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Alexander Kumjian
*University of Nevada - Reno*

David A. Pask
*University of Wollongong*, dpask@uow.edu.au

Aidan Sims
*University of Wollongong*

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## Abstract

A covering of k-graphs (in the sense of Pask-Quigg- Raeburn) induces an embedding of universal C∗-algebras. We show how to build a (k + 1)-graph whose universal algebra encodes this embedding. More generally we show how to realise a direct limit of k-graph algebras under embeddings induced from coverings as the universal algebra of a (k +1)-graph. Our main focus is on computing the K-theory of the (k+1)-graph algebra from that of the component k-graph algebras.

Examples of our construction include a realisation of the Kirchberg algebra Pn whose K-theory is opposite to that of On, and a class of AT-algebras that can naturally be regarded as higher-rank Bunce- Deddens algebras.

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# $C^*$-Algebras Associated to Coverings of $k$-Graphs

Alex Kumjian, David Pask, Aidan Sims*

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Abstract. A covering of $k$-graphs (in the sense of Pask-Quigg-Raeburn) induces an embedding of universal $C^*$-algebras. We show how to build a $(k+1)$-graph whose universal algebra encodes this embedding. More generally we show how to realise a direct limit of $k$-graph algebras under embeddings induced from coverings as the universal algebra of a $(k+1)$-graph. Our main focus is on computing the $K$-theory of the $(k+1)$-graph algebra from that of the component $k$-graph algebras.

Examples of our construction include a realisation of the Kirchberg algebra $\mathcal{P}_n$ whose $K$-theory is opposite to that of $\mathcal{O}_n$, and a class of A$\mathbf{T}$-algebras that can naturally be regarded as higher-rank Bunce-Deddens algebras.

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## 1. Introduction

A directed graph $E$ consists of a countable collection $E^0$ of vertices, a countable collection $E^1$ of edges, and maps $r, s : E^1 \to E^0$ which give the edges their direction; the edge $e$ points from $s(e)$ to $r(e)$. Following the convention established in [30], the associated graph algebra $C^*(E)$ is the universal $C^*$-algebra generated by partial isometries $\{s_e : e \in E^1\}$ together with mutually orthogonal projections $\{p_v : v \in E^0\}$ such that $p_{s(e)} = s_e^* s_e$ for all $e \in E^1$, and $p_v \geq \sum_{e \in F} s_e s_e^*$ for all $v \in E^0$ and finite $F \subset r^{-1}(v)$, with equality when $F = r^{-1}(v)$ is finite and nonempty.
Graph algebras, introduced in [13, 23], have been studied intensively in recent years because much of the structure of $C^*(E)$ can be deduced from elementary

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features of $E$. In particular, graph $C^*$-algebras are an excellent class of models for Kirchberg algebras, because it is easy to tell from the graph $E$ whether $C^*(E)$ is simple and purely infinite [22]. Indeed, a Kirchberg algebra can be realised up to Morita equivalence as a graph $C^*$-algebra if and only if its $K_1$-group is torsion-free [39]. It is also true that every AF algebra can be realised up to Morita equivalence as a graph algebra; the desired graph is a Bratteli diagram for the AF algebra in question (see [11] or [40]). However, this is the full extent to which graph algebras model simple classifiable $C^*$-algebras due to the following dichotomy: if $E$ is a directed graph and $C^*(E)$ is simple, then $C^*(E)$ is either AF or purely infinite (see [22, Corollary 3.10], [2, Remark 5.6]).

Higher-rank graphs, or $k$-graphs, and their $C^*$-algebras were originally developed by the first two authors [20] to provide a graphical framework for the higher-rank Cuntz-Krieger algebras of Robertson and Steger [35]. A $k$-graph $\Lambda$ is a kind of $k$-dimensional graph, which one can think of as consisting of vertices $\Lambda^0$ together with $k$ collections of edges $\Lambda^{e_1}, \ldots, \Lambda^{e_k}$ which we think of as lying in $k$ different dimensions. As an aid to visualisation, we often distinguish the different types of edges using $k$ different colours.

Higher-rank graphs and their $C^*$-algebras are generalisations of directed graphs and their algebras. Given a directed graph $E$, its path category $E^*$ is a 1-graph, and the 1-graph $C^*$-algebra $C^*(E^*)$ as defined in [20] is canonically isomorphic to the graph algebra $C^*(E)$ as defined in [23]. Furthermore, every 1-graph arises this way, so the class of graph algebras and the class of 1-graph algebras are one and the same. For $k \geq 2$, there are many $k$-graph algebras which do not arise as graph algebras. For example, the original work of Robertson and Steger on higher-rank Cuntz-Krieger algebras describes numerous 2-graphs $\Lambda$ for which $C^*(\Lambda)$ is a Kirchberg algebra and $K_1(C^*(\Lambda))$ contains torsion.

Recent work of Pask, Raeburn, Rørdam and Sims has shown that one can also realise a substantial class of A$\mathbf{T}$-algebras as 2-graph algebras, and that one can tell from the 2-graph whether or not the resulting $C^*$-algebra is simple and has real-rank zero [27]. The basic idea of the construction in [27] is as follows. One takes a Bratteli diagram in which the edges are coloured red, and replaces each vertex with a blue simple cycle (there are technical restrictions on the relationship between the lengths of the blue cycles and the distribution of the red edges joining them, but this is the gist of the construction). The resulting 2-graph is called a *rank-2 Bratteli diagram*. The associated $C^*$-algebra is A$\mathbf{T}$ because the $C^*$-algebra of a simple cycle of length $n$ is isomorphic to $M_n(C(\mathbf{T}))$ [17]. The results of [27] show how to read off from a rank-2 Bratteli diagram the $K$-theory, simplicity or otherwise, and real-rank of the resulting A$\mathbf{T}$ algebra.

The construction explored in the current paper is motivated by the following example of a rank-2 Bratteli diagram. For each $n \in \mathbf{N}$, let $L_{2^n}$ be the simple directed loop graph with $2^n$ vertices labelled $0, \ldots, 2^n - 1$ and $2^n$ edges $f_0, \ldots, f_{2^n-1}$, where $f_i$ is directed from the vertex labelled $i+1 \pmod{2^n}$ to the vertex labelled $i$. We specify a rank-2 Bratteli diagram $\Lambda(2^\infty)$ as follows. The $n^{\text{th}}$ level of $\Lambda(2^\infty)$ consists of a single blue copy of $L_{2^{n-1}}$ $(n = 1, 2, \cdots)$. For $0 \leq i \leq 2^n - 1$, there is a single red edge from the vertex labelled $i$ at the

$(n+1)^{\text{st}}$ level to the vertex labelled $i \pmod{2^n}$ at the $n^{\text{th}}$ level. The $C^*$-algebra of the resulting 2-graph is Morita equivalent to the Bunce-Deddens algebra of type $2^\infty$, and this was one of the first examples of a 2-graph algebra which is simple but neither purely infinite nor AF (see [27, Example 6.7]).

The purpose of this paper is to explore the observation that the growing blue cycles in $\Lambda(2^\infty)$ can be thought of as a tower of *coverings* of 1-graphs (roughly speaking, a covering is a locally bijective surjection — see Definition 2.1), where the red edges connecting levels indicate the covering maps.
In Section 2, we describe how to construct $(k+1)$-graphs from coverings. In its simplest form, our construction takes $k$-graphs $\Lambda$ and $\Gamma$ and a covering map $p : \Gamma \to \Lambda$, and produces a $(k+1)$-graph $\Lambda \xleftarrow{p} \Gamma$ in which each edge in the $(k+1)^{\text{st}}$ dimension points from a vertex $v$ of $\Gamma$ to the vertex $p(v)$ of $\Lambda$ which it covers[†]. Building on this construction, we show how to take an infinite tower of coverings $p_n : \Lambda_{n+1} \to \Lambda_n$, $n = 1, 2, \ldots$ and construct from it an infinite $(k+1)$-graph $\varprojlim(\Lambda_n, p_n)$ with a natural inductive structure (Corollary 2.11).
The next step, achieved in Section 3, is to determine how the universal $C^*$-algebra of $\Lambda \xleftarrow{p} \Gamma$ relates to those of $\Lambda$ and $\Gamma$. We show that $C^*(\Lambda \xleftarrow{p} \Gamma)$ is Morita equivalent to $C^*(\Gamma)$ and contains an isomorphic copy of $C^*(\Lambda)$ (Proposition 3.2). We then show that given a system of coverings $p_n : \Lambda_{n+1} \to \Lambda_n$, the $C^*$-algebra $C^*(\varprojlim(\Lambda_n, p_n))$ is Morita equivalent to a direct limit of the $C^*(\Lambda_n)$ (Theorem 3.8).
In Section 4, we use results of [34] to characterise simplicity of $C^*(\varprojlim(\Lambda_n, p_n))$, and we also give a sufficient condition for this $C^*$-algebra to be purely infinite.
In Section 5, we show how various existing methods of computing the $K$-theory of the $C^*(\Lambda_n)$ can be used to compute the $K$-theory of $C^*(\varprojlim(\Lambda_n, p_n))$. Our results boil down to checking that each of the existing $K$-theory computations for the $C^*(\Lambda_n)$ is natural in the appropriate sense. Given that $K$-theory for higher-rank graph $C^*$-algebras has proven quite difficult to compute in general (see [14]), our $K$-theory computations are an important outcome of the paper.
We conclude in Section 6 by exploring some detailed examples which illustrate the covering-system construction, and show how to apply our $K$-theory calculations to the resulting higher-rank graph $C^*$-algebras. For integers $3 \le n < \infty$, we obtain a 3-graph algebra realisation of Kirchberg algebra $\mathcal{P}_n$ whose $K$-theory is opposite to that of $\mathcal{O}_n$ (see Section 6.3). We also obtain, using 3-graphs, a class of simple $A\mathbf{T}$-algebras with real-rank zero which cannot be obtained from the rank-2 Bratteli diagram construction of [27] (see Section 6.4), and which we can describe in a natural fashion as higher-rank analogues of the Bunce-Deddens algebras. These are, to our knowledge, the first explicit computations of $K$-theory for infinite classes of 3-graph algebras.

[†]In its full generality, our construction is more complicated (see Proposition 2.14), enabling us to recover the important example of the irrational rotation algebras discussed in [27]. To keep technical detail in this introduction to a minimum, we discuss only the basic construction here.

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## 2. Covering systems of $k$-graphs

For $k$-graphs we adopt the conventions of [20, 25, 31]; briefly, a *$k$-graph* is a countable small category $\Lambda$ equipped with a functor $d : \Lambda \to \mathbf{N}^k$ satisfying the *factorisation property*: for all $\lambda \in \Lambda$ and $m, n \in \mathbf{N}^k$ such that $d(\lambda) = m + n$ there exist unique $\mu, \nu \in \Lambda$ such that $d(\mu) = m$, $d(\nu) = n$, and $\lambda = \mu\nu$. When $d(\lambda) = n$ we say $\lambda$ has *degree* $n$. By abuse of notation, we will use $d$ to denote the degree functor in every $k$-graph in this paper; the domain of $d$ is always clear from context.

The standard generators of $\mathbf{N}^k$ are denoted $e_1, \dots, e_k$, and for $n \in \mathbf{N}^k$ and $1 \le i \le k$ we write $n_i$ for the $i^{\text{th}}$ coordinate of $n$.

If $\Lambda$ is a $k$-graph, the *vertices* are the morphisms of degree 0. The factorisation property implies that these are precisely the identity morphisms, and so can be identified with the objects. For $\alpha \in \Lambda$, the *source* $s(\alpha)$ is the domain of $\alpha$, and the *range* $r(\alpha)$ is the codomain of $\alpha$ (strictly speaking, $s(\alpha)$ and $r(\alpha)$ are the identity morphisms associated to the domain and codomain of $\alpha$).

For $n \in \mathbf{N}^k$, we write $\Lambda^n$ for $d^{-1}(n)$. In particular, $\Lambda^0$ is the vertex set. For $u, v \in \Lambda^0$ and $E \subset \Lambda$, we write $uE := E \cap r^{-1}(u)$ and $Ev := E \cap s^{-1}(v)$. For $n \in \mathbf{N}^k$, we write

$$\Lambda^{\le n} := \{\lambda \in \Lambda : d(\lambda) \le n, s(\lambda)\Lambda^{e_i} = \emptyset \text{ whenever } d(\lambda) + e_i \le n\}.$$

We say that $\Lambda$ is *connected* if the equivalence relation on $\Lambda^0$ generated by $\{(v, w) \in \Lambda^0 \times \Lambda^0 : v\Lambda w \neq \emptyset\}$ is the whole of $\Lambda^0 \times \Lambda^0$. A *morphism* between $k$-graphs is a degree-preserving functor.

We say that $\Lambda$ is *row-finite* if $v\Lambda^n$ is finite for all $v \in \Lambda^0$ and $n \in \mathbf{N}^k$. We say that $\Lambda$ is *locally convex* if whenever $1 \le i < j \le k$, $e \in \Lambda^{e_i}$, $f \in \Lambda^{e_j}$ and $r(e) = r(f)$, we can extend both $e$ and $f$ to paths $ee'$ and $ff'$ in $\Lambda^{e_i+e_j}$.

We next introduce the notion of a covering of one $k$-graph by another. For a more detailed treatment of coverings of $k$-graphs, see [25].

Definition 2.1. A *covering* of a $k$-graph $\Lambda$ is a surjective $k$-graph morphism $p : \Gamma \to \Lambda$ such that for all $v \in \Gamma^0$, $p$ maps $\Gamma v$ 1-1 onto $\Lambda p(v)$ and $v\Gamma$ 1-1 onto $p(v)\Lambda$. A covering $p : \Gamma \to \Lambda$ is *connected* if $\Gamma$, and hence also $\Lambda$, is connected. A covering $p : \Gamma \to \Lambda$ is *finite* if $p^{-1}(v)$ is finite for all $v \in \Lambda^0$.

*Remarks* 2.2. (1) A covering $p : \Gamma \to \Lambda$ has the unique path lifting property: for every $\lambda \in \Lambda$ and $v \in \Gamma^0$ with $p(v) = s(\lambda)$ there exists a unique $\gamma$ such that $p(\gamma) = \lambda$ and $s(\gamma) = v$; likewise, if $p(v) = r(\lambda)$ there is a unique $\zeta$ such that $p(\zeta) = \lambda$ and $r(\zeta) = v$.

(2) If $\Lambda$ is connected then surjectivity of $p$ is implied by the unique path-lifting property.

(3) If there is a fixed integer $n$ such that $|p^{-1}(v)| = n$ for all $v \in \Lambda^0$, $p$ is said to be an *$n$-fold covering*. If $\Gamma$ is connected, then $p$ is automatically an $n$-fold covering for some $n$.

*Notation* 2.3. For $m \in \mathbf{N} \setminus \{0\}$, we write $S_m$ for the group of permutations of the set $\{1, \dots, m\}$. We denote both composition of permutations in $S_m$, and the action of a permutation in $S_m$ on an element of $\{1, \dots, m\}$ by juxtaposition; so for $\phi, \psi \in S_m$, $\phi\psi \in S_m$ is the permutation $\phi \circ \psi$, and for $\phi \in S_m$ and $j \in \{1, \dots, m\}$, $\phi j \in \{1, \dots, m\}$ is the image of $j$ under $\phi$. When convenient, we regard $S_m$ as (the morphisms of) a category with a single object.

Definition 2.4. Fix $k, m \in \mathbf{N} \setminus \{0\}$, and let $\Lambda$ be a $k$-graph. A *cocycle* $\mathfrak{s} : \Lambda \to S_m$ is a functor $\lambda \mapsto \mathfrak{s}(\lambda)$ from the category $\Lambda$ to the category $S_m$. That is, whenever $\alpha, \beta \in \Lambda$ satisfy $s(\alpha) = r(\beta)$ we have $\mathfrak{s}(\alpha)\mathfrak{s}(\beta) = \mathfrak{s}(\alpha\beta)$.

We are now ready to describe the data needed for our construction.

Definition 2.5. A *covering system of $k$-graphs* is a quintuple $(\Lambda, \Gamma, p, m, \mathfrak{s})$ where $\Lambda$ and $\Gamma$ are $k$-graphs, $p : \Lambda \to \Gamma$ is a covering, $m$ is a nonzero positive integer, and $\mathfrak{s} : \Gamma \to S_m$ is a cocycle. We say that the covering system is *row finite* if the covering map $p$ is finite and both $\Lambda$ and $\Gamma$ are row finite. When $m = 1$ and $\mathfrak{s}$ is the identity cocycle, we drop references to $m$ and $\mathfrak{s}$ altogether, and say that $(\Lambda, \Gamma, p)$ is a covering system of $k$-graphs.

Given a covering system $(\Lambda, \Gamma, p, m, \mathfrak{s})$ of $k$-graphs, we will define a $(k+1)$-graph $\Lambda \overset{p,\mathfrak{s}}{\leftarrow} \Gamma$ which encodes the covering map. Before the formal statement of this construction, we give an intuitive description of $\Lambda \overset{p,\mathfrak{s}}{\leftarrow} \Gamma$. The idea is that $\Lambda \overset{p,\mathfrak{s}}{\leftarrow} \Gamma$ is a $(k+1)$-graph containing disjoint copies $\imath(\Lambda)$ and $\jmath(\Gamma)$ of the $k$-graphs $\Lambda$ and $\Gamma$ in the first $k$ dimensions. The image $\jmath(v)$ of a vertex $v \in \Gamma$ is connected to the image $\imath(p(v))$ of the vertex it covers in $\Lambda$ by $m$ parallel edges $e(v, 1), \dots, e(v, m)$ of degree $e_{k+1}$. Factorisations of paths in $\Lambda \overset{p,\mathfrak{s}}{\leftarrow} \Gamma$ involving edges $e(v, l)$ of degree $e_{k+1}$ are determined by the unique path-lifting property and the cocycle $\mathfrak{s}$.
It may be helpful on the first reading to consider the case where $m = 1$ so that $\mathfrak{s}$ is necessarily trivial. To state the result formally, we first establish some notation.

*Notation* 2.6. Fix $k > 0$. For $n \in \mathbf{N}^k$ we denote by $(n, 0_1)$ the element $\sum_{i=1}^k n_i e_i \in \mathbf{N}^{k+1}$ and for $m \in \mathbf{N}$, we denote by $(0_k, m)$ the element $m e_{k+1} \in \mathbf{N}^{k+1}$. We write $(\mathbf{N}^k, 0_1)$ for $\{(n, 0_1) : n \in \mathbf{N}^k\}$ and $(0_k, \mathbf{N})$ for $\{(0_k, m) : m \in \mathbf{N}\}$.
Given a $(k+1)$-graph $\Xi$, we write $\Xi^{(0_k, \mathbf{N})}$ for $\{\xi \in \Xi : d(\xi) \in (0_k, \mathbf{N})\}$, and we write $\Xi^{(\mathbf{N}^k, 0_1)}$ for $\{\xi \in \Xi : d(\xi) \in (\mathbf{N}^k, 0_1)\}$. When convenient, we regard $\Xi^{(0_k, \mathbf{N})}$ as a 1-graph and $\Xi^{(\mathbf{N}^k, 0_1)}$ as a $k$-graph, ignoring the distinctions between $\mathbf{N}$ and $(0_k, \mathbf{N})$ and between $\mathbf{N}^k$ and $(\mathbf{N}^k, 0_1)$.

Proposition 2.7. *Let $(\Lambda, \Gamma, p, m, \mathfrak{s})$ be a covering system of $k$-graphs. There is a unique $(k+1)$-graph $\Lambda \overset{p,\mathfrak{s}}{\leftarrow} \Gamma$ such that:*

(1) *there are injective functors* $\imath : \Lambda \to \Lambda \overset{p,\mathfrak{s}}{\leftarrow} \Gamma$ *and* $\jmath : \Gamma \to \Lambda \overset{p,\mathfrak{s}}{\leftarrow} \Gamma$ *such that* $d(\imath(\alpha)) = (d(\alpha), 0_1)$ *and* $d(\jmath(\beta)) = (d(\beta), 0_1)$ *for all* $\alpha \in \Lambda$ *and* $\beta \in \Gamma$*;*
(2) $\imath(\Lambda) \cap \jmath(\Gamma) = \emptyset$ *and* $\imath(\Lambda) \cup \jmath(\Gamma) = \{\tau \in \Lambda \overset{p,\mathfrak{s}}{\leftarrow} \Gamma : d(\tau)_{k+1} = 0\}$*;*
(3) *there is a bijection* $e : \Gamma^0 \times \{1, \dots, m\} \to (\Lambda \overset{p,\mathfrak{s}}{\leftarrow} \Gamma)^{e_{k+1}}$*;*
(4) $s(e(v,l)) = \jmath(v)$ *and* $r(e(v,l)) = \imath(p(v))$ *for all* $v \in \Gamma^0$ *and* $1 \leq l \leq m$*; and*
(5) $e(r(\lambda), l)\jmath(\lambda) = \imath(p(\lambda))e(s(\lambda), \mathfrak{s}(\lambda)^{-1}l)$ *for all* $\lambda \in \Gamma$ *and* $1 \leq l \leq m$*.*

*If the covering system* $(\Lambda, \Gamma, p, m, \mathfrak{s})$ *is row finite, then* $\Lambda \overset{p,\mathfrak{s}}{\leftarrow} \Gamma$ *is row finite. Moreover,* $\Lambda$ *is locally convex if and only if* $\Gamma$ *is locally convex, and in this case* $\Lambda \overset{p,\mathfrak{s}}{\leftarrow} \Gamma$ *is also locally convex.*

*Notation* 2.8. If $m = 1$ so that $\mathfrak{s}$ is necessarily trivial, we drop all reference to $\mathfrak{s}$. We denote $\Lambda \overset{p,\mathfrak{s}}{\leftarrow} \Gamma$ by $\Lambda \overset{p}{\leftarrow} \Gamma$, and write $(\Lambda \overset{p}{\leftarrow} \Gamma)^{e_{k+1}} = \{e(v) : v \in \Gamma^0\}$. In this case, the factorisation property is determined by the unique path-lifting property alone.

The main ingredient in the proof of Proposition 2.7 is the following fact from [15, Remark 2.3] (see also [31, Section 2]).

Lemma 2.9. *Let* $E_1, \dots, E_k$ *be* 1*-graphs with the same vertex set* $E^0$*. For distinct* $i, j \in \{1, \dots, k\}$*, let* $E_{i,j} := \{(e,f) \in E_i^1 \times E_j^1 : s(e) = r(f)\}$*, and write* $r\big((e,f)\big) = r(e)$ *and* $s\big((e,f)\big) = s(f)$*. For distinct* $h, i, j \in \{1, \dots, k\}$*, let* $E_{h,i,j} := \{(e,f,g) \in E_h^1 \times E_i^1 \times E_j^1 : (e,f) \in E_{h,i}, (f,g) \in E_{i,j}\}$*.*
*Suppose we have bijections* $\theta_{i,j} : E_{i,j} \to E_{j,i}$ *such that* $r \circ \theta_{i,j} = r$*,* $s \circ \theta_{i,j} = s$ *and* $\theta_{i,j} \circ \theta_{j,i} = \mathrm{id}$*, and such that*

$$(2.1) \qquad (\theta_{i,j} \times \mathrm{id})(\mathrm{id} \times \theta_{h,j})(\theta_{h,i} \times \mathrm{id}) = (\mathrm{id} \times \theta_{h,i})(\theta_{h,j} \times \mathrm{id})(\mathrm{id} \times \theta_{i,j})$$

*as bijections from* $E_{h,i,j}$ *to* $E_{j,i,h}$*.*
*Then there is a unique* $k$*-graph* $\Lambda$ *such that* $\Lambda^0 = E^0$*,* $\Lambda^{e_i} = E_i^1$ *for* $1 \leq i \leq k$*, and for distinct* $i, j \in \{1, \dots, k\}$ *and* $(e,f) \in E_{i,j}$*, the pair* $(f', e') \in E_{j,i}$ *such that* $(f', e') = \theta_{i,j}(e,f)$ *satisfies* $ef = f'e'$ *as morphisms in* $\Lambda$*.*

*Remark* 2.10. Every $k$-graph arises in this way: Given a $k$-graph $\Lambda$, let $E^0 := \Lambda^0$, and $E_i^1 := \Lambda^{e_i}$ for $1 \leq i \leq k$, and define $r, s : E_i^1 \to E^0$ by restriction of the range and source maps in $\Lambda$. Define bijections $\theta_{i,j} : E_{i,j} \to E_{j,i}$ via the factorisation property: $\theta_{i,j}(e,f)$ is equal to the unique pair $(f', e') \in E_{j,i}$ such that $ef = f'e'$ in $\Lambda$. Then condition (2.1) holds by the associativity of the category $\Lambda$, and the uniqueness assertion of Lemma 2.9 implies that $\Lambda$ is isomorphic to the $k$-graph obtained from the $E_i$ and the $\theta_{i,j}$ using Lemma 2.9.

Lemma 2.9 tells us how to describe a $k$-graph pictorially. As in [31, 27], the *skeleton* of a $k$-graph $\Lambda$ is the directed graph $E_\Lambda$ with vertices $E_\Lambda^0 = \Lambda^0$, edges $E_\Lambda^1 = \bigcup_{i=1}^k \Lambda^{e_i}$, range and source maps inherited from $\Lambda$, and edges of different degrees in $\Lambda$ distinguished using $k$ different colours in $E_\Lambda$: in this paper, we will often refer to edges of degree $e_1$ as "blue" and edges of degree $e_2$ as "red." Lemma 2.9 implies that the skeleton $E_\Lambda$ together with the factorisation rules $fg = g'f'$ where $f, f' \in \Lambda^{e_i}$ and $g, g' \in \Lambda^{e_j}$ completely specify $\Lambda$. In practise, we draw $E_\Lambda$ using solid, dashed and dotted edges to distinguish the different colours, and list the factorisation rules separately.

