying assumption $F^2 = 1$ in the odd case.

Given a pre-Fredholm module $(\mathcal{H}, F)$ relative to $(\mathcal{N}, \tau)$ for a separable $*$-algebra $\mathcal{A}$, we obtain a Kasparov module $({}_AC_C, F)$, just as we did for a spectral triple in Corollary 2.6. Here A is the norm completion of $\mathcal{A}$ and $C \subset \mathcal{K}_\mathcal{N}$ is given by the norm closure of the algebra defined in Definition 2.5, using the operator F for the commutators, and polynomials in $1 - F^2$ in place of $\varphi(\mathcal{D})$, $\varphi \in C_0(\mathbb{R})$. Also, given $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ relative to $(\mathcal{N}, \tau)$, the following diagram commutes

$$\begin{tikzcd} (\mathcal{A},\mathcal{H},\mathcal{D}) \arrow[r] \arrow[d] & ({}_AC_C, F) \\ (\mathcal{H},F) \arrow[ur] & \end{tikzcd} .$$

Thus we have a single well-defined Kasparov class arising from either the spectral triple or the associated pre-Fredholm module. Now we show how to obtain a representative of this class with $F^2 = 1$, so simplifying the index pairing. This reduces to showing that if our spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is such that $\mathcal{D}$ is not invertible, we can replace it by a new spectral triple for which the unbounded operator is invertible and has the same KK-class. We learned this trick from [**20**, p. 68].

Definition 2.9. Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a semifinite spectral triple relative to $(\mathcal{N}, \tau)$. For any $\mu > 0$, define the 'double' of $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ to be the semifinite spectral triple $(\mathcal{A}, \mathcal{H}^2, \mathcal{D}_\mu)$ relative to $(M_2(\mathcal{N}), \tau \otimes \mathrm{tr}_2)$, with $\mathcal{H}^2 := \mathcal{H} \oplus \mathcal{H}$ and the action of $\mathcal{A}$ and $\mathcal{D}_\mu$ given by

$$\mathcal{D}_\mu := \begin{pmatrix} \mathcal{D} & \mu \\ \mu & -\mathcal{D} \end{pmatrix}, \qquad a \mapsto \hat{a} := \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}, \quad \text{for all } a \in \mathcal{A}.$$

If $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is graded by γ, then the double is even and graded by $\hat{\gamma} := \gamma \oplus -\gamma$.

Remark. Whether $\mathcal{D}$ is invertible or not, $\mathcal{D}_\mu$ always is invertible, and $F_\mu = \mathcal{D}_\mu |\mathcal{D}_\mu|^{-1}$ has square 1. This is the chief reason for introducing this construction.

We also need to extend the action of $M_n(\mathcal{A}^\sim)$ on $(\mathcal{H} \oplus \mathcal{H}) \otimes \mathbb{C}^n$, in a compatible way with the extended action of $\mathcal{A}$ on $\mathcal{H} \oplus \mathcal{H}$. So, for a generic element $b \in M_n(\mathcal{A}^\sim)$, we let

$$\hat{b} := \begin{pmatrix} b & 0 \\ 0 & \mathbf{1}_b \end{pmatrix} \in M_{2n}(\mathcal{N}), \tag{2.2}$$

with $\mathbf{1}_b := \pi^n(b) \otimes \mathrm{Id}_\mathcal{N}$, where $\pi^n : M_n(\mathcal{A}^\sim) \to M_n(\mathbb{C})$ is the quotient map.

It is known (see for instance [**20**, Proposition 12, p. 443]), that up to an addition of a degenerate module, any Kasparov module is operator homotopic to a normalised Kasparov module, i.e. one with $F^2 = 1$. The following makes it explicit.

Lemma 2.10. *When $\mathcal{A}$ is separable, the KK-classes associated with $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ and $(\mathcal{A}, \mathcal{H}^2, \mathcal{D}_\mu)$ coincide. A representative of this class is $({}_A(C \oplus C)_C, F_\mu)$ with $F_\mu = \mathcal{D}_\mu |\mathcal{D}_\mu|^{-1}$ and C the norm closure of the $*$-subalgebra of $\mathcal{K}(\mathcal{N}, \tau)$ given in Definition 2.5.*

PROOF. The KK-class of $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is represented (by an application of Corollary 2.6) by $({}_AC_C, F_\varepsilon)$ with $F_\varepsilon = \mathcal{D}(\varepsilon+\mathcal{D}^2)^{-1/2}$, $\varepsilon > 0$, while the class of $(\mathcal{A}, \mathcal{H}^2, \mathcal{D}_\mu)$ is represented by the Kasparov module $({}_AM_2(C)_{M_2(C)}, F_{\mu,\varepsilon})$ with operator defined by $F_{\mu,\varepsilon} = \mathcal{D}_\mu(\varepsilon + \mathcal{D}_\mu^2)^{-1/2}$. By Morita equivalence, this module has the same class as the module $({}_A(C \oplus C)_C, F_{\mu,\varepsilon})$, since ${}_{M_2(C)}(C \oplus C)_C$ is a Morita equivalence bimodule. The one-parameter family $({}_A(C \oplus C)_C, F_{m,\varepsilon})_{0\le m\le\mu}$ is a continuous operator homotopy, [**36**], from $({}_A(C \oplus C)_C, F_{\mu,\varepsilon})$ to the direct sum of two Kasparov modules

$$({}_AC_C, F_\varepsilon) \oplus ({}_AC_C, -F_\varepsilon).$$

In the odd case the second Kasparov module is operator homotopic to $({}_AC_C, \mathrm{Id}_\mathcal{N})$ by the straight line path since $\mathcal{A}$ is represented by zero on this module. In the even case we find the second Kasparov module is homotopic to

$$\left({}_AC_C, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right),$$

the matrix decomposition being with respect to the $\mathbb{Z}_2$-grading of $\mathcal{H}$ which provides a $\mathbb{Z}_2$-grading of $C \subset \mathcal{K}_\mathcal{N}$. Thus in both the even and odd cases the second module is degenerate, i.e. $F^2 = 1$, $F = F^*$ and $[F, a] = 0$ for all $a \in \mathcal{A}$, and so the KK-class of $({}_A(C \oplus C)_C, F_{\mu,\varepsilon})$, written $[({}_A(C \oplus C)_C, F_{\mu,\varepsilon})]$, is the KK-class of $({}_AC_C, F_\varepsilon)$. In addition, the Kasparov module $({}_A(C \oplus C)_C, F_\mu)$ with $F_\mu = \mathcal{D}_\mu|\mathcal{D}_\mu|^{-1}$ is operator homotopic to $({}_A(C \oplus C)_C, F_{\mu,\varepsilon})$ via

$$t \mapsto \mathcal{D}_\mu(t\varepsilon + \mathcal{D}_\mu^2)^{-1/2}, \quad 0 \le t \le 1.$$

This provides the desired representative. $\square$

The next result records what is effectively a tautology, given our definitions. Namely we define the $K_0(C)$-valued index pairing of $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ with $K_*(\mathcal{A})$ in terms of the associated Kasparov module. Similarly, the associated pre-Fredholm module has an index pairing defined in terms of the associated Kasparov module.

COROLLARY 2.11. *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a spectral triple relative to $(\mathcal{N}, \tau)$ with $\mathcal{A}$ separable. Let $(\mathcal{A}, \mathcal{H}^2, \mathcal{D}_\mu)$ be the double of $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ relative to $(M_2(\mathcal{N}), \tau \otimes \mathrm{tr}_2)$ and $({}_A(C\oplus C)_C, F_\mu)$ the associated Fredholm module. Then the $K_0(C)$-valued index pairings defined by the two spectral triples and the semifinite Fredholm module all agree: for $x \in K_*(\mathcal{A})$ of the appropriate parity and $\mu > 0$*

$$\begin{aligned} x \otimes_A [(\mathcal{A}, \mathcal{H}, \mathcal{D})] &= x \otimes_A [({}_AC_C, F_\varepsilon)] \\ &= x \otimes_A \left[(\mathcal{A}, \mathcal{H}^2, \mathcal{D}_\mu)\right] \\ &= x \otimes_A [({}_A(C \oplus C)_C, F_\mu)] \in K_0(C). \end{aligned}$$

Looking ahead to the numerical index, we recall that we noted after Corollary 2.6 that the trace τ induces a homomorphism $\tau_* : K_0(C) \to \mathbb{R}$.

An important feature of the double construction is that it allows us to make pairings in the nonunital case explicit. To be precise, if $e \in M_n(\mathcal{A}^\sim)$ is a projection and $\pi^n : M_n(\mathcal{A}^\sim) \to M_n(\mathbb{C})$ is the quotient map (by $M_n(\mathcal{A})$), we set as in (2.2)

$$\mathbf{1}_e := \pi^n(e) \in M_n(\mathbb{C}). \tag{2.3}$$

Then in the double e is represented on $\mathcal{H}\otimes\mathbb{C}^n\oplus\mathcal{H}\otimes\mathbb{C}^n$ (this is the spectral triple picture, but similar comments hold for Kasparov modules) via

$$e\mapsto\hat{e}:=\begin{pmatrix} e & 0\\ 0 & \mathbf{1}_e\end{pmatrix}.$$

Thus $\hat{e}(\mathcal{D}_\mu\otimes\mathrm{Id}_n)\hat{e}$ is $\tau\otimes\mathrm{tr}_{2n}$-Fredholm in $M_{2n}(\mathcal{N})$, with the understanding that the matrix units $e_{ij}\in M_{2n}(\mathbb{C})$ sit in $M_{2n}(\mathcal{N})$ as $e_{ij}\,\mathrm{Id}_{\mathcal{N}}$.

Example. Let $p_B\in M_2(C_0(\mathbb{C})^\sim)$ be the Bott projector, given explicitly by [**30**, p. 76-77]

$$p_B(z)=\frac{1}{1+|z|^2}\begin{pmatrix}1 & \bar{z}\\ z & |z|^2\end{pmatrix},\quad\text{then}\quad \mathbf{1}_{p_B}=\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}. \tag{2.4}$$

We are now ready to define the numerical index paring for semifinite spectral triples.

Definition 2.12. Let $(\mathcal{A},\mathcal{H},\mathcal{D})$ be a semifinite spectral triple relative to $(\mathcal{N},\tau)$ of parity $\bullet\in\{0,1\}$, $\bullet=0$ for an even triple, $\bullet=1$ for an odd triple and with $\mathcal{A}$ separable. We define the numerical index pairing of $(\mathcal{A},\mathcal{H},\mathcal{D})$ with $K_\bullet(\mathcal{A})$ as follows:

1. Take the Kasparov product with the KK-class defined by the doubled up spectral triple
$$\cdot\otimes_A[({}_A(C\oplus C)_C,F_\mu)]\,:\,K_\bullet(A)\to K_0(C),$$
2. Apply the homomorphism $\tau_*:K_0(C)\to\mathbb{R}$ to the resulting class.

We will denote this pairing by

$$\langle[e]-[\mathbf{1}_e],(\mathcal{A},\mathcal{H},\mathcal{D})\rangle\in\mathbb{R},\ \text{even case},\quad\langle[u],(\mathcal{A},\mathcal{H},\mathcal{D})\rangle\in\mathbb{R},\ \text{odd case}.$$

If, in the even case, $[e]-[f]\in K_0(\mathcal{A})$ then $[\mathbf{1}_e]=[\mathbf{1}_f]\in K_0(\mathbb{C})$ and we may define

$$\langle[e]-[f],(\mathcal{A},\mathcal{H},\mathcal{D})\rangle:=\langle[e]-[\mathbf{1}_e],(\mathcal{A},\mathcal{H},\mathcal{D})\rangle-\langle[f]-[\mathbf{1}_f],(\mathcal{A},\mathcal{H},\mathcal{D})\rangle\in\mathbb{R}.$$

From Corollary 2.11 we may deduce the following important result, which justifies the name 'numerical index pairing' for the map given in the previous Definition, as well as our notations.

Proposition 2.13. *Let $(\mathcal{A},\mathcal{H},\mathcal{D})$ be a semifinite spectral triple relative to $(\mathcal{N},\tau)$, of parity $\bullet\in\{0,1\}$ and with $\mathcal{A}$ separable. Let e be a projector in $M_n(\mathcal{A}^\sim)$ which represents $[e]\in K_0(\mathcal{A})$, for $\bullet=0$ (resp. u be a unitary in $M_n(\mathcal{A}^\sim)$ which represents $[u]\in K_1(\mathcal{A})$, for $\bullet=1$). Then with $F_\mu:=\mathcal{D}_\mu/|\mathcal{D}_\mu|$ and $P_\mu:=(1+F_\mu)/2$, we have*

$$\langle[e]-[\mathbf{1}_e],(\mathcal{A},\mathcal{H},\mathcal{D})\rangle=\mathrm{Index}_{\tau\otimes\mathrm{tr}_{2n}}\big(\hat{e}(F_{\mu_+}\otimes\mathrm{Id}_n)\hat{e}\big),\quad\textit{even case},$$
$$\langle[u],(\mathcal{A},\mathcal{H},\mathcal{D})\rangle=\mathrm{Index}_{\tau\otimes\mathrm{tr}_{2n}}\big((P_\mu\otimes\mathrm{Id}_n)\hat{u}(P_\mu\otimes\mathrm{Id}_n)\big),\quad\textit{odd case}.$$

2.4. Smoothness and summability for spectral triples

In this Section we discuss the notions of finitely summable spectral triple, QC^∞ spectral triple and most importantly *smoothly summable spectral triples* for nonunital $*$-algebras. We then examine how these notions fit with our discussion of summability and the pseudodifferential calculus introduced in the previous Chapter. One of the main technical difficulties that we have to overcome in the nonunital case is the issue of finding the appropriate definition of a smooth algebra stable under the holomorphic functional calculus.

We begin by considering possible notions of summability for spectral triples. There are two basic tasks that we need some summability for:
1) To obtain a well-defined Chern character for the associated Fredholm module,
2) To obtain a local index formula.

Even in the case where $\mathcal{A}$ is unital, point 2) requires extra smoothness assumptions, discussed below, in addition to the necessary summability. Thus we expect point 2) to require more assumptions on the spectral triple than point 1). For point 1) we have the following answer.

PROPOSITION 2.14. *Let $(\mathcal{A},\mathcal{H},\mathcal{D})$ be a semifinite spectral triple relative to $(\mathcal{N},\tau)$. Suppose further that there exists $p\geq 1$ such that $a(1+\mathcal{D}^2)^{-s/2}\in\mathcal{L}^1(\mathcal{N},\tau)$ for all $s>p$ and all $a\in\mathcal{A}$. Then $(\mathcal{H},F_\varepsilon=\mathcal{D}(\varepsilon+\mathcal{D}^2)^{-1/2})$ defines a $\lfloor p\rfloor+1$-summable pre-Fredholm module for $\mathcal{A}^2$ whose KK-class is independent of $\varepsilon>0$ (or even $\varepsilon\geq 0$ if $\mathcal{D}$ is invertible). If in addition we have $[\mathcal{D},a](1+\mathcal{D}^2)^{-s/2}\in\mathcal{L}^1(\mathcal{N},\tau)$ for all $s>p$ and all $a\in\mathcal{A}$, then $(\mathcal{H},F_\varepsilon=\mathcal{D}(\varepsilon+\mathcal{D}^2)^{-1/2})$ defines a $\lfloor p\rfloor+1$-summable pre-Fredholm module for $\mathcal{A}$ whose KK-class is independent of $\varepsilon>0$ (or even $\varepsilon\geq 0$ if $\mathcal{D}$ is invertible).*

Remark. Here $\mathcal{A}^2$ means the algebra given by the finite linear span of products ab, a, $b\in\mathcal{A}$.

PROOF. First we employ Lemma 1.37 to deduce that for all $\delta>0$ we have

$$a(1-F_\varepsilon^2)=\varepsilon\, a(\varepsilon+\mathcal{D}^2)^{-1}\in\mathcal{L}^{p/2+\delta}(\mathcal{N},\tau).$$

The same lemma tells us that for all $a\in\mathcal{A}$ and $\delta>0$

$$a(\varepsilon+\mathcal{D}^2)^{-\frac{\lfloor p\rfloor+\delta}{2(\lfloor p\rfloor+1)}}\in\mathcal{L}^{\lfloor p\rfloor+1}(\mathcal{N},\tau).$$

We use the integral formula for fractional powers and [**12**, Lemma 3.3] to obtain

$$\begin{aligned}[F_\varepsilon,a]&=\frac{-1}{\pi}\int_0^\infty\lambda^{-1/2}\mathcal{D}(\varepsilon+\lambda+\mathcal{D}^2)^{-1}[\mathcal{D},a]\mathcal{D}(\varepsilon+\lambda+\mathcal{D}^2)^{-1}d\lambda\\&-\frac{1}{\pi}\int_0^\infty\lambda^{-1/2}(\varepsilon+\lambda+\mathcal{D}^2)^{-1}[\mathcal{D},a]\mathcal{D}^2(\varepsilon+\lambda+\mathcal{D}^2)^{-1}d\lambda+(\varepsilon+\mathcal{D}^2)^{-1/2}[\mathcal{D},a].\end{aligned}$$

Now we multiply on the left by $b\in\mathcal{A}$, and estimate the $\lfloor p\rfloor+1$-norm. Since

$$(\varepsilon+\mathcal{D}^2+\lambda)^{-1}=(\varepsilon+\mathcal{D}^2+\lambda)^{-\frac{\lfloor p\rfloor+\delta}{2(\lfloor p\rfloor+1)}}(\varepsilon+\mathcal{D}^2+\lambda)^{-\frac{1}{2}-\frac{(1-\delta)}{2(\lfloor p\rfloor+1)}},$$

and

$$\|\mathcal{D}(\varepsilon+\mathcal{D}^2+\lambda)^{-\frac{1}{2}-\frac{(1-\delta)}{2(\lfloor p\rfloor+1)}}\|_\infty\leq(\varepsilon+\lambda)^{-\frac{(1-\delta)}{2(\lfloor p\rfloor+1)}},$$

by spectral theory, we find that for $1>\delta>0$

$$\begin{aligned}\|b[F_\varepsilon,a]\|_{\lfloor p\rfloor+1}&\leq 2\|[\mathcal{D},a]\|\,\|b(\varepsilon+\mathcal{D}^2)^{-\frac{\lfloor p\rfloor+\delta}{2(\lfloor p\rfloor+1)}}\|_{\lfloor p\rfloor+1}\\&\qquad\times\int_0^\infty\lambda^{-1/2}(\varepsilon+\lambda)^{-\frac{1}{2}-\frac{(1-\delta)}{2(\lfloor p\rfloor+1)}}\,d\lambda<\infty.\end{aligned}$$

Hence $b[F_\varepsilon,a]\in\mathcal{L}^{\lfloor p\rfloor+1}(\mathcal{N},\tau)$, and taking adjoints yields $[F_\varepsilon,a]b\in\mathcal{L}^{\lfloor p\rfloor+1}(\mathcal{N},\tau)$ for all a, $b\in\mathcal{A}$ also. Now we observe that $[F_\varepsilon,ab]=a[F_\varepsilon,b]+[F_\varepsilon,a]b$, and so $[F_\varepsilon,ab]\in\mathcal{L}^{\lfloor p\rfloor+1}(\mathcal{N},\tau)$ for all $ab\in\mathcal{A}^2$. This completes the proof of the first part. The second claim follows from a similar estimate without the need to multiply by $b\in\mathcal{A}$. The independence of the class on $\varepsilon>0$ is established as in Lemma 2.4. □

The previous proposition shows that we have sufficient conditions on a spectral triple in order to obtain a finitely summable pre-Fredholm module for $\mathcal{A}^2$ or $\mathcal{A}$. These two conditions are not equivalent. Here is a counterexample for $p = 1$.

Consider the function $f : x \mapsto \sin(x^3)/(1 + x^2)$ on the real line, and the operator $\mathcal{D} = -i(d/dx)$ on $L^2(\mathbb{R})$. Then the operator $f(1 + \mathcal{D}^2)^{-s/2}$ is trace class for $\Re(s) > 1$, by [**56**, Theorem 4.5], while $[\mathcal{D}, f](1 + \mathcal{D}^2)^{-s/2}$ is not trace class for any $\Re(s) > 1$, by [**56**, Proposition 4.7]. To see the latter, it suffices to show that with $g(x) = x^2/(1 + x^2)$, we have that $g(1 + \mathcal{D}^2)^{-s/2}$ is not trace class. However this follows from $g(1 + \mathcal{D}^2)^{-s/2} = (1 + \mathcal{D}^2)^{-s/2} - h(1 + \mathcal{D}^2)^{-s/2}$ with $h = \frac{1}{1+x^2}$. The second operator is trace class, however $(1 + \mathcal{D}^2)^{-s/2}$ is well-known to be noncompact, and so not trace class.

We investigate the weaker of these two summability conditions first, relating it to our integration theory from Chapter 1. Indeed the following two propositions show that finite summability, in the sense of the next definition, almost uniquely determines where $\mathcal{A}$ must sit inside $\mathcal{N}$, and justifies the introduction of the Fréchet algebras $\mathcal{B}_1^k(\mathcal{D}, p)$.

Definition 2.15. A semifinite spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is said to be finitely summable if there exists $s > 0$ such that for all $a \in \mathcal{A}$, $a(1 + \mathcal{D}^2)^{-s/2} \in \mathcal{L}^1(\mathcal{N}, \tau)$. In such a case, we let

$$p := \inf\big\{s > 0 \,:\, \text{for all } a \in \mathcal{A},\ \tau\big(|a|(1 + \mathcal{D}^2)^{-s/2}\big) < \infty\big\},$$

and call p the spectral dimension of $(\mathcal{A}, \mathcal{H}, \mathcal{D})$.

Remark. For the definition of the spectral dimension above to be meaningful, one needs two facts. First, if $\mathcal{A}$ is the algebra of a finitely summable spectral triple, we have $|a|(1 + \mathcal{D}^2)^{-s/2} \in \mathcal{L}^1(\mathcal{N}, \tau)$ for all $a \in \mathcal{A}$, which follows by using the polar decomposition $a = v|a|$ and writing

$$|a|(1 + \mathcal{D}^2)^{-s/2} = v^* a(1 + \mathcal{D}^2)^{-s/2}.$$

Observe that we are *not* asserting that $|a| \in \mathcal{A}$, which is typically *not* true in examples.

The second fact we require is that $\tau\big(a(1+\mathcal{D}^2)^{-s/2}\big) \geq 0$ for $a \geq 0$, which follows from [**6**, Theorem 3], quoted here as Proposition 1.5.

In contrast to the unital case, checking the finite summability condition for a nonunital spectral triple can be difficult. This is because our definition relies on control of the trace norm of the non-self-adjoint operators $a(1+\mathcal{D}^2)^{-s/2}$, $a \in \mathcal{A}$. The next two results show that for a spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ to be finitely summable with spectral dimension p, it is necessary that $\mathcal{A} \subset \mathcal{B}_1(\mathcal{D}, p)$ and this condition is *almost* sufficient as well.

Proposition 2.16. *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a semifinite spectral triple. If for some $p \geq 1$ we have $\mathcal{A} \subset \mathcal{B}_1^\infty(\mathcal{D}, p)$, then $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is finitely summable with spectral dimension given by the infimum of such p's. More generally, if for some $p \geq 1$ we have $\mathcal{A} \subset \mathcal{B}_2(\mathcal{D}, p)\mathcal{B}_2^{\lfloor p \rfloor + 1}(\mathcal{D}, p) \subset \mathcal{B}_1(\mathcal{D}, p)$, then $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is finitely summable with spectral dimension given by the infimum of such p's.*

Proof. The first statement is an immediate consequence of Corollary 1.30. For the second statement, let $a \in \mathcal{A}$. We need to prove that $a(1 + \mathcal{D}^2)^{-s/2}$ is trace class for $a = bc$ with $b \in \mathcal{B}_2(\mathcal{D}, p)$ and $c \in \mathcal{B}_2^{\lfloor p \rfloor + 1}(\mathcal{D}, p)$. Thus, for all $k \leq \lfloor p \rfloor + 1$

and all $s > p$ we have

$$b(1+\mathcal{D}^2)^{-s/4},\ (1+\mathcal{D}^2)^{-s/4}\delta^k(c) \in \mathcal{L}^2(\mathcal{N},\tau).$$

We start from the identity

$$(-1)^k \frac{\Gamma(s+k)}{\Gamma(s)\Gamma(k+1)}(1+|\mathcal{D}|)^{-s-k} = \frac{1}{2\pi i}\int_{\Re(\lambda)=1/2} \lambda^{-s}(\lambda-1-|\mathcal{D}|)^{-k-1}d\lambda,$$

and then by induction we have

$$\begin{aligned}[(\lambda-1-|\mathcal{D}|)^{-1},c] = &\sum_{k=1}^{\lfloor p\rfloor}(-1)^{k+1}(\lambda-1-|\mathcal{D}|)^{-k-1}\delta^k(c)\\ &+(-1)^{\lfloor p\rfloor}(\lambda-1-|\mathcal{D}|)^{-\lfloor p\rfloor-1}\delta^{\lfloor p\rfloor+1}(c)(\lambda-1-|\mathcal{D}|)^{-1}.\end{aligned}$$

It follows that

$$\begin{aligned}[(1+|\mathcal{D}|)^{-s},c] &= \frac{1}{2\pi i}\int_{\Re\lambda=1/2}\lambda^{-s}[(\lambda-1-|\mathcal{D}|)^{-1},c]\,d\lambda\\ &= -\sum_{k=1}^{\lfloor p\rfloor}\frac{\Gamma(s+k)}{\Gamma(s)\Gamma(k+1)}(1+|\mathcal{D}|)^{-s-k}\delta^k(c)\\ &\quad+\frac{(-1)^{\lfloor p\rfloor}}{2\pi i}\int_{\Re(\lambda)=1/2}\lambda^{-s}(\lambda-1-|\mathcal{D}|)^{-\lfloor p\rfloor-1}\delta^{\lfloor p\rfloor+1}(c)(\lambda-1-|\mathcal{D}|)^{-1}d\lambda.\end{aligned}$$

Since $\big|\lambda-1-|\mathcal{D}|\big| \geq |\lambda|$ and since the $\|\cdot\|_2$–norms of the operators

$$b(\lambda-1-|\mathcal{D}|)^{-(\lfloor p\rfloor+1)/2},\quad (\lambda-1-|\mathcal{D}|)^{-(\lfloor p\rfloor+1)/2}\delta^{\lfloor p\rfloor+1}(c),$$

are bounded uniformly over λ, we find that

$$\left\|b\frac{(-1)^{\lfloor p\rfloor}}{2\pi i}\int_{\Re\lambda=1/2}\lambda^{-s}(\lambda-1-|\mathcal{D}|)^{-\lfloor p\rfloor-1}\delta^{\lfloor p\rfloor+1}(c)(\lambda-1-|\mathcal{D}|)^{-1}d\lambda\right\|_1,$$

is bounded by $C(b,c)\int_{\Re\lambda=1/2}\frac{|d\lambda|}{|\lambda|^{1+s}}$, which is finite. Hence we have $b[(1+|\mathcal{D}|)^{-s},c] \in \mathcal{L}^1(\mathcal{N},\tau)$ and since

$$b(1+|\mathcal{D}|)^{-s}c = (b(1+|\mathcal{D}|)^{-s/2})\cdot((1+|\mathcal{D}|)^{-s/2}c) \in \mathcal{L}^1(\mathcal{N},\tau),$$

we conclude that $a(1+|\mathcal{D}|)^{-s} \in \mathcal{L}^1(\mathcal{N},\tau)$, and so $a(1+\mathcal{D}^2)^{-s/2} \in \mathcal{L}^1(\mathcal{N},\tau)$. The claim about the spectral dimension follows immediately. □

Proposition 2.17. *Let $(\mathcal{A},\mathcal{H},\mathcal{D})$ be a finitely summable semifinite spectral triple of spectral dimension p. Then $\mathcal{A}$ is a subalgebra of $\mathcal{B}_1(\mathcal{D},p)$.*

Proof. Since $\mathcal{A}$ is a $*$-algebra, it suffices to consider self-adjoint elements. For $a = a^* \in \mathcal{A}$, we have by assumption that $a(1+\mathcal{D}^2)^{-s/2} \in \mathcal{L}^1(\mathcal{N},\tau)$, for all $s > p$. Now let $a = v|a| = |a|v^*$ be the polar decomposition. Observe that neither v nor $|a|$ need be in $\mathcal{A}$. However,

$$|a|(1+\mathcal{D}^2)^{-s/2} = v^*a(1+\mathcal{D}^2)^{-s/2} \in \mathcal{L}^1(\mathcal{N},\tau)\ \text{ for all }\ s > p.$$

Proposition 1.5, from [**6**, Theorem 3], implies that $|a|^{1/2}(1+\mathcal{D}^2)^{-s/4} \in \mathcal{L}^2(\mathcal{N},\tau)$, for all $s > p$, and so $|a|^{1/2} \in \mathcal{B}_2(\mathcal{D},p)$. In addition, $v|a|^{1/2} \in \mathcal{B}_2(\mathcal{D},p)$, since $v|a|^{1/2} = |a|^{1/2}v^*$ by the functional calculus, and

$$v|a|v^* = |a|^{1/2}v^*v|a|^{1/2} = |a|,$$

and $(1+\mathcal{D}^2)^{-s/4}|a|^{1/2}v^*v|a|^{1/2}(1+\mathcal{D}^2)^{-s/4} = (1+\mathcal{D}^2)^{-s/4}|a|(1+\mathcal{D}^2)^{-s/4}$. From this we can conclude that $a = v|a|^{1/2}\cdot|a|^{1/2} \in (\mathcal{B}_2(\mathcal{D},p))^2 \subset \mathcal{B}_1(\mathcal{D},p)$. □

Remark. The previous two results tell us that a finitely summable spectral triple must have $\mathcal{A} \subset \mathcal{B}_1(\mathcal{D},p)$. However the last result does *not* imply that for a finitely summable spectral triple $(\mathcal{A},\mathcal{H},\mathcal{D})$ and $a = a^* \in \mathcal{A}$ we have a_+, a_-, $|a|$ in $\mathcal{A}$. On the other hand, the previous proof shows that $|a|$ does belong to $\mathcal{B}_1(\mathcal{D},p)$, and so for a finitely summable spectral triple, we can improve on the result of Proposition 1.14, at least for elements of $\mathcal{A}$.

In addition to the summability of a spectral triple $(\mathcal{A},\mathcal{H},\mathcal{D})$ relative to $(\mathcal{N},\tau)$, we need to consider smoothness, and the two notions are much more tightly related in the nonunital case. One reason for smoothness is that we need to be able to control commutators with $\mathcal{D}^2$ to obtain the local index formula. Another reason is that we need to be able to show that we have a spectral triple for a (possibly) larger algebra $\mathcal{B} \supset \mathcal{A}$ where $\mathcal{B}$ is Fréchet and stable under the holomorphic functional calculus, and has the same norm closure as $\mathcal{A}$: $A = \overline{\mathcal{A}} = \overline{\mathcal{B}}$.

The next definition recalls how the problem of finding suitable $\mathcal{B} \supset \mathcal{A}$ is solved in the unital case.

Definition 2.18. Let $(\mathcal{A},\mathcal{H},\mathcal{D})$ be a semifinite spectral triple, relative to $(\mathcal{N},\tau)$. With $\delta = [|\mathcal{D}|,\cdot]$ we say that $(\mathcal{A},\mathcal{H},\mathcal{D})$ is QC^k if for all $b \in \mathcal{A}\cup[\mathcal{D},\mathcal{A}]$ we have $\delta^j(b) \in \mathcal{N}$ for all $0 \le j \le k$. We say that $(\mathcal{A},\mathcal{H},\mathcal{D})$ is QC^∞ if it is QC^k for all $k \in \mathbb{N}_0$.

Remark. For a QC^∞ spectral triple $(\mathcal{A},\mathcal{H},\mathcal{D})$ with $T_0,\dots,T_m \in \mathcal{A}\cup[\mathcal{D},\mathcal{A}]$, we see by iteration of the relation $T^{(1)} = \delta^2(T) + 2\delta(T)|\mathcal{D}|$, that

$$T_0^{(k_0)}\cdots T_m^{(k_m)}(1+\mathcal{D}^2)^{-|k|/2} \in \mathcal{N},$$

where $|k| := k_0 + \cdots + k_m$ and $T^{(n)}$ is given in Definition 1.20.

For $(\mathcal{A},\mathcal{H},\mathcal{D})$ a QC^∞ spectral triple, unital or not, we may endow the algebra $\mathcal{A}$ with the topology determined by the family of norms

$$\mathcal{A} \ni a \mapsto \|\delta^k(a)\| + \|\delta^k([\mathcal{D},a])\|, \quad k \in \mathbb{N}_0. \tag{2.5}$$

We call this topology the δ-*topology* and observe that by [**49**, Lemma 16], $(\mathcal{A}_\delta,\mathcal{H},\mathcal{D})$ is also a QC^∞ spectral triple, where $\mathcal{A}_\delta$ is the completion of $\mathcal{A}$ in the δ-topology. Thus we may, without loss of generality, suppose that $\mathcal{A}$ is complete in the δ-topology by completing if necessary. This completion is Fréchet and stable under the holomorphic functional calculus. So, with A the C^*-completion of $\mathcal{A}$, $K_*(\mathcal{A}) \simeq K_*(A)$ via inclusion.

However, and this is crucial in the remaining text, in the nonunital case the completion $\mathcal{A}_\delta$ may not satisfy the same summability conditions as $\mathcal{A}$ (as classical examples show). Thus we will define and use a finer topology which takes into account the summability of the spectral triple, to which we now return.

Keeping in mind Propositions 2.14, 2.16, 2.17, and incorporating smoothness in the picture, we see that the natural condition for a smooth and finitely summable spectral triple is to require that $\mathcal{A}\cup[\mathcal{D},\mathcal{A}] \subset \mathcal{B}_1^\infty(\mathcal{D},p)$. The extra benefit is that our algebra $\mathcal{A}$ sits inside a Fréchet algebra which is stable under the holomorphic functional calculus.

Definition 2.19. Let $(\mathcal{A},\mathcal{H},\mathcal{D})$ be a semifinite spectral triple relative to $(\mathcal{N},\tau)$. Then we say that $(\mathcal{A},\mathcal{H},\mathcal{D})$ is QC^k *summable* if $(\mathcal{A},\mathcal{H},\mathcal{D})$ is finitely summable with

spectral dimension p and
$$\mathcal{A} \cup [\mathcal{D}, \mathcal{A}] \subset \mathcal{B}_1^k(\mathcal{D}, p).$$
We say that $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is *smoothly summable* if it is QC^k summable for all $k \in \mathbb{N}_0$ or, equivalently, if
$$\mathcal{A} \cup [\mathcal{D}, \mathcal{A}] \subset \mathcal{B}_1^\infty(\mathcal{D}, p).$$
If $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is smoothly summable with spectral dimension p, the δ-φ-topology on $\mathcal{A}$ is determined by the family of norms
$$\mathcal{A} \ni a \mapsto \mathcal{P}_{n,k}(a) + \mathcal{P}_{n,k}([\mathcal{D}, a]),\ n \in \mathbb{N},\ k \in \mathbb{N}_0,$$
where the norms $\mathcal{P}_{n,k}$ are those of Definition 1.21,
$$\mathcal{N} \ni T \mapsto \mathcal{P}_{n,k}(T) := \sum_{j=0}^{k} \mathcal{P}_n(\delta^j(T)).$$

Remark. The δ-φ-topology generalises the δ-topology. Indeed, if $(1+\mathcal{D}^2)^{-s/2}$ belongs to $\mathcal{L}^1(\mathcal{N}, \tau)$ for $s > p$, then the norm $\mathcal{P}_{n,k}$ is equivalent to the norm defined in Equation (2.5).

The following result shows that given a smoothly summable spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$, we may without loss of generality assume that the algebra $\mathcal{A}$ is complete with respect to the δ-φ-topology, by completing if necessary. Moreover the completion of $\mathcal{A}$ in the δ-φ-topology is stable under the holomorphic functional calculus.

Proposition 2.20. *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a smoothly summable semifinite spectral triple with spectral dimension p, and let $\mathcal{A}_{\delta,\varphi}$ denote the completion of $\mathcal{A}$ for the δ-φ topology. Then $(\mathcal{A}_{\delta,\varphi}, \mathcal{H}, \mathcal{D})$ is also a smoothly summable semifinite spectral triple with spectral dimension p, and moreover $\mathcal{A}_{\delta,\varphi}$ is stable under the holomorphic functional calculus.*

Proof. First observe that a sequence $(a_i)_{i\geq 1} \subset \mathcal{A}$ converges in the δ-φ topology if and only if both $(a_i)_{i\geq 1}$ and $([\mathcal{D}, a_i])_{i\geq 1}$ converge in $\mathcal{B}_1^\infty(\mathcal{D}, p)$. As $\mathcal{B}_1^\infty(\mathcal{D}, p)$ is a Fréchet space, both $\mathcal{A}_{\delta,\varphi}$ and $[\mathcal{D}, \mathcal{A}_{\delta,\varphi}]$ are contained in $\mathcal{B}_1^\infty(\mathcal{D}, p)$.

Next, let us show that $(\mathcal{A}_{\delta,\varphi}, \mathcal{H}, \mathcal{D})$ is finitely summable with spectral dimension still given by p. Let $a \in \mathcal{A}_{\delta,\varphi}$ and $s > p$. By definition of tame pseudodifferential operators and Corollary 1.30, we have
$$a(1+\mathcal{D}^2)^{-s/2} \in \mathrm{OP}_0^{-s} \subset \mathcal{L}^1(\mathcal{N}, \tau),$$
as needed. Since $\mathcal{A} \subset \mathcal{A}_{\delta,\varphi}$, p is the smallest number for which this property holds.

Last, it remains to show that $\mathcal{A}_{\delta,\varphi}$ is stable under the holomorphic functional calculus inside its (operator) norm completion. For each k, $N \geq 1$, we complete $\mathcal{A}$ in the norm $\|\cdot\|_{N,k} := \sum_{n=1}^{N} \sum_{j=0}^{k} \mathcal{P}_{n,j}(\cdot) + \mathcal{P}_{n,j}([\mathcal{D}, \cdot])$ to obtain a Banach algebra $\mathcal{A}_{N,k}$. We claim that $\mathcal{A}_{\delta,\varphi} = \bigcap_{N\geq 1, k\geq 0} \mathcal{A}_{N,k}$. The inclusion $\mathcal{A}_{\delta,\varphi} \subset \bigcap_{N\geq 1,k\geq 0} \mathcal{A}_{N,k}$ is straightforward. For the inclusion $\mathcal{A}_{\delta,\varphi} \supset \bigcap_{N\geq 1,k\geq 0} \mathcal{A}_{N,k}$, suppose that a is an element of the intersection. Then for each N, k there is a sequence $(a_i^{N,k})_{i\geq 1}$ contained in $\mathcal{A}$ which converges to a in the norm $\|\cdot\|_{N,k}$. Now we make the observation that if $N' \leq N$ and $k' \leq k$ then $(a_i^{N,k})_{i\geq 1}$ converges in $\mathcal{A}_{N',k'}$ to the same limit. Thus, in this situation, for all $\varepsilon > 0$ there is $l \in \mathbb{N}$ such that $i > l$ implies that $\|a_i^{N,k} - a\|_{N',k'} < \varepsilon$. Thus for such an $\varepsilon > 0$ and l we have $\|a_N^{N,N} - a\|_{N',k'} < \varepsilon$

whenever $N > \max\{N', k', l\}$. Hence the sequence $(a_N^{N,N})_{N\geq 1}$ converges in all of the norms $\|\cdot\|_{N',k'}$ and hence the limit a lies in $\mathcal{A}_{\delta,\varphi}$. Hence an element of $\mathcal{A}_{\delta,\varphi}$ is an element of A which lies in each $\mathcal{A}_{N,k}$. Moreover the norm completions of $\mathcal{A}$, $\mathcal{A}_{\delta,\varphi}$ and $\mathcal{A}_{N,k}$, for each N, k, are all the same since the δ-φ and $\|\cdot\|_{N,k}$ topologies are finer than the norm topology. We denote the latter by A. Now let $a \in \mathcal{A}_{\delta,\varphi}$ and $\lambda \in \mathbb{C}$ be such that $a+\lambda$ is invertible in $A^{\sim}$. Then with $b = (a+\lambda)^{-1} - \lambda^{-1}$ we have

$$(a+\lambda)(b+\lambda^{-1}) = 1 = 1 + ab + \lambda b + \lambda^{-1}a \quad\Rightarrow\quad b = -\lambda^{-1}ab - \lambda^{-2}a. \tag{2.6}$$

Rearranging Equation (2.6) shows that $b = -\lambda^{-1}(\lambda+a)^{-1}a$. Now as $\mathcal{B}_1^\infty(\mathcal{D},p)$ is stable under the holomorphic functional calculus, $b \in \mathcal{B}_1^\infty(\mathcal{D},p) \oplus \mathbb{C}$, but this formula shows that in fact $b \in \mathcal{B}_1^\infty(\mathcal{D},p)$. Now we would like to apply $[\mathcal{D},\cdot]$ to Equation (2.6). Since $b \in \mathcal{B}_1^\infty(\mathcal{D},p)$, b preserves $\operatorname{dom}\mathcal{D} = \operatorname{dom}|\mathcal{D}| \subset \mathcal{H}$, and so it makes sense to apply $[\mathcal{D},\cdot]$ to b. Then

$$[\mathcal{D},b] = -\lambda^{-1}[\mathcal{D},a]b - \lambda^{-1}a[\mathcal{D},b] - \lambda^{-2}[\mathcal{D},a] \quad\Rightarrow\quad [\mathcal{D},b] = -(\lambda+a)^{-1}[\mathcal{D},a](\lambda+a)^{-1}.$$

Thus we see that $[\mathcal{D},b] \in \mathcal{B}_1^\infty(\mathcal{D},p)$ since $(\lambda+a)^{-1} \in \mathcal{B}_1^\infty(\mathcal{D},p) \oplus \mathbb{C}$ and $[\mathcal{D},a] \in \mathcal{B}_1^\infty(\mathcal{D},p)$. Hence $b \in \mathcal{A}_{N,k}$ for all $N \geq 1$ and $k \geq 0$ and so $b \in \mathcal{A}_{\delta,\varphi}$. $\square$

We close this Section by giving a sufficient condition for a finitely summable spectral triple to be smoothly summable. We stress that this condition is easy to check, as shown in all of our examples.

PROPOSITION 2.21. *Let $(\mathcal{A},\mathcal{H},\mathcal{D})$ be a finitely summable spectral triple of spectral dimension p relative to $(\mathcal{N},\tau)$. If for all $T \in \mathcal{A} \cup [\mathcal{D},\mathcal{A}]$, $k \in \mathbb{N}_0$ and all $s > p$ we have*

$$(1+\mathcal{D}^2)^{-s/4}L^k(T)(1+\mathcal{D}^2)^{-s/4} \in \mathcal{L}^1(\mathcal{N},\tau), \tag{2.7}$$

then $(\mathcal{A},\mathcal{H},\mathcal{D})$ is smoothly summable. Here $L(T) = (1+\mathcal{D}^2)^{-1/2}[\mathcal{D}^2,T]$.

PROOF. We need to show that condition (2.7) guarantees that $\mathcal{A} \cup [\mathcal{D},\mathcal{A}] \subset \mathcal{B}_1^\infty(\mathcal{D},p)$, that is, for all $a \in \mathcal{A}$, the operators $\delta^k(a)$ and $\delta^k([\mathcal{D},a])$, $k \in \mathbb{N}_0$, all belong to $\mathcal{B}_1(\mathcal{D},p)$. From $\delta^k(a)^* = (-1)^k\delta^k(a^*)$ (respectively $\delta^k([\mathcal{D},a])^* = (-1)^{k+1}\delta^k([\mathcal{D},a^*])$) and since the norms $\mathcal{P}_m$, $m \in \mathbb{N}$, are $*$-invariant, we see that $\delta^k(a) \in \mathcal{B}_1(\mathcal{D},p)$ (resp. $\delta^k([\mathcal{D},a]) \in \mathcal{B}_1(\mathcal{D},p)$) if and only if $\delta^k(\Re(a))$ and $\delta^k(\Im(a))$ (resp. $\delta^k([\mathcal{D},\Re(a)])$ and $\delta^k([\mathcal{D},\Im(a)])$) belong to $\mathcal{B}_1(\mathcal{D},p)$. Thus, we may assume that $a = a^*$.

