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A family of 2-graphs arising from two-dimensional subshifts

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Abstract. Higher-rank graphs (or k -graphs) were introduced by Kumjian and Pask to provide combinatorial models for the higher-rank Cuntz–Krieger C^* -algebras of Robertson and Steger. Here we consider a family of finite 2-graphs whose path spaces are dynamical systems of algebraic origin, as studied by Schmidt and others. We analyse the C^* -algebras of these 2-graphs, find criteria under which they are simple and purely infinite, and compute their K -theory. We find examples whose C^* -algebras satisfy the hypotheses of the classification theorem of Kirchberg and Phillips, but are not isomorphic to the C^* -algebras of ordinary directed graphs.

1. Introduction

Higher-rank graphs (or k -graphs) were introduced by Kumjian and Pask [9] to provide combinatorial models for the higher-rank Cuntz–Krieger C^* -algebras of Robertson and Steger [23]. They have since provided a fertile source of examples in non-commutative geometry [16, 17, 19], and many important operator algebras can be realized as the C^* -algebras of higher-rank graphs [2, 8, 15]. There has therefore been continuing interest in finding new families of k -graphs and analysing the structure of their C^* -algebras.

Every shift of finite type is equivalent to the backward shift σ on the two-sided infinite path space of a finite directed graph [12, Theorem 2.3.2]. The two-sided infinite path space Λ^Δ of a finite k -graph Λ introduced in [10] carries a set of k commuting shifts σ_i , and these are examples of the higher-dimensional shifts of finite type studied by dynamicists. In this paper we consider a family of finite 2-graphs Λ for which the path spaces $(\Lambda^\Delta, \sigma_i)$ are dynamical systems of algebraic origin, as studied by Schmidt and others [26]. (A particular motivating example for us was the system introduced by Ledrappier in [11].)

We analyse the C^* -algebras of these 2-graphs, find criteria under which they are simple and purely infinite, and compute their K -theory.

Each of our graphs Λ is associated to a *tile*, which is a finite hereditary subset T of \mathbb{N}^2 containing the origin. We picture T as a collection of boxes into which we can put elements of the commutative ring $\mathbb{Z}/q\mathbb{Z}$, which we think of as an alphabet: for example, we picture the *sock* $T := \{0, e_1, e_2\}$ as



The vertices in Λ are copies of T in which each box is filled with elements of $\mathbb{Z}/q\mathbb{Z}$ which together satisfy a fixed equation in $\mathbb{Z}/q\mathbb{Z}$; for example, the vertices in the *Ledrappier graph* underlying Ledrappier’s system are copies of the sock filled with 0s and 1s such that sum of the entries is 0 (mod 2). Paths in Λ^* are diagrams covered by translates of T , filled in so that each translate of T is a valid vertex. Thus for example,

$$\begin{array}{|c|c|c|c|} \hline 0 & 0 & 1 & 1 \\ \hline 1 & 1 & 1 & 0 & 1 \\ \hline 0 & 1 & 0 & 1 & 1 \\ \hline 0 & 0 & 1 & 1 & 0 \\ \hline \end{array} \tag{1.1}$$

represents a path λ of degree $(3, 2)$ in the Ledrappier graph

$$\text{from } s(\lambda) = \begin{array}{|c|c|} \hline 1 & \\ \hline 0 & 1 \\ \hline \end{array} \text{ (the top RH one) to } r(\lambda) = \begin{array}{|c|c|} \hline 0 & \\ \hline 0 & 0 \\ \hline \end{array} \text{ (the bottom LH one).}$$

The infinite path space Λ^Δ consists of similar diagrams covering the entire plane, and the shifts σ_1 and σ_2 simply move the diagram one row left and one column down respectively. So it is easy to construct paths in these graphs, and thereby determine properties of their C^* -algebras.

We begin with a short section in which we recall the basic properties of 2-graphs and their C^* -algebras. In §3, we fix a set of ‘basic data’, which consists of a tile T , an integer q determining the alphabet, another integer t and a function $w : T \rightarrow \mathbb{Z}/q\mathbb{Z}$ which determines the equation relating the entries. We describe the vertices and paths in our 2-graph as functions from T and translates of T into $\mathbb{Z}/q\mathbb{Z}$, and then we have to prove that they form the objects and morphisms in a category satisfying the axioms of a 2-graph $\Lambda = \Lambda(T, q, t, w)$ (see Theorem 3.4). In §4, we show that the two-sided infinite path space of $\Lambda(T, q, t, w)$ is the underlying space for a higher-dimensional shift of the sort studied in dynamical systems (Theorem 4.1).

In §5, we show that, provided certain key values of the function w are invertible, the 2-graph $\Lambda(T, q, t, w)$ is aperiodic in the sense of Kumjian and Pask, so that the Cuntz–Krieger uniqueness theorem applies. We prove this using the recent formulation of Robertson and Sims [22], and as an intermediate step in the proof we show that $\Lambda(T, q, t, w)$ is always strongly connected in the sense that there are paths joining any

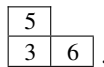
two vertices. In §6, we show that, under the same invertibility hypothesis on w , the C^* -algebra $C^*(\Lambda(T, q, 0, w))$ is nuclear, simple and purely infinite. After our work in §5, this follows quickly from general results in [22] and [27] about the structure of k -graph algebras. The main result, Theorem 6.1, implies that $C^*(\Lambda(T, q, 0, w))$ is a Kirchberg algebra, and hence by the theorem of Kirchberg and Phillips is classifiable by its K -groups.

In §7, we compute the K -theory of $C^*(\Lambda(T, q, t, w))$, using the techniques developed by Robertson and Steger [24] and Evans [4], which identify $K_0(C^*(\Lambda(T, q, t, w)))$ and $K_1(C^*(\Lambda(T, q, t, w)))$ in terms of the kernels and cokernels of certain integer matrices. Our 2-graphs are finite but large, so we have used the computational algebra system Magma [1] to compute these kernels and cokernels. We have presented some of these results in Table 1. These results have led us to make some general conjectures about the K -theory of our 2-graphs, and in §8 we prove two of these conjectures. Perhaps the most surprising result is that, under mild hypotheses, K_0 and K_1 have the same finite cardinality—though our proof of this is indirect, and gives us no hint of whether the groups are actually isomorphic.

1.1. *Visualization convention.* We visualize a subset S of \mathbb{N}^2 as the union of the unit squares whose bottom left-hand corners belong to S , and a function $f : S \rightarrow \mathbb{Z}$ as a diagram in which the number $f(i)$ is placed in the square with bottom left-hand corner i . Thus the sock $T = \{0, e_1, e_2\}$ is visualized as



and the function $f : T \rightarrow \mathbb{Z}$ defined by $f(0) = 3$, $f(e_1) = 6$ and $f(e_2) = 5$ as



1.2. *Notation.* Let $\mathbb{N} = \{0, 1, 2, 3, \dots\}$ denote the monoid of natural numbers under addition and let \mathbb{Z} be the group of integers. For $k \geq 1$, we view \mathbb{N}^k as the set of morphisms in a category with one object and composition given by addition. We write n_i for the i th coordinate of $n \in \mathbb{Z}^k$, and $\{e_i\}$ for the usual basis of \mathbb{Z}^k . For $m, n \in \mathbb{Z}^k$ we say $m \leq n$ if $m_i \leq n_i$ for each i , and write $m \vee n$ and $m \wedge n$ for the coordinate-wise maximum and minimum.