*Proof of Proposition 2.7.* The idea is to apply Lemma 2.9 to obtain the $(k+1)$-graph $\Lambda \overset{p,\mathfrak{s}}{\leftarrow} \Gamma$. We first define sets $E^0$ and $E^1_i$ for $1 \le i \le k+1$. As a set, $E^0$ is a copy of the disjoint union $\Lambda^0 \sqcup \Gamma^0$. We denote the copy of $\Lambda^0$ in $E^0$ by $\{\imath(v) : v \in \Lambda^0\}$ and the copy of $\Gamma^0$ in $E^0$ by $\{\jmath(w) : w \in \Gamma^0\}$ where as yet the $\imath(v)$ and $\jmath(w)$ are purely formal symbols. So

$$E^0 = \{\imath(v) : v \in \Lambda^0\} \sqcup \{\jmath(w) : w \in \Gamma^0\}.$$

For $1 \le i \le k$, we define, in a similar fashion,

$$E^1_i := \{\imath(f) : f \in \Lambda^{e_i}\} \sqcup \{\jmath(g) : g \in \Gamma^{e_i}\}$$

to be a copy of the disjoint union $\Lambda^{e_i} \sqcup \Gamma^{e_i}$. We define $E^1_{k+1}$ to be a copy of $\Gamma^0 \times \{1, \dots, m\}$ which is disjoint from $E^0$ and each of the other $E^1_i$, and use formal symbols $\{e(v, l) : v \in \Gamma^0, 1 \le l \le m\}$ to denote its elements. For $1 \le i \le k$, define range and source maps $r, s : E^1_i \to E^0$ by $r(\imath(f)) := \imath(r(f))$, $s(\imath(f)) := \imath(s(f))$, $r(\jmath(g)) := \jmath(r(g))$ and $s(\jmath(g)) := \jmath(s(g))$. Define $r, s : E^1_{k+1} \to E^0$ as in Proposition 2.7(4).
For distinct $i, j \in \{1, \dots, k+1\}$, define $E_{i,j}$ as in Lemma 2.9. Define bijections $\theta_{i,j} : E_{i,j} \to E_{j,i}$ as follows:

- For $1 \le i, j \le k$ and $(e, f) \in E_{i,j}$, we must have either $e = \imath(a)$ and $f = \imath(b)$ for some composable pair $(a, b) \in \Lambda^{e_i} \times_{\Lambda^0} \Lambda^{e_j}$, or else $e = \jmath(a)$ and $f = \jmath(b)$ for some composable pair $(a, b) \in \Gamma^{e_i} \times_{\Gamma^0} 3\Gamma^{e_j}$. If $e = \imath(a)$ and $f = \imath(b)$, the factorisation property in $\Lambda$ yields a unique pair $b' \in \Lambda^{e_j}$, $a' \in \Lambda^{e_i}$ such that $ab = b'a'$, and we then define $\theta_{i,j}(e, f) = (\imath(b'), \imath(a'))$. If $e = \jmath(a)$ and $f = \jmath(b)$, we define $\theta_{i,j}(e, f)$ similarly using the factorisation property in $\Gamma$.
- For $1 \le i \le k$, and $(e, f) \in E_{k+1,i}$, we have $f = \jmath(b)$ and $e = e(r(b), l)$ for some $b \in \Gamma^{e_i}$ and $1 \le l \le m$. Define $\theta_{k+1,i}(e, f) := (\imath(p(b)), e(s(f), \mathfrak{s}(f)^{-1}l))$.
- For $1 \le i \le k$, to define $\theta_{i,k+1}$, first note that if $(f', e') = \theta_{k+1,i}(e, f)$, then $e' = e(w, l)$ for some $w \in \Gamma^0$ and $l \in \{1, \dots, m\}$ such that $p(w) = s(f')$, $f$ is the unique lift of $f'$ such that $s(f) = \jmath(w)$, and $e = e(r(f), \mathfrak{s}(f)l)$. It follows that $\theta_{k+1,i}$ is a bijection and we may define $\theta_{i,k+1} := \theta^{-1}_{k+1,i}$.

Since $\Lambda$ and $\Gamma$ are $k$-graphs, the maps $\theta_{i,j}$, $1 \le i, j \le k$ are bijections with $\theta_{j,i} = \theta^{-1}_{i,j}$, and we have $\theta_{i,k+1} = \theta^{-1}_{k+1,i}$ by definition, so to invoke Lemma 2.9, we just need to establish equation (2.1).
Equation (2.1) holds when $h, i, j \le k$ because $\Lambda$ and $\Gamma$ are both $k$-graphs. Suppose one of $h, i, j = k+1$. Fix edges $f_h \in E^1_h$, $f_i \in E^1_i$ and $f_j \in E^1_j$. First suppose that $h = k+1$; so $f_h = e(r(f_i), l)$ for some $l$, and $f_i$ and $f_j$ both belong to $\jmath(\Gamma)$. Apply the factorisation property for $\Gamma$ to obtain $f'_j$ and $f'_i$ such that $f'_i \in E^1_i$, $f'_j \in E^1_j$ and $f'_j f'_i = f_i f_j$. We then have $\theta_{i,j}(f_i, f_j) = (f'_j, f'_i)$. If we write $\tilde{p}$ for the map from $\{\jmath(f) : f \in \bigcup_{i=1}^k \Gamma^{e_i}\}$ to $\{\imath(f) : f \in \bigcup_{i=1}^k \Lambda^{e_i}\}$

given by $\tilde{p}(\jmath(\lambda)) := \imath(p(\lambda))$, then the properties of the covering map imply that $\theta_{i,j}(\tilde{p}(f_i), \tilde{p}(f_j)) = (\tilde{p}(f_j'), \tilde{p}(f_i)')$. Now

$$\begin{aligned}
&(\theta_{i,j} \times \mathrm{id})(\mathrm{id} \times \theta_{h,j})(\theta_{h,i} \times \mathrm{id})(f_h, f_i, f_j) \\
&\qquad = (\theta_{i,j} \times \mathrm{id})(\mathrm{id} \times \theta_{h,j})(\tilde{p}(f_i), e(s(f_i), \mathfrak{s}(f_i)^{-1}l), f_j) \\
&\qquad = (\theta_{i,j} \times \mathrm{id})(\tilde{p}(f_i), \tilde{p}(f_j), e(s(f_j), \mathfrak{s}(f_j)^{-1}(\mathfrak{s}(f_i)^{-1})l)) \\
&\qquad = (\tilde{p}(f_j'), \tilde{p}(f_i'), e(s(f_j), \mathfrak{s}(f_i f_j)^{-1}l), \qquad (2.2)
\end{aligned}$$

where, in the last equality, $\mathfrak{s}(f_j)^{-1}\mathfrak{s}(f_i)^{-1} = \mathfrak{s}(f_i f_j)^{-1}$ by the cocycle property. On the other hand,

$$\begin{aligned}
&(\mathrm{id} \times \theta_{h,i})(\theta_{h,j} \times \mathrm{id})(\mathrm{id} \times \theta_{i,j})(f_h, f_i, f_j) \\
&\qquad = (\mathrm{id} \times \theta_{h,i})(\theta_{h,j} \times \mathrm{id})(f_h, f_j', f_i') \\
&\qquad = (\mathrm{id} \times \theta_{h,i})(\tilde{p}(f_j'), e(s(f_j), \mathfrak{s}(f_j')^{-1}l), f_i') \\
&\qquad = \big(\tilde{p}(f_j'), \tilde{p}(f_i'), e(s(f_i), \mathfrak{s}(f_i')^{-1}(\mathfrak{s}(f_j')^{-1}l))\big) \\
&\qquad = \big(\tilde{p}(f_j'), \tilde{p}(f_i'), e(s(f_i), \mathfrak{s}(f_j' f_i')^{-1}l)\big).
\end{aligned}$$

Since $f_j' f_i' = f_i f_j$, this establishes (2.1) when $h = k+1$ and $1 \leq i, j \leq k$. Similar calculations establish (2.1) when $i = k+1$ and when $j = k+1$.

By Lemma 2.9, there is a unique $(k+1)$-graph $\Lambda \overset{p,\mathfrak{s}}{\leftarrow} \Gamma$ with $(\Lambda \overset{p,\mathfrak{s}}{\leftarrow} \Gamma)^0 = E^0$, $(\Lambda \overset{p,\mathfrak{s}}{\leftarrow} \Gamma)^{e_i} = E_i^1$ for all $i$ and with commuting squares determined by the $\theta_{i,j}$. Since the $\theta_{i,j}$, $1 \leq i, j \leq k$ agree with the factorisation properties in $\Gamma$ and $\Lambda$, the uniqueness assertion of Lemma 2.9 applied to paths consisting of edges in $E_1^1 \cup \cdots \cup E_k^1$ shows that $\imath$ and $\jmath$ extend uniquely to injective functors from $\Lambda$ and $\Gamma$ to $(\Lambda \overset{p,\mathfrak{s}}{\leftarrow} \Gamma)^{(\mathbf{N}^k, 0_1)}$ which satisfy Proposition 2.7(2). Assertions (3) and (4) of Proposition 2.7 follow from the definition of $E_{k+1}^1$, and the last assertion (5) is established by factorising $\lambda$ into edges from the $E_i^1$, $1 \leq i \leq k$ and then performing calculations like (2.2).

Now suppose that $p$ is finite. Then $\Gamma$ is row-finite if and only if $\Lambda$ is, and in this case, $\Lambda \overset{p,\mathfrak{s}}{\leftarrow} \Gamma$ is also row-finite because $p$ is locally bijective and $m < \infty$. That $p$ is locally bijective shows that $\Lambda$ is locally convex if and only if $\Gamma$ is. Suppose that $\Gamma$ is locally convex. Fix $1 \leq i < j \leq k+1$, $a \in (\Lambda \overset{p,\mathfrak{s}}{\leftarrow} \Gamma)^{e_i}$ and $b \in (\Lambda \overset{p,\mathfrak{s}}{\leftarrow} \Gamma)^{e_j}$ with $r(a) = r(b)$. If $j < k+1$ then $a$ and $b$ can be extended to paths of degree $e_i + e_j$ because $\Lambda$ and $\Gamma$ are locally convex. If $j = k+1$, then $b = e(v, l)$ for some $v \in \Gamma^0$ and $1 \leq l \leq m$. Let $a'$ be the lift of $a$ such that $r(a') = s(v)$, then $ae(s(a'), l)$ and $ba'$ extend $a$ and $b$ to paths of degree $e_i + e_j$. It follows that $\Lambda \overset{p}{\leftarrow} \Gamma$ is locally convex. $\square$

Corollary 2.11. *Fix $N \geq 2$ in $\mathbf{N} \cup \{\infty\}$. Let $(\Lambda_n, \Lambda_{n+1}, p_n, m_n, \mathfrak{s}_n)_{n=1}^{N-1}$ be a sequence of covering systems of $k$-graphs. Then there is a unique $(k+1)$-graph $\mathbf{\Lambda}$ such that $\mathbf{\Lambda}^{e_i} = \bigsqcup_{n=1}^{N} \Lambda_n^{e_i}$ for $1 \leq i \leq k$, $\mathbf{\Lambda}^{e_{k+1}} = \bigsqcup_{n=1}^{N-1} (\Lambda_n \overset{p_n,\mathfrak{s}_n}{\leftarrow} \Lambda_{n+1})^{e_{k+1}}$, and such that range, source and composition are all inherited from the $\Lambda_n \overset{p_n,\mathfrak{s}_n}{\leftarrow} \Lambda_{n+1}$. If each $(\Lambda_n, \Lambda_{n+1}, p_n, m_n, \mathfrak{s}_n)$ is row-finite then $\mathbf{\Lambda}$ is row-finite. If each $\Lambda_n$ is locally convex, so is $\mathbf{\Lambda}$, and if each $\Lambda_n$ is connected, so is $\mathbf{\Lambda}$.*

*Proof.* For the first part we just apply Lemma 2.9; the hypotheses follow automatically from the observation that if $h, i, j$ are distinct elements of $\{1, \dots, k+1\}$ then each path of degree $e_h + e_i + e_j$ lies in some $\Lambda_n \overset{p_n, \mathfrak{s}_n}{\leftharpoondown} \Lambda_{n+1}$, and these are all $(k+1)$-graphs by Proposition 2.7.
The arguments for row-finiteness, local convexity and connectedness are the same as those in Proposition 2.7. $\square$

*Notation* 2.12. When $N$ is finite, the $(k+1)$-graph $\mathbf{\Lambda}$ of the previous lemma will henceforth be denoted $\Lambda_1 \overset{p_1, \mathfrak{s}_1}{\leftharpoondown} \cdots \overset{p_{N-1}, \mathfrak{s}_{N-1}}{\leftharpoondown} \Lambda_N$. If $N = \infty$, we instead denote $\mathbf{\Lambda}$ by $\varprojlim(\Lambda_n; p_n, \mathfrak{s}_n)$.

2.1. Matrices of covering systems. In this subsection, we generalise our construction to allow for a different covering system $(\Lambda_j, \Gamma_i, p_{i,j}, m_{i,j}, \mathfrak{s}_{i,j})$ for each pair of connected components $\Lambda_j \subset \Lambda$ and $\Gamma_i \subset \Gamma$. The objective is to recover the example of the irrational rotation algebras [27, Example 6.5].

Definition 2.13. Fix nonnegative integers $c_\Lambda, c_\Gamma \in \mathbf{N} \setminus \{0\}$. A *matrix of covering systems* $(\Lambda_j, \Gamma_i, m_{i,j}, p_{i,j}, \mathfrak{s}_{i,j})_{i,j=1}^{c_\Gamma, c_\Lambda}$ consists of:

(1) $k$-graphs $\Lambda$ and $\Gamma$ which decompose into connected components $\Lambda = \bigsqcup_{j=1,\dots,c_\Lambda} \Lambda_j$ and $\Gamma = \bigsqcup_{i=1,\dots,c_\Gamma} \Gamma_i$;
(2) a matrix $(m_{i,j})_{i,j=1}^{c_\Gamma, c_\Lambda} \in M_{c_\Gamma, c_\Lambda}(\mathbf{N})$ with no zero rows or columns; and
(3) for each $i, j$ such that $m_{i,j} \neq 0$, a covering system $(\Lambda_i, \Gamma_j, p_{i,j}, m_{i,j}, \mathfrak{s}_{i,j})$ of $k$-graphs.

Proposition 2.14. *Fix nonnegative integers $c_\Lambda, c_\Gamma \in \mathbf{N} \setminus \{0\}$ and a matrix of covering systems $(\Lambda_j, \Gamma_i, m_{i,j}, p_{i,j}, \mathfrak{s}_{i,j})_{i,j=1}^{c_\Gamma, c_\Lambda}$. Then there is a unique $(k+1)$-graph*

$$\big(\textstyle\bigsqcup \Lambda_j\big) \overset{p, \mathfrak{s}}{\leftharpoondown} \big(\bigsqcup \Gamma_i\big)$$

*such that*

$$\Big(\big(\textstyle\bigsqcup \Lambda_j\big) \overset{p, \mathfrak{s}}{\leftharpoondown} \big(\bigsqcup \Gamma_i\big)\Big)^{e_{k+1}} = \textstyle\bigsqcup_{i,j} (\Lambda_j \overset{p_{i,j}, \mathfrak{s}_{i,j}}{\leftharpoondown} \Gamma_i)^{e_{k+1}},$$

*each $\big(\big(\bigsqcup \Lambda_j\big) \overset{p, \mathfrak{s}}{\leftharpoondown} \big(\bigsqcup \Gamma_i\big)\big)^{e_l}$ for $1 \leq l \leq k$ is equal to $\Lambda^{e_l} \sqcup \Gamma^{e_l}$ and the commuting squares are inherited from the $\Lambda_j \overset{p_{i,j}, \mathfrak{s}_{i,j}}{\leftharpoondown} \Gamma_i$.*
*If each $(\Lambda_i, \Gamma_j, p_{i,j}, m_{i,j}, \mathfrak{s}_{i,j})$ is row finite then $\big(\bigsqcup \Lambda_j\big) \overset{p, \mathfrak{s}}{\leftharpoondown} \big(\bigsqcup \Gamma_i\big)$ is row finite. If $\Lambda$ and $\Gamma$ are locally convex, so is $\big(\bigsqcup \Lambda_j\big) \overset{p, \mathfrak{s}}{\leftharpoondown} \big(\bigsqcup \Gamma_i\big)$.*

*Proof.* We apply Lemma 2.9; since the commuting squares are inherited from the $\Lambda_j \overset{p_{i,j}, \mathfrak{s}_{i,j}}{\leftharpoondown} \Gamma_i$, they satisfy the associativity condition (2.1) because each $\Lambda_j \overset{p_{i,j}, \mathfrak{s}_{i,j}}{\leftharpoondown} \Gamma_i$ is a $(k+1)$-graph. $\square$

Corollary 2.15. *Fix $N \geq 2$ in $\mathbf{N} \cup \{\infty\}$. Let $(c_n)_{n=1}^N \subset \mathbf{N} \setminus \{0\}$ be a sequence of positive integers. For $1 \leq n < N$, let $(\Lambda_{n,j}, \Lambda_{n+1,i}, p_{i,j}^n, m_{i,j}^n, \mathfrak{s}_{i,j}^n)_{i,j=1}^{c_{n+1}, c_n}$ be a matrix of covering systems. Then there exists a unique $(k+1)$-graph $\mathbf{\Lambda}$ such* *that*

$$\mathbf{\Lambda}^{e_i} = \bigcup_{n=1}^{N} \bigcup_{j=1}^{c_n} \Lambda_{n,j}^{e_i} \qquad \textit{for } 1 \le i \le k,$$
$$\mathbf{\Lambda}^{e_{k+1}} = \bigcup_{n=1}^{N-1} \Big( \big( \bigsqcup_{j=1}^{c_n} \Lambda_{n,j} \big) \overset{p^n, \mathfrak{s}^n}{\leftarrow} \big( \bigsqcup_{i=1}^{c_{n+1}} \Lambda_{n+1,i} \big) \Big)^{e_{k+1}},$$

*and the range, source and composition functions are all inherited from the $(k+1)$-graphs $\big( \bigsqcup_{j=1}^{c_n} \Lambda_{n,j} \big) \overset{p^n, \mathfrak{s}^n}{\leftarrow} \big( \bigsqcup_{i=1}^{c_{n+1}} \Lambda_{n+1,i} \big)$.*
*If each $(\Lambda_{n,j}, \Lambda_{n+1,i}, p_{i,j}^n, m_{i,j}^n, \mathfrak{s}_{i,j}^n)$ is row finite, then $\mathbf{\Lambda}$ is row finite. If each $\Lambda_n$ is locally convex, so is $\mathbf{\Lambda}$.*

*Example* 2.16 (The Irrational Rotation algebras). Fix $\theta \in [0,1] \setminus \mathbf{Q}$. Let $[a_1, a_2, \dots]$ be the simple continued fraction expansion of $\theta$. For each $n$, let $c_n = 2$, let $\phi_n := \left(\begin{smallmatrix} a_n & 1 \\ 1 & 0 \end{smallmatrix}\right)$, and let $m^n := (m_{i,j}^n)_{i,j=1}^2$ be the matrix product $\phi_{T(n+1)} \cdots \phi_{T(n)+1}$ where $T(n) := n(n+1)/2$ is the $n^{\text{th}}$ triangular number. Of all the integers $m_{i,j}^n$ obtained this way, only $m_{1,2}^1$ is equal to zero, so the matrices $m^n$ have no zero rows or columns. Whenever $m_{i,j}^n \neq 0$, let $\mathfrak{s}_{i,j}^n$ be the permutation of the set $\{1, \dots, m_{i,j}^n\}$ given by $\mathfrak{s}_{i,j}^n l = l+1$ if $1 \le l < m_{i,j}^n$, and $\mathfrak{s}_{i,j}^n m_{i,j}^n = 1$.
Let $\Lambda_{n,i}$, $n \in \mathbf{N} \setminus \{0\}$, $i = 1, 2$ be mutually disjoint copies of the 1-graph $T_1$ whose skeleton consists of a single vertex and a single directed edge. For each $n$, let $\Lambda_n$ be the 1-graph $\Lambda_{n,1} \sqcup \Lambda_{n,2}$ so that for each $n$, $(\Lambda_{n,j}, \Lambda_{n+1,i}, p_{i,j}^n, m_{i,j}^n, \mathfrak{s}_{i,j}^n)_{i,j=1}^2$ is a matrix of covering systems.

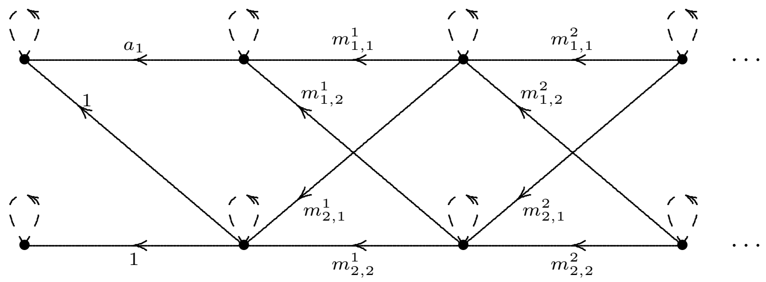


Figure 1. A tower of coverings with multiplicities

Modulo relabelling the generators of $\mathbf{N}^2$, the 2-graph $\varprojlim \Big( \bigsqcup_{j=1}^{c_n} \Lambda_{n,j}; p_{i,j}^n, \mathfrak{s}_{i,j}^n \Big)$ obtained from this data as in Corollary 2.15 is precisely the rank-2 Bratteli diagram of [27, Example 6.5] whose $C^*$-algebra is Morita equivalent to the irrational rotation algebra $A_\theta$. Figure 1 is an illustration of its skeleton (parallel edges drawn as a single edge with a label indicating the multiplicity). The factorisation rules are all of the form $fg = \sigma(g)f'$ where $f$ and $f'$ are the dashed loops at either end of a solid edge in the diagram, and $\sigma$ is a transitive permutation of the set of edges with the same range and source as $g$.
More generally, Section 7 of [27] considers in some detail the structure of the $C^*$-algebras associated to rank-2 Bratteli diagrams with length-1 cycles. All such rank-2 Bratteli diagrams can be recovered as above from Corollary 2.15.

## 3. $C^*$-algebras associated to covering systems of $k$-graphs

In this section, we describe how a covering system $(\Lambda, \Gamma, p, m, \mathfrak{s})$ induces an inclusion of $C^*$-algebras $C^*(\Lambda) \hookrightarrow M_m(C^*(\Gamma))$ and hence a homomorphism of $K$-groups $K_*(C^*(\Lambda)) \to K_*(C^*(\Gamma))$. The main result of the section is Theorem 3.8 which shows how to use these maps to compute the $K$-theory of $C^*(\varprojlim(\Lambda_n; p_n, \mathfrak{s}_n))$ from the data in a sequence $(\Lambda_n, \Lambda_{n+1}, p_n, m_n, \mathfrak{s}_n)_{n=1}^{\infty}$ of covering systems.

The following definition of the Cuntz-Krieger algebra of a row-finite locally convex $k$-graph $\Lambda$ is taken from [31, Definition 3.3].

Given a row-finite, locally convex $k$-graph $(\Lambda, d)$, a Cuntz-Krieger $\Lambda$-family is a collection $\{t_\lambda : \lambda \in \Lambda\}$ of partial isometries satisfying

(CK1) $\{t_v : v \in \Lambda^0\}$ is a collection of mutually orthogonal projections;
(CK2) $t_\lambda t_\mu = t_{\lambda\mu}$ whenever $s(\lambda) = r(\mu)$;
(CK3) $t_\lambda^* t_\lambda = t_{s(\lambda)}$ for all $\lambda \in \Lambda$; and
(CK4) $t_v = \sum_{\lambda \in v\Lambda^{\leq n}} t_\lambda t_\lambda^*$ for all $v \in \Lambda^0$ and $n \in \mathbf{N}^k$.

The *Cuntz-Krieger algebra* $C^*(\Lambda)$ is the $C^*$-algebra generated by a Cuntz-Krieger $\Lambda$-family $\{s_\lambda : \lambda \in \Lambda\}$ which is universal in the sense that for every Cuntz-Krieger $\Lambda$-family $\{t_\lambda : \lambda \in \Lambda\}$ there is a unique homomorphism $\pi_t$ of $C^*(\Lambda)$ satisfying $\pi_t(s_\lambda) = t_\lambda$ for all $\lambda \in \Lambda$.

*Remarks* 3.1. If $\Lambda$ has no sources (that is $v\Lambda^n \neq \emptyset$ for all $v \in \Lambda^0$ and $n \in \mathbf{N}^k$), then $\Lambda$ is automatically locally convex, and the definition of $C^*(\Lambda)$ given above reduces to the original definition [20, Definition 1.5].

By [31, Theorem 3.15] there is a Cuntz-Krieger $\Lambda$-family $\{t_\lambda : \lambda \in \Lambda\}$ such that $t_\lambda \neq 0$ for all $\lambda \in \Lambda$. The universal property of $C^*(\Lambda)$ therefore implies that the generating partial isometries $\{s_\lambda : \lambda \in \Lambda\} \subset C^*(\Lambda)$ are all nonzero.

Let $\Xi$ be a $k$-graph. The universal property of $C^*(\Xi)$ gives rise to an action $\gamma$ of $\mathbf{T}^k$ on $C^*(\Xi)$, called the *gauge-action* (see, for example [31, §4.1]), such that $\gamma_z(s_\xi) = z^{d(\xi)} s_\xi$ for all $z \in \mathbf{T}^k$ and $\xi \in \Xi$.