Let us treat first the case of $\delta^k(a)$ and for $a = a^*$. Consider the polar decomposition $\delta^k(a) = u_k|\delta^k(a)|$. Depending on the parity of k, the partial isometry u_k is self-adjoint or skew-adjoint, and in both cases it commutes with $|\delta^k(a)|$. This implies that

$$\delta^k(a) = |\delta^k(a)|^{1/2}u_k|\delta^k(a)|^{1/2}.$$

Thus, the condition

$$\delta^k(a) \in \mathcal{B}_1(\mathcal{D},p), \text{ for all } k \in \mathbb{N}_0,$$

will follow if

$$|\delta^k(a)|^{1/2},\ u_k|\delta^k(a)|^{1/2} \in \mathcal{B}_2(\mathcal{D},p), \text{ for all } k \in \mathbb{N}_0. \tag{2.8}$$

Since u_k commutes with $|\delta^k(a)|^{1/2}$, and using the definition of the space $\mathcal{B}_2(\mathcal{D},p)$, the condition (2.8) is equivalent to

$$|\delta^k(a)|^{1/2}(1+\mathcal{D}^2)^{-s/4},\ u_k|\delta^k(a)|^{1/2}(1+\mathcal{D}^2)^{-s/4} \in \mathcal{L}^2(\mathcal{N},\tau), \tag{2.9}$$

for all $k \in \mathbb{N}_0$ and $s > p$. The conditions in (2.9) are equivalent to a single condition

$$|\delta^k(a)|^{1/2}(1+\mathcal{D}^2)^{-s/4} \in \mathcal{L}^2(\mathcal{N},\tau), \text{ for all } k \in \mathbb{N}_0 \text{ and } s > p,$$

which is equivalent to

$$(1+\mathcal{D}^2)^{-s/4}|\delta^k(a)|(1+\mathcal{D}^2)^{-s/4} \in \mathcal{L}^1(\mathcal{N},\tau), \text{ for all } k \in \mathbb{N}_0 \text{ and } s > p. \tag{2.10}$$

Now, by [**6**, Theorem 3], see Proposition 1.5, the condition (2.10) is satisfied if

$$|\delta^k(a)|(1+\mathcal{D}^2)^{-s/2} \in \mathcal{L}^1(\mathcal{N},\tau), \text{ for all } k \in \mathbb{N}_0 \text{ and } s > p,$$

which in turn is equivalent to

$$\delta^k(a)(1+\mathcal{D}^2)^{-s/2} \in \mathcal{L}^1(\mathcal{N},\tau), \text{ for all } k \in \mathbb{N}_0 \text{ and } s > p. \tag{2.11}$$

Next, since

$$\delta^k(a)(1+\mathcal{D}^2)^{-s/2} = (1+\mathcal{D}^2)^{-s/4}\delta^k(\sigma^{s/4}(a))(1+\mathcal{D}^2)^{-s/4},$$

by an application of the same ideas leading to Lemmas 1.25 and 1.26, we see then that condition (2.11) is equivalent to

$$(1+\mathcal{D}^2)^{-s/4}\delta^k(a)(1+\mathcal{D}^2)^{-s/4} \in \mathcal{L}^1(\mathcal{N},\tau), \text{ for all } k \in \mathbb{N}_0 \text{ and } s > p. \tag{2.12}$$

Finally, using $L = (1+\sigma^{-1})\circ\delta$, given in Lemma 1.29, we see that condition (2.12) is equivalent to

$$(1+\mathcal{D}^2)^{-s/4}L^k(a)(1+\mathcal{D}^2)^{-s/4} \in \mathcal{L}^1(\mathcal{N},\tau), \text{ for all } k \in \mathbb{N}_0 \text{ and } s > p.$$

In an entirely similar way, we see that $\delta^k([\mathcal{D},a]) \in \mathcal{B}_1(\mathcal{D},p)$ if

$$(1+\mathcal{D}^2)^{-s/4}L^k([\mathcal{D},a])(1+\mathcal{D}^2)^{-s/4} \in \mathcal{L}^1(\mathcal{N},\tau), \text{ for all } k \in \mathbb{N}_0 \text{ and } s > p.$$

This completes the proof. □

2.5. Some cyclic theory

In the following discussion we recall cyclic theory, sufficient for the purposes of this memoir. More information about the complexes and bicomplexes underlying our definitions is contained in [**15**, **17**], and much more can be found in [**21**, **40**]. When we discuss tensor products of algebras we always use the projective tensor product.

Let $\mathcal{A}$ be a *unital* Fréchet algebra. A cyclic m-cochain on $\mathcal{A}$ is a multilinear functional ψ such that

$$\psi(a_0,\dots,a_m) = (-1)^m\psi(a_m,a_0,\dots,a_{m-1}).$$

The set of all cyclic cochains is denoted C_λ^m. We say that ψ is a cyclic cocycle if for all $a_0,\dots,a_{m+1} \in \mathcal{A}$ we have $(b\psi)(a_0,\dots,a_{m+1}) = 0$ where b is the Hochschild coboundary in Equation (2.13) below. The cyclic cochain is normalised if $\psi(a_0,a_1,\dots,a_m) = 0$ whenever any of $a_1,\dots,a_m$ is the unit of $\mathcal{A}$.

A (b,B)-cochain ϕ for $\mathcal{A}$ is a finite collection of multilinear functionals,

$$\phi = (\phi_m)_{m=0,1,\dots,M}, \quad \phi_m : \mathcal{A}^{\otimes(m+1)} \to \mathbb{C}.$$

An odd cochain has $\phi_m = 0$ for even m, while an even cochain has $\phi_m = 0$ for odd m. Thought of as functionals on the projective tensor product $\mathcal{A}^{\otimes(m+1)}$, a normalised cochain will satisfy $\phi(a_0,a_1,\dots,a_m) = 0$ whenever for $k \geq 1$, any $a_k = 1_{\mathcal{A}}$. A normalised cochain is a (b, B)-cocycle if, for all m, $b\phi_m + B\phi_{m+2} = 0$ where b is the Hochschild coboundary operator given by

$$\begin{aligned}(b\phi_m)(a_0, a_1, \ldots, a_{m+1}) = &\sum_{k=0}^{m}(-1)^k \phi_m(a_0, a_1, \ldots, a_k a_{k+1}, \ldots, a_{m+1}) \\ &+ (-1)^{m+1}\phi_m(a_{m+1}a_0, a_1, \ldots, a_m),\end{aligned} \tag{2.13}$$

and B is Connes' coboundary operator

$$\begin{aligned}&(B\phi_m)(a_0, a_1, \ldots, a_{m-1}) \\ &\quad= \sum_{k=0}^{m-1}(-1)^{(m-1)j}\phi_m(1_{\mathcal{A}}, a_k, a_{k+1}, \ldots, a_{m-1}, a_0, \ldots, a_{k-1}).\end{aligned} \tag{2.14}$$

We write $(b+B)\phi = 0$ for brevity, and observe that this formula for B is only valid on the normalised complex, [**40**]. As we will only consider normalised cochains, this will be sufficient for our purposes.

For a *nonunital* Fréchet algebra $\mathcal{A}$, a reduced (b, B)-cochain $(\phi_n)_{n=\bullet,\bullet+2,\ldots,M}$ for $\mathcal{A}^\sim$ and of parity $\bullet \in \{0, 1\}$, is a normalised (b, B)-cochain such that if $\bullet = 0$ we have $\phi_0(1_{\mathcal{A}^\sim}) = 0$. The formulae for the operators b, B are the same. By [**40**, Proposition 2.2.16], the reduced cochains come from a suitable bicomplex called the reduced (b, B)-bicomplex, which gives a cohomology theory for $\mathcal{A}$.

Thus far, our discussion has been algebraic. We now remind the reader that when working with a Fréchet algebra, we complete the algebraic tensor product in the projective tensor product topology. Given a spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$, we may without loss of generality complete $\mathcal{A}$ in the δ-φ-topology using Proposition 2.20. Then the algebraic discussion above carries through. This follows because the operators b and B are defined using multiplication, which is continuous, and insertion of $1_{\mathcal{A}^\sim}$ in the first slot. This latter is also continuous, and one just needs to check that $B : C^1(\mathcal{A}) \to C^0(\mathcal{A})$ maps *normalised* cochains to cochains vanishing on the unit $1_{\mathcal{A}^\sim} \in \mathcal{A}^\sim$. This follows from the definitions.

Finally, an $(n+1)$-linear functional on an algebra $\mathcal{A}$ is cyclic if and only if it is the character of a cycle, [**21**, Chapter III], [**30**, Proposition 8.12], and so the Chern character of a Fredholm module over $\mathcal{A}$, defined in the next section, will always define a reduced cyclic cocyle for $\mathcal{A}^\sim$.

2.6. The Kasparov product, numerical index and Chern character

First we discuss the Chern character of semifinite Fredholm modules and then relate the Chern character to our analytic index pairing and the Kasparov product.

Definition 2.22. Let $(\mathcal{H}, F)$ be a Fredholm module relative to $(\mathcal{N}, \tau)$. We define the 'conditional trace' τ' by

$$\tau'(T) = \tfrac{1}{2}\tau\big(F(FT + TF)\big),$$

provided $FT + TF \in \mathcal{L}^1(\mathcal{N})$ (as it will be in our case, see [**21**, p. 293] and (2.15) below). Note that if $T \in \mathcal{L}^1(\mathcal{N})$, using the trace property and $F^2 = 1$, we find $\tau'(T) = \tau(T)$.

The **Chern character**, $[\mathrm{Ch}_F]$, of a $(p+1)$-summable ($p \geq 1$) semifinite Fredholm module $(\mathcal{H}, F)$ relative to $(\mathcal{N}, \tau)$ is the class in periodic cyclic cohomology of the single normalized and reduced cyclic cocycle

$$\lambda_m \tau'\big(\gamma a_0[F,a_1]\dots[F,a_m]\big), \quad a_0,\dots,a_m \in \mathcal{A}, \quad m \geq \lfloor p \rfloor,$$

where m is even if and only if $(\mathcal{H},F)$ is even. Here λ_m are constants ensuring that this collection of cocycles yields a well-defined periodic class, and they are given by

$$\lambda_m = \begin{cases} (-1)^{m(m-1)/2}\Gamma(\frac{m}{2}+1), & m \quad \text{even}, \\ \sqrt{2i}(-1)^{m(m-1)/2}\Gamma(\frac{m}{2}+1), & m \quad \text{odd}. \end{cases}$$

For $p = n \in \mathbb{N}$, the Chern character of an $(n+1)$-summable Fredholm module of the same parity as n, is represented by the cyclic cocycle in dimension n, $\mathrm{Ch}_F \in C^n_\lambda(\mathcal{A})$, given by

$$\mathrm{Ch}_F(a_0,\dots,a_n) = \lambda_n \tau'(\gamma a_0[F,a_1]\dots[F,a_n]), \qquad a_0,\dots,a_n \in \mathcal{A}.$$

The latter makes good sense since

$$\begin{aligned} &F\gamma a_0[F,a_1]\dots[F,a_n] + \gamma a_0[F,a_1]\dots[F,a_n]F \\ &\quad = (-1)^n \gamma[F,a_0][F,a_1]\dots[F,a_n], \end{aligned} \tag{2.15}$$

belongs to $\mathcal{L}^1(\mathcal{N},\tau)$ by the $(p+1)$-summability assumption. We will always take the cyclic cochain Ch_F (or its (b,B) analogue; see below) as representative of $[\mathrm{Ch}_F]$, and will often refer to Ch_F as the Chern character.

Since the Chern character is a cyclic cochain, it lies in the image of the operator B, [**21**, Corollary 20, III.1.β], and as $B^2 = 0$ we have $B\,\mathrm{Ch}_F = 0$. Since $b\,\mathrm{Ch}_F = 0$, we may regard the Chern character as a one term element of the (b,B)-bicomplex. However, the correct normalisation is (taking the Chern character in degree n)

$$C^n_\lambda \ni \mathrm{Ch}_F \mapsto \frac{(-1)^{\lfloor n/2 \rfloor}}{n!}\mathrm{Ch}_F \in C^n.$$

Thus instead of λ_n defined above, we use $\mu_n := \frac{(-1)^{\lfloor n/2 \rfloor}}{n!}\lambda_n$. The difference in normalisation between periodic and (b,B) is due to the way the index pairing is defined in the two cases, [**21**], and compatibility with the periodicity operator. From now on we will use the (b,B)-normalisation, and so make the following definition.

Definition 2.23. Let $(\mathcal{H},F)$ be a semifinite $(n+1)$-summable, $n \in \mathbb{N}$, Fredholm module for a nonunital algebra $\mathcal{A}$, relative to $(\mathcal{N},\tau)$, and suppose the parity of the Fredholm module is the same as the parity of n. Then we define the Chern character $[\mathrm{Ch}_F]$ to be the cyclic cohomology class of the single term (b,B)-cocycle defined for $a_0,\dots,a_n \in \mathcal{A}$ by

$$\mathrm{Ch}^n_F(a_0,a_1,\dots,a_n) := \begin{cases} \frac{\Gamma(\frac{n}{2}+1)}{n!}\tau'(\gamma a_0[F,a_1]\dots[F,a_n]), & n \quad \text{even}, \\[2ex] \sqrt{2i}\frac{\Gamma(\frac{n}{2}+1)}{n!}\tau'(a_0[F,a_1]\dots[F,a_n]), & n \quad \text{odd}. \end{cases}$$

If $e \in \mathcal{A}^\sim$ is a projection we define $\mathrm{Ch}_0(e) = e \in \mathcal{A}^\sim$ and for $k \geq 1$

$$\mathrm{Ch}_{2k}(e) = (-1)^k \frac{(2k)!}{k!}(e - 1/2) \otimes e \otimes \cdots \otimes e \in (\mathcal{A}^\sim)^{\otimes 2k+1}.$$

If $u \in \mathcal{A}^\sim$ is a unitary then we define for $k \geq 0$

$$\mathrm{Ch}_{2k+1}(u) = (-1)^k\, k!\, u^* \otimes u \otimes \cdots \otimes u^* \otimes u \in (\mathcal{A}^\sim)^{\otimes 2k+2}.$$

In order to prove the equality of our numerical index with the Chern character pairing, we need the cyclicity of the trace on a semifinite von Neumann algebra from [**8**, Theorem 17], quoted here as Proposition 1.4.

PROPOSITION 2.24. *Let $(\mathcal{A},\mathcal{H},\mathcal{D})$ be a semifinite spectral triple, with $\mathcal{A}$ separable, which is smoothly summable with spectral dimension $p \geq 1$, and such that $\lfloor p \rfloor$ has the same parity as the spectral triple. Then for a class $[e] \in K_0(\mathcal{A})$, with e a projection in $M_n(\mathcal{A}^\sim)$ (resp. for a class $[u] \in K_1(\mathcal{A})$, with u a unitary in $M_n(\mathcal{A}^\sim)$) we have for any $\mu > 0$*

$$\begin{aligned}\langle [e]-[\mathbf{1}_e],(\mathcal{A},\mathcal{H},\mathcal{D})\rangle &= \mathrm{Ch}^{\lfloor p\rfloor}_{F_\mu\otimes \mathrm{Id}_n}\big(\mathrm{Ch}_{\lfloor p\rfloor}(\hat{e})\big), \quad \textit{even case},\\ \langle [u],(\mathcal{A},\mathcal{H},\mathcal{D})\rangle &= -(2i\pi)^{-1/2}\,\mathrm{Ch}^{\lfloor p\rfloor}_{F_\mu\otimes \mathrm{Id}_n}\big(\mathrm{Ch}_{\lfloor p\rfloor}(\hat{u})\big), \quad \textit{odd case}.\end{aligned}$$

PROOF. The first thing to prove is that $[F_\mu,\hat{a}] \in \mathcal{L}^{\lfloor p\rfloor+1}(\mathcal{N},\tau)$ for all $a \in \mathcal{A}$. This will follow if we have $[F_\varepsilon, a] \in \mathcal{L}^{\lfloor p\rfloor+1}(\mathcal{N},\tau)$ for all $a \in \mathcal{A}$. By the smooth summability assumption, we have a, $[\mathcal{D},a] \in \mathcal{B}_1^\infty(\mathcal{D},p) = \mathrm{Op}^0_0$ for all $a \in \mathcal{A}$. Thus the Schatten class property we need follows from Proposition 2.14.

For the even case the remainder of the proof is just as in [**21**, Proposition 4, IV.1.γ]. The strategy in the odd case is the same. However, we present the proof in the odd case in order to clarify some sign conventions. To simplify the notation, we let u be a unitary in $\mathcal{A}^\sim$ and suppress the matrices $M_n(\mathcal{A}^\sim)$. In this case the operator $P_\mu\hat{u}P_\mu : P_\mu(\mathcal{H}\oplus\mathcal{H}) \to P_\mu(\mathcal{H}\oplus\mathcal{H})$, is $\tau\otimes\mathrm{tr}_2$-Fredholm with parametrix $P_\mu\hat{u}^*P_\mu$, where $u \in \mathcal{A}^\sim$ unitary and $P_\mu = (F_\mu+1)/2 \in M_2(\mathcal{N})$. To obtain our result, we need [**45**, Lemma 3.5] which shows that with $Q_\mu := \hat{u}P_\mu\hat{u}^*$ we have

$$|(1-Q_\mu)P_\mu|^{2n} = [P_\mu(1-Q_\mu)(1-Q_\mu)P_\mu]^n = [P_\mu - P_\mu Q_\mu P_\mu]^n = (P_\mu - P_\mu\hat{u}P_\mu\hat{u}^*P_\mu)^n.$$

One ingredient in the proof that connects this to odd summability is the identity

$$(Q_\mu - P_\mu)^{2n+1} = |(1-P_\mu)Q_\mu|^{2n} - |(1-Q_\mu)P_\mu|^{2n},$$

proved by induction in [**45**, Lemma 3.4]. It is then shown in [**13**, Theorem 3.1] that if f is any odd function with $f(1) \neq 0$ and $f(Q_\mu - P_\mu)$ trace-class, we have

$$\mathrm{Index}_{\tau\otimes\mathrm{tr}_2}(P_\mu Q_\mu) = \frac{1}{f(1)}\tau\otimes\mathrm{tr}_2\big(f(Q_\mu - P_\mu)\big).$$

Putting these ingredients together we have

$$\begin{aligned}\mathrm{Index}_{\tau\otimes\mathrm{tr}_2}(P_\mu\hat{u}P_\mu) &= \mathrm{Index}_{\tau\otimes\mathrm{tr}_2}(P_\mu\hat{u}P_\mu\hat{u}^*) = \mathrm{Index}_{\tau\otimes\mathrm{tr}_2}(P_\mu Q_\mu)\\ &= \tau\otimes\mathrm{tr}_2\big((P_\mu - P_\mu\hat{u}^*P_\mu\hat{u}P_\mu)^n - (P_\mu - P_\mu\hat{u}P_\mu\hat{u}^*P_\mu)^n\big),\end{aligned}$$

where $n = (\lfloor p\rfloor + 1)/2$ is an integer, since $\lfloor p\rfloor$ is assumed odd. First we observe that $P_\mu - P_\mu\hat{u}^*P_\mu\hat{u}P_\mu = -P_\mu[\hat{u}^*,P_\mu]\hat{u}P_\mu$, and by replacing P_μ by $(1+F_\mu)/2$ we have

$$P_\mu[\hat{u}^*,P_\mu]\hat{u}P_\mu = [F_\mu,\hat{u}^*]\,[F_\mu,\hat{u}](1+F_\mu)/8.$$

Since $F_\mu[F_\mu, \hat{a}] = -[F_\mu, \hat{a}]F_\mu$ for all $a \in \mathcal{A}$, cycling a single $[F_\mu, \hat{u}^*]$ around using Proposition 1.4 yields

$$\begin{aligned}
&\mathrm{Index}_{\tau\otimes\mathrm{tr}_2}(P_\mu\hat{u}P_\mu) = \tau\otimes\mathrm{tr}_2\big((P_\mu - P_\mu\hat{u}^*P_\mu\hat{u}P_\mu)^n\big) - \tau\otimes\mathrm{tr}_2\big((P_\mu - P_\mu\hat{u}P_\mu\hat{u}^*P_\mu)^n\big)\\
&= \tau\otimes\mathrm{tr}_2\Big(\Big(-\frac{1}{4}[F_\mu,\hat{u}^*]\,[F_\mu,\hat{u}]\,\frac{1+F_\mu}{2}\Big)^n\Big)\\
&\qquad - \tau\otimes\mathrm{tr}_2\Big(\Big(-\frac{1}{4}[F_\mu,\hat{u}]\,[F_\mu,\hat{u}^*]\,\frac{1+F_\mu}{2}\Big)^n\Big)\\
&= (-1)^n\frac{1}{4^n}\tau\otimes\mathrm{tr}_2\Big(\frac{1+F_\mu}{2}([F_\mu,\hat{u}^*][F_\mu,\hat{u}])^n\\
&\qquad - [F_\mu,\hat{u}^*][F_\mu,\hat{u}][F_\mu,\hat{u}^*]\frac{1+F_\mu}{2}[F_\mu,\hat{u}][F_\mu,\hat{u}^*]\dots\frac{1+F_\mu}{2}[F_\mu,\hat{u}]\frac{1-F_\mu}{2}\Big).
\end{aligned}$$

Thus

$$\begin{aligned}
\mathrm{Index}_{\tau\otimes\mathrm{tr}_2}(P_\mu\hat{u}P_\mu) &= (-1)^n\frac{1}{4^n}\tau\otimes\mathrm{tr}_2\Big(\Big(\frac{1+F_\mu}{2}-\frac{1-F_\mu}{2}\Big)\big([F_\mu,\hat{u}^*][F_\mu,\hat{u}]\big)^n\Big)\\
&= (-1)^n\frac{1}{4^n}\tau\otimes\mathrm{tr}_2\big(F_\mu([F_\mu,\hat{u}^*][F_\mu,\hat{u}])^n\big)\\
&= (-1)^n\frac{1}{2^{2n-1}}(\tau\otimes\mathrm{tr}_2)'\big(\hat{u}^*[F_\mu,\hat{u}]\dots[F_\mu,\hat{u}^*][F_\mu,\hat{u}]\big),
\end{aligned}$$

where in the last line there are $2n-1 = \lfloor p\rfloor$ commutators. Comparing the normalisation of the formulae above with the Chern characters using the duplication formula for the Gamma function, we find

$$\mathrm{Index}_{\tau\otimes\mathrm{tr}_2}(P_\mu\hat{u}P_\mu) = \frac{-1}{\sqrt{2\pi i}}\mathrm{Ch}^{\lfloor p\rfloor}_{F_\mu}(\mathrm{Ch}_{\lfloor p\rfloor}(\hat{u})),$$

as needed. □

Remark. When the parity of $\lfloor p\rfloor$ does not agree with the parity of the spectral triple, we apply the same proof to $\lfloor p\rfloor+1$, and so use $\mathrm{Ch}^{\lfloor p\rfloor+1}_{F_\mu\otimes\mathrm{Id}_n}$ to represent the class of the Chern character.

Remark. An independent check of the sign can be made on the circle, using the unitary $u = e^{i\theta}$ and the Dirac operator $\frac{1}{i}\frac{d}{d\theta}$. In this case $\mathrm{Index}(PuP) = -1$. To arrive at this sign we have retained the usual definition of the Chern character and introduced an additional minus sign in the normalisation. In [**15**] the signs used are all correct, however in [**17**] we introduced an additional minus sign (in error) in the formula for spectral flow. This disguised the fact that we were not taking a homotopy to the Chern character (as defined above) but rather to minus the Chern character. This is of some relevance, as our strategy for proving the local index formula in the nonunital case is based on the homotopy arguments of [**17**].

2.7. Digression on the odd index pairing for nonunital algebras

To emphasise that the introduction of the double is only a technical device to enable us to work with invertible operators, we explain a different approach to handling the problem of constructing an involutive Fredholm module in the odd case.

Assume that we have an odd Fredholm module $(\mathcal{H}, F)$ over a nonunital C^*-algebra A, with $F^2 = 1$. Then, as mentioned previously, it is straightforward to check that with $P = (1+F)/2$ and $u \in A^\sim$ a unitary, the operator PuP is Fredholm with parametrix Pu^*P (as operators on $P\mathcal{H}$).

Now we have constructed a doubled up version of a spectral triple $(\mathcal{A}, \mathcal{H}^2, \mathcal{D}_\mu)$, and so obtained a Fredholm module $(\mathcal{H}^2, F_\mu)$ with $F_\mu^2 = 1$. By Lemma 2.10, this Fredholm module represents the class of our spectral triple. In this brief digression we show that the odd index pairing can be defined in terms of the original data with no doubling.

So assume that we have a spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$. First we can decompose $P := \chi_{[0,\infty)}(\mathcal{D})$ as the kernel projection P_0 plus the positive spectral projection P_+. We will use P_- for the negative spectral projection so that $P_- + P_0 + P_+$ is the identity of $\mathcal{N}$. We let $F = 2P - 1$ and we want to prove that F can be used to construct a Fredholm module for $\mathcal{A}$ that is in the same Kasparov class as that given by $F_\varepsilon := \mathcal{D}(\varepsilon + \mathcal{D}^2)^{-1/2}$. If we can show that $[F, a]$ is compact for all $a \in \mathcal{A}$ then we are done because the straight-line path $F_t = tF + (1-t)F_\varepsilon$ provides a homotopy of Kasparov modules. To prove compactness of the commutators we use the method of [**11**].

PROPOSITION 2.25. *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a semifinite spectral triple relative to $(\mathcal{N}, \tau)$ with $\mathcal{A}$ separable. With $F = 2\chi_{[0,\infty)}(\mathcal{D}) - 1$, the pair $(\mathcal{H}, F)$ is a Fredholm module for $\mathcal{A}$ and (F, C_C) (with C the C^*-completion of the subalgebra of $\mathcal{K}(\mathcal{N}, \tau)$ given in Definition 2.5) provides a bounded representative for the Kasparov class of the spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$.*

PROOF. Our proof uses the doubled spectral triple $(\mathcal{A}, \mathcal{H}^2, \mathcal{D}_\mu)$. Let $P_\mu = (1 + F_\mu)/2$ and use the notation Q for the operator obtained by taking the strong limit $\lim_{\mu\to 0} P_\mu$ as $\mu \to 0$. We note that

$$Q = \begin{pmatrix} P_+ + \frac{1}{2}P_0 & \frac{1}{2}P_0 \\ \frac{1}{2}P_0 & P_- + \frac{1}{2}P_0 \end{pmatrix} \text{ and } P_\mu = \begin{pmatrix} A & A^{1/2}(1-A)^{1/2} \\ A^{1/2}(1-A)^{1/2} & 1-A \end{pmatrix},$$

where $A = \frac{1}{2}\big((\mu^2 + \mathcal{D}^2)^{1/2} + \mathcal{D}\big)(\mu^2 + \mathcal{D}^2)^{-1/2}$. Next a short calculation shows that

$$2Q - 1 = \begin{pmatrix} F & 0 \\ 0 & -F \end{pmatrix} + \begin{pmatrix} -P_0 & P_0 \\ P_0 & -P_0 \end{pmatrix}.$$

Recall that in the double spectral triple

$$a \mapsto \hat{a} = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}, \quad \text{for all } a \in \mathcal{A}.$$

Thus to show that $[F, a]$ is compact for all $a \in \mathcal{A}$, it suffices to show that $[Q, \hat{a}]$ is compact, since for any $s > 0$ we have $P_0 a = P_0(1 + \mathcal{D}^2)^{-s}a$ and so both $P_0 a$ and aP_0 are compact for all $a \in \mathcal{A}$. This follows since $a(1 + \mathcal{D}^2)^{-1/2}$ is compact. Consider

$$[P_\mu, \hat{a}] - [Q, \hat{a}] = [P_\mu - Q, \hat{a}],$$

and the individual matrix elements in $(P_\mu - Q)\hat{a}$ for example. We have two terms to deal with: the diagonal one

$$\tfrac{1}{2}\big((\mu^2 + \mathcal{D}^2)^{1/2} + \mathcal{D} - 2(P_+ + \tfrac{1}{2}P_0)(\mu^2 + \mathcal{D}^2)^{1/2}\big)(\mu^2 + \mathcal{D}^2)^{-1/2}a,$$

and the off-diagonal one

$$\tfrac{1}{2}\mu(\mu^2 + \mathcal{D}^2)^{-1/2}a - \tfrac{1}{2}P_0 a.$$

We have already observed that since we have a spectral triple, the off-diagonal terms are compact. For the diagonal terms, we first observe that

$$(\mu^2 + \mathcal{D}^2)^{1/2} + \mathcal{D} - 2(P_+ + \tfrac{1}{2}P_0)(\mu^2 + \mathcal{D}^2)^{1/2} = \mathcal{D} - (2P - 1)(\mu^2 + \mathcal{D}^2)^{1/2} - P_0\mu,$$

is a bounded operator. This follows from the functional calculus applied to the function $f(x) = x - \mathrm{sign}(x)(\mu^2 + x^2)^{1/2}$, where $\mathrm{sign}(0)$ is defined to be 1. This can be checked for all μ in $[0,1]$. This boundedness, together with the compactness of $(\mu^2 + \mathcal{D}^2)^{-1/2}a$, shows that

$$\tfrac{1}{2}\big((\mu^2 + \mathcal{D}^2)^{1/2} + \mathcal{D} - 2(P_+ + \tfrac{1}{2}P_0)(\mu^2 + \mathcal{D}^2)^{1/2}\big)(\mu^2 + \mathcal{D}^2)^{-1/2}a,$$

is compact for all $\mu \in [0,1]$. This establishes that $[Q, \hat{a}]$ is compact for all $a \in \mathcal{A}$. The second statement now follows immediately. $\square$

Combining this with Proposition 2.13 proves the following result.

Corollary 2.26. *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be an odd semifinite smoothly summable spectral triple relative to $(\mathcal{N}, \tau)$ with spectral dimension $p \geq 1$ and with $\mathcal{A}$ separable. Let u be a unitary in $M_n(\mathcal{A}^\sim)$ representing a class $[u]$ in $K_1(\mathcal{A})$ and $P = \chi_{[0,\infty)}(\mathcal{D})$. Then*

$$\langle [u], (\mathcal{A}, \mathcal{H}, \mathcal{D}) \rangle = \mathrm{Index}_{\tau \otimes \mathrm{tr}_n}\big((P \otimes \mathrm{Id}_n)u(P \otimes \mathrm{Id}_n)\big).$$

CHAPTER 3

The Local Index Formula for Semifinite Spectral Triples

We have now come to the proof of the local index formula in noncommutative geometry for semifinite smoothly summable spectral triples. This proof is modelled on that in [**17**] in the unital case, which in turn was inspired by Higson's proof in [**32**].

We have opted to present the proof 'almost in full', though sometimes just sketching the algebraic parts of the argument, referring to [**17**] for more details. This means we have some repetition of material from [**17**] in order that the proof be comprehensible. Due to the nonunital subtleties, we include detailed proofs of the analytic statements, deferring the lengthier proofs to the Appendix so as not to distract from the main argument.

In the unital case we constructed two (b, B)-cocycles, the resolvent and residue cocycles. The proof in [**17**] shows that the residue cocycle is cohomologous to the Chern character, while the resolvent cocycle is 'almost' cohomologous to the Chern character, in a sense we make precise later. The aim now is to show that for smoothly summable semifinite spectral triples:

1) the resolvent and residue cocycles are still defined as elements of the reduced (b, B)-complex in the nonunital setting;

2) the homotopies from the Chern character to the resolvent and residue cocycles are still well-defined and continuous in the nonunital setting. In particular, various intermediate cocycles must be shown to be well-defined and continuous.

3.1. The resolvent and residue cocycles and other cochains

In order to deal with the even and odd cases simultaneously, we need to introduce some further notation to handle the differences in the formulae between the two cases.

In the following, we fix $(\mathcal{A}, \mathcal{H}, \mathcal{D})$, a semifinite, smoothly summable, spectral triple, with spectral dimension $p \geq 1$ and parity $\bullet \in \{0, 1\}$ ($\bullet = 0$ for an even spectral triple and $\bullet = 1$ for odd triples). We will use the notation $da := [\mathcal{D}, a]$ for commutators in order to save space. We further require that A, the norm closure of $\mathcal{A}$, be separable in order that we can apply the Kasparov product to define the numerical index pairings given in Definition 2.12. Finally, we have seen in Proposition 2.20 that we may assume, without loss of generality, that $\mathcal{A}$ is complete in the δ-φ-topology.

We define a (partial) $\mathbb{Z}_2$-grading on OP^*, by declaring that $|\mathcal{D}|$ and the elements of $\mathcal{A}$ have degree zero, while $\mathcal{D}$ has degree one. When the triple is even, this coincides with the degree defined by the grading γ. When defined, we denote the grading degree of an element $T \in \mathrm{OP}^*$ by $\deg(T)$. We also let $M := 2\lfloor (p + \bullet + 1)/2 \rfloor - \bullet$, the greatest integer of parity $\bullet$ in $[0, p+1]$. In particular, $M = p$ when p is an integer of parity $\bullet$ and $M = p+1$ if p is an integer of parity $1-\bullet$. The grading degree allows us to define the graded commutator of S, $T \in \mathrm{OP}^*$ of definite grading degree, by

$$[S,T]_\pm := ST - (-1)^{\deg(S)\deg(T)}TS.$$

We will begin by defining the various cocycles and cochains we need on $\mathcal{A}^{\otimes(m+1)}$ for appropriate m. In order to work in the reduced (b,B)-bicomplex for $\mathcal{A}^\sim$, we will need to extend the definitions of all these cochains to $\mathcal{A}^\sim \otimes \mathcal{A}^{\otimes m}$. We will carry out this extension in the next Section.

3.1.1. The residue cocycle. In order to define the residue cocycle, we need a condition on the singularities of zeta functions constructed from $\mathcal{D}$ and $\mathcal{A}$.

Definition 3.1. Let $(\mathcal{A},\mathcal{H},\mathcal{D})$ be a smoothly summable spectral triple of spectral dimension p. We say that the spectral dimension is isolated, if for any element $b \in \mathcal{N}$, of the form[1]

$$b = a_0\, da_1^{(k_1)} \dots da_m^{(k_m)}(1+\mathcal{D}^2)^{-|k|-m/2}, \quad a_0,\dots,a_m \in \mathcal{A},$$

with $k \in \mathbb{N}_0^m$ a multi-index and $|k| = k_1 + \cdots + k_m$, then the zeta function

$$\zeta_b(z) := \tau\big(b(1+\mathcal{D}^2)^{-z}\big),$$

has an analytic continuation to a deleted neighbourhood of $z=0$. In this case, we define the numbers

$$\tau_j(b) := \mathrm{res}_{z=0}\ z^j\,\zeta_b(z), \quad j \in \{-1\}\cup\mathbb{N}_0. \tag{3.1}$$

Remark. The isolated spectral dimension condition is implied by the much stronger notion of *discrete dimension spectrum*, [**25**]. We say that a smoothly summable spectral triple $(\mathcal{A},\mathcal{H},\mathcal{D})$, has discrete dimension spectrum $\mathrm{Sd} \subset \mathbb{C}$, if Sd is a discrete set and for all b in the polynomial algebra generated by $\delta^k(a)$ and $\delta^k(da)$, with $a \in \mathcal{A}$ and $k \in \mathbb{N}_0$, the function $\zeta_b(z)$ is defined and holomorphic for $\Re(z)$ large, and analytically continues to $\mathbb{C}\setminus\mathrm{Sd}$.

For a multi-index $k \in \mathbb{N}_0^m$, we define

$$\alpha(k)^{-1} := k_1!\dots k_m!(k_1+1)(k_1+k_2+2)\dots(|k|+m), \tag{3.2}$$

and we let $\sigma_{n,l}$ be the non-negative rational numbers defined by the identities

$$\prod_{j=0}^{n-1}(z+j+\tfrac{1}{2}) = \sum_{j=0}^{n} z^j\,\sigma_{n,j}, \quad \text{when } \bullet = 1,$$

$$\prod_{j=0}^{n-1}(z+j) = \sum_{j=1}^{n} z^j\sigma_{n,j}, \quad \text{when } \bullet = 0. \tag{3.3}$$

Definition 3.2. Assume that $(\mathcal{A},\mathcal{H},\mathcal{D})$ is a semifinite smoothly summable spectral triple with isolated spectral dimension $p \geq 1$. For $m = \bullet, \bullet+2, \dots, M$, with τ_j defined in Definition 3.1, and for a multi-index k setting $h = |k| + (m-\bullet)/2$, the m-th component of the **residue cocycle** $\phi_m : \mathcal{A}\otimes\mathcal{A}^{\otimes m} \to \mathbb{C}$ is defined by

$$\phi_0(a_0) = \tau_{-1}(a_0),$$

[1] Recall $T^{(n)} = [\mathcal{D}^2, T^{(n-1)}]$; see equation (1.10).

and for $m = 1, \ldots, M$

$$\phi_m(a_0, \ldots, a_m) =$$

$$(\sqrt{2i\pi})^{\bullet} \sum_{|k|=0}^{M-m} (-1)^{|k|} \alpha(k) \sum_{j=1-\bullet}^{h} \sigma_{h,j}\, \tau_{j-1+\bullet}\Big(\gamma a_0\, da_1^{(k_1)} \ldots da_m^{(k_m)} (1+\mathcal{D}^2)^{-|k|-m/2}\Big),$$

with $\alpha(k)$ given in Equation (3.2).

3.2. The resolvent cocycle and variations

In this Section, we do not assume that our spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ has isolated spectral dimension, however several of the cochains defined here require invertibility of $\mathcal{D}$. The issue of invertibility will be discussed in the next Section, and we will show in Section 3.8 how this assumption is removed.

For the invertibility we assume that there exists $\mu > 0$ such that $\mathcal{D}^2 \geq \mu^2$. For such an invertible $\mathcal{D}$, we may define

$$\mathcal{D}_u := \mathcal{D}|\mathcal{D}|^{-u} \quad \text{for} \quad u \in [0,1], \quad \text{and for } a \in \mathcal{A}, \quad d_u(a) := [\mathcal{D}_u, a].$$

Thus $\mathcal{D}_0 = \mathcal{D}$ and $\mathcal{D}_1 = F$. Note that d_u maps $\mathcal{A}$ to OP_0^0. This follows from the estimates given in the proof of Lemma 1.38 with $|\mathcal{D}|$ instead of $(1+\mathcal{D}^2)^{1/2}$ when $\mathcal{D}$ is invertible. Note also that the family of derivations $\{d_u,\, u \in [0,1]\}$, interpolates between the two natural notions of differential in quantised calculus, that is $d_0 a = da = [\mathcal{D}, a]$ and $d_1 a = [F, a]$. We also set

$$\dot{\mathcal{D}}_u := -\mathcal{D}_u \log |\mathcal{D}|,$$

the formal derivative of $\mathcal{D}_u$ with respect to the parameter $u \in [0,1]$. We define the shorthand notations

$$R_{s,t,u}(\lambda) := (\lambda - (t + s^2 + \mathcal{D}_u^2))^{-1}, \tag{3.4}$$

$$R_{s,t}(\lambda) := R_{s,t,0}(\lambda), \qquad R_{s,u}(\lambda) := R_{s,0,u}(\lambda), \qquad R_s(\lambda) := R_{s,1,0}(\lambda).$$

The range of the parameters is $\lambda \in \mathbb{C}$, with $0 < \Re(\lambda) < \mu^2/2$, $s \in [0, \infty)$, and $t, u \in [0,1]$. Recall that for a multi-index $k \in \mathbb{N}^m$, we set $|k| := k_1 + \cdots + k_m$.

The parameters s, λ constitute an essential part of the definition of our cocycles, while the parameters t, u will be the parameters of homotopies which will eventually take us from the resolvent cocycle to the Chern character.

Next we have the analogue of [**15**, Lemma 7.2]. This is the lemma which will permit us to demonstrate that the resolvent cococycle introduced below is well defined. We refer to the Appendix, Section A.2.1, for the proof of this important but technical result.

LEMMA 3.3. *Let ℓ be the vertical line $\{a + iv : v \in \mathbb{R}\}$ for some $a \in (0, \mu^2/2)$. Also let $A_j \in \mathrm{OP}^{k_j}$, $j = 1, \ldots, m$ and $A_0 \in \mathrm{OP}_0^{k_0}$. For $s > 0$, $r \in \mathbb{C}$ and $t \in [0,1]$, the operator-valued function*[2]

$$B_{r,t}(s) = \frac{1}{2\pi i} \int_\ell \lambda^{-p/2-r} A_0\, R_{s,t}(\lambda)\, A_1\, R_{s,t}(\lambda) \ldots R_{s,t}(\lambda)\, A_m\, R_{s,t}(\lambda)\, d\lambda,$$

is trace class valued for $\Re(r) > -m + |k|/2 > 0$. Moreover, the function $[s \mapsto s^\alpha \|B_{r,t}(s)\|_1]$, $\alpha > 0$, is integrable on $[0,\infty)$ when $\Re(r) > -m + (|k| + \alpha + 1)/2$.

[2]we define λ^{-r} using the principal branch of log.

Remark. In Corollary 3.11, we will generalize this result to the case where any one of the A_j's belongs to $\mathrm{OP}_0^{k_j}$. From Lemma 3.3 and Corollary 3.11, it follows that the expectations and cochains introduced below are well-defined, for $\Re(r)$ sufficiently large, whenever one of its entries belongs to OP_0^k.

DEFINITION 3.4. For $a \in (0, \mu^2/2)$, let ℓ be the vertical line $\ell = \{a+iv : v \in \mathbb{R}\}$. Given $m \in \mathbb{N}$, $s \in \mathbb{R}^+$, $r \in \mathbb{C}$ and operators $A_j \in \mathrm{OP}^{k_j}$, $j = 0, \dots, m$, with $A_0 \in \mathrm{OP}_0^{k_0}$, such that $|k| - 2m < 2\Re(r)$, we define

$$\langle A_0, \dots, A_m \rangle_{m,r,s,t} := \frac{1}{2\pi i} \tau\Big(\gamma \int_\ell \lambda^{-p/2-r} A_0 \, R_{s,t}(\lambda) \dots A_m \, R_{s,t}(\lambda) \, d\lambda\Big). \tag{3.5}$$

Here γ is the $\mathbb{Z}_2$-grading in the even case and the identity operator in the odd case. When $|k| - 2m - 1 < 2\Re(r)$ and when the operators A_j have definite grading degree, we use the fact that $\mathcal{D} \in \mathrm{OP}^1$ to allow us to define

$$\begin{aligned} &\langle\langle A_0, \dots, A_m \rangle\rangle_{m,r,s,t} \\ &:= \sum_{j=0}^m (-1)^{\deg(A_j)} \langle A_0, \dots, A_j, \mathcal{D}, A_{j+1}, \dots, A_m \rangle_{m+1,r,s,t}. \end{aligned} \tag{3.6}$$

We now state the definition of the resolvent cocycle in terms of the expectations $\langle \cdot, \dots, \cdot \rangle_{m,r,s,t}$.

DEFINITION 3.5. For $m = \bullet, \bullet + 2, \dots, M$, we introduce the constants η_m by

$$\eta_m = \left(-\sqrt{2i}\right)^\bullet 2^{m+1} \frac{\Gamma(m/2+1)}{\Gamma(m+1)}.$$

Then for $t \in [0,1]$ and $\Re(r) > (1-m)/2$, the m-th component of the **resolvent cocycles** $\phi_m^r, \phi_{m,t}^r : \mathcal{A} \otimes \mathcal{A}^{\otimes m} \to \mathbb{C}$ are defined by $\phi_m^r := \phi_{m,1}^r$ and

$$\phi_{m,t}^r(a_0, \dots, a_m) := \eta_m \int_0^\infty s^m \langle a_0, da_1, \dots, da_m \rangle_{m,r,s,t} \, ds, \tag{3.7}$$

Remark. It is important to note that the resolvent cocycle ϕ_m^r is well defined even when $\mathcal{D}$ is not invertible.

Our proof of the local index formula involves constructing cohomologies and homotopies in the reduced (b,B)-bicomplex. This involves the use of 'transgression' cochains, as well as some other auxiliary cochains. The transgression cochains $\Phi_{m,t}^r$ and auxiliary cochains $B\Phi_{M+1,0,u}^r$, $\Psi_{M,u}^r$ (see below) are defined similarly to the resolvent cochains. However, the cochains $\Phi_{m,t}^r$ are of the *opposite parity* to ϕ_m^r. Thus, if we have an even spectral triple, we will only have $\Phi_{m,t}^r$ with m odd.

DEFINITION 3.6. For $t \in [0,1]$, $r \in \mathbb{C}$ with $\Re(r) > (1-m)/2$ and with $\mathcal{D}$ invertible, the m-th component, $m = 1-\bullet, 1-\bullet+2, \dots, M+1$, of the **transgression cochains** $\Phi_{m,t}^r : \mathcal{A} \otimes \mathcal{A}^{\otimes m} \to \mathbb{C}$ are defined by

$$\Phi_{m,t}^r(a_0, \dots, a_m) := \eta_{m+1} \int_0^\infty s^{m+1} \langle\langle a_0, da_1, \dots, da_m \rangle\rangle_{m,r,s,t} \, ds. \tag{3.8}$$

By specialising the parameter t to $t = 1$, we define $\Phi_m^r := \Phi_{m,1}^r$.

Finally we need to consider $B\Phi_{M+1,0,u}^r$ and another auxiliary cochain $\Psi_{M,u}^r$ for $u \neq 0$. We define $\Psi_{M,u}^r$ below, and the definition of $B\Phi_{M+1,0,u}^r$ is the same as $B\Phi_{M+1,0}^r$ with every appearance of $\mathcal{D}$ replaced by $\mathcal{D}_u := \mathcal{D}|\mathcal{D}|^{-u}$, including in the resolvents.

To show that these cochains are well-defined when $u \neq 0$ requires additional argument beyond power counting and Lemma 3.3. We outline the argument briefly, beginning with the case $p \geq 2$. We start from the identity for $a \in \mathcal{A}$,

$$d_u(a) = [\mathcal{D}_u, a] = [F|\mathcal{D}|^{1-u}, a] = F[|\mathcal{D}|^{1-u}, a] + \big(da - F\delta(a)\big)|\mathcal{D}|^{-u},$$

and we note that $da - F\delta(a) \in \mathrm{OP}_0^0$. Applying the second part of Lemma 1.38 and Lemma 1.37 now shows that $d_u(a) \in \mathcal{L}^q(\mathcal{N}, \tau)$ for all $q > p/u$. Next, we find that

$$R_{s,u}(\lambda) = (\lambda - s^2 - \mathcal{D}_u^2)^{-1} = |\mathcal{D}|^{-2(1-u)}\mathcal{D}_u^2(\lambda - s^2 - \mathcal{D}_u^2)^{-1} =: |\mathcal{D}|^{-2(1-u)}B(u),$$

where $B(u)$ is uniformly bounded. Then Lemma 1.37 and the Hölder inequality show that $d_u(a_i)\, R_{s,u}(\lambda) \in \mathcal{L}^q(\mathcal{N}, \tau)$ for all q with $(2-u)q > p \geq 2$ and $i = 0, 1, \ldots, j, j+2, \ldots, M-1, M$, while $R_{s,u}(\lambda)^{1/2}\, d_u(a_{j+1})\, R_{s,u}(\lambda) \in \mathcal{L}^q(\mathcal{N}, \tau)$ for all q satisfying $(3-2u)q > p \geq 2$. An application of the Hölder inequality now shows that $B\Phi^r_{M+1,0,u}$ is well-defined. To see that $\Psi^r_{M,u}$ is well-defined requires the arguments above, as well as Lemma 1.38 to deal with the extra $\log(|\mathcal{D}|)$ factor appearing in $\dot{\mathcal{D}}_u$. More details can be found in the proof of Lemma 3.26 in Section A.2.4. For $2 > p \geq 1$ the algebra is a little more complicated, and we again refer to the proof of Lemma 3.26 in Section A.2.4 for more details.

Definition 3.7. For $t \in [0,1]$, $r \in \mathbb{C}$ with $\Re(r) > (1-M)/2$ and with $\mathcal{D}$ invertible, the **auxiliary cochain** $\Psi^r_{M,u} : \mathcal{A} \otimes \mathcal{A}^{\otimes M} \to \mathbb{C}$ is defined by

$$\Psi^r_{M,u}(a_0, \ldots, a_M) := -\frac{\eta_M}{2}\int_0^\infty s^M \langle\langle a_0\dot{\mathcal{D}}_u, d_u(a_1), \ldots, d_u(a_M)\rangle\rangle_{M,r,s,0}\, ds, \tag{3.9}$$

where the expectation uses the resolvent $R_{s,t,u}(\lambda)$ for $\mathcal{D}_u$.