2. 2-graphs

Let k be a positive integer. A *graph of rank k* or *k -graph* is a pair (Λ, d) consisting of a countable category Λ and a functor $d : \Lambda \rightarrow \mathbb{N}^k$, called the *degree map*, satisfying the *factorization property*: for every $\lambda \in \Lambda$ and $m, n \in \mathbb{N}^k$ with $d(\lambda) = m + n$, there exist unique elements $\mu, \nu \in \Lambda$ such that $d(\mu) = m$, $d(\nu) = n$ and $\lambda = \mu\nu$. In practice, we drop the degree map from the notation.

We refer to the morphisms in Λ as *paths* and the objects as *vertices*. If $\lambda \in \Lambda$ satisfies $d(\lambda) = n$ we say λ has *degree n* ; we write $\Lambda^n := \{\lambda \in \Lambda \mid d(\lambda) = n\}$. All k -graphs in this

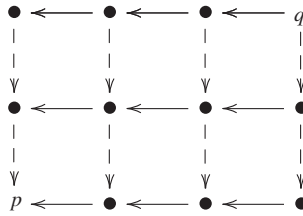


FIGURE 1. A path of degree (3,2).

paper are *finite* in the sense that each Λ^n is a finite set, and have no sources in the sense that for each vertex v and each $n \in \mathbb{N}^k$, there is at least one $\lambda \in \Lambda^n$ with $r(\lambda) = v$.

The factorization property has several consequences. First, it implies that for each vertex v the identity morphism ι_v is the only morphism of degree 0 from v to v , so that we can identify $\text{Obj}(\Lambda)$ with Λ^0 . We then write $v\Lambda$, for example, to mean the set of paths λ with $r(\lambda) = v$. Second, it implies that for every triple $m, n, p \in \mathbb{N}^k$ satisfying $0 \leq m \leq n \leq p$ and $\lambda \in \Lambda^p$ with $d(\lambda) = p$, there are unique *segments* $\lambda(0, m) \in \Lambda^m$, $\lambda(m, n) \in \Lambda^{n-m}$, $\lambda(n, p) \in \Lambda^{p-n}$ such that $\lambda = \lambda(0, m)\lambda(m, n)\lambda(n, p)$. The paths $\lambda(m, m)$ have degree 0, and hence are vertices; in the literature it is common to write $\lambda(m) := \lambda(m, m)$, but we will refrain from doing this as $\lambda(m)$ will have another more natural meaning.

In this paper we are primarily interested in 2-graphs, so k is usually 2. We visualize a 2-graph as a directed bicoloured graph with vertex set Λ^0 in which the elements $\beta \in \Lambda^{e_1}$ are represented by blue edges from $s(\beta) \in \Lambda^0$ to $r(\beta) \in \Lambda^0$, and elements of Λ^{e_2} as red edges. (In print we use black curves to represent blue edges and dashed curves for red edges.) This bicoloured graph is called the *skeleton* of Λ . Applying the factorization property to $(1, 1) = e_1 + e_2 = e_2 + e_1$ gives a bijection between the blue-red paths of length 2 and the red-blue paths of length 2. We then visualize a path of degree $(1, 1)$ as a square

$$\begin{array}{ccc}
 \bullet & \xleftarrow{f} & \bullet \\
 | & & | \\
 e | & & | h \\
 \Downarrow & & \Downarrow \\
 \bullet & \xleftarrow{g} & \bullet
 \end{array} \tag{2.1}$$

in which the bijection matches up the blue-red path gh with the red-blue path ef , so that $gh = ef$ are the two factorizations of the path of degree $(1, 1)$. It turns out (though we shall not rely on this fact in this paper) that a 2-graph is completely determined by a collection C of squares (2.1) in which each blue-red and each red-blue path occur exactly once. The paths of degree $(3, 2)$ from w to v , for example, then consist of copies of the rectangle in Figure 1 pasted round the blue-red graph, so that q lands on w , p lands on v , and each constituent square is one of the given collection C . Composition of paths involves taking the convex hull: if $d(\lambda) = (1, 1)$ and $d(\mu) = (1, 2)$, for example, then $\lambda\mu$ is obtained by filling in the corners of the diagram in Figure 2 with squares from C , which can be done in exactly one way (there is only one square fitting ef , for example, so this has to be the square in the top left corner).

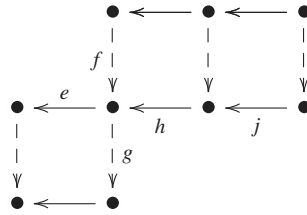


FIGURE 2. Composing paths.

If Λ is a finite k -graph with no sources, then a *Cuntz–Krieger Λ -family* is a collection of partial isometries $\{S_\lambda \mid \lambda \in \Lambda\}$ (either operators in a Hilbert space or elements of an abstract C^* -algebra) satisfying the following *Cuntz–Krieger relations*:

- (1) $\{S_v \mid v \in \Lambda^0\}$ is a family of mutually orthogonal projections;
- (2) $S_{\lambda\mu} = S_\lambda S_\mu$ for all $\lambda, \mu \in \Lambda$ such that $s(\lambda) = r(\mu)$;
- (3) $S_\lambda^* S_\lambda = S_{s(\lambda)}$ for all $\lambda \in \Lambda$;
- (4) $S_v = \sum_{\lambda \in v\Lambda^n} S_\lambda S_\lambda^*$ for all $v \in \Lambda^0$ and $n \in \mathbb{N}^k$.

The C^* -algebra of Λ is the C^* -algebra $C^*(\Lambda)$ generated by a universal Cuntz–Krieger Λ -family $\{s_\lambda \mid \lambda \in \Lambda\}$. The basic facts about these C^* -algebras are discussed in [9, 21] and [20, Ch. 10], for example.

3. Tiles and 2-graphs

There are four variables in our main construction of 2-graphs. Recall that a subset T of \mathbb{N}^2 is *hereditary* if $j \in T$ and $0 \leq i \leq j$ imply $i \in T$. The variables are:

- a *tile* T , which is a hereditary subset of \mathbb{N}^2 with finite cardinality $|T|$;
- an *alphabet* $\{0, 1, \dots, q - 1\}$, where $q \geq 2$ is an integer (we view the alphabet as a commutative ring by identifying it with $\mathbb{Z}/q\mathbb{Z}$ in the obvious way);
- an element t of the alphabet, called the *trace*; and
- a function $w : T \rightarrow \{0, 1, \dots, q - 1\}$ called the *rule*.

For the rest of the section, we fix the *basic data* (T, q, t, w) .

Example 3.1. For the 2-graph underlying the Ledrappier system, the basic data consists of the sock tile $T = \{0, e_1, e_2\}$, $q = 2$, $t = 0$ and the constant function $w \equiv 1$.