Proposition 3.2. *Let $(\Lambda, \Gamma, p, m, \mathfrak{s})$ be a row-finite covering system of locally convex $k$-graphs. Let $\gamma_\Lambda$ and $\gamma_\Gamma$ denote the gauge actions of $\mathbf{T}^k$ on $C^*(\Lambda)$ and $C^*(\Gamma)$, and let $\gamma$ denote the gauge action of $\mathbf{T}^{k+1}$ on $C^*(\Lambda \overset{p,\mathfrak{s}}{\leftarrow} \Gamma)$.*

(1) *The inclusions $\imath : \Lambda \to \Lambda \overset{p,\mathfrak{s}}{\leftarrow} \Gamma$ and $\jmath : \Gamma \to \Lambda \overset{p,\mathfrak{s}}{\leftarrow} \Gamma$ induce embeddings of $C^*(\Lambda)$ and $C^*(\Gamma)$ in $C^*(\Lambda \overset{p,\mathfrak{s}}{\leftarrow} \Gamma)$ characterised by*
$$\imath_*(s_\alpha) = s_{\imath(\alpha)} \text{ and } \jmath_*(s_\beta) = s_{\jmath(\beta)} \quad \text{for } \alpha \in \Lambda \text{ and } \beta \in \Gamma.$$
(2) *The sum $\sum_{v \in \jmath(\Gamma^0)} s_v$ converges in the strict topology to a full projection $Q \in \mathcal{M}(C^*(\Lambda \overset{p,\mathfrak{s}}{\leftarrow} \Gamma))$, and the range of $\jmath_*$ is $QC^*(\Lambda \overset{p,\mathfrak{s}}{\leftarrow} \Gamma)Q$.*
(3) *For $1 \leq i \leq m$, the sum $\sum_{v \in \Gamma^0} s_{e(v,i)}$ converges strictly to a partial isometry $V_i \in \mathcal{M}(C^*(\Lambda \overset{p,\mathfrak{s}}{\leftarrow} \Gamma))$. The sum $\sum_{v \in \imath(\Lambda^0)} s_v$, converges strictly to the full projection $P := \sum_{i=1}^m V_i V_i^* \in \mathcal{M}(C^*(\Lambda \overset{p,\mathfrak{s}}{\leftarrow} \Gamma))$. Moreover, $\imath_*$ is a nondegenerate homomorphism into $PC^*(\Lambda \overset{p,\mathfrak{s}}{\leftarrow} \Gamma)P$.*

(4) *There is an isomorphism* $\phi : M_m(C^*(\Gamma)) \to PC^*(\Lambda \overset{p,\mathfrak{s}}{\leftarrow} \Gamma)P$ *such that*

$$\phi\big((a_{i,j})_{i,j=1}^m\big) = \sum_{i,j=1}^m V_i \jmath_*(a_{i,j}) V_j^*.$$

(5) *There is an embedding* $\iota_{p,\mathfrak{s}} : C^*(\Lambda) \to M_m(C^*(\Gamma))$ *such that* $\phi \circ \iota_{p,\mathfrak{s}} = \imath_*$. *The embedding* $\iota_{p,\mathfrak{s}}$ *is equivariant in* $\gamma_\Lambda$ *and the action* $\mathrm{id}_m \otimes \gamma_\Gamma$ *of* $\mathbf{T}^k$ *on* $M_m(C^*(\Gamma))$ *by coordinate-wise application of* $\gamma_\Gamma$.

(6) *If we identify* $K_*(C^*(\Gamma))$ *with* $K_*(M_m(C^*(\Gamma)))$*, then the induced homomorphism* $(\iota_{p,\mathfrak{s}})_*$ *may be viewed as a map from* $K_*(C^*(\Lambda)) \to K_*(C^*(\Gamma))$. *When applied to* $K_0$*-classes of vertex projections, this map satisfies*

$$(\iota_{p,\mathfrak{s}})_*([s_v]) = \sum_{p(u)=v} m \cdot [s_u] \in K_0(C^*(\Gamma)).$$

The proofs of the last three statements require the following general Lemma. This is surely well-known but we include it for completeness.

Lemma 3.3. *Let* $A$ *be a* $C^*$*-algebra, let* $q \in \mathcal{M}(A)$ *be a projection, and suppose that* $v_1, \dots, v_n \in \mathcal{M}(A)$ *satisfy* $v_i^* v_j = \delta_{i,j} q$ *for* $1 \le i, j \le n$. *Then* $p = \sum_{i=1}^n v_i v_i^*$ *is a projection and* $pAp \cong M_n(qAq)$.

*Proof.* That $v_i^* v_j = \delta_{i,j} q$ implies that the $v_i$ are partial isometries with mutually orthogonal range projections $v_i v_i^*$. Hence $p$ is a projection in $\mathcal{M}(A)$. Define a map $\phi$ from $pAp$ to $M_n(qAq)$ as follows: for $a \in pAp$ and $1 \le i, j \le n$, let $a_{i,j} := v_i^* a v_j$, and define $\phi(a)$ to be the matrix $\phi(a) = (a_{i,j})_{i,j=1}^n$.
It is straightforward to check using the properties of the $v_i$ that $\phi$ is a $C^*$-homomorphism. It is an isomorphism because the homomorphism $\psi : M_n(qAq) \to pAp$ defined by

$$\psi\big((a_{i,j})_{i,j=1}^n\big) := \sum_{i,j=1}^n v_i a_{ij} v_j^* \in qAq$$

is an inverse for $\phi$. □

*Proof of Proposition 3.2.* (1) The collection $\{s_{\imath(\lambda)} : \lambda \in \Lambda\}$ forms a Cuntz-Krieger $\Lambda$-family in $C^*(\Lambda \overset{p,\mathfrak{s}}{\leftarrow} \Gamma)$, and so by the universal property of $C^*(\Lambda)$ induces a homomorphism $\imath_* : C^*(\Lambda) \to C^*(\Lambda \overset{p,\mathfrak{s}}{\leftarrow} \Gamma)$. For $z \in \mathbf{T}^k$, write $(z, 1)$ for the element $(z_1, \dots, z_k, 1) \in \mathbf{T}^{k+1}$. Recall that $\gamma$ denotes the gauge action of $\mathbf{T}^{k+1}$ on $C^*(\Lambda \overset{p,\mathfrak{s}}{\leftarrow} \Gamma)$. Then the action $z \mapsto \gamma_{(z,1)}$ of $\mathbf{T}^k$ on $C^*(\Lambda \overset{p,\mathfrak{s}}{\leftarrow} \Gamma)$ satisfies

$$\imath_*((\gamma_\Lambda)_z(a)) = \gamma_{(z,1)}(\imath_*(a))$$

for all $a \in C^*(\Lambda)$ and $z \in \mathbf{T}^k$. Since $\imath_*(s_v) = s_{\imath(v)} \ne 0$ for all $v \in \Lambda^0$ it follows from the gauge-invariant uniqueness theorem [20, Theorem 2.1] that $\imath_*$ is injective. A similar argument applies to $\jmath_*$.
(2) As the projections $s_v$, $v \in \jmath(\Gamma^0)$ are mutually orthogonal, a standard argument shows that the sum $\sum_{v \in \jmath(\Gamma^0)} s_v$ converges to a projection $Q$ in the multiplier algebra (see [30, Lemma 2.1]). The range of $\jmath_*$ is equal to $QC^*(\Lambda \overset{p,\mathfrak{s}}{\leftarrow} \Gamma)Q$

because $\jmath(\Gamma^0)(\Lambda \overset{p,\mathfrak{s}}{\leftarrow} \Gamma)\jmath(\Gamma^0) = \jmath(\Gamma)$. To see that $Q$ is full, it suffices to show that every generator of $C^*(\Lambda \overset{p,\mathfrak{s}}{\leftarrow} \Gamma)$ belongs to the ideal $I(Q)$ generated by $Q$. So let $\alpha \in \Lambda \overset{p,\mathfrak{s}}{\leftarrow} \Gamma$. Either $s(\alpha) \in \jmath(\Gamma^0)$ or $s(\alpha) \in \imath(\Lambda^0)$. If $s(\alpha) \in \jmath(\Gamma^0)$, then $s_\alpha = s_\alpha Q \in I(Q)$. On the other hand, if $s(\alpha) \in \imath(\Lambda^0)$, the Cuntz-Krieger relation ensures that

$$s_\alpha = \sum_{p(w)=s(\alpha)} \sum_{i=1}^m s_\alpha s_{e(w,i)} Q s_{e(w,i)}^*,$$

which also belongs to $I(Q)$.

(3) For fixed $i$, the partial isometries $s_{e(v,i)}$ have mutually orthogonal range projections and mutually orthogonal source projections. Hence an argument similar that of [30, Lemma 2.1] shows that $\sum_{v \in \Gamma^0} s_{e(v,i)}$ converges strictly to a multiplier $V_i \in \mathcal{M}(C^*(\Lambda \overset{p,\mathfrak{s}}{\leftarrow} \Gamma))$. A simple calculation shows that $V_i^* V_j = \delta_{i,j} Q$ for all $i, j$. Hence each $V_i$ is a partial isometry, and $P$ is full because $Q$ is full. The homomorphism $\imath_*$ is nondegenerate because the net

$$\Big(\imath_*\big(\sum_{v \in F} s_v\big)\Big)_{F \subset \Lambda^0 \text{ finite}}$$

converges strictly to $P \in \mathcal{M}(C^*(\Lambda \overset{p,\mathfrak{s}}{\leftarrow} \Gamma))$.

(4) This follows directly from Part (3) and Lemma 3.3.

(5) We define $\iota_{p,\mathfrak{s}} := \phi^{-1} \circ \imath_*$. For the gauge-equivariance, recall that $\imath_*$ (respectively $\jmath_*$) are equivariant in $\gamma|_{(\mathbf{T}^k,1)}$ and $\gamma_\Lambda$ (respectively $\gamma_\Gamma$). By definition, $\phi$ is equivariant in $(\mathrm{id} \otimes \gamma)$ and $\gamma_{(\mathbf{T}^k,1)} \circ \jmath_*$. The equivariance of $\iota_{p,\mathfrak{s}}$ follows.

(6) By (CK4), for $v \in \Lambda^0$ we have $s_{\imath(v)} = \sum_{f \in v(\Lambda \overset{p,\mathfrak{s}}{\leftarrow} \Gamma)^{e_{k+1}}} s_f s_f^*$, so the $K_0$-class $[s_{\imath(v)}]$ is equal to $\sum_{f \in v(\Lambda \overset{p,\mathfrak{s}}{\leftarrow} \Gamma)^{e_{k+1}}} [s_f s_f^*]$. We can write $v(\Lambda \overset{p,\mathfrak{s}}{\leftarrow} \Gamma)^{e_{k+1}}$ as the disjoint union

$$v(\Lambda \overset{p,\mathfrak{s}}{\leftarrow} \Gamma)^{e_{k+1}} = \bigsqcup_{p(u)=v} \{e(u,i) : 1 \le i \le m\}.$$

In $K_0(C^*(\Lambda \overset{p,\mathfrak{s}}{\leftarrow} \Gamma))$, we have $[s_{e(u,i)} s_{e(u,i)}^*] = [s_{e(u,i)}^* s_{e(u,i)}] = [s_{\jmath(u)}]$, and the result follows. □

*Notation* 3.4. As in Notation 2.8, when $m = 1$ so that $\mathfrak{s}$ is trivial, we continue to drop references to $\mathfrak{s}$ at the level of $C^*$-algebras. So Proposition 3.2(5) gives an inclusion $\iota_p : C^*(\Lambda) \to C^*(\Gamma)$ and the induced homomorphism of $K$-groups obtained from Proposition 3.2(6) is denoted $(\iota_p)_* : K_*(C^*(\Lambda)) \to K_*(C^*(\Gamma))$. This homomorphism satisfies

$$(\iota_p)_*([s_v]) = \sum_{p(u)=v} [s_u].$$

When no confusion is likely to occur, we will suppress the maps $\imath$, $\jmath$, $\imath_*$ and $\jmath_*$ and regard $\Lambda$ and $\Gamma$ as subsets of $\Lambda \overset{p,\mathfrak{s}}{\leftarrow} \Gamma$ and $C^*(\Lambda)$ and $C^*(\Gamma)$ as $C^*$-subalgebras of $C^*(\Lambda \overset{p,\mathfrak{s}}{\leftarrow} \Gamma)$.

*Remark* 3.5. (1) The isomorphism $\phi$ of Proposition 3.2(4) extends to an isomorphism $\tilde{\phi} : M_{m+1}(C^*(\Gamma)) \to C^*(\Lambda \overset{p,\mathfrak{s}}{\leftarrow} \Gamma)$ which takes the block diagonal matrix $\left(\begin{smallmatrix} 0_{m\times m} & 0_{m\times 1} \\ 0_{1\times m} & a \end{smallmatrix}\right)$ to $\jmath_*(a)$. To see this, let $V, \dots, V_m$ be as in Proposition 2.7(3), let $V_{m+1} = Q$, and apply Lemma 3.3.

(2) If $m = 1$ then $\phi$ is an isomorphism of $C^*(\Gamma)$ onto $PC^*(\Lambda \overset{p}{\leftarrow} \Gamma)P$, and $\iota_p : C^*(\Lambda) \hookrightarrow C^*(\Gamma)$ satisfies

$$\iota_p(s_\lambda) = \textstyle\sum_{p(\tilde\lambda)=\lambda} s_{\tilde\lambda}.$$

Fix $N \geq 2$ in $\mathbf{N}$. Let $(\Lambda_n, \Lambda_{n+1}, p_n, m_n, \mathfrak{s}_n)_{n=1}^{N-1}$ be a sequence of row-finite covering systems of locally convex $k$-graphs. Recall that in Corollary 2.11 we obtained from such data a $(k+1)$-graph $\Lambda_1 \overset{p_1,\mathfrak{s}_1}{\leftarrow} \cdots \overset{p_{N-1},\mathfrak{s}_{N-1}}{\leftarrow} \Lambda_N$, which for convenience we will denote $\mathbf{\Lambda}_N$ (the subscript is unnecessary here, but will be needed in Proposition 3.7). We now examine the structure of $C^*(\mathbf{\Lambda}_N)$ using Proposition 3.2.

Proposition 3.6. *Continue with the notation established in the previous paragraph. For each $v \in \Lambda_N^0$, list $\mathbf{\Lambda}_N^{Ne_{k+1}} v$ as $\{\alpha(v,i) : 1 \leq i \leq M\}$ where $M = m_1 m_2 \cdots m_{N-1}$.*

(1) *For $1 \leq n \leq N$, the sum $\sum_{v\in\Lambda_n^0} s_v$ converges strictly to a full projection $P_n \in \mathcal{M}(C^*(\mathbf{\Lambda}_N))$.*
(2) *For $1 \leq i \leq M$, the sum $\sum_{v\in\Lambda_N^0} s_{\alpha(v,i)}$ converges strictly to a partial isometry $V_i \in \mathcal{M}(C^*(\mathbf{\Lambda}_N))$ such that $V_i^* V_i = P_N$.*
(3) *We have $\sum_{i=1}^M V_i V_i^* = P_1$, and there is an isomorphism*

$$\phi : M_M(C^*(\Lambda_N)) \to P_1 C^*(\mathbf{\Lambda}_N) P_1$$

*such that $\phi((a_{i,j})_{i,j=1}^M) = \sum_{i,j=1}^M V_i a_{i,j} V_j^*$.*

*Proof.* Calculations like those in parts (2) and (3) of Proposition 3.2 show that the sums defining the $P_n$ and the $V_i$ converge in the multiplier algebra of $C^*(\mathbf{\Lambda}_N)$ and that each $P_n$ is full.
Since distinct paths in $\mathbf{\Lambda}_N^{Ne_{k+1}}$ have orthogonal range projections and since paths in $\mathbf{\Lambda}_N^{Ne_{k+1}}$ with distinct sources have orthogonal source projections, each $V_i^* V_i = P_N$, and $\sum_{i=1}^M V_i V_i^* = P_1$.
One checks as in Proposition 3.2(1) that the inclusions $\imath_n : \Lambda_n \hookrightarrow \mathbf{\Lambda}_N$ induce inclusions $(\imath_n)_* : C^*(\Lambda_n) \hookrightarrow P_n C^*(\mathbf{\Lambda}_N) P_n$, and in particular that $(\imath_N)_* : C^*(\Lambda_N) \to P_N C^*(\mathbf{\Lambda}_N) P_N$ is an isomorphism. The final statement follows from Lemma 3.3. □

We now describe the inclusions of the corners determined by $P_1$ as $N$ increases. To do this, we first need some notation. Given a $C^*$-algebra $A$, and positive integers $m, n$, we denote by $\pi_{m,n} \otimes \mathrm{id}_A : M_m(M_n(A)) \to M_{mn}(A)$ the canonical isomorphism which takes the matrix $a = \big((a_{i,j,j',i'})_{j,j'=1}^n\big)_{i,i'=1}^m$ to the matrix $\pi(a)$ satisfying

$$\pi(a)_{j+n(i-1),j'+n(i'-1)} = a_{i,j,j',i'} \quad \text{for } 1 \leq i, i' \leq m,\ 1 \leq j, j' \leq n.$$

Given $C^*$-algebras $A$ and $B$, a positive integer $m$, and a $C^*$-homomorphism $\psi : A \to B$, we write $\mathrm{id}_m \otimes \psi : M_m(A) \to M_m(B)$ for the $C^*$-homomorphism

$$(\mathrm{id}_m \otimes \psi)\big((a_{i,j})_{i,j=1}^m\big) = \big(\psi(a_{i,j})\big)_{i,j=1}^m.$$

Finally, given a matrix algebra $M_m(A)$ over a $C^*$-algebra $A$, and given $1 \leq i, i' \leq m$ and $a \in A$, we write $\theta_{i,i'}a$ for the matrix

$$(\theta_{i,i'}a)_{j,j'} = \begin{cases} a & \text{if } j = i \text{ and } j' = i' \\ 0 & \text{otherwise.} \end{cases}$$

PROPOSITION 3.7. *Fix $N \geq 2$ in $\mathbf{N}$. Let $(\Lambda_n, \Lambda_{n+1}, p_n, m_n, \mathfrak{s}_n)_{n=1}^N$ be a sequence of row-finite covering systems of locally convex k-graphs. We view the $(k+1)$-graph $\mathbf{\Lambda}_N := \Lambda_1 \xleftarrow{p_1,\mathfrak{s}_1} \cdots \xleftarrow{p_{N-1},\mathfrak{s}_{N-1}} \Lambda_N$ as a subcategory of $\mathbf{\Lambda}_{N+1} := \Lambda_1 \xleftarrow{p_1,\mathfrak{s}_1} \cdots \xleftarrow{p_N,\mathfrak{s}_N} \Lambda_{N+1}$ and likewise regard $C^*(\mathbf{\Lambda}_N)$ as a $C^*$-subalgebra of $C^*(\mathbf{\Lambda}_{N+1})$. In particular, we view $P_1 = \sum_{v \in \Lambda_1^0} s_v$ as a projection in both $\mathcal{M}(C^*(\mathbf{\Lambda}_N))$ and $\mathcal{M}(C^*(\mathbf{\Lambda}_{N+1}))$.*
*Let $M := m_1 m_2 \dots m_{N-1}$, and let $\phi_N : M_M(C^*(\Lambda_N)) \to P_1(C^*(\mathbf{\Lambda}_N)P_1$ and $\phi_{N+1} : M_{Mm_N}(C^*(\Lambda_{N+1})) \to P_1 C^*(\mathbf{\Lambda}_{N+1})P_1$ be the isomorphisms obtained from Proposition 3.6. Then the following diagram commutes.*

$$\begin{tikzcd}[column sep=huge, row sep=large]
P_1 C^*(\mathbf{\Lambda}_N) P_1 \arrow[r, hook, "\subseteq"] & P_1 C^*(\mathbf{\Lambda}_{N+1}) P_1 \\
M_M(C^*(\Lambda_N)) \arrow[u, "\phi_N"'] \arrow[r, "(\pi_{M,m_N} \otimes \mathrm{id}_{C^*(\Lambda_{N+1})}) \circ (\mathrm{id}_M \otimes \iota_{p_N,\mathfrak{s}_N})"] & M_{Mm_N}(C^*(\Lambda_{N+1})) \arrow[u, "\phi_{N+1}"']
\end{tikzcd}$$

*Proof.* As in Proposition 3.6, write $\mathbf{\Lambda}_N^{Ne_{k+1}} = \{\alpha(v,i) : v \in \Lambda_N^0, i \in \{1, \cdots, M\}\}$. For $i = 1, \dots, M$, let $V_i := \sum_{v \in \Lambda_N^0} s_{\alpha(v,i)}$. For $j = 1, \dots, m_N$, let

$$W_j := \sum_{w \in \Lambda_{N+1}^0} \sum_{i=1}^M s_{\alpha(p_N(w),i)} s_{e(w,j)}.$$

For $(i,j)$ in the cartesian product $\{1, \dots, M\} \times \{1, \dots, m_N\}$, let $U_{j+m_N(i-1)} := \sum_{u \in \Lambda_{N+1}^0} s_{\alpha(p_N(u),i)e(u,j)}$. In what follows, we suppress canonical inclusion maps, and regard $C^*(\Lambda_N)$ as a subalgebra of $C^*(\mathbf{\Lambda}_N)$, and both $C^*(\mathbf{\Lambda}_N)$ and $C^*(\Lambda_{N+1})$ as subalgebras of $C^*(\mathbf{\Lambda}_{N+1})$. The corner $P_1 C^*(\mathbf{\Lambda}_N) P_1$ is equal to the closed span of elements of the form $V_i a V_{i'}^*$ where $a \in C^*(\Lambda_N)$ and $i, i' \in \{1, \dots, M\}$, and $P_1 C^*(\mathbf{\Lambda}_{N+1}) P_1$ is equal to the closed span of elements of the form $U_l b U_{l'}^*$ where $b \in C^*(\Lambda_{N+1})$, $l, l' \in \{1, \dots, Mm_N\}$.
We have $\phi_N\big((a_{i,i'})_{i,i'=1}^M\big) = \sum_{i,i'=1}^M V_i a_{i,i'} V_{i'}^*$ by definition. The isomorphism $\phi_{N+1}$ from $M_{Mm_N}(C^*(\Lambda_{N+1}))$ to $P_1 C^*(\mathbf{\Lambda}_{N+1}) P_1$ described in Proposition 3.6 satisfies

$$\phi_{N+1}\big(\textstyle\sum_{l,l'=1}^{Mm_N} U_l b_{l,l'} U_{l'}^*\big) = \big(b_{l,l'}\big)_{l,l'=1}^{Mm_N}.$$

The Cuntz-Krieger relations show that

$$V_iV_{i'}^*W_jW_{j'}^* = U_{j+m_N(i-1)}U_{j'+m_N(i'-1)}^* = W_jW_{j'}^*V_iV_{i'}^*$$

for $1 \le i, i' \le M$, $1 \le j, j' \le m_N$, and this decomposition of the matrix units $U_lU_{l'}^*$ implements $\pi_{M,m_N}$. Hence $\phi_{N+1} \circ (\pi_{M,m_N} \otimes \mathrm{id}_{C^*(\Lambda_{N+1})})$ satisfies

$$\begin{aligned}\phi_{N+1} \circ (\pi_{M,m_N} \otimes \mathrm{id}_{C^*(\Lambda_{N+1})})\Big(\big((b_{i,j,j',i'})_{j,j'=1}^{m_N}\big)_{i,i'=1}^{M}\Big) \\ = \textstyle\sum_{i,i'=1}^{M}\sum_{j,j'=1}^{m_N} U_{j+m_N(i-1)}b_{i,j,j',i'}U_{j'+m_N(i'-1)}^*.\end{aligned} \tag{3.1}$$

The Cuntz-Krieger relations also show that $V_i = \sum_{j=1}^{m_N} W_jW_j^*V_i$ for all $i$, and hence $V_iaV_{i'}^* = \sum_j U_{j+m_N(i-1)}W_j^*aW_jU_{j+m_N(i'-1)}^*$ for all $a \in P_1C^*(\mathbf{\Lambda}_N)P_1$. One now checks that for $\lambda \in \Lambda_N$, we have

$$W_j^*s_\lambda W_j = \textstyle\sum_{p_N(\lambda')=\lambda} s_{e(r(\lambda'),j)}^*s_{e(r(\lambda),\mathfrak{s}_N(\lambda')j)}s_{\lambda'},$$

and hence that $V_is_\lambda V_{i'}^* = \sum_j\sum_{p_N(\lambda')=\lambda} U_{\mathfrak{s}_N(\lambda')j+m_N(i-1)}s_{\lambda'}U_{j+m_N(i'-1)}^*$. Recall that $\theta_{i,i'}s_\lambda \in M_M(C^*(\Lambda_N))$ denotes the matrix

$$(\theta_{i,i'}s_\lambda)_{j,j'} = \begin{cases} s_\lambda & \text{if } j = i \text{ and } j' = i' \\ 0 & \text{otherwise.}\end{cases}$$

Then $V_is_\lambda V_{i'}^* = \phi_N(\theta_{i,i'}s_\lambda)$ by definition of $\phi_N$, so

$$\phi_N(\theta_{i,i'}s_\lambda) = \textstyle\sum_j\sum_{p_N(\lambda')=\lambda} U_{\mathfrak{s}_N(\lambda')j+m_N(i-1)}s_{\lambda'}U_{j+m_N(i'-1)}^*.$$

Since $(\mathrm{id}_M \otimes \iota_{p_N,\mathfrak{s}_N})(\theta_{i,i'}s_\lambda) = \theta_{i,i'}\sum_{p_N(\lambda')=\lambda} s_{\lambda'}$, we may therefore apply (3.1) to see that

$$\phi_N(\theta_{i,i'}s_\lambda) = \phi_{N+1} \circ (\pi_{M,m_N} \otimes \mathrm{id}_{C^*(\Lambda_{N+1})}) \circ (\mathrm{id}_M \otimes \iota_{p_N,\mathfrak{s}_N})(\theta_{i,i'}s_\lambda).$$

Since elements of the form $\theta_{i,i'}s_\lambda$ generate $M_M(C^*(\Lambda_N))$ this proves the result. □

Theorem 3.8. *Let $(\Lambda_n, \Lambda_{n+1}, p_n, m_n, \mathfrak{s}_n)_{n=1}^\infty$ be a sequence of row-finite coverings of locally convex $k$-graphs. For each $n$, let $\mathbf{\Lambda}_n := \Lambda_1 \overset{p_1,\mathfrak{s}_1}{\longleftarrow} \cdots \overset{p_{n-1},\mathfrak{s}_{n-1}}{\longleftarrow} \Lambda_n$, identify $\mathbf{\Lambda}_n$ with the corresponding subset of $\varprojlim(\Lambda_n; p_n, \mathfrak{s}_n)$, and likewise identify $C^*(\mathbf{\Lambda}_n)$ with the corresponding $C^*$-subalgebra of $C^*(\varprojlim(\Lambda_n; p_n, \mathfrak{s}_n))$. Then*

$$C^*(\varprojlim(\Lambda_n; p_n, \mathfrak{s}_n)) = \overline{\textstyle\bigcup_{n=1}^\infty C^*(\mathbf{\Lambda}_n)}. \tag{3.2}$$

*Let $P_1 := \sum_{v\in\Lambda_1^0} s_v$, and for each $n$, let $M_n := m_1m_2\cdots m_{n-1}$. Then $P_1$ is a full projection in each $\mathcal{M}(C^*(\mathbf{\Lambda}_n))$, and we have*

$$P_1C^*(\varprojlim(\Lambda_n; p_n, \mathfrak{s}_n))P_1 \cong \varinjlim\big(M_{M_n}(C^*(\Lambda_n)), \mathrm{id}_{M_n} \otimes \iota_{p_n,\mathfrak{s}_n}\big). \tag{3.3}$$

*In particular,*

$$\begin{aligned}K_*(C^*(\varprojlim(\Lambda_n; p_n, \mathfrak{s}_n))) &= K_*(P_1C^*(\varprojlim(\Lambda_n; p_n, \mathfrak{s}_n))P_1) \\ &\cong \varinjlim(K_*(C^*(\Lambda_n)), (\iota_{p_n,\mathfrak{s}_n})_*).\end{aligned}$$

*Proof.* For the duration of the proof, let $\mathbf{\Lambda} := \varprojlim(\Lambda_n; p_n, \mathfrak{s}_n)$. We have $C^*(\mathbf{\Lambda}) = \overline{\operatorname{span}}\{s_\mu s_\nu^* : \mu, \nu \in \mathbf{\Lambda}\}$, so for the first statement, we need only show that

$$\operatorname{span}\{s_\mu s_\nu^* : \mu, \nu \in \mathbf{\Lambda}\} \subset \bigcup_{n=1}^\infty C^*(\mathbf{\Lambda}_n).$$

To see this we simply note that for any finite $F \subset \mathbf{\Lambda}$, the integer $N := \max\{n \in \mathbf{N} : s(F) \cap \Lambda_n^0 \neq \emptyset\}$ satisfies $F \subset \mathbf{\Lambda}_N$.