These are all the cochains that will appear in our homotopy arguments connecting the resolvent and residue cocycles to the Chern character. However, we still need to ensure that we can extend all these cochains to $\mathcal{A}^\sim \otimes \mathcal{A}^{\otimes m}$, in such a way that we obtain reduced cochains. This extension must also allow us to remove the invertibility assumption on $\mathcal{D}$ when we reach the end of the argument. We deal with these two related issues next.

3.3. The double construction, invertibility and reduced cochains

The cochains $\phi^r_{m,t}$, $B\Phi^r_{m,t,u}$ and $\Psi^r_{M,u}$ require the invertibility of $\mathcal{D}$ for $u \neq 0$ and $t = 0$. Thus we will need to assume the invertibility of $\mathcal{D}$ for the main part of our proof, and show how to remove the assumption at the end.

More importantly, we need to know that all our cochains and cocycles lie in the reduced (b, B)-bicomplex. The good news is that the same mechanism we employ to deal with invertibility also ensures that our homotopy to the Chern character takes place within the reduced bicomplex.

The mechanism we employ is the double spectral triple $(\mathcal{A}, \mathcal{H}^2, \mathcal{D}_\mu, \hat{\gamma})$ (see Definition 2.9), with invertible operator $\mathcal{D}_\mu$. We know that this spectral triple defines the same index pairing with $K_*(\mathcal{A})$ as $(\mathcal{A}, \mathcal{H}, \mathcal{D}, \gamma)$. Now we show how the various cochains associated to the double spectral triple extend naturally to $\mathcal{A}^\sim \otimes \mathcal{A}^{\otimes m}$. Recall that this is really only an issue when $m = 0$, and in particular does not affect any odd cochains.

To distinguish the residue and resolvent cocycles associated with the double spectral triple $(\mathcal{A}, \mathcal{H}^2, \mathcal{D}_\mu, \hat{\gamma})$, we use for them the notations $\phi_{\mu,m}$, $\phi^r_{\mu,m}$, and similarly for the other cochains.

Let $\overline{\mathrm{OP}_0^0}$ be the C^*-closure of OP_0^0 (*defined using the operator* $\mathcal{D}_\mu$*!*), and let $\{\psi_\lambda\}_{\lambda\in\Lambda} \subset \mathrm{OP}_0^0$ be a net forming an approximate unit for $\overline{\mathrm{OP}_0^0}$. Such an approximate unit always exists by the density of OP_0^0. In terms of the two-by-two matrix picture of our doubled spectral triple, we can suppose that there is an approximate unit $\{\tilde{\psi}_\lambda\}_{\lambda\in\Lambda}$ for the OP_0^0 algebra defined by $\mathcal{D}$ (rather than $\mathcal{D}_\mu$) such that $\psi_\lambda = \tilde{\psi}_\lambda \otimes \mathrm{Id}_2$. Then we define for $m > 0$ and $c_0, c_1, \ldots, c_m \in \mathbb{C}$

$$\begin{aligned} &\phi_{\mu,m}(a_0 + c_0\mathrm{Id}_{\mathcal{A}^\sim}, a_1 + c_1\mathrm{Id}_{\mathcal{A}^\sim}, \ldots, a_m + c_m\mathrm{Id}_{\mathcal{A}^\sim}) \\ &\qquad := \phi_{\mu,m}(a_0 + c_0, a_1, \ldots, a_m). \end{aligned} \tag{3.10}$$

This makes sense as the residue cocycle is already normalised. For $m > 0$ this is well-defined since $[\mathcal{D}_\mu, \hat{a}_1]^{(k_1)} \ldots [\mathcal{D}_\mu, \hat{a}_m]^{(k_m)}(1 + \mathcal{D}_\mu^2)^{-|k|/2} \in \mathrm{OP}_0^0$, by Lemma 1.33. Then by definition of isolated spectral dimension, we see that for $m > 0$ the components of the residue cocycle take finite values on $\mathcal{A}^\sim \otimes \mathcal{A}^{\otimes m}$. For $m = \bullet = 0$, we define

$$\begin{aligned} \phi_{\mu,0}(1_{\mathcal{A}^\sim}) &:= \lim_{\lambda\to\infty} \mathrm{res}_{z=0} \frac{1}{z}\tau \otimes \mathrm{tr}_2 \begin{pmatrix} \gamma\tilde{\psi}_\lambda(1 + \mu^2 + \mathcal{D}^2)^{-z} & 0 \\ 0 & -\gamma\tilde{\psi}_\lambda(1 + \mu^2 + \mathcal{D}^2)^{-z} \end{pmatrix} \\ &= 0. \end{aligned}$$

Thus this extension of the residue cocycle for $\mathcal{D}_\mu$ defines a reduced cochain for $\mathcal{A}$.

The resolvent cochains $\phi^r_{\mu,m}$, $m = \bullet, \bullet + 2, \ldots$, are normalised cochains by definition. We extend all of these cochains to $\mathcal{A}^\sim \otimes \mathcal{A}^{\otimes m}$ just as we did for the residue cocycle in Equation (3.10). The resulting cochains are then reduced cochains. For $\Psi^r_{\mu,M,u}$ and $B\Phi^r_{\mu,M+1,0,u}$ there is no issue since $M \geq 1$ in all cases.

For $\Phi^r_{\mu,m,t}$ the situation is different as we will employ an even version of Φ when $\bullet = 1$, and so there is no grading. However, when $m = 0$ we can perform the Cauchy integral in the definition of $\Phi^r_{\mu,0,t}$, and so we obtain for $\Re(r) > 1/2$ a constant C such that

$$\begin{aligned} &\Phi^r_{\mu,0,t}(1_{\mathcal{A}^\sim}) = \\ &\lim_{\lambda\to\infty} C \int_0^\infty s\ \tau \otimes \mathrm{tr}_2 \left(\begin{pmatrix} \tilde{\psi}_\lambda & 0 \\ 0 & \tilde{\psi}_\lambda \end{pmatrix} \begin{pmatrix} \mathcal{D} & \mu \\ \mu & -\mathcal{D} \end{pmatrix} \right) (t + \mu^2 + s^2 + \mathcal{D}^2)^{-p/2-r}\, ds = 0. \end{aligned}$$

These arguments prove the following:

LEMMA 3.8. *Let $t \in [0,1]$ and $r \in \mathbb{C}$. Provided $\Re(r) > (1-m)/2$, the components of the residue $(\phi_{\mu,m})_{m=\bullet,\bullet+2,\ldots,M}$, the resolvent cochain $(\phi^r_{\mu,m,t})_{m=\bullet,\bullet+2,\ldots,M}$, the transgression cochain $(\Phi^r_{\mu,m,t})_{m=1-\bullet,1-\bullet+2,\ldots,M+1}$ and the auxiliary cochains $\Psi^r_{\mu,M,u}$ and $B\Phi^r_{\mu,M+1,0,u}$ are finite on $\mathcal{A}^\sim \otimes \mathcal{A}^{\otimes m}$ and, moreover, define cochains in the reduced (b,B)-bicomplex for $\mathcal{A}^\sim$.*

Thus all the relevant cochains defined using the double live in the reduced bicomplex for $\mathcal{A}^\sim$, and $\mathcal{D}_\mu$ is invertible. For the central part of our proof, from Section 3.4 until the beginning of Section 3.8, we shall simply assume that our smoothly summable spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ has $\mathcal{D}$ invertible with $\mathcal{D}^2 \geq \mu^2 > 0$. In Section 3.8 we will complete the proof by relating cocycles for the double, for which our arguments are valid, to cocycles for our original spectral triple.

3.4. Algebraic properties of the expectations

Here we develop some of the properties of the expectations given in Definition 3.4. These properties are the same as those stated in [**17**], but some of the proofs require extra care in the nonunital setting.

We refer to the following two lemmas as the s-trick and the λ-trick, respectively. Their proofs are given in the Appendix, Sections A.2.2 and A.2.3 respectively. Both the s-trick and the λ-trick provide a way of integrating by parts. Unfortunately, justifying these tricks is somewhat technical.

Formally, the s-trick follows by integrating $\frac{d}{ds}\left(s^\alpha\langle\cdot,\dots,\cdot\rangle_{m,r,s,t}\right)$ and using the fundamental Theorem of calculus.

LEMMA 3.9. *Let $m\in\mathbb{N}$, $\alpha>0$, $t\in[0,1]$ and let $r\in\mathbb{C}$ be such that $2\Re(r)>1+\alpha+|k|-2m$. Also let $A_j\in\mathrm{OP}^{k_j}$, $j=1,\dots,m$ and $A_0\in\mathrm{OP}^{k_0}_0$. Then*

$$\alpha\int_0^\infty s^{\alpha-1}\langle A_0,\dots,A_m\rangle_{m,r,s,t}\,ds = \\ -2\sum_{j=0}^m\int_0^\infty s^{\alpha+1}\langle A_0,\dots,A_j,\mathrm{Id}_{\mathcal{N}},A_{j+1},\dots,A_m\rangle_{m+1,r,s,t}\,ds,$$

and if $2\Re(r)>\alpha+|k|-2m$ then

$$\alpha\int_0^\infty s^{\alpha-1}\langle\langle A_0,\dots,A_m\rangle\rangle_{m,r,s,t}\,ds = \\ -2\sum_{j=0}^m\int_0^\infty s^{\alpha+1}\langle\langle A_0,\dots,A_j,\mathrm{Id}_{\mathcal{N}},A_{j+1},\dots,A_m\rangle\rangle_{m+1,r,s,t}\,ds.$$

Differentiating the λ-parameter under the Cauchy integral, we obtain in a similar manner:

LEMMA 3.10. *Let $m\in\mathbb{N}$, $\alpha>0$, $t\in[0,1]$, $s>0$ and $r\in\mathbb{C}$ such that $2\Re(r)>|k|-2m$. Let also $A_j\in\mathrm{OP}^{k_j}$, $j=1,\dots,m$ and $A_0\in\mathrm{OP}^{k_0}_0$. Then*

$$-(p/2+r)\langle A_0,\dots,A_m\rangle_{m,r+1,s,t}=\sum_{j=0}^m\langle A_0,\dots,A_j,\mathrm{Id}_{\mathcal{N}},A_{j+1},\dots,A_m\rangle_{m+1,r,s,t},$$

and if $2\Re(r)>|k|-2m-1$, then

$$-(p/2+r)\langle\langle A_0,\dots,A_m\rangle\rangle_{m,r+1,s,t}=\sum_{j=0}^m\langle\langle A_0,\dots,A_j,\mathrm{Id}_{\mathcal{N}},A_{j+1},\dots,A_m\rangle\rangle_{m+1,r,s,t}.$$

COROLLARY 3.11. *Let $A_j\in\mathrm{OP}^{k_j}$, $j=0,\dots,m$, have definite grading degree, and suppose that there exists $j_0\in\{0,\dots,m\}$ with $A_{j_0}\in\mathrm{OP}^{k_{j_0}}_0$. Then, for $\Re(r)$ sufficiently large and with $1-\bullet$ the anti-parity, the signed expectations*

$$(-1)^{(1-\bullet)\sum_{k=j}^m\deg(A_k)}\langle A_j,A_{j+1},\dots,A_0,\dots,A_m,\dots,A_{j-1}\rangle_{m,r,s,t},\quad j=0,\dots,m,$$

are all finite and coincide, and similarly for the expectations (3.6). In particular, Lemmas 3.3, 3.9 and 3.10 remain valid if one assumes that $A_{j_0}\in\mathrm{OP}^{k_{j_0}}_0$, for any $j_0\in\{0,\dots,m\}$.

Proof. Formally, the proof is to integrate by parts until the integrand is trace-class, and then apply cyclicity of the trace. To make such a formal argument rigorous, we employ the λ-trick. We assume first $A_0 \in \mathrm{OP}_0^{k_0}$. From the same reasoning as at the beginning of the proof of Lemma 3.3, one can further assume that $A_m \in \mathrm{OP}^0$, at the price that A_{m-1} will be in $\mathrm{OP}^{k_{m-1}+k_m}$. Then, we repeat the λ-trick (Lemma 3.10) until the integrand of

$$\langle A_0, 1, \dots, 1, A_2, 1, \dots, 1, A_m, 1, \dots, 1\rangle_{M+1,r,s,t},$$

is trace class. We then move the bounded (by [**15**, Lemma 6.10], see the Appendix Lemma A.2) operator $R^{-k} A_m R^k$ (k is the number of resolvents on the right of A_m) to the front, using the trace property. This gives after recombination

$$\langle A_0, \dots, A_m\rangle_{m,r,s,t} = (-1)^{(1-\bullet)\deg(A_m)}\langle A_m, A_0, \dots, A_{m-1}\rangle_{m,r,s,t}.$$

The sign comes from the relation $A_m\gamma = (-1)^{(1-\bullet)\deg(A_m)}\gamma A_m$. One concludes iteratively. The proof for the expectations (3.6) is entirely similar. □

We quote several results from [**17**] which carry over to our setting with no substantial change in their proofs.

Lemma 3.12. *Let $m \geq 0$, $A_j \in \mathrm{OP}^{k_j}$, $j = 0, \dots, m$, with definite grading degree and with $|k| - 2m - 1 < 2\Re(r)$, and suppose there exists $j_0 \in \{0, \dots, m\}$ with $A_{j_0} \in \mathrm{OP}_0^{k_{j_0}}$. Then for $1 \leq j < m$ we have*

$$\begin{aligned}\langle A_0, \dots, A_{j-1}, [\mathcal{D}^2, A_j], A_{j+1}, \dots, A_m\rangle_{m,r,s,t} = \\ - \langle A_0, \dots, A_{j-2}, A_{j-1}A_j, A_{j+1}, \dots, A_m\rangle_{m-1,r,s,t} \\ + \langle A_0, \dots, A_{j-1}, A_jA_{j+1}, A_{j+2}, \dots, A_m\rangle_{m-1,r,s,t},\end{aligned}$$

while for $j = m$ we have

$$\begin{aligned}\langle A_0, \dots, A_{m-1}, [\mathcal{D}^2, A_m]\rangle_{m,r,s,t} = \\ - \langle A_0, \dots, A_{m-2}, A_{m-1}A_m\rangle_{m-1,r,s,t} \\ + (-1)^{(1-\bullet)\deg(A_m)}\langle A_mA_0, A_1, \dots, A_{m-1}\rangle_{m-1,r,s,t}.\end{aligned}$$

For $k \geq 1$ we have

$$\int_0^\infty s^k\langle \mathcal{D}A_0, A_1, \dots, A_m\rangle_{m,r,s,t}ds = \tag{3.11}$$
$$(-1)^{1-\bullet}\int_0^\infty s^k\langle A_0, A_1, \dots, A_m\mathcal{D}\rangle_{m,r,s,t}ds.$$

If furthermore $\sum_{j=0}^m \deg(A_j) \equiv 1 - \bullet \pmod 2$, we define

$$\deg_{-1} = 0 \ \text{ and } \ \deg_k = \deg(A_0) + \deg(A_1) + \dots + \deg(A_k),$$

then

$$\sum_{j=0}^m (-1)^{\deg_{j-1}}\int_0^\infty s^k\langle A_0, \dots, A_{j-1}, [\mathcal{D}, A_j]_\pm, A_{j+1} \dots, A_m\rangle_{m,r,s,t}ds = 0. \tag{3.12}$$

Lemma 3.13. *Let $m \geq 0$, $A_j \in \mathrm{OP}^{k_j}$, $j = 0, \dots, m$, with definite grading degree and with $|k| - 2m - 2 < 2\Re(r)$, and suppose there exists $j_0 \in \{0, \dots, m\}$ with*

$A_{j_0} \in \mathrm{OP}_0^{k_{j_0}}$. *Then for* $1 \le j < m$ *we have the identity*

$$\begin{aligned}\langle\langle A_0, \dots, A_{j-1}, [\mathcal{D}^2, A_j], A_{j+1}, \dots, A_m\rangle\rangle_{m,r,s,t} = \\ - (-1)^{\deg_{j-1}}\langle A_0, \dots, A_{j-1}, [\mathcal{D}, A_j]_\pm, A_{j+1}, \dots, A_m\rangle_{m,r,s,t} \\ - \langle\langle A_0, \dots, A_{j-2}, A_{j-1}A_j, A_{j+1} \dots, A_m\rangle\rangle_{m-1,r,s,t} \\ + \langle\langle A_0, \dots, A_{j-1}, A_jA_{j+1}, A_{j+2} \dots, A_m\rangle\rangle_{m-1,r,s,t}.\end{aligned} \tag{3.13}$$

For $j = m$ *we also have*

$$\begin{aligned}\langle\langle A_0, \dots, A_{m-1}, [\mathcal{D}^2, A_m]\rangle\rangle_{m,r,s,t} = \\ - (-1)^{\deg_{m-1}}\langle A_0, \dots, A_{m-1}, [\mathcal{D}, A_m]_\pm\rangle_{m,r,s,t} \\ - \langle\langle A_0, \dots, A_{m-2}, A_{m-1}A_m\rangle\rangle_{m-1,r,s,t} \\ + (-1)^{\bullet\deg(A_m)}\langle\langle A_mA_0, A_1, \dots, A_{m-1}\rangle\rangle_{m-1,r,s,t}.\end{aligned}$$

If $\sum_{j=0}^m \deg(A_j) \equiv \bullet \pmod 2$ *and* $\alpha \ge 1$, *then we also have*

$$\begin{aligned}\sum_{j=0}^m (-1)^{\deg_{j-1}}\int_0^\infty s^\alpha\langle\langle A_0, \dots, A_{j-1}, [\mathcal{D}, A_j]_\pm, A_{j+1}, \dots, A_m\rangle\rangle_{m,r,s,t}ds = \\ 2\sum_{j=0}^m \int_0^\infty s^\alpha\langle A_0, \dots, A_j, \mathcal{D}^2, A_{j+1}, \dots, A_m\rangle_{m+1,r,s,t}ds.\end{aligned} \tag{3.14}$$

On the other hand, if $\sum_{j=0}^m \deg(A_j) \equiv 1 - \bullet \pmod 2$ *and* $\alpha \ge 1$ *then* $\langle\langle \dots\rangle\rangle$ *satisfies the cyclic property*

$$\int_0^\infty s^\alpha\langle\langle A_0, \dots, A_m\rangle\rangle_{m,r,s,t}ds = (-1)^{\bullet\deg(A_m)}\int_0^\infty s^\alpha\langle\langle A_m, A_0, \dots, A_{m-1}\rangle\rangle_{m,r,s,t}ds.$$

From these various algebraic identities and $\mathcal{D}^2R_{s,t}(\lambda) = -1 + (\lambda - (t+s^2))R_{s,t}(\lambda)$ we deduce the following important relationship between powers of $\mathcal{D}$ and the values of our parameters.

LEMMA 3.14. *Let* $m, \alpha \ge 0$, $A_j \in \mathrm{OP}^{k_j}$, $j = 1, \dots, m$, *with definite grading degree,* $r \in \mathbb{C}$ *be such that* $2\Re(r) > 1 + \alpha - 2m + |k|$, *and suppose there exists* $j_0 \in \{0, \dots, m\}$ *with* $A_{j_0} \in \mathrm{OP}_0^{k_{j_0}}$. *Then*

$$\begin{aligned}\sum_{j=0}^m \int_0^\infty s^\alpha\langle A_0, \dots, A_j, \mathcal{D}^2, A_{j+1}, \dots, A_m\rangle_{m+1,r,s,t}ds = \\ - (m+1)\int_0^\infty s^\alpha\langle A_0, \dots, A_m\rangle_{m,r,s,t}ds \\ + (1 - p/2 - r)\int_0^\infty s^\alpha\langle A_0, \dots, A_m\rangle_{m,r,s,t}ds \\ + \frac{(\alpha+1)}{2}\int_0^\infty s^\alpha\langle A_0, \dots, A_m\rangle_{m,r,s,t}ds \\ - t\sum_{j=0}^m \int_0^\infty s^\alpha\langle A_0, \dots, A_j, 1, A_{j+1}, \dots, A_m\rangle_{m+1,r,s,t}ds.\end{aligned}$$

3.5. Continuity of the resolvent cochain

In this Section, we demonstrate the continuity, differentiability and holomorphy properties, allowing us to prove that the resolvent cocycle represents the Chern character.

DEFINITION 3.15. Let $\mathcal{O}_m$ be the set of holomorphic functions on the open half-plane $\{z \in \mathbb{C} : \Re(z) > (1-m)/2\}$. We endow $\mathcal{O}_m$ with the topology of uniform convergence on compact sets.

LEMMA 3.16. *Let $m = \bullet, \bullet+2, \dots, M$ and $t \in [0,1]$. For $A_0, \dots, A_m \in \mathrm{OP}^0$ such that there exists $j \in \{0, \dots, m\}$ with $A_j \in \mathrm{OP}^0_0$, both the functions*

$$\Big[r \mapsto \int_0^\infty s^m \langle A_0, \dots, A_m\rangle_{m,r,s,t}\, ds\Big], \quad \Big[r \mapsto \int_0^\infty s^{m+1} \langle\langle A_0, \dots, A_m\rangle\rangle_{m,r,s,t}\, ds\Big],$$

are elements of $\mathcal{O}_m$.

PROOF. We prove a stronger result, namely that the operator-valued function

$$2\pi i\, B_{r,t}(s,\varepsilon) = \int_\ell \lambda^{-p/2-r}\Big(\varepsilon^{-1}(\lambda^{-\varepsilon}-1)+\log\lambda\Big) A_0\, R_{s,t}(\lambda)\, A_1\, R_{s,t}(\lambda) \dots R_{s,t}(\lambda)\, A_m\, R_{s,t}(\lambda)\, d\lambda,$$

satisfies $\lim_{\varepsilon\to 0}\int_0^\infty s^m \|B_{r,t}(s,\varepsilon)\|_1 ds = 0$, whenever $\Re(r) > (1-m)/2$. (Here ℓ is the vertical line $\ell = \{a+iv : v \in \mathbb{R}\}$ with $0 < a < \mu^2/2$ and $\mu \in (0,\infty)$ is such that $\mathcal{D}^2 \geq \mu^2$.) By Corollary 3.11, we can assume that $A_0 \in \mathrm{OP}^0_0$. The proof then follows by a minor modification of the arguments of the proof of Lemma 3.3 (see the Appendix Section A.2.1), so that we only outline it. (We use the shorthand notation $R := R_{s,t}(\lambda)$.) We start by writing for any $L \in \mathbb{N}$, using Lemma A.3 (see [**15**, Lemma 6.11])

$$A_0\, R\, A_1\, R \dots R\, A_m\, R = \sum_{|n|=0}^{L} C(n) A_0\, A_1^{(n_1)} \dots A_m^{(n_m)}\, R^{m+|n|+1} + A_0\, P_{L,m},$$

with $P_{L,m} \in \mathrm{OP}^{-2m-L-3}$. The conclusion for the remainder term follows then from the estimate

$$\Big|\lambda^{-p/2-r}\Big(\varepsilon^{-1}(\lambda^{-\varepsilon}-1)+\log(\lambda)\Big)\Big| \leq C\,|\varepsilon|\,|\lambda|^{-p/2-\Re(r)},$$

together with the same techniques as those used in the proof of Lemma 3.3. A more detailed account can be found in [**15**, Lemma 7.4]. For the non-remainder terms, we perform the Cauchy integrals

$$\begin{aligned}
&\frac{1}{2\pi i}\int_\ell \lambda^{-p/2-r}\Big(\varepsilon^{-1}(\lambda^{-\varepsilon}-1)+\log\lambda\Big) A_0 A_1^{(n_1)} \dots A_m^{(n_m)} R^{m+1+|n|} d\lambda = \\
&\quad (-1)^{m+|n|}\frac{\Gamma(p/2+r+m+|n|)}{\Gamma(p/2+r)} A_0 A_1^{(n_1)} \dots A_m^{(n_m)} (t+s^2+\mathcal{D}^2)^{-p/2-r-m-|n|} \\
&\qquad\qquad \times \Big(\varepsilon^{-1}((t+s^2+\mathcal{D}^2)^{-\varepsilon}-1) + \log\big(t+s^2+\mathcal{D}^2\big)\Big) \\
&\quad + \sum_{k=0}^{m+|n|-1}\binom{m+|n|}{k}(-1)^{m+|n|}\frac{\Gamma(p/2+r+k)}{\Gamma(p/2+r)} A_0 A_1^{(n_1)} \dots A_m^{(n_m)} \\
&\quad \times (t+s^2+\mathcal{D}^2)^{-p/2-r-m-|n|}\Big(\frac{\Gamma(\varepsilon+m+|n|-k)}{\Gamma(\varepsilon+1)}(t+s^2+\mathcal{D}^2)^{-\varepsilon} - \Gamma(m+|n|-k)\Big).
\end{aligned}$$

Let $\rho > 0$ such that $\Re(z) > (1-m)/2 + \rho$. Call $T_k(s)$ the terms with no logarithm. Using the estimates of Lemma 3.3 and

$$(t+s^2+\mathcal{D}^2)^{-\rho}\big((t+s^2+\mathcal{D}^2)^{-\varepsilon}-1\big) \to 0 \quad \text{as} \quad \varepsilon \to 0,$$

in norm, we see that $\lim_{\varepsilon\to 0}\int_0^\infty s^m \,\|T_k(s)\|_1\, ds = 0$. For the first term (with a logarithm), one concludes using the fact that for any $\rho > 0$

$$\left\|(t+s^2+\mathcal{D}^2)^{-\rho}\left(\frac{(t+s^2+\mathcal{D}^2)^{-\varepsilon}-1}{\varepsilon} + \log\big(t+s^2+\mathcal{D}^2\big)\right)\right\| \le C\,\varepsilon,$$

where the constant C is independent of s (and of t). $\square$

We finally arrive at the main result of this Section.

Proposition 3.17. *Let $m = \bullet, \bullet+2, \dots, M$ for the resolvent cocycle, and $m = 1-\bullet, 1-\bullet+2, \dots, M+1$ for the transgression cochain. Also let $t \in [0,1]$. Then the maps*

$$a_0\otimes\cdots\otimes a_m \mapsto \big[r \mapsto \phi^r_{m,t}(a_0,\dots,a_m)\big], \quad a_0\otimes\cdots\otimes a_m \mapsto \big[r \mapsto \Phi^r_{m,t}(a_0,\dots,a_m)\big],$$

are continuous multilinear maps from $\mathcal{A}\otimes\mathcal{A}^{\otimes m}$ to $\mathcal{O}_m$.

Proof. We only give the proof for the resolvent cocycle, the case of the transgression cochain being similar. So let us first fix $r \in \mathbb{C}$ with $\Re(r) > (1-m)/2$. Since Lemma 3.8 ensures that our functionals are finite for these values of r, all that we need to do is to improve the estimates of Lemma 3.3 to prove continuity. We do this using the s- and λ-tricks. We recall that we have defined $M = 2\lfloor (p+\bullet+1)/2\rfloor - \bullet$ (which is the biggest integer of parity $\bullet$ less than or equal to $p+1$). By applying successively the s- and λ-tricks (which commute) $(M-m)/2$ times, we obtain

$$\phi^r_{m,t}(a_0,\dots,a_m) = 2^{(M-m)/2}(M-n)! \prod_{j_1=1}^{(M-m)/2}\frac{1}{p/2+r-j_1}\prod_{j_2=1}^{(M-m)/2}\frac{1}{m+j_2}$$

$$\times \sum_{|k|=M-m}\int_0^\infty s^M \langle a_0, 1^{k_0}, da_1, 1^{k_1},\dots, da_m, 1^{k_m}\rangle_{M,r-(M-m)/2,s,t}ds, \tag{3.15}$$

where $1^{k_i} = 1,1,\dots,1$ with k_i entries. Since $M \le p+1$, the poles associated to the prefactors are outside the region $\{z\in\mathbb{C} : \Re(z) > (1-m)/2\}$. Ignoring the prefactors, setting $n_i = k_i + 1$ and $R := R_{s,t}(\lambda)$, we need to deal with the integrals

$$\int_0^\infty s^M \tau\Big(\gamma\int_\ell \lambda^{-p/2-r-(M-m)/2} a_0R^{n_0}da_1R^{n_1}\dots da_mR^{n_m}d\lambda\Big)ds, \qquad |n| = M+1,$$

where ℓ is the vertical line $\ell = \{a+iv : v\in\mathbb{R}\}$ with $a \in (0,\mu^2/2)$ and $\mu\in(0,\infty)$ is such that $\mathcal{D}^2 \ge \mu^2$. Let $p_j := (M+1)/n_j$, so that $\sum_{j=0}^m p_j^{-1} = 1$. The Hölder inequality gives

$$\|a_0R^{n_0}da_1R^{n_1}\dots da_mR^{n_m}\|_1 \le \|a_0R^{n_0}\|_{p_0}\|da_1R^{n_1}\|_{p_1}\dots\|da_mR^{n_m}\|_{p_m}.$$

By Lemma 1.39, we obtain for $\varepsilon > 0$, and with $A_0 = a_0$, $A_j = da_j$, $j = 1,\dots,m$,

$$\|A_jR^{n_j}\|_{p_j} \le \|A_j(\mathcal{D}^2-\mu^2/2)^{-(\frac{p}{p_j}+\frac{\varepsilon}{m+1})/2}\|_{p_j}\big((s^2+a)^2+v^2\big)^{-\frac{n_j}{2}+(\frac{p}{p_j}+\frac{\varepsilon}{m+1})/4}.$$

Since $\sum_{j=0}^m n_j = M+1$, this gives

$$\|a_0R^{n_0}da_1R^{n_1}\dots da_mR^{n_m}\|_1 \le C(a_0,\dots,a_m)\ \big((s^2+a)^2+v^2\big)^{-(M+1)/2+(p+\varepsilon)/4},$$

which is enough to show the absolute convergence of the iterated integrals (see [**15**, Lemma 5.4]). Now observe that the constant $C(a_0,\dots,a_m)$ above, is equal to

$$\|a_0(\mathcal{D}^2-\mu^2/2)^{-(p/p_0+\varepsilon/(m+1))/2}\|_{p_0}\dots\|da_m(\mathcal{D}^2-\mu^2/2)^{-(p/p_m+\varepsilon/(m+1))/2}\|_{p_m}.$$

Note also that the explicit interpolation inequality of Lemma 1.37 reads

$$\|A(\mathcal{D}^2-\mu^2/2)^{-\alpha/2}\|_q\le\|A(\mathcal{D}^2-\mu^2/2)^{-\alpha q/2}\|_1^{1/q}\,\|A\|^{1-1/q},\quad A\in\mathrm{OP}^0_0,\quad q>p/\alpha,$$

and the latter is bounded by $\mathcal{P}_{n,k}(A)$ for $n=\lfloor(\alpha q-p)^{-1}\rfloor$ and $k=3\lfloor\alpha q/4\rfloor+1$, by a simultaneous application of Lemma 1.26 and Corollary 1.30. Thus, with the same notations as above, we find for $j\neq 0$ and some constant $C>0$

$$\begin{aligned}&\|da_j(\mathcal{D}^2-\mu^2/2)^{-(p/p_j+\varepsilon/(m+1))/4}\|_{p_j}\\&\qquad\le\|da_j(\mathcal{D}^2-\mu^2/2)^{-(p+p_j\varepsilon/(m+1))/4}\|_2^{1/p_j}\|da_j\|^{1-1/p_j}\le C\,\mathcal{P}_{n,k}(da_j),\end{aligned}$$

for suitable $n,k\in\mathbb{N}$. For $j=0$ we have a similar but easier calculation. This proves the joint continuity of the resolvent cocycle for the δ-φ-topology.

The proof that the map $r\mapsto\phi^r_{m,t}(a_0,\dots,a_m)$ is holomorphic in the region $\Re(r)>(1-m)/2$ follows from Lemma 3.16. □

PROPOSITION 3.18. *For each $m=\bullet,\bullet+2,\dots,M$, the map*

$$[0,1]\ni t\mapsto\big[r\mapsto\phi^r_{m,t}\big]\in\mathrm{Hom}(\mathcal{A}^{\otimes(m+1)},\mathcal{O}_m),$$

is continuously differentiable and

$$\frac{d}{dt}\Big[t\mapsto\big[r\mapsto\phi^r_{m,t}\big]\Big]=\Big[t\mapsto\big[r\mapsto-(p/2+r)\,\phi^{r+1}_{m,t}\big]\Big].$$

PROOF. We do the case $m<M$ where we must use some initial trickery to reduce to a computable situation. For $m=M$ such tricks are not needed. We proceed as in the proof of Proposition 3.17, applying the s- and λ- tricks to obtain (3.15). Keeping the same notations as in the cited proposition, in particular $p_j=(M+1)/n_j$, $j=0,\dots,m$, and ignoring the prefactors, we are left with the integrals

$$\int_0^\infty s^M\tau\Big(\gamma\int_\ell\lambda^{-p/2-r-(M-m)/2}a_0\,R^{n_0}_{s,t}\,da_1\,R^{n_1}_{s,t}\dots da_m\,R^{n_m}_{s,t}\,d\lambda\Big)ds.$$

(Here ℓ is the vertical line $\ell=\{a+iv:v\in\mathbb{R}\}$ with $0<a<\mu^2/2$.) Now each integrand is not only trace class, but also t-differentiable in trace norm. This is a consequence of the product rule, Hölder's inequality and the following argument showing the Schatten norm differentiability of $AR^n_{s,t}$ for $A\in\mathrm{OP}^0_0$. By adding and substracting suitable terms, the resolvent identity gives

$$A\Big(\varepsilon^{-1}(R^n_{s,t+\varepsilon}-R^n_{s,t})+nR^{n+1}_{s,t}\Big)=nAR^n_{s,t}\Big(R_{s,t}-\frac{1}{n}\sum_{k=1}^n R^{-k+1}_{s,t}R^k_{s,t+\varepsilon}\Big).$$

The term in brackets on the right hand side converges to zero in operator norm since $R^{-k+1}_{s,t}R^{k-1}_{s,t+\varepsilon}$ is uniformly bounded. Thus as $\varepsilon\to 0$, and for suitable $q\ge 1$,

$$\|A\left(\varepsilon^{-1}(R^n_{s,t+\varepsilon}-R^n_{s,t})+nR^{n+1}_{s,t}\right)\|_q\le\|nAR^n_{s,t}\|_q\,\Big\|R_{s,t}-\frac{1}{n}\sum_{k=1}^n R^{-k+1}_{s,t}R^k_{s,t+\varepsilon}\Big\|\to 0.$$

Choosing $A=a_0$ or $A=da_j$ and $q=p_0$ or $q=p_j$ respectively proves the differentiability of each term $AR^n_{s,t}$ in the integrand in the appropriate q-norm, and so an application of Hölder's inequality completes the proof of trace norm differentiability. The existence of the integrals can now be deduced from the formula for the derivative of the integrand and Lemma 3.3. This proves differentiability, and so the t-derivative of $\phi^r_{m,t}(a_0,\ldots,a_m)$ exists and (reinstating the prefactors) equals

$$\eta_m 2^{\frac{M-m}{2}}(M-m)!\prod_{b=1}^{\frac{(M-m)}{2}}\frac{1}{p/2+r-b}\prod_{j=1}^{\frac{(M-m)}{2}}\frac{1}{m+j}\sum_{|k|=M-m}\sum_{j=0}^{m}$$
$$\times\int_0^\infty s^M(k_j+1)\langle a_0,1^{k_0},\ldots,da_j,1^{k_j+1},\ldots,da_m,1^{k_m}\rangle_{M+1,r-(M-m)/2,s,t}ds.$$

Now undoing our applications of the s-trick and the λ-trick gives

$$\frac{d}{dt}\phi^r_{m,t}(a_0,\ldots,a_m)=\eta_m\sum_{j=0}^{m}\int_0^\infty s^m\langle a_0,\ldots,da_j,1,da_{j+1},\ldots,da_m\rangle_{m+1,r,s,t}ds,$$

and a final application of the λ-trick yields our final formula,

$$\frac{d}{dt}\phi^r_{m,t}(a_0,\ldots,a_m)=-(p/2+r)\phi^{r+1}_{m,t}(a_0,\ldots,a_m).$$

We note that by our estimates the convergence is uniform in r, for r in a compact subset of a suitable right half-plane. $\square$

3.6. Cocyclicity of the resolvent and residue cocycles

We start by explaining why the resolvent cocycle is is indeed a (b,B)-cocycle.

PROPOSITION 3.19. *Provided $\Re(r)>1/2$, there exists $\delta\in(0,1)$ such that the resolvent cochain $(\phi^r_{m,t})^M_{m=\bullet}$ is a reduced (b,B)-cocycle of parity $\bullet\in\{0,1\}$ for $\mathcal{A}$, modulo functions holomorphic in the half plane $\Re(r)>(1-p)/2-\delta$.*

PROOF. Since $(\phi^r_{m,t})^M_{m=\bullet}$ is a reduced cochain, the proof of the first claim will follow from the same algebraic arguments as in [**15**, Proposition 7.10] (odd case) and [**16**, Proposition 6.2] (even case). We reproduce the main elements of the proof for the odd case here.

We start with the computation of the coboundaries of the $\phi^r_{m,t}$. The definition of the operator B and $\phi^r_{m+2,t}$ gives

$$\begin{aligned}(B\phi^r_{m+2,t})(a_0,\ldots,a_{m+1})&=\sum_{j=0}^{m+1}\phi^r_{m+2,t}(1,a_j,\ldots,a_{m+1},a_0,\ldots,a_{j-1})\\&=\sum_{j=0}^{m+1}\eta_{m+2}\int_0^\infty s^{m+2}\langle 1,[\mathcal{D},a_j],\ldots,[\mathcal{D},a_{j-1}]\rangle_{m+2,r,s,t}ds.\end{aligned}$$

Using Lemma 3.11 and Lemma 3.9, this is equal to

$$\sum_{j=0}^{m+1}\eta_{m+2}\int_0^\infty s^{m+2}\langle[\mathcal{D},a_0],\ldots,[\mathcal{D},a_{j-1}],1,[\mathcal{D},a_j],\ldots,[\mathcal{D},a_{m+1}]\rangle_{m+2,r,s,t}ds=$$
$$-\eta_{m+2}\frac{(m+1)}{2}\int_0^\infty s^m\langle[\mathcal{D},a_0],\ldots,[\mathcal{D},a_{m+1}]\rangle_{m+1,r,s,t}ds.$$

We observe at this point that $\eta_{m+2}(m+1)/2=\eta_m$, using the functional equation for the Gamma function. Next we write $[\mathcal{D},a_0]=\mathcal{D}a_0-a_0\mathcal{D}$ and anticommute the second $\mathcal{D}$ through the remaining $[\mathcal{D}, a_j]$ using $\mathcal{D}[\mathcal{D}, a_j] + [\mathcal{D}, a_j]\mathcal{D} = [\mathcal{D}^2, a_j]$. This gives, after some algebra and an application of Equation (3.11) from Lemma 3.12,

$$(B\phi^r_{m+2,t})(a_0, \dots, a_{m+1}) = \\ -\eta_m \int_0^\infty s^m \sum_{j=1}^{m+1} (-1)^j \langle a_0, [\mathcal{D}, a_1], \dots, [\mathcal{D}^2, a_j], \dots, [\mathcal{D}, a_{m+1}]\rangle_{m+1,s,r,t} ds. \tag{3.16}$$

Observe that for $\phi^r_{1,t}$ we have

$$(B\phi^r_{1,t})(a_0) = \frac{\eta_1}{2\pi i} \int_0^\infty s\tau \left(\int_\ell \lambda^{-p/2-r} R_{s,t}(\lambda)[\mathcal{D}, a_0] R_{s,t}(\lambda) d\lambda \right) ds = 0,$$

by a variant of Lemma 3.12. We now compute the Hochschild coboundary of $\phi^r_{m,t}$. From the definitions we have

$$(b\phi^r_{m,t})(a_0, \dots, a_{m+1}) = \phi^r_{m,t}(a_0 a_1, a_2, \dots, a_{m+1}) \\ + \sum_{i=1}^m (-1)^i \phi^r_{m,t}(a_0, \dots, a_i a_{i+1}, \dots, a_{m+1}) + \phi^r_{m,t}(a_{m+1} a_0, a_1, \dots, a_m),$$

but this is equal to

$$\eta_m \int_0^\infty s^m \Big(\langle a_0 a_1, [\mathcal{D}, a_2], \dots, [\mathcal{D}, a_{m+1}]\rangle_{m,r,s,t} + \langle a_{m+1} a_0, [\mathcal{D}, a_1], \dots, [\mathcal{D}, a_m]\rangle_{m,r,s,t} \\ + \sum_{j=1}^m (-1)^i \langle a_0, [\mathcal{D}, a_1], \dots, a_j[\mathcal{D}, a_{j+1}] + [\mathcal{D}, a_j]a_{j+1}, \dots, [\mathcal{D}, a_{m+1}]\rangle_{m,r,s,t} \Big) ds.$$

We now reorganise the terms so that we can employ the first identity of Lemma 3.12. So

$$(b\phi^r_{m,t})(a_0, \dots, a_{m+1}) = \\ \sum_{j=1}^{m+1} (-1)^j \eta_m \int_0^\infty s^m \langle a_0, [\mathcal{D}, a_1], \dots, [\mathcal{D}^2, a_j], \dots, [\mathcal{D}, a_{m+1}]\rangle_{m+1,r,s,t} ds. \tag{3.17}$$

For $m = 1, 3, 5, \dots, M + \bullet - 3$ comparing Equations (3.17) and (3.16) shows that

$$(B\phi^r_{m+2,t} + b\phi^r_{m,t})(a_0, \dots, a_{m+1}) = 0.$$

So we just need to check the claim that $b\phi^r_{M+\bullet-1}$ is holomorphic for $\Re(r) > -p/2+\delta$ for some suitable δ. From the computation given above, we have (up to a constant)

$$b\phi^r_{M,t}(a_0, \dots, a_{M+1}) = \\ C(M) \sum_{j=1}^{M+1} (-1)^j \int_0^\infty s^M \langle a_0, da_1, \dots, [\mathcal{D}^2, a_j], \dots, da_{M+1}\rangle_{M+1,r,s,t}\, ds.$$

Now, since the total order $|k|$ of the pseudodifferential operator entries of the expectation is equal to one, we obtain by Lemma 3.3 that $b\phi^r_{M,t}(a_0, \dots, a_{M+1})$ is finite for ($\varepsilon > 0$ is arbitrary)

$$\Re(r) > -M - 1 + (1 + M + 1)/2 + \varepsilon = (1 - p)/2 + (p - M - 1 + 2\varepsilon)/2.$$

Since $p - M - 1 < 0$, one can always find $\varepsilon > 0$ such that $-\delta := p - M - 1 + 2\varepsilon \in (-1, 0)$. The holomorphy follows from Lemma 3.16. $\square$

We can now relate the resolvent and residue cocycles.

PROPOSITION 3.20. *Suppose the smoothly summable spectral triple* $(\mathcal{A},\mathcal{H},\mathcal{D})$ *has isolated spectral dimension. Then for* $m=\bullet,\bullet+2,\dots,M$, $a_0,a_1\dots,a_m\in\mathcal{A}$, *the map* $\big[r\mapsto\phi_m^r(a_0,\dots,a_m)\big]\in\mathcal{O}_m$, *analytically continues to a deleted neighbourhood of the critical point* $r=(1-p)/2$. *Keeping the same notation for this continuation, we have*

$$\mathrm{res}_{r=(1-p)/2}\ \phi_m^r(a_0,\dots,a_m)=\phi_m(a_0,\dots,a_m),\quad m=\bullet,\bullet+2,\dots,M.$$

PROOF. For the even case and $m=0$, we can explicitly compute

$$\phi_0^r(a_0)=\frac{1}{r-(1-p)/2}\tau(\gamma a_0(1+\mathcal{D}^2)^{-(r-(1-p)/2)}),$$

modulo a function of r holomorphic at $r=(1-p)/2$. So we need only consider the case $m\ge 1$. We start with the expansion, described in detail in the Appendix, Lemma A.3, with $L=M-m$ and $R:=R_s(\lambda)$

$$a_0\,R\,da_1\,R\dots R\,da_m\,R=\sum_{|n|=0}^{M-m}C(n)a_0\,da_1^{(n_1)}\dots da_m^{(n_m)}\,R^{m+|n|+1}+a_0\,P_{M-m,m}.$$

Ignoring for a moment the remainder term $P_{M-m,m}$, performing the Cauchy integrals gives

$$\phi_m^r(a_0,\dots,a_m)=\sum_{|n|=0}^{M-m}C'(n,m,r)$$
$$\times\int_0^\infty s^m\tau\Big(\gamma a_0\,da_1^{(n_1)}\dots da_m^{(n_m)}\,(1+s^2+\mathcal{D}^2)^{-m-|n|-p/2-r}\Big)\,ds.$$

Setting $h=|n|+(m-\bullet)/2$, and for $\Re(r)>(1-m)/2$, one can perform the s-integral to obtain (after some manipulation of the constants as in [**16**, Theorem 6.4]) for $m>0$

$$\phi_m^r(a_0,\dots,a_m)=(\sqrt{2i\pi})^\bullet\sum_{|n|=0}^{M-m}(-1)^{|n|}\alpha(n)\sum_{j=1-\bullet}^{h}\sigma_{h,j}\,\big(r-(1-p)/2\big)^{j-1+\bullet}$$
$$\times\,\tau\Big(\gamma a_0\,da_1^{(n_1)}\dots da_m^{(n_m)}(1+\mathcal{D}^2)^{-|n|-m/2-r+1/2-p/2}\Big).\tag{3.18}$$

From this the result will be clear if the remainder term is holomorphic for $\Re(r)>(1-p)/2$, since under the isolated spectral dimension assumption the residues of the right hand side of the previous expression are individually well defined. This can be shown using the estimate of the remainder term given in the proof of Lemma 3.3 presented in A.2.1. □

3.7. The homotopy to the Chern character

We explain here the sequence of results that leads to the fact that the Chern character in degree M is cohomologous to the residue cocycle.