The vertex set in our 2-graph will be

$$\Lambda^0 = \left\{ v : T \rightarrow \mathbb{Z}/q\mathbb{Z} \mid \sum_{i \in T} w(i)v(i) = t \pmod{q} \right\}.$$

The Ledrappier graph, for example, has four vertices a, b, c, d visualized as

$$\begin{array}{|c|} \hline 0 \\ \hline 0 \quad 0 \\ \hline \end{array} \quad \begin{array}{|c|} \hline 1 \\ \hline 0 \quad 1 \\ \hline \end{array} \quad \begin{array}{|c|} \hline 1 \\ \hline 1 \quad 0 \\ \hline \end{array} \quad \begin{array}{|c|} \hline 0 \\ \hline 1 \quad 1 \\ \hline \end{array}. \tag{3.1}$$

To describe the paths, we need some notation. Let $(c_1, c_2) := \bigvee \{i \mid i \in T\}$, so that the longest row in T (the bottom one) has $c_1 + 1$ boxes and the highest column in T (the left-hand one) has $c_2 + 1$ boxes. For $S \subset \mathbb{Z}^2$ and $n \in \mathbb{Z}^2$, we let $S + n = \{i + n \mid i \in S\}$ denote

the translate of S by n , and we set $T(n) := \bigcup_{0 \leq m \leq n} T + m$. When we visualize $T(n)$ using our convention, it looks like a $(c_1 + 1 + n_1) \times (c_2 + 1 + n_2)$ rectangle of boxes with a bite taken out of the top right-hand corner. If $f : S \rightarrow \mathbb{Z}/q\mathbb{Z}$ is a function defined on a subset S of \mathbb{N}^2 containing $T + n$, then we define $f|_{T+n} : T \rightarrow \mathbb{Z}/q\mathbb{Z}$ by

$$f|_{T+n}(i) = f(i + n) \quad \text{for } i \in T. \tag{3.2}$$

A *path of degree n* is a function $\lambda : T(n) \rightarrow \mathbb{Z}/q\mathbb{Z}$ such that $\lambda|_{T+m}$ is a vertex for $0 \leq m \leq n$; then λ has *source* $s(\lambda) = \lambda|_{T+n}$ and *range* $r(\lambda) = \lambda|_T$. Thus, for example, the diagram (1.1) is the visualization of a path of degree $(3, 2)$ in the Ledrappier graph based on the sock tile.

Notice that the function $f|_{T+n}$ defined in (3.2) is not a simple restriction: because our tiles all have their bottom left-hand corner at the origin, we need to translate by n on the right-hand side. We need to use a similar convention when we define the segments appearing in the factorizations of paths. For $\lambda \in \Lambda^p$ and $0 \leq m \leq n \leq p$, the *segment* $\lambda(m, n)$ is the path of degree $n - m$ defined by

$$\lambda(m, n)(i) = \lambda(m + i) \quad \text{for } i \in T(n - m).$$

In particular, $\lambda(m, m)$ is the vertex $\lambda|_{T+m}$.

We want $\Lambda^* := \bigcup_{n \in \mathbb{N}^2} \Lambda^n$ to be the morphisms in a category, and so we have to define composition. To make this work, we need to make an assumption on the rule w . We say that the rule w has *invertible corners* if $w(c_1e_1)$ and $w(c_2e_2)$ are invertible elements of the ring $\mathbb{Z}/q\mathbb{Z}$. The next proposition tells us that there is exactly one candidate for the composition of two paths.

PROPOSITION 3.2. *Suppose we have basic data (T, q, t, w) and the rule w has invertible corners. Suppose $\mu \in \Lambda^m$ and $\nu \in \Lambda^n$ satisfy $s(\mu) = r(\nu)$. Then there is a unique path $\lambda \in \Lambda^{m+n}$ such that*

$$\lambda(0, m) = \mu \quad \text{and} \quad \lambda(m, m + n) = \nu. \tag{3.3}$$

Notice that equation (3.3) defines λ uniquely on $T(m) \cup (T(n) + m)$, so our problem is to show that there is a unique function $\lambda' : T(m + n) \rightarrow \mathbb{Z}/q\mathbb{Z}$ such that $\lambda'|_{T(m) \cup (T(n) + m)} = \lambda$ and $\lambda'|_{T+k}$ belongs to Λ^0 for every k such that $0 \leq k \leq m + n$; since μ and ν are paths and λ' extends λ , we already know this for k such that $T + k \subset T(m) \cup (T(n) + m)$.

Our strategy is to extend λ from $T(m) \cup (T(n) + m)$ to $T(m + n)$ by adding one point at a time in such a way that there is only one possible value for λ at the new point. The next lemma tells us how to do this. It depends crucially on the assumption that the rule has invertible corners (see Remark 3.7), and it fails spectacularly for tiles in \mathbb{N}^k when $k > 2$ (see Remark 3.8).

LEMMA 3.3. *Suppose that $l \in \mathbb{N}^2$ and S is a subset of \mathbb{N}^2 containing $T + l - e_2$ and $T + l + e_1$, and $\lambda : S \rightarrow \mathbb{Z}/q\mathbb{Z}$ is a function such that $\lambda|_{T+k}$ belongs to Λ^0 for every $k \in \mathbb{N}^2$ such that $T + k \subset S$. Then there is a unique function*

$$\lambda' : S' := S \cup \{l + e_1 - e_2 + c_1e_1\} \rightarrow \mathbb{Z}/q\mathbb{Z}$$

such that $\lambda'|_S = \lambda$ and $\lambda'|_{T+k}$ belongs to Λ^0 for every $k \in \mathbb{N}^2$ such that $T + k \subset S'$.

Proof. If $l + e_1 - e_2 + c_1e_1$ belongs to S , there is nothing to do. So we suppose $l + e_1 - e_2 + c_1e_1$ is not in S . Let $i \in T \setminus \{c_1e_1\}$. Then either $i_2 = 0$ and $i_1 < c_1$, in which case $i + e_1$ belongs to T and $l + e_1 - e_2 + i$ belongs to $T + l - e_2$, or $i_2 > 0$, in which case $i - e_2 \in T$ and $l + e_1 - e_2 + i$ belongs to $T + l + e_1$. So $(T \setminus \{c_1e_1\}) + l + e_1 - e_2$ is contained in the domain S of λ , and $l + e_1 - e_2 + c_1e_1$ is the only point of $T + l + e_1 - e_2$ which is not in S . Thus we can define $\lambda'|_S = \lambda$ and

$$\lambda'(l + e_1 - e_2 + c_1e_1) := w(c_1e_1)^{-1} \left(t - \sum_{i \in T \setminus \{c_1e_1\}} w(i)\lambda(l + e_1 - e_2 + i) \right). \quad (3.4)$$

If $T + k \subset S'$, then either $T + k \subset S$, in which case $\lambda'|_{T+k} = \lambda|_{T+k} \in \Lambda^0$, or $k = l + e_1 - e_2$, in which case (3.4) implies $\lambda'|_{T+k} \in \Lambda^0$. No other value of $\lambda'(l + e_1 - e_2 + c_1e_1)$ would give $\lambda'|_{T+l+e_1-e_2} \in \Lambda^0$, so this function λ' is the only one with the required property. \square