Since $P_1$ is full in each $C^*(\mathbf{\Lambda}_n)$ by Proposition 3.2(3), it is full in $C^*(\mathbf{\Lambda})$ by (3.2). Equation 3.3 follows from Proposition 3.7. The final statement then follows from continuity of the $K$-functor. □

*Remark* 3.9. Note that if we let $\gamma$ denote the restriction of the gauge action to $P_1 C^*(\varprojlim(\Lambda_n; p_n, \mathfrak{s}_n))P_1$ then $\gamma_{(1,\cdots,1,z)}$ is trivial for all $z \in \mathbf{T}$. Indeed, if $s_\mu s_\nu^*$ is a nonzero element $P_1 C^*(\varprojlim(\Lambda_n; p_n, \mathfrak{s}_n))P_1$, then $d(\mu)_{n+1} = d(\nu)_{n+1}$. So $\gamma$ may be regarded as an action by $\mathbf{T}^k$ rather than $\mathbf{T}^{k+1}$.

We can extend Theorem 3.8 to the situation of matrices of covering systems as discussed in Section 2.1 as follows.

Proposition 3.10. *Resume the notation of Corollary 2.15. Each $C^*(\Lambda_n)$ is canonically isomorphic to $\bigoplus_{j=1}^{c_n} C^*(\Lambda_{n,j})$. There are homomorphisms $(\iota_n)_* : K_*(C^*(\Lambda_n)) \to K_*(C^*(\Lambda_{n+1}))$ such that the partial homomorphism which maps the $j^{\text{th}}$ summand in $K_*(C^*(\Lambda_n))$ to the $i^{\text{th}}$ summand in $K_*(C^*(\Lambda_{n+1}))$ is equal to 0 if $m_{i,j}^n = 0$, and is equal to $(\iota_{p_{i,j}^n, \mathfrak{s}_{i,j}^n})_*$ otherwise. The sum $\sum_{v \in \Lambda_1^0} s_v$ converges strictly to a full projection $P_1 \in \mathcal{M}(C^*(\mathbf{\Lambda}))$. Furthermore,*

$$K_*(P_1 C^*(\mathbf{\Lambda})P_1) \cong \varinjlim\Big(\bigoplus_{j=1}^{c_n} K_*(C^*(\Lambda_{n,j})), (\iota_n)_*\Big).$$

*Proof.* For each $\lambda \in \Lambda_n = \bigsqcup_{j=1}^{c_n} \Lambda_{n,j}$, define a partial isometry $t_\lambda \in \bigoplus_{j=1}^{c_n} C^*(\Lambda_{n,j})$ by $t_\lambda := (0, \ldots, 0, s_\lambda, 0, \ldots, 0)$ (the nonzero term is in the $j^{\text{th}}$ coordinate when $\lambda \in \Lambda_{n,j}$). These nonzero partial isometries form a Cuntz-Krieger $\Lambda_n$-family consisting of nonzero partial isometries. The universal property of $C^*(\Lambda_n)$ gives a homomorphism $\pi_t^n : C^*(\Lambda_n) \to \bigoplus_{j=1}^{c_n} C^*(\Lambda_{n,j})$ which intertwines the direct sum of the gauge actions on the $C^*(\Lambda_{n,j})$ and the gauge action on $C^*(\Lambda_n)$. The gauge-invariant uniqueness theorem [20, Theorem 3.4], and the observation that each generator of each summand in $\bigoplus_{j=1}^{c_n} C^*(\Lambda_{n,j})$ is nonzero and belongs to the image of $\pi_t^n$ therefore shows that $\pi_t^n$ is an isomorphism.

The individual covering systems $(\Lambda_{n,j}, \Lambda_{n+1,i}, p^n, m^n, \mathfrak{s}^n)$ induce inclusions $\iota_{p_{i,j}^n, \mathfrak{s}_{i,j}^n} : C^*(\Lambda_{n,j}) \to M_{m_{i,j}^n}(C^*(\Lambda_{n+1,i}))$ as in Proposition 3.2(5). We therefore obtain homomorphisms $(\iota_{p_{i,j}^n, \mathfrak{s}_{i,j}^n})_* : K_*(C^*(\Lambda_{n,j})) \to K_*(C^*(\Lambda_{n+1,i}))$. The statement about the partial homomorphisms of $K$-groups then follows from the properties of the isomorphism $K_*(\bigoplus_i A_i) \cong \bigoplus_i K_*(A_i)$ for $C^*$-algebras $A_i$.

The final statement can then be deduced from arguments similar to those of Theorem 3.8. □

## 4. Simplicity and pure infiniteness

Theorem 3.1 of [34] gives a necessary and sufficient condition for simplicity of the $C^*$-algebra of a row-finite $k$-graph with no sources. Specifically, $C^*(\Lambda)$ is simple if and only if $\Lambda$ is cofinal and every vertex of $\Lambda$ receives an aperiodic infinite path (see below for the definitions of cofinality and aperiodicity). In this section we present some means of deciding whether $\varprojlim(\Lambda_n; p_n, \mathfrak{s}_n)$ is cofinal (Lemma 4.7), and whether an infinite path in $\varprojlim(\Lambda_n; p_n, \mathfrak{s}_n)$ is aperiodic (Lemma 4.3). We also present a condition under which $C^*(\varprojlim(\Lambda_n; p_n, \mathfrak{s}_n))$ is purely infinite (Proposition 4.8).

We begin by recalling the notation and definitions required to make sense of the hypotheses of [34, Theorem 3.1]. For more detail, see Section 2 of [31].

*Notation* 4.1. We write $\Omega_k$ for the $k$-graph such that $\Omega_k^q := \{(m,n) \in \mathbf{N}^k \times \mathbf{N}^k : n - m = q\}$ for each $q \in \mathbf{N}^k$, with $r(m,n) = (m,m)$ and $s(m,n) = (n,n)$. We identify $\Omega_k^0 = \{(m,m) : m \in \mathbf{N}^k\}$ with $\mathbf{N}^k$. An infinite path in a $k$-graph $\Xi$ is a graph morphism $x : \Omega_k \to \Xi$, and we denote the image $x(0)$ of the vertex $0 \in \Omega_k^0$ by $r(x)$. We write $\Xi^\infty$ for the collection of all infinite paths in $\Xi$, and for $v \in \Xi^0$ we denote by $v\Xi^\infty$ the collection $\{x \in \Xi^\infty : r(x) = v\}$. For $x \in \Xi^\infty$ and $q \in \mathbf{N}^k$, there is a unique infinite path $\sigma^q(x) \in \Xi^\infty$ such that $\sigma^q(x)(m,n) = x(m+q, n+q)$ for all $m \le n \in \mathbf{N}^k$.

Definition 4.2. We say that a row-finite $k$-graph $\Xi$ with no sources is *aperiodic* if for each vertex $v \in \Xi^0$ there is an infinite path $x \in v\Xi^\infty$ such that $\sigma^q(x) \neq \sigma^{q'}(x)$ for all $q \neq q' \in \mathbf{N}^k$. We say that $\Xi$ is *cofinal* if for each $v \in \Xi^0$ and $x \in \Xi^\infty$ there exists $m \in \mathbf{N}^k$ such that $v\Xi x(m) \neq \emptyset$.

We continue to make use in the following of the notation established earlier (see Notation 2.6) for the embeddings of $\mathbf{N}^k$ and of $\mathbf{N}$ in $\mathbf{N}^{k+1}$.

If $y$ is an infinite path in the $(k+1)$-graph $\Xi$, we write $\alpha_y$ for the infinite path in $\Xi^{(0_k, \mathbf{N})}$ defined by $\alpha_y(p,q) := y((0_k, p), (0_k, q))$ for $p \le q \in \mathbf{N}$, and we write $x_y$ for the infinite path in $\Xi^{(\mathbf{N}^k, 0_1)}$ defined by $x_y(p,q) := y((p, 0_1), (q, 0_1))$ where $p \le q \in \mathbf{N}^k$.

Proposition 4.3. *Let $(\Lambda_n, \Lambda_{n+1}, p_n, m_n, \mathfrak{s}_n)_{n=1}^\infty$ be a sequence of row-finite covering systems of $k$-graphs with no sources. For $a, b \in \mathbf{N}^{k+1}$, an infinite path $y \in \big(\varprojlim(\Lambda_n; p_n, \mathfrak{s}_n)\big)^\infty$ satisfies $\sigma^a(y) = \sigma^b(y)$ if and only if $x_{\sigma^a(y)} = x_{\sigma^b(y)}$ and $\alpha_{\sigma^a(y)} = \alpha_{\sigma^b(y)}$.*

*Proof.* The "only if" implication is trivial. For the "if" implication, note that the factorisation property implies that an infinite path $z$ of $\varprojlim(\Lambda_n; p_n, \mathfrak{s}_n)$ is uniquely determined by $x_z$ and the paths $\alpha_{\sigma^{(n,0_1)}(z)}$, $n \in \mathbf{N}^k$. So it suffices to show that each $\alpha_{\sigma^{(n,0_1)}(z)}$ is uniquely determined by $x_z(0_k, n)$ and $\alpha_z$. Fix $n \in \mathbf{N}^k$ and let $\lambda := x_z(0_k, n) = z(0_{k+1}, (n, 0_1))$. Fix $i \in \mathbf{N}$. We will show that $\alpha_{\sigma^{(n,0_1)}(z)}(0_1, i)$ is uniquely determined by $\alpha_z(0_1, i)$ and $\lambda$. Let $v = r(z)$, and let $N \in \mathbf{N}$ be the element such that $v \in \Lambda_N^0$. For $1 \le j \le i$, let $w_j = \alpha_z(i) \in \Lambda_{N+j}^0$, and let $1 \le l_j \le m_{N+j-1}$ be the integer such that $\alpha_z(j-1, j) = e(w_i, l_j)$. We have $p_N(w_1) = v$, and $p_{N+j-1}(w_j) = w_{j-1}$ for $2 \leq j \leq i$. For each $j$, let $\lambda_j$ be the unique lift of $\lambda$ such that $r(\lambda_j) = w_j$. By definition of the $(k+1)$-graph $\varprojlim(\Lambda_n; p_n, \mathfrak{s}_n)$, the path

$$\lambda e(s(\lambda_1), \mathfrak{s}(\lambda_1)^{-1} l_1) e(s(\lambda_2), \mathfrak{s}(\lambda_2)^{-1} l_2) \dots e(s(\lambda_i), \mathfrak{s}(\lambda_i)^{-1} l_i) = \alpha_z(0_1, i)\lambda_i$$

is the unique minimal common extension of $\lambda$ and $\alpha_z(0_1, i)$ in $\varprojlim(\Lambda_n; p_n, \mathfrak{s}_n)$. Hence

$$\alpha_{\sigma^{(n,0_1)}(z)}(0_1, i) = e(s(\lambda_1), \mathfrak{s}(\lambda_1)^{-1} l_1) e(s(\lambda_2), \mathfrak{s}(\lambda_2)^{-1} l_2) \dots e(s(\lambda_i), \mathfrak{s}(\lambda_i)^{-1} l_i)$$

which is uniquely determined by $\lambda$ and $\alpha_z(0_1, i)$. □

Corollary 4.4. *Let $(\Lambda_n, \Lambda_{n+1}, p_n, m_n, \mathfrak{s}_n)_{n=1}^\infty$ be a sequence of row-finite covering systems of $k$-graphs with no sources. Suppose that $\Lambda_n$ is aperiodic for some $n$. Then so is $\varprojlim(\Lambda_n; p_n, \mathfrak{s}_n)$.*

*Proof.* Since each vertex in $\Lambda_n$ receives an aperiodic path in $\Lambda_n$, Proposition 4.3, guarantees that each vertex in $\Lambda_n$ receives an aperiodic path in $\varprojlim(\Lambda_n; p_n, \mathfrak{s}_n)$. Since the $p_n$ are coverings, it follows that every vertex of $\varprojlim(\Lambda_n; p_n, \mathfrak{s}_n)$ receives an infinite path of the form $\lambda y$ or of the form $\sigma^p(y)$ where $y$ is an aperiodic path with range in $\Lambda_n$. If $y$ is aperiodic, then $\lambda y$ is aperiodic for any $\lambda$ and $\sigma^a(y)$ is aperiodic for any $a$ and the result follows. □

Lemma 4.5. *Let $(\Lambda_n, \Lambda_{n+1}, p_n, m_n, \mathfrak{s}_n)_{n=1}^\infty$ be a sequence of row-finite covering systems of $k$-graphs with no sources. Fix $y \in \big(\varprojlim(\Lambda_n; p_n, \mathfrak{s}_n)\big)^\infty$, with $y(0) \in \Lambda_n$ and $a, b \in \mathbf{N}^{k+1}$. Let $\tilde{a}$ and $\tilde{b}$ denote the elements of $\mathbf{N}^k$ determined by the first $k$ coordinates of $a$ and $b$. For each $m \geq n$, let $v_m$ and $i_m$ be the unique pair such that $\alpha_y(m, m+1) = e(v_m, i_m)$. For each $m \geq n$, let $\mu_m$ and $\nu_m$ be the unique lifts of $x_y(0, \tilde{a})$ and $x_y(0, \tilde{b})$ such that $r(\mu_m) = r(\nu_m) = v_m$. Then $\alpha_{\sigma^a(y)} = \alpha_{\sigma^b(y)}$ if and only if the following three conditions hold:*

1. *$a_{k+1} = b_{k+1}$;*
2. *$s(\mu_m) = s(\nu_m)$ for all $m \geq n$; and*
3. *$\mathfrak{s}_m(\mu_m) i_m = \mathfrak{s}_m(\nu_m) i_m$ for all $m \geq n$.*

*Proof.* We have $\alpha_{\sigma^a(y)}(m, m+1) = e(s(\mu_{m+a_{k+1}}), \mathfrak{s}_m(\mu_{m+a_{k+1}}) i_{m+a_{k+1}})$ for all $m$, and likewise for $b$ and $\nu$. □

*Remark* 4.6. Lemma 5.4 of [27] implies that an infinite path in a rank-2 Bratteli diagram $\Lambda$ is aperiodic if and only if the factorisation permutations of its red coordinate-paths are of unbounded order. Lemma 4.5 is the analogue of this result for general systems of coverings. To see the analogy, note that in a rank-2 Bratteli diagram, every $x_y$ is of the form $\lambda\lambda\lambda\dots$ for some blue cycle $\Lambda$, so that condition (3) fails for all $a \neq b$ precisely when the orders of the permutations $\mathfrak{s}_m(\mu_m)$ grow arbitrarily large with $m$.

Lemma 4.7. *Let $(\Lambda_n, \Lambda_{n+1}, p_n, m_n, \mathfrak{s}_n)_{n=1}^\infty$ be a sequence of row-finite coverings of $k$-graphs with no sources. If infinitely many of the $\Lambda_n$ are cofinal, then $\varprojlim(\Lambda_n; p_n, \mathfrak{s}_n)$ is also cofinal.*

*Proof.* Fix a vertex $v$ and an infinite path $z \in (\varprojlim(\Lambda_n; p_n, \mathfrak{s}_n))^\infty$. Let $n_1, n_2 \in \mathbf{N}$ be the elements such that $v \in \Lambda^0_{n_1}$ and $r(z) \in \Lambda^0_{n_2}$. Choose $N \geq n_1, n_2$ such that $\Lambda_N$ is cofinal. Fix $w \in \Lambda^0_N$ such that $p_n \circ p_{n+1} \circ \cdots \circ p_{N-1}(w) = v$; so $v(\varprojlim(\Lambda_n; p_n, \mathfrak{s}_n))w \neq \emptyset$. We have $x_{\sigma^{(0_k, N-n_2)}(z)} \in \Lambda^\infty_N$, and since $\Lambda_N$ is cofinal, it follows that $w\Lambda_N x_{\sigma^{(0_k,N-n_2)}(z)}(q) \neq \emptyset$ for some $q \in \mathbf{N}^k$. Since $x_{\sigma^{(0_k,N-n_2)}(z)}(q) = z(q, N - n_2)$, this completes the proof. □

As in [38], we say that a path $\lambda$ in a $k$-graph $\Lambda$ is a *cycle with an entrance* if $s(\lambda) = r(\lambda)$, and there exists $\mu \in r(\lambda)\Lambda$ with $d(\mu) \leq d(\lambda)$ and $\lambda(0, d(\mu)) \neq \mu$.

Proposition 4.8. *Let $(\Lambda_n, \Lambda_{n+1}, p_n, m_n, \mathfrak{s}_n)_{n=1}^\infty$ be a sequence of row-finite coverings of $k$-graphs with no sources. There exists $n$ such that $\Lambda_n$ contains a cycle with an entrance if and only if every $\Lambda_n$ contains a cycle with an entrance. Moreover, if $C^*(\varprojlim(\Lambda_n; p_n, \mathfrak{s}_n))$ is simple and $\Lambda_1$ contains a cycle with an entrance, then $C^*(\mathbf{\Lambda})$ is purely infinite.*

*Proof.* That the presence of a cycle with an entrance in $\Lambda_1$ is equivalent to the presence of a cycle with an entrance in every $\Lambda_n$ follows from the properties of covering maps. Now the result follows from [38, Proposition 8.8] □

## 5. K-Theory

In this section, we consider the $K$-theory of $C^*(\Lambda \overset{p,\mathfrak{s}}{\leftarrow} \Gamma)$. Specifically, we show how the homomorphism from $K_*(C^*(\Lambda))$ to $K_*(C^*(\Gamma))$ obtained from Proposition 3.2 behaves with respect to existing calculations of $K$-theory for various classes of higher-rank graph $C^*$-algebras. We will use these results later to compute the $K$-theory of $C^*(\varprojlim(\Lambda_n; p_n, \mathfrak{s}_n))$ for a number of sequences of covering systems.

Throughout this section, given a $k$-graph $\Lambda$, we view the ring $\mathbf{Z}\Lambda^0$ as the collection of finitely supported functions $f : \Lambda^0 \to \mathbf{Z}$. For $v \in \Lambda^0$, we denote the point-mass at $v$ by $\delta_v$. Given a finite covering $p : \Gamma \to \Lambda$ of row-finite $k$-graphs, we define $p^* : \mathbf{Z}\Lambda^0 \to \mathbf{Z}\Gamma^0$ by $p^*(\delta_v) = \sum_{p(u)=v} \delta_w$; equivalently, $p^*(f)(w) = f(p(w))$.

### 5.1. Coverings of 1-graphs and the Pimsner-Voiculescu exact sequence.

It is shown in [26, 32] how to compute the $K$-theory of a graph $C^*$-algebra using the Pimsner-Voiculescu exact sequence. In this subsection, we show how this calculation interacts with the inclusion of $C^*$-algebras arising from a covering of 1-graphs.

The $K$-theory computations for arbitrary graph $C^*$-algebras [12, 1] are somewhat more complicated than for the $C^*$-algebras of row-finite graphs with no sources. Moreover, every graph $C^*$-algebra is Morita equivalent to the $C^*$-algebra of a row-finite graph with no sources [12]. We therefore restrict out attention here to the simpler setting.

Theorem 5.1. *Let $(E^*, F^*, p, m, \mathfrak{s})$ be a row-finite covering system of 1-graphs with no sources. Let $A, B$ be the vertex connectivity matrices of the underlying* *graphs $E$ and $F$ respectively. Then the diagram*

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & K_1(C^*(E^*)) & \longrightarrow & \mathbf{Z}E^0 & \xrightarrow{1-A^t} & \mathbf{Z}E^0 & \longrightarrow & K_0(C^*(E^*)) & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle (\iota_{p,\mathfrak{s}})_*} & & \Big\downarrow{\scriptstyle m\cdot p^*} & & \Big\downarrow{\scriptstyle m\cdot p^*} & & \Big\downarrow{\scriptstyle (\iota_{p,\mathfrak{s}})_*} & & \\ 0 & \longrightarrow & K_1(C^*(F^*)) & \longrightarrow & \mathbf{Z}F^0 & \xrightarrow{1-B^t} & \mathbf{Z}F^0 & \longrightarrow & K_0(C^*(F^*)) & \longrightarrow & 0 \end{array} \tag{5.1}$$

*commutes and the rows are exact.*

The proof of this theorem occupies the remainder of Section 5.1. We fix, for the duration, a finite covering $p : F^* \to E^*$ of row-finite 1-graphs with no sources, a multiplicity $m$ and a cocycle $\mathfrak{s} : F^* \to S_m$.
It is relatively straightforward to prove that the right-hand two squares of (5.1) commute and that the rows are exact.

Lemma 5.2. *Resume the notation of Theorem 5.1. We have $(1 - B^t)p^* = p^*(1 - A^t)$, the right-hand two squares of* (5.1) *commute, and the rows are exact.*

*Proof.* For the first statement, consider a generator $\delta_v \in \mathbf{Z}E^0$. We have

$$\big(p^* \circ (1 - A^t)\big)(\delta_v) = p^*(\delta_v - \sum_{e\in vE^1} \delta_{s(e)}) = \sum_{p(u)=v} \delta_u - \sum_{e\in vE^1} \sum_{p(f)=e} \delta_{s(f)}.$$

On the other hand,

$$\big((1 - B^t) \circ p^*\big)(\delta_v) = (1 - B^t) \sum_{p(u)=v} \delta_u = \sum_{p(u)=v} \Big(\delta_u - \sum_{f\in uF^1} \delta_{s(f)}\Big).$$

Since $p$ is a covering the double-sums occurring in these two equations each contain exactly one term for each edge $f \in F^1$ such that $p(r(f)) = v$, and it follows that the two are equal.
Multiplying by $m$ throughout the above calculation shows that the middle square of (5.1) commutes.
The identification of $K_0(C^*(E^*))$ with $\operatorname{coker}(1 - A^t)$ takes the class of the projection $s_v \in C^*(E^*)$ to the class of the corresponding generator $\delta_v \in \mathbf{Z}E^0$ (see [30]). That the right-hand square commutes then follows from Proposition 3.2(6).
Exactness of the rows is precisely the computation of $K$-theory for 1-graph $C^*$-algebras [8, 26, 32]. □

It remains to prove that the left-hand square of (5.1) commutes. The strategy is to assemble the eight-term commuting diagrams which describe the $K$-theory of each of $C^*(E^*)$ and $C^*(F^*)$ (see equation (5.3) below) into a sixteen-term diagram, one face of which is the left-hand square of (5.1). We then focus on the cube in the sixteen-term diagram which contains left-hand square of (5.1) as one of its faces, and show that the remaining five faces of this cube commute. A diagram-chase then establishes that the sixth face commutes as well. The majority of the work involved goes into defining the connecting maps needed to write down the sixteen-term diagram in the first place. The proof that the various squares in it commute is then relatively straightforward.

To begin, we recall how one shows that the rows of (5.1) are exact. Let $E^* \times_d \mathbf{Z}$ be the skew-product of $E^*$ by the length functor $d$ (see [20, Section 5]). Let $\gamma$ be the gauge action of $\mathbf{T}$ on $C^*(E^*)$ satisfying $\gamma_z(s_e) = zs_e$ for $e \in E^1$ and $z \in \mathbf{T}$. Let $(i_{\mathbf{T}}, i_{C^*(E^*)})$ be the universal covariant representation of $(C^*(E^*), \mathbf{T}, \gamma)$ in the crossed product $C^*(E^*) \times_\gamma \mathbf{T}$. By [32, Lemma 3.1], there is an isomorphism

$$\psi_E : C^*(E^* \times_d \mathbf{Z}) \to C^*(E^*) \times_\gamma \mathbf{T} \tag{5.2}$$

satisfying $\psi_E(s_{(\lambda,n)}) = i_{\mathbf{T}}(z)^n i_{C^*(E^*)}(s_\lambda)$.