LEMMA 3.21. *Let* $t\in[0,1]$, $\Re(r)>1/2$ *and* $m\equiv\bullet \bmod 2$. *Then we have*

$$B\Phi_{m+1,t}^r+b\Phi_{m-1,t}^r=\Big(\frac{p-1}{2}+r\Big)\phi_{m,t}^r-t\,\frac{p+2r}{2}\phi_{m,t}^{r+1}.$$

Proof. By Proposition 3.17, we see that both sides are well defined as continuous multi-linear maps from $\mathcal{A}^{\otimes(m+1)}$ to the set of holomorphic functions on the half plane $\Re(r) > (m-1)/2$. We include the following argument from [**17**, Proposition 5.14] for completeness.

First, using the cyclic property of $\langle\langle \dots \rangle\rangle$ of Lemma 3.13 and the fact that $m \equiv \bullet \,(\mathrm{mod}\ 2)$, we have

$$\begin{aligned}
&B\Phi^r_{m+1,t}(a_0,\dots,a_m)\\
&\quad= \frac{\eta_{m+2}}{2}\sum_{j=0}^{m}\int_0^\infty s^{m+2}(-1)^{mj}\langle\langle 1, da_j, da_{j+1}\dots, da_{j-2}, da_{j-1}\rangle\rangle_{m+1,r,s,t}ds\\
&\quad= \frac{\eta_{m+2}}{2}\sum_{j=0}^{m}\int_0^\infty s^{m+2}\langle\langle da_0,\dots, da_{j-1}, 1, da_j,\dots, da_m\rangle\rangle_{m+1,r,s,t}ds.
\end{aligned}$$

Using the s-trick (Lemma 3.9), we deduce

$$\begin{aligned}
B\Phi^r_{m+1,t}(a_0,\dots,a_m) &= -\frac{\eta_{m+2}(m+1)}{4}\int_0^\infty s^m\langle\langle da_0,\dots,da_m\rangle\rangle_{m,r,s,t}ds\\
&= -\frac{\eta_m}{2}\int_0^\infty s^m\langle\langle da_0,\dots,da_m\rangle\rangle_{m,r,s,t}ds. \qquad (3.19)
\end{aligned}$$

The computation for $b\Phi^r_{m-1,t}$ is the same as for $b\phi^r_{m-1,t}$ in Equation (3.17), except we need to take account of the extra term in Equation (3.13). This gives

$$\begin{aligned}
b\Phi^r_{m-1,t}(a_0,\dots,a_m) &= \frac{\eta_m}{2}\sum_{j=1}^{m}(-1)^j\int_0^\infty s^m\langle\langle a_0, da_1,\dots,[\mathcal{D}^2,a_j],\dots,da_m\rangle\rangle_{m,s,r,t}ds\\
&\qquad - \frac{\eta_m}{2}\sum_{j=1}^{m}\int_0^\infty s^m\langle a_0, da_1,\dots,da_j,\dots,da_m\rangle_{m,s,r,t}ds\\
&= \frac{\eta_m}{2}\sum_{j=1}^{m}(-1)^j\int_0^\infty s^m\langle\langle a_0, da_1\dots,[\mathcal{D}^2,a_j],\dots,da_m\rangle\rangle_{m,s,r,t}ds\\
&\qquad - \frac{\eta_m m}{2}\int_0^\infty s^m\langle a_0, da_1,\dots,da_m\rangle_{m,s,r,t}ds.
\end{aligned}$$

Now put them together. First, using $\eta_{m+2}(m+1)/2 = \eta_m$ we have

$$\begin{aligned}
(B\Phi^r_{m+1,t} + b\Phi^r_{m-1,t})(a_0,\dots,a_m) &= -\frac{\eta_m}{2}\int_0^\infty s^m\langle\langle da_0,\dots,da_m\rangle\rangle_{m,s,r,t}ds\\
&+ \frac{\eta_m}{2}\sum_{j=1}^{m}(-1)^j\int_0^\infty s^m\langle\langle a_0, da_1,\dots,[\mathcal{D}^2,a_j],\dots,da_m\rangle\rangle_{m,s,r,t}ds\\
&- \frac{\eta_m m}{2}\int_0^\infty s^m\langle a_0, da_1,\dots,da_m\rangle_{m,s,r,t}ds,
\end{aligned}$$

and then applying $[\mathcal{D}^2, a_j] = [\mathcal{D}, [\mathcal{D}, a_j]]_\pm$ to the previous equality (with $\deg(a_0) = 0$) yields

$$\begin{aligned}
&- \frac{\eta_m}{2}(-1)^{\deg(a_0)} \int_0^\infty s^m \langle\langle [\mathcal{D}, a_0]_\pm, da_1, \dots, da_m \rangle\rangle_{m,s,r,t} ds \\
&+ \frac{-\eta_m}{2} \sum_{j=1}^m (-1)^{\deg(a_0)+\deg(da_1)+\cdots+\deg(da_{j-1})} \\
&\qquad\qquad \times \int_0^\infty s^m \langle\langle a_0, da_1 \dots, [\mathcal{D}, da_j]_\pm, \dots, da_m \rangle\rangle_{m,s,r,t} ds \\
&- \frac{\eta_m m}{2} \int_0^\infty s^m \langle a_0, da_1, \dots, da_m \rangle_{m,s,r,t} ds.
\end{aligned}$$

Then identity (3.14) of Lemma 3.13 shows that this is equal to

$$\begin{aligned}
\frac{-2\eta_m}{2} \int_0^\infty s^m \Big(&\sum_{j=0}^m \langle a_0, \dots, da_j, \mathcal{D}^2, da_{j+1}, \dots, da_m \rangle_{m+1,s,r,t} \\
&+ \frac{m}{2} \langle a_0, da_1, \dots, da_m \rangle_{m,s,r,t} \Big) ds,
\end{aligned}$$

where for $j = 0$ we mean $\langle a_0, \mathcal{D}^2, da_1, \dots, da_m \rangle_{m+1,s,r,t}$. Then, applying Lemma 3.14 gives us finally

$$\begin{aligned}
(B\Phi^r_{m+1,t} + b\Phi^r_{m-1,t})(a_0, \dots, a_m) &= \eta_m \frac{p+2r-1}{2} \int_0^\infty s^m \langle a_0, da_1, \dots, da_m \rangle_{m,s,r,t} ds \\
&\quad + t\, \eta_m \sum_{j=0}^m \int_0^\infty s^m \langle a_0, \dots, da_j, 1, da_{j+1}, \dots, da_m \rangle_{m+1,s,r,t} ds \\
&= \frac{p+2r-1}{2} \phi^r_{m,t}(a_0, \dots, a_m) - t \frac{p+2r}{2} \phi^{r+1}_{m,t}(a_0, \dots, a_m), \qquad (3.20)
\end{aligned}$$

where we used the λ-trick (Lemma 3.10) in the last line. Again, for $j = 0$ we mean $\langle a_0, 1, da_1, \dots, da_m \rangle_{m+1,s,r,t}$. □

PROPOSITION 3.22. *Viewed as a cochain with non-trivial components for $m = M$ only,*

$$(r - (1-p)/2)^{-1} B\Phi^r_{M+1,0},$$

is a (b, B)-cocycle modulo cochains with values in functions holomorphic at $r = (1-p)/2$ and is cohomologous to the resolvent cocycle $(\phi^r_{m,0})^M_{m=\bullet}$.

PROOF. By Proposition 3.21, applying (B, b) to the finitely supported cochain

$$\Big(\frac{1}{(r - (1-p)/2)} \Phi^r_{1-\bullet,0}, \dots, \frac{1}{(r - (1-p)/2)} \Phi^r_{M-1,0}, 0, 0, \dots \Big),$$

yields

$$\Big(\phi^r_{\bullet,0}, \phi^r_{\bullet+2,0}, \dots, \phi^r_{M,0} - \frac{B\Phi^r_{M+1,0}}{(r-(1-p)/2)}, 0, 0, \dots\Big) = \Big((\phi^r_{m,0})^M_{m=\bullet} - \frac{B\Phi^r_{M+1,0}}{(r-(1-p)/2)}\Big).$$

That is, $(\phi^r_{m,0})^M_{m=\bullet}$ is cohomologous to $(r-(1-p)/2)^{-1} B\Phi^r_{M+1,0}$. Observe that because it is in the image of B, $(r-(1-p)/2)^{-1} B\Phi^r_{M+1,0}$ is cyclic. It is also a b-cyclic cocycle modulo cochains with values in the functions holomorphic at $r = (1-p)/2$. This follows from

$$b\Phi^r_{M-1,0} + B\Phi^r_{M+1,0} = (r - (1-p)/2)\phi^r_{M,0},$$

by applying b and recalling that $b\phi^r_{M,0}$ is holomorphic at $r=(1-p)/2$. $\square$

Taking residues at $r=(1-p)/2$ and applying Proposition 3.20, together with the two preceding results, leads directly to

COROLLARY 3.23. *If the spectral triple $(\mathcal{A},\mathcal{H},\mathcal{D})$ has isolated dimension spectrum, then the residue cocycle $(\phi_{m,0})^M_{m=\bullet}$ is cohomologous to $B\Phi^{(1-p)/2}_{M+1,0}$ (viewed as a single term cochain).*

PROPOSITION 3.24. *Let $R,\,T\in[0,1]$. Then, modulo coboundaries and cochains yielding holomorphic functions at the critical point $r=(1-p)/2$, we have the equality $(\phi^r_{m,R})^M_{m=\bullet}=(\phi^r_{m,T})^M_{m=\bullet}$.*

PROOF. Replacing r by $r+k$ in Proposition 3.21 yields the formula

$$\phi^{r+k}_{m,t}=\frac{1}{r+k+(p-1)/2}\Big(B\Phi^{r+k}_{m+1,t}+b\Phi^{r+k}_{m-1,t}+\Big(\frac{p}{2}+r+k\Big)t\phi^{r+k+1}_{m,t}\Big). \tag{3.21}$$

Recall from Proposition 3.18 that for $\mathcal{D}$ invertible, $\phi^r_{m,t}$ is defined and holomorphic for $\Re(r)>(1-m)/2$ for all $t\in[0,1]$. As $[0,1]$ is compact, the integral

$$\int_0^1\phi^r_{m,t}(a_0,\dots,a_m)dt,$$

is holomorphic for $\Re(r)>(1-m)/2$ and any $a_0,\dots,a_m\in\mathcal{A}$. Now we make some simple observations, omitting the variables $a_0,\dots,a_m$ to lighten the notation. For $T,\,R\in[0,1]$ we have

$$\phi^r_{m,T}-\phi^r_{m,R}=\int_R^T\frac{d}{dt}\phi^r_{m,t}dt=-(p/2+r)\int_R^T\phi^{r+1}_{m,t}dt. \tag{3.22}$$

Now apply the formula of Equation (3.21) iteratively. At the first step we have

$$\phi^r_{m,T}-\phi^r_{m,R}=\frac{-(p/2+r)}{r+1+(p-1)/2}\int_R^T\Big(B\Phi^{r+1}_{m+1,t}+b\Phi^{r+1}_{m-1,t}+\Big(\frac{p}{2}+r+1\Big)t\phi^{r+2}_{m,t}\Big)dt.$$

Observe that the numerical factors are holomorphic at $r=(1-p)/2$. Iterating this procedure L times gives us

$$\phi^r_{m,T}-\phi^r_{m,R}=\frac{-(p/2+r)\dots(p/2+r+L)}{(r+1+(p-1)/2)\dots(r+L+(p-1)/2)}\int_R^T t^L\phi^{r+L+1}_{m,t}dt$$
$$+\sum_{j=1}^{L}\frac{-(p/2+r)\dots(p/2+r+j-1)}{(r+1+(p-1)/2)\dots(r+j+(p-1)/2)}\int_R^T\Big(B\Phi^{r+j}_{m+1,t}+b\Phi^{r+j}_{m-1,t}\Big)t^{j-1}dt.$$

In fact the smallest L guaranteeing that $\phi^{r+L+1}_{m,t}$ is holomorphic at $r=(1-p)/2$ for all m is $(M-\bullet)/2$. See [**17**, Lemma 5.20] for a proof. With this choice of $L=(M-\bullet)/2$, we have modulo cochains yielding functions holomorphic in a half plane containing $(1-p)/2$,

$$\phi^r_{m,T}-\phi^r_{m,R}=$$
$$\sum_{j=1}^{L}\frac{-(p/2+r)\dots(p/2+r+j-1)}{(r+1+(p-1)/2)\dots(r+j+(p-1)/2)}\int_R^T\Big(B\Phi^{r+j}_{m+1,t}+b\Phi^{r+j}_{m-1,t}\Big)t^{j-1}dt.$$

Thus, a simple rearrangement yields the cohomology, valid for $\Re(r) > (1-\bullet)/2$,

$$(\phi^r_{m,T} - \phi^r_{m,R})^M_{m=\bullet} - B\sum_{j=1}^{L} \frac{-(p/2+r)\dots(p/2+r+j-1)}{(r+1+(p-1)/2)\dots(r+j+(p-1)/2)} \int_R^T \Phi^{r+j}_{M+1,t} t^{j-1} dt = (B+b)\left(\sum_{j=1}^{L} \frac{-(p/2+r)\dots(p/2+r+j-1)}{(r+1+(p-1)/2)\dots(r+j+(p-1)/2)} \int_R^T \Phi^{r+j}_{m,t} t^{j-1} dt\right)^{M-1}_{m=1-\bullet}.$$

Hence, modulo coboundaries and cochains yielding functions holomorphic at $r = (1-p)/2$, the cochain $(\phi^r_{m,T} - \phi^r_{m,R})^M_{m=\bullet}$ is equal to

$$B\sum_{j=1}^{L} \frac{-(p/2+r)\dots(p/2+r+j-1)}{(r+1+(p-1)/2)\dots(r+j+(p-1)/2)} \int_R^T \Phi^{r+j}_{M+1,t} t^{j-1} dt.$$

However, an application of Lemma 3.3 now shows that this term is holomorphic at $r = (1-p)/2$, since $j \geq 1$ in all cases. Hence, modulo coboundaries and cochains yielding functions holomorphic at $r = (1-p)/2$, we have

$$(\phi^r_{m,T})^M_{m=\bullet} = (\phi^r_{m,R})^M_{m=\bullet},$$

which is the equality we were looking for. □

Corollary 3.25. *Modulo coboundaries and cochains yielding functions holomorphic in a half plane containing $r = (1-p)/2$, we have the equality*

$$(\phi^r_m)^M_{m=\bullet} := (\phi^r_{m,1})^M_{m=\bullet} = B\Phi^r_{M+1,0}.$$

At this point we have shown that the resolvent cocycle is (b,B)-cohomologous to the cocycle $(r-(1-p)/2)^{-1}B\Phi^r_{M+1,0}$ (modulo functions holomorphic at $r = (1-p)/2$), while the residue cocycle is (b,B)-cohomologous to $B\Phi^{(1-p)/2}_{M+1,0}$. We remark that $B\Phi^{(1-p)/2}_{M+1,0}$ is well-defined (i.e. finite) by an application of Lemma 3.3.

Our aim now is to use the map $[0,1] \ni u \to \mathcal{D}|\mathcal{D}|^{-u}$ to obtain a homotopy from $B\Phi^{(1-p)/2}_{M+1,0}$ to the Chern character. This is the most technically difficult part of the proof, and we defer the proof of the next lemma to the Appendix, Lemma A.2.4. This lemma proves a trace class differentiability result.

Lemma 3.26. *For $a_0,\dots,a_M \in \mathcal{A}$ and $j = 0,\dots,M$, we define $T_{s,\lambda,j}(u)$ to be*

$$d_u(a_0)\, R_{s,u}(\lambda)\dots d_u(a_j)\, R_{s,u}(\lambda)\, \mathcal{D}_u\, R_{s,u}(\lambda)\, d_u(a_{j+1})\, R_{s,u}(\lambda)\dots d_u(a_M)\, R_{s,u}(\lambda).$$

Then the map $\big[u \mapsto T_{s,\lambda,j}(u)\big]$ *is continuously differentiable for the trace norm topology. Moreover, with* $R_u := R_{s,u}(\lambda)$ *and* $\dot{\mathcal{D}}_u = -\mathcal{D}_u \log|\mathcal{D}|$*, we obtain*

$$
\begin{aligned}
&\frac{dT_{s,\lambda,j}}{du}(u) = \\
&\sum_{k=0}^{M} d_u(a_0)\,R_u \ldots R_u\, d_u(a_k)\,(2R_u\,\mathcal{D}_u\,\dot{\mathcal{D}}_u\,R_u)\,d_u(a_{k+1})\,R_u \ldots d_u(a_M)\,R_u \\
&+ d_u(a_0)\,R_u \ldots R_u\, d_u(a_j)\,R_u\,\mathcal{D}_u\,(2R_u\,\mathcal{D}_u\,\dot{\mathcal{D}}_u\,R_u)\,d_u(a_{j+1}) \ldots R_u\, d_u(a_M)\,R_u \\
&+ \sum_{k=0}^{M} d_u(a_0)\,R_u\,d_u(a_1)\,R_u \ldots R_u[\dot{\mathcal{D}}_u, a_k]\,R_u \ldots R_u\,d_u(a_M)\,R_u \\
&+ d_u(a_0)\,R_u\,d_u(a_1)\,R_u \ldots R_u\,d_u(a_j)\,R_u\,\dot{\mathcal{D}}_u\,R_u\,d_u(a_{j+1}) \ldots R_u\,d_u(a_M)\,R_u.
\end{aligned}
$$

LEMMA 3.27. *For* $a_0, \ldots, a_M \in \mathcal{A}$ *and for* $r > (1-M)/2$*, we have*

$$
\begin{aligned}
(bB\Psi^r_{M,u})(a_0,\ldots,a_M) &= \frac{d}{du}(B\Phi^r_{M+1,0,u})(a_0,\ldots,a_M) - \eta_M \frac{r+(p-1)}{2} \\
\times \sum_{j=0}^{M}(-1)^j \int_0^\infty & s^M \langle [\mathcal{D}_u, a_0], \ldots, [\mathcal{D}_u, a_j], \dot{\mathcal{D}}_u, [\mathcal{D}_u, a_{j+1}], \ldots, [\mathcal{D}_u, a_M]\rangle_{M+1,r,s,0}\, ds,
\end{aligned}
$$

where the expectation uses the resolvent for $\mathcal{D}_u$*, that is* $R_{s,0,u}(\lambda)$*. Moreover,*

$$
r \mapsto -\eta_M \sum_{j=0}^{M}(-1)^j \int_0^\infty s^M \langle [\mathcal{D}_u, a_0], \ldots, [\mathcal{D}_u, a_j], \dot{\mathcal{D}}_u, \ldots, [\mathcal{D}_u, a_M]\rangle_{M+1,r,s,0}\, ds,
$$

is a holomorphic function of r *in a right half plane containing the critical point* $r = (1-p)/2$*.*

PROOF. Lemma 3.26, and together with arguments of a similar nature, show that $\Psi^r_{M,u}$ and $\frac{d}{du}\Phi^r_{M+1,0,u}$ are well-defined and continuous. The proof of Lemma 3.26 also shows that the formal differentiations given below are in fact justified. First of all, using the $\mathcal{D}_u$ version of Equation 3.19 of Lemma 3.21 and the R_u version of Definition 3.4 to expand $(B\Phi^r_{M+1,0,u})(a_0,\ldots,a_M)$, we see that it is the sum of the $T_{s,\lambda,j}(u)$ and so its derivative is the sum over j of the derivatives in Lemma 3.26. Using the R_u version of Definition 3.4 again to rewrite this in terms of $\langle\langle \ldots \rangle\rangle$ where possible, shows that

$$
\begin{aligned}
&\frac{d}{du}(B\Phi^r_{M+1,0,u})(a_0,\ldots,a_M) = \\
&-\frac{\eta_M}{2}\int_0^\infty s^M \sum_{j=0}^{M}\Big(\langle\langle [\mathcal{D}_u, a_0], \ldots, [\mathcal{D}_u, a_j], 2\mathcal{D}_u\dot{\mathcal{D}}_u, \ldots, [\mathcal{D}_u, a_M]\rangle\rangle_{M+1,s,r,0} \\
&\qquad\qquad + \langle\langle [\mathcal{D}_u, a_0], \ldots, [\dot{\mathcal{D}}_u, a_j], \ldots, [\mathcal{D}_u, a_M]\rangle\rangle_{M,s,r,0}\Big)\, ds \\
&-\frac{\eta_M}{2}\int_0^\infty s^M \sum_{j=0}^{M}(-1)^j \langle [\mathcal{D}_u, a_0], \ldots, [\mathcal{D}_u, a_j], \dot{\mathcal{D}}_u, \ldots, [\mathcal{D}_u, a_M]\rangle_{M+1,s,r,0}\, ds.
\end{aligned}
$$

For the next step we compute $Bb\Psi^r_{M,u}$, and then use $bB = -Bb$. First we apply b

$$
\begin{aligned}
&(b\Psi^r_{M,u})(a_0,\dots,a_{M+1}) = -\frac{\eta_M}{2}\int_0^\infty s^M\langle\langle a_0a_1\dot{\mathcal{D}}_u,[\mathcal{D}_u,a_2],\dots,[\mathcal{D}_u,a_{M+1}]\rangle\rangle_{M,s,r,0}ds\\
&-\frac{\eta_M}{2}\sum_{j=1}^{M}(-1)^j\int_0^\infty s^M\langle\langle a_0\dot{\mathcal{D}}_u,\dots,[\mathcal{D}_u,a_ja_{j+1}],\dots,[\mathcal{D}_u,a_{M+1}]\rangle\rangle_{M,s,r,0}ds\\
&-(-1)^{M+1}\frac{\eta_M}{2}\int_0^\infty s^M\langle\langle a_{M+1}a_0\dot{\mathcal{D}}_u,[\mathcal{D}_u,a_1],\dots,[\mathcal{D}_u,a_M]\rangle\rangle_{M,s,r,0}ds\\
&=-\frac{\eta_M}{2}\int_0^\infty s^M\sum_{j=1}^{M+1}(-1)^j\langle\langle a_0\dot{\mathcal{D}}_u,[\mathcal{D}_u,a_1],\dots,[\mathcal{D}_u^2,a_j],\dots,[\mathcal{D}_u,a_{M+1}]\rangle\rangle_{M+1,s,r,0}ds\\
&-\frac{\eta_M}{2}\int_0^\infty s^M\sum_{j=1}^{M+1}(-1)^j(-1)^{\deg(a_0\dot{\mathcal{D}}_u)+\cdots+\deg([\mathcal{D}_u,a_{j-1}])}\\
&\qquad\qquad\times\langle a_0\dot{\mathcal{D}}_u,[\mathcal{D}_u,a_1],\dots,[\mathcal{D}_u,a_{M+1}]\rangle_{M+1,s,r,0}ds\\
&+\frac{\eta_M}{2}\int_0^\infty s^M\langle\langle a_0[\dot{\mathcal{D}}_u,a_1],\dots,[\mathcal{D}_u,a_{M+1}]\rangle\rangle_{M,s,r,0}ds.
\end{aligned}
$$

The last equality follows from the R_u version of Lemma 3.13. In the above, we note that $\deg(a_0\dot{\mathcal{D}}_u) = 1 = \deg([\mathcal{D}_u,a_k])$ for all k so that $\deg(a_0\dot{\mathcal{D}}_u)+\cdots+\deg([\mathcal{D}_u,a_{j-1}]) = j$ and $\deg(a_0\dot{\mathcal{D}}_u)+\cdots+\deg([\mathcal{D}_u,a_{M+1}]) = M+2 \equiv \bullet(\mathrm{mod}\ 2)$. We also note the commutator identity $[\mathcal{D}_u^2,a_j] = \{\mathcal{D}_u,[\mathcal{D}_u,a_j]\} = [\mathcal{D}_u,[\mathcal{D}_u,a_j]]_\pm$ so in order to apply the $\mathcal{D}_u$ version of Equation (3.14) of Lemma 3.13 we first add and substract

$$-\frac{\eta_M}{2}\int_0^\infty s^M\langle\langle\{\mathcal{D}_u,a_0\dot{\mathcal{D}}_u\},[\mathcal{D}_u,a_1],\dots,[\mathcal{D}_u,a_{M+1}]\rangle\rangle_{M+1,s,r,0}ds,$$

and then an application of Equation (3.14) yields

$$
\begin{aligned}
&(b\Psi^r_{M,u})(a_0,\dots,a_{M+1}) =\\
&-2\frac{\eta_M}{2}\int_0^\infty s^M\sum_{j=0}^{M+1}\langle a_0\dot{\mathcal{D}}_u,\dots,[\mathcal{D}_u,a_j],\mathcal{D}_u^2,\dots,[\mathcal{D}_u,a_{M+1}]\rangle_{M+2,s,r,0}ds\\
&+\frac{\eta_M}{2}\int_0^\infty s^M\langle\langle a_0\{\mathcal{D}_u,\dot{\mathcal{D}}_u\}+[\mathcal{D}_u,a_0]\dot{\mathcal{D}}_u,[\mathcal{D}_u,a_1],\dots,[\mathcal{D}_u,a_{M+1}]\rangle\rangle_{M+1,s,r,0}ds\\
&-\frac{\eta_M}{2}(M+1)\int_0^\infty s^M\langle a_0\dot{\mathcal{D}}_u,[\mathcal{D}_u,a_1],\dots,[\mathcal{D}_u,a_{M+1}]\rangle_{M+1,s,r,0}ds\\
&+\frac{\eta_M}{2}\int_0^\infty s^M\langle\langle a_0[\dot{\mathcal{D}}_u,a_1],\dots,[\mathcal{D}_u,a_{M+1}]\rangle\rangle_{M,s,r,0}ds.
\end{aligned}
$$

Then we apply the $\mathcal{D}_u$ version of Lemma 3.14 to obtain

$$(b\Psi^r_{M,u})(a_0,\dots,a_{M+1}) =$$
$$\frac{\eta_M}{2}(p+2r)\int_0^\infty s^M\langle a_0\dot{\mathcal{D}}_u,[\mathcal{D}_u,a_1],\dots,[\mathcal{D}_u,a_{M+1}]\rangle_{M+1,s,r,0}ds$$
$$+\frac{\eta_M}{2}\int_0^\infty s^M\langle\langle a_0\{\mathcal{D}_u,\dot{\mathcal{D}}_u\}+[\mathcal{D}_u,a_0]\dot{\mathcal{D}}_u,[\mathcal{D}_u,a_1],\dots,[\mathcal{D}_u,a_{M+1}]\rangle\rangle_{M+1,s,r,0}ds$$
$$+\frac{\eta_M}{2}\int_0^\infty s^M\langle\langle a_0[\dot{\mathcal{D}}_u,a_1],\dots,[\mathcal{D}_u,a_{M+1}]\rangle\rangle_{M,s,r,0}ds.$$

The next step is to apply B to these three terms, producing (with $a_{-1}:=a_M$)

$$(Bb\Psi^r_{M,u})(a_0,\dots,a_M) =$$
$$(p+2r)\frac{\eta_M}{2}\sum_{j=0}^M(-1)^{(M+1)j}\int_0^\infty s^M\langle\dot{\mathcal{D}}_u,[\mathcal{D}_u,a_j],\dots,[\mathcal{D}_u,a_{j-1}]\rangle_{M+1,s,r,0}ds$$
$$+\frac{\eta_M}{2}\sum_{j=0}^M(-1)^{(M+1)j}\int_0^\infty s^M\langle\langle\{\mathcal{D}_u,\dot{\mathcal{D}}_u\},[\mathcal{D}_u,a_j],\dots,[\mathcal{D}_u,a_{j-1}]\rangle\rangle_{M+1,s,r,0}ds$$
$$+\frac{\eta_M}{2}\sum_{j=0}^M(-1)^{(M+1)j}\int_0^\infty s^M\langle\langle[\dot{\mathcal{D}}_u,a_j],\dots,[\mathcal{D}_u,a_{j-1}]\rangle\rangle_{M,s,r,0}ds,$$

which is identical to

$$\frac{(p+2r)\eta_M}{2}\sum_{j=0}^M(-1)^{(M+1)j+(1-\bullet)j}$$
$$\times\int_0^\infty s^M\langle[\mathcal{D}_u,a_0],\dots,[\mathcal{D}_u,a_{j-1}],\dot{\mathcal{D}}_u,\dots,[\mathcal{D}_u,a_M]\rangle_{M+1,s,r,0}ds$$
$$+\frac{\eta_M}{2}\sum_{j=0}^M(-1)^{(M+1)j+(2-\bullet)j}$$
$$\times\int_0^\infty s^M\langle\langle[\mathcal{D}_u,a_0],\dots,\{\mathcal{D}_u,\dot{\mathcal{D}}_u\},[\mathcal{D}_u,a_j],\dots,[\mathcal{D}_u,a_M]\rangle\rangle_{M+1,s,r,0}ds$$
$$+\frac{\eta_M}{2}\sum_{j=0}^M(-1)^{(M+1)j+(2-\bullet)j}$$
$$\times\int_0^\infty s^M\langle\langle[\mathcal{D}_u,a_0],\dots,[\mathcal{D}_u,a_{j-1}],[\dot{\mathcal{D}}_u,a_j],\dots,[\mathcal{D}_u,a_M]\rangle\rangle_{M,s,r,0}ds.$$

This last expression equals

$$(p+2r)\frac{\eta_M}{2}\sum_{j=0}^M(-1)^j\int_0^\infty s^M\langle[\mathcal{D}_u,a_0],\dots,[\mathcal{D}_u,a_{j-1}],\dot{\mathcal{D}}_u,\dots,[\mathcal{D}_u,a_M]\rangle_{M+1,s,r,0}ds$$
$$+\frac{\eta_M}{2}\sum_{j=0}^M\int_0^\infty s^M\langle\langle[\mathcal{D}_u,a_0],\dots,2\mathcal{D}_u\dot{\mathcal{D}}_u,[\mathcal{D}_u,a_j],\dots,[\mathcal{D}_u,a_M]\rangle\rangle_{M+1,s,r,0}ds$$
$$+\frac{\eta_M}{2}\sum_{j=0}^M\int_0^\infty s^M\langle\langle[\mathcal{D}_u,a_0],\dots,[\mathcal{D}_u,a_{j-1}],[\dot{\mathcal{D}}_u,a_j],\dots,[\mathcal{D}_u,a_M]\rangle\rangle_{M,s,r,0}ds.$$

Using $bB = -Bb$, and our formula for $\frac{d}{du}(B\Phi^r_{M+1,0,u})(a_0,\dots,a_M)$ gives

$$(bB\Psi^r_{M,u})(a_0,\dots,a_M) = -(p+2r)\frac{\eta_M}{2}\sum_{j=0}^{M}(-1)^j\int_0^\infty s^M\langle[\mathcal{D}_u,a_0],\dots,$$
$$\dots,[\mathcal{D}_u,a_{j-1}],\dot{\mathcal{D}}_u,[\mathcal{D}_u,a_j],\dots,[\mathcal{D}_u,a_M]\rangle_{M+1,s,r,0}ds$$
$$+\frac{\eta_M}{2}\sum_{j=0}^{M}(-1)^j\int_0^\infty s^M\langle[\mathcal{D}_u,a_0],\dots,$$
$$\dots,[\mathcal{D}_u,a_j],\dot{\mathcal{D}}_u,[\mathcal{D}_u,a_{j+1}],\dots,[\mathcal{D}_u,a_M]\rangle_{M+1,s,r,0}ds$$
$$+\frac{d}{du}(B\Phi^r_{M+1,0,u})(a_0,\dots,a_M).$$

This proves the result. □

Thus we have proven the following key statement.

COROLLARY 3.28. *We have*

$$\frac{1}{(r+(p-1)/2)}(bB\Psi^r_{M,U})(a_0,\dots,a_M) =$$
$$\frac{1}{(r+(p-1)/2)}\frac{d}{du}(B\Phi^r_{M+1,0,u})(a_0,\dots,a_M)+holo(r),$$

where holo is analytic for $\Re(r) > -M/2$, and by taking residues

$$(bB\Psi^{(1-p)/2}_{M,u})(a_0,\dots,a_M) = \frac{d}{du}(B\Phi^{(1-p)/2}_{M+1,0,u})(a_0,\dots,a_M).$$

We now have the promised cohomologies.

THEOREM 3.29. *Let $(\mathcal{A},\mathcal{H},\mathcal{D})$ be a smoothly summable spectral triple relative to $(\mathcal{N},\tau)$ and of spectral dimension $p \geq 1$, parity $\bullet \in \{0,1\}$, with $\mathcal{D}$ invertible and $\mathcal{A}$ separable. Then*
(1) In the (b,B)-bicomplex with coefficients in the set of holomorphic functions on the right half plane $\Re(r) > 1/2$, the resolvent cocycle $(\phi^r_m)^M_{m=\bullet}$ is cohomologous to the single term cocycle

$$(r-(1-p)/2)^{-1}\mathrm{Ch}^M_F,$$

modulo cochains with values in the set of holomorphic functions on a right half plane containing the critical point $r = (1-p)/2$. Here $F = \mathcal{D}\,|\mathcal{D}|^{-1}$.
(2) If, moreover, the spectral triple $(\mathcal{A},\mathcal{H},\mathcal{D})$ has isolated spectral dimension, then the residue cocycle $(\phi_m)^M_{m=\bullet}$ is cohomologous to the Chern character Ch^M_F.

PROOF. Up to cochains holomorphic at the critical point (the integral on a compact domain does not modify the holomorphy property), Lemma 3.27 gives

$$\frac{1}{r-(1-p)/2}\int_0^1(bB\Psi^r_{M,u})(a_0,\dots,a_M)\,du =$$
$$\frac{1}{r-(1-p)/2}\int_0^1\frac{d}{du}(B\Phi^r_{M+1,0,u})(a_0,\dots,a_M)\,du.$$

Since $\frac{1}{r-(1-p)/2}\int_0^1 bB\Psi^r_{M,u}$ is a coboundary, we obtain the following equality in cyclic cohomology (up to coboundaries and a cochain holomorphic at the critical point)

$$\frac{1}{r-(1-p)/2}(B\Phi^r_{M+1,0,1}) = \frac{1}{r-(1-p)/2}(B\Phi^r_{M+1,0,0}).$$

One can now compute directly to see that the left hand side is $(r-(1-p)/2)^{-1}\mathrm{Ch}_F^M$ as follows. Recalling that $F^2=1$ and using our previous formula for $B\Phi^r_{M+1,0,u}$ (the $\mathcal{D}_u$ version of Proposition 3.21 with $u=1$) we have

$$\begin{aligned}
&(B\Phi^r_{M+1,0,u})(a_0,\dots,a_M)|_{u=1}\\
&= -\frac{\eta_M}{2}\sum_{j=0}^{M}(-1)^{j+1}\int_0^\infty s^M\langle [F,a_0],\dots,[F,a_j],F,[F,a_{j+1}],\dots,[F,a_M]\rangle_{M+1,s,r,0}ds\\
&= -\frac{\eta_M}{2}\sum_{j=0}^{M}\int_0^\infty s^M\frac{1}{2\pi i}\tau\left(\gamma\int_\ell \lambda^{-p/2-r}F[F,a_0]\dots[F,a_M](\lambda-(s^2+1))^{-M-2}d\lambda\right)ds\\
&= \frac{\eta_M}{2}\frac{(-1)^M}{M!}\frac{\Gamma(M+1+p/2+r)}{\Gamma(p/2+r)}\\
&\qquad\times\int_0^\infty s^M\tau\big(\gamma F[F,a_0]\dots[F,a_M](s^2+1)^{-M-1-p/2-r}\big)ds.
\end{aligned}$$

In the second equality we anticommuted F past the commutators, and pulled all the resolvents to the right (they commute with everything, since they involve only scalars). In the last equality we used the Cauchy integral formula to do the contour integral, and performed the sum. Now we pull out $(s^2+1)^{-M-1-p/2-r}$ from the trace, leaving the identity behind. The s-integral is given by

$$\int_0^\infty s^M(s^2+1)^{-M-1-p/2-r}ds = \frac{\Gamma((M+1)/2)\Gamma(p/2+r+M/2+1/2)}{2\Gamma(M+1+p/2+r)}.$$

Putting the pieces together gives

$$\begin{aligned}
&(B\Phi^r_{M+1,0,u})(a_0,\dots,a_M)|_{u=1} =\\
&\frac{\eta_M}{2}(-1)^M\frac{\Gamma((M+1)/2)}{\Gamma(p/2+r)}\frac{\Gamma(((p-1)/2+r)+M/2+1)}{2M!}\tau(\gamma F[F,a_0]\dots[F,a_M]).
\end{aligned}$$

Now $\eta_M = \sqrt{2i}^{\bullet}(-1)^M 2^{M+1}\Gamma(M/2+1)/\Gamma(M+1)$, and the duplication formula for the Gamma function tells us that $\Gamma((M+1)/2)\Gamma(M/2+1)2^M = \sqrt{\pi}\Gamma(M+1)$. Hence

$$\begin{aligned}
&(B\Phi^r_{M+1,0,u})(a_0,\dots,a_M)|_{u=1} =\\
&\qquad\frac{\sqrt{\pi}\sqrt{2i}^{\bullet}\Gamma(((p-1)/2+r)+M/2+1)}{2M!\Gamma(p/2+r)}\tau(\gamma F[F,a_0][F,a_1]\dots[F,a_M]).
\end{aligned}$$

Now we use the functional equation for the Gamma function

$$\Gamma(((p-1)/2+r)+M/2+1) = \Gamma((p-1)/2+r)\prod_{j=0}^{(M-\bullet)/2}((p-1)/2+r+j+\bullet/2),$$

to write this as

$$(B\Phi^r_{M+1,0,u})(a_0,\dots,a_M)|_{u=1} =$$
$$\frac{C_{p/2+r}\sqrt{2i}^{\bullet}}{2\cdot M!}\sum_{j=1-\bullet}^{(M-\bullet)/2+1}(r+(p-1)/2)^j\sigma_{(M-\bullet)/2,j}\tau(\gamma F[F,a_0][F,a_1]\dots[F,a_M]),$$

where the $\sigma_{(M-\bullet)/2,j}$ are the elementary symmetric functions of the integers $1,2,\dots,M/2$ (even case) or of the half integers $1/2,3/2,\dots,M/2$ (odd case). The 'constant'

$$C_{p/2+r} := \frac{\sqrt{\pi}\Gamma((p-1)/2+r)}{\Gamma(p/2+r)},$$

has a simple pole at $r=(1-p)/2$ with residue equal to 1, and in both even and odd cases we have $\sigma_{M/2,1-\bullet}=\Gamma(M/2+1)$. So, recalling Definition 2.22 of τ' we see that

$$\frac{1}{(r-(1-p)/2)}(B\Phi^r_{M+1,0,u})(a_0,\dots,a_M)|_{u=1} =$$
$$\frac{1}{(r-(1-p)/2)}\mathrm{Ch}_F(a_0,a_1,\dots,a_M)+holo(r),$$

where *holo* is a function holomorphic at $r=(1-p)/2$, and on the right hand side the Chern character appears with its (b,B) normalisation. As the left hand side is cohomologous to the resolvent cocycle by Proposition 3.22, the first part is proven. The proof of the second part is now a consequence of Proposition 3.20. $\square$

3.8. Removing the invertibility of $\mathcal{D}$

We can now apply Theorem 3.29 to the double of a smoothly summable spectral triple of spectral dimension $p\geq 1$. In this case, the resolvent and residue cocycles extend to the reduced (b,B)-bicomplex for $\mathcal{A}^\sim$, and it is simple to check that they are still cocycles there. Moreover, as noted in Lemma 3.8, all of our cohomologies can be considered to take place in the reduced complex for $\mathcal{A}^\sim$.

Thus, under the isolated spectral dimension assumption, the residue cocycle for $(\mathcal{A},\mathcal{H}\oplus\mathcal{H},\mathcal{D}_\mu,\hat{\gamma})$ is cohomologous to the Chern character $\mathrm{Ch}^M_{F_\mu}$, and similarly for the resolvent cocycle. We now show how to obtain a residue and resolvent formula for the index in terms of the original spectral triple.

In the following we write $\{\phi^r_{\mu,m}\}_{m=\bullet,\bullet+2,\dots,M}$ for the resolvent cocycle for $\mathcal{A}$ defined using the double spectral triple and $\{\phi^r_m\}_{m=\bullet,\bullet+2,\dots,M}$ for the resolvent cocycle for $\mathcal{A}$ defined by using original spectral triple, according to the notations introduced in Section 3.3.

The formula for $\mathrm{Ch}^M_{F_\mu}$ is scale invariant, in that it remains unchanged if we replace $\mathcal{D}_\mu$ by $\lambda\mathcal{D}_\mu$ for any $\lambda>0$. This scale invariance is the main tool we employ.

In the double up procedure we will start with $0<\mu<1$. We are interested in the relationship between $(1+\mathcal{D}^2)\otimes\mathrm{Id}_2$ and $1+\mathcal{D}^2_\mu$, given by

$$1+\mathcal{D}^2_\mu = \begin{pmatrix} 1+\mu^2+\mathcal{D}^2 & 0 \\ 0 & 1+\mu^2+\mathcal{D}^2\end{pmatrix}.$$

If we perform the scaling $\mathcal{D}_\mu\mapsto(1-\mu^2)^{-1/2}\mathcal{D}_\mu$, then

$$(1+\mathcal{D}^2_\mu)^{-s}\mapsto(1-\mu^2)^s(1+\mathcal{D}^2)^{-s}\otimes\mathrm{Id}_2.$$

This algebraic simplification is not yet enough. We need to scale every appearance of $\mathcal{D}$ in the formula for the resolvent cocycle. Now Proposition 3.20 provides the following formula for the resolvent cocycle in terms of zeta functions, modulo functions holomorphic at $r = (1-p)/2$:

$$\phi^r_{\mu,m}(a_0, \dots, a_m) = \tag{3.23}$$
$$(\sqrt{2i\pi})^{\bullet} \sum_{|k|=0}^{M-m} (-1)^{|k|} \alpha(n) \sum_{j=1-\bullet}^{(M-\bullet)/2+|k|} \sigma_{h,j} \left(r - (1-p)/2\right)^{j-1+\bullet}$$
$$\times\, \tau \otimes \mathrm{tr}_2\Big(\gamma a_0\, [\mathcal{D}_\mu, a_1]^{(k_1)} \dots [\mathcal{D}_\mu, a_m]^{(k_m)} (1+\mathcal{D}_\mu^2)^{-|k|-m/2-r+1/2-p/2}\Big).$$

So we require the scaling properties of the coefficient operators

$$\omega_{\mu,m,k} = [\mathcal{D}_\mu, a_1]^{(k_1)} \dots [\mathcal{D}_\mu, a_m]^{(k_m)},$$

that appear in this zeta function representation of the resolvent cocycle. In order to study these coefficient operators, it is useful to introduce the following operations (arising from the periodicity operator in cyclic cohomology, see [**14**,**21**]). We define $\hat{S} : \mathcal{A}^{\otimes m} \to \mathrm{OP}^0_0$, for any $m \geq 0$ by

$$\hat{S}(a_1) = 0, \quad \hat{S}(a_1, \dots, a_m) = a_1 a_2 da_3 \dots da_m + da_1 a_2 a_3 da_4 \dots da_m$$
$$\sum_{j=3}^{m-1} da_1 \dots (da_{j-1}) a_j a_{j+1} da_{j+2} \dots da_m,$$

and extend it by linearity to the tensor product $\mathcal{A}^{\otimes m}$. As usual, we write $da = [\mathcal{D}, a]$. To define 'powers' of $\hat{S}$, we recursively set

$$\hat{S}^k(a_1, \dots, a_m) = \sum_{j=0}^{k-1} \binom{k-1}{j} \sum_{i=1}^{m-1} \hat{S}^l(a_1, \dots, a_{i-1}) \hat{S}^{k-j-1}(a_i a_{i+1}, \dots, a_m).$$

The following lemma is proven in [**14**, Appendix].

Lemma 3.30. *The maps $\hat{S}^j$ satisfy the following relations:*

$$\hat{S}(a_1, \dots, a_{m-1}) da_m = \hat{S}(a_1, \dots, a_m) - da_1 \dots (da_{m-2}) a_{m-1} a_m, \tag{3.24}$$

and for $j > 1$

$$\hat{S}^j(a_1, \dots, a_{m-1}) da_m = \hat{S}^j(a_1, \dots, a_m) - j\, \hat{S}^{j-1}(a_1, \dots, a_{m-2}) a_{m-1} a_m,$$
$$j\, \hat{S}^{j-1}(a_1, \dots, a_{2j-2}) a_{2j-1} a_{2j} = \hat{S}^j(a_1, \dots, a_{2j}), \qquad \hat{S}^j(a_1, \dots, a_{2j-1}) = 0.$$

As a last generalisation, we note that if k is now a multi-index then we can define analogues of the operations $\hat{S}^j$ by

$$\hat{S}_k(a_1) := 0,$$
$$\hat{S}_k(a_1, \dots, a_m) := (a_1)^{(k_1)} (a_2)^{(k_2)} (da_3)^{(k_3)} \dots (da_m)^{(k_m)}$$
$$+ (da_1)^{(k_1)} (a_2)^{(k_2)} (a_3)^{(k_3)} (da_4)^{(k_4)} \dots (da_m)^{(k_m)}$$
$$+ \sum_{j=3}^{m-1} (da_1)^{(k_1)} \dots (da_{j-1})^{(k_{j-1})} a_j^{(k_j)} a_{j+1}^{(k_{j+1})} (da_{j+2})^{(k_{j+2})} \dots (da_m)^{(k_m)}.$$

With these operations in hand we can state the result.