Proof of Proposition 3.2. The region $T(m + n)$ is obtained from $T(m) \cup (T(n) + m)$ by adding two rectangles

$$BR := \{j \in \mathbb{N}^2 \mid c_1 + m_1 < j_1 \leq c_1 + m_1 + n_1, 0 \leq j_2 < m_2\}, \quad \text{and} \\ UL := \{j \in \mathbb{N}^2 \mid 0 \leq j_1 < m_1, c_2 + m_2 < j_2 \leq c_2 + m_2 + n_2\}.$$

We order the bottom right rectangle BR lexicographically, first going down the column $j_1 = c_1 + m_1 + 1$, then down the column $j_1 = c_1 + m_1 + 2$, and so on. We then apply Lemma 3.3 to each j in order: when we come to define $\lambda(j)$, we have already defined $\lambda(i)$ for every i above and to the left of j , and with $l := j - e_1 + e_2 - c_1e_1$, λ is defined on both $T + l - e_2$ and $T + l + e_1$. Since there is only one possible value of $\lambda(j)$ at each stage, there is only one way to extend λ to $T(m) \cup (T(n) + m) \cup BR$.

To see that λ extends uniquely to UL , we can either run the mirror image of this argument in the rectangle UL , or reflect everything in the line $n_1 = n_2$ and apply what we have just proved. \square

THEOREM 3.4. *Suppose we have basic data (T, q, t, w) and the rule w has invertible corners. Say that $\mu \in \Lambda^m$ and $\nu \in \Lambda^n$ are composable if $s(\mu) = r(\nu)$, and define the composition $\mu\nu$ to be the unique path λ satisfying (3.3). Define $d : \Lambda \rightarrow \mathbb{N}^2$ by $d(\lambda) = n$ for $\lambda \in \Lambda^n$. Then, with Λ^0, Λ^*, r and s defined at the beginning of the section, $\Lambda(T, q, t, w) := ((\Lambda^0, \Lambda^*, r, s), d)$ is a 2-graph.*

Proof. We can view a vertex $v \in \Lambda^0$ as a path of degree 0; then λ has the property (3.3) which characterizes $r(\lambda)\lambda$ and $\lambda s(\lambda)$, so v has the properties required of the identity morphism at v . For $\mu \in \Lambda^m, \nu \in \Lambda^n$ with $s(\mu) = r(\nu)$, (3.3) implies that $r(\mu\nu) = (\mu\nu)|_T = \mu|_T = r(\mu)$ and

$$s(\mu\nu) = (\mu\nu)|_{T+m+n} = (\mu\nu)(m, m + n)|_{T+n} = \nu|_{T+n} = s(\nu).$$

To prove that Λ is a category, it remains to show that composition is associative.

Suppose $\mu \in \Lambda^m$, $\nu \in \Lambda^n$ and $\rho \in \Lambda^p$ satisfy $s(\mu) = r(\nu)$ and $s(\nu) = r(\rho)$. For $i \in T(n)$, we have

$$\begin{aligned} ((\mu\nu)\rho)(m, m + n + p)(i) &= ((\mu\nu)\rho)(i + m) = ((\mu\nu)\rho)(0, m + n)(i + m) \\ &= (\mu\nu)(i + m) = (\mu\nu)(m, m + n)(i) = \nu(i), \end{aligned}$$

and for $i \in T(p)$ we have

$$\begin{aligned} ((\mu\nu)\rho)(m, m + n + p)(i + n) &= ((\mu\nu)\rho)(i + m + n) \\ &= ((\mu\nu)\rho)(m + n, m + n + p)(i) = \rho(i). \end{aligned}$$

Thus

$$((\mu\nu)\rho)(m, m + n + p)(0, n) = \nu$$

and

$$((\mu\nu)\rho)(m, m + n + p)(n, n + p) = \rho,$$

and hence

$$((\mu\nu)\rho)(m, m + n + p) = \nu\rho.$$

On the other hand, for $i \in T(m)$, we have

$$\begin{aligned} ((\mu\nu)\rho)(0, m)(i) &= ((\mu\nu)\rho)(i) = ((\mu\nu)\rho)(0, m + n)(i) \\ &= (\mu\nu)(i) = (\mu\nu)(0, m)(i) = \mu(i), \end{aligned}$$

so $((\mu\nu)\rho)(0, m) = \mu$. Thus $(\mu\nu)\rho$ has the property which characterizes $\mu(\nu\rho)$, and we have $(\mu\nu)\rho = \mu(\nu\rho)$.

We have now shown that Λ is a category, and it is countable because each Λ^n is finite. The map $d : \Lambda \rightarrow \mathbb{N}^2$ satisfies $d(\mu\nu) = d(\mu) + d(\nu)$ and hence is a functor, so it remains to verify that d has the factorization property. But this is easy: given $\lambda \in \Lambda^{m+n}$, the paths $\mu := \lambda(0, m)$ and $\nu := \lambda(m, m + n)$ are the only ones which can satisfy $\lambda = \mu\nu$. \square

To visualize the 2-graph $\Lambda(T, q, t, w)$, we draw its skeleton. This skeleton has a few special properties.

PROPOSITION 3.5. *Suppose we have basic data (T, q, t, w) and the rule w has invertible corners. Then $\Lambda = \Lambda(T, q, t, w)$ satisfies*

- (a) $|\Lambda^0| = q^{|T|-1}$;
- (b) for $v, u \in \Lambda^0$, $v\Lambda^{e_i}u$ is non-empty if and only if

$$v(m) = u(m - e_i) \quad \text{for every } m \in T \cap (T + e_i), \tag{3.5}$$

in which case $|v\Lambda^{e_i}u| = 1$;

- (c) $|v\Lambda^{e_1}| = |\Lambda^{e_1}v| = q^{c_2}$ and $|v\Lambda^{e_2}| = |\Lambda^{e_2}v| = q^{c_1}$ for every $v \in \Lambda^0$.

Proof. There are $q^{|T|-1}$ functions $v : T \setminus \{c_1e_1\} \rightarrow \mathbb{Z}/q\mathbb{Z}$, and each defines a unique vertex v by setting

$$v(c_1e_1) = w(c_1e_1)^{-1} \left(t - \sum_{i \in T \setminus \{c_1e_1\}} w(i)v(i) \right).$$

This gives (a). For (b), note that if $\beta \in v\Lambda^{e_i}u$ and $m \in T \cap (T + e_i)$, then

$$v(m) = \beta|_T(m) = \beta|_{T+e_i}(m - e_i) = u(m - e_i).$$

Conversely, if u, v satisfy (3.5), then we can define $\beta : T(e_i) \rightarrow \mathbb{Z}/q\mathbb{Z}$ by

$$\beta(m) = \begin{cases} v(m) & \text{for } m \in T, \\ u(m - e_i) & \text{for } m \in (T + e_i) \setminus T, \end{cases}$$

and (3.5) says that $\beta|_{T+e_i} = u$. The constraints $\beta|_T = v$ and $\beta|_{T+e_i} = u$ completely determine β , so $|v\Lambda^{e_i}u| = 1$.