The $C^*$-algebra $C^*(E^* \times_d \mathbf{Z})$ is AF with $K_0$-group $\varinjlim(\mathbf{Z}E^0, A^t)$ (see [26, 32]). Hence one may apply the dual Pimsner-Voiculescu sequence [4, Section 10.6] to the crossed product algebra $C^*(E^*) \times_\gamma \mathbf{T}$ to show that the top row of (5.1) is exact (the bottom row is the same after replacing $E$ with $F$).

From the point of view of coverings, the skew-product graph $E^* \times_d \mathbf{Z}$ and its $C^*$-algebra are more natural to work with than the crossed product $C^*(E^*) \times_\gamma \mathbf{T}$. Before proving that the final square of (5.1) commutes, we therefore detail first how coverings $p : F^* \to E^*$ interact with the isomorphisms $\psi_E : C^*(E^* \times_d \mathbf{Z}) \to C^*(E^*) \times_\gamma \mathbf{T}$.

Lemma 5.3. *With the above notation, let $E^* \times_d \mathbf{Z}$ and $F^* \times_d \mathbf{Z}$ be the skew-product graphs by the length functors $d$, and let $\psi_E$ and $\psi_F$ be the isomorphisms described in* (5.2). *Let $\gamma_E$ and $\gamma_F$ denote the gauge actions of $\mathbf{T}$ on $C^*(E^*)$ and $C^*(F^*)$.*

(1) *the formulae $\tilde{p}(\lambda, n) := (p(\lambda), n)$ and $\tilde{\mathfrak{s}}(\lambda, n) := \mathfrak{s}(\lambda)$ determine a covering $\tilde{p} : F^* \times_d \mathbf{Z} \to E^* \times_d \mathbf{Z}$ and a cocycle $\tilde{\mathfrak{s}} : F^* \times_d \mathbf{Z} \to S_m$.*

(2) *the inclusion $\iota_{p,\mathfrak{s}} : C^*(E^*) \to M_m(C^*(F^*))$ is equivariant in the actions $\gamma_E$ and $\mathrm{id}_m \otimes \gamma_F$, and induces an inclusion $\widetilde{\iota_{p,\mathfrak{s}}} : C^*(E^*) \times_{\gamma_E} \mathbf{T} \to M_m(C^*(F^*)) \times_{\mathrm{id}_m \otimes \gamma_F} \mathbf{T}$.*

(3) *The following diagram commutes.*

$$\begin{tikzcd}
C^*(E^* \times_d \mathbf{Z}) \arrow[r, "\iota_{\tilde{p},\tilde{\mathfrak{s}}}"] \arrow[d, "\psi_E"'] & M_m(C^*(F^* \times_d \mathbf{Z})) \arrow[d, "\mathrm{id}_m \otimes \psi_F"'] \\
C^*(E^*) \times_{\gamma_E} \mathbf{T} \arrow[r, "\widetilde{\iota_{p,\mathfrak{s}}}"] & M_m(C^*(F^*)) \times_{\mathrm{id}_m \otimes \gamma_F} \mathbf{T}
\end{tikzcd}$$

*Proof.* (1) It is straightforward to check that $\tilde{p}$ is a covering. To see that $\tilde{\mathfrak{s}}$ is a cocycle, note that $(\mu, m)$ and $(\nu, n)$ are composable in the skew-product precisely when $\mu$ and $\nu$ are composable, and $n = m - d(\nu)$. So for $i \in \{1, \dots, m\}$ we may calculate

$$\tilde{\mathfrak{s}}(\mu, m)(\tilde{\mathfrak{s}}(\nu, m - d(\nu))i) = \mathfrak{s}(\mu)(\mathfrak{s}(\nu)i) = \mathfrak{s}(\mu\nu)i = \tilde{\mathfrak{s}}(\mu\nu, m - d(\nu))i.$$

(2) That $\iota_{p,\mathfrak{s}}$ is equivariant in $\gamma_E$ and $\mathrm{id}_m \otimes \gamma_F$ follows from Proposition 3.2(5). That it induces the desired inclusion $\widetilde{\iota_{p,\mathfrak{s}}}$ of crossed-products follows from the universal properties of the crossed-product algebras.

(3) That the diagram commutes follows from a simple calculation using the definitions of the maps involved. $\square$

*Proof of Theorem 5.1.* Lemma 5.2 establishes everything except that the left-hand square in the diagram (5.1) commutes. To establish this last claim, recall from [32, Theorem 3.2] (see also [26]) that there is a homomorphism $\phi_E : \mathbf{Z}E^0 \to K_0(C^*(E^*) \times_{\gamma_E} \mathbf{T})$ satisfying $\phi_E(\delta_v) = [i_{\mathbf{T}}(1)i_{C^*(E^*)}(s_v)]$. Moreover, the rows of the following commutative diagram are exact and the left- and right-most vertical maps are isomorphisms (see [30, Lemma 7.15], [26]).

$$\tag{5.3}
\begin{tikzcd}
0 \arrow[r] & \ker(1-A^t) \arrow[r] \arrow[d, "\cong"] & \mathbf{Z}E^0 \arrow[r, "1-A^t"] \arrow[d, "\phi_E"] & \mathbf{Z}E^0 \arrow[r] \arrow[d, "\phi_E"] & \operatorname{coker}(1-A^t) \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & K_1(C^*(E^*)) \arrow[r] & K_0(C^*(E^*) \times_{\gamma_E} \mathbf{T}) \arrow[r, "1-\hat{\gamma}_*^{-1}"] & K_0(C^*(E^*) \times_{\gamma_E} \mathbf{T}) \arrow[r] & K_0(C^*(E^*)) \arrow[r] & 0
\end{tikzcd}$$

A similar commutative diagram holds for $F^*$, and using the standard isomorphism of $K_*(M_m(C^*(F^*)))$ with $K_*(C^*(F^*))$, we may assemble these two diagrams can into a single three-dimensional diagram by connecting each term in the diagram for $E^*$ to the corresponding term in the diagram for $F^*$ using the appropriate maps induced from $(p, \mathfrak{s})$. The map connecting the $K_0$-groups of the skew-product graph algebras is induced from the connecting map in the bottom row of the commuting diagram in Lemma 5.3(3) by applying the $K$-functor and using the canonical isomorphisms

$$K_*(M_m(C^*(F^*) \times_{\gamma_F} \mathbf{T})) \cong K_*(C^*(F^*) \times_{\gamma_F} \mathbf{T}) \quad \text{and}$$
$$M_m(C^*(F^*) \times_{\gamma_F} \mathbf{T}) \cong M_m(C^*(F^*)) \times_{\mathrm{id}_m \otimes \gamma_F} \mathbf{T}.$$

Let $\eta$ denote the unlabelled inclusion $K_1(C^*(F^*)) \hookrightarrow K_0(C^*(F^*) \times_{\gamma_F} \mathbf{T}))$ in the bottom row of the diagram of the form (5.3) for $F^*$. Notice that injectivity of the map $m \cdot p^* : \mathbf{Z}E^0 \to \mathbf{Z}F^0$ together with the first statement of Lemma 5.2 ensures that $m \cdot p^*$ restricts to a map from $\ker(1 - A^t)$ to $\ker(1 - B^t)$; abusing notation, we denote this map $m \cdot p^*$ too. With this notation the diagram (5.4) below is the left-hand cube of the three-dimensional diagram described in the previous paragraph.

$$\tag{5.4}
\begin{tikzcd}
& \ker(1-A^t) \arrow[rr, hook] \arrow[dl, "m\cdot p^*"'] \arrow[dd, "\cong"] & & \mathbf{Z}E^0 \arrow[dl, "m\cdot p^*"'] \arrow[dd, "\phi_E"] \\
\ker(1-B^t) \arrow[rr, hook] \arrow[dd, "\cong"] & & \mathbf{Z}F^0 \arrow[dd, "\phi_E"] & \\
& K_1(C^*(E^*)) \arrow[rr] \arrow[dl, "(\iota_{p,\mathfrak{s}})_*"'] & & K_0(C^*(E^*) \times_{\gamma_E} \mathbf{T}) \arrow[dl, "(\widetilde{\iota_{p,\mathfrak{s}}})_*"'] \\
K_1(C^*(F^*)) \arrow[rr, "\eta"] & & K_0(C^*(F^*) \times_{\gamma_F} \mathbf{T}) &
\end{tikzcd}$$

We have shown the whole cube because we prove that the left-hand face — which is none other than the left-hand square of (5.1) — commutes by showing that the other five faces commute.

To see why this suffices, suppose that the other five faces do indeed commute. Since $\eta$ is an injection by the exactness of the rows of (5.3), we just need to show that the two maps from $\ker(1 - A^t)$ into $K_0(C^*(F^*) \times_\gamma \mathbf{T}))$ obtained from the maps in the left-hand face of the cube followed by $\eta$ agree. A diagram chase shows that this is the case.

It therefore remains only to show that the top, bottom, front, back and right-hand faces of (5.4) commute. The top square commutes by definition. The bottom square commutes by the naturality of the dual Pimsner-Voiculescu exact sequence (see the argument at the beginning of [32, Section 3]). The back and front faces commute because (5.3) commutes.

To see that the right-hand face commutes, recall that $C^*(E^* \times_d \mathbf{Z})$ is AF with $K_0$-group $\varinjlim(\mathbf{Z}E^0, 1 - A^t)$. Hence there is an inclusion $\varepsilon_E : \mathbf{Z}E^0 \to K_0(C^*(E^* \times_d \mathbf{Z}))$ which takes $\delta_v$ to the $K_0$-class of the vertex projection $s_{(v,0)}$, and likewise for $F$. Consider the map $\psi_E$ defined in (5.2) and the map $\phi_E$ appearing in (5.3). It is clear that $\phi_E = (\psi_E)_* \circ \varepsilon_E$ and similarly for $F$. So it suffices to show that the following diagram commutes.

$$\begin{tikzcd}
\mathbf{Z}E^0 \arrow[r, "m\cdot p^*"] \arrow[d, "\varepsilon_E"'] & \mathbf{Z}F^0 \arrow[d, "\varepsilon_F"'] \\
K_0(C^*(E^* \times_d \mathbf{Z})) \arrow[r, "(\iota_{\tilde{p},\tilde{\mathfrak{s}}})_*"] \arrow[d, "(\psi_E)_*"'] & K_0(C^*(F^* \times_d \mathbf{Z})) \arrow[d, "(\psi_F)_*"'] \\
K_0(C^*(E^*) \times_{\gamma_E} \mathbf{T}) \arrow[r, "(\widetilde{\iota_{p,\mathfrak{s}}})_*"] & K_0(C^*(F^*) \times_{\gamma_F} \mathbf{T})
\end{tikzcd} \tag{5.5}$$

If one applies the $K$-functor to all terms and maps in the diagram of Lemma 5.3(3), and then applies the natural isomorphism

$$K_*(M_m(C^*(E^*) \times_{\gamma_E} \mathbf{T})) \cong K_*(C^*(E^*) \times_{\gamma_E} \mathbf{T})$$

to the terms on the right, one obtains precisely the bottom rectangle of (5.5). The bottom rectangle of (5.5) therefore commutes by naturality of the $K$-functor together with Lemma 5.3(3).

To see that the top rectangle of (5.5) commutes, recall that $\varepsilon_E$ takes the image of the point-mass $\delta_v$ in the direct limit $\varinjlim(\mathbf{Z}E^0, A^t)$ to the class of the projection $s_{(v,0)}$. The image of $s_{(v,0)}$ under the homomorphism $\iota_{\tilde{p},\tilde{\mathfrak{s}}}$ is the diagonal matrix in $M_m(C^*(F^* \times_d \mathbf{Z}))$ whose diagonal entries are all equal to $\sum_{p(w)=v} s_{(w,0)}$. Under the standard isomorphism $K_0(M_m(C^*(F^* \times_d \mathbf{Z}))) \cong K_0(C^*(F^* \times_d \mathbf{Z}))$, we therefore obtain the following equality in $K_0(C^*(F^* \times_d \mathbf{Z}))$:

$$[\iota_{\tilde{p},\tilde{\mathfrak{s}}}(s_{(v,0)})] = \sum_{p(w)=v} m \cdot [s_{(w,0)}] = m \cdot \Big( \sum_{p(w)=v} [s_{(w,0)}] \Big).$$

Using once again the characterisation of the maps $\varepsilon_E$ and $\varepsilon_F$, we see that this is precisely the statement that the bottom rectangle of (5.5) commutes. □

5.2. Coverings of higher-rank graphs and Kasparov's spectral sequence theorem. We turn to the case where $k > 1$. We invoke the $K$-theory computations of [14] which are based on Kasparov's spectral sequence theorem for the computation of the $K$-theory of crossed products by groups for which the Baum-Connes conjecture holds (see [18, Theorem 6.10], [14, Lemma 3.3] and [35]). We are grateful to Gennadi Kasparov for pointing out that the spectral sequence is natural.

The standard notation for spectral sequences is that a spectral sequence $(E^r, d^r)$ has terms $E^r_{p,q}$ and differentials $d^r : E^r_{p,q} \to E^r_{p-r,q+r-1}$ where $r > 0$ and $p, q \in \mathbf{Z}$. This however is problematic in the current situation because $p$ clashes with our notation for a covering map. To avoid this, we replace the indexing variables $p, q$ in the spectral sequence with $a, b$. That is, our spectral sequences have terms $E^r_{a,b}$ and differentials $d^r : E^r_{a,b} \to E^r_{a-r,b+r-1}$ where $r > 0$ and $a, b \in \mathbf{Z}$.

Since each higher rank graph $C^*$-algebra $C^*(\Lambda)$ is Morita equivalent to a crossed product by $\mathbf{Z}^k$ [21, Theorem 5.6], Kasparov's result applies to give a spectral sequence which converges to $K_*(C^*(\Lambda))$ with $E^2$ terms given by the homology of $\mathbf{Z}^k$ with appropriately chosen coefficients. In [14] Evans computes these homology groups using a resolution related to the Koszul complex. It follows that the above spectral sequence may be extended so that the terms of the resolution become the terms $E^1_{a,b}$ for $b$ even.

The main result of this subsection is to show that given a finite covering $p : \Gamma \to \Lambda$ of row-finite $k$-graphs with no sources, a multiplicity $m$ and a cocycle $\mathfrak{s} : \Gamma \to S_m$, there is a natural morphism of spectral sequences defined on $E^1$ terms using $m \cdot p^* : \mathbf{Z}\Lambda^0 \to \mathbf{Z}\Gamma^0$ which is compatible (see [41, p. 126]) with $(\iota_{p,\mathfrak{s}})_*$ the induced map on $K$-theory. This result is specialised to the case $k = 2$ with a view to applications in Section 6.

The following is an immediate Corollary of [18, Theorem 6.10] (see [14, Lemma 3.3] and [35]). For more detail on spectral sequences used in this context, see [35, 14].

Proposition 5.4. *Let $\mathcal{F}$ be a $C^*$-algebra and let $\alpha : \mathbf{Z}^k \to Aut\mathcal{F}$ be an action of $\mathbf{Z}^k$ on $\mathcal{F}$. Then there is a spectral sequence $(E^r, d^r)$ with differentials $d^r : E^r_{a,b} \to E^r_{a-r,b+r-1}$ which converges to $K_*(\mathcal{F} \times_\alpha \mathbf{Z}^k)$ with $E^2_{a,b} = H_a(\mathbf{Z}^k, K_b(\mathcal{F}))$. Moreover, the spectral sequence is natural with respect to equivariant maps of $C^*$-algebras.*

*Proof.* As noted in the proof of [14, Lemma 3.3] this follows immediately from [18, Theorem 6.10] since $\mathbf{Z}^k$ is amenable and the Baum-Connes conjecture is known to hold for amenable groups [16, Theorem 1.1], so the $\gamma$ part of $K_*(\mathcal{F} \times_\alpha \mathbf{Z}^k)$ exhausts. The naturality of the spectral sequence with respect to equivariant maps follows from the construction in the proof of [18, Theorem 6.10], since every step is functorial. □

Naturality means that given $\mathbf{Z}^k$ actions $\alpha_i$ on $\mathcal{F}_i$, a $\mathbf{Z}^k$-equivariant map $\varphi : \mathcal{F}_1 \to \mathcal{F}_2$ induces a morphism of spectral sequences and this morphism is compatible with

$$\widehat{\varphi}_* : K_*(\mathcal{F}_1 \times_{\alpha_1} \mathbf{Z}^k) \to K_*(\mathcal{F}_2 \times_{\alpha_2} \mathbf{Z}^k)$$

where $\widehat{\varphi} : \mathcal{F}_1 \times_{\alpha_1} \mathbf{Z}^k \to \mathcal{F}_2 \times_{\alpha_2} \mathbf{Z}^k$ is the natural map.

Evans applies this when $\mathcal{F} = \mathcal{F}_\Lambda$ is the crossed product $C^*(\Lambda) \times_\gamma \mathbf{T}^k$ of $C^*(\Lambda)$ by the gauge action, and $\alpha$ is the dual action $\hat{\gamma}$ of $\mathbf{Z}^k$. Hence, by Takai duality we have $K_*(C^*(\Lambda)) = K_*(\mathcal{F}_\Lambda \times_\alpha \mathbf{Z}^k)$. In this case we have more specific results (see [14, Lemma 3.3]):

$$E^2_{a,b} = \begin{cases} H_a(\mathbf{Z}^k, K_0(\mathcal{F}_\Lambda)) & \text{if } 0 \le a \le k \text{ and } b \text{ is even,} \\ 0 & \text{otherwise.} \end{cases}$$

In [14, Theorem 3.14]), Evans shows that these homology groups may be computed as the homology of the complex $D^\Lambda_* = \bigwedge^* \mathbf{Z}^k \otimes \mathbf{Z}\Lambda^0$. That is, $D^\Lambda_a = \bigwedge^a \mathbf{Z}^k \otimes \mathbf{Z}\Lambda^0$ for $0 \le a \le k$ and $D^\Lambda_a = 0$ for $a > k$. For $1 \le j \le k$ let $M_j$ denote the vertex connectivity matrix of the coordinate graph $(\Lambda^0, \Lambda^{e_j}, r, s)$. For $1 \le a \le k$ define the differential $\partial_a : D^\Lambda_a \to D^\Lambda_{a-1}$ by

$$\partial_a(\epsilon_{i_1} \wedge \cdots \wedge \epsilon_{i_a} \otimes e_v) = \sum_{j=1}^{a} (-1)^{j+1} \epsilon_{i_1} \wedge \cdots \wedge \widehat{\epsilon}_{i_j} \wedge \cdots \wedge \epsilon_{i_a} \otimes (1 - M^t_j) e_v$$

where $\epsilon_1, \ldots, \epsilon_k$ constitute the canonical basis for $\mathbf{Z}^k$, $1 \le i_1 < \cdots < i_a \le k$ and $v \in \Lambda^0$. It is straightforward to verify that $D^\Lambda_*$ is a complex. The first part of the following theorem is a restatement of [14, Theorem 3.15]).

Theorem 5.5. *Fix $k > 1$. Let $\Lambda$ be a row-finite $k$-graph with no sources. With notation as above there is a spectral sequence $(E^r, d^r)$ with differentials $d^r : E^r_{a,b} \to E^r_{a-r,b+r-1}$ which converges to $K_*(C^*(\Lambda)) = K_*(\mathcal{F}_\Lambda \times_\alpha \mathbf{Z}^k)$ with*

$$E^1_{a,b} = D^\Lambda_a := \bigwedge^a \mathbf{Z}^k \otimes \mathbf{Z}\Lambda^0,$$

*if $0 \le a \le k$ and $b$ is even, and $0$ otherwise. The differential $d^1 : E^1_{a,b} \to E^1_{a-1,b}$ is given by $\partial_a$ if $b$ is even.*

*Let $(\Lambda, \Gamma, p, m, \mathfrak{s})$ be a row-finite covering system of $k$-graphs with no sources. There is a morphism $f$ of spectral sequences which is compatible with $(\iota_{p,\mathfrak{s}})_* : K_*(C^*(\Lambda)) \to K_*(C^*(\Gamma))$ such that $f^1 : D^\Lambda_a \to D^\Gamma_a$ is given by $\mathrm{id} \otimes (m \cdot p^*)$.*

*Proof.* Evans computes the homology groups using a Koszul complex (see [41, §4.5]). Set $G = \mathbf{Z}^k = \langle s_1, \ldots s_k \rangle$, $R = \mathbf{Z}G$ and let $I$ be the ideal in $R$ generated by $\{1 - s_a^{-1} : 1 \le a \le k\}$. Let $\epsilon_1, \ldots, \epsilon_k$ constitute the canonical basis for $R^k$. For each $a$, define $\partial_a : \bigwedge^a R^k \to \bigwedge^{a-1} R^k$ as follows: for $1 \le i_1 < \cdots < i_a \le k$ so that $\epsilon_{i_1} \wedge \cdots \wedge \epsilon_{i_a} \in \bigwedge^a R^k$, define

$$\partial_a(\epsilon_{i_1} \wedge \cdots \wedge \epsilon_{i_a}) = \sum_{j=1}^{a} (-1)^{j+1} (1 - {s_j}^{-1}) \epsilon_{i_1} \wedge \cdots \wedge \widehat{\epsilon}_{i_j} \wedge \cdots \wedge \epsilon_{i_a}$$

where the symbol "$\widehat{\cdot}$" denotes deletion of an element (note that $\partial_1(\epsilon_i) = 1 - s_i^{-1}$).

Then $R/I \cong \mathbf{Z}$ and the following sequence of $R$-modules is exact (see [41, Corollary 4.5.5])

$$0 \to \bigwedge^k R^k \to \cdots \to \bigwedge^1 R^k \to \bigwedge^0 R^k \to \mathbf{Z} \to 0.$$

Note that $\bigwedge^0 R^k = R$ and $\bigwedge^a R^k$ is a free $R$-module with basis

$$\{\epsilon_{i_1} \wedge \cdots \wedge \epsilon_{i_a} : 1 \le i_1 < \cdots < i_a \le k\}.$$

Hence, $\bigwedge^* R^k$ yields a projective resolution of $\mathbf{Z}$. Thus, by [6, §III.1] we have

$$H_*(G, K_0(\mathcal{F}_\Lambda)) \cong H_*(\bigwedge^* R^k \otimes_G K_0(\mathcal{F}_\Lambda)).$$

We follow Evans here but have adopted slightly different notation to make naturality more apparent (see [14, Definition 3.11] and following). Under the isomorphism $\bigwedge^a R^k \otimes_G K_0(\mathcal{F}_\Lambda) \cong \bigwedge^a \mathbf{Z}^k \otimes K_0(\mathcal{F}_\Lambda)$ (as abelian groups), the boundary map $\partial_a : \bigwedge^a \mathbf{Z}^k \otimes K_0(\mathcal{F}_\Lambda) \to \bigwedge^{a-1} \mathbf{Z}^k \otimes K_0(\mathcal{F}_\Lambda)$ is given by

$$\partial_a(\epsilon_{i_1} \wedge \cdots \wedge \epsilon_{i_a} \otimes x) = \sum_{j=1}^{a} (-1)^{a+1} \epsilon_{i_1} \wedge \cdots \wedge \widehat{\epsilon}_{i_j} \wedge \cdots \wedge \epsilon_{i_a} \otimes (1 - s_j)x$$

where $1 \leq i_1 < \cdots < i_a \leq k$ and $x \in K_0(\mathcal{F}_\Lambda)$.

Let $D_a^\Lambda$ be given as above. There is a natural map $\varepsilon^\Lambda : C_0(\Lambda^0) \hookrightarrow \mathcal{F}_\Lambda$ which induces a map $\varepsilon_*^\Lambda : \mathbf{Z}\Lambda^0 \to K_0(\mathcal{F}_\Lambda)$. Moreover (see [14, Theorem 3.14]) the natural map

$$\operatorname{id} \otimes \varepsilon_*^\Lambda : \bigwedge^* \mathbf{Z}^k \otimes \mathbf{Z}\Lambda^0 \to \bigwedge^* \mathbf{Z}^k \otimes K_0(\mathcal{F}_\Lambda)$$

is a map of complexes which induces an isomorphism on homology and hence

$$H_*(G, K_0(\mathcal{F}_\Lambda)) \cong H_*(\bigwedge^* \mathbf{Z}^k \otimes \mathbf{Z}\Lambda^0).$$

Therefore, setting

$$E^1_{a,b} = \begin{cases} \bigwedge^a \mathbf{Z}^k \otimes \mathbf{Z}\Lambda^0 & \text{if } 0 \leq a \leq k \text{ and } b \text{ is even,} \\ 0 & \text{otherwise} \end{cases}$$

and defining $d^1 : E^1_{a,b} \to E^1_{a-1,b}$ to be $\partial_a$ if $b$ is even (and 0 otherwise), yields

$$E^2_{a,b} \cong \begin{cases} H_p(G, K_0(\mathcal{F}_\Lambda)) & \text{if } 0 \leq a \leq k \text{ and } b \text{ is even,} \\ 0 & \text{otherwise.} \end{cases}$$

It follows by [14, Lemma 3.3] that the spectral sequence converges to $K_*(C^*(\Lambda)) = K_*(\mathcal{F}_\Lambda \times_\alpha \mathbf{Z}^k)$ as required.