Lemma 3.31. *With $\mathcal{D}$ and $\mathcal{D}_\mu$ as above, and for $m > 1$, the product of commutators $[\mathcal{D}_\mu, a_1]^{(k_1)} \dots [\mathcal{D}_\mu, a_m]^{(k_m)}$ is given by*

$$\begin{pmatrix} \omega_{m,k} + \sum_{j=1}^{\lfloor m/2 \rfloor} c_j\, \hat{S}^j(a_1, \dots, a_m) & \begin{matrix} -\mu\omega_{m-1,k} a_m^{(k_m)} - \\ \mu \sum_{j=1}^{\lfloor (m-1)/2 \rfloor} c_j\, \hat{S}^j(a_1, \dots, a_{m-1}) a_m^{(k_m)} \end{matrix} \\ \\ \begin{matrix} \mu a_1^{(k_1)} \tilde{\omega}_{m-1,k} + \\ \mu \sum_{j=1}^{\lfloor (m-1)/2 \rfloor} c_j\, a_1^{(k_1)} \hat{S}^j(a_2, \dots, a_m) \end{matrix} & \begin{matrix} -\mu^2 a_1^{(k_1)}\, \widehat{\omega}_{m-2,k} a_m^{(k_m)} - \\ \mu^2 \sum_{j=1}^{\lfloor m/2 \rfloor - 1} c_j\, a_1^{(k_1)} \hat{S}^j(a_2, \dots, a_{m-1}) a_m^{(k_m)} \end{matrix} \end{pmatrix}.$$

In this expression

$$\omega_{m,k} = (da_1)^{(k_1)} \dots (da_m)^{(k_m)}, \qquad \omega_{m-1,k} = (da_1)^{(k_1)} \dots (da_{m-1})^{(k_{m-1})},$$

$$\tilde{\omega}_{m-1,k} = (da_2)^{(k_2)} \dots (da_m)^{(k_m)}, \qquad \widehat{\omega}_{m-2,k} = (da_2)^{(k_2)} \dots (da_{m-1})^{(k_{m-1})},$$

the superscript (k_j)'s refer to commutators with $\mathcal{D}^2$ (Definition 1.20), and $c_j = (-1)^j \mu^{2j}/j!$.

Proof. This is proved by induction using

$$[\mathcal{D}_\mu, a_{n+1}]^{(k_{n+1})} = [\mathcal{D}_\mu, a_{n+1}^{(k_{n+1})}] = \begin{pmatrix} da_{n+1}^{(k_{n+1})} & -\mu\, a_{n+1}^{(k_{n+1})} \\ \mu\, a_{n+1}^{(k_{n+1})} & 0 \end{pmatrix}.$$

It is important to note that the formulae for the $\hat{S}$ operation are unaffected by the commutators with $\mathcal{D}_\mu^2$, since $\mathcal{D}_\mu^2$ is diagonal. A similar calculation in [**14**, Appendix], where there is a sign error corrected here, indicates how the proof proceeds. $\square$

Multiplying the operator in Lemma 3.31 by $\hat{a}_0 = \begin{pmatrix} a_0 & 0 \\ 0 & 0 \end{pmatrix}$ gives us $a_0 \omega_{m,\mu,k}$. Having identified the μ dependence of $\omega_{m,\mu,k}(1 + \mathcal{D}_\mu^2)^{-|k|-m/2-r-(p-1)/2}$ arising from the coefficient operators $\omega_{m,\mu,k}$, we now identify the remaining μ dependence in $a_0\omega_{m,\mu,k}(1 + \mathcal{D}_\mu^2)^{-|k|-m/2-r-(p-1)/2}$ coming from $(1 + \mathcal{D}_\mu^2)^{-|k|-m/2-r-(p-1)/2}$. So replacing $\mathcal{D}_\mu$ by $(1-\mu^2)^{-1/2}\mathcal{D}_\mu$, our calculations give for $m > 0$

$$a_0\omega_{m,\mu,k}(1 + \mathcal{D}_\mu^2)^{-|k|-m/2-r-(p-1)/2} \longmapsto$$
$$(1-\mu^2)^{-r-(p-1)/2} a_0 \omega_{m,k}(1 + \mathcal{D}^2)^{-|k|-m/2-r-(p-1)/2} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + O(\mu),$$

where the $O(\mu)$ terms are those arising from Lemma 3.31. Of course at $r = (1-p)/2$ the numerical factor $(1-\mu^2)^{-r-(p-1)/2}$ is equal to one, and contributes nothing when we take residues. For $m = 0$ there are no additional $O(\mu)$ terms. Ignoring the factor of $(1-\mu^2)^{-r-(p-1)/2}$, we collect all terms in the cochain $\{\phi^r_{\mu,m}\}_{m=\bullet,\bullet+2,\dots,M}$ with the same power of μ arising from $a_0\omega_{m,k,\mu}$. This gives us a finite family of (b, B)-cochains of different lengths but the same parity, one for each power of μ in the expansion of $a_0\omega_{m,k,\mu}$. Denote these new cochains by $\psi_i^r = (\psi^r_{i,m})_{m=\bullet,\bullet+2,\dots}$, where ψ_i^r is assembled as the coefficient cochain for μ^i, $i \in \mathbb{N}_0$. To simplify the notation, we will consider the cochains ψ_i^r as functionals on suitable elements in OP^*. With these conventions, and modulo functions holomorphic at $r = (1-p)/2$, we have

$$\phi^r_{\mu,m}(a_0, \dots, a_m) = (1-\mu^2)^{-r+(1-p)/2} \Big(\sum_{i=0}^{2\lfloor \frac{m}{2} \rfloor + 1} \psi^r_{i,m}(a_0\omega_{m,k,i})\, \mu^i \Big),$$

where $\omega_{m,k,i}$ are some coefficient operators depending on $a_1, \ldots, a_m$, but not on μ, and $\omega_{m,k,0} = \omega_{m,k}$, as defined in Lemma 3.31.

Let $\alpha = (\alpha_m)_{m=\bullet,\bullet+2,\ldots}$ be a (b,B)-boundary in the reduced complex for $\mathcal{A}^\sim$. Then as $\mathrm{Ch}^M_{F_\mu}$ is a (b,B)-cocycle, we find by performing the pseudodifferential expansion that there are reduced (b,B)-cochains $C_0, \ldots, C_{2\lfloor M/2\rfloor+\bullet}$ such that

$$\begin{aligned} 0 = \mathrm{Ch}^M_{F_\mu}(\alpha_M) &= \mathrm{res}_{r=(1-p)/2} \sum_{m=\bullet}^{M} \phi^r_{\mu,m}(\alpha_m) \\ &= C_0(\alpha) + C_1(\alpha)\,\mu + \cdots + C_{2\lfloor M/2\rfloor+\bullet}(\alpha)\,\mu^{2\lfloor M/2\rfloor+\bullet}. \end{aligned}$$

The class of $\mathrm{Ch}^M_{F_\mu}$ is independent of $\mu > 0$, and as we can vary $\mu \in (0,1)$, we see that each of the coefficients $C_i(\alpha) = 0$. As the $C_i(\alpha)$ arise as the result of pairing a (b,B)-cochain with the (b,B)-boundary α, and α is an arbitrary boundary, we see that all the ψ^r_i are (reduced) cocycles modulo functions holomorphic at $r = (1-p)/2$.

Now let β be a (b,B)-cycle. Then by performing the pseudodifferential expansion we find that

$$\begin{aligned} \mathrm{Ch}^M_{F_\mu}(\beta_M) &= \mathrm{res}_{r=(1-p)/2} \sum_{m=\bullet}^{M} \phi^r_{\mu,m}(\beta_m) \\ &= C_0(\beta) + C_1(\beta)\,\mu + \cdots + C_{2\lfloor M/2\rfloor+\bullet}(\beta)\,\mu^{2\lfloor M/2\rfloor+\bullet}. \end{aligned}$$

The left hand side is independent of μ, and so taking the derivative with respect to μ yields

$$0 = C_1(\beta) + \cdots + (2\lfloor M/2\rfloor + \bullet) C_{2\lfloor M/2\rfloor+\bullet}(\beta)\,\mu^{2\lfloor M/2\rfloor+\bullet-1}.$$

Again, by varying μ we see that each coefficient $C_i(\beta)$, $i > 0$, must vanish. As β is an arbitrary (b,B)-cycle, for $i \neq 0$, ψ^r_i is a coboundary modulo functions holomorphic at $r = (1-p)/2$. The conclusion is that $\mathrm{res}\,\psi^r_0$ represents the Chern character. We now turn to making this representative explicit.

The cocycle ψ^r_0 is given, in terms of the original spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$, in all degrees except zero, by $\{\phi^r_m\}_{m=\bullet,\bullet+2,\ldots,M}$, that is the formula for the resolvent cocycle presented in Definition 3.5 with $\mathcal{D}$ in place of $\mathcal{D}_\mu$. In degree zero we need some care, and after a computation we find that for $b \in \mathcal{A}^\sim$ and $\mu \in (0,1)$, $\phi^r_{\mu,0}(b)$ is given by

$$\begin{aligned} \phi^r_{\mu,0}(b) = \lim_{\lambda\to\infty} & \frac{\Gamma(r-(1-p)/2)\sqrt{\pi}(1-\mu^2)^{-(r-(1-p)/2)}}{\Gamma(p/2+r)} \tau \otimes \mathrm{tr}_2 \\ & \begin{pmatrix} \gamma(b-\mathbf{1}_b)(1+\mathcal{D}^2)^{-z} + \gamma\tilde{\psi}_\lambda \mathbf{1}_b(1+\mathcal{D}^2)^{-(r-(1-p)/2)} & 0 \\ 0 & -\gamma\tilde{\psi}_\lambda\mathbf{1}_b(1+\mathcal{D}^2)^{-(r-(1-p)/2)} \end{pmatrix}, \end{aligned}$$

where $\mathbf{1}_b$ is defined after Equation (2.2). Canceling the $\mathbf{1}_b$ terms and taking the limit shows that $\phi^r_{\mu,0}(b)$ is given by

$$\frac{\Gamma(r-(1-p)/2)\sqrt{\pi}(1-\mu^2)^{-(r-(1-p)/2)}}{\Gamma(p/2+r)} \tau\Big(\gamma(b-\mathbf{1}_b)(1+\mathcal{D}^2)^{-(r-(1-p)/2)}\Big).$$

The function of r outside the trace has a simple pole at $r = (1-p)/2$ with residue equal to 1, and can be replaced by any other such function, like $(r-(1-p)/2)^{-1}$. Thus modulo functions holomorphic at the critical point, we have

$$\phi^r_{\mu,0}(b) = \phi^r_0(b - \mathbf{1}_b).$$

Thus, we have proved the following proposition.

PROPOSITION 3.32. *Let $(\mathcal{A},\mathcal{H},\mathcal{D})$ be a smoothly summable spectral triple of spectral dimension $p \geq 1$ and of parity $\bullet \in \{0,1\}$. Let also $a_0 \otimes a_1 \otimes \cdots \otimes a_m \in \mathcal{A}^\sim \otimes \mathcal{A}^{\otimes m}$. Let $\{\phi^r_{\mu,m}\}_{m=\bullet,\bullet+2,\dots,M}$ and $\{\phi^r_m\}_{m=\bullet,\bullet+2,\dots,M}$ be the resolvent cocycles defined respectively by the double and the original spectral triple. Then $\{\phi^r_m - \phi^r_{\mu,m}\}_{m=\bullet,\bullet+2,\dots,M}$ is a reduced (b,B)-coboundary modulo functions holomorphic at $r=(1-p)/2$.*

If, moreover, the spectral dimension of $(\mathcal{A},\mathcal{H},\mathcal{D})$ is isolated, we have

$$\mathrm{res}_{r=(1-p)/2}\,\phi^r_{\mu,m}(a_0,\dots,a_m) = \mathrm{res}_{r=(1-p)/2}\,\phi^r_m(a_0,\dots,a_m), \quad m>0,$$
$$\mathrm{res}_{r=(1-p)/2}\,\phi^r_{\mu,0}(a_0) = \mathrm{res}_{r=(1-p)/2}\,\phi^r_0(a_0 - \mathbf{1}_{a_0}).$$

3.9. The local index formula

Let $u \in M_n(\mathcal{A}^\sim)$ be a unitary and let $e \in M_n(\mathcal{A}^\sim)$ be a projection. Set $\mathbf{1}_e = \pi^n(e) \in M_n(\mathbb{C})$ as in Equation (2.3). We also observe that inflating a smoothly summable spectral triple $(\mathcal{A},\mathcal{H},\mathcal{D})$ to $(M_n(\mathcal{A}),\mathcal{H}\otimes\mathbb{C}^n,\mathcal{D}\otimes\mathrm{Id}_n)$ yields a smoothly summable spectral triple for $M_n(\mathcal{A})$, with the same spectral dimension. Then we can summarise the results of Chapters 2 and 3 as follows.

THEOREM 3.33. *Let $(\mathcal{A},\mathcal{H},\mathcal{D})$ be a semifinite spectral triple of parity $\bullet \in \{0,1\}$, which is smoothly summable with spectral dimension $p \geq 1$ and with $\mathcal{A}$ separable. Let also $M = 2\lfloor (p+\bullet+1)/2 \rfloor - \bullet$ be the largest integer of parity $\bullet$ less than or equal to $p+1$. Let $\mathcal{D}_{\mu,n}$ denote the operator coming from the double of the inflation $(M_n(\mathcal{A}),\mathcal{H}\otimes\mathbb{C}^n,\mathcal{D}\otimes\mathrm{Id}_n)$ of $(\mathcal{A},\mathcal{H},\mathcal{D})$, with phase $F_\mu\otimes\mathrm{Id}_n$ and $\mathcal{D}_n$ be the operator coming from the inflation of $(\mathcal{A},\mathcal{H},\mathcal{D})$. Then with the notations introduced above:*
(1) The Chern character in cyclic homology computes the numerical index pairing:

$$\langle [u], [(\mathcal{A},\mathcal{H},\mathcal{D})]\rangle = \frac{-1}{\sqrt{2\pi i}}\mathrm{Ch}^M_{F_\mu\otimes\mathrm{Id}_n}\big(\mathrm{Ch}^M(\hat{u})\big), \quad \textit{(odd case)},$$
$$\langle [e]-[\mathbf{1}_e], [(\mathcal{A},\mathcal{H},\mathcal{D})]\rangle = \mathrm{Ch}^M_{F_\mu\otimes\mathrm{Id}_n}\big(\mathrm{Ch}^M(\hat{e})\big), \quad \textit{(even case)}.$$

(2) The numerical index pairing can also be computed with the resolvent cocycle of $\mathcal{D}_n$ via

$$\langle [u], [(\mathcal{A},\mathcal{H},\mathcal{D})]\rangle = \frac{-1}{\sqrt{2\pi i}}\mathrm{res}_{r=(1-p)/2}\sum_{m=1,\,odd}^{M}\phi^r_m\big(\mathrm{Ch}^m(u)\big), \qquad \textit{(odd case)},$$

$$\langle [e]-[\mathbf{1}_e], [(\mathcal{A},\mathcal{H},\mathcal{D})]\rangle = \mathrm{res}_{r=(1-p)/2}\sum_{m=0,\,even}^{M}\phi^r_m\big(\mathrm{Ch}^m(e)-\mathrm{Ch}^m(\mathbf{1}_e)\big), \ \textit{(even case)},$$

and, in particular, for $x=u$ or $x=e$, depending on the parity, $\sum_{m=\bullet}^M \phi^r_m(\mathrm{Ch}_m(x))$ analytically continues to a deleted neighborhood of the critical point $r=(1-p)/2$ with at worst a simple pole at that point.

(3) If, moreover, the triple $(\mathcal{A},\mathcal{H},\mathcal{D})$ has isolated spectral dimension, then the numerical index can also be computed with the residue cocycle for $\mathcal{D}_n$, via

$$\langle [u],[(\mathcal{A},\mathcal{H},\mathcal{D})]\rangle = \frac{-1}{\sqrt{2\pi i}}\sum_{m=1,\,odd}^{M}\phi_m\big(\mathrm{Ch}^m(u)\big),\quad \textit{(odd case)},$$

$$\langle [e]-[\mathbf{1}_e],[(\mathcal{A},\mathcal{H},\mathcal{D})]\rangle = \sum_{m=0,\,even}^{M}\phi_m\big(\mathrm{Ch}^m(e)-\mathrm{Ch}^m(\mathbf{1}_e)\big),\quad \textit{(even case)}.$$

3.10. A nonunital McKean-Singer formula

To illustrate this theorem, we prove a nonunital version of the McKean-Singer formula. To the best knowledge of the authors, there is no other version of McKean-Singer formula which is valid without the assumption that $f(\mathcal{D}^2)$ is trace class for some function f. Our assumptions are quite different from the usual McKean-Singer formula.

Let $(\mathcal{A},\mathcal{H},\mathcal{D})$ be an even semifinite smoothly summable spectral triple relative to $(\mathcal{N},\tau)$ with spectral dimension $p\geq 1$. Also, let $e\in M_n(\mathcal{A}^\sim)$ be a projection with $\pi^n(e)=\mathbf{1}_e\in M_n(\mathbb{C})\subset M_n(\mathcal{N})$. Then using the well known homotopy (with $\mathcal{D}_n=\mathcal{D}\otimes \mathrm{Id}_n$)

$$\begin{aligned}\mathcal{D}_n &= e\mathcal{D}_n e+(1-e)\mathcal{D}_n(1-e)+t\big(e\mathcal{D}_n(1-e)+(1-e)\mathcal{D}_n e\big)\\ &= e\mathcal{D}_n e+(1-e)\mathcal{D}_n(1-e)+t\big((1-e)[\mathcal{D}_n,e]-e[\mathcal{D}_n,e]\big)\\ &=: \mathcal{D}_e-t(2e-1)[\mathcal{D}_n,e],\end{aligned}\tag{3.25}$$

we see that we have an equality of the KK-classes associated to the spectral triples

$$[(M_n(\mathcal{A}),\mathcal{H}\otimes\mathbb{C}^n,\mathcal{D}_n)]=[(M_n(\mathcal{A}),\mathcal{H}\otimes\mathbb{C}^n,\mathcal{D}_e)]\in KK^0(\mathcal{A},C),$$

where C is the (separable) C^*-algebra generated by the τ-compact operators listed in Definition 2.5. However, the property of smooth summability may not be preserved by this homotopy. The next lemma shows that the summability part is preserved.

LEMMA 3.34. *Let $(\mathcal{A},\mathcal{H},\mathcal{D})$ be a smoothly summable semifinite spectral triple relative to $(\mathcal{N},\tau)$ with spectral dimension $p\geq 1$. Let $A\in\mathrm{OP}^0_0$ be a self-adjoint element. Then*

$$\mathcal{B}_2(\mathcal{D}+A,p)=\mathcal{B}_2(\mathcal{D},p)\quad\text{and}\quad \mathcal{B}_1(\mathcal{D}+A,p)=\mathcal{B}_1(\mathcal{D},p).$$

PROOF. For $K\in\mathbb{N}$ arbitrary, Cauchy's formula and the resolvent expansion gives

$$\begin{aligned}&(1+(\mathcal{D}+A)^2)^{-s/2}-(1+\mathcal{D}^2)^{-s/2}=\\ &\quad\sum_{m=1}^{K}\frac{1}{2\pi i}\int_\ell \lambda^{-s/2}\left(R(\lambda)(\{\mathcal{D},A\}+A^2)\right)^m R(\lambda)d\lambda\\ &\quad+\frac{1}{2\pi i}\int_\ell \lambda^{-s/2}\left(R(\lambda)(\{\mathcal{D},A\}+A^2)\right)^{K+1}R_A(\lambda)d\lambda,\end{aligned}$$

where $R(\lambda)=(\lambda-(1+\mathcal{D}^2))^{-1}$, $R_A(\lambda)=(\lambda-(1+(\mathcal{D}+A)^2))^{-1}$ and $\{\cdot,\cdot\}$ denotes the anticommutator. Now since $\{\mathcal{D},A\}+A^2$ is in OP^1_0, Lemma 3.3 can be applied to all terms except the last, to see that each is trace-class for $s>p-m$. Using

Lemma 1.39, the Hölder inequality and estimating $R_A(\lambda)$ in norm, we see that the integrand of the remainder term has trace norm

$$\|\big(R(\lambda)(\{\mathcal{D},A\}+A^2)\big)^{K+1}R_A(\lambda)\|_1 \leq C_\varepsilon(a^2+v^2)^{-(K+1)/4+(K+1)p/4q+(K+1)\varepsilon-1/2},$$

where $q>p$ and $\varepsilon>0$. Choosing $q=p+\delta$ for some $\delta>0$, we may choose K large enough so that the integral over $v=\Re(\lambda)$ converges absolutely whenever $s>p-1$. Hence, we can suppose that the remainder term is trace-class for $s>p-1$. Now let $T\in\mathcal{B}_2(\mathcal{D},p)$ and use the tracial property to see that

$$\begin{aligned}\tau((1+(\mathcal{D}+A)^2)^{-s/4}&T^*T(1+(\mathcal{D}+A)^2)^{-s/4})\\ &=\tau(|T|(1+(\mathcal{D}+A)^2)^{-s/2}|T|)\\ &=\tau(|T|(1+\mathcal{D}^2)^{-s/2}|T|)+C_s\\ &=\tau((1+\mathcal{D}^2)^{-s/4}T^*T(1+\mathcal{D}^2)^{-s/4})+C_s,\end{aligned}$$

where $C_s=\tau(|T|\left((1+(\mathcal{D}+A)^2)^{-s/2}-(1+\mathcal{D}^2)^{-s/2}\right)|T|)$ is finite for $s>p-1$ by the previous considerations. By repeating the argument for T^* we have $T\in\mathcal{B}_2(\mathcal{D}+A,p)$. As $\mathcal{D}=(\mathcal{D}+A)-A$, the argument is symmetric, and we see that $\mathcal{B}_2(\mathcal{D},p)=\mathcal{B}_2(\mathcal{D}+A,p)$. This entails by construction that $\mathcal{B}_1(\mathcal{D},p)=\mathcal{B}_1(\mathcal{D}+A,p)$. □

Unfortunately, there is no reason to suppose that the smoothness properties of the spectral triple $(M_n(\mathcal{A}),\mathcal{H}^n,\mathcal{D}_n)$ are preserved by the homotopy from $\mathcal{D}_n$ to $\mathcal{D}_e$. Instead, consider $(\mathcal{A}_e,\mathcal{H}^n,\mathcal{D}_e)$, where $\mathcal{A}_e$ is the algebra of polynomials in $e-\mathbf{1}_e\in M_n(\mathcal{A})$. Then by Lemma 3.34 and $[\mathcal{D}_e,e-\mathbf{1}_e]=[\mathcal{D}_e,e]=0$ (which implies since $\mathcal{D}_e$ is self-adjoint that $[|\mathcal{D}_e|,e-\mathbf{1}_e]=[|\mathcal{D}_e|,e]=0$ too) and we easily check that $(\mathcal{A}_e,\mathcal{H}^n,\mathcal{D}_e)$ is a smoothly summable spectral triple. Now employing the resolvent cocycle of $(\mathcal{A}_e,\mathcal{H}^n,\mathcal{D}_e)$ yields

$$\begin{aligned}\mathrm{Index}_{\tau\otimes\mathrm{tr}_{2n}}\big(\hat{e}(F_{\mu,+}\otimes\mathrm{Id}_n)\hat{e}\big)&=\mathrm{res}_{r=(1-p)/2}\Big(\sum_{m=2,even}^{M}\phi^r_{\mu,m}\big(\mathrm{Ch}_m(\hat{e})\big)\\ &+\ \frac{1}{(r-(1-p)/2)}\tau\otimes\mathrm{tr}_n\big(\gamma(e-\mathbf{1}_e)(1+\mathcal{D}_e^2)^{-(r-(1-p)/2)}\big)\Big).\end{aligned}$$

This equality follows from Proposition 3.32 and the explicit computation of the zero degree term. Now since $[\mathcal{D}_e,e]=0$, $\phi^r_m(\mathrm{Ch}_m(e))=0$ for all $m\geq 2$. This proves the following nonunital McKean-Singer formula.

THEOREM 3.35. *Let $(\mathcal{A},\mathcal{H},\mathcal{D})$ be an even semifinite smoothly summable spectral triple relative to $(\mathcal{N},\tau)$ with spectral dimension $p\geq 1$ and with $\mathcal{A}$ separable. Also, let $e\in M_n(\mathcal{A}^\sim)$ be a projection. Then*

$$\begin{aligned}\langle[e]-[\mathbf{1}_e],&[(\mathcal{A},\mathcal{H},\mathcal{D})]\rangle=\langle[e]-[\mathbf{1}_e],[(\mathcal{A}_e,\mathcal{H},\mathcal{D})]\rangle\\ &=\mathrm{res}_{r=(1-p)/2}\frac{1}{(r-(1-p)/2)}\tau\otimes\mathrm{tr}_n\big(\gamma(e-\mathbf{1}_e)(1+\mathcal{D}_e^2)^{-(r-(1-p)/2)}\big).\end{aligned}$$

This gives a nonunital analogue of the McKean-Singer formula. Observe that the formula has $\mathcal{D}_e$ *not* $\mathcal{D}_n$.

Remark. We have also proved a nonunital version of the Carey-Phillips spectral flow formula for paths $(\mathcal{D}_t)_{t\in[0,1]}$ with unitarily equivalent endpoints and with $\dot{\mathcal{D}}_t$ satisfying suitable summability constraints. The proof is quite lengthy, and so we will present this elsewhere.

3.11. A classical example with weaker integrability properties

Perhaps surprisingly, given the difficulty of the nonunital case, we gain a little more freedom in choosing representatives of K-theory classes than we might have expected. We do not formulate a general statement, but instead illustrate with an example. This example involves a projection which does not live in a matrix algebra over (the unitisation of) our 'integrable algebra' $\mathcal{B}_1(\mathcal{D}, p)$, but we may still use the local index formula to compute index pairings.

We will employ the uniform Sobolev algebra $W^{\infty,1}(\mathbb{R}^2)$, i.e. the Fréchet completion of $C_c^\infty(\mathbb{R}^2)$ for the seminorms $\mathfrak{q}_n(f) := \max_{n_1+n_2\leq n} \|\partial_1^{n_1}\partial_2^{n_2} f\|_1$. By the Sobolev Lemma, $W^{\infty,1}(\mathbb{R}^2)$ is continuously embedded in $L^\infty(\mathbb{R}^2)$, and is separable for the uniform topology as it contains $C_c^\infty(\mathbb{R}^2)$ as a dense subalgebra, and $C_c^\infty(\mathbb{R}^2)$ is separable for the uniform norm topology. The spin Dirac operator on $\mathbb{R}^2 \simeq \mathbb{C}$ is $\not\partial := \begin{pmatrix} 0 & \partial_1 + i\partial_2 \\ -\partial_1 + i\partial_2 & 0 \end{pmatrix}$, with grading $\gamma := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Identifying a function with the operator of pointwise multiplication by it, an element $f \in W^{\infty,1}(\mathbb{R}^2)$ is represented as $f \otimes \mathrm{Id}_2$ on $L^2(\mathbb{R}^2, \mathbb{C}^2)$. Anticipating the results of the next Chapter, we know by Proposition 4.9 that the triple $\big(W^{\infty,1}(\mathbb{R}^2), L^2(\mathbb{R}^2,\mathbb{C}^2), \not\partial\big)$ is smoothly summable, relative to the pair $\big(\mathcal{B}(L^2(\mathbb{R}^2,\mathbb{C}^2)), \mathrm{Tr}\big)$ whose spectral dimension is 2 and is isolated. Thus, we can employ the residue cocycle to compute indices.

Let $p_B \in M_2(C_0(\mathbb{C})^\sim)$ be the Bott projector

$$p_B(z) := \frac{1}{1+|z|^2}\begin{pmatrix} 1 & \bar z \\ z & |z|^2 \end{pmatrix}, \quad \mathbf{1}_{p_B} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{3.26}$$

It is important to observe that $p_B - \mathbf{1}_{p_B}$ is *not* in $\mathcal{B}_1(\not\partial, 2)$ since the off-diagonal terms are not even L^2-functions.

Since the fibre trace of $p_B - \mathbf{1}_{p_B}$ is identically zero, the zero degree term of the local index formula does not contribute to the index pairing. This observation holds in general for commutative algebras since elements of K_0 then correspond to virtual bundles of virtual rank zero. Thus, there is only one term to consider in the local index formula, in degree 2. More generally, for even dimensional manifolds we will only ever need to consider the terms in the local index formula with $m \geq 2$. This means that all we really require is that $[\not\partial \otimes \mathrm{Id}_2, p_B][\not\partial \otimes \mathrm{Id}_2, p_B]$ lies in $M_2(W^{\infty,1}(\mathbb{R}^2))$, and this is straightforward to check. Indeed, the routine computation

$$(p_B - 1/2)[\not\partial\otimes\mathrm{Id}_2, p_B][\not\partial\otimes\mathrm{Id}_2, p_B] = \frac{-4}{(1+|z|^2)^3}\begin{pmatrix} 1/2 & \bar z/2 & 0 & 0 \\ z/2 & |z|^2/2 & 0 & 0 \\ 0 & 0 & -|z|^2/2 & \bar z/2 \\ 0 & 0 & z/2 & -1/2 \end{pmatrix},$$

shows that $(p_B - 1/2)[\not\partial \otimes \mathrm{Id}_2, p_B][\not\partial \otimes \mathrm{Id}_2, p_B]$ is a matrix over $W^{\infty,1}(\mathbb{R}^2)$. The fibrewise trace gives

$$\mathrm{tr}_2\big((p_B - 1/2)[\not\partial\otimes\mathrm{Id}_2, p_B][\not\partial\otimes\mathrm{Id}_2, p_B]\big) = \frac{-2}{(1+|z|^2)^2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Applying [**50**, Corollary 14], we find (the prefactor of $1/2$ comes from the coefficients in the local index formula)

$$\frac{1}{2}\mathrm{Tr}\otimes\mathrm{tr}_2\big(\gamma(p_B-1/2)[\not\partial\otimes\mathrm{Id}_2,p_B][\not\partial\otimes\mathrm{Id}_2,p_B](1+\mathcal{D}^2)^{-1-\xi}\big)$$
$$=-\frac{\Gamma(\xi)}{\Gamma(1+\xi)}\int_0^\infty\frac{r}{(1+r^2)^2}dr=-\frac{1}{2\xi}.$$

Recalling that the second component of the Chern character of p_B introduces a factor of -2, we arrive at the numerical index

$$\langle[p_B]-[\mathbf{1}_{p_B}],\big[\big(W^{\infty,1}(\mathbb{R}^2),L^2(\mathbb{R}^2,\mathbb{C}^2),\not\partial\big)\big]\rangle=1,$$

as expected. This indicates that the resolvent cocycle extends by continuity to a larger complex, defined using norms of iterated projective tensor product type associated to the norms $\mathcal{P}_n$. We leave a more thorough discussion of this to another place.

CHAPTER 4

Applications to Index Theorems on Open Manifolds

This Chapter contains a discussion of some of what the noncommutative residue formula implies for the classical situation of a noncompact manifold. The main contribution of the noncommutative approach that we have endeavoured to explain here, is the extent to which compact support assumptions such as those in [**29**] may be avoided. However, we do not exhaust all of the applications of the residue formula in the classical case in this memoir.

Our aim is to write an account of our results in a relatively complete fashion. We recall the basic definitions of spin geometry, [**39**], and heat kernel estimates for manifolds of bounded geometry. Using this data we construct a smoothly summable spectral triple for manifolds of bounded geometry. Having done this, we use results of Ponge and Greiner to obtain an Atiyah-Singer formula for the index pairing on manifolds of bounded geometry. Then we utilise the semifinite framework to obtain an L^2-index theorem for covers of manifolds of bounded geometry.

4.1. A spectral triple for manifolds of bounded geometry

4.1.1. Dirac-type operators and Dirac bundles. Let (M, g) be a (finite dimensional, paracompact, second countable) geodesically complete Riemannian manifold. We let $n \in \mathbb{N}$ be the dimension of M and μ_g be the Riemannian volume form. Unless otherwise specified, the measure involved in the definition of the Lebesgue function spaces $L^q(M)$, $1 \le q \le \infty$, is the one associated with μ_g.

We let $\mathcal{D}_S$ be a *Dirac-type operator* in the sense of [**29**, **39**]. Such operators are of the following form. Let $S \to M$, be a vector bundle, complex for simplicity, of rank $m \in \mathbb{N}$ and $(\cdot|\cdot)$, a fiber-wise Hermitian form. We suppose that S is a bundle of left modules over the Clifford bundle algebra $\mathrm{Cliff}(M) := \mathrm{Cliff}(T^*M, g)$ which is such that for each unit vector e_x of T_x^*M, the Clifford module multiplication $c(e_x) : S_x \to S_x$ is a (smoothly varying) isometry. It is further equipped with a metric compatible connection ∇^S, such that for any smooth sections $\sigma \in \Gamma^\infty(S)$ and $\varphi \in \Gamma^\infty(\mathrm{Cliff}(M))$, it satisfies

$$\nabla^S(c(\varphi)\sigma) = c(\nabla\varphi)\sigma + c(\varphi)\nabla^S(\sigma). \tag{4.1}$$

Here, ∇ is the Levi-Civita connection naturally extended to a (metric compatible) connection on $\mathrm{Cliff}(M)$ which satisfies, for φ, $\psi \in \mathrm{Cliff}(M)$, $\nabla(\varphi \cdot \psi) = \nabla(\varphi) \cdot \psi + \varphi \cdot \nabla(\psi)$ (the dot here is the Clifford multiplication). We call such a bundle a *Dirac bundle*, [**39**, Definition 5.2]. Then, $\mathcal{D}_S$ is defined as the composition

$$\Gamma^\infty(S) \to \Gamma^\infty(T^*M \otimes S) \to \Gamma^\infty(S),$$

where the first arrow is given by ∇^S and the second by the Clifford multiplication.

For any orthonormal basis $\{e^\mu\}_{\mu=1,\dots,n}$ of T_x^*M, at each point $x \in M$ and $\{e_\mu\}_{\mu=1,\dots,n}$ the dual basis of T_xM, with Einstein summation convention understood, we therefore have

$$\mathcal{D}_S = c(e^\mu)\,\nabla^S_{e_\mu}.$$

Let $\langle\sigma_1,\sigma_2\rangle_S = \int_M(\sigma_1|\sigma_2)(x)\,\mu_g(x)$ be the L^2-inner product on $\Gamma_c^\infty(S)$, with $(\cdot|\cdot)$ the Hermitian form on S. As usual $L^2(M,S)$ is the associated Hilbert space completion of $\Gamma_c^\infty(S)$. Recall that under the assumption of geodesic completeness, $\mathcal{D}_S$ is essentially self-adjoint and $\Gamma_c^\infty(S)$ is a core for $\mathcal{D}_S$, [**33**, Corollary 10.2.6] and [**29**, Theorem 1.17]. Moreover, if the Dirac bundle $S \to M$ is a $\mathbb{Z}_2$-graded $\mathrm{Cliff}(M)$-module, then $\mathcal{D}_S$ is odd, and in the usual matrix decomposition, it reads

$$\mathcal{D}_S = \begin{pmatrix} 0 & \mathcal{D}_S^+ \\ \mathcal{D}_S^- & 0 \end{pmatrix}, \quad \text{with} \quad (\mathcal{D}_S^\pm)^* = \mathcal{D}_S^\mp.$$

We identify $L^\infty(M)$ with a subalgebra of the bounded Borel sections of $\mathrm{Cliff}(M)$ in the usual way. We thus have a left action $L^\infty(M)\times L^2(M,S)\to L^2(M,S)$ given by $(f,\sigma)\mapsto c(f)\sigma$. In a local trivialization of S, this action is given by the diagonal point-wise multiplication. It, moreover, satisfies $\|c(f)\| = \|f\|_\infty$.

We recall now the important Bochner-Weitzenböck-Lichnerowicz formula for the square of a Dirac-type operator:

$$\mathcal{D}_S^2 = \Delta_S + \tfrac{1}{2}\mathcal{R}, \qquad \mathcal{R} := c(e^\mu)\,c(e^\nu)\;F(e_\mu,e_\nu), \tag{4.2}$$

where $\Delta_S := (\nabla^S)^*\,\nabla^S$ is the Laplacian on S and $F : \Lambda^2T^*M \to \mathrm{End}(S)$ is the curvature tensor of ∇^S.

Remark. Using the formula (4.2), Gromov and Lawson [**29**, Theorem 3.2] have proven that if there exists a compact set $K \subset M$ such that

$$\inf_{x\in M\setminus K}\sup\{\kappa\in\mathbb{R} : \mathcal{R}(x)\geq \kappa\,\mathrm{Id}_{S_x}\} > 0,$$

then $\mathcal{D}_S$ (and thus $\mathcal{D}_S^\pm$ in the graded case) is Fredholm in the ordinary sense.

Note that the Leibniz-type relation (4.1) shows that for any $f \in C_c^\infty(M)$, the commutator $[\mathcal{D}_S, c(f)]$ extends to a bounded operator since an explicit computation gives

$$[\mathcal{D}_S, c(f)] = c(df). \tag{4.3}$$

4.1.2. The case of a manifold with bounded geometry. Recall that the injectivity radius $r_{\mathrm{inj}} \in [0,\infty)$, is defined as

$$r_{\mathrm{inj}} := \inf_{x\in M}\sup\{r_x > 0\},$$

where $r_x \in (0,\infty)$ is such that the exponential map $\exp_x$ is a diffeomorphism from $B(0,r_x) \subset T_xM$ to $U_{r,x}$, an open neighborhood of $x \in M$. We call *canonical coordinates* the coordinates given by $\exp_x^{-1} : U_{r,x} \to B(0,r_x) \subset T_xM \simeq \mathbb{R}^n$. Note that $r_{\mathrm{inj}} > 0$ implies that (M,g) is geodesically complete.

With these preliminaries, we recall the definition of bounded geometry.

Definition 4.1. A Riemannian manifold (M,g) is said to have bounded geometry if it has strictly positive injectivity radius and all the covariant derivatives of the curvature tensor are bounded on M. A Dirac bundle on M is said to have bounded geometry if, in addition, all the covariant derivatives of F, the curvature tensor of the connection ∇^S, are bounded on M. For brevity, we simply say that (M,g,S) has bounded geometry.

We summarise some facts about manifolds of bounded geometry. Bounded geometry allows the construction of canonical coordinates which are such that the transition functions have bounded derivatives of all orders, uniformly on M, [**51**, Proposition 2.10]. Moreover, for all $\varepsilon \in (0, r_{\rm inj}/3)$, there exist countably many points $x_i \in M$, such that $M = \cup B(x_i, \varepsilon)$ and such that the covering of M by the balls $B(x_i, 2\varepsilon)$ has finite order. (Recall that the order of a covering of a topological space, is the least integer k, such that the intersection of any $k+1$ open sets of this covering is empty.) Subordinate to the covering by the balls $B(x_i, 2\varepsilon)$, there exists a partition of unity, $\sum_i \varphi_i = 1$, with $\operatorname{supp} \varphi_i \in B(x_i, 2\varepsilon)$ and such that their derivatives of all orders and in normal coordinates are bounded, uniformly in the covering index i. See [**55**, Lemmas 1.2, 1.3, Appendix 1] for details and proofs of all these assertions. Also, a differential operator is said to have *uniform C^∞-bounded coefficients*, if for any atlas consisting of charts of normal coordinates, the derivatives of all order of the coefficients are bounded on the chart domain and the bounds are uniform on the atlas.

The next proposition follows from results of Kordyukov [**37**] and Greiner [**31**], and records everything that we need to know about the heat semi-group with generator $\mathcal{D}_S^2$.

Proposition 4.2. *Let (M, g) be a Riemannian manifold of dimension n with bounded geometry. Let $\mathcal{D}_S$ be a Dirac type operator acting on the sections of a Dirac bundle S of bounded geometry and P a differential operator on $\Gamma_c^\infty(S)$ of order $\alpha \in \mathbb{N}$, with uniform C^∞-bounded coefficients. Let then $K_{t,P}^S(x,y) \in \operatorname{Hom}(S_x, S_y)$ be the operator kernel of $P\, e^{-t\mathcal{D}_S^2}$. Then:*
(1) We have the global off-diagonal gaussian upper bound

$$\big|K_{t,P}^S(x,y)\big|_\infty \leq C\, t^{-(n+\alpha)/2} \exp\Big(-\frac{d_g^2(x,y)}{4(1+c)t}\Big), \quad t > 0,$$

where $|\cdot|_\infty$ denotes the operator norm on $\operatorname{Hom}(S_x, S_y)$ and d_g the geodesic distance function.
(2) We have the short-time asymptotic expansion

$$\operatorname{tr}\big(K_{t,P}^S(x,x)\big) \sim_{t\to 0^+} t^{-\lfloor \alpha/2\rfloor - n/2} \sum_{i\geq 0} t^i b_{P,i}(x), \quad \textit{for all } x \in M,$$

where the functions $b_{P,i}(x)$ are determined by a finite number of jets of the principal symbol of $P(\partial_t + \mathcal{D}_S^2)^{-1}$.
(3) Moreover, this local asymptotic expansion carries through to give a global one: For any $f \in L^1(M)$, we have

$$\int_M f(x) \operatorname{tr}\big(K_{t,P}^S(x,x)\big)\, d\mu_g(x) \sim_{t\to 0^+} t^{-\lfloor \alpha/2\rfloor - n/2} \sum_{i\geq 0} t^i \int_M f(x)\, b_{P,i}(x)\, d\mu_g(x).$$

Proof. When M is compact, the first two results can be found in [**31**, Chapter I]. When M is noncompact but has bounded geometry, Kordyukov has proven in [**37**, Section 5.2] that all the relevant gaussian bounds used in [**31**] to construct a fundamental solution, via the Levi method, of a parabolic equation associated with an elliptic differential operator, remains valid for any uniformly elliptic differential operator with C^∞-bounded coefficients, which is the case for $\mathcal{D}_S^2$. The only restriction for us is that Kordyukov treats the scalar case only. However, a careful inspection of his arguments shows that the same bounds still hold for a uniformly elliptic differential operator acting on the smooth sections of a vector bundle of bounded geometry, as far as the operators under consideration have C^∞-bounded coefficients. With these gaussian bounds at hand (for the approximating solution and for the remainder term), one can then repeat word for word the arguments of Greiner to conclude for (1) and (2). For (3) one uses Kordyukov's bounds extended to the vector bundle case, [**37**, Proposition 5.4], to see that for all $k \in \mathbb{N}_0$, one has

$$\Big|\mathrm{tr}\big(K^S_{t,P}(x,x)\big) - t^{-\lfloor \alpha/2 \rfloor - n/2} \sum_{i=0}^{k} t^i b_{P,i}(x)\Big| \leq C\, t^{-\lfloor \alpha/2 \rfloor - n/2 + k + 1},$$

for a constant $C > 0$, independent of $x \in M$. This is enough to conclude. $\square$

Given ω, a weight function (positive and nowhere vanishing) on M, we denote by $W^{k,l}(M,\omega)$, $1 \leq k \leq \infty$, $0 \leq l < \infty$, the weighted uniform Sobolev space. That is to say, the completion of $C^\infty_c(M)$ for the topology associated to the norm

$$\|f\|_{k,l,\omega} := \Big(\int_M |\Delta^{l/2} f|^k \, \omega \, d\mu_g \Big)^{1/k},$$

where, Δ denotes the scalar Laplacian on M. For $\omega = 1$ we simply denote this space by $W^{k,l}(M)$ and the associated norm by $\| \cdot \|_{k,l}$. We also write $W^{k,\infty}(M,\omega) := \bigcap_{l \geq 0} W^{k,l}(M,\omega)$ endowed with the projective limit topology.

When M has strictly positive injectivity radius (thus in particular for manifolds of bounded geometry), the standard Sobolev embedding

$$W^{k,l}(M) \subset L^\infty(M),$$

holds for any $1 \leq k \leq \infty$ and $l > n/k$, with n the dimension of M (see [**2**, Chapter 2]). In particular, if $\varepsilon > 0$ then $W^{k,n/k+\varepsilon}(M)$ is not only a Fréchet space but a Fréchet algebra. Moreover, $W^{k,l}(M) \subset C_0(M)$ for $1 \leq k \leq \infty$ and $0 \leq l \leq \infty$, so that it is separable for the uniform topology as M is metrisable. The next lemma gives equivalent norms for the weighted Sobolev spaces $W^{k,l}(M,\omega)$.

Lemma 4.3. *Let $\sum \varphi_i = 1$ be a partition of unity subordinate to a covering of M by balls of radius $\varepsilon \in (0, r_{\mathrm{inj}}/3)$. Then the norm $\| \cdot \|_{k,l,\omega}$ on $W^{k,l}(M,\omega)$, $1 \leq k \leq \infty$, $l \in \mathbb{N}_0$, is equivalent to*

$$f \mapsto \sum_{i=1}^{\infty} \|\varphi_i f\|_{k,l,\omega}.$$

Proof. This is the weighted version of the discussion which follows [**55**, Lemma 1.3, Appendix 1], which is a consequence of the fact that the normal derivatives of φ_i are bounded uniformly in the covering index and because this covering has finite order. $\square$

In the following lemma, we examine first the question of (ordinary) smoothness before turning to smooth summability.

Lemma 4.4. *Let (M, g, S) have bounded geometry. For T an operator on $L^2(M,S)$ preserving the domain of $\mathcal{D}_S$, define $\delta(T) = [|\mathcal{D}_S|, T]$. Then for any $f \in W^{\infty,\infty}(M)$, the operators $c(f)$ and $c(df)$ on $L^2(M,S)$ belong to $\bigcap_{j=0}^{\infty} \mathrm{dom}\, \delta^j$.*

Proof. By the discussion following Definition 1.20, it suffices to show that for $f \in W^{\infty,\infty}(M)$, $c(f)$ belongs to $\bigcap_{j=0}^{\infty} \operatorname{dom} R^j$, with $R(T) = [\mathcal{D}_S^2, T](1+\mathcal{D}_S^2)^{-1/2}$. Next observe that since $[c(f), \mathcal{R}] = 0$, with $\mathcal{R}$ the zero-th order operator appearing in (4.2), we have

$$\begin{aligned} R^k\big(c(f)\big) &= [\mathcal{D}_S^2, [\ldots, [\mathcal{D}_S^2, [\mathcal{D}_S^2, c(f)]] \ldots]](1+\mathcal{D}_S^2)^{-k/2} \\ &= [\Delta_S + \tfrac{1}{2}\mathcal{R}, [\ldots, [\Delta_S + \tfrac{1}{2}\mathcal{R}, [\Delta_S, c(f)]] \ldots]](1+\mathcal{D}_S^2)^{-k/2}, \end{aligned}$$

with k commutators. Define

$$B_k := [\Delta_S + \tfrac{1}{2}\mathcal{R}, [\ldots, [\Delta_S + \tfrac{1}{2}\mathcal{R}, [\Delta_S, c(f)]] \ldots]],$$

so that $R^k\big(c(f)\big) = B_k\,(1+\mathcal{D}_S^2)^{-k/2}$. Since the principal symbol of Δ_S is $|\xi|^2 \mathrm{Id}_{S_x}$, a local computation shows that B_k is a differential operator of order k. With the bounded geometry assumption, we see moreover that B_k has uniform C^∞-bounded coefficients. (This follows because the covariant derivatives of $\mathcal{R}$ will appear in the expression of the coefficients of B_k and since $\mathcal{R}(x) = c(e_x^\mu)\, c(e_x^\nu)\, F(e_{\mu,x}, e_{\nu,x}) \in \mathrm{End}(S_x)$.) In particular, B_k is a properly supported pseudodifferential operator with bounded symbol (in the sense of [**37**, Definition 2.1]) of order k. While the pseudodifferential operator $(1+\mathcal{D}_S^2)^{-k/2}$ is not properly supported, it can be written as the sum of a properly supported pseudodifferential operator of order $-k$ and an infinitely smoothing operator; see [**37**, Theorem 3.3] for more information. Hence, by [**37**, Proposition 2.7], $R^k\big(c(f)\big)$ is properly supported with bounded symbol of zeroth order. Then one concludes using [**37**, Proposition 2.9], where one needs [**55**, Theorem 3.6, Appendix] instead of [**37**, Lemma 2.2] used in that proof, to extend the result to the case of a vector bundle of bounded geometry. The proof for $c(df)$ is entirely similar. □

As before, we let K_t^S, $t > 0$, be the Schwartz kernel of the heat semigroup with generator $\mathcal{D}_S^2$. When it exists, we let k_s, $s > 0$, be the restriction to the diagonal of the fibre-wise trace of the distributional kernel of $(1+\mathcal{D}_S^2)^{-s/2}$. That is for $s > 0$ and $x \in M$, we set

$$k_s(x) = \operatorname{tr}\big([(1+\mathcal{D}_S^2)^{-s/2}]_{x,x}\big),$$

where the trace tr is the matrix trace on $\mathrm{End}(S_x)$ and for A a bounded operator on $L^2(M,S)$ we denote by $[A]_{x,y}$ its distributional kernel.