To see (c), note that an edge $\beta \in \Lambda^{e_1}v$ has $\beta|_{T+e_1}$ determined by v . The remainder $T(e_1) \setminus (T + e_1)$ is the first column of $T(e_1)$, which has $c_2 + 1$ entries. There are q^{c_2} ways of filling in the bottom c_2 squares, and then the top entry is determined by

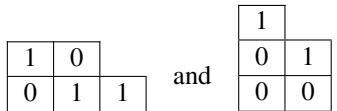
$$\beta(c_2e_2) = w(c_2e_2)^{-1} \left(t - \sum_{i \in T \setminus \{c_2e_2\}} w(i)\beta(i) \right).$$

Thus $|\Lambda^{e_1}v| = q^{c_2}$. On the other hand, an edge $\beta \in v\Lambda^{e_1}$ has $\beta|_T = v$, and $T(e_1) \setminus T$ also has $c_2 + 1$ squares. We can fill in all the squares except $c_1e_1 + e_1$ arbitrarily in q^{c_2} ways, and then $\beta(c_1e_1 + e_1)$ is determined by

$$\beta(c_1e_1 + e_1) = w(c_1e_1)^{-1} \left(t - \sum_{i \in T \setminus \{c_1e_1\}} w(i)\beta(i + e_1) \right).$$

The facts about the red edges follow by symmetry. □

Example 3.6. The *Ledrappier graph* $L(\boxplus)$ is the 2-graph constructed from the basic data consisting of the sock tile T , $q = 2$, $t = 0$ and $w \equiv 1$. It has four vertices a, b, c, d listed in (3.1). Examples of a blue edge (with range b and source d) and a red edge (with range a and source b) are visualized by



The skeleton of $L(\boxplus)$ is the 2-coloured graph in Figure 3.

Remark 3.7. If we start with a rule which does not have invertible corners, then we can still draw a bicoloured graph, which may or may not be the skeleton of a 2-graph. For example, suppose T is the sock, $w(0) = w(e_1) = 1$ and $w(e_2) = 0$. Then there is exactly one blue-red path between each pair of vertices, but there are sometimes two and sometimes no red-blue paths, so the bicoloured graph cannot be the skeleton of a 2-graph. On the other hand, if we use the zero rule $w \equiv 0$ on the sock, then there are two blue-red paths and two red-blue paths between each pair of vertices, so there are bijections between the sets of blue-red and red-blue paths, each of which determines a potentially different 2-graph. This is reminiscent of the 2-graphs \mathcal{F}_θ^+ studied in [2], which have a single vertex, and are thus completely determined by a permutation θ of a finite set $\{1, \dots, m\} \times \{1, \dots, n\}$.

We will observe in Remark 4.4 that when we have to make choices to define a factorization property, the correspondence between 2-graphs and shifts breaks down.

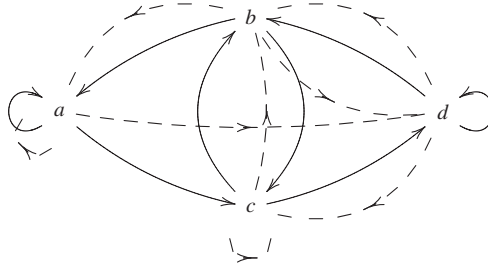


FIGURE 3. The skeleton of the Ledrappier graph.

Remark 3.8. When we start with a tile T which is a finite hereditary subset of \mathbb{N}^3 , we can construct a tricoloured graph, but this will not completely determine a 3-graph because Lemma 3.3 fails. The crux of the proof of Lemma 3.3 (when $l = e_2$) is that the set $T(e_1 + e_2) \setminus (T(e_2) \cup (T(e_1) + e_2))$, consists of the single point $(c_1 + 1)e_1$. When we consider the tile $T = \{0, e_1, e_2, e_3\}$, which is a natural 3-dimensional analogue of the sock, we have

$$T(e_1 + e_2) \setminus (T(e_2) \cup (T(e_1) + e_2)) = \{2e_1, e_1 + e_3\}.$$

If we use a single rule w with invertible corners to define our vertices and paths, then there is still more than one way to fill in the two empty cubes. So one would have to impose more than one rule to get a uniquely defined red-blue factorization of a blue-red path. However, the number of empty cubes to be filled depends on the dimensions of the original tile, so just one extra rule is not enough in general.

4. *Connections with shift spaces*

To make contact with the dynamics literature, we consider basic data $(T, q, 0, w)$. We denote by $R_2^q = \mathbb{Z}/q\mathbb{Z}[u_1^{\pm 1}, u_2^{\pm 1}]$ the commutative ring of Laurent polynomials in u_1, u_2 over the ring $\mathbb{Z}/q\mathbb{Z}$, and define $g = g_{T,w} \in R_2^q$ by

$$g_{T,w} = \sum_{m \in T} w(m)u^m.$$

The shift space $\Omega = \Omega^{R_2^q/(g)}$ is defined in [7, p. 719] as

$$\begin{aligned} \Omega &= \left\{ f = (f(n)) \in (\mathbb{Z}/q\mathbb{Z})^{\mathbb{Z}^2} \mid \sum_{i \in T} w(i) f(i + n) = 0 \pmod{q} \text{ for } n \in \mathbb{Z}^2 \right\} \\ &= \left\{ f : \mathbb{Z}^2 \rightarrow \mathbb{Z}/q\mathbb{Z} \mid \sum_{i \in T} w(i) f|_{T+n}(i) = 0 \pmod{q} \text{ for } n \in \mathbb{Z}^2 \right\}. \end{aligned} \tag{4.1}$$

This is a compact subspace of $(\mathbb{Z}/q\mathbb{Z})^{\mathbb{Z}^2}$ in the product topology, and carries an action of \mathbb{Z}^2 defined by $(\alpha_p f)(n) = f(n + p)$.

The two-sided path space of a k -graph Λ was introduced in [10, §3]; here we consider the case where Λ is a finite 2-graph. Let

$$\Delta = \{(m, n) \mid m, n \in \mathbb{Z}^2, m \leq n\};$$

with $r(m, n) = m$, $s(m, n) = n$ and $d(m, n) = n - m$, (Δ, d) becomes a 2-graph. The two-sided infinite path space of Λ is

$$\Lambda^\Delta = \{x : \Delta \rightarrow \Lambda \mid x \text{ is a degree-preserving functor}\}.$$

It is shown in [10] that Λ^Δ has a locally compact (metric) topology with basic open sets

$$Z(\lambda, n) = \{x \in \Lambda^\Delta \mid x(n, n + d(\lambda)) = \lambda\},$$

for $\lambda \in \Lambda$ and $n \in \mathbb{Z}^2$. Since we are assuming that Λ is finite, Λ^Δ is compact.

Now for each $p \in \mathbb{Z}^2$ we define $\sigma_p : \Lambda^\Delta \rightarrow \Lambda^\Delta$ by

$$\sigma_p(x)(m, n) = x(m + p, n + p).$$

Observe that for all $n, p \in \mathbb{Z}^2$ and $\lambda \in \Lambda$ we have $\sigma_p(Z(\lambda, n)) = Z(\lambda, n + p)$, so σ_p is a homeomorphism for every $p \in \mathbb{Z}^2$, and σ is an action of \mathbb{Z}^2 on Λ^Δ .