For the second part of the theorem, fix $(\Lambda, \Gamma, p, m, \mathfrak{s})$. The embedding $\iota_{p,\mathfrak{s}} : C^*(\Lambda) \to M_m(C^*(\Gamma))$ induces an embedding $\widetilde{\iota_{p,\mathfrak{s}}} : \mathcal{F}_\Lambda \to M_m(\mathcal{F}_\Gamma)$. Functoriality yields a map of complexes

$$\operatorname{id} \otimes (\widetilde{\iota_{p,\mathfrak{s}}})_* : \bigwedge^* \mathbf{Z}^k \otimes K_0(\mathcal{F}_\Lambda) \to \bigwedge^* \mathbf{Z}^k \otimes K_0(\mathcal{F}_\Gamma).$$

Since group homology is a covariant functor of its coefficient module we obtain the functorial maps for each $n = 0, 1, \ldots, k$

$$H_n((\widetilde{\iota_{p,\mathfrak{s}}})_*) : H_n(\mathbf{Z}^k, K_0(\mathcal{F}_\Lambda)) \to H_n(\mathbf{Z}^k, K_0(\mathcal{F}_\Gamma)).$$

Then arguing as in Lemma 5.2 with $p^* : \mathbf{Z}\Lambda^0 \to \mathbf{Z}\Gamma^0$ defined as above we see that

$$(1 - (M_j^\Gamma)^t)(m \cdot p^*) = (m \cdot p^*)(1 - (M_j^\Lambda)^t)$$

for all $j = 1, \ldots, k$. It follows that the natural map

$$\operatorname{id} \otimes (m \cdot p^*) : \bigwedge^* \mathbf{Z}^k \otimes \mathbf{Z}\Lambda^0 \to \bigwedge^* \mathbf{Z}^k \otimes \mathbf{Z}\Gamma^0$$

is a map of complexes.

Arguing as in the proof of Theorem 5.1, we see that $(\widetilde{\iota_{p,\mathfrak{s}}})_* \circ \varepsilon_*^\Lambda = \varepsilon_*^\Gamma \circ (m \cdot p^*)$, so the map on homology induced by $\operatorname{id} \otimes (m \cdot p^*)$ coincides with the functorial map above (under the identifications of the homology groups induced by $\operatorname{id} \otimes \varepsilon_*^\Lambda$ and $\operatorname{id} \otimes \varepsilon_*^\Gamma$). This combined with the naturality of Proposition 5.4 yields a morphism $f$ of spectral sequences compatible with the map

$$\left(\widehat{\iota_{p,\mathfrak{s}}}\right)_* : K_*(\mathcal{F}_\Lambda \times_\alpha \mathbf{Z}^k) \to K_*(\mathcal{F}_\Gamma \times_\alpha \mathbf{Z}^k)$$

such that $f^1 : D_a^\Lambda \to D_a^\Gamma$ is given by $\operatorname{id} \otimes (m \cdot p^*)$. Under the identifications $K_*(C^*(\Lambda)) = K_*(\mathcal{F}_\Lambda \times_\alpha \mathbf{Z}^k)$ and $K_*(C^*(\Gamma)) = K_*(\mathcal{F}_\Gamma \times_\alpha \mathbf{Z}^k)$, we have $\left(\widehat{\iota_{p,\mathfrak{s}}}\right)_* = (\iota_{p,\mathfrak{s}})_*$. □

The following corollary is an immediate consequence of the above theorem restricted to the case $k = 2$; for the first assertion see [14, Proposition 3.16] and its proof (see also [35]).

Given a 2-graph $\Lambda$, recall that $M_1$ and $M_2$ denote the vertex connectivity matrices of the coordinate graphs $(\Lambda^0, \Lambda^{e_1}, r, s)$ and $(\Lambda^0, \Lambda^{e_2}, r, s)$.

Corollary 5.6. *Suppose that $(\Lambda, \Gamma, p, m, \mathfrak{s})$ is a row-finite covering system of 2-graphs with no sources. With the notation of Theorem 5.5, the complex $D_a^\Lambda = \bigwedge^a \mathbf{Z}^2 \otimes \mathbf{Z}\Lambda^0$ may be written as follows:*

$$0 \leftarrow \mathbf{Z}\Lambda^0 \xleftarrow{\partial_1} \mathbf{Z}\Lambda^0 \oplus \mathbf{Z}\Lambda^0 \xleftarrow{\partial_2} \mathbf{Z}\Lambda^0 \leftarrow 0 \tag{5.6}$$

*where $\partial_1 = (1 - M_1^t, 1 - M_2^t)$ and $\partial_2 = \begin{pmatrix} M_2^t - 1 \\ 1 - M_1^t \end{pmatrix}$. We have $E_{a,b}^2 = E_{a,b}^\infty$, and*

$$\begin{aligned} K_0(C^*(\Lambda)) &\cong \operatorname{coker} \partial_1 \oplus \ker \partial_2 \\ K_1(C^*(\Lambda)) &\cong \ker \partial_1 / \operatorname{Im} \partial_2 \cong H_1(\mathbf{Z}^k, K_0(\mathcal{F}_\Lambda)). \end{aligned} \tag{5.7}$$

*Moreover, the following diagram commutes*

$$\begin{array}{ccccccccc} 0 & \longleftarrow & \mathbf{Z}\Lambda^0 & \xleftarrow{\partial_1^\Lambda} & \mathbf{Z}\Lambda^0 \oplus \mathbf{Z}\Lambda^0 & \xleftarrow{\partial_2^\Lambda} & \mathbf{Z}\Lambda^0 & \longleftarrow & 0 \\ & & \Big\downarrow{\scriptstyle m \cdot p^*} & & \Big\downarrow{\scriptstyle m \cdot p^* \oplus m \cdot p^*} & & \Big\downarrow{\scriptstyle m \cdot p^*} & & \\ 0 & \longleftarrow & \mathbf{Z}\Gamma^0 & \xleftarrow{\partial_1^\Gamma} & \mathbf{Z}\Gamma^0 \oplus \mathbf{Z}\Gamma^0 & \xleftarrow{\partial_2^\Gamma} & \mathbf{Z}\Gamma^0 & \longleftarrow & 0 \end{array} \tag{5.8}$$

*and by naturality induces $(\iota_{p,\mathfrak{s}})_* : K_*(C^*(\Lambda)) \to K_*(C^*(\Gamma))$.*

The inclusion of $\operatorname{coker} \partial_1$ into $K_0(C^*(\Lambda))$ obtained from (5.7) takes the equivalence class (in the quotient group $\operatorname{coker} \partial_1 = \mathbf{Z}\Lambda^0 / \operatorname{Im}(\partial_1)$) of the generator $\delta_v$ of $\mathbf{Z}\Lambda^0$ to the $K_0$-class of the vertex projection $[s_v]$ in $C^*(\Lambda)$. The proof of this fact can be obtained from the proof of [14, Proposition 4.4]. We thank Gwion Evans for pointing this out to us.

5.3. Product coverings and the Künneth formula. In this section we consider covering systems $(\Lambda_n, p_n)$ in which each $k$-graph $\Lambda_n$ is a cartesian product of two lower-dimensional graphs, and the covering maps $p_n$ respect the product decomposition.

Recall from [20, Proposition 1.8] that given a $k$-graph $(\Lambda, d)$ and a $k'$-graph $(\Lambda', d')$, the cartesian-product category $\Lambda \times \Lambda'$ becomes a $(k + k')$-graph when we endow it with the degree functor $d \times d' : (\lambda, \lambda') \mapsto (d(\lambda)_1, \ldots, d(\lambda)_k, d'(\lambda')_1, \ldots, d'(\lambda')_{k'})$.

Proposition 5.7. *Fix $k, k' \in \mathbf{N} \setminus \{0\}$. Let $(\Lambda, \Gamma, p, m, \mathfrak{s})$ and $(\Lambda', \Gamma', p', m', \mathfrak{s}')$ be row-finite covering systems of $k$- and $k'$-graphs with no sources. Then*

$$p \times p' : \Gamma \times \Gamma' \to \Lambda \times \Lambda'$$

*is a finite covering of row-finite $(k + k')$-graphs with no sources. Let $f : \{1, \ldots, m\} \times \{1 \ldots, m'\} \to \{1, \ldots, mm'\}$ denote the bijection $f(j, j') := j + (j' - 1)m$. There is a cocycle $\mathfrak{s} \times \mathfrak{s}' : \Gamma \times \Gamma' \to S_{mm'}$ determined by $\big((\mathfrak{s} \times \mathfrak{s}')(\alpha, \alpha')\big) f(j, j') := f\big(\mathfrak{s}(\alpha) j, \mathfrak{s}'(\alpha') j'\big)$. Moreover, the following diagram commutes.*

$$\begin{array}{ccc}
C^*(\Lambda \times \Lambda') & \xrightarrow{\cong} & C^*(\Lambda) \otimes C^*(\Lambda') \\
\Big\downarrow{\scriptstyle \iota_{p \times p', \mathfrak{s} \times \mathfrak{s}'}} & & \Big\downarrow{\scriptstyle \iota_{p,\mathfrak{s}} \otimes \iota_{p', \mathfrak{s}'}} \\
M_{mm'}(C^*(\Gamma \times \Gamma')) & \xrightarrow{\cong} & M_m(C^*(\Gamma)) \otimes M_{m'}(C^*(\Gamma'))
\end{array}$$

*Suppose that at least one of $K_*(C^*(\Lambda))$, $K_*(C^*(\Lambda'))$ and at least one of $K_*(C^*(\Gamma))$, $K_*(C^*(\Gamma'))$ are torsion-free. Then the following diagram commutes and the horizontal connecting maps are zero-graded isomorphisms:*

$$\begin{array}{ccc}
K_*(C^*(\Lambda)) \otimes K_*(C^*(\Lambda')) & \xrightarrow{\cong} & K_*(C^*(\Lambda \times \Lambda')) \\
\Big\downarrow{\scriptstyle (\iota_{p,\mathfrak{s}})_* \otimes (\iota_{p',\mathfrak{s}'})_*} & & \Big\downarrow{\scriptstyle (\iota_{p \times p', \mathfrak{s} \times \mathfrak{s}'})_*} \\
K_*(C^*(\Gamma)) \otimes K_*(C^*(\Gamma')) & \xrightarrow{\cong} & K_*(C^*(\Gamma \times \Gamma'))
\end{array}$$

*If $\Gamma^0$ and $(\Gamma')^0$ (and hence also $\Lambda^0$ and $(\Lambda')^0$) are finite then the $C^*$-algebras are unital, and the horizontal isomorphisms take $[1] \otimes [1]$ to $[1]$.*

*Proof.* It is straightforward to check that $p \times p'$ is a covering using the properties of the covering maps $p$ and $p'$ and the definition of the cartesian-product graph. A simple calculation shows that $\mathfrak{s} \times \mathfrak{s}'$ defines a cocycle.

Theorem 5.5 of [20] shows that $C^*(\Lambda)$, $C^*(\Lambda')$, $C^*(\Gamma)$ and $C^*(\Gamma')$ are nuclear, and so there is just one tensor-product $C^*$-algebra $C^*(\Lambda) \otimes C^*(\Lambda')$. Corollary 3.5(iv) of [20] shows that the map $s_{(\lambda,\mu)} \mapsto s_\lambda \otimes s_\mu$ is an isomorphism of $C^*(\Lambda \times \Lambda')$ onto $C^*(\Lambda) \otimes C^*(\Lambda')$, and similarly for $C^*(\Gamma)$ and $C^*(\Gamma')$. It is easy to check using the formulae for the maps $\iota_{p,\mathfrak{s}}$, $\iota_{p',\mathfrak{s}'}$, and $\iota_{p \times p', \mathfrak{s} \times \mathfrak{s}'}$ and using the chain of isomorphisms

$$\begin{aligned} M_{mm'}(C^*(\Gamma \times \Gamma')) &\cong M_{mm'}(\mathbf{C}) \otimes C^*(\Gamma \times \Gamma') \\ &\cong M_m(\mathbf{C}) \otimes C^*(\Gamma) \otimes M_{m'}(\mathbf{C}) \otimes C^*(\Gamma') \\ &\cong M_m(C^*(\Gamma)) \otimes M_{m'}(C^*(\Gamma')) \end{aligned}$$

that the first diagram commutes.

In the presence of the additional hypothesis concerning torsion-free $K$-groups, the Künneth Theorem of [37] (see also Theorem 23.1.3 of [4]) implies: (1) that

$$K_*(C^*(\Lambda)) \otimes K_*(C^*(\Lambda')) \cong K_*(C^*(\Lambda) \otimes C^*(\Lambda'))$$

and similarly for $\Gamma, \Gamma'$; (2) that these isomorphisms are natural and are zero-graded; and (3) that these isomorphisms take $[1]\otimes[1]$ to $[1]$. The result therefore follows from the naturality of the $K$-functor. □

Note that in general when no assumption is made about torsion, the Künneth Theorem of [37] gives a short exact sequence which is still natural. The analogue of Proposition 5.7 still holds and gives a (fairly complicated) commuting diagram in which the rows are short exact sequences.

## 6. Examples

In this section we discuss a number of examples. A recurring theme will be supernatural numbers and the associated dimension groups, so we pause here to establish some notation.

We will think of a supernatural number as an infinite product $\alpha = \prod_{n=1}^{\infty} \alpha_n$ where each $\alpha_n$ is an integer greater than 1. Any two such expressions in which the same prime factors occur with the same cardinality correspond to the same supernatural number. Given supernatural numbers $\alpha, \beta$, we will abuse notation and write $\alpha\beta$ for the supernatural number $\prod_{n=1}^{\infty} \alpha_n\beta_n$. We write $\alpha[1, n]$ for the product $\prod_{i=1}^{n} \alpha_i$ of the first $n$ terms in $\alpha$.

For $z_1, \ldots, z_n \in \mathbf{C}$, we write $\mathbf{Z}[z_1, \ldots, z_n]$ for the ring obtained by adjoining $z_1, \ldots, z_n$ to $\mathbf{Z}$; we regard $\mathbf{Z}[z_1, \ldots, z_n]$ as a group under addition. Abusing notation, for a supernatural number $\alpha$, we write $\mathbf{Z}\big[\frac{1}{\alpha}\big]$ for the dimension group $\varinjlim(\mathbf{Z}, \times\alpha_n)$ which we identify with the group

$$\bigcup_{n=1}^{\infty} \mathbf{Z}\Big[\frac{1}{\alpha[1, n]}\Big] \subset \mathbf{Q}$$

consisting of all fractions $p/q$ where $p, q \in \mathbf{Z}$, and $q$ is a divisor of some $\alpha[1, n]$.

### 6.1. Rank-2 Bratteli diagrams.

A rank-2 Bratteli diagram is a 2-graph in which the blue edges form a Bratteli diagram and the red edges determine simple cycles so that every vertex lies on precisely one red cycle, and all vertices on a given red cycle are at the same level in the blue Bratteli diagram.

The $C^*$-algebras of these 2-graphs were studied in [27] and provided the initial motivation for the covering construction. A rank-2 Bratteli diagram $\Lambda$ can be constructed using Proposition 2.14 and Corollary 2.15 precisely when the length of each red cycle at level $n$ of $\Lambda$ is divisible by the lengths of all the cycles at level $n-1$ to which it connects. In particular, the 2-graphs whose $C^*$-algebras are Morita equivalent to the Bunce-Deddens algebras [27, Example 6.7] and the irrational rotation algebras [27, Example 6.5] arise in this fashion.

6.2. Coverings of dihedral graphs $\boldsymbol{D_n}$. For $n \in \mathbf{N} \setminus \{0\}$, let $D_n$ be the directed graph with $n$ vertices $\{v_0, \ldots, v_{n-1}\}$ and edges $\{x_i, y_i : 0 \leq i \leq n-1\}$ where $r(x_i) = v_i = s(y_i)$ and $s(x_i) = v_{i+1} = r(y_i)$ (throughout this section, addition in the subscripts is understood to be evaluated modulo $n$). More descriptively, $D_n$ is a ring of $n$ vertices, each of which connects to both of its neighbours (see Figure 2). Let $D_n^*$ be the path-category of $D_n$, regarded as a 1-graph. Note that for $n \in \mathbf{N} \setminus \{0\}$, the graph $D_{2n}$ is the Cayley graph for the

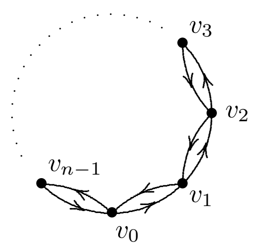


Figure 2. The 1-graph $D_n$

dihedral group with $2n$ elements.

*Example* 6.1. For $n, m \geq 1$ there are $m$-fold covering maps $p_{n,mn} : D_{mn}^* \to D_n^*$ as follows: for $0 \leq i \leq mn-1$ let $i' = i \mod n$ and define

$$p_{n,mn}(v_i) := v_{i'}, \quad p_{n,mn}(x_i) := x_{i'} \quad \text{and} \quad p_{n,mn}(y_i) := y_{i'}.$$

Hence for each pair of positive integers $n, m$, we obtain a row-finite covering system $(D_n^*, D_{mn}^*, p_{n,mn})$ of 1-graphs with no sources (see Notation 2.8).
Fix an infinite supernatural number $\alpha = \prod_{i=1}^\infty \alpha_i$. Consider the sequence of covering systems $(D^*_{6\alpha[1,n]}, D^*_{6\alpha[1,n+1]}, p_{6\alpha[1,n],6\alpha[1,n+1]})_{n=1}^\infty$ as in Notation 2.8. Applying Corollary 2.11, we obtain a 2-graph

$$D := \varprojlim(D^*_{6\alpha[1,n]}, p_{6\alpha[1,n],6\alpha[1,n+1]}).$$

Proposition 6.2. *Consider the situation discussed in Example 6.1. We have* $K_0(C^*(D)) = \mathbf{Z}[\frac{1}{\alpha}] \oplus \mathbf{Z}[\frac{1}{\alpha}]$ *and* $K_1(C^*(D)) = \mathbf{Z} \oplus \mathbf{Z}$. *Let* $P_1 := \sum_{v \in D_6^0} s_v$. *Then* $[P_1]$ *is the 0 element of* $K_0(P_1 C^*(D) P_1)$. *Moreover,* $C^*(D)$ *is simple and purely infinite.*

Before proving the proposition, we describe the $K$-theory of $C^*(D_n^*)$ in general.

Lemma 6.3. (1) $K_0(C^*(D_n^*))$ *is generated by* $[s_{v_0}]$ *and* $[s_{v_1}]$, *and for each* $i$, *we have* $[s_{v_i}] = -[s_{v_{i+3}}]$ *in* $K_0(C^*(D_n^*))$.
(2) $K_1(C^*(D_n^*)) \cong \{(a_1, \ldots, a_n) \in \mathbf{Z}^n : a_{i+2} = a_{i+1} - a_i$ *for all* $i\}$.

(3) *the following table describes the $K$-theory of each $C^*(D_n^*)$.*

| $n \bmod 6$ | $K_0(C^*(D_n^*))$ | $K_1(C^*(D_n^*))$ |
|---|---|---|
| 0 | $\mathbf{Z}^2$ | $\mathbf{Z}^2$ |
| 1 | 0 | 0 |
| 2 | $\mathbf{Z}/3\mathbf{Z}$ | 0 |
| 3 | $\mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z}$ | 0 |
| 4 | $\mathbf{Z}/3\mathbf{Z}$ | 0 |
| 5 | 0 | 0 |

*Proof.* (1) The $K_0$ group is generated by the classes $[s_{v_0}], \dots, [s_{v_{n-1}}]$ subject to the relations $[s_{v_i}] = [s_{v_{i+1}}] + [s_{v_{i-1}}]$. This relation forces $[s_{v_{i+2}}] = [s_{v_{i+1}}] - [s_{v_i}]$, from which we conclude first that $K_0$ is generated by $[s_{v_0}]$ and $[s_{v_1}]$ and second that

$$[s_{v_{i+3}}] = [s_{v_{i+2}}] - [s_{v_{i+1}}] = ([s_{v_{i+1}}] - [s_{v_i}]) - [s_{v_{i+1}}] = -[s_{v_i}].$$

(2) Let $A_n$ denote the vertex connectivity matrix of $D_n$; so $A_n(i,j) = 1$ when $i = j \pm 1 \pmod n$ and zero otherwise. As in Theorem 5.1, we have $K_1(C^*(D_n^*)) \cong \ker(1 - A_n^t)$. For $m \in \mathbf{Z}^n$, $((1 - A_n^t)m)_i = -m_{i-1} + m_i - m_{i+1}$ by definition of $A_n$, and this establishes (2).

(3) If $E$ is a finite 1-graph with no sinks or sources, then $C^*(E)$ is isomorphic to the Cuntz-Krieger algebra of the adjacency matrix $A_E$ of $E$ [23]. In particular, $K_1(C^*(E))$ is torsion-free and has the same rank as $K_0(C^*(E))$ [9]. Hence it suffices to verify that the first column of the table is correct. To calculate $K_0$, we use (1) to check by hand that the cases $n = 1, 2, \dots 6$ are as claimed. If $n > 6$, then applying the relations we find that $[s_{v_{i+6}}] = [s_{v_i}]$ for all $i$ which accounts for all remaining cases. □

*Proof of Proposition 6.2.* Lemma 6.3(1) shows that $K_0(C^*(D^*_{6\alpha[1,n]})$ is generated by $[s_{v_0^n}]$ and $[s_{v_1^n}]$ where the $v_i^n$ are the vertices of $D^*_{6\alpha[1,n]}$. Fix $i \in \{1, 2\}$. We have

$$(\iota_{p_n})_*[s_{v_i^n}] = [s_{v_i^{n+1}}] + [s_{v^{n+1}_{i+6\alpha[1,n]}}] + \cdots + [s_{v^{n+1}_{i+6(\alpha_{n+1}-1)\alpha[1,n]}}]. \tag{6.1}$$

By Lemma 6.3(1), each $[s_{v_i^{n+1}+6k}] = [s_{v_i^{n+1}}]$ in $K_0(C^*(D^*_{6\alpha[1,n+1]}))$, so (6.1) implies $(\iota_{p_n})_*[s_{v_i^n}] = \alpha_n \cdot [s_{v_i^{n+1}}]$. Hence $K_0(\iota_{p_n}) : \mathbf{Z}^2 \to \mathbf{Z}^2$ is multiplication by $\alpha_n$.

Fix $m \in \mathbf{N} \setminus \{0\}$. By Lemma 6.3(2), $K_1(C^*(D^*_{6m}))$ is identified with the set of sequences $(a_1, \dots, a_{6m})$ which satisfy $a_{i+2} = a_{i+1} - a_i$ for all $i$. By Lemma 6.3(2), this forces $a_{i+2} = a_{i+1} - a_i$ for all $i$. Consequently, the map $a = (a_1, \dots, a_{6m}) \mapsto (a_1, a_2)$ yields an isomorphism $\zeta_m : K_1(C^*(D^*_{6m})) \to \mathbf{Z}^2$. As $\zeta_{\alpha[1,n+1]} \circ K_1(\iota_{p_{6\alpha[1,n],6\alpha[1,n+1]}}) = \zeta_{\alpha[1,n]}$, it follows that $K_1(\iota_{p_n}) : \mathbf{Z}^2 \to \mathbf{Z}^2$ is the identity map.

Recall that $D$ denotes $\varprojlim(D^*_{6\alpha[1,n]}, p_{6\alpha[1,n],6\alpha[1,n+1]})$. By Theorem 5.1 the $K$-groups of $C^*(D)$ are as claimed. To compute the class of the identity, let $P_1 \in C^*(D)$ be the sum of the six vertex projections in the bottom level. The final statement of Lemma 6.3(1) shows that the classes of the vertex projections

in $K_0(C^*(D_6^*))$ cancel, so that the class of the identity in $K_0(C^*(D_6^*))$ is the zero element. It follows that the class of the identity $P_1$ in $K_0(P_1C^*(D)P_1)$ is also the zero element.
Each $D_n^*$ is aperiodic and cofinal (see Definition 4.2), so we may conclude from Corollary 4.4 and Lemma 4.7 that $D$ is aperiodic and cofinal. Hence Proposition 4.8 of [20] implies that $C^*(D)$ is simple. The path $x_1y_1$ is a cycle with an entrance (namely $y_0$) in $D_1^*$. Proposition 4.8 now shows that $C^*(D)$ is purely infinite. □

6.3. Direct limits of $\mathcal{O}_n \otimes C(\mathbf{T})$.

*Example* 6.4. Fix $n \geq 3$, and let $B_n$ be the bouquet of $n$ loops. For $m \geq 1$, let $L_m$ denote the loop with $m$ vertices, and let $\Lambda_m$ be the cartesian-product 2-graph $\Lambda_m = L^*_{(n-1)^m} \times B_n^*$ obtained from the path categories of $L_{(n-1)^m}$ and $B_n$.
For each $m$, Let $p_m$ denote the obvious $(n-1)$-fold covering of $L^*_{(n-1)^m}$ by $L^*_{(n-1)^{m+1}}$, and let $p'$ be the identity covering of $B_n$ by $B_n$.