Now assuming the geodesic completeness of M, the heat kernel K_t^S, $t > 0$, is a smooth section of the endomorphism bundle of S. Combining this with the Laplace transform representation

$$k_s(x) = \frac{1}{\Gamma(s/2)} \int_0^\infty t^{s/2-1}\, e^{-t} \operatorname{tr}\big(K_t^S(x,x)\big)\, dt, \quad \text{for all } x \in M,$$

we see that the question of existence of k_s is uniquely determined by the integrability of the on-diagonal fibre-wise trace of the Dirac heat kernel with respect to the parameter t. More precisely, Proposition 4.2 (1) gives

Lemma 4.5. *Let $\mathcal{D}_S$ be a Dirac type operator operating on the sections of a Dirac bundle S of bounded geometry. Then, for $s > n$, the function k_s is uniformly bounded on M.*

As a corollary of the lemma above, we see that $W^{r,t}(M) \subset W^{r,t}(M, k_s)$ with $\|\cdot\|_{r,t,k_s} \le C(s)\|\cdot\|_{r,t}$, for some constant $C(s)$ independent of $r \in [1,\infty]$ and of $t \in \mathbb{R}$.

LEMMA 4.6. *Let $\mathcal{D}_S$ be a Dirac type operator operating on the sections of a Dirac bundle S of bounded geometry. Then provided $f \in W^{2,0}(M, k_s)$ and $s > n$, the operator $c(f)(1+\mathcal{D}_S^2)^{-s/4}$ is Hilbert-Schmidt on $L^2(M,S)$, with*

$$\|c(f)(1+\mathcal{D}_S^2)^{-s/4}\|_2 = \Big(\int_M |f|^2(x)\, k_s(x)\, d\mu_g(x)\Big)^{1/2} = \|f\|_{2,0,k_s}.$$

PROOF. By Lemma 4.5, the function k_s is well defined and uniformly bounded on M. Now let A be a bounded operator acting on $L^2(M,S)$, with distributional kernel $[A]_{x,y}$. Then for $f \in L^\infty(M)$, a calculation shows that $A\,c(f)$ has distributional kernel $f(y)[A]_{x,y}$. We then have the following expression for the Hilbert-Schmidt norm of $A\,c(f)$:

$$\begin{aligned}
\|Ac(f)\|_2^2 &= \int_{M\times M} \mathrm{tr}\big(|[A\,c(f)]_{x,y}|^2\big)\, d\mu_g(x)\, d\mu_g(y)\\
&= \int_{M\times M} |f(y)|^2 \mathrm{tr}\big(|[A]_{x,y}|^2\big)\, d\mu_g(x)\, d\mu_g(y)\\
&= \int_{M\times M} |f(y)|^2 \mathrm{tr}\big([A^*]_{y,x}[A]_{x,y}\big)\, d\mu_g(x)\, d\mu_g(y)\\
&= \int_M |f(y)|^2 \mathrm{tr}\big([A^*A]_{y,y}\big)\, d\mu_g(y),
\end{aligned}$$

where in the last equality we used the operator-kernel product rule. Then, the proof follows by setting $A = (1+\mathcal{D}_S^2)^{-s/4}$. □

As explained above, we identify the von Neumann algebra generated by the operators $\{c(f),\ f \in C_c^\infty(M)\}$ acting on $L^2(M,S)$ with $L^\infty(M)$. Then, from the previous Hilbert-Schmidt norm computation, we can determine the weights φ_s of Definition 1.1, constructed with $\mathcal{D}_S$.

COROLLARY 4.7. *Let $\mathcal{D}_S$ be a Dirac type operator operating on the sections of a Dirac bundle S of bounded geometry. For $s > n$, let φ_s be the faithful normal semifinite weight of Definition 1.1, on the type I von Neumann algebra $\mathcal{B}(L^2(M,S))$ with operator trace. When restricted to $L^\infty(M)$, φ_s coincides with the integral on M with respect to the Borel measure $k_s\, d\mu_g$.*

We now examine which functions on the manifold are in $\mathcal{B}_1^\infty(\mathcal{D}_S, n)$. Combining Proposition 1.19 with Lemma 4.6 allows us to determine the norms $\mathcal{P}_m$ restricted to $L^\infty(M)$.

COROLLARY 4.8. *Let $\mathcal{D}_S$ be a Dirac type operator operating on the sections of a Dirac bundle S of bounded geometry. Then*

$$\mathcal{B}_1(\mathcal{D}_S, n) \bigcap L^\infty(M) = L^\infty(M) \bigcap_{m\in\mathbb{N}} L^1(M, k_{s+1/m} d\mu_g).$$

Moreover, we have the equality

$$\mathcal{P}_m\big(c(f)\big) = \|f\|_\infty + 2\|f\|_{1,k_{n+1/m}}, \quad m \in \mathbb{N}.$$

By Lemma 4.5, we see that $\bigcap_{m\in\mathbb{N}} L^1(M, k_{s+1/m} d\mu_g)$ contains $L^1(M)$. Note also that if a uniform on-diagonal lower bound for the Dirac heat kernel of the form

$$\big|K_t^S(x,x)\big|_\infty \geq c t^{-n/2},$$

holds (with $|\cdot|_\infty$ the operator norm on $\mathrm{End}(S_x)$), then $\bigcap_{m\in\mathbb{N}} L^1(M, k_{s+1/m}d\mu_g) = L^1(M)$. Such an estimate holds for the spin Dirac operator on Euclidean spaces, for example, and for the scalar heat kernel for any manifold of bounded geometry.

We now arrive at the main statement of this Section.

Proposition 4.9. *Let $\mathcal{D}_S$ be a Dirac type operator operating on the sections of a Dirac bundle S of bounded geometry on a manifold M of dimension n. Relative to the I_∞ factor $\mathcal{B}(L^2(M,S))$ with operator trace, the spectral triple*

$$\big(W^{\infty,1}(M), L^2(M,S), \mathcal{D}_S\big),$$

is smoothly summable and of spectral dimension n. Moreover, the spectral dimension is isolated in the sense of Definition 3.1.

Proof. We first show that for any $f \in W^{\infty,1}(M)$, the operators $\delta^k(c(f))$ and $\delta^k(c(df))$, $k \in \mathbb{N}_0$, all belong to $\mathcal{B}_1(\mathcal{D}_S, n)$. That $c(f) \in \mathcal{B}_1(\mathcal{D}_S, n)$ for $f \in W^{\infty,1}(M)$ has already been proven in Corollary 4.8 since $\bigcap_m W^{\infty,1}(M, k_{n+1/m}) \supset W^{\infty,1}(M)$. For the rest, we know by Proposition 2.21 that it is sufficient to prove that

$$(1+\mathcal{D}_S^2)^{-s/4} R^k(c(f))(1+\mathcal{D}_S^2)^{-s/4} \in \mathcal{L}^1\big(L^2(M,S)\big), \text{ for all } k \in \mathbb{N}_0, \text{ and all } s > n,$$

and similarly for $c(df)$. By the proof of Lemma 4.4, we also know that for $f \in W^{\infty,1}(M) \subset W^{\infty,\infty}(M)$, the operators $R^k(c(f))$ and $R^k(c(df))$ are of the form $B_k(1+\mathcal{D}_S^2)^{-k/2}$, where B_k is a differential operator of order k, with uniform $W^{\infty,1}(M)$-coefficients. This means that for any covering of $M = \cup B(x_i, \varepsilon)$ of balls of radius $\varepsilon \in (0, r_{\mathrm{inj}}/3)$ and partition of unity $\sum \varphi_i = 1$ subordinate to the covering, there exist elements $f_\alpha \in \mathrm{End}(S_x)$ with $B_k|_{B(x_i,\varepsilon)} = \sum_{|\alpha|\le k} f_\alpha \partial^\alpha$ in normal coordinates. Moreover, $\sum_{i=0}^\infty \|\varphi_i |f_\alpha|_\infty\|_1 < \infty$, where $|\cdot|_\infty$ is the operator norm on $\mathrm{End}(S_x)$, each φ_i has bounded derivatives of all order, uniformly in the covering index i. Now take $\sum \psi_i = 1$ a second partition of unity subordinate to the covering $M = \cup B(x_i, 2\varepsilon)$ (recall that the latter has finite order), with $\psi_i(x) = 1$ in a neighbourhood of $\mathrm{supp}(\varphi_i)$. We then have

$$B_k = \sum_{i=0}^\infty \psi_i B_k \varphi_i = \sum_{i=0}^\infty \sum_{|\alpha|\le k} \psi_i f_\alpha \partial^\alpha \varphi_i = \sum_{i=0}^\infty \sum_{|\alpha|,|\beta|\le k} \psi_i f_\alpha \partial^\beta(\varphi_i)\partial^\alpha.$$

Let $\psi_i f_\alpha \partial^\beta(\varphi_i) = u_{i,\alpha,\beta}|\psi_i f_\alpha \partial^\beta(\varphi_i)|$ be the polar decomposition. Define

$$C_{i,\alpha,\beta} := u_{i,\alpha,\beta}|\psi_i f_\alpha \partial^\beta(\varphi_i)|^{1/2}, \quad D_{i,\alpha,\beta} := |\psi_i f_\alpha \partial^\beta(\varphi_i)|^{1/2}\partial^\alpha,$$

so that

$$(1+\mathcal{D}_S^2)^{-s/4} B_k (1+\mathcal{D}_S^2)^{-(s+2k)/4} = \sum_{i=0}^\infty \sum_{|\alpha|,|\beta|\le k} (1+\mathcal{D}_S^2)^{-s/4} C_{i,\alpha,\beta}\, D_{i,\alpha,\beta}(1+\mathcal{D}_S^2)^{-(s+2k)/4}.$$

The fibre-wise trace of the on-diagonal operator kernel of $C^*_{i,\alpha,\beta}(1+\mathcal{D}_S^2)^{-s/2}C_{i,\alpha,\beta}$ being given by $|\psi_i(x) f_\alpha(x)\partial^\beta(\varphi_i)(x)|_1 k_s(x)$ (with $|\cdot|_1$ the trace-norm on $\mathrm{End}(S_x)$), we have for $s > n$

$$\mathrm{Tr}\big(C^*_{i,\alpha,\beta}(1+\mathcal{D}_S^2)^{-s/2}C_{i,\alpha,\beta}\big) = \int_{B(x_i,2\varepsilon)} |\psi_i(x) f_\alpha(x)\partial^\beta(\varphi_i)(x)|_1 k_s(x) d\mu_g(x),$$

so that

$$\|(1+\mathcal{D}_S^2)^{-s/4} C_{i,\alpha,\beta}\|_2 = \|\psi_i |f_\alpha|_1 \partial^\beta(\varphi_i)\|^{1/2}_{1,0,k_s} \le C_{\alpha,\beta}\|\psi_i |f_\alpha|_\infty\|_1^{1/2}.$$

For $D_{i,\alpha,\beta}$, note that the off-diagonal kernel of $D_{i,\alpha,\beta}(1+\mathcal{D}_S^2)^{-(s+2k)/2}D^*_{i,\alpha,\beta}$ reads up to a Γ-function factor

$$i^{|\alpha|}|\psi_i f_\alpha \partial^\beta(\varphi_i)|(x)^{1/2}\int_0^\infty t^{(s+2k)/2-1}\,e^{-t}\,\partial_x^\alpha\partial_y^\alpha K_t^S(x,y)\,dt\;|\psi_i f_\alpha\partial^\beta(\varphi_i)|(y)^{1/2}.$$

But Proposition 4.2 (1) gives

$$|\partial_x^\alpha\partial_y^\beta K_t^S(x,y)|_\infty \le C'(\alpha,\beta)t^{-(n+|\alpha|+|\beta|)/2}\exp\Big(-\frac{d_g^2(x,y)}{4(1+c)t}\Big),\quad t>0.$$

Since $|\alpha|$, $|\beta|\le k$, we finally obtain the inequality

$$\begin{aligned}\|D_{i,\alpha,\beta}(1+\mathcal{D}_S^2)^{-(s+2k)/4}\|_2^2 &\le C'(\alpha)\int_{B(x_i,2\varepsilon)}|\psi_i f_\alpha\partial^\beta(\varphi_i)|_\infty(x)d\mu_g(x)\\ &\le C''(\alpha,\beta)\int_{B(x_i,2\varepsilon)}|\psi_i|\,|f_\alpha|_\infty(x)d\mu_g(x)=C''(\alpha,\beta)\|\psi_i|f_\alpha|_\infty\|_1.\end{aligned}$$

Thus,

$$\begin{aligned}&\|(1+\mathcal{D}_S^2)^{-s/4}B_k(1+\mathcal{D}_S^2)^{-(s+2k)/4}\|_1\\ &\qquad\le\sum_{i=0}^\infty\sum_{|\alpha|,|\beta|\le k}\|(1+\mathcal{D}_S^2)^{-s/4}C_{i,\alpha,\beta}\|_2\,\|D_{i,\alpha,\beta}(1+\mathcal{D}_S^2)^{-(s+2k)/4}\|_2\\ &\qquad\le C\sum_{i=0}^\infty\sum_{|\alpha|\le k}\|\psi_i|f_\alpha|_\infty\|_1,\end{aligned}$$

which is finite by Lemma 4.3. This proves that for all $k\in\mathbb{N}_0$, $\delta^k(c(f))$ and $\delta^k(c(df))$ are in $\mathcal{B}_1(\mathcal{D}_S,n)$. We also have proven that the triple $\big(W^{\infty,1}(M),L^2(M,S),\mathcal{D}_S\big)$ is finitely summable.

That n is the smallest number such that $c(f)(1+\mathcal{D}_S^2)^{-s/2}$ is trace class for all $s>n$ follows from Proposition 4.2 (3), since

$$\mathrm{Tr}\big(c(f)(1+\mathcal{D}_S^2)^{-s/2}\big)=\frac{1}{\Gamma(s/2)}\int_0^\infty t^{s/2-1}\,e^{-t}\int_M f(x)\,\mathrm{tr}\big(K_t^S(x,x)\big)\,d\mu_g(x)\,dt,$$

and

$$\mathrm{tr}\big(K_t^S(x,x)\big)\sim_{t\to0}t^{-n/2}\sum_{i\ge0}t^i\,b_i(x).$$

Thus, the spectral dimension is n.

Last, that the spectral dimension is isolated follows from the fact that it has discrete dimension spectrum, which follows from Proposition 4.2 (3) and the trace computation above, since for any $f_0,f_1,\dots,f_m\in W^{\infty,1}(M)$, the operator

$$c(f_0)c(df_1)^{(k_1)}\dots c(df_m)^{(k_m)},$$

is a differential operator of order $|k|=k_1+\dots+k_m$ with uniform C^∞-bounded coefficients. □

4.2. An index formula for manifolds of bounded geometry

4.2.1. Extension of the Ponge approach. We still consider (M,g), a complete Riemannian manifold of dimension n, but now suppose that (M,g) is spin. We fix S to be the spinor bundle endowed with a connection ∇^S which is the usual lift of the Levi-Civita connection. We let $\mathcal{D}_S$ be the associated Dirac operator. We still assume that (M, g, S) has bounded geometry, in the sense of Definition 4.1.

Now we need to explain how to use the asympotic expansions of Proposition 4.2 (3), to deduce the Atiyah-Singer local index formula from the residue cocycle formula for the index. (Recall that by Proposition 4.9, the spectral triple $\big(W^{\infty,1}(M), L^2(M,S), \mathcal{D}_S\big)$ has isolated spectral dimension, so that we can use the last version of Theorem 3.33 to compute the index.) The key tool is Ponge's adaptation of Getzler's arguments, [**47**].

As Ponge and Roe explain, [**47, 51**], the arguments that Gilkey uses to prove that the coefficients in the asymptotic expansion of the Dirac Laplacian are universal polynomials, carries over to the noncompact situation and produces universal polynomials, identical to those of the compact case. Moreover, Ponge's argument is purely local; that is, it proceeds by choosing a single point in M and checking what the asymptotic expansion gives for the terms in the residue cocycle formula at that point. As such there is no change needed in Ponge's argument to handle complete manifolds of bounded geometry.

Thus, both the following results are proven just as in Ponge, and the only work is in checking that the constants are consistent with our conventions.

4.2.2. The odd case. We treat the odd case first, which is not affected by our 'doubling up' construction.

THEOREM 4.10. *Let (M, g, S) be a Riemannian spin manifold with bounded geometry and of odd dimension n. Let $\big(W^{\infty,1}(M), L^2(M,S), \mathcal{D}_S\big)$ be the smoothly summable spectral triple of spectral dimension n described in the last Section. The components of the odd residue cocycle are given by*

$$\phi_{2m+1}(f^0, f^1, \ldots, f^{2m+1}) = \frac{(-1)^m\sqrt{2\pi i}}{(2\pi i)^{\frac{n+1}{2}}(2m+1)!\,m!}\int_M f^0 df^1 \wedge \cdots \wedge df^{2m+1} \wedge \hat{A}(R)^{(n-2m-1)},$$

for $f^0, f^1, \ldots, f^{2m+1} \in W^{\infty,1}(M)$, $m \geq 0$, R being the curvature tensor of M.

Remark. The A-roof genus, $\hat{A}(R)$, is computed here with no normalisation of the Pontryagin classes by factors of $2\pi i$. To obtain the index formula in the next result, one should use the (b, B)-Chern character of a unitary $u \in M_N\big(W^{\infty,1}(M)^\sim\big)$, antisymmetrising after taking the matrix trace.

COROLLARY 4.11. *For any unitary $u \in M_N\big(W^{\infty,1}(M)^\sim\big)$ and with $2P_\mu - 1$ being the phase of $\mathcal{D}_{S,\mu} \otimes \mathrm{Id}_N$ and $P = \chi_{[0,\infty)}(\mathcal{D}_S) \otimes \mathrm{Id}_N$, we have the odd index pairing given by*

$$\mathrm{Ind}(PuP) = \mathrm{Ind}(P_\mu \hat{u} P_\mu) = -\frac{1}{(2\pi i)^{\frac{n+1}{2}}}\sum_{m=0}^{\frac{n-1}{2}} \frac{(-1)^m}{(2m+1)!\,m!}\int_M \mathrm{Ch}_{2m+1}(u) \wedge \hat{A}(R)^{(n-2m-1)}.$$

4.2.3. The even case. Now as the rank of a projection $f \in M_N\big(W^{\infty,1}(M)^\sim\big)$ is constant on connected components and equal to the rank of $\mathbf{1}_f$, the contribution of the zeroth term to the local index formula is zero. It remains therefore to compute ϕ_{2m} for $m \geq 1$ evaluated on the Chern character of a projection f.

THEOREM 4.12. *Let (M, g, S) be a Riemannian spin manifold with bounded geometry and of even dimension n. Let $\big(W^{\infty,1}(M), L^2(M,S), \mathcal{D}_S\big)$ be the smoothly summable spectral triple of spectral dimension n described in the last Section. The non-zero components of the even residue cocycle are given by*

$$\phi_{2m}(f^0, f^1, \dots, f^{2m}) = \frac{(-1)^m}{(2\pi i)^{n/2}(2m)!}\int_M f^0 df^1 \wedge \cdots \wedge df^{2m} \wedge \hat{A}(R)^{(n-2m)}, \quad m \geq 1,$$

for $f^0, f^1, \dots, f^{2m} \in W^{\infty,1}(M)$, R being the curvature tensor of M.

Again the A-roof genus is defined without $2\pi i$ normalisations, and in the following result one uses the (b,B)-Chern character of $f \in M_N\big(W^{\infty,1}(M)^\sim\big)$, antisymmetrising after taking the trace.

COROLLARY 4.13. *For any projector $f \in M_N\big(W^{\infty,1}(M)^\sim\big)$ and with F_μ being the phase of $\mathcal{D}_{S,\mu} \otimes \mathrm{Id}_N$, we have*

$$\mathrm{Ind}(\hat{f}\, F_{\mu,+}\, \hat{f}) = (2\pi i)^{-n/2} \sum_{m=1}^{\frac{n}{2}} \frac{(-1)^m}{(2m)!} \int_M \mathrm{Ch}_{2m}(f) \wedge \hat{A}(R)^{(n-2m)}.$$

4.3. An L^2-index theorem for manifolds of bounded geometry

In this Section we show how a version of the relative L^2-index (see [**59**] for another version) which generalises that in [**1**], can be obtained from our residue formula.

As above, we fix $(\tilde{M}, \tilde{g})$, a Riemannian manifold of dimension n and of bounded geometry. Let also G be a countable discrete group acting freely and properly on $\tilde{M}$ by (smooth) isometries. Note that *we do not assume* $\tilde{M}$ to be G-compact and we let $M := G \setminus \tilde{M}$ be the possibly noncompact manifold (by properness) of right cosets. It is then natural to think of $\tilde{M}$ as the total space of a principal G-bundle with noncompact base M. We denote by $q : \tilde{M} \to M$ the projection map. Note that the metric $\tilde{g}$ on $\tilde{M}$ then naturally yields a metric g on M given by $g_x(v_1, v_2) = \tilde{g}_{\tilde{x}}(\tilde{v}_1, \tilde{v}_2)$, if $x = q(\tilde{x}) \in M$ and $v_i = q(\tilde{v}_i) \in T_x M$ where we have identified $T_x M \simeq G.(T_{\tilde{x}}\tilde{M})$, since the action of G naturally extends to $T\tilde{M}$. In particular, (M, g) also has bounded geometry.

An important class of examples is given by universal coverings. In this case, G is the fundamental group of a manifold of bounded geometry M and $\tilde{M}$ is its universal cover. Also, in this case $q : \tilde{M} \to M$ is the covering map and $\tilde{g}$ is the lifted metric on $\tilde{M}$ by $\tilde{g}_{\tilde{x}} = g_{q(\tilde{x})}$.

Let now $\mathcal{D}_S$ be a Dirac type operator acting on the sections of a Dirac bundle S of bounded geometry on M. To simplify the notations, we denote by $(\mathcal{A}, \mathcal{H}, \mathcal{D}_S) := \big(W^{\infty,1}(M), L^2(M,S), \mathcal{D}_S\big)$ the smoothly summable spectral triple constructed in Section 4.1.1. If the triple is either even or odd, then we have various formulae for

$$\mathrm{Index}(\hat{e}F_{\mu,+}\hat{e}) \quad \text{even case}, \quad \mathrm{Index}(P_\mu \hat{u} P_\mu) \quad \text{odd case},$$

where F_μ is the phase of $\mathcal{D}_{S,\mu}$, is the double of $\mathcal{D}_S$ (see Definition 2.9), and $P_\mu = (F_\mu + 1)/2$. We lift the bundle S to a bundle $\tilde{S}$ on $\tilde{M}$ (pullback by q) and we also lift the operator $\mathcal{D}_S$ to an equivariant operator $\tilde{\mathcal{D}}_S$ on sections of $\tilde{S}$. This requires that the action of G on $\tilde{M}$ lifts to an action on $\tilde{S}$, and we assume that this is the case. We also denote by $\tilde{c}$ the Clifford action of $\mathrm{Cliff}(\tilde{M})$ on $\tilde{S}$. We let $\tilde{\mathcal{H}} = L^2(\tilde{M}, \tilde{S})$

and observe that $\mathcal{A}$ acts on $\tilde{\mathcal{H}}$ by $(\tilde{c}(f)\xi)(\tilde{x}) = c(f(x))\xi(\tilde{x})$, for $f \in \mathcal{A}$, $\xi \in \tilde{\mathcal{H}}$, and $\tilde{x} \in \tilde{M}$ with $x = q(\tilde{x})$.

We now briefly review the setting for L^2-index theory referring, for example, to the review [**53**] for some details and references to the original literature. Since the action of G on $\tilde{M}$ is free and proper, we have an isometric identification $L^2(\tilde{M}, \tilde{S}) \cong L^2(M, S) \otimes \ell^2(G)$. This allows us to define the von Neumann algebra $\mathcal{N}_G = G' \cong \mathcal{B}(\mathcal{H}) \otimes R(G)''$, where $R(G)$ is the group algebra consisting of the span of the unitaries giving the right action of G on $\ell^2(G)$. There is a canonical semifinite faithful normal trace τ_G defined on elementary tensors $T \otimes U \in \mathcal{B}(\mathcal{H}) \otimes R(G)''$ by

$$\tau_G(T \otimes U) = \mathrm{Tr}_{\mathcal{H}}(T)\, \tau_e(U),$$

where $\mathrm{Tr}_{\mathcal{H}}$ is the operator trace on $\mathcal{H}$ and τ_e is the usual finite faithful normal trace on $R(G)''$ given by evaluation at the neutral element. Let now $\tilde{T}$ be a pseudodifferential operator on $\tilde{\mathcal{H}}$ with smooth kernel $[\tilde{T}] \in \Gamma^\infty(\tilde{S} \boxtimes \tilde{S})$. Then, $\tilde{T}$ is G-equivariant if and only if

$$[\tilde{T}](h \cdot \tilde{x}, h \cdot \tilde{y}) = e_{\tilde{x}}(h)\, [T](\tilde{x}, \tilde{y})\, e_{\tilde{y}}(h)^{-1}, \quad \text{for all } (h, \tilde{x}, \tilde{y}) \in G \times \tilde{M}^2,$$

where $e_{\tilde{x}} : G \to \mathrm{Aut}(\tilde{S}_{\tilde{x}})$ is the fibre-wise lift of the action of G to $\tilde{S}$. For such G-equivariant pseudodifferential operators on $\tilde{\mathcal{H}}$ which belongs to $\mathcal{L}^1(\mathcal{N}_G, \tau_G)$, we have

$$\tau_G(\tilde{T}) = \int_F \mathrm{tr}\big([T](\tilde{x}, \tilde{x})\big)\, d\mu_{\tilde{g}}(\tilde{x}), \tag{4.4}$$

where F is a fundamental domain in $\tilde{M}$ and tr is the fibre-wise trace on $\mathrm{End}(\tilde{S}_{\tilde{x}})$. This latter formulation is the natural one, and was initially defined by Atiyah [**1**]. It is clear from its definition that τ_G is faithful so that the algebra $\mathcal{N}_G$ is semifinite. It need not be a factor because (as it is well known) the algebra $R(G)''$ has a non-trivial centre precisely when the group G has finite conjugacy classes [**53**].

We note that when T is a pseudodifferential operator of trace class on $L^2(M, S)$ with Schwartz kernel $[T]$ (and, thus, order less than $-n$ and with L^1-coefficients), and $U \in R(G)''$, we have, using the identification above,

$$\tau_G(T \otimes U) := \int_M \mathrm{tr}\big([T](x, x)\big)\, \mu_g(x) \,\times\, \tau_e(U).$$

When the original triple $(\mathcal{A}, \mathcal{H}, \mathcal{D}_S)$ on M is even with grading γ, we denote by $\tilde{\gamma} := \gamma \otimes \mathrm{Id}_{\ell^2(G)}$ the grading lifted to $\tilde{\mathcal{H}}$.

Remark. The ideal of τ_G-compact operators $\mathcal{K}_{\mathcal{N}_G} = \mathcal{K}(\mathcal{N}_G, \tau_G)$ is given by the norm closure of the G-equivariant pseudodifferential operators of strictly negative order and with integral kernel vanishing at infinity inside a fundamental domain.

LEMMA 4.14. *Let $(\tilde{M}, \tilde{g})$ be a Riemannian manifold of bounded geometry endowed with a free and proper action of a countable group G. Let also P be a differential operator of order $\alpha \in \mathbb{N}_0$ and of uniform C^∞-bounded coefficients, acting on the sections of S and let $\tilde{P}$ be its lift as a G-equivariant operator on $\tilde{S}$ (which has also uniform C^∞-bounded coefficients). Assume further that*

$$\kappa := \inf\big\{d_{\tilde{g}}(\tilde{x}, h \cdot \tilde{x}) \,:\, \tilde{x} \in \tilde{M},\ h \in G \setminus \{e\}\big\} > 0.$$

Then there exist two constants $C >$ and $c > 0$, such that for any $(\tilde{x}, x) \in \tilde{M} \times M$, with $x = q(\tilde{x})$ we have

$$\big|[\tilde{P}e^{-t\tilde{\mathcal{D}}_S^2}](\tilde{x}, \tilde{x}) - [Pe^{-t\mathcal{D}_S^2}](x, x)\big|_\infty \le C\, t^{-(n+\alpha)/2} e^{-c/t},$$

where $|\cdot|_\infty$ is the operator norm on $\mathrm{End}(\tilde{S}_x)$.

Proof. Note first that for any $(\tilde{x}, x), (\tilde{y}, y) \in \tilde{M} \times M$, with $x = q(\tilde{x})$, $y = q(\tilde{y})$, we have

$$[Pe^{-t\mathcal{D}_S^2}](x, y) = \sum_{h \in G} [\tilde{P}e^{-t\tilde{\mathcal{D}}_S^2}](\tilde{x}, h \cdot \tilde{y}),$$

which is proven using the uniqueness of solutions of the heat equation on $\tilde{M}$ and on M. Thus,

$$[Pe^{-t\mathcal{D}_S^2}](x, x) - [\tilde{P}e^{-t\tilde{\mathcal{D}}_S^2}](\tilde{x}, \tilde{x}) = \sum_{h \in G,\, h \neq e} [\tilde{P}e^{-t\tilde{\mathcal{D}}_S^2}](\tilde{x}, h \cdot \tilde{x}).$$

From Proposition 4.2, we immediately deduce

$$\big|[\tilde{P}e^{-t\tilde{\mathcal{D}}_S^2}](\tilde{x}, \tilde{x}) - [Pe^{-t\mathcal{D}_S^2}](x, x)\big|_\infty \leq C\, t^{-(n+\alpha)/2} \sum_{h \in G,\, h \neq e} e^{-d_g^2(\tilde{x}, h \cdot \tilde{x})/4(1+c)t}.$$

Since $(\tilde{M}, \tilde{g})$ has bounded geometry, the sectional curvature is bounded below, by say $-K^2$ with $K > 0$. From [**41**], we have for any $\rho > 0$ the existence of a uniform (over $\tilde{M}$) constant $C' > 0$ such that

$$N_{\tilde{x}}(\rho) := \mathrm{Card}\big\{h \in G \,:\, d_{\tilde{g}}(\tilde{x}, h \cdot \tilde{x}) \leq \rho\big\} \leq C' e^{(n-1)K\rho}.$$

Then the assumption that $\kappa := \inf\big\{d_{\tilde{g}}(\tilde{x}, h \cdot \tilde{x}) \,:\, \tilde{x} \in \tilde{M}, h \in G \setminus \{e\}\big\} > 0$, yields the inequality

$$\big|[\tilde{P}e^{-t\tilde{\mathcal{D}}_S^2}](\tilde{x}, \tilde{x}) - [Pe^{-t\mathcal{D}_S^2}](x, x)\big|_\infty \leq C''\, t^{-(n+\alpha)/2} \int_\kappa^\infty e^{-\rho^2/4(1+c)t} dN_{\tilde{x}}(\rho),$$

which after an integration by parts, gives the proof. $\square$

Lemma 4.15. *Under the hypotheses of Lemma 4.14 and for $f \in \mathcal{A}$ and P a differential operator on S with uniform C^∞-bounded coefficients (and $\tilde{P}$ its lift on $\tilde{S}$ as a G-equivariant operator), the functions*

$$\mathbb{C} \ni z \mapsto \tau_G\Big(\tilde{c}(f)\tilde{P}\int_1^\infty t^z e^{-t(1+\tilde{\mathcal{D}}_S^2)}dt\Big), \quad \mathbb{C} \ni z \mapsto \mathrm{Tr}\Big(c(f)P\int_1^\infty t^z e^{-t(1+\mathcal{D}_S^2)}dt\Big),$$

are entire.

Proof. From Proposition 4.2 and Equation (4.4), we see that the integral is absolutely convergent. We thus may differentiate under the integral sign with respect to z and since the resulting integral is again absolutely convergent, we are done. $\square$

Proposition 4.16. *Under the hypotheses of Lemma 4.14, for $f \in \mathcal{A}$, P is a differential operator of uniform C^∞-bounded coefficients and $\Re(z) > n$, there is an equality*

$$\tau_G\big(\tilde{\gamma}\tilde{c}(f)\tilde{P}(1 + \tilde{\mathcal{D}}_S^2)^{-z/2}\big) = \mathrm{Tr}\big(\gamma c(f)P(1 + \mathcal{D}_S^2)^{-z/2}\big),$$

modulo an entire function of z.

Proof. This is a combinations of Lemmas 4.14 and 4.15 together with the usual Laplace transform representation for the operators concerned. $\square$

The following result, whose proof follows from the previous discussion and the same arguments as in Section 4.1.1, is key.

COROLLARY 4.17. *The triple $(\mathcal{A}, \tilde{\mathcal{H}}, \tilde{\mathcal{D}})$ is a smoothly summable semifinite spectral triple with respect to $(\mathcal{N}_G, \tau_G)$, of isolated spectral dimension n.*

PROOF. This follows from Proposition 4.16 combined with Proposition 4.9 together with similar arguments as those used in Proposition 4.9 to prove that the operators $\delta^k(\tilde{c}(f))$ and $\delta^k(\tilde{c}(df))$, $k \in \mathbb{N}_0$, all belong to $\mathcal{B}_1(\mathcal{D}_S, n)$ for $f \in \mathcal{A}$. □

We arrive at the main result of this Section.

THEOREM 4.18. *The numerical pairing of $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ with $K_*(\mathcal{A})$ coincides with the numerical pairing of $(\mathcal{A}, \tilde{\mathcal{H}}, \tilde{\mathcal{D}})$ with $K_*(\mathcal{A})$ (which is thus integer-valued).*

PROOF. Since both spectral triples $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ and $(\mathcal{A}, \tilde{\mathcal{H}}, \tilde{\mathcal{D}})$ have isolated spectral dimension, one can use the last version of Theorem 3.33 to compute the index pairing, i.e. we can use the residue cocycle. Then the result follows from Proposition 4.16. □

CHAPTER 5

Noncommutative Examples

In this Chapter, we apply our results to purely noncommutative examples. The first source of examples comes from torus actions on C^*-algebras and the construction follows [**42**] and [**43**] where explicit special cases for graph and k-graph algebras were studied. The second describes the Moyal plane and uses the results of [**27**].

5.1. Torus actions on C^*-algebras

We are interested here in spectral triples arising from an action of a compact abelian Lie group $\mathbb{T}^p = (\mathbb{R}/2\pi\mathbb{R})^p$ on a separable C^*-algebra A, which we denote by $\sigma_\cdot : \mathbb{T}^p \to \mathrm{Aut}(A)$. We suppose that A possesses a $\mathbb{T}^p$-invariant norm lower-semicontinuous faithful semifinite trace τ. Recall that τ is norm lower-semicontinuous if whenever we have a norm convergent sequence of positive elements, $A \ni a_j \to a \in A$, then $\tau(a) \leq \liminf \tau(a_j)$, and the tracial property says that $\tau(a^*a) = \tau(aa^*)$ for all $a \in A$.

We show that with this data we obtain a smoothly summable spectral triple, even if we dispense with the assumption that the algebra has local units employed in [**42, 43, 60**].

We begin by setting $\mathcal{H}_1 = L^2(A, \tau)$, the GNS space for A constructed using the trace τ. The action of $\mathbb{T}^p$ on our algebra A gives a $\mathbb{Z}^p$-grading on A by the spectral subspaces

$$A = \overline{\bigoplus_{m \in \mathbb{Z}^p} A_m}, \quad A_m = \{a \in A : \sigma_z(a) = z^m a = z_1^{m_1} \dots z_p^{m_p} a\}.$$

So for all $a \in A$ we can write a as a sum of elements a_m homogenous for the action of $\mathbb{T}^p$

$$a = \sum_{m \in \mathbb{Z}^p} a_m, \qquad t \cdot a_m = e^{i\langle m, t\rangle} a_m, \quad m \in \mathbb{Z}^p,\ t \in \mathbb{T}^p.$$

The invariance of the trace τ implies that the $\mathbb{T}^p$ action extends to a unitary action U on $\mathcal{H}_1$ which implements the action on A. As a consequence there exist pairwise orthogonal projections $\Phi_m \in \mathcal{B}(\mathcal{H}_1)$, $m \in \mathbb{Z}^p$, such that $\sum_{m \in \mathbb{Z}^p} \Phi_m = \mathrm{Id}_{\mathcal{H}_1}$ (strongly) and $a_m \Phi_k = \Phi_{m+k} a_m$ for a homogenous algebra element $a_m \in A_m$. Moreover, we say that A has full spectral subspaces if for all $m \in \mathbb{Z}^p$ we have $\overline{A_m A_m^*} = A_0$. Observe that A_0 coincides with $A^{\mathbb{T}^p}$, the fixed point algebra of A for the action of $\mathbb{T}^p$.

Let $\mathcal{H} := \mathcal{H}_1 \otimes_{\mathbb{C}} \mathcal{H}_f$, where $\mathcal{H}_f := \mathbb{C}^{2^{\lfloor p/2 \rfloor}}$. We define our operator $\mathcal{D}$ as the operator affiliated to $\mathcal{B}(\mathcal{H})$, given by the 'push-forward' of the flat Dirac operator on $\mathbb{T}^p$ to the Hilbert space $\mathcal{H}$. More precisely, we first define the domain $\mathrm{dom}(\mathcal{D})$ by

$$\mathrm{dom}(\mathcal{D}) := \mathcal{H}_1^\infty \otimes \mathcal{H}_f, \quad \mathcal{H}_1^\infty := \big\{\psi \in \mathcal{H}_1 : [t \mapsto t \cdot \psi] \in C^\infty(\mathbb{T}^p, \mathcal{H}_1)\big\}.$$

Then we define $\mathcal{D}$ on $\mathrm{dom}(\mathcal{D})$ by

$$\mathcal{D} = \sum_{n\in\mathbb{Z}^p} \Phi_n \otimes \gamma(in),$$

where $\gamma(in) = i\sum_{j=1}^{p} \gamma_j n_j$, $n = (n_1, \ldots, n_p)$, and the γ_j are Clifford matrices acting on $\mathcal{H}_f$ with

$$\gamma_j\gamma_l + \gamma_l\gamma_j = -2\,\delta_{jl}\,\mathrm{Id}_{\mathcal{H}_f}.$$

In future we will abuse notation by letting Φ_n denote the projections acting on $\mathcal{H}_1$, on A, and also the projections $\Phi_n \otimes \mathrm{Id}_{\mathcal{H}_f}$ acting on $\mathcal{H}$. Similarly we will speak of A and A_0 acting on $\mathcal{H}$, by tensoring the GNS representation on $\mathcal{H}_1$ by $\mathrm{Id}_{\mathcal{H}_f}$. To simplify the notations, we just identify A with its image in the GNS representation.

We let $\mathcal{N} \subset \mathcal{B}(\mathcal{H})$ be the commutant of the *right* multiplication action of the fixed point algebra A_0 on $\mathcal{H}$. Then it can be checked that the left multiplication representation of A is in $\mathcal{N}$ and $\mathcal{D}$ is affiliated to $\mathcal{N}$. To obtain a faithful normal semifinite trace, which we call Tr_τ, on $\mathcal{N}$, we have two possible routes, which both lead to the same trace, and which yield different and complementary information about the trace.

The first approach is to let Tr_τ be the dual trace on $\mathcal{N} = (A_0)'$. The dual trace is defined using spatial derivatives and is a faithful normal semifinite trace on $\mathcal{N}$. A detailed discussion of this construction and its equivalence with our next construction, can be found in [**38**, p. 471-478]. The discussion referred to in [**38**] is in the context of KMS weights, but by specializing the β-KMS weights to the case $\beta = 0$, the particular case of invariant traces, we obtain the description we want. Alternatively, the reader may examine [**38**, Theorem 1.1] for a trace specific description of our next construction.

In fact, the article [**38**] is, in part, concerned with inducing traces from the coefficient algebra of a C^*-module to traces on the algebra of compact endomorphisms on that module. To make contact with [**38**], we make $A \otimes \mathcal{H}_f$ a right inner product module over A_0 via the inner product

$$(a\otimes\xi|b\otimes\eta) := \Phi_0(a^*b)\langle\xi,\eta\rangle_{\mathcal{H}_f}, \quad a,\, b\in A, \quad \xi,\, \eta \in \mathcal{H}_f.$$

Calling the completed right A_0-C^*-module by X, it can be shown, see [**38**], that $\mathrm{End}_{A_0}(X)$ acts on $\mathcal{H}$ and that $\mathcal{N} = \mathrm{End}_{A_0}(X)''$. We introduce this additional structure because we can compute Tr_τ on all rank one endomorphisms on X. Given $x,\, y,\, z \in X$, the rank one endomorphism $\Theta_{x,y}$ acts on z by $\Theta_{x,y}z := x(y|z)$. Then by [**38**, Lemma 3.1 & Theorem 3.2] specialised to invariant traces, see also [**38**, Theorem 1.1], we have

$$\mathrm{Tr}_\tau(\Theta_{x,y}) = \tau((y|x)) := \sum_{i=1}^{2^{\lfloor p/2\rfloor}} \tau((y_i|x_i)), \tag{5.1}$$

where $x = \sum_i x_i \otimes e_i$, the e_i are the standard basis vectors of $\mathcal{H}_f$, and similarly $y = \sum_i y_i \otimes e_i$. Moreover, Tr_τ restricted to the compact endomorphisms of X is an $\mathrm{Ad}\,U(\mathbb{T}^P)$-invariant norm lower-semicontinuous trace, [**38**, Theorem 3.2], where U is the action of $\mathbb{T}^p$ on $\mathcal{H}$.

LEMMA 5.1. *Let $0 \le a \in \mathrm{dom}\,\tau \subset A \subset \mathcal{N}$. Then,*

$$\mathrm{Tr}_\tau\big(a\,\Phi_0\big) = 2^{\lfloor p/2\rfloor}\,\tau(a),$$

More generally, for $m \in \mathbb{Z}^p$ we have

$$0 \leq \mathrm{Tr}_\tau\big(a\,\Phi_m\big) \leq 2^{\lfloor p/2\rfloor}\,\tau(a), \tag{5.2}$$

and when A has full spectral subspaces, we have an equality:

$$0 \leq \mathrm{Tr}_\tau\big(a\,\Phi_m\big) = 2^{\lfloor p/2\rfloor}\,\tau(a).$$

PROOF. We prove the statement for $a \in A_0$, and then proceed to general elements of A. We begin with the case of full spectral subspaces. Consider first $a = bb^*$ for $b \in A_k \cap \mathrm{dom}^{1/2}\,\tau$ homogenous of degree k, so that $a \in A_0 \cap \mathrm{dom}\,\tau$ (since τ is a trace). Then a short calculation shows that $\Phi_k a \Phi_k = a\Phi_k = \sum_{i=1}^{2^{\lfloor p/2\rfloor}} \Theta_{b\otimes e_i, b\otimes e_i}$ where the e_i are the standard basis vectors in $\mathcal{H}_f$. Hence

$$\mathrm{Tr}_\tau(a\Phi_k) = \sum_{i=1}^{2^{\lfloor p/2\rfloor}} \tau(b^*b) = \sum_{i=1}^{2^{\lfloor p/2\rfloor}} \tau(bb^*) = 2^{\lfloor p/2\rfloor}\tau(a).$$

Therefore $\mathrm{Tr}_\tau(a\Phi_k) = 2^{\lfloor p/2\rfloor}\tau(a)$ if a is a finite sum of elements of the form bb^*, $b \in A_k$. Thus, if $\overline{A_k A_k^*} = A_0$ for all $k \in \mathbb{Z}^p$ we get equality for all $\mathrm{dom}\,\tau \cap A_0^+ \ni a$ and $k \in \mathbb{Z}^p$. In particular, we always have $\mathrm{Tr}_\tau(a\Phi_0) = 2^{\lfloor p/2\rfloor}\tau(a)$.