THEOREM 4.1. *Suppose we have basic data $(T, q, 0, w)$ and w has invertible corners, let $\Lambda := \Lambda(T, q, 0, w)$ be the associated 2-graph, and define Ω as in (4.1). Then there is a homeomorphism $h : \Lambda^\Delta \rightarrow \Omega$ such that $\alpha_p \circ h = h \circ \sigma_p$.*

Proof. Define $h : \Lambda^\Delta \rightarrow (\mathbb{Z}/q\mathbb{Z})^{\mathbb{Z}^2}$ by

$$h(x)(i) = x(i, i)(0) \quad \text{for } x \in \Lambda^\Delta, i \in \mathbb{Z}^2.$$

Let $j \in T$. Since $x(i - j, i)$ is a path in Λ , it is a well-defined function on $T(j)$ and

$$x(i, i)(0) = x(i - j, i)|_{T+j}(0) = x(i - j, i)(j + 0) = x(i - j, i - j)(j). \quad (4.2)$$

Then $h(x) \in \Omega$ since for all $j \in \mathbb{Z}^2$, (4.2) gives

$$\begin{aligned} \sum_{i \in T} w(i)h(x)|_{T+j}(i) &= \sum_{i \in T} w(i)h(x)(i + j) \\ &= \sum_{i \in T} w(i)x(i + j, i + j)(0) \\ &= \sum_{i \in T} w(i)x(j, j)(i), \end{aligned}$$

which is 0 (mod q) since $x \in \Lambda^\Delta$.

To see that $h : \Lambda^\Delta \rightarrow \Omega$ is a homeomorphism it suffices to show h is a continuous bijection. For $f \in \Omega$ define a function $k(f) : \Delta \rightarrow \Lambda$ by

$$k(f)(m, n) = f|_{T(n-m)+m} \quad \text{for } m \leq n.$$

Condition (4.1) implies that $k(f)(m, n)$ is a path in Λ of degree $n - m$, so k is degree preserving. An application of Proposition 3.2 says that $k(f)(m, p) = f|_{T(p-m)+m}$ factors as

$$f|_{T(p-m)+m} = f|_{T(n-m)+m} f|_{T(p-n)+n},$$

which gives

$$k(f)(m, p) = k(f)(m, n)k(f)(n, p),$$

so $k(f)$ is a functor.

We claim that $k : \Omega \rightarrow \Lambda^\Delta$ is the inverse of h . We have $h(k(f)) = f$ since for $i \in \mathbb{Z}^2$

$$h(k(f))(i) = k(f)(i, i)(0) = f|_{T(0)+i}(0) = f(i + 0) = f(i).$$

Now we must show $k(h(x)) = x$. Let $i \in T(n - m)$. Then $i = j + l$ for some $j \in T$ and $0 \leq l \leq n - m$. We have

$$\begin{aligned} (k(h(x))(m, n))(i) &= h(x)|_{T(n-m)+m}(i) \\ &= h(x)(i + m) \\ &= h(x)(j + l + m) \\ &= x(j + l + m, j + l + m)(0) \\ &= x(l + m, l + m)(j) \text{ by (4.2)} \\ &= x(m, n)|_{T+l}(j) \\ &= x(m, n)(j + l) \\ &= x(m, n)(i), \end{aligned}$$

so $k(h(x)) = x$.

To see that h is continuous, suppose $x_\gamma \rightarrow x$ in Λ^Δ . Since Ω has the product topology, it suffices to prove that $h(x_\gamma)(i) \rightarrow h(x)(i)$ for all $i \in \mathbb{Z}^2$. Let $i \in \mathbb{Z}^2$. Then $Z(x(i, i), 0)$ is an open neighbourhood of x in Λ^Δ , so for large γ , $x_\gamma \in Z(x(i, i), 0)$. But then for large γ we have $h(x_\gamma)(i) = x_\gamma(i, i)(0) = x(i, i)(0) = h(x)(i)$, so certainly $h(x_\gamma(i)) \rightarrow h(x)(i)$.

For the last part we have $h(\sigma_p(x)) = \alpha_p(h(x))$ since

$$h(\sigma_p(x))(i) = (\sigma_p x)(i, i)(0) = x(i + p, i + p)(0) = h(x)(i + p) = \alpha_p(h(x))(i). \quad \square$$

Remark 4.2. Theorem 4.1 implies in particular that the shift space Ω associated to the Ledrappier graph $L(\boxplus)$ is the 2-dimensional Markov system known as Ledrappier’s example (see [13, Examples 1.8, 2.4] and [11]).

Remark 4.3. There is a one-sided version of Theorem 4.1. The space

$$\Omega^+ := \left\{ f : \mathbb{N}^2 \rightarrow \mathbb{Z}/q\mathbb{Z} \mid \sum_{i \in T} w(i) f|_{T+n}(i) = 0 \pmod{q} \text{ for all } n \in \mathbb{N}^2 \right\}$$

has a natural action of \mathbb{N}^2 , and the \mathbb{Z}^2 action on Λ^Δ restricts to an \mathbb{N}^2 action on the one-sided infinite path space Λ^∞ . Then the argument of Theorem 4.1 gives a homeomorphism of Λ^∞ onto Ω^+ which commutes with the actions of \mathbb{N}^2 .

Remark 4.4. We saw in Remarks 3.7 and 3.8 that relaxing our hypotheses on the rule or using higher-dimensional tiles would lead to situations where we have to nominate blue-red to red-blue factorizations to define a k -graph Λ . In the two-dimensional case, this would mean that if $d(\lambda) = e_1 + e_2$ then $\lambda((c_2 + 1)e_2)$ will depend on the choice of $\lambda((c_1 + 1)e_1)$ as well as the values of λ on $T \cup (T + e_1 + e_2)$. So the homeomorphism of Remark 4.3 will carry the infinite path space of Λ onto a proper subspace of the shift space Ω^+ .

5. Aperiodicity

Aperiodicity is the property of a k -graph Λ which ensures that Λ has a Cuntz–Krieger uniqueness theorem which says that all Cuntz–Krieger Λ -families generate isomorphic C^* -algebras. We will use a formulation of aperiodicity due to Robertson and Sims: Λ is *aperiodic* if for every $v \in \Lambda^0$ and $m, n \in \mathbb{N}^2$ with $m \neq n$, there is a path $\lambda \in \Lambda$ satisfying $r(\lambda) = v$, $d(\lambda) \geq m \vee n$ and

$$\lambda(m, m + d(\lambda) - (m \vee n)) \neq \lambda(n, n + d(\lambda) - (m \vee n)). \tag{5.1}$$

It is shown in [22, Lemma 3.2] that this is equivalent to the aperiodicity hypotheses used in [9] and [21] (which phrase aperiodicity as properties of the shifts on the one-sided path space Λ^∞).