Proposition 6.5. *Consider the situation of Example 6.4. Let $v$ be a vertex of $\Lambda_1$. Then $s_vC^*(\varprojlim(\Lambda_m, p_m \times p'))s_v$ is isomorphic to the Kirchberg algebra $\mathcal{P}_n$ (see* [5]*) whose $K$-theory is opposite to that of $\mathcal{O}_n$.*

*Proof.* Since $C^*(B_n)$ is generated by $n$ isometries whose range projections sum to the identity, $C^*(B_n)$ is canonically isomorphic to $\mathcal{O}_n$ [7]. Hence

$$C^*(\Lambda_m) \cong C^*(L^*_{(n-1)^m}) \otimes \mathcal{O}_n$$

by [20, Corollary 3.5(iv)]. As in [17, Lemma 2.4], there is an isomorphism $C^*(L^*_{(n-1)^m}) \cong M_{(n-1)^m}(C(\mathbf{T}))$ for each $m$, and in particular we have $K_*(C^*(L^*_{(n-1)^m})) \cong (\mathbf{Z}, \mathbf{Z})$. Since $K_*(\mathcal{O}_n) = (\mathbf{Z}/(n-1)\mathbf{Z}, 0)$ [9], the Künneth theorem implies that $K_*(C^*(\Lambda_m)) \cong (\mathbf{Z}/(n-1)\mathbf{Z}, \mathbf{Z}/(n-1)\mathbf{Z})$.
A special case of [27, Equation (4.7)] implies that the covering map $p_m$ induces multiplication by $n-1$ from $K_0(C^*(L^*_{(n-1)^m}))$ to $K_0(C^*(L^*_{(n-1)^{m+1}}))$, and the identity homomorphism from $K_1(C^*(L^*_{(n-1)^m}))$ to $K_1(C^*(L^*_{(n-1)^{m+1}}))$. Clearly $p'$ induces the identity map on $K_*(\mathcal{O}_n)$.
Let $\Lambda = \varprojlim(\Lambda_m, p_m \times p')$. Theorem 3.8 and Proposition 5.7 combine to show that

$$K_*(C^*(\Lambda)) \cong \varinjlim((\mathbf{Z}/(n-1)\mathbf{Z}, \mathbf{Z}/(n-1)\mathbf{Z}), (\times(n-1), \mathrm{id})).$$

Since multiplication by $n-1$ is the 0 homomorphism from $\mathbf{Z}/(n-1)\mathbf{Z}$ to $\mathbf{Z}/(n-1)\mathbf{Z}$, it follows that $K_*(C^*(\Lambda)) \cong (0, \mathbf{Z}/(n-1)\mathbf{Z})$.
Lemma 4.7 proves that $\Lambda$ is cofinal. For an infinite path $y \in \Lambda^\infty$, Lemma 4.5 combined with the observation that the cycles in the $L^*_{(n-1)^m}$ grow with $m$ shows that if $a, b \in \mathbf{N}^3$ and $\sigma^a(y) = \sigma^b(y)$, then $a$ and $b$ differ only in their first coordinates. It follows from Proposition 4.3 that the aperiodicity of $\Lambda$ is implied by the well-known aperiodicity of $B_n$. Hence $C^*(\Lambda)$ is simple by [20, Proposition 4.8]. Moreover, since every vertex of $\Lambda$ hosts a cycle with an

entrance, $C^*(\Lambda)$ is also purely infinite (see [20, Proposition 4.9], [38, Proposition 8.8]). The result therefore follows from the Kirchberg-Phillips classification theorem [28]. □

6.4. Higher-rank Bunce-Deddens algebras. In this subsection we describe a class of simple $A\mathbf{T}$ algebras with real-rank 0 which arise from sequences of covering systems of 2-graphs and which cannot in general be obtained from the construction of [27] (see Example 6.6 and Theorem 6.7). We indicate in Remark 6.12 why we think of these algebras as higher-rank analogues of the Bunce-Deddens algebras.

For $k \geq 1$, let $\Delta_k$ be the $k$-graph with vertices $\mathbf{Z}^k$, morphisms $\{(m,n) \in \mathbf{Z}^k \times \mathbf{Z}^k : m \leq n\}$ where $r(m,n) = m$, $s(m,n) = n$ and $d(m,n) = n-m$. There is a free action of $\mathbf{Z}^k$ on $\Delta_k$ given by translation; that is $m\cdot(p,q) = (p+m, q+m)$ for $m \in \mathbf{Z}^k$ and $(p,q) \in \Delta_k$.

Given a finite-index subgroup $H$ of $\mathbf{Z}^k$, we denote by $\Delta_k/H$ the quotient of $\Delta_k$ by the action of $H$. That is, for $q \in \mathbf{N}^k$, $(\Delta_k/H)^q = \{[g, g+q] : g \in \mathbf{Z}^k\}$; in particular, $(\Delta_k/H)^0 = \{[g,g] : g \in \mathbf{Z}^k\}$, and we henceforth identify $(\Delta_k/H)^0$ with $\mathbf{Z}^k/H$ via the map $[g,g] \mapsto [g]$ where $[g]$ denotes the class $g+H$ of $g$ in $\mathbf{Z}^k/H$. The range and source maps in $\Delta_k/H$ are then given by $r([g,g+q]) = [g]$ and $s([g,g+q]) = [g+q]$. If $H' \subset H$ is a finite-index subgroup of $H$, then it also has finite index in $\mathbf{Z}^k$, and there is a natural surjection $p : \mathbf{Z}^k/H' \to \mathbf{Z}^k/H$ which induces a finite covering map, also denoted $p$ of $\Delta_k/H$ by $\Delta_k/H'$.

Most of the remainder of this section is concerned with the following example of a sequence of covering systems.

*Example* 6.6. Let $H_1 \supset H_2 \supset H_3 \supset \dots$ be a chain of finite-index subgroups of $\mathbf{Z}^2$. For each $n$, let $p_n : \Delta_2/H_{n+1} \to \Delta_2/H_n$ be the canonical covering induced by the quotient maps described above, let $m_n = 1$, and let $\mathfrak{s}_n : \Delta_2/H_{n+1} \to S_1$ be the trivial cocycle. This data specifies a sequence $(\Delta_2/H_n, \Delta_2/H_{n+1}, p_n)_{n=1}^\infty$ of row-finite covering systems of 2-graphs with no sources. Applying Corollary 2.11, we obtain a 3-graph $\varprojlim(\Delta_2/H_n; p_n))$. As always, $P_1$ denotes $\sum_{v \in (\Delta_2/H_1)^0} s_v \in C^*(\Delta_2/H_1) \subset C^*(\varprojlim(\Delta_2/H_n; p_n))$.

Theorem 6.7. *Consider the situation of Example 6.6.*

(1) *We have*

$$K_0(P_1 C^*(\varprojlim(\Delta_2/H_n; p_n))P_1) \cong \varinjlim(\mathbf{Z}, \times[H_n : H_{n+1}]) \oplus \mathbf{Z},$$

*and this isomorphism takes $[P_1]$ to $(g,0)$ where $g$ is the image of $[\mathbf{Z}^2 : H_1]$ in the direct limit $\varinjlim(\mathbf{Z}, \times[H_n : H_{n+1}])$.*

(2) *For each $n$ the homomorphism from $\mathbf{Z}^2$ to $\mathbf{Z}^2$ determined by coordinate-wise multiplication by the integer $[H_n : H_{n+1}]$ restricts to a homomorphism $m_{H_n,H_{n+1}} : H_n \to H_{n+1}$. Moreover,*

$$K_1(P_1 C^*(\varprojlim(\Delta_2/H_n; p_n))P_1) \cong \varinjlim(H_n, m_{H_n,H_{n+1}}).$$

(3) *$C^*(\varprojlim(\Delta_2/H_n; p_n))$ is simple if and only if $\bigcap_{n=1}^\infty H_n = \{0\}$, and is an $A\mathbf{T}$ algebra with real-rank 0 when it is simple.*

The proof of this result will occupy the bulk of this section. Before presenting it, we state a Corollary and use it to formulate some concrete examples.

Corollary 6.8. *Consider the situation of Example 6.6. There are sequences $(h_1^n)_{n=1}^\infty$ and $(h_2^n)_{n=1}^\infty$ in $\mathbf{Z}^2$ such that: (1) for each $n$, the elements $h_1^n$ and $h_2^n$ generate $H_n$; and (2) the matrix $M_n = \begin{pmatrix} m_{1,1}^n & m_{1,2}^n \\ m_{2,1}^n & m_{2,2}^n \end{pmatrix}$ satisfying $h_1^{n+1} = m_{1,1}^n h_1^n + m_{1,2}^n h_2^n$ and $h_2^{n+1} = m_{2,1}^n h_1^n + m_{2,2}^n h_2^n$ has positive determinant for all $n$. Moreover, if $M_n^{\mathrm{ca}}$ denotes the classical adjoint $\begin{pmatrix} m_{2,2}^n & -m_{1,2}^n \\ -m_{2,1}^n & m_{1,1}^n \end{pmatrix}$ of $M_n$ for each $n$, and if we regard these matrices as homomorphisms of $\mathbf{Z}^2$, then*

$$K_1(P_1 C^*(\varprojlim(\Delta_2/H_n; p_n))P_1) \cong \varinjlim(\mathbf{Z}^2, M_n^{\mathrm{ca}}). \tag{6.2}$$

*Proof.* That we can choose the $h_i^n$ so that the matrices $M_n$ all have positive determinant follows from an inductive argument based on the observation that replacing $h_i^{n+1}$ with $-h_i^{n+1}$ reverses the sign of $\det(M_n)$.
For each $n$, let $\psi_n$ be the isomorphism of $\mathbf{Z}^2$ onto $H_n$ satisfying $\psi_n(e_i) = h_i^n$, and let $m_{H_n,H_{n+1}} : H_n \to H_{n+1}$ be the homomorphism described in Theorem 6.7(2). We claim that $\psi_{n+1} \circ M_n^{\mathrm{ca}} = m_{H_n,H_{n+1}} \circ \psi_n$.
To see this, observe that $m_{H_n,H_{n+1}}$ is multiplication by the determinant of $M_n$. Hence, as rational transformations, $m_{H_n,H_{n+1}}^{-1} \circ M_n^{\mathrm{ca}} = M_n^{-1}$. Since $m_{H_n,H_n+1}$ commutes with $\psi_{n+1}$, the desired equality $\psi_{n+1} \circ M_n^{\mathrm{ca}} = m_{H_n,H_{n+1}} \circ \psi_n$ is therefore equivalent to $\psi_{n+1} = \psi_n \circ M_n$, which follows from the definitions of the maps involved. This establishes the claim.
The claim guarantees that $\varinjlim(H_n, m_{H_n,H_{n+1}}) \cong \varinjlim(\mathbf{Z}^2, M_n^{\mathrm{ca}})$, and (6.2) then follows from Theorem 6.7(2). □

*Examples* 6.9. (1) Let $\alpha$ and $\beta$ be supernatural numbers. For $n \in \mathbf{N} \setminus \{0\}$, let $\phi_n$ be the homomorphism of $\mathbf{Z}^2$ determined by the diagonal matrix $M_n := \begin{pmatrix} \alpha_n & 0 \\ 0 & \beta_n \end{pmatrix}$.
For each $n$, let

$$H_n := \alpha[1,n]\mathbf{Z} \times \beta[1,n]\mathbf{Z} = \phi_n(\mathbf{Z}^2) \subset \mathbf{Z}^2.$$

We deduce from Theorem 6.7 that

$$K_*(P_1 C^*(\varprojlim(\Delta_2/H_n; p_n))P_1) = \Big(\mathbf{Z}\big[\tfrac{1}{\alpha\beta}\big] \oplus \mathbf{Z},\ \mathbf{Z}\big[\tfrac{1}{\alpha}\big] \oplus \mathbf{Z}\big[\tfrac{1}{\beta}\big]\Big),$$

that the position of the unit in $K_0$ corresponds to the element $(\alpha_1, 0)$, and that $P_1 C^*(\varprojlim(\Delta_2/H_n; p_n))P_1$ is a simple A$\mathbf{T}$ algebra of real-rank 0.

We claim that this is an example of an A$\mathbf{T}$ algebra which cannot be realised using a rank-2 Bratteli diagram as in [27]. To see this, suppose otherwise. Then [27, Theorem 6.1] implies that there exists an injective homomorphism $\phi : \mathbf{Z}\big[\frac{1}{\alpha}\big] \oplus \mathbf{Z}\big[\frac{1}{\beta}\big] \to \mathbf{Z}\big[\frac{1}{\alpha\beta}\big] \oplus \mathbf{Z}$ such that each element of $\operatorname{coker}(\phi)$ has finite order. Hence there exists $(x,y) \in \mathbf{Z}\big[\frac{1}{\alpha}\big] \oplus \mathbf{Z}\big[\frac{1}{\beta}\big]$ such that $\phi(x,y) = (z,m)$ with $m \neq 0$. Since $\mathbf{Z}\big[\frac{1}{\alpha}\big] \oplus \mathbf{Z}\big[\frac{1}{\beta}\big]$ is generated by elements of the form $(x,0)$ and $(0,y)$, we may in fact assume without loss of generality that there is an element $x \in \mathbf{Z}\big[\frac{1}{\alpha}\big]$ such that $\phi(x,0) = (z,m)$. Since $\alpha$ is infinite, there exist $n > m$ and $x' \in \mathbf{Z}\big[\frac{1}{\alpha}\big]$ such that $n \cdot x' = x$, and this forces $n \cdot \phi(x', 0) = (z, m)$ which is impossible by our choice of $n$.

Since each $\Delta_2/H_n \cong L^*_{\alpha[1,n]} \times L^*_{\beta[1,n]}$, the $K$-theory calculations for this example can also be verified using the Künneth formula (Theorem 3.8 and Proposition 5.7).

(2) Let $\phi$ be the homomorphism of $\mathbf{Z}^2$ determined by the integer matrix $M := \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. Suppose that $M$ is diagonalisable as a real $2 \times 2$ matrix, and that its eigenvalues are greater than 1 in modulus. Let $D := ad - bc$ be the determinant of $M$. For $n \geq 1$, let $H_n := M^n \mathbf{Z}^2$ and $\Lambda_n := \Delta_2/H_n$. Our assumption regarding the eigenvalues of $M$ ensures that $\bigcap_{n=1}^\infty H_n = \{0\}$, so Theorem 6.7 and Corollary 6.8 imply that $C^*(\varprojlim(\Delta_2/H_n; p_n))$ is a simple A$\mathbf{T}$ algebra of real rank zero with

$$K_*(P_1 C^*(\varprojlim(\Delta_2/H_n; p_n))P_1) \cong \Big(\mathbf{Z}\big[\tfrac{1}{D}\big] \oplus \mathbf{Z},\ \varinjlim\big(\mathbf{Z}^2, \big(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\big)\big)\Big).$$

In particular, let $M = \left(\begin{smallmatrix} a & -b \\ b & a \end{smallmatrix}\right)$ with $a^2 + b^2 > 1$. We may identify $\mathbf{Z}^2$ with the group of Gaussian integers $\mathbf{Z}[i]$ by $(m, n) \mapsto m + in$, and then the group homomorphism of $\mathbf{Z}^2$ obtained from multiplication by $M$ coincides with the group homomorphism of $\mathbf{Z}[i]$ obtained from multiplication by $a + ib$. Likewise $M^{\mathrm{ca}}$ implements multiplication by the conjugate $a - ib$. With $D := a^2 + b^2$ and $\zeta := \frac{1}{a - ib} = \frac{a + ib}{a^2 + b^2}$, we have

$$K_*(P_1 C^*(\varprojlim(\Delta_2/H_n; p_n))P_1) \cong \Big(\mathbf{Z}\big[\tfrac{1}{D}\big] \oplus \mathbf{Z},\ \mathbf{Z}\big[i, \tfrac{1}{\zeta}\big]\Big).$$

by Theorem 6.7.

(3) More generally, a sequence of Gaussian integers $\zeta_j := a_j + b_j i$ with $|\zeta_j| > 1$ for all $j$ gives rise to a natural notion of a Gaussian supernatural number $\zeta = \prod_{j=1}^\infty \zeta_j$. Generalising the construction of the latter part of example (2) above, let $H_n := (\prod_{j=1}^n \overline{\zeta_j})\mathbf{Z}[i]$ for each $n$, and identify $\mathbf{Z}[i]$ with $\mathbf{Z}^2$ as a group to obtain a decreasing chain of subgroups of $H_n$ of $\mathbf{Z}^2$ with trivial intersection.

Let $\alpha$ be the supernatural number $\alpha = \prod_{j=1}^\infty |\zeta_j|^2$. Then

$$K_*(P_1 C^*(\varprojlim(\Delta_2/H_n; p_n))P_1) \cong \Big(\mathbf{Z}\big[\tfrac{1}{\alpha}\big] \oplus \mathbf{Z},\ \mathbf{Z}\big[i, \tfrac{1}{\zeta}\big]\Big).$$

by Theorem 6.7 and Corollary 6.8.

We now turn to the proof of Theorem 6.7; in particular, we adopt the notation and conventions of Example 6.6. Our first step is to describe explicitly the $K$-theory of $C^*(\Delta_2/H_n)$ for a fixed $n \in \mathbf{N} \setminus \{0\}$. We do this using the results of Section 5.2.

For $q \in \mathbf{Z}^k$ we write $q_+$ and $q_-$ for the positive and negative parts of $q$. That is to say that $q_+$ and $q_-$ are the unique elements of $\mathbf{N}^k$ whose coordinate-wise minimum $q_+ \wedge q_-$ is equal to 0, and which satisfy $q = q_+ - q_-$.

For $q \in \mathbf{Z}^k$, a *cycle of degree* $q$ in a $k$-graph $\Lambda$ is a pair $(\mu, \nu)$ where $\mu \in \Lambda^{q_+}$ and $\nu \in \Lambda^{q_-}$ such that $r(\mu) = r(\nu)$ and $s(\mu) = s(\nu)$. When $q \in \mathbf{N}^k$, $q = q_+$

and $q_- = 0$, so $\nu$ is a vertex, and $\mu$ is a cycle in the usual sense: a path whose range and source coincide.

Let $H \subset \mathbf{Z}^2$ be a finite-index subgroup of $\mathbf{Z}^2$. Let $G = \mathbf{Z}^2/H$. We view the ring $\mathbf{Z}G$ as the collection of functions $f : G \to \mathbf{Z}$. For $X \subseteq G$ we denote the indicator function of $X$ by $1_X$. We denote the point-mass at $g \in G$ by $\delta_g$.

Let $\Lambda := \Delta_2/H$. Let $E$ be the skeleton of $\Lambda$. That is $E$ is the directed graph with the same vertices as $\Lambda$, and edges $\Lambda^{e_1} \cup \Lambda^{e_2}$, with range and source inherited from $\Lambda$. The degree map from $\Lambda$ restricts to a map from $E^1$ to $\{e_1, e_2\}$. As in [31, 27] we call edges in $E$ *blue* when they are of degree $e_1$ in $\Lambda$, and *red* when they are of degree $e_2$. We often blur the distinction between concatenation of edges in $E$ and the corresponding factorisation of a path in $\Lambda$.

Recall that we are identifying $\Lambda^0$ with $G = \mathbf{Z}^2/H$. Hence, given a path $\alpha = a_0 a_1 \cdots a_n$ in $E$, we define functions $f^b_\alpha$ and $f^r_\alpha$ in $\mathbf{Z}G$ by

$$\begin{aligned} f^b_\alpha(g) &= \#\{0 \le j \le n : r(a_j) = g, d(a_j) = e_1\} \\ f^r_\alpha(h) &= \#\{0 \le k \le n : r(a_k) = h, d(a_k) = e_2\}. \end{aligned}$$

The idea is that $f^b_\alpha(g)$ counts the number of blue edges in $\alpha$ whose range is $g$, and $f^r_\alpha(g)$ does the same thing for red edges.

We define $f_\alpha \in \mathbf{Z}G \oplus \mathbf{Z}G$ by $f_\alpha = f^b_\alpha \oplus f^r_\alpha$. For a vertex $g \in \Lambda^0 = G$, we define $f^b_g$ and $f^r_g$ to be the zero element of $\mathbf{Z}G$, and $f_g = f^b_g \oplus f^r_g$ is then the zero element of $\mathbf{Z}G \oplus \mathbf{Z}G$.

As $\Lambda = \Delta_2/H$, for each $g \in \Lambda^0 = G$ there is a unique path $[g, g + (1,1)]$ of degree $(1,1)$ with range $g$. Using the factorisation property, we can express this path as $b_g r_{g+[e_1]} = r_g b_{g+[e_2]}$ where $r_g$ and $b_g$ denote the unique red and blue edges in $E$ with range $g$ (for $n \in \mathbf{Z}^2$, $[n]$ denotes the class of $n$ in the quotient group $G = \mathbf{Z}^2/H$). We write $z_g$ for the function $(\delta_{g+[e_2]} - \delta_g) \oplus (\delta_g - \delta_{g+[e_1]})$ in $\mathbf{Z}G \oplus \mathbf{Z}G$.

Given paths $\alpha = a_0 \cdots a_m$ and $\beta = b_0 \cdots b_n$ in the skeleton $E$ of $\Lambda$ such that $r(a_0) = r(b_0)$ and $s(a_m) = s(b_n)$, let $f_{\alpha,\beta} := f_\alpha - f_\beta \in \mathbf{Z}G \oplus \mathbf{Z}G$. Fix generators $h_1, h_2$ for $H$; so $[h_i] = [0]$ in $G$. By definition of $\Lambda$, there are unique paths $\mu_1^+ \in \Lambda^{(h_1)_+}$ and $\mu_1^- \in \Lambda^{(h_1)_-}$ with $r(\mu_1^\pm) = 0$. Fix factorisations $\alpha_1^\pm$ of $\mu_1^\pm$ into edges from the skeleton $E$. Since

$$s(\mu_1^+) = [(h_1)_+] = [(h_1)_-] = s(\mu_1^-)$$

in $G$, the pair $(\mu_1^+, \mu_1^-)$ is a cycle of degree $h_1$ in $\Lambda$ with range $[0]$. The same construction for $h_2$ gives a cycle $(\mu_2^+, \mu_2^-)$ of degree $h_2$ with range $[0]$ and fixed factorisations $\alpha_2^\pm$ of $\mu_2^\pm$ into edges from the skeleton $E$.

Lemma 6.10. *With the notation established in the preceding paragraphs, the chain complex* (5.6) *can be described as follows:*

(1) *for each $g \in G$, $\partial_1(\delta_g \oplus 0) = \delta_g - \delta_{g+[e_1]}$, $\partial_1(0 \oplus \delta_g) = \delta_g - \delta_{g+[e_2]}$, and*

$$\partial_2(\delta_g) = (\delta_{g+[e_2]} - \delta_g) \oplus (\delta_g - \delta_{g+[e_1]}) = z_g.$$

(2) $\operatorname{coker}(\partial_1) \cong \mathbf{Z}$ *is generated by* $\delta_0 + \operatorname{Im}(\partial_1)$*;*

(3) $\ker(\partial_2) \cong \mathbf{Z}$ *is generated by* $1_G$*;*

(4) *For each $h \in G$, the set $\{z_g : g \in G \setminus \{h\}\}$ is a basis for* $\mathrm{Im}(\partial_2) \cong \mathbf{Z}^{|G|-1}$.

(5) *Fix any two factorisations $\alpha$ and $\beta$ of a path $\mu$ in $\Lambda$ into edges from $E$. Then $f_\alpha - f_\beta \in \mathrm{Im}(\partial_2)$, and $\partial_1(f_\alpha) = \partial_1(f_\beta) = \delta_{r(\alpha)} - \delta_{s(\alpha)}$.*

(6) $\ker(\partial_1)$ *is the subgroup of $\mathbf{Z}G \oplus \mathbf{Z}G$ generated by the elements $f_{\alpha,\beta}$ where $\alpha$ and $\beta$ are paths in the skeleton $E$ with $r(\alpha) = r(\beta)$ and $s(\alpha) = s(\beta)$.*

(7) *There is an isomorphism $\psi$ of $H$ onto* $\ker(\partial_1)/\mathrm{Im}(\partial_2)$ *which takes $d(\mu) - d(\nu)$ to $f_{\alpha,\beta} + \mathrm{Im}(\partial_2)$ for each cycle $(\mu, \nu)$ in $\Lambda$ and pair of factorisations $\alpha$ of $\mu$ and $\beta$ of $\nu$. In particular, for any basis $B$ for* $\mathrm{Im}(\partial_2)$*, the set $B \cup \{f_{\alpha_1^+,\alpha_1^-}, f_{\alpha_2^+,\alpha_2^-}\}$ is a basis for* $\ker(\partial_1) \cong \mathbf{Z}^{|G|+1}$ *(where $\alpha_i^\pm$ are the fixed factorisations of the paths $\mu_i^\pm$ of degree $(h_i)_\pm$ described above).*

*In particular, $K_*(C^*(\Lambda)) \cong (\mathbf{Z}^2, H)$ where the class of the identity in $K_0$ is identified with the element $(|G|, 0)$ of $\mathbf{Z}^2$.*

*Proof.* (1) The adjacency matrix $M_1$ associated to $(\Lambda^0, \Lambda^{e_1}, r, s)$ is the permutation matrix determined by translation by $[e_1]$ in $G$ and similarly for $M_2$. The first statement then follows from the formulae for $\partial_1$ and $\partial_2$ in terms of $M_1$ and $M_2$.

(2) The formulae for $\partial_1(\delta_g \oplus 0)$ and $\partial_1(0 \oplus \delta_g)$ show that $\delta_g + \mathrm{Im}(\partial_1) = \delta_{g+[e_i]} + \mathrm{Im}(\partial_1)$ in $\mathrm{coker}(\partial_1)$ for $i = 1, 2$ and $g \in G$. Since the action of $\mathbf{Z}^2$ on $G$ by translation is transitive, this establishes (2).

(3) Using the formula for $\partial_2$ established in (1), one can see that for $f \in \mathbf{Z}G$, $\partial_2(f) = f_1 \oplus f_2$ where

$$f_1(g) = -f(g) + f(g - [e_1]) \qquad \text{and} \qquad f_2(g) = f(g) - f(g - [e_2])).$$

Hence $f \in \ker(\partial_2)$ if and only if $f(g) = f(g - [e_1]) = f(g - [e_2])$ for all $g \in G$, and since the action of $\mathbf{Z}^2$ on $G$ is transitive, this establishes (3).

(4) Part (1) establishes that $\mathrm{Im}(\partial_2)$ is generated by $\{z_g : g \in G\}$. A simple calculation shows that $\sum_{g \in G} z_g = 0$ in $\mathbf{Z}G \oplus \mathbf{Z}G$, and it follows that for any $h \in G$, the set $\{z_g : g \in G \setminus \{h\}\}$ generates $\mathrm{Im}(\partial_2) \cong \mathbf{Z}^{|G|-1}$. Since $\ker(\partial_2)$ has rank 1, the rank of its image is $|G| - 1$, establishing (4).

(5) By part (4), the image of $\partial_2$ is generated by elements of the form $f_\alpha - f_\beta$ where $\alpha$ and $\beta$ are the two possible factorisations of a path in $\Lambda^{(1,1)}$. Since $f_{\alpha\beta} = f_\alpha + f_\beta$ when $\alpha$ and $\beta$ are paths in $E$ which can be concatenated, this establishes the first claim. The second statement follows from a straightforward calculation using that

$$\partial_1(f^b \oplus f^r)(g) = f^b(g) - f^b(g - [e_1]) + f^r(g) - f^r(g - [e_2]). \tag{6.3}$$

(6) If $\alpha, \beta$ are paths in the skeleton with $r(\alpha) = r(\beta)$ and $s(\alpha) = s(\beta)$ then $f_{\alpha,\beta}$ belongs to $\ker(\partial_1)$ by (5).