In the more general situation consider the closed ideal $\overline{A_k A_k^*}$ in A_0, which is σ-unital by the separability of A, and of $\overline{A_k A_k^*}$. Choose a positive approximate unit $\{\psi_n\}_{n\geq 1} \subset A_k A_k^*$ for $\overline{A_k A_k^*}$. Since $A_k A_k^* A_k$ is dense in A_k, we have $\psi_n x \to x$ for any $x \in X_k = A_k \otimes \mathcal{H}_f$. Hence, when n goes to ∞, $\psi_n a \psi_n \in A_k A_k^*$ converges strongly to the action of a on X_k for any $a \in A_0$. Since Tr_τ is strictly lower semicontinuous, [**38**, Theorem 3.2], for $A_0 \cap \mathrm{dom}\,\tau \ni a \geq 0$ we therefore get

$$\begin{aligned}\mathrm{Tr}_\tau(a\Phi_k) \leq \liminf_n \mathrm{Tr}_\tau(\psi_n a \psi_n \Phi_k) &= \liminf_n 2^{\lfloor p/2\rfloor}\tau(\psi_n a \psi_n)\\ &= \liminf_n 2^{\lfloor p/2\rfloor}\tau(a^{1/2}\psi_n^2 a^{1/2}) \leq 2^{\lfloor p/2\rfloor}\tau(a).\end{aligned}$$

This proves the Lemma for $a \in A_0 \cap \mathrm{dom}\,\tau$. Now for general $0 \leq a \in \mathrm{dom}\,\tau$, we may use the Ad$U$-invariance of Tr_τ to see that

$$\mathrm{Tr}_\tau(a\Phi_k) = \mathrm{Tr}_\tau(\Phi_0(a)\Phi_k) \leq 2^{\lfloor p/2\rfloor}\tau(\Phi_0(a)),$$

with equality for $k = 0$ or for all $k \in \mathbb{Z}^p$ if A has full spectral subspaces. Thus, if we write $a = \sum_{m\in\mathbb{Z}^p} a_m$ as a sum of homogenous components,

$$\mathrm{Tr}_\tau(a\Phi_k) = \mathrm{Tr}_\tau(a_0\Phi_k) \leq 2^{\lfloor p/2\rfloor}\tau(a_0) = 2^{\lfloor p/2\rfloor}\tau(a),$$

with equality if $k = 0$ or for all $k \in \mathbb{Z}^p$ if A has full spectral subspaces. □

COROLLARY 5.2. *Let $A, \mathcal{H}, \mathcal{D}, \mathcal{N}, \mathrm{Tr}_\tau$ be as above. Use $\mathcal{D}$ and Tr_τ to construct the weights φ_s, $s > p$, on $\mathcal{N}$ via Definition 1.1. Consider the restrictions ψ_s of the weights φ_s to the domain of τ in A. Then*

$$\psi_s(a) \leq 2^{\lfloor p/2\rfloor}\Big(\sum_{m\in\mathbb{Z}^p}(1+|m|^2)^{-s/2}\Big)\,\tau(a)\,, \quad a \in A_+ \cap \mathrm{dom}\,\tau\,,\ s > p,$$

with equality if A has full spectral subspaces.

PROOF. Note first that

$$(1+\mathcal{D}^2)^{-s/2} = \sum_{m\in\mathbb{Z}^p}(1+|m|^2)^{-s/2}\Phi_m,$$

so that for $a \in A_+$ and $s > p$, by definition of the weights φ_s, we have that

$$\varphi_s(a) = \mathrm{Tr}_\tau\big((1+\mathcal{D}^2)^{-s/4} a (1+\mathcal{D}^2)^{-s/4}\big),$$

which by traciality of Tr_τ implies

$$\varphi_s(a) = \mathrm{Tr}_\tau\big(\sqrt{a}(1+\mathcal{D}^2)^{-s/2}\sqrt{a}\big) = \mathrm{Tr}_\tau\Big(\sum_{m\in\mathbb{Z}^p}(1+|m|^2)^{-s/2}\sqrt{a}\Phi_m\sqrt{a}\Big).$$

The normality of Tr_τ allows us to permute the sum and the trace

$$\begin{aligned}\varphi_s(a) &= \sum_{m\in\mathbb{Z}^p}(1+|m|^2)^{-s/2}\mathrm{Tr}_\tau\big(\sqrt{a}\,\Phi_m\,\sqrt{a}\big) = \sum_{m\in\mathbb{Z}^p}(1+|m|^2)^{-s/2}\mathrm{Tr}_\tau\big(\Phi_m\, a\, \Phi_m\big)\\ &= \sum_{m\in\mathbb{Z}^p}(1+|m|^2)^{-s/2}\mathrm{Tr}_\tau\big(\Phi_0(a)\,\Phi_m\big) \leq 2^{\lfloor p/2\rfloor}\Big(\sum_{m\in\mathbb{Z}^p}(1+|m|^2)^{-s/2}\Big)\,\tau(a),\end{aligned} \tag{5.3}$$

the last inequality following from Lemma 5.1, and it is an equality if A has full spectral subspaces. □

Let $\mathcal{A}\subset A$ be the algebra of smooth vectors for the action of $\mathbb{T}^p$

$$\begin{aligned}\mathcal{A} &:= \big\{a\in A\,:\,[t\mapsto t\cdot a]\in C^\infty(\mathbb{T}^p, A)\big\}\\ &= \Big\{a=\sum_{m\in\mathbb{Z}^p}a_m\in\bigoplus_{m\in\mathbb{Z}^p}A_m\,:\,\sum_{m\in\mathbb{Z}^p}|m|^k\|a_m\|<\infty \text{ for all } k\in\mathbb{N}_0\Big\}.\end{aligned}$$

Then, as expected, $\mathcal{A}$ is contained in OP^0. We let $\delta(T) = [|\mathcal{D}|, T]$ for $T\in\mathcal{N}$ preserving $\mathcal{H}_\infty$.

LEMMA 5.3. *The subalgebra $\mathcal{A}$ of smooth vectors in A for the action of $\mathbb{T}^p$ is contained in $\bigcap_k \mathrm{dom}(\delta^k)$. More explicitly, for $a = \sum_{m\in\mathbb{Z}^p}a_m\in\bigoplus_{m\in\mathbb{Z}^p}A_m$ we have the bound*

$$\|\delta^k(a)\|\leq C_k\sum_{m\in\mathbb{Z}^p}|m|^{2k}\,\|a_m\|.$$

PROOF. By the discussion following Definition 1.20, the claim is equivalent to $\mathcal{A}\subset\cap_k\mathrm{dom}(R^k)$, where $R(T) = [\mathcal{D}^2, T](1+\mathcal{D}^2)^{-1/2}$. Recall that for $a\in\mathcal{A}$ and $k\in\mathbb{N}$, we have

$$R^k(a) = [\mathcal{D}^2,\dots[\mathcal{D}^2, a]\dots](1+\mathcal{D}^2)^{-k/2},$$

with k commutators. For $j = 1,\dots,p$, denote by ∂_j the generators of the $\mathbb{T}^p$-action on both $\mathcal{A}$ and $\mathcal{H}_1$. For $\alpha\in\mathbb{N}^p$, let $\partial^\alpha := \partial_1^{\alpha_1}\dots\partial_p^{\alpha_p}$. Since $\mathcal{D}^2 = -(\sum_{j=1}^p\partial_j^2)\otimes\mathrm{Id}_{\mathcal{H}_\mathfrak{f}}$, an elementary computation shows that

$$R^k(a) = \sum_{|\alpha|\leq 2k, |\beta|\leq k} C_{\alpha,\beta}\,\partial^\alpha(a)\,\partial^\beta\otimes\mathrm{Id}_{\mathcal{H}_\mathfrak{f}}\,(1+\mathcal{D}^2)^{-k/2}.$$

This is enough to conclude, since $a\in\mathcal{A}$ implies that $\|\partial^\alpha(a)\|<\infty$, and elementary spectral theory of p pairwise commuting operators shows that for $|\beta|\leq k$, $\partial^\beta\otimes\mathrm{Id}_{\mathcal{H}_\mathfrak{f}}\,(1+\mathcal{D}^2)^{-k/2}$ is bounded too. The bound then follows from

$$\partial^\alpha(a_m) = i^{|\alpha|}m^\alpha\,a_m\,,\quad a_m\in A_m,$$

which delivers the proof. □

Define the algebras $\mathcal{B}, \mathcal{C} \subset \mathcal{A} \subset A$ by

$$\mathcal{B} = \Big\{ a = \sum_{m\in\mathbb{Z}^p} a_m \in \mathcal{A} \,:\, \sum_{m\in\mathbb{Z}^p} |m|^k\,\tau(a_m^* a_m) < \infty \text{ for all } k \in \mathbb{N}_0 \Big\},$$
$$\mathcal{C} = \Big\{ a = \sum_{m\in\mathbb{Z}^p} a_m \in \mathcal{A} \,:\, \sum_{m\in\mathbb{Z}^p} |m|^k \tau(|a_m|) < \infty \text{ for all } k \in \mathbb{N}_0 \Big\}.$$

The following is the main result of this Section.

PROPOSITION 5.4. *Let $\mathbb{T}^p$ be a torus acting on a C^*-algebra A with a norm lower-semicontinuous faithful $\mathbb{T}^p$-invariant trace τ. Then $(\mathcal{C}, \mathcal{H}, \mathcal{D})$ defined as above is a semifinite spectral triple relative to $(\mathcal{N}, \mathrm{Tr}_\tau)$. Moreover, $(\mathcal{C}, \mathcal{H}, \mathcal{D})$ is smoothly summable with spectral dimension p. The square integrable and integrable elements of A satisfy*

$$\mathcal{B}_2(\mathcal{D}, p) \bigcap A = (\mathrm{dom}(\tau))^{1/2}, \quad \mathcal{B}_1(\mathcal{D}, p) \bigcap A = \mathrm{dom}(\tau),$$

The space of smooth square integrable and the space of smooth integrable elements of A contain $\mathcal{B}$ and $\mathcal{C}$ respectively. More precisely,

$$\mathcal{B}_2^\infty(\mathcal{D}, p) \supset \mathcal{B} \cup [\mathcal{D}, \mathcal{B}], \quad \mathcal{B}_1^\infty(\mathcal{D}, p) \supset \mathcal{C} \cup [\mathcal{D}, \mathcal{C}].$$

Furthermore, if $0 \le a \in \mathrm{dom}(\tau)$ and A has full spectral subspaces, then

$$\mathrm{res}_{z=0} \mathrm{Tr}_\tau(a(1+\mathcal{D}^2)^{-p/2-z}) = 2^{\lfloor p/2 \rfloor - 1}\,\mathrm{Vol}(S^{p-1})\,\tau(a).$$

PROOF. We begin by proving that $\mathcal{B}_2(\mathcal{D}, p) \bigcap A \supset (\mathrm{dom}(\tau))^{1/2}$. Lemma 5.1 shows that for all $a \in \mathrm{dom}(\tau)$ with $a \ge 0$ and all $m \in \mathbb{Z}^p$ we have

$$\mathrm{Tr}_\tau(a\Phi_m) \le 2^{\lfloor p/2 \rfloor}\,\tau(a)\,, \tag{5.4}$$

and equality holds when we have full spectral subspaces or $m = 0$. Thus, using the normality of Tr_τ and the same arguments as in Equation (5.3), for $a \in (\mathrm{dom}(\tau))^{1/2}$ and $\Re(s) > p$ we see that

$$\begin{aligned}\mathrm{Tr}_\tau((1+\mathcal{D}^2)^{-s/4} a^* a (1+\mathcal{D}^2)^{-s/4}) &= \sum_{n\in\mathbb{Z}^p} (1+|n|^2)^{-s/2}\,\mathrm{Tr}_\tau(a^* a \Phi_n) \\ &\le \tau(a^*a)\,2^{\lfloor p/2 \rfloor} \sum_{n\in\mathbb{Z}^p} (1+|n|^2)^{-s/2} < \infty.\end{aligned}$$

Hence $(\mathrm{dom}(\tau))^{1/2} \subset \mathcal{B}_2(\mathcal{D}, p)$. Conversely, if $a \in A$ lies in $\mathcal{B}_2(\mathcal{D}, p)$ we have $a(1+\mathcal{D}^2)^{-s/4} \in \mathcal{L}^2(\mathcal{N}, \mathrm{Tr}_\tau)$ for all s with $\Re(s) > p$. Then

$$a\Phi_0 a^* \le a(1+\mathcal{D}^2)^{-s/2} a^* \in \mathcal{L}^1(\mathcal{N}, \mathrm{Tr}_\tau), \quad \Re(s) > p,$$

and so $a\Phi_0 a^* \in \mathcal{L}^1(\mathcal{N}, \mathrm{Tr}_\tau)$. Then

$$\tau(a^*a) = \mathrm{Tr}_\tau(\Phi_0 a^* a \Phi_0) = \mathrm{Tr}_\tau(a\Phi_0 a^*) < \infty.$$

Thus, $a^*a \in \mathrm{dom}(\tau)$, and so $a \in \mathrm{dom}(\tau)^{1/2}$. Since $\mathcal{B}_2(\mathcal{D}, p)$ is a $*$-algebra, we also have $a^*(1+\mathcal{D}^2)^{-s/4} \in \mathcal{L}^2(\mathcal{N}, \mathrm{Tr}_\tau)$, and so $a^* \in \mathrm{dom}(\tau)^{1/2}$ as expected. Now for $0 \le a \in A$, Lemma 1.13 tells us that $a \in \mathcal{B}_1(\mathcal{D}, p)$ if and only if $a^{1/2} \in \mathcal{B}_2(\mathcal{D}, p)$. So $a \in \mathrm{dom}(\tau)_+$ if and only if $a^{1/2} \in (\mathrm{dom}(\tau))_+^{1/2} = (\mathcal{B}_2(\mathcal{D}, p) \cap A)_+$, proving that $\mathrm{dom}(\tau)_+ = \mathcal{B}_1(\mathcal{D}, p)_+ \bigcap A_+$. Since $\mathcal{B}_1(\mathcal{D}, p)$ is the span of its positive cone by Proposition 1.14, we have

$$\mathcal{B}_1(\mathcal{D}, p) \bigcap A = \mathrm{span}(\mathcal{B}_1(\mathcal{D}, p)_+ \bigcap A_+) = \mathrm{span}(\mathrm{dom}(\tau)_+) = \mathrm{dom}(\tau).$$

Now we turn to the smooth subalgebras. The definitions show that for $k \in \mathbb{Z}^p$, and a homogeneous element $a_m \in A_m$, we have

$$\delta(a_m)\Phi_k = (|m+k| - |k|)a_m\Phi_k.$$

Since $\delta(a_m)$ is also homogenous of degree m, which follows since $|\mathcal{D}|$ is invariant, we find that for all $\alpha \in \mathbb{N}_0$

$$\delta^\alpha(a_m)\Phi_k = (|m+k| - |k|)^\alpha a_m\Phi_k.$$

Hence for $a = \sum_m a_m \in \mathcal{B}$ and $s > p$ we have

$$\begin{aligned}
(5.5)\quad & \mathrm{Tr}_\tau\Big((1+\mathcal{D}^2)^{-s/4}|\delta^\alpha(a)|^2(1+\mathcal{D}^2)^{-s/4}\Big)\\
&= \sum_{m,n,k\in\mathbb{Z}^p} (1+|k|^2)^{-s/2}\mathrm{Tr}_\tau\left(\Phi_k\delta^\alpha(a_m)^*\delta^\alpha(a_n)\Phi_k\right)\\
&= \sum_{m,n,k\in\mathbb{Z}^p} (|m+k|-|k|)^\alpha(|n+k|-|k|)^\alpha(1+|k|^2)^{-s/2}\mathrm{Tr}_\tau\big(\Phi_k a_m^* a_n\Phi_k\big).
\end{aligned}$$

Now, using $a_m\Phi_k = \Phi_{m+k}a_m$ for $a_m \in A_m$ we have

$$\Phi_k a_m^* a_n\Phi_k = a_m^* a_n\Phi_{k-n+m}\Phi_k = \delta_{n,m}a_m^* a_n\Phi_k.$$

Inserting this equality into the last line of Equation (5.5) yields

$$\begin{aligned}
&\sum_{m,k\in\mathbb{Z}^p} \big||m+k|-|k|\big|^{2\alpha}(1+|k|^2)^{-s/2}\mathrm{Tr}_\tau\left(a_m^* a_m\Phi_k\right)\\
&\qquad\le \sum_{k\in\mathbb{Z}^p}(1+|k|^2)^{-s/2}\sum_{m\in\mathbb{Z}^p}|m|^{2\alpha}\mathrm{Tr}_\tau\left(a_m^* a_m\Phi_k\right)\\
&\qquad\le 2^{\lfloor p/2\rfloor}\sum_{k\in\mathbb{Z}^p}(1+|k|^2)^{-s/2}\sum_{m\in\mathbb{Z}^p}|m|^{2\alpha}\tau\left(a_m^* a_m\right),
\end{aligned}$$

where we used Lemma 5.1 in the last step and the latter is finite by definition of $\mathcal{B}$. Since

$$\begin{aligned}
\mathcal{Q}_n(\delta^\alpha(a))^2 = \|\delta^\alpha(a)\|^2 &+ \mathrm{Tr}_\tau\Big((1+\mathcal{D}^2)^{-p/4-1/n}|\delta^\alpha(a)|^2(1+\mathcal{D}^2)^{-p/4-1/n}\Big)\\
&+ \mathrm{Tr}_\tau\Big((1+\mathcal{D}^2)^{-p/4-1/n}|\delta^\alpha(a)^*|^2(1+\mathcal{D}^2)^{-p/4-1/n}\Big),
\end{aligned}$$

we deduce that $\mathcal{B} \subset \mathcal{B}_2^\infty(\mathcal{D},p)$. Finally, for $m \in \mathbb{Z}^p$ and $a_m \in \mathcal{B}$ homogenous of degree m, we have

$$[\mathcal{D}, a_m] = a_m\,\mathrm{Id}_{\mathcal{H}_1}\otimes\gamma(im).$$

Then by the same arguments as above, we deduce that $[\mathcal{D}, a_m] \in \mathcal{B}_2(\mathcal{D},p)$, and thus $[\mathcal{D},\mathcal{B}] \subset \mathcal{B}_2(\mathcal{D},p)$. By combining the estimates for $[\mathcal{D},a]$ and $\delta^\alpha(a)$, we see that $\mathcal{B}\cup[\mathcal{D},\mathcal{B}] \subset \mathcal{B}_2^\infty(\mathcal{D},p)$.

Now let $a = \sum_m a_m \in \mathcal{C}$, so that, in particular, $|a_m|, |a_m^*| \in \mathrm{dom}(\tau)$. Then $v_m|a_m|^{1/2}$, $|a_m|^{1/2} \in (\mathrm{dom}(\tau))^{1/2} \subset \mathcal{B}_2(\mathcal{D},p)$ where $a_m = v_m|a_m|$ is the polar decomposition in $\mathcal{N}$.

To deal with smooth summability, we need another operator inequality. For $a_m \in A_m$, $k \in \mathbb{Z}^p$ we have the simple computation

$$\begin{aligned}
&\delta^\alpha(a_m)^*\delta^\alpha(a_m)\Phi_k = (-1)^\alpha\delta^\alpha(a_m^*)\delta^\alpha(a_m)\Phi_k\\
&\quad= (-1)^\alpha(|k|-|m+k|)^\alpha(|m+k|-|k|)^\alpha a_m^* a_m\Phi_k = (|m+k|-|k|)^{2\alpha}a_m^* a_m\Phi_k.
\end{aligned}$$

Since $0 \leq (|m+k|-|k|)^{2\alpha} \leq |m|^{2\alpha}$ for all $k \in \mathbb{Z}^p$, we deduce that

$$0 \leq \delta^\alpha(a_m)^*\delta^\alpha(a_m) \leq |m|^{2\alpha} a_m^* a_m.$$

With this inequality in hand, and using $a \in \mathcal{C}$, we use the polar decomposition as above to see that for all $\alpha \in \mathbb{N}_0$, the decomposition

$$\delta^\alpha(a) = \sum_m \delta^\alpha(a_m) = \sum_m v_{\alpha,m}|\delta^\alpha(a_m)|^{1/2}\,|\delta^\alpha(a_m)|^{1/2} \in \mathcal{B}_1(\mathcal{D},p),$$

gives a representation of $\delta^\alpha(a_m)$ as an element of $\mathcal{B}_1(\mathcal{D},p)$. To see this we first check that $|\delta^\alpha(a_m)|^{1/2} \in \mathcal{B}_2(\mathcal{D},p)$, which follows from

$$\begin{aligned}
&\mathrm{Tr}_\tau\Big((1+\mathcal{D}^2)^{-p/4-1/n}|\delta^\alpha(a_m)|(1+\mathcal{D}^2)^{-p/4-1/n}\Big)\\
&= \sum_{k\in\mathbb{Z}^p}(1+k^2)^{-p/2-1/2n}\,\mathrm{Tr}_\tau(\Phi_k\sqrt{\delta^\alpha(a_m)^*\delta^\alpha(a_m)}\Phi_k)\\
&\leq \sum_{k\in\mathbb{Z}^p}(1+k^2)^{-p/2-1/2n}|m|^\alpha\tau(\sqrt{a_m^*a_m}) = |m|^\alpha\,\tau(|a_m|)\sum_{k\in\mathbb{Z}^p}(1+k^2)^{-p/2-1/2n}.
\end{aligned}$$

Since

$$\big(v_{\alpha,m}|\delta^\alpha(a_m)|^{1/2}\big)^* v_{\alpha,m}|\delta^\alpha(a_m)|^{1/2} = |\delta^\alpha(a_m)|,$$

the corresponding term is handled in the same way. Finally we have

$$\begin{aligned}
&\mathrm{Tr}_\tau\Big((1+\mathcal{D}^2)^{-p/4-1/n}v_{\alpha,m}|\delta^\alpha(a_m)|v_{\alpha,m}^*(1+\mathcal{D}^2)^{-p/4-1/n}\Big)\\
&\quad= \sum_{k\in\mathbb{Z}^p}(1+k^2)^{-p/2-1/2n}\,\mathrm{Tr}_\tau(\Phi_k v_{\alpha,m}|\delta^\alpha(a_m)|v_{\alpha,m}^*\Phi_k)\\
&\quad= \sum_{k\in\mathbb{Z}^p}(1+k^2)^{-p/2-1/2n}\mathrm{Tr}_\tau(|\delta^\alpha(a_m)|^{1/2}v_{\alpha,m}^*\Phi_k v_{\alpha,m}|\delta^\alpha(a_m)|^{1/2})\\
&\quad= \sum_{k\in\mathbb{Z}^p}(1+k^2)^{-p/2-1/2n}\mathrm{Tr}_\tau(|\delta^\alpha(a_m)|^{1/2}\Phi_{k-m}v_{\alpha,m}^*v_{\alpha,m}|\delta^\alpha(a_m)|^{1/2})\\
&\quad= \sum_{k\in\mathbb{Z}^p}(1+k^2)^{-p/2-1/2n}\mathrm{Tr}_\tau(|\delta^\alpha(a_m)|^{1/2}\Phi_{k-m}v_{\alpha,m}^*v_{\alpha,m}\Phi_{k-m}|\delta^\alpha(a_m)|^{1/2})\\
&\quad\leq \sum_{k\in\mathbb{Z}^p}(1+k^2)^{-p/2-1/2n}\mathrm{Tr}_\tau(|\delta^\alpha(a_m)|^{1/2}\Phi_{k-m}|\delta^\alpha(a_m)|^{1/2})\\
&\quad= \sum_{k\in\mathbb{Z}^p}(1+k^2)^{-p/2-1/2n}\,\mathrm{Tr}_\tau(\Phi_{k-m}|\delta^\alpha(a_m)|\Phi_{k-m})\\
&\quad\leq \sum_{k\in\mathbb{Z}^p}(1+k^2)^{-p/2-1/2n}|m|^\alpha\,\mathrm{Tr}_\tau(\Phi_{k-m}|a_m|\Phi_{k-m})\\
&\quad\leq |m|^\alpha\,\tau(|a_m|)\sum_{k\in\mathbb{Z}^p}(1+k^2)^{-p/2-1/2n}.
\end{aligned}$$

In the third equality we again used $v_{\alpha,m}^*\Phi_k = \Phi_{k-m}v_{\alpha,m}^*$, which is true since $\delta^\alpha(a_m)$ is homogenous of degree m and $|\delta^\alpha(a_m)|$ is homogenous of degree zero. In the forth equality, we used this again for both $v_{\alpha,m}$ and $v_{\alpha,m}^*$. In the last equality, we again used this trick, and the fact that $|\delta^\alpha(a_m)|$ is homogenous of degree zero. The last two inequalities follow just as above. So $\mathcal{Q}_n(|\delta^\alpha(a_m)|^{1/2})$ is bounded by

$$|m|^{\alpha/2}(\|a_m\|+\tau(|a_m|)+\tau(|a_m^*|))^{1/2}\Big(\sum_{k\in\mathbb{Z}^p}(1+k^2)^{-p/2-1/2n}\Big)^{1/2} = |m|^{\alpha/2}(\|a_m\|+2\tau(|a_m|))^{1/2}\Big(\sum_{k\in\mathbb{Z}^p}(1+k^2)^{-p/2-1/2n}\Big)^{1/2},$$

and similarly for $v_{\alpha,m}|\delta^\alpha(a_m)|^{1/2}$. Hence,

$$\begin{aligned}\mathcal{P}_{n,\beta}(a) &\le \sum_{\alpha=0}^{\beta}\sum_m \mathcal{Q}_n(v_{\alpha,m}|\delta^\alpha(a_m)|^{1/2})\,\mathcal{Q}_n(|\delta^\alpha(a_m)|^{1/2})\\ &\le \sum_{k\in\mathbb{Z}^p}(1+k^2)^{-p/2-1/2n}\sum_{\alpha=0}^{\beta}\sum_m |m|^\alpha(\|a_m\|+2\tau(|a_m|)),\end{aligned}$$

which is enough to show that $\delta^\alpha(a)\in\mathcal{B}_1(\mathcal{D},p)$. Since similar arguments show that $\delta^\alpha([\mathcal{D},a])\in\mathcal{B}_1(\mathcal{D},p)$, we see that $\mathcal{C}\cup[\mathcal{D},\mathcal{C}]\subset\mathcal{B}_1^\infty(\mathcal{D},p)$.

The computation of the zeta function is straightforward using Lemma 5.1, once one realises that $\sum_{k\in\mathbb{Z}^p}(1+k^2)^{-p/2-z}$ is just $(2\pi)^p$ times the trace of the Laplacian on a flat torus. This precise value of the residue can be deduced from the Dixmier trace calculation for the torus in [**30**, Example 7.1, p. 291], and the relationship between residues of zeta functions and Dixmier traces in [**18**, Lemma 5.1]. This also proves that the spectral dimension is p. $\square$

Semifinite spectral triples for more general compact group actions on C^*-algebras have been constructed in [**60**]. These spectral triples are shown to satisfy some summability conditions, but it is not immediately clear that they satisfy our definition of smooth summability. We leave this investigation to another place.

For torus actions we can give a simple description of the index formula. First we observe that elementary Clifford algebra considerations, [**3**, Appendix] and [**42**,**43**], reduce the resolvent cocycle to a single term in degree p. This means that we automatically obtain the analytic continuation of the single zeta function which arises, and so the spectral dimension is isolated, and there is at worst a simple pole at $r=(1-p)/2$. Hence, the residue cocycle is given by the single functional, defined on $a_0,\dots,a_p\in\mathcal{C}$ by

$$\phi_p(a_0,\dots,a_p)=\begin{cases}\sqrt{2i\pi}\,\frac{1}{p!}\,\mathrm{res}_{s=0}\mathrm{Tr}_\tau\Big(a_0\,[\mathcal{D},a_1]\dots[\mathcal{D},a_p](1+\mathcal{D}^2)^{-p/2-s}\Big), & p \text{ odd},\\[2ex] \frac{1}{p!}\,\mathrm{res}_{s=0}\mathrm{Tr}_\tau\Big(\gamma a_0\,[\mathcal{D},a_1]\dots[\mathcal{D},a_p](1+\mathcal{D}^2)^{-p/2-s}\Big), & p \text{ even}.\end{cases}$$

Applications of this formula to graph and k-graph algebras appear in [**42**,**43**]. Both these papers show that the index is sensitive to the group action, by presenting an algebra with two different actions of the same group which yield different indices.

5.2. Moyal plane

5.2.1. Definition of the Moyal product. Recall that the Moyal product of a pair of functions (or distributions) f, g on $\mathbb{R}^{2d}$, is given by

$$f\star_\theta g(x) := (\pi\theta)^{-2d}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{\frac{2i}{\theta}\omega_0(x-y,x-z)}f(y)g(z)\,dy\,dz. \tag{5.6}$$

The parameter θ lies in $\mathbb{R} \setminus \{0\}$ and plays the role of the Planck constant. The quadratic form ω_0 is the canonical symplectic form of $\mathbb{R}^{2d} \simeq T^*\mathbb{R}^d$. With basic Fourier analysis one shows that the Schwartz space, $\mathcal{S}(\mathbb{R}^{2d})$, endowed with this product is a (separable) Fréchet $*$-algebra with jointly continuous product (the involution being given by the complex conjugation). For instance, when $f, g \in \mathcal{S}(\mathbb{R}^{2d})$, we have the relations

$$\int_{\mathbb{R}^2} f \star_\theta g(x)\, dx = \int_{\mathbb{R}^2} f(x)\, g(x)\, dx, \tag{5.7}$$
$$\partial_j(f \star_\theta g) = \partial_j(f) \star_\theta g + f \star_\theta \partial_j(g), \qquad \overline{f \star_\theta g} = \overline{g} \star_\theta \overline{f}.$$

This noncommutative product is nothing but the composition law of symbols in the framework of the Weyl pseudodifferential calculus on $\mathbb{R}^d$. Indeed, let Op_W be the Weyl quantization map:

$$\mathrm{Op}_W : T \in S'(\mathbb{R}^{2d}) \mapsto \Big[\varphi \in S(\mathbb{R}^d) \mapsto$$
$$\big[q_0 \in \mathbb{R}^d \mapsto (2\pi)^{-d} \int_{\mathbb{R}^{2d}} T\big((q_0+q)/2, p\big)\varphi(q_0) e^{i(q_0-q)p}\, d^dq\, d^dp\big] \quad \in S'(\mathbb{R}^d)\Big].$$

Again, Fourier analysis shows that Op_W restricts to a unitary operator from the Hilbert space $L^2(\mathbb{R}^{2d})$ (the L^2-symbols) to the Hilbert space of Hilbert-Schmidt operators acting on $L^2(\mathbb{R}^d)$, with

$$\|\mathrm{Op}_W(f)\|_2 = (2\pi)^{-d/2} \|f\|_2\,, \tag{5.8}$$

where the first 2-norm is the Hilbert-Schmidt norm on $L^2(\mathbb{R}^d)$ while the second is the Lebesgue 2-norm on $L^2(\mathbb{R}^{2d})$. Thus, the algebra $(L^2(\mathbb{R}^{2d}), \star_\theta)$ turns out to be a full Hilbert-algebra. It is then natural to use the GNS construction (associated with the operator trace on $L^2(\mathbb{R}^d)$ in the operator picture, or with the Lebesgue integral in the symbolic picture) to represent this algebra. To keep track of the dependence on the deformation parameter θ, the left regular representation is denoted by L^θ. With this notation we have (see [**27**, Lemma 2.12])

$$L^\theta(f)g := f \star_\theta g, \quad \|L^\theta(f)\| \le (2\pi\theta)^{-d/2}\|f\|_2, \qquad f, g \in L^2(\mathbb{R}^{2d}). \tag{5.9}$$

Note the singular nature of this estimate in the commutative $\theta \to 0$ limit. Since the operator norm of a bounded operator on a Hilbert space $\mathcal{H}$ coincides (via the left regular representation) with the operator norm of the same bounded operator acting by left multiplication on the Hilbert space $\mathcal{L}^2(\mathcal{B}(\mathcal{H}))$ of Hilbert-Schmidt operators, we have

$$\|L^\theta(f)\| = (2\pi)^{d/2} \|\mathrm{Op}_W(f)\|, \tag{5.10}$$

where the first norm is the operator norm on $L^2(\mathbb{R}^{2d})$ and the second is the operator norm on $L^2(\mathbb{R}^d)$. In particular, the Weyl quantization gives the identification of von Neumann algebras:

$$\mathcal{B}\big(L^2(\mathbb{R}^{2d})\big) \supset \big\{L^\theta(f),\, f \in L^2(\mathbb{R}^{2d})\big\}'' \simeq \mathcal{B}\big(L^2(\mathbb{R}^2)\big). \tag{5.11}$$

The following Hilbert-Schmidt norm equality on $L^2(\mathbb{R}^{2d})$, is proven in [**27**, Lemma 4.3] (this is the analogue of Lemma 4.6 in this context):

$$\|L^\theta(f)g(\nabla)\|_2 = (2\pi)^{-d} \|g\|_2 \|f\|_2. \tag{5.12}$$

Note the independence of θ on the right hand side.

5.2.2. A smoothly summable spectral triple for the Moyal plane. In this paragraph, we generalize the result of [**27**]. For simplicity, we restrict ourself to the simplest $d = 2$ case, despite the fact that our analysis can be carried out in any even dimension. Here we let $\mathcal{H} := L^2(\mathbb{R}^2) \otimes \mathbb{C}^2$ be the Hilbert space of square integrable sections of the trivial spinor bundle on $\mathbb{R}^2$. In Cartesian coordinates, the flat Dirac operator reads

$$\mathcal{D} := \begin{pmatrix} 0 & i\partial_1 - \partial_2 \\ i\partial_1 + \partial_2 & 0 \end{pmatrix}.$$

Elements of the algebra $(\mathcal{S}(\mathbb{R}^2), \star_\theta)$ are represented on $\mathcal{H}$ via $L^\theta \otimes \mathrm{Id}_2$, the diagonal left regular representation. In [**27**], it is proven that $\big((\mathcal{S}(\mathbb{R}^2), \star_\theta), \mathcal{H}, \mathcal{D}\big)$ is an even QC^∞ finitely summable spectral triple with spectral dimension 2 and with grading

$$\gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

In particular, the Leibniz rule in the first display of Equation (5.7) gives

$$[\mathcal{D}, L^\theta(f) \otimes \mathrm{Id}_2] = \begin{pmatrix} 0 & iL^\theta(\partial_1 f) - L^\theta(\partial_2 f) \\ iL^\theta(\partial_1 f) + L^\theta(\partial_2 f) & 0 \end{pmatrix}, \tag{5.13}$$

which together with (5.9) shows that for f a Schwartz function the commutator $[\mathcal{D}, L^\theta(f) \otimes \mathrm{Id}_2]$ extends to a bounded operator.

Then, from the Hilbert-Schmidt norm computation of Equation (5.12), we can determine the weights φ_s of Definition 1.1, constructed with the flat Dirac operator on $\mathbb{R}^2$.

LEMMA 5.5. *For $s > 2$, let φ_s be the faithful normal semifinite weight of Definition 1.1 determined by $\mathcal{D}$ on the type I von Neumann algebra $\mathcal{B}(\mathcal{H})$ with operator trace. When restricted to the von Neumann subalgebra of $\mathcal{B}(\mathcal{H})$ generated by $L^\theta(f) \otimes \mathrm{Id}_2$, φ_s is a tracial weight and for $f \in L^2(\mathbb{R}^2)$ we have*

$$\varphi_s\big(L^\theta(f)^* L^\theta(f) \otimes \mathrm{Id}_2\big) = (\pi(s-2))^{-1} \int_{\mathbb{R}^2} \bar{f}(x) \star_\theta f(x) dx = 2(s-2)^{-1} \|\mathrm{Op}_W(f)\|_2^2.$$

PROOF. Since $\mathcal{D}^2 = \Delta \otimes \mathrm{Id}_2$, with $0 \le \Delta$ the usual Laplacian on $\mathbb{R}^2$, we have

$$\varphi_s\big(L^\theta(f)^* L^\theta(f) \otimes \mathrm{Id}_2\big) = 2\mathrm{Tr}_{L^2(\mathbb{R}^2)}\big((1+\Delta)^{-s/4} L^\theta(f)^* L^\theta(f)(1+\Delta)^{-s/4}\big).$$

Thus, the result follows from Equations (5.7), (5.8) and (5.12). □

We turn now to the question of which elements of the von Neumann algebra generated by $L^\theta(f) \otimes \mathrm{Id}_2$ are in $\mathcal{B}_1^\infty(\mathcal{D}, 2)$. The next result follows by combining Proposition 1.19 with Lemma 5.5.

COROLLARY 5.6. *Identifying the von Neumann subalgebra of $\mathcal{B}(L^2(\mathbb{R}^2))$ generated by $L^\theta(f) \otimes \mathrm{Id}_2$, $f \in L^2(\mathbb{R}^2)$, with $\mathcal{B}(L^2(\mathbb{R}))$ as in Equation (5.11) yields the identifications*

$$\mathcal{B}_1(\mathcal{D}, 2) \bigcap \mathcal{B}(L^2(\mathbb{R})) \simeq L^2(\mathbb{R}^2) \star_\theta L^2(\mathbb{R}^2) \simeq \mathcal{L}^1\big(L^2(\mathbb{R})\big).$$

Moreover, for all $m \in \mathbb{N}$, the norms on $L^2(\mathbb{R}^2) \star_\theta L^2(\mathbb{R}^2)$

$$f \mapsto \mathcal{P}_m\big(L^\theta(f) \otimes \mathrm{Id}_2\big),$$

are equivalent to the single norm

$$f \mapsto \|\mathrm{Op}_W(f)\|_1.$$

Proof. The identification $L^2(\mathbb{R}^2) \star_\theta L^2(\mathbb{R}^2) \simeq \mathcal{L}^1\big(L^2(\mathbb{R})\big)$ follows from the identification $L^2(\mathbb{R}^2) \simeq \mathcal{L}^2\big(L^2(\mathbb{R})\big)$ given by the unitarity of the Weyl quantization map, and the equality

$$\mathcal{L}^2\big(L^2(\mathbb{R})\big) \cdot \mathcal{L}^2\big(L^2(\mathbb{R})\big) = \mathcal{L}^1\big(L^2(\mathbb{R})\big).$$

By Proposition 1.19 we know that $\mathcal{B}_1(\mathcal{D}, 2) \bigcap \mathcal{B}(L^2(\mathbb{R}))$ is identified with

$$\bigcap_{n\geq 1} \mathcal{L}^1\big(\mathcal{B}(L^2(\mathbb{R})), \varphi_{2+1/n}\big).$$

Lemma 5.5 says that restricted to $\mathcal{B}(L^2(\mathbb{R}))$, all the weights $\varphi_{2+1/n}$ are proportional to the operator trace of $\mathcal{B}(L^2(\mathbb{R}))$. This gives the final identification. Moreover, Proposition 1.19 also gives the equality

$$\mathcal{P}_n(.) = 2\|\cdot\|_{\tau_n} + \|\cdot\|,$$

where $\|\cdot\|_{\tau_n}$ is the trace norm associated to the tracial weight $\varphi_{2+1/n}$ restricted to $\mathcal{B}(L^2(\mathbb{R}))$. As the latter is proportional to the operator trace on $\mathcal{B}(L^2(\mathbb{R}))$, which dominates the operator norm since we are in the I_∞ factor case, we get the equivalence of the norms

$$f \mapsto \mathcal{P}_n\big(L^\star(f) \otimes \mathrm{Id}_2\big),\ n \in \mathbb{N}, \quad \text{and} \quad \|\mathrm{Op_W}(f)\|_1,$$

and we are done. $\square$

On the basis of the previous result, we construct a Fréchet algebra yielding a smoothly summable spectral triple of spectral dimension 2, for the Moyal product.

Lemma 5.7. *Endowed with the set of seminorms*

$$f \mapsto \|f\|_{1,\alpha} := \|\mathrm{Op}_W(\partial^\alpha f)\|_1, \quad \alpha \in \mathbb{N}_0^2,$$

the set

$$\mathcal{A} := \big\{f \in C^\infty(\mathbb{R}^2)\ :\ \forall n = (n_1, n_2) \in \mathbb{N}_0^2,\ \exists f_1, f_2 \in L^2(\mathbb{R}^2)\ :\ \partial_1^{n_1}\partial_2^{n_2} f = f_1 \star_\theta f_2\big\},$$

is a Fréchet algebra for the Moyal product.

Proof. From the Leibniz rule for the Moyal product (see Equation (5.7) second display) and the fact that $L^2(\mathbb{R}^2)\star_\theta L^2(\mathbb{R}^2) \subset L^2(\mathbb{R}^2)$, the set $\mathcal{A}$ is an algebra for the Moyal product. Since $L^2(\mathbb{R}^2) \star_\theta L^2(\mathbb{R}^2) \simeq \mathcal{L}^1\big(L^2(\mathbb{R})\big)$, the seminorms $\|\cdot\|_{1,\alpha}$, $\alpha \in \mathbb{N}_0^2$, take finite values on $\mathcal{A}$. It remains to show that $\mathcal{A}$ is complete for the topology induced by these seminorms. So let $(f_k)_{k\in\mathbb{N}}$ be a Cauchy sequence on $\mathcal{A}$, i.e. Cauchy for each seminorm $\|\cdot\|_{1,\alpha}$. Since $\mathcal{L}^1(L^2(\mathbb{R}))$ is complete, for each $\alpha \in \mathbb{N}_0^2$, $\big(\mathrm{Op}_W(\partial^\alpha f_k)\big)_{k\in\mathbb{N}}$ converges to A_α, a trace-class operator on $L^2(\mathbb{R})$. But since $\mathcal{L}^1(L^2(\mathbb{R})) \simeq L^2(\mathbb{R}^2) \star_\theta L^2(\mathbb{R}^2)$, via the Weyl map, $A_\alpha = \mathrm{Op}_W(f_\alpha)$ for some element $f_\alpha \in L^2(\mathbb{R}^2) \star_\theta L^2(\mathbb{R}^2)$. In particular, for $\alpha = (0, 0)$, the sequence $(f_k)_{k\in\mathbb{N}}$ converges to an element $f \in L^2(\mathbb{R}^2) \star_\theta L^2(\mathbb{R}^2)$. But we need to show that $f \in \mathcal{A}$, that is, we need to show that $\|\mathrm{Op}_W(\partial^\alpha f)\|_1 < \infty$ for all $\alpha \in \mathbb{N}_0^2$. This will be the case if $\partial^\alpha f = f_\alpha$. Note that $f \in L^2(\mathbb{R}^2) \star_\theta L^2(\mathbb{R}^2) \subset L^2(\mathbb{R}^2) \subset \mathcal{S}'(\mathbb{R}^2)$, so that $\partial^\alpha f \in \mathcal{S}'(\mathbb{R}^2)$ too. With $\langle\cdot|\cdot\rangle$ denoting the duality brackets $\mathcal{S}'(\mathbb{R}^2) \times \mathcal{S}(\mathbb{R}^2) \to \mathbb{C}$, we have for any $k \in \mathbb{N}$ and any $\psi \in \mathcal{S}(\mathbb{R}^2)$

$$\begin{aligned}
\big|\langle(\partial^\alpha f - f_\alpha)|\psi\rangle\big| &= \big|\langle(\partial^\alpha f - \partial^\alpha f_k)|\psi\rangle - \langle(f_\alpha - \partial^\alpha f_k)|\psi\rangle\big| \\
&= \big|(-1)^{|k|}\langle(f - f_k)|\partial^\alpha\psi\rangle - \langle(f_\alpha - \partial^\alpha f_k)|\psi\rangle\big| \\
&\leq \|f - f_k\|_2\,\|\partial^\alpha\psi\|_2 + \|f_\alpha - \partial^\alpha f_k\|_2\,\|\psi\|_2 \\
&= (2\pi)^{1/2}\|\partial^\alpha\psi\|_2\,\|\mathrm{Op}_W(f - f_k)\|_2 + (2\pi)^{1/2}\|\psi\|_2\,\|\mathrm{Op}_W(f_\alpha - \partial^\alpha f_k)\|_2,
\end{aligned}$$

where we have used Equation (5.12). Now, since the trace-norm dominates the Hilbert-Schmidt norm, we find

$$\big|\langle(\partial^\alpha f - f_\alpha)|\psi\rangle\big| \leq C(\psi)\big(\|\mathrm{Op}_W(f) - \mathrm{Op}_W(f_k)\|_1 + \|\mathrm{Op}_W(f_\alpha) - \mathrm{Op}_W(\partial^\alpha f_k)\|_1\big).$$

However, since $\mathrm{Op}_W(\partial^\alpha f_k) \to \mathrm{Op}_W(f_\alpha)$ in the trace-norm for all $\alpha \in \mathbb{N}_0^2$, we see that $\big|\langle(\partial^\alpha f - f_\alpha)|\psi\rangle\big| \leq \varepsilon$ for all $\varepsilon > 0$ and thus $\langle(\partial^\alpha f - f_\alpha)|\psi\rangle = 0$ for all $\psi \in \mathcal{S}(\mathbb{R}^2)$. Hence, $\partial^\alpha f = f_\alpha$ in $\mathcal{S}'(\mathbb{R}^2)$, but since $f_\alpha \in L^2(\mathbb{R}^2) \star_\theta L^2(\mathbb{R}^2)$, $\partial^\alpha f \in L^2(\mathbb{R}^2) \star_\theta L^2(\mathbb{R}^2)$ too. This completes the proof. □

Remark. Note that the C^*-completion of $(\mathcal{A}, \star_\theta)$, is isomorphic to the C^*-algebra of compact operators acting on $L^2(\mathbb{R})$ and that $\mathcal{A}$ contains $\mathcal{S}(\mathbb{R}^2)$.

Combining all these preliminary statements, we now improve the results of [**27**].