To prove aperiodicity of our 2-graphs we need to make another restriction on the rule w . We say that w has *three invertible corners* if $w(0)$, $w(c_1 e_1)$ and $w(c_2 e_2)$ are all invertible in $\mathbb{Z}/q\mathbb{Z}$ (implicitly demanding that $c_1 \geq 1$ and $c_2 \geq 1$). We show in Example 5.1 that aperiodicity may fail if $w(0)$ is not invertible. We will also simplify things by assuming that the trace t is zero, and we will discuss this hypothesis after the proof of Theorem 5.2.

Example 5.1. Consider the data consisting of the sock tile $T = \{0, e_1, e_2\}$, $q = 2$, $t = 0$ and rule defined by $w(0) = 0$, $w(e_1) = w(e_2) = 1$. The vertices are:



Since every vertex v has $v(e_1) = v(e_2)$, every path is constant along the short diagonals $n_1 + n_2 = c$. In other words, for every path λ and every n , we have $\lambda(n) = \lambda(n + e_1 - e_2)$ whenever n and $n + e_1 - e_2$ lie in the domain of λ . This implies in particular that for every path λ with $d(\lambda) \geq (1, 1)$, we have

$$\lambda(e_2, d(\lambda) - e_1) = \lambda(e_1, d(\lambda) - e_2),$$

so (5.1) fails for every v with $m = e_2$ and $n = e_1$.

THEOREM 5.2. *If the rule w in the basic data $(T, q, 0, w)$ has $c_1 \geq 1$, $c_2 \geq 1$ and three invertible corners, then the associated 2-graph Λ is aperiodic.*

For the proof of the theorem we need to know that Λ is *strongly connected* in the sense that every $v\Lambda^*u$ is non-empty.

PROPOSITION 5.3. *Suppose that $k \in \mathbb{N}$ satisfies $(k - 1)(e_1 + e_2) \in T$ and $k(e_1 + e_2) \notin T$. Then for every $v, u \in \Lambda^0$ there exists $\lambda \in \Lambda^{k(e_1+e_2)}$ such that $r(\lambda) = v$ and $s(\lambda) = u$.*

The proof of Proposition 5.3 depends on the following variant of Proposition 3.5(b).

LEMMA 5.4. *If $v, u \in \Lambda^0$ satisfy*

$$v(j) = u(j - e_1 - e_2) \quad \text{for } j \in T \cap (T + e_1 + e_2), \tag{5.2}$$

then there is a unique path $\mu \in \Lambda^{e_1+e_2}$ such that $r(\mu) = v$ and $s(\mu) = u$.

Proof. We define

$$\mu(j) = \begin{cases} v(j) & \text{if } j \in T \\ u(j - (e_1 + e_2)) & \text{if } j \in (T + e_1 + e_2) \setminus T; \end{cases}$$

then we obviously have $\mu|_T = v$, and (5.2) says that $\mu|_{T+e_1+e_2} = u$. Now we observe that because the corners $w(c_i e_i)$ are invertible, there are unique values of $\mu((c_i + 1)e_i)$ such that $\mu|_{T+e_i}$ belongs to Λ^0 . So there is exactly one path μ with the required property. \square

Proof of Proposition 5.3. We will prove by induction on p that for $0 \leq p \leq k$, there exists $\mu^p \in \Lambda^{p(e_1+e_2)}$ such that $r(\mu^p) = v$ and

$$s(\mu^p)(j) = u(j - (k - p)(e_1 + e_2)) \quad \text{for } j \in T \cap (T + (k - p)(e_1 + e_2)). \quad (5.3)$$

Then $\mu := \mu^k$ is the required path.

For $p = 0$, we take $\mu^0 := v$. Suppose that $0 \leq p < k$ and we have μ^p with the required properties. Now we define

$$v^{p+1}(i) = \begin{cases} s(\mu^p)(i + e_1 + e_2) & \text{for } i \in T \cap (T - e_1 - e_2) \\ u(i - (k - p - 1)(e_1 + e_2)) & \text{for } i \in T \cap (T + (k - p - 1)(e_1 + e_2)); \end{cases} \quad (5.4)$$

if j belongs to both sets on the right-hand side, then we can apply (5.3) with $j = i + e_1 + e_2$ and deduce that the two possible values for $v^{p+1}(i)$ coincide. We now define $v^{p+1}(i)$ arbitrarily for other points i in $T \setminus \{c_1 e_1\}$, and set

$$v^{p+1}(c_1 e_1) := w(c_1 e_1)^{-1} \left(\sum_{i \in T \setminus \{c_1 e_1\}} w(i) v^{p+1}(i) \right),$$

so that $v^{p+1} \in \Lambda^0$. The first option in (5.4) implies that the pair $s(\mu^p)$ and v^{p+1} satisfy (5.2), and hence by Lemma 5.4 there exists a path $\nu \in \Lambda^{e_1+e_2}$ with $r(\nu) = s(\mu^p)$ and $s(\nu) = v^{p+1}$. Now we take μ^{p+1} to be the composition $\mu^p \nu$, and the second option in (5.4) implies that $s(\mu^{p+1})$ satisfies (5.3). \square

Proof of Theorem 5.2. We fix $v \in \Lambda^0$ and $m, n \in \mathbb{N}^2$ with $m \neq n$. We choose a path μ with $r(\mu) = v$ and $d(\mu) = m \vee n$. We aim to extend μ to a path λ satisfying (5.1). If the vertices $\mu|_{T+m}$ and $\mu|_{T+n}$ are different, then $\lambda := \mu$ will do. So we suppose that $\mu|_{T+m} = \mu|_{T+n}$. We deal separately with the cases where m and n are comparable in the sense that either $m \leq n$ or $n \leq m$, and where they are not comparable.

Suppose first that m and n are comparable, say $m \leq n$. Since $m \neq n$, there exists i such that $m + e_i \leq n$, and then we have

$$r(\mu(m, m + e_i)) = \mu|_{T+m} = \mu|_{T+n} = s(\mu).$$

Since $c_i \geq 1$, Proposition 3.5(c) implies that $|s(\mu)\Lambda^{e_i}| > 1$ and so there exists $\nu \in s(\mu)\Lambda^{e_i}$ such that $\nu \neq \mu(m, m + e_i)$. Then $\lambda := \mu\nu$ has the required properties:

$$\lambda(m, m + d(\lambda) - (m \vee n)) = \lambda(m, m + e_i) = \mu(m, m + e_i)$$

is not equal to

$$\lambda(n, n + d(\lambda) - (m \vee n)) = \lambda(n, n + e_i) = v.$$

Now suppose that m and n are not comparable, say $m_1 > n_1$ and $m_2 < n_2$. This is where we use the extra hypotheses on the rule w and the trace t . Since $t = 0$, the identically zero function $v_0 : T \rightarrow \mathbb{Z}/q\mathbb{Z}$ defines a vertex v_0 , and the identically zero function $x : \mathbb{N}^2 \rightarrow \mathbb{Z}/q\mathbb{Z}$ defines an infinite path $x \in \Lambda^\infty$ (via the homeomorphism of Remark 4.3). Since $\Lambda^{e_1} v_0$ has more than one element, and there is just one blue edge from v_0 to v_0 (see Proposition 3.5), there must be a blue edge β with $s(\beta) = v_0$ and $r(\beta) \neq v_0$. By Proposition 5.3, there is a path α with $r(\alpha) = s(\mu)$ and $s(\alpha) = r(\beta)$. We claim that

$$\lambda := \mu\alpha\beta x(0, (m \vee n) - (m \wedge n) - e_1) = \mu\alpha v, \text{ say,}$$

satisfies (5.1); indeed, we claim that the two paths in (5.1) have different sources. Since $d(\lambda) = d(\mu\alpha) + (m \vee n) - (m \wedge n)$ and

$$\begin{aligned} \lambda|_{T+m+d(\lambda)-(m \vee n)} &= \lambda|_{T+m+d(\mu\alpha)-(m \wedge n)} \\ &= v|_{T+m-(m \wedge n)} \\ &= x|_{T+(m_1-n_1-1)e_1} \\ &= v_0, \end{aligned}$$

it suffices to show that

$$\lambda|_{T+n+d(\lambda)-(m \vee n)} = v|_{T+n-(m \wedge n)} = v|_{T+(n_2-m_2)e_2}$$

is not equal to v_0 .