We must show that every $f \in \ker(\partial_1)$ can be written as a $\mathbf{Z}$-linear combination of elements of the form $f_{\alpha,\beta}$. First note that it suffices to treat the case where $f$ takes only nonnegative values (this is because $1_G \oplus 1_G$ can be so expressed). So suppose that $f$ takes nonnegative values, and write $f = f^b \oplus f^r$. Let $E_f$ be the directed graph with vertices $G$ and which contains $f^b(g)$ parallel copies of the blue edge in $E$ with range $g$ and $f^r(g)$ copies of the red edge in $E$ with range

$g$. If $E_f$ contains a *terminal vertex* $g$ which receives at least one edge but emits no edges at all, then $f^b(g) + f^r(g) \neq 0$, but $f^b(g - [e_1]) = f^r(g - [e_2]) = 0$, and (6.3) shows that $\partial_1(f)(g) \neq 0$. Hence $E_f$ contains no such vertex, and therefore must either contain a cycle $\alpha$ or contain no edges at all. In the latter case, the claim is trivial, and in the former case, $f \geq f_\alpha$, and removing the cycle $\alpha$ from $E_f$ produces the graph $E_{f-f_\alpha}$ for the function $f - f_\alpha$. After finitely many such steps, we must obtain a forest with no terminal vertex. The only such forest is the empty graph which corresponds to the function $0 \oplus 0$. That is $f - \sum_{\alpha\in L} f_\alpha = 0 \oplus 0$ for some collection $L$ of cycles, and this proves (6).
(7) Suppose that $(\mu, \nu)$ is a cycle in $\Lambda$. Then

$$s(\mu) - [d(\mu)] = r(\mu) = r(\nu) = s(\nu) - [d(\nu)] = s(\mu) - [d(\nu)]$$

in $G = \Lambda^0 = \mathbf{Z}^2/H$, so $d(\mu) - d(\nu) \in H$. It is clear from the definition of $\Lambda$ that each element of $H$ arises as $d(\mu) - d(\nu)$ for some cycle $(\mu, \nu)$ in $\Lambda$.
To see that the assignment $d(\mu) - d(\nu) \mapsto f_{\alpha,\beta} + \operatorname{Im}(\partial_2)$ is well defined, we must show two things. First that for distinct factorisations $\alpha$ and $\alpha'$ of $\mu$ and distinct factorisations $\beta$ and $\beta'$ of $\nu$, the difference $f_{\alpha,\beta} - f_{\alpha',\beta'}$ lies in the image of $\partial_2$. This follows from (5). Second, we must show that if $(\mu, \nu)$ and $(\mu', \nu')$ are cycles in $\Lambda$ with $d(\mu) - d(\nu) = d(\mu') - d(\nu')$, then there exist factorisations $\alpha$ of $\mu$, $\beta$ of $\nu$, $\alpha'$ of $\mu'$, and $\beta'$ of $\nu'$ such that $f_{\alpha,\beta} - f_{\alpha',\beta'}$ is in $\operatorname{Im}(\partial_2)$. To see this, first note that by factorising $\mu = \mu'\tau$ and $\nu = \nu'\tau$ where $d(\tau) = d(\mu) \wedge d(\nu)$, we can reduce to the case where $d(\mu) \wedge d(\nu) = 0$. Next we claim that it suffices to consider the case where $r(\mu) = r(\nu) = r(\mu') = r(\nu') = [0]$. To see this, fix $\eta$ in $[0]\Lambda r(\mu)$ and note that the cycle $(\eta\mu, \eta\nu)$ corresponds to the same class as $(\mu, \nu)$ in $\ker(\partial_1)/\operatorname{Im}(\partial_2)$. Factorise $\eta\mu = \xi\rho$ and $\eta\nu = \omega\sigma$ where $d(\xi) = d(\mu)$, $d(\omega) = d(\nu)$ and $d(\rho) = d(\sigma) = d(\eta)$. Since each $g\Lambda^n$ is a singleton and since $\mathbf{Z}^2$ acts on $\Lambda$ by translation, $(\xi, \omega)$ is a cycle with range $[0]$, and $\rho = \sigma$. Hence the cycle $(\xi, \omega)$ corresponds to the same class in $\ker(\partial_1)/\operatorname{Im}(\partial_2)$ as $(\mu, \nu)$. After shifting $(\mu', \nu')$ in a similar way we may assume that both cycles have range $[0]$. We now have cycles $(\mu, \nu)$ and $(\mu', \nu')$ with range $[0]$ and such that $d(\mu) - d(\nu) = d(\mu') - d(\nu')$ and $d(\mu) \wedge d(\nu) = 0 = d(\mu') \wedge d(\nu')$. Since $[0]\Lambda^n$ is a singleton for any $n \in \mathbf{Z}^2$, this forces $\mu = \mu'$ and $\nu = \nu'$. This completes the proof that $d(\mu) - d(\nu) \mapsto f_{\alpha,\beta} + \operatorname{Im}(\partial_2)$ is well defined.
That $f_{\alpha\beta} = f_\alpha + f_\beta$ ensures that $\psi(g+h) = \psi(g) + \psi(h)$, and that $f_{\beta,\alpha} = -f_{\alpha,\beta}$ shows that $\psi(-g) = -\psi(g)$. Hence $\psi$ is a homomorphism. By part (6), to see that $\psi$ is surjective, we just need to show that each $f_{\alpha,\beta} + \operatorname{Im}(\partial_2)$ is in the range of $\psi$. This is clear because $f_{\alpha,\beta} + \operatorname{Im}(\partial_2)$ is precisely $\psi(d(\mu) - d(\nu))$ where $\mu$ factorises as $\alpha$ and $\nu$ factorises as $\beta$. Finally, to see that $\psi$ is injective, note that if $f_{\alpha,\beta} \in \operatorname{Im}(\partial_2)$, then $d(\mu) = d(\nu)$ where $\mu$ factorises as $\alpha$ and $\nu$ factorises as $\beta$. This completes the proof that $\psi : H \to \ker(\partial_1)/\operatorname{Im}(\partial_2)$ is an isomorphism. The remaining statement follows from (4) and that $(\mu_1^+, \mu_1^-)$ and $(\mu_2^+, \mu_2^-)$ are cycles whose degrees form a basis for $H$. This proves (7).
The final statement of the Lemma follows from (5.7). □

We now consider two consecutive graphs in the sequence of covering systems described in Example 6.6, and describe the homomorphism of $K$-invariants obtained from Proposition 3.2(6).

Theorem 6.11. *Consider the situation described in Example 6.6, and fix $n \in \mathbf{N} \setminus \{0\}$. For $i = n, n+1$, let $\Lambda_i := \Delta_2/H_i$, and consider the commuting diagram*

$$\begin{array}{ccccccccc} 0 & \longleftarrow & \mathbf{Z}\Lambda_n^0 & \xleftarrow{\partial_1^{\Lambda_n}} & \mathbf{Z}\Lambda_n^0 \oplus \mathbf{Z}\Lambda_n^0 & \xleftarrow{\partial_2^{\Lambda_n}} & \mathbf{Z}\Lambda_n^0 & \longleftarrow & 0 \\ & & \Big\downarrow{\scriptstyle p_n^*} & & \Big\downarrow{\scriptstyle p_n^* \oplus p_n^*} & & \Big\downarrow{\scriptstyle p_n^*} & & \\ 0 & \longleftarrow & \mathbf{Z}\Lambda_{n+1}^0 & \xleftarrow{\partial_1^{\Lambda_{n+1}}} & \mathbf{Z}\Lambda_{n+1}^0 \oplus \mathbf{Z}\Lambda_{n+1}^0 & \xleftarrow{\partial_2^{\Lambda_{n+1}}} & \mathbf{Z}\Lambda_{n+1}^0 & \longleftarrow & 0 \end{array}$$

(1) *The right-hand vertical map $p_n^* : \mathbf{Z}\Lambda_n^0 \to \mathbf{Z}\Lambda_{n+1}^0$ restricts to a homomorphism $p_n^*|_{\ker(\partial_2^{\Lambda_n})} : \ker(\partial_2^{\Lambda_n}) \to \ker(\partial_2^{\Lambda_{n+1}})$ which is characterised by $p_n^*|_{\ker(\partial_2^{\Lambda_n})}(1_{G_n}) = 1_{G_{n+1}}$.*

(2) *The left-hand vertical map $p_n^* : \mathbf{Z}\Lambda_n^0 \to \mathbf{Z}\Lambda_{n+1}^0$ induces a homomorphism $\widetilde{p_n^*} : \operatorname{coker}(\partial_1^{\Lambda_n}) \to \operatorname{coker}(\partial_1^{\Lambda_{n+1}})$ characterised by*

$$\widetilde{p_n^*}(\delta_0 + \operatorname{Im}(\partial_1^{\Lambda_n})) = [H_n : H_{n+1}] \cdot \delta_0 + \operatorname{Im}(\partial_1^{\Lambda_{n+1}}).$$

(3) *The middle vertical map $p_n^* \oplus p_n^* : \mathbf{Z}\Lambda_n^0 \oplus \mathbf{Z}\Lambda_n^0 \to \mathbf{Z}\Lambda_{n+1}^0 \oplus \mathbf{Z}\Lambda_{n+1}^0$ induces a homomorphism $(p_n^* \oplus p_n^*)^\sim : \ker(\partial_1^{\Lambda_n})/\operatorname{Im}(\partial_2^{\Lambda_n}) \to \ker(\partial_1^{\Lambda_{n+1}})/\operatorname{Im}(\partial_2^{\Lambda_{n+1}})$ such that the following diagram commutes.*

$$\begin{array}{ccc} H_n & \xrightarrow{\psi_n} & \ker(\partial_1^{\Lambda_n})/\operatorname{Im}(\partial_2^{\Lambda_n}) \\ \Big\downarrow{\scriptstyle m_{H_n,H_{n+1}}} & & \Big\downarrow{\scriptstyle (p_n^*\oplus p_n^*)^\sim} \\ H_{n+1} & \xrightarrow{\psi_{n+1}} & \ker(\partial_1^{\Lambda_{n+1}})/\operatorname{Im}(\partial_2^{\Lambda_{n+1}}) \end{array}$$

*where $\psi_n$ and $\psi_{n+1}$ are the isomorphisms obtained from Lemma 6.10(7), and $m_{H_n,H_{n+1}}$ is as in Theorem 6.7(2).*

*Under the isomorphism*

$$K_*(C^*(\Lambda_i)) \cong \big(\operatorname{coker}(\partial_1^{\Lambda_i}) \oplus \ker(\partial_2^{\Lambda_i}),\ \ker(\partial_1^{\Lambda_i})/\operatorname{Im}(\partial_2^{\Lambda_i})\big)$$

*obtained from Corollary 5.6, the maps described in (1), (2) and (3) determine the map $(\iota_{p_n})_* : K_*(C^*(\Lambda_n)) \to K_*(C^*(\Lambda_{n+1}))$ obtained from Proposition 3.2(6).*

*Proof.* Lemma 6.10(3) ensures that $1_{G_i}$ generates $\ker(\partial_2^{\Lambda_i})$ for $i = n, n+1$. The formula for $p_n^*$ shows that $p_n^*(1_{G_n}) = 1_{G_{n+1}}$, which gives (1). Statement (2) follows from the formula for $p_n^*$ combined with the observation that for $i = n, n+1$, the $\delta_g$, $g \in G_i$ are all equivalent modulo $\operatorname{Im}(\partial_1^{\Lambda_i})$.

It remains only to prove (3). We first consider the case where $H_n = \mathbf{Z}^2$, so $G_n = \{0\}$ and $\Lambda_n$ is a copy of the 2-graph $T_2 \cong \mathbf{N}^2$ (as a category) with one vertex and one morphism $\lambda_m$ of each degree $m \in \mathbf{N}^2$. In this case, $\psi_n$ is just the identity map from $\mathbf{Z}^2$ to $\mathbf{Z} \oplus \mathbf{Z}$. Let $h_1, h_2$ be a pair of generators for $H_{n+1}$.

Since $H_{n+1}$ has finite index in $\mathbf{Z}^2$, the assignments $(1,0) \mapsto h_1$ and $(0,1) \mapsto h_2$ determine an endomorphism of $H_n$ which is a rational isomorphism. Hence it suffices to show that $(p_n^* \oplus p_n^*)^\sim \circ \psi_n(h_i) = \psi_{n+1}([\mathbf{Z}^2 : H_{n+1}] \cdot h_i)$ for $i = 1, 2$. We just argue that this happens for $i = 1$ (the case $i = 2$ follows from a symmetric argument).

Writing $h_1 = (x, y)$ where $x, y \in \mathbf{Z}$, the formula for $p_n^*$ ensures that $(p_n^* \oplus p_n^*)^\sim$ takes $\psi_n(h_1)$ to the class of $x1_{G_{n+1}} \oplus y1_{G_{n+1}}$. To see that this is $\psi_{n+1}([\mathbf{Z}^2 : H_{n+1}] \cdot h_1)$, let $f := f_{\alpha_1^+, \alpha_1^-} = \psi_{n+1}(h_1)$ be the function in $\mathbf{Z}G_{n+1} \oplus \mathbf{Z}G_{n+1}$ obtained from Lemma 6.10(7). By definition of $f$, we have $f = f_b \oplus f_r$ where the entries of $f_b$ sum to $x$ and the entries of $f_r$ sum to $y$. For $g \in G_{n+1}$, let $g \cdot f_b$ be the function determined by $g \cdot f_b(h) = f_b(h - g)$, and similarly for $f_r$. Since $G_{n+1}$ acts freely and transitively on $\Lambda_{n+1}^0 = G_{n+1}$, it follows that

$$\sum_{g \in G_{n+1}} g \cdot f = x1_{G_{n+1}} \oplus y1_{G_{n+1}} = (p_n^* \oplus p_n^*)^\sim \circ \psi_n(h_1). \tag{6.4}$$

The proof of statement (7) in Lemma 6.10 shows that each $g \cdot f := g \cdot f_b \oplus g \cdot f_r$ represents the same class as $f$ in $\ker(\partial_1^{\Lambda_{n+1}}) / \operatorname{Im}(\partial_2^{\Lambda_{n+1}})$. Hence the left-hand side of (6.4) has the same class in $\ker(\partial_1^{\Lambda_{n+1}}) / \operatorname{Im}(\partial_2^{\Lambda_{n+1}})$ as $\psi_{n+1}(|G_{n+1}| \cdot h_1)$ as required.

For the general case, first note that we may assume without loss of generality that $H_1 = \mathbf{Z}^2$ so that $\Lambda_1 = T_2$. Let $p_{[1,n]} := p_1 \circ \cdots \circ p_{n-1}$ and $p_{[1,n+1]} := p_1 \circ \cdots \circ p_n$ be the coverings of $\Lambda_1 = T_2$ by $\Lambda_n$ and $\Lambda_{n+1}$ obtained by composing the first $n$ and $n+1$ levels of the covering system; we may apply the argument of the previous paragraph to these coverings. Then $p_{[1,n+1]} = p_{[1,n]} \circ p_n$, so $p_{[1,n+1]}^* \oplus p_{[1,n+1]}^* = (p_{[1,n]}^* \oplus p_{[1,n]}^*) \circ (p_n^* \oplus p_n^*)$, and since these maps induce homomorphisms between $\ker(\partial_1^{T_2}) / \operatorname{Im}(\partial_2^{T_2})$ and $\ker(\partial_1^{\Lambda_{n+1}}) / \operatorname{Im}(\partial_2^{\Lambda_{n+1}})$ which are rational isomorphisms, it follows that $(p_n^* \oplus p_n^*)^\sim$ behaves as claimed.

The final statement follows from Corollary 5.6. □

We are now ready to prove Theorem 6.7.

*Proof of Theorem 6.7.* Proposition 3.2 shows that $P_1$ is full so that compression by $P_1$ induces an isomorphism on $K$-theory. The formulae for the $K$-groups in statements (1) and (2) follow from Lemma 6.10 and Theorem 6.11 and the continuity of the $K$-functor.

Since $v(\Delta_2/H_n)w \neq \emptyset$ for all $n \in \mathbf{N} \setminus \{0\}$, and $v, w \in \Delta_2^0/H_n$, the 3-graph $\varprojlim(\Delta_2/H_n, p_n)$ is cofinal. Moreover a given infinite path $x$ in $\varprojlim(\Delta_2/H_n, p_n)$ is periodic with period $m \in \mathbf{Z}^2$ if and only if every infinite path in $\varprojlim(\Delta_2/H_n, p_n)$ is periodic with period $m$, which in turn is equivalent to the condition that $m \in \bigcap_{n=1}^\infty H_n$. It follows from Lemma 4.5 that $\varprojlim(\Delta_2/H_n, p_n)$ is simple if and only if $\bigcap H_n = \{0\}$; moreover, in this case, the argument of the second part of [27, Section 5] shows that $C^*(\varprojlim(\Delta_2/H_n, p_n))$ has unique trace.

We next claim that each $C^*(\Delta_2/H_n) \cong M_{[\mathbf{Z}^2 : H_n]}(C(\mathbf{T}^2))$. To verify this, one first checks that $h \mapsto s_{[(0,h_+)]} s_{[(0,h_-)]}^*$ is a group isomorphism $H_n \to \mathcal{U}(s_{[0]} C^*(\Delta_2/H_n) s_{[0]})$ for each $n$. The standard argument used in [27, Lemma 3.9] shows that each $s_{[(0,h_+)]} s_{[(0,h_-)]}^*$ has full spectrum. One can then deduce that $s_{[0]}C^*(\Delta_2/H_n)s_{[0]} \cong C^*(H_n) \cong C^*(\mathbf{Z}^2) \cong C(\mathbf{T}^2)$. For $m \in \mathbf{Z}^2/H_n$, define $V_m := s^*_{[0,m]} \in C^*(\Delta_2/H_n)$. Applying Lemma 3.3 to these partial isometries with $p = s_{[0]}$ and $q = 1_{C^*(\Delta_2/H_n)}$proves that $C^*(\Delta_2/H_n) \cong M_{[\mathbf{Z}^2:H_n]}(C(\mathbf{T}^2))$.

It now follows from [3, Theorem 1.3] that $C^*(\varprojlim(\Delta_2/H_n;p_n))$ has real-rank 0. The classification of such algebras of Dădărlat-Elliott-Gong (see [36, Theorem 3.3.1]), and the $K$-theory calculations above complete the proof. □

*Remark* 6.12. *Higher-rank Bunce-Deddens algebras and generalised odometer actions.* We consider a slightly more general version of the situation described in Example 6.6. Let $H_1 \supset H_2 \supset H_3 \supset \dots$ be a chain of finite-index subgroups of $\mathbf{Z}^k$ such that $\bigcap_n H_n = \{0\}$. For each $n$, let $p_n : \Delta_k/H_{n+1} \to \Delta_k/H_n$ be the canonical covering induced by the quotient maps described above, let $m_n = 1$, and let $\mathfrak{s}_n : \Delta_k/H_{n+1} \to S_1$ be the trivial cocycle. This data specifies a sequence $(\Delta_k/H_n, \Delta_k/H_{n+1}, p_n)_{n=1}^\infty$ of row-finite covering systems of $k$-graphs with no sources. Applying Corollary 2.11, we obtain a $(k+1)$-graph $\varprojlim(\Delta_k/H_n;p_n)$.

We claim that the corner $P_1C^*(\varprojlim(\Delta_k/H_n;p_n))P_1$ can be thought of as a higher-rank Bunce-Deddens algebra. We justify this by giving a description of $P_1C^*(\varprojlim(\Delta_k/H_n;p_n))P_1$ as a crossed product by a generalised odometer action. We assume here that $H_1 = \mathbf{Z}^k$ so that $\Delta_k/H_1$ is a copy of the $k$-graph $T_k \cong \mathbf{N}^k$ (as a category) with one vertex and one morphism $\lambda_m$ of each degree $m \in \mathbf{N}^k$.

One way to realise the Bunce-Deddens algebras is as crossed products of algebras of continuous functions on Cantor sets by generalised odometer actions. Given a supernatural number $\alpha = \alpha_1\alpha_2\cdots$, let $G_n := \mathbf{Z}/\alpha[1,n]\mathbf{Z}$ for all $n$. Then for each $n$, since $\alpha[1,n+1]\mathbf{Z} \supset \alpha[1,n]\mathbf{Z}$, there is a natural surjective group homomorphism from $G_{n+1}$ to $G_n$. Hence, we may form the projective limit group $\varprojlim(G_n,p_n)$. The automorphism $\tau(g_1,g_2,\dots) = (g_1+[1], g_2+[1],\dots)$ for $(g_1,g_2,\dots) \in \varprojlim(G_n,p_n)$ can then naturally be regarded as an odometer action on $\varprojlim(G_n,p_n)$. The Bunce-Deddens algebra of type $\alpha$ is the crossed product $C(\varprojlim(G_n,p_n)) \rtimes_{\tilde\tau} \mathbf{Z}$ where $\tilde\tau$ is the automorphism of $C(\varprojlim(G_n,p_n))$ induced by $\tau$ (see [33, Examples 1(3)]).

There is an analogous realisation of $P_1C^*(\varprojlim(\Delta_k/H_n,p_n))P_1$ as follows. Let $\Lambda := \varprojlim(\Delta_k/H_n,p_n)$. Let $F$ denote the fixed-point algebra of $C^*(\Lambda)$ for the gauge action $\gamma$ of $\mathbf{T}^{k+1}$. Note that by Remark 3.9, the restriction of the gauge action to $P_1C^*(\Lambda)P_1$ is trivial on the last coordinate of $\mathbf{T}^{k+1}$ and therefore becomes an action by $\mathbf{T}^k$ denoted $\tilde\gamma$. Recall that $\Lambda^\infty$ denotes the collection of infinite paths in $\Lambda$ (see Notation 4.1). It is not hard to see that $P_1FP_1$ is canonically isomorphic to $C(v\Lambda^\infty)$ where $v$ is the unique vertex of $\Delta_k/H_1 \cong T_k$. Let $G_n := \mathbf{Z}^k/H_n$ for each $n$, and let $p_n : G_{n+1} \to G_n$ be the induced map $p_n(m+H_{n+1}) := m+H_n$. Observe that $G = \varprojlim(G_n,p_n)$ is a compact abelian group. By functoriality of the projective limit the quotient maps $\mathbf{Z}^k \to \mathbf{Z}^k/H_n$ induce a homomorphism $j : \mathbf{Z}^k \to G$; injectivity of $j$ follows from the fact that $\bigcap_n H_n = \{0\}$. There is an action $\tau$ of $\mathbf{Z}^k$ on $G$ given by $\tau_m(g_1, g_2, \dots) = (g_1 + [m], g_2 + [m], \dots)$, which generalises the odometer action discussed above. Since there is just one infinite path in $T_k$, the arguments of Section 4 show that $v\Lambda^\infty \cong G$ as a topological space. Note that for every $m \in \mathbf{N}^k$, the generator $s_{\lambda_m}$ associated to the unique path $\lambda_m \in T_k^m$ is a unitary in $P_1 C^*(\Lambda) P_1$ and that under the identification of $P_1 F P_1$ with $C(v\Lambda^\infty) = C(G)$ conjugation by $s_{\lambda_m}$ implements the automorphism induced by the homeomorphism $\tau_m$ of $G$. It follows that the reduction of the path groupoid (see [20, Section 2]) of $\Lambda$ to $v\Lambda^\infty$ is isomorphic to the semidirect product groupoid $G \rtimes_\tau \mathbf{Z}^k$. Therefore, standard arguments show that

$$P_1 C^*(\Lambda) P_1 \cong C(G) \rtimes_{\tilde{\tau}} \mathbf{Z}^k$$

where $\tilde{\tau}$ is the action induced by $\tau$. Note that under this identification the restricted gauge action $\tilde{\gamma}$ coincides with the dual action of $\mathbf{T}^k = \widehat{\mathbf{Z}^k}$.
The action of $G$ on $C(G)$ induced by translation in $G$ yields an action of $G$ on $C(G) \rtimes_{\tilde{\tau}} \mathbf{Z}^k$ which commutes with the dual action of $\mathbf{T}^k = \widehat{\mathbf{Z}^k}$. Thus we obtain an action $\alpha$ by the compact abelian group $G \times \mathbf{T}^k$ with fixed point algebra isomorphic to $\mathbf{C}$. Hence, $C(G) \rtimes_{\tilde{\tau}} \mathbf{Z}^k$ (and thus $P_1 C^*(\Lambda) P_1$) admits an ergodic action of a compact abelian group. Such ergodic actions have been classified in [24, 4.5, 6.1]; the invariant is a symplectic bicharacter $\chi_\alpha$ on $\widehat{G} \times \mathbf{Z}^k$, the dual of $G \times \mathbf{T}^k$. This gives rise to an alternative description of the $C^*$-algebra as a twisted group $C^*$-algebra with the group $\widehat{G} \times \mathbf{Z}^k$ and a 2-cocycle associated to the bicharacter $\chi_\alpha$ (only its cohomology class is determined by the bicharacter). It follows that

$$C(G) \rtimes_{\tilde{\tau}} \mathbf{Z}^k \cong C(\mathbf{T}^k) \rtimes \widehat{G}$$

where the action of $\widehat{G}$ on $C(\mathbf{T}^k)$ arises by translation from the embedding $\widehat{G} \to \mathbf{T}^k$ dual to $j : \mathbf{Z}^k \to G$.

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Alex Kumjian
Department of Mathematics (084)
University of Nevada
Reno NV 89557-0084
USA
alex@unr.edu

David Pask
School of Mathematics and
Applied Statistics
University of Wollongong
NSW 2522
AUSTRALIA
dpask@uow.edu.au

Aidan Sims
School of Mathematics and
Applied Statistics
University of Wollongong
NSW 2522
AUSTRALIA
asims@uow.edu.au