Proposition 5.8. *The data $(\mathcal{A}, \mathcal{H}, \mathcal{D}, \gamma)$ defines an even smoothly summable spectral triple with spectral dimension 2.*

Proof. We first need to prove that $(\mathcal{A}, \mathcal{H}, \mathcal{D}, \gamma)$ (which is even) is finitely summable, that is, we need to show that

$$\delta^k\big(L^\theta(f) \otimes \mathrm{Id}_2\big)\,(1 + \mathcal{D}^2)^{-s/2} \in \mathcal{L}^1(\mathcal{H}),\ \text{for all } f \in \mathcal{A},\ \text{for all } s > 2,\ \text{for all } k \in \mathbb{N}_0.$$

This will follow from the proof of Proposition 2.21, if

$$(1 + \mathcal{D}^2)^{-s/4} R^k\big(L^\theta(f) \otimes \mathrm{Id}_2\big)(1 + \mathcal{D}^2)^{-s/4} \in \mathcal{L}^1(\mathcal{N}, \tau),$$

for all $f \in \mathcal{A}$, $s > 2$ and $k \in \mathbb{N}_0$. Now, by the Leibniz rule (Equation 5.7 first display), with $\Delta = -\partial_1^2 - \partial_2^2$, we have

$$[\Delta, L^\theta(f)] = L^\theta(\Delta f) + 2L^\theta(\partial_1 f)\partial_1 + 2L^\theta(\partial_2 f)\partial_2,$$

so that since $\mathcal{D}^2 = \Delta \otimes \mathrm{Id}_2$, for all $k \in \mathbb{N}_0$, we have

$$R^k\big(L^\theta(f) \otimes \mathrm{Id}_2\big) = \sum_{|\alpha|,|\beta| \leq k} C_{\alpha,\beta} L^\theta(\partial^\alpha f)\partial^\beta(1 + \Delta)^{-k/2} \otimes \mathrm{Id}_2,$$

and thus,

$$\begin{aligned}
&(1 + \mathcal{D}^2)^{-s/4} R^k\big(L^\theta(f) \otimes \mathrm{Id}_2\big)(1 + \mathcal{D}^2)^{-s/4} \\
&\quad = \sum_{|\alpha|,|\beta| \leq k} C_{\alpha,\beta}(1 + \Delta)^{-s/4} L^\theta(\partial^\alpha f)(1 + \Delta)^{-s/4}\partial^\beta(1 + \Delta)^{-k/2} \otimes \mathrm{Id}_2,
\end{aligned}$$

which is trace class because $\partial^\beta(1 + \Delta)^{-k/2}$ is bounded and by definition of $\mathcal{A}$, $\partial^\alpha f = f_1 \star_\theta f_2$ with $f_1, f_2 \in L^2(\mathbb{R}^2)$, so that this operator appears as the product of two Hilbert-Schmidt by Equation (5.12). Thus, the spectral triple is finitely summable, and the spectral dimension is 2 by [**27**, Lemma 4.14], which gives for any $f \in \mathcal{A}$

$$\mathrm{Tr}\big(L^\theta(f) \otimes \mathrm{Id}_2(1 + \mathcal{D}^2)^{-s/2}\big) = \frac{1}{\pi(s - 2)} \int_{\mathbb{R}^2} f(x)\,dx.$$

From Proposition 2.21, we also have verified one of the condition ensuring that $\mathcal{A} \cup [\mathcal{D}, \mathcal{A}] \subset \mathcal{B}_1^\infty(\mathcal{D}, 2)$. The second is to verify that for all $s > p$

$$(1+\mathcal{D}^2)^{-s/4} R^k\big([\mathcal{D}, L^\theta(f) \otimes \mathrm{Id}_2]\big)(1+\mathcal{D}^2)^{-s/4} \in \mathcal{L}^1(\mathcal{N}, \tau), \text{ for all } k \in \mathbb{N}_0.$$

This can be done as for $R^k\big(L^\theta(f) \otimes \mathrm{Id}_2\big)$ by noticing that

$$R^k\big([\mathcal{D}, L^\theta(f) \otimes \mathrm{Id}_2]\big) = \sum_{|\alpha| \le k} \sum_{|\beta_1|, |\beta_2| \le k+1} C_{\alpha,\beta_1,\beta_2} \begin{pmatrix} 0 & L^\theta(\partial^{\beta_1} f) \\ L^\theta(\partial^{\beta_2} f) & 0 \end{pmatrix} \partial^\alpha (1+\Delta)^{-k/2} \otimes \mathrm{Id}_2,$$

and the proof is complete. □

5.2.3. An index formula for the Moyal plane. In order to obtain an explicit index formula out of the spectral triple previously constructed, we need to introduce a suitable family of projectors.

Let $H := \frac{1}{2}(x_1^2 + x_2^2)$, $x_1, x_2 \in \mathbb{R}$, be the (classical) Hamiltonian of the one-dimensional harmonic oscillator. Let also $a := 2^{-1/2}(x_1 + ix_2)$, $\bar{a} := 2^{-1/2}(x_1 - ix_2)$ be the annihilation and creation functions. Define next

$$f_{m,n} := \frac{1}{\sqrt{\theta^{n+m} n! m!}} \, \bar{a}^{\star_\theta m} \star_\theta f_{0,0} \star_\theta a^{\star_\theta n}, \quad \text{where} \quad f_{0,0} := 2e^{-\frac{2}{\theta} H}, \qquad m, n \in \mathbb{N}_0.$$

The family $\{f_{m,n}\}_{m,n \in \mathbb{N}_0}$ is an orthogonal basis of $L^2(\mathbb{R}^2)$, consisting of Schwartz functions. They constitute an important tool in the analysis of [**27**], since they allow to construct local units. In fact, they are the Weyl symbols of the rank one operators $\varphi \mapsto \langle \varphi_m | \varphi \rangle \, \varphi_n$, with $\{\varphi_n\}_{n \in \mathbb{N}_0}$ the basis of $L^2(\mathbb{R})$ consisting of eigenvectors for the one-dimensional quantum harmonic oscillator. The proof of the next lemma can be found in [**27**, subsection 2.3 and Appendix].

Lemma 5.9. *The following relations hold true.*

$$\overline{f_{m,n}} = f_{n,m}, \qquad f_{m,n} \star_\theta f_{k,l} = \delta_{n,k} \, f_{m,l}, \qquad \int_{\mathbb{R}^2} f_{m,n}(x)\, dx = 2\pi\theta \, \delta_{m,n},$$

so, in particular, $\{f_{n,n}\}_{n \in \mathbb{N}_0}$, is a family of pairwise orthogonal projectors. Moreover, we have

$$\big[\mathcal{D}, L^\theta(f_{m,n}) \otimes \mathrm{Id}_2\big] = -i\sqrt{\frac{2}{\theta}} \begin{pmatrix} 0 & \sqrt{m}\, L^\theta(f_{m-1,n}) - \sqrt{n+1}\, L^\theta(f_{m,n+1}) \\ \sqrt{n}\, L^\theta(f_{m,n-1}) - \sqrt{m+1}\, L^\theta(f_{m+1,n}) & 0 \end{pmatrix},$$

with the convention that $f_{m,n} \equiv 0$ whenever $n < 0$ or $m < 0$.

We are in the situation where the projectors $f_{n,n}$ belong to the algebra (not its unitization, nor a matrix algebra over it). Thus, if we set $F = \mathcal{D}(1+\mathcal{D}^2)^{-1/2}$, then $L^\theta(f_{n,n}) F_\pm L^\theta(f_{n,n})$ is a Fredholm operator from $L^2(\mathbb{R}^2)$ to itself, according to the discussion at the beginning of the Section 2.3. Thus, we do not need the 'double picture' here. In particular, $[f_{n,n}] \in K_0(\mathcal{A})$. The next result computes the numerical index pairing between $(\mathcal{A}, L^2(\mathbb{R}^2, \mathbb{C}^2), \mathcal{D})$ and $K_0(\mathcal{A})$.

Proposition 5.10. *For J a finite subset of $\mathbb{N}_0$, let $p_J := \sum_{n \in J} L^\theta(f_{n,n})$. Setting $F = \mathcal{D}(1+\mathcal{D}^2)^{-1/2}$, we have the integer-valued index paring*

$$\mathrm{Index}\big(p_J F_+ p_J\big) = \big\langle [p_J], [(\mathcal{A}, L^2(\mathbb{R}^2, \mathbb{C}^2), \mathcal{D})] \big\rangle = \mathrm{Card}(J).$$

In particular, the index map gives an explicit isomorphism between $K_0\big(\mathcal{K}(L^2(\mathbb{R}))\big)$ and $\mathbb{Z}$.

Proof. Assume first that $J = \{n\}$, $n \in \mathbb{N}_0$. The degree zero term is not zero in this case as the projection lies in our algebra. Hence, including all the constants from the local index formula and the Chern character of $f_{n,n}$ gives

$$\mathrm{Index}\big(L^\theta(f_{n,n})F_+L^\theta(f_{n,n})\big) = \mathrm{res}_{z=0}\Big(\frac{1}{z}\mathrm{Tr}\big(\gamma L^\theta(f_{n,n})(1+\mathcal{D}^2)^{-z}\big) -$$
$$\mathrm{Tr}\Big(\gamma\big(L^\theta(f_{n,n})\otimes \mathrm{Id}_2 - 1/2\big)[\mathcal{D}, L^\theta(f_{n,n})\otimes \mathrm{Id}_2][\mathcal{D}, L^\theta(f_{n,n})\otimes \mathrm{Id}_2](1+\mathcal{D}^2)^{-1-z}\Big)\Big).$$

The second term is computed with the help of Lemma 5.9. First we have

$$\gamma\big(L^\theta(f_{n,n})\otimes \mathrm{Id}_2 - 1/2\big)[\mathcal{D}, L^\theta(f_{n,n})\otimes \mathrm{Id}_2][\mathcal{D}, L^\theta(f_{n,n})\otimes \mathrm{Id}_2]$$
$$= \frac{1}{\theta}\begin{pmatrix} n\,L^\theta(f_{n-1,n-1}) - (n+1)\,L^\theta(f_{n,n}) & 0 \\ 0 & -(n+1)\,L^\theta(f_{n+1,n+1}) + n\,L^\theta(f_{n,n})\end{pmatrix}.$$

Since $\mathcal{D}^2 = \Delta\otimes \mathrm{Id}_2$, with here $\Delta = -\partial_1^2 - \partial_2^2$, we find that

$$\begin{aligned}
&\mathrm{Tr}\big(\gamma(L^\theta(f_{n,n})\otimes \mathrm{Id}_2 - 1/2)[\mathcal{D}, L^\theta(f_{n,n})]\otimes \mathrm{Id}_2[\mathcal{D}, L^\theta(f_{n,n})\otimes \mathrm{Id}_2](1+\mathcal{D}^2)^{-1-z}\big)\\
&\quad= \frac{1}{\theta}\mathrm{Tr}\Big(\big(-L^\theta(f_{n,n}) - (n+1)L^\theta(f_{n+1,n+1}) + nL^\theta(f_{n-1,n-1})\big)(1+\Delta)^{-1-z}\Big)\\
&\quad= \frac{1}{\theta}\frac{1}{(2\pi)^2}\int_{\mathbb{R}^2}\big(-f_{n,n}(x) - (n+1)f_{n+1,n+1}(x) + nf_{n-1,n-1}(x)\big)dx\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \times\int_{\mathbb{R}^2}(1+|\xi|^2)^{-1-z}\,d\xi\\
&\quad= \frac{1}{\theta}\frac{1}{(2\pi)^2}\big(-1-(n+1)+n\big)\,(2\pi\theta)\,\frac{2\pi}{2z} = -\frac{1}{z}.
\end{aligned}$$

In the second equality we have used [**27**, Lemma 4.14]. Note that the factor $(2\pi)^{-2}$ can also be deduced from (5.12). Finally, we have used Lemma 5.9 to obtain the last line and this is where the factor $2\pi\theta$ comes from. Thus, the residue from the second term gives us 1. For the first term we compute

$$\mathrm{res}_{z=0}\frac{1}{z}\mathrm{Tr}\big(\gamma L^\theta(f_{n,n})\otimes \mathrm{Id}_2\,(1+\mathcal{D}^2)^{-z}\big) = 0,$$

because the grading γ cancels the traces on each half of the spinor space. This gives the result in this elementary case, $\mathrm{Index}\big(L^\theta(f_{n,n})F_+L^\theta(f_{n,n})\big) = 1$. For the general case, note that, since for $n \neq m$, $f_{m,m}$ and $f_{n,n}$ are orthogonal projectors, we have $[f_{m,m} + f_{n,n}] = [f_{m,m}] + [f_{n,n}] \in K_0(\mathcal{A})$ and the final result follows immediately. Last, to see that the index paring is an isomorphism, it suffices to observe that if I and J are subsets of $\mathbb{N}_0$ with the same cardinality, then $[P_I] = [P_J] \in K_0(\mathcal{A})$, since the norm completion of $\mathcal{A}$ is just the compact operators. □

APPENDIX A

Estimates and Technical Lemmas

A.1. Background material on the pseudodiferential expansion

To aid the reader, this Appendix recalls five Lemmas from [**15**] which are used repeatedly in Chapter 1 and in Chapter 3. All were proved in the unital setting, however, all norm estimates remain unchanged, and in the pseudodifferential expansion in Lemmas A.1, A.3, if the operators A_i lie in OP_0^*, then so does the remainder, by the invariance of OP_0^* under the one parameter group σ (see Proposition 1.28). The integral estimate in Lemma A.5 is unaffected by any changes.

We begin by giving the algebraic version of the pseudodifferential expansion developed by Higson. This expansion gives simple formulae, and sharp estimates on remainders. In the statement $Q = t + s^2 + \mathcal{D}^2$, $t \in [0,1]$, $s \in [0,\infty)$.

Lemma A.1. *(see [**15**, Lemma 6.9]) Let m, n, k be non-negative integers and $T \in \mathrm{OP}_0^m$ (resp. $T \in \mathrm{OP}^m$). Then for λ in the resolvent set of Q*

$$(\lambda - Q)^{-n}T = \sum_{j=0}^{k} \binom{n+j-1}{j} T^{(j)}(\lambda - Q)^{-n-j} + P(\lambda),$$

where the remainder $P(\lambda)$ belongs to $\mathrm{OP}_0^{-(2n+k-m+1)}$ (resp. $\mathrm{OP}^{-(2n+k-m+1)}$) and is given by

$$P(\lambda) = \sum_{j=1}^{n} \binom{j+k-1}{k} (\lambda - Q)^{j-n-1} T^{(k+1)} (\lambda - Q)^{-j-k}.$$

In the following lemmas, we let $R_s(\lambda) = (\lambda - (1 + \mathcal{D}^2 + s^2))^{-1}$.

Lemma A.2. *(see [**15**, Lemma 6.10]) Let k, n be non-negative integers, $s \geq 0$, and suppose $\lambda \in \mathbb{C}$, $0 < \Re(\lambda) < 1/2$. Then for $A \in \mathrm{OP}^k$, we have*

$$\|R_s(\lambda)^{n/2+k/2} A R_s(\lambda)^{-n/2}\| \leq C_{n,k} \quad \text{and} \quad \|R_s(\lambda)^{-n/2} A R_s(\lambda)^{n/2+k/2}\| \leq C_{n,k},$$

where $C_{n,k}$ is constant independent of s and λ (square roots use the principal branch of log.*)*

Lemma A.3. *(see [**15**, Lemma 6.11]) Let $A_i \in \mathrm{OP}_0^{n_i}$ (resp. $A_i \in \mathrm{OP}^{n_i}$) for $i = 1, \ldots, m$ and let $0 < \Re(\lambda) < 1/2$ as above. We consider the operator*

$$R_s(\lambda)A_1R_s(\lambda)A_2R_s(\lambda)\ldots R_s(\lambda)A_mR_s(\lambda),$$

Then for all $M \geq 0$

$$R_s(\lambda)A_1R_s(\lambda)A_2\ldots A_mR_s(\lambda) = \sum_{|k|=0}^{M} C(k)A_1^{(k_1)}\ldots A_m^{(k_m)}R_s(\lambda)^{m+|k|+1} + P_{M,m},$$

where $P_{M,m} \in \mathrm{OP}_0^{|n|-2m-M-3}$ (resp. $P_{M,m} \in \mathrm{OP}^{|n|-2m-M-3}$), and k and n are multi-indices with $|k| = k_1 + \cdots + k_m$ and $|n| = n_1 + \cdots + n_m$. The constant $C(k)$ is given by

$$C(k) = \frac{(|k|+m)!}{k_1!k_2!\dots k_m!(k_1+1)(k_1+k_2+2)\dots(|k|+m)}.$$

LEMMA A.4. *(see [**15**, Lemma 6.12]) With the assumptions and notation of the last Lemma including the assumption that $A_i \in \mathrm{OP}^{n_i}$ for each i, there is a positive constant C such that*

$$\|(\lambda - (1 + \mathcal{D}^2 + s^2))^{m+M/2+3/2-|n|/2} P_{M,m}\| \le C,$$

and Cindependent of s and λ (though it depends on M and m and the A_i).

LEMMA A.5. *(see [**15**, Lemma 5.4]) Let $0 < a < 1/2$ and $0 \le c \le \sqrt{2}$ and $j = 0$ or 1. Let J,K and M be nonegative constants. Then the integral*

$$\int_0^\infty \int_{-\infty}^\infty s^J \left(a^2 + v^2\right)^{-M/2} \left((s^2 + 1/2 - a)^2 + v^2\right)^{-K/2} \left((s^2 + 1 - a - sc)^2 + v^2\right)^{-j/2} dvds,$$

converges provided $J - 2K - 2j < -1$ and $J - 2K - 2j + 1 - 2M < -2$.

A.2. Estimates for Chapter 2

In this Section, we collect the proofs of the key lemmas in our homotopy arguments which are essentially nonunital variations of proofs appearing in [**17**].

The first result we prove is the analogue of [**15**, Lemma 7.2], needed to prove that the expectations used to define our various cochains are well-defined and holomorphic.

A.2.1. Proof of Lemma 3.3. Most of the proof relies on the same algebraic arguments and norm estimates as in [**15**, Lemma 7.2]. We just need to adapt the arguments which use some trace norm estimates. To simplify the notations for $0 \le t \le 1$, we use the shorthand

$$R := R_{s,t}(\lambda) = (\lambda - (t + s^2 + \mathcal{D}^2))^{-1},$$

as in Equation (3.4). We first remark that we can always assume $A_0 \in \mathrm{OP}_0^0$, at the price that A_1 will be in $\mathrm{OP}^{k_0+k_1}$, so that the global degree $|k|$ remains unchanged. Indeed, we can write

$$A_0 \, R \, A_1 \, R \dots R \, A_m \, R = A_0 (1 + \mathcal{D}^2)^{-k_0/2} \, R \, (1 + \mathcal{D}^2)^{k_0/2} A_1 \, R \dots R \, A_m \, R,$$

and this remark follows from the change

$$A_0 \in \mathrm{OP}_0^{k_0} \mapsto A_0(1+\mathcal{D}^2)^{-k_0/2} \in \mathrm{OP}_0^0, \quad A_1 \in \mathrm{OP}^{k_1} \mapsto (1+\mathcal{D}^2)^{k_0/2} A_1 \in \mathrm{OP}^{k_0+k_1}.$$

From Lemma A.3, we know that for any $L \in \mathbb{N}$, there exists a regular pseudodifferential operator $P_{L,m}$ of order (at most) $|k| - 2m - L - 3$ (i.e. $P_{L,m} \in \mathrm{OP}^{|k|-2m-L-3}$), such that

$$A_0 \, R \, A_1 \, R \dots R \, A_m \, R = \sum_{|n|=0}^{L} C(n) A_0 \, A_1^{(n_1)} \dots A_m^{(n_m)} \, R^{m+|n|+1} + A_0 \, P_{L,m}.$$

Regarding the remainder term $P_{L,m}$, by Lemma A.4 we know that it satisfies the norm inequality

$$\|R_{s,t}(\lambda)^{-m-L/2-3/2+|k|/2} \, P_{L,m}\| \le C,$$

where the constant C is uniform in s and λ. (Here the complex square root function is defined with its principal branch.) Using Lemma 1.39 and $A_0 \in \mathrm{OP}_0^0$, we obtain the trace norm bound

$$\begin{aligned}\|A_0\, P_{L,m}\|_1 &\le C\|A_0 R_{s,t}(\lambda)^{m+L/2+3/2-|k|/2}\|_1\\ &\le C'((s^2+a)^2+v^2)^{-m/2-L/4-3/4+|k|/4+(p+\varepsilon)/4}.\end{aligned}$$

Thus, the corresponding s-integral of the trace-norm of $B_{r,t}(s)$ is bounded by

$$\begin{aligned}&\int_0^\infty s^\alpha \Big\| \int_\ell \lambda^{-p/2-r}\, A_0\, P_{L,m} d\lambda \Big\|_1 ds \le \int_0^\infty s^\alpha \int_\ell |\lambda|^{-p/2-r} \|A_0\, P_{L,m}\|_1 |d\lambda| ds\\ &\le C\!\int_0^\infty\! s^\alpha\!\int_{-\infty}^\infty\!(a^2+v^2)^{-p/4-\Re(r)/2}((s^2+a)^2+v^2)^{-m/2-L/4-3/4+|k|/4+(p+\varepsilon)/4} dv ds,\end{aligned}$$

where ℓ is the vertical line $\ell = \{a+iv : v \in \mathbb{R}\}$ with $a \in (0, \mu^2/2)$. By Lemma A.5, the latter integral is finite for $L > |k| + \alpha + p + \varepsilon - 2 - 2m$, which can always be arranged. To perform the Cauchy integrals

$$\frac{1}{2\pi i}\int_\ell \lambda^{-p/2-r} A_0 A_1^{(n_1)} \dots A_m^{(n_m)} R^{m+1+|n|} d\lambda,$$

we refer to [**15**, Lemma 7.2] for the precise justifications. This gives a multiple of

$$A_0 A_1^{(n_1)} \dots A_m^{(n_m)} (t + s^2 + \mathcal{D}^2)^{-p/2-r-m-|n|}.$$

By Lemmas 1.31 and 1.33, we see that $A_0 A_1^{(n_1)} \dots A_m^{(n_m)} \in \mathrm{OP}_0^{|k|+|n|}$, so that

$$B := A_0 A_1^{(n_1)} \dots A_m^{(n_m)} |\mathcal{D}|^{-|n|-|k|} \in \mathrm{OP}_0^0.$$

(Remember that in this setting we assume $\mathcal{D}^2 \ge \mu^2$). Thus, for $\varepsilon > 0$, Equation (1.22) gives

$$\begin{aligned}&\big\|A_0 A_1^{(n_1)} \dots A_m^{(n_m)} (t+s^2+\mathcal{D}^2)^{-p/2-r-m-|n|}\big\|_1 =\\ &\big\|B|\mathcal{D}|^{|n|+|k|}(t+s^2+\mathcal{D}^2)^{-p/2-r-m-|n|}\big\|_1 \le\\ &\big\|B(t+s^2+\mathcal{D}^2)^{-p/2-r-m-|n|/2+|k|/2}\big\|_1 \big\||\mathcal{D}|^{|n|+|k|}(t+s^2+\mathcal{D}^2)^{-|n|/2-|k|/2}\big\| \le\\ &C(\mu/2+s^2)^{-\Re(r)-m-|n|/2+|k|/2+\varepsilon/2}.\end{aligned}$$

In particular, the constant C is uniform in s. The worst term being that with $|n| = 0$, we obtain that the corresponding s-integral is convergent for all $r \in \mathbb{C}$ with $\Re(r) > -m + (|k| + \alpha + 1)/2 + \varepsilon$. $\square$

A.2.2. Proof of Lemma 3.9. We give the proof for the first type of expectation $\langle A_0, \dots, A_m \rangle_{m,r,s,t}$. The proof for $\langle\langle A_0, \dots, A_m \rangle\rangle_{m,r,s,t}$ is similar with suitable modification of the domain of the parameters. From Lemma 3.3, we first see that each term of the equality is well defined, provided $2\Re(r) > 1+\alpha+|k|-2m$, and, since $2m+2 > \alpha > 0$, Lemma 3.3 also shows that $\langle\langle A_0, \dots, A_m \rangle\rangle_{m,r,s,t}$ vanishes at $s = 0$ and $s = \infty$. All we have to do is to show that the map $[s \mapsto \langle A_0, \dots, A_m \rangle_{m,r,s,t}]$ is differentiable, with derivative given by

$$2s \sum_{j=0}^m \langle A_0, \dots, A_j, 1, A_{j+1}, \dots, A_m \rangle_{m+1,r,s,t},$$

since then the result will follow by integrating between 0 and $+\infty$ the following total derivative

$$\frac{d}{ds}s^{\alpha}\langle A_0,\dots,A_m\rangle_{m,r,s,t} = \alpha\, s^{\alpha-1}\langle A_0,\dots,A_m\rangle_{m,r,s,t} + 2\sum_{j=0}^{m} s^{\alpha+1}\langle A_0,\dots,A_j,1,A_{j+1},\dots,A_m\rangle_{m+1,r,s,t}.$$

As $\frac{1}{\varepsilon}\big(R_{s+\varepsilon,t}(\lambda)-R_{s,t}(\lambda)\big) = -R_{s+\varepsilon,t}(\lambda)(2s+\varepsilon)R_{s,t}(\lambda)$, we see that the resolvent is continuously norm-differentiable in the s-parameter, with norm derivative given by $2sR_{s,t}(\lambda)^2$. We then write

$$2\pi i\tfrac{1}{\varepsilon}\big(\langle A_0,\dots,A_m\rangle_{m,r,s+\varepsilon,t}-\langle A_0,\dots,A_m\rangle_{m,r,s,t}\big) = \sum_{j=0}^{m}\tau\Big(\gamma\int_\ell \lambda^{-p/2-r}A_0\,R_{s+\varepsilon,t}(\lambda)\dots A_j\,R_{s+\varepsilon,t}(\lambda)(2s+\varepsilon) \times R_{s,t}(\lambda)\,A_{j+1}\dots R_{s,t}(\lambda)\,A_m\,R_{s,t}(\lambda)\,d\lambda\Big),$$

where ℓ is the vertical line $\ell=\{a+iv : v\in\mathbb{R}\}$ with $a\in(0,\mu^2/2)$. This leads to

$$\begin{aligned}
&\tfrac{1}{\varepsilon}\big(\langle A_0,\dots,A_m\rangle_{m,r,s+\varepsilon,t}-\langle A_0,\dots,A_m\rangle_{m,r,s,t}\big) - 2s\sum_{j=0}^{m}\langle A_0,\dots,A_j,1,A_{j+1},\dots,A_m\rangle_{m+1,r,s,t} = \\
&\frac{\varepsilon}{2\pi i}\sum_{j=0}^{m}\tau\Big(\gamma\int_\ell \lambda^{-p/2-r}A_0\,R_{s+\varepsilon,t}(\lambda)\dots A_j\,R_{s+\varepsilon,t}(\lambda)^2 \times A_{j+1}\dots R_{s,t}(\lambda)\,A_m\,R_{s,t}(\lambda)\,d\lambda\Big)\\
&+\frac{2s\varepsilon}{2\pi i}\sum_{k\le j=0}^{m}\tau\Big(\gamma\int_\ell \lambda^{-p/2-r}A_0\,R_{s+\varepsilon,t}(\lambda)\dots A_k\,R_{s+\varepsilon,t}(\lambda)(2s+\varepsilon)\\
&\qquad\times R_{s,t}(\lambda)\,A_{j+k}\dots A_j\,R_{s,t}(\lambda)^2A_{j+1}\dots R_{s,t}(\lambda)\,A_m\,R_{s,t}(\lambda)\,d\lambda\Big).
\end{aligned}$$

We now proceed as in Lemma 3.3. We write each integrand (of the first or second type) as

$$A_0\,R\,A_1\,R\dots R\,A_{m+j}\,R = \sum_{|n|=0}^{M} C(k)A_0\,A_1^{(n_1)}\dots A_{m+j}^{(n_{m+j})}\,R^{m+j+|n|+1} + A_0\,P_{M,m+j},$$

where $j\in\{1,2\}$ depending the type of term we are looking at, the A_j's have been redefined and now R stands for $R_{s,t}(\lambda)$ or $R_{s+\varepsilon,t}(\lambda)$. To treat the non-remainder terms, before applying the Cauchy formula, one needs to perform a resolvent expansion

$$R_{s+\varepsilon,t}(\lambda) = \sum_{j=0}^{M}(-\varepsilon(2s+\varepsilon))^{j-1}R_{s,t}(\lambda)^j + (-\varepsilon(2s+\varepsilon))^M R_{s,t}(\lambda)^M R_{s+\varepsilon,t}(\lambda).$$

We can always choose M big enough so that the integrand associated with the remainder term in the resolvent expansion is integrable in trace norm, by Lemma 3.3. Provided $\Re(r)+m-|k|/2>0$, one sees with the same estimates as in Lemma

3.3, that the corresponding term in the difference-quotient goes to zero with ε. For the non-remainder terms of the resolvent expansion, we can use the Cauchy formula as in Lemma 3.3, and obtain the same conclusion. All that is left is to treat the remainder term. The main difference with the corresponding term in Lemma 3.3 is that $P_{M,m+j}$ is now ε-dependent. But the ε-dependence only occurs in $R_{s+\varepsilon,t}(\lambda)$ and since the estimate of Lemma A.2 is uniform in s, we still have

$$\|R_{s,t}(\lambda)^{-m-M/2-3/2+|k|/2}\, P_{M,m+j}\| \leq C,$$

where the constant is uniform in s, λ and ε. This is enough (see again the proof of Lemma 3.3) to show that the corresponding term in the difference-quotient goes to zero with ε, provided $\Re(r)+m-|k|/2 > 0$. Thus, $\langle A_0, \ldots, A_m\rangle_{m,r,s,t}$ is differentiable in s, concluding the proof. $\square$

A.2.3. Proof of Lemma 3.10. According to our assumptions, one first notes from Lemma 3.3, that all the terms involved in the equalities above are well defined. From

$$\tfrac{1}{\varepsilon}\big(R_{s,t}(\lambda+\varepsilon) - R_{s,t}(\lambda)\big) + R_{s,t}(\lambda)^2 = \varepsilon R_{s,t}(\lambda+\varepsilon)R_{s,t}(\lambda)^2,$$

we readily conclude that the map $\lambda \mapsto R_{s,t}(\lambda)$ is norm-continuously differentiable, with norm derivatives given by $-R_{s,t}(\lambda)^2$. We deduce that for $A_j \in \mathrm{OP}^{k_j}$, the map $\lambda \mapsto A_j R_{s,t}(\lambda)$ is continuously differentiable for the topology of OP^{k_j-2}, with derivative given by $-A_j R_{s,t}(\lambda)^2$. Thus, $A_0 R \ldots A_m R$ is continuously differentiable for the topology of $\mathrm{OP}_0^{|k|-2m}$, with derivative given by

$$-\sum_{j=0}^{m} A_0\, R_{s,t}(\lambda) \ldots A_j\, R_{s,t}(\lambda)^2\, A_{j+1} \ldots A_m\, R_{s,t}(\lambda).$$

We thus arrive at the identity in $\mathrm{OP}_0^{|k|-2m}$:

$$\begin{aligned}
&\frac{d}{d\lambda}\Big(\lambda^{-q/2-r} A_0\, R_{s,t}(\lambda) \ldots A_m\, R_{s,t}(\lambda)\Big)\\
&\qquad = -\,(p/2-r)\lambda^{-q/2-r-1} A_0\, R_{s,t}(\lambda) \ldots A_m\, R_{s,t}(\lambda)\\
&\qquad\quad - \sum_{j=0}^{m} \lambda^{-q/2-r} A_0\, R_{s,t}(\lambda) \ldots A_j\, R_{s,t}(\lambda)^2\, A_{j+1} \ldots A_m\, R_{s,t}(\lambda)\\
&\qquad = -\,(p/2-r)\lambda^{-q/2-r-1} A_0\, R_{s,t}(\lambda) \ldots A_m\, R_{s,t}(\lambda)\\
&\qquad\quad - \sum_{j=0}^{m} \lambda^{-q/2-r} A_0\, R_{s,t}(\lambda) \ldots A_j\, R_{s,t}(\lambda)\, 1\, R_{s,t}(\lambda)\, A_{j+1} \ldots A_m\, R_{s,t}(\lambda).
\end{aligned}$$

By Lemma 3.3, the λ-integral of the right hand side of the former equality is well defined as a trace class operator for $2\Re(r) > |k| - 2m$. Performing the integration gives the result, since $\langle\langle A_0, \ldots, A_m\rangle\rangle_{m,r+1,s,t}$ vanishes at the endpoints of the integration domain. $\square$

We now present the proof of the trace norm differentiability result, Lemma 3.26, needed to complete the homotopy to the Chern character.

A.2.4. Proof of Lemma 3.26. Recall that $a_0, \dots, a_M \in \mathcal{A}^\sim$ so that each of da_i, $\delta(a_i) \in \mathrm{OP}^0_0$ for $i = 0, \dots, M$. This means we can use the result of Lemma 1.38. We first assume $p \geq 2$. We start from the identity,

$$d_u(a) = [\mathcal{D}_u, a] = [F|\mathcal{D}|^{1-u}, a] = F[|\mathcal{D}|^{1-u}, a] + \big(da - F\delta(a)\big)|\mathcal{D}|^{-u},$$

and we note that $da - F\delta(a) \in \mathrm{OP}^0_0$. Applying the second part of Lemma 1.38 and Lemma 1.37 now shows that $d_u(a) \in \mathcal{L}^q(\mathcal{N}, \tau)$ for all $q > p/u$. Next, we find that

$$R_{s,u}(\lambda) = (\lambda - s^2 - \mathcal{D}_u^2)^{-1} = |\mathcal{D}|^{-2(1-u)}\mathcal{D}_u^2(\lambda - s^2 - \mathcal{D}_u^2)^{-1} =: |\mathcal{D}|^{-2(1-u)}B(u),$$

where $B(u)$ is uniformly bounded. Then Lemma 1.37 and the Hölder inequality show that

$$d_u(a_i)\, R_{s,u}(\lambda) \in \mathcal{L}^q(\mathcal{N}, \tau), \qquad \text{for all} \quad q > p/(2-u) \geq p/2 \geq 1,$$

and for each $i = 0, \dots, j, j+2, \dots, M$, while

$$R_{s,u}(\lambda)^{1/2}\, d_u(a_{j+1}) R_{s,u}(\lambda) \in \mathcal{L}^q(\mathcal{N}, \tau) \qquad \text{for all} \quad q \geq 2 \quad \text{with} \quad (3-2u)q > p.$$

The worst case is $u = 1$ for which we find $2 \leq p \leq q$, allowing us to use the first and simplest case of Lemma 1.37. Since $T_{s,\lambda,j}(u)$ contains M terms $d_u(a_i)\, R_{s,u}(\lambda)$ and contains one term $R_{s,u}(\lambda)^{1/2}\, d_u(a_{j+1})\, R_{s,u}(\lambda)$ and one bounded term $\mathcal{D}_u R_{s,u}(\lambda)^{1/2}$, the Hölder inequality gives

$$T_{s,\lambda,j}(u) \in \mathcal{L}^q(\mathcal{N}, \tau), \quad \text{for all} \quad q > p/(M(2-u)+(3-2u)) = p/(2M+3-u(M+2)).$$

Since $u \in [0,1]$ and $M > p - 1$, we obtain

$$p/(2M + 3 - u(M+2)) < p/(M+1) < 1,$$

that is $T_{s,\lambda,j}(u) \in \mathcal{L}^1(\mathcal{N}, \tau)$. The proof then proceeds by showing that

$$\big[u \mapsto d_u(a_i) R_{s,u}(\lambda)\big] \in C^1\big([0,1], \mathcal{L}^q(\mathcal{N}, \tau)\big), \quad q > p/(2-u),$$

for each $i = 0, \dots, j, j+2, \dots, M$, and

$$\big[u \mapsto \mathcal{D}_u R_{s,u}(\lambda)\, d_u(a_{j+1}) R_{s,u}(\lambda)\big] \in C^1\big([0,1], \mathcal{L}^q(\mathcal{N}, \tau)\big), \quad q > p/(3-2u),$$

with derivatives given respectively by

$$[\dot{\mathcal{D}}_u, a_i] R_{s,u}(\lambda) + 2 d_u(a_i) R_{s,u}(\lambda) \dot{\mathcal{D}}_u \mathcal{D}_u R_{s,u}(\lambda),$$

and

$$\begin{aligned}
&\dot{\mathcal{D}}_u R_{s,u}(\lambda)\, d_u(a_{j+1}) R_{s,u}(\lambda) + 2\mathcal{D}_u R_{s,u}(\lambda) \dot{\mathcal{D}}_u \mathcal{D}_u R_{s,u}(\lambda)\, d_u(a_{j+1}) R_{s,u}(\lambda) \\
&\quad + \mathcal{D}_u R_{s,u}(\lambda)\, [\dot{\mathcal{D}}_u, a_{j+1}] R_{s,u}(\lambda) + 2\mathcal{D}_u R_{s,u}(\lambda)\, d_u(a_{j+1}) R_{s,u}(\lambda) \dot{\mathcal{D}}_u \mathcal{D}_u R_{s,u}(\lambda).
\end{aligned}$$

This will eventually imply the statement of the lemma. We only treat the first term, the arguments for the second term being similar but algebraically more involved. We write,

$$\begin{aligned}
&\varepsilon^{-1}(d_{u+\varepsilon}(a_i) R_{s,u+\varepsilon}(\lambda) - d_u(a_i) R_{s,u}(\lambda)) \\
&\qquad\qquad - [\dot{\mathcal{D}}_u, a_i] R_{s,u}(\lambda) - 2 d_u(a_i) R_{s,u}(\lambda) \dot{\mathcal{D}}_u \mathcal{D}_u R_{s,u}(\lambda) = \\
&\Big(\varepsilon^{-1}(d_{u+\varepsilon}(a_i) - d_u(a_i)) - [\dot{\mathcal{D}}_u, a_i]\Big) R_{s,u}(\lambda) \\
&\qquad\qquad + \big(d_{u+\varepsilon}(a_i) - d_u(a_i)\big)\varepsilon^{-1}(R_{s,u+\varepsilon}(\lambda) - R_{s,u}(\lambda)) \\
&\qquad\qquad + d_u(a_i)\Big(\varepsilon^{-1}(R_{s,u+\varepsilon}(\lambda) - R_{s,u}(\lambda)) - 2R_{s,u}(\lambda) \dot{\mathcal{D}}_u \mathcal{D}_u R_{s,u}(\lambda)\Big).
\end{aligned} \tag{A.1}$$

The first term of Equation (A.1) is the most involved. We start by writing

$$\begin{aligned}\varepsilon^{-1}(d_{u+\varepsilon}(a_i)-d_u(a_i))-[\dot{\mathcal{D}}_u,a_i] &= \Big[\varepsilon^{-1}(\mathcal{D}_{u+\varepsilon}-\mathcal{D}_u)+\mathcal{D}_u\log|\mathcal{D}|,a_i\Big]\\ &= \Big[F|\mathcal{D}|^{1-u}\Big(\varepsilon^{-1}(|\mathcal{D}|^{-\varepsilon}-1)+\log|\mathcal{D}|\Big),a_i\Big]\\ &= F\Big[|\mathcal{D}|^{1-u}\Big(\varepsilon^{-1}(|\mathcal{D}|^{-\varepsilon}-1)+\log|\mathcal{D}|\Big),a_i\Big]\\ &\quad+\big(da_i-F\delta(a_i)\big)|\mathcal{D}|^{-u}\Big(\varepsilon^{-1}(|\mathcal{D}|^{-\varepsilon}-1)+\log|\mathcal{D}|\Big).\end{aligned}$$

We are seeking convergence for the Schatten norm $\|\cdot\|_q$ with $q>p/(2-u)$. So, let $\rho>0$, be such that for $A\in \mathrm{OP}_0^0$, $A|\mathcal{D}|^{-2+u+\rho}\in\mathcal{L}^q(\mathcal{N},\tau)$. Thus, the last term of the previous expression, multiplied by $R_{s,u}(\lambda)$ can be estimated in q-norm by:

$$\begin{aligned}&\Big\|\big(da_i-F\delta(a_i)\big)|\mathcal{D}|^{-u}\Big(\varepsilon^{-1}(|\mathcal{D}|^{-\varepsilon}-1)+\log|\mathcal{D}|\Big)R_{s,u}(\lambda)\Big\|_q\\ &\quad\le\big\|\big(da_i-F\delta(a_i)\big)|\mathcal{D}|^{-2+u+\rho}\big\|_q\,\big\||\mathcal{D}|^{-2(1-u)}R_{s,u}(\lambda)\big\|\\ &\qquad\times\Big\|\Big(\varepsilon^{-1}(|\mathcal{D}|^{-\varepsilon}-1)+\log|\mathcal{D}|\Big)\mathcal{D}^{-\rho}\Big\|,\end{aligned}$$

which treats this term since the last operator norm goes to zero with ε. We now show that

$$\Big[|\mathcal{D}|^{1-u}\Big(\varepsilon^{-1}(|\mathcal{D}|^{-\varepsilon}-1)+\log|\mathcal{D}|\Big),a_i\Big],\tag{A.2}$$

converges to zero in q-norm (for the same values of q as before). We first remark that we can assume $u>0$. Indeed, when $u=0$, we can use (as before) the little room left between q and $p/2$, find $\rho>0$ such that $a|\mathcal{D}|^{-2+\rho}\in\mathcal{L}^q(\mathcal{N},\tau)$ and write

$$\begin{aligned}\Big[|\mathcal{D}|\Big(\varepsilon^{-1}(|\mathcal{D}|^{-\varepsilon}-1)+\log|\mathcal{D}|\Big),a_i\Big]|\mathcal{D}|^{-\rho} &= \Big[|\mathcal{D}|^{1-\rho}\Big(\varepsilon^{-1}(|\mathcal{D}|^{-\varepsilon}-1)+\log|\mathcal{D}|\Big),a_i\Big]\\ &\quad-|\mathcal{D}|^{1-\rho}\Big(\varepsilon^{-1}(|\mathcal{D}|^{-\varepsilon}-1)+\log|\mathcal{D}|\Big)\big[|\mathcal{D}|^{\rho},a_i\big]|\mathcal{D}|^{-\rho},\end{aligned}$$

and use an estimate of the previous type plus the content of Lemma 1.38. To take care of the term (A.2) (for $u>0$), we use the integral formula for fractional powers. After some rearrangements, this gives the following expression for (A.2):

$$\begin{aligned}&\int_0^\infty\lambda^{u-1}(\pi\varepsilon)^{-1}\Big\{(\sin\pi(1-u-\varepsilon)-\sin\pi(1-u))(\lambda^\varepsilon-1)\\ &+\sin\pi(1-u)(\lambda^\varepsilon-1-\varepsilon\log\lambda)+\cos\pi(1-u)\\ &+\Big((\pi\varepsilon)^{-1}(\sin\pi(1-u-\varepsilon)-\sin\pi(1-u))\Big)\Big\}(1+\lambda|\mathcal{D}|)^{-1}\delta(a_i)(1+\lambda|\mathcal{D}|)^{-1}\,d\lambda.\end{aligned}$$

The last term can be recombined as

$$\Big((\pi\varepsilon)^{-1}(\sin\pi(1-u-\varepsilon)-\sin\pi(1-u))+\cos\pi(1-u)\Big)\pi(\sin\pi(1-u))^{-1}\big[|\mathcal{D}|^{1-u},a_i\big],$$

and one concludes (for this term) using Lemma 1.38 together with an (ordinary) Taylor expansion for the pre-factor. Since $\mathcal{D}^2\ge\mu^2>0$, the first term (multiplied by $R_{s,u}(\lambda)$) is estimated (up to a constant) in q-norm by

$$\big|\sin\pi(1-u-\varepsilon)-\sin\pi(1-u)\big|\big\|\delta(a_i)R_{s,u}(\lambda)\big\|_q\int_0^\infty\lambda^{u-1}\varepsilon^{-1}(\lambda^\varepsilon-1)(1+\lambda\,\mu^{1/2})^{-2}\,d\lambda,$$

which goes to zero with ε, as seen by a Taylor expansion of the prefactor and since $(\lambda^\varepsilon - 1)/\varepsilon$ is uniformly bounded in ε for $\lambda \in [0,1]$, while between 1 in ∞, we use

$$\int_1^\infty \lambda^{u-1}\varepsilon^{-1}(\lambda^\varepsilon - 1)(1+\lambda\,\mu^{1/2})^{-2}\,d\lambda \le (\mu\,\varepsilon)^{-1}\int_1^\infty \left(\lambda^{u-3+\varepsilon} - \lambda^{u-3}\right)d\lambda$$
$$= (\mu(2-u-\varepsilon))^{-1} \le (\mu(1-u))^{-1}.$$

For the middle term, we obtain instead the bound (up to a constant depending on u only)

$$\left\|\delta(a_i)R_{s,u}(\lambda)\right\|_q \int_0^\infty \lambda^{u-1}\varepsilon^{-1}(\lambda^\varepsilon - 1 - \varepsilon\log(\lambda))(1+\lambda\,\mu^{1/2})^{-2}\,d\lambda,$$

and one concludes using the same kind of arguments as employed previously. Similar (and easier) arguments show that the two other terms in (A.1) converge to zero in q-norm. That the derivative of $T_{s,\lambda,l}(u)$ is continuous for the trace norm topology follows from analogous arguments.

Now we consider the case $1 \le p < 2$. In this case $M = 1$ in the odd case and $M = 2$ in the even case. For the odd case we have two terms to consider,

$$T_{s,\lambda,0}(u) = d_u(a_0)R_{s,u}(\lambda)\mathcal{D}_u R_{s,u}(\lambda)d_u(a_1)R_{s,u}(\lambda),$$

and

$$T_{s,\lambda,1}(u) = d_u(a_0)R_{s,u}(\lambda)d_u(a_1)R_{s,u}(\lambda)\mathcal{D}_u R_{s,u}(\lambda).$$

We write $T_{s,\lambda,0}(u)$ as

$$\underbrace{d_u(a_0)|\mathcal{D}|^{-\frac{5}{2}(1-u)}}_{A}\;\underbrace{R_{s,u}(\lambda)\mathcal{D}_u R_{s,u}(\lambda)|\mathcal{D}|^{3(1-u)}}_{B}\;\underbrace{|\mathcal{D}|^{-\frac{1}{2}(1-u)}d_u(a_1)R_{s,u}(\lambda)}_{C}.$$

Now the operator B is uniformly bounded in $u \in [0,1]$, while Lemma 1.37 shows that both A and C lie in $\mathcal{L}^q(\mathcal{N},\tau)$ for all $q \ge p$. Since $1 > p/2$, the Hölder inequality now shows that $T_{s,\lambda,0}(u)$ lies in $\mathcal{L}^1(\mathcal{N},\tau)$ for each $u \in [0,1]$. Now the strict inequality $1 > p/2$ allows us to handle the difference quotients as in the $p \ge 2$ case above to obtain the trace norm differentiability of $T_{s,\lambda,0}(u)$. For $T_{s,\lambda,1}(u)$ we write

$$\underbrace{d_u(a_0)R_{s,u}(\lambda)|\mathcal{D}|^{-2(1-u)}}_{A}\;\underbrace{d_u(\sigma^{(1-u)/2}(a_1))|\mathcal{D}|^{-2(1-u)}R_{s,u}(\lambda)\mathcal{D}_u R_{s,u}(\lambda)}_{B}.$$

Applying Lemma 1.37 and the Hölder inequality again shows that $T_{s,\lambda,1}(u) \in \mathcal{L}^1(\mathcal{N},\tau)$. The strict inequality $1 > p/2$ again allows us to prove trace norm differentiability. For the even case where $M = 2$ we have more terms to consider, but the pattern is now clear. We break up $T_{s,\lambda,j}(u)$, $j = 0,1$, into a product of terms whose Schatten norms we can control, and obtain a strict inequality allowing us to control the logarithms arising in the formal derivative. This completes the proof. □

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