We suppose that there exists $p \in \mathbb{N}$ such that $v|_{T+pe_2} = v_0$, and look for a contradiction. Then there is a smallest such p , and since $v|_T = r(\beta) \neq v_0$, we then have $p > 0$. Now $v|_{T+(p-1)e_2} \in \Lambda^0$ implies

$$w(0)v((p-1)e_2) = - \sum_{i \in T \setminus \{0\}} w(i)v(i + (p-1)e_2); \tag{5.5}$$

since we have $v(l) = x(l) = 0$ whenever $l_1 > 0$, (5.5) implies that

$$w(0)v((p-1)e_2) = - \sum_{j=1}^{c_2} w(je_2)v((j+p-1)e_2), \tag{5.6}$$

which is 0 because $v((k+p)e_2) = v|_{T+pe_2}(ke_2) = v_0(ke_2) = 0$ for $k \geq 0$. Since $w(0)$ is invertible, this implies that $v((p-1)e_2) = 0$. Thus we have $v|_{T+(p-1)e_2} = v_0$, and this contradicts the choice of p . Thus for every p , $v|_{T+pe_2}$ is not equal to v_0 , and in particular $v|_{T+(n_2-m_2)e_2}$ is not equal to v_0 , as required. \square

Remark 5.5. The preceding proof also works when $t \neq 0$ provided there is a vertex v_0 which is constant, say $v_0(m) = c$ for all $m \in T$. There is such a vertex if and only if there exists $c \in \mathbb{Z}/q\mathbb{Z}$ such that

$$c \left(\sum_{i \in T} w(i) \right) = t \pmod{q}. \tag{5.7}$$

However, we do not obtain any new 2-graphs this way: if there is such a c , then $\Lambda(T, q, t, w)$ is isomorphic to $\Lambda(T, q, 0, w)$. To see this, note that for every path λ in $\Lambda(T, q, 0, w)$, $\lambda_t : i \mapsto \lambda(i) + c \pmod{q}$ is a path in $\Lambda(T, q, t, w)$, and the map $\lambda \mapsto \lambda_t$ is an isomorphism of $\Lambda(T, q, 0, w)$ onto $\Lambda(T, q, t, w)$.

It is easy to find examples where (5.7) has no solution c . For example, if $|T|$ is even, $q = 2, t = 1$ and $w \equiv 1$, we have $\sum w(i) = |T|$ and $c|T| \equiv 1 \pmod{2}$ has no solutions. We do not have general criteria for aperiodicity when (5.7) has no solution.

6. The C^* -algebras

We now summarize the properties of the C^* -algebras of the 2-graphs $\Lambda(T, q, 0, w)$.

THEOREM 6.1. *Suppose $(T, q, 0, w)$ is basic data with $c_1 \geq 1$ and $c_2 \geq 1$, and the rule w has three invertible corners. Then $C^*(\Lambda(T, q, 0, w))$ is unital, nuclear, simple and purely infinite, and belongs to the bootstrap class \mathcal{N} .*

Proof. We write Λ for $\Lambda(T, q, 0, w)$. We begin by observing that $C^*(\Lambda)$ is unital because Λ^0 is finite, and is nuclear and belongs to the bootstrap class by [9, Theorem 5.5]. It follows easily from Proposition 5.3 that Λ is cofinal: if $x \in \Lambda^\infty$ and $v \in \Lambda^0$, then there is a path from $r(x)$ to v . Since we know from Theorem 5.2 that Λ is aperiodic (that is, satisfies property (iv) of [22, Lemma 3.2]), it follows from [22, Theorem 3.1 and Lemma 3.2] that $C^*(\Lambda)$ is simple.

To see that Λ is purely infinite, we need to check that every vertex v can be reached from a ‘loop with an entrance’ (see [27, Proposition 8.8]). But we know that v receives at least two blue edges α, β , and then Proposition 5.3 implies that there is a path ν from v to $s(\alpha)$, so there is a path $\mu = \alpha\nu$ with $d(\mu) \neq 0$ such that $r(\mu) = s(\mu) = v$. Since β is an entrance to μ , we have verified the hypothesis of [27, Proposition 8.8], and can deduce that $C^*(\Lambda)$ is purely infinite. □

Remark 6.2. We have appealed to [27, Proposition 8.8] rather than [9, Proposition 4.9] because the latter is not correct as it stands. For example, the 2-graphs in [15, Figures 3 and 4] satisfy the hypothesis of [9, Proposition 4.9], but their C^* -algebras are $A\mathbb{T}$ -algebras and hence are not purely infinite.

7. K -theory

Theorem 6.1 implies that, when the rule has three invertible corners, the C^* -algebra falls into the class which is classified by the celebrated theorem of Kirchberg and Phillips, which says that $C^*(\Lambda)$ is determined up to isomorphism by its K -theory [6, 18, 25]. So we want to compute the K -groups of $C^*(\Lambda)$.

Suppose we have basic data satisfying the hypotheses of Proposition 3.5, so that in particular the associated 2-graph Λ is finite with no sources, and the methods of [4] apply. Let B and R be the vertex matrices of Λ , defined for $u, v \in \Lambda^0$ by

$$B(u, v) = \#\{\lambda \in \Lambda^{e_1} \mid r(\lambda) = u, s(\lambda) = v\}$$

$$R(u, v) = \#\{\lambda \in \Lambda^{e_2} \mid r(\lambda) = u, s(\lambda) = v\};$$

the matrices B and R are the vertex matrices of the blue graph $B\Lambda := (\Lambda^0, \Lambda^{e_1}, r, s)$ and the red graph $R\Lambda := (\Lambda^0, \Lambda^{e_2}, r, s)$. The entries $BR(u, v)$ in the product BR are the numbers of blue-red paths from v to u , which the factorization property implies are the same as the entries $RB(u, v)$ in RB ; in other words, $BR = RB$. Let $\delta_1 : \mathbb{Z}^{\Lambda^0} \oplus \mathbb{Z}^{\Lambda^0} \rightarrow \mathbb{Z}^{\Lambda^0}$ and $\delta_2 : \mathbb{Z}^{\Lambda^0} \rightarrow \mathbb{Z}^{\Lambda^0} \oplus \mathbb{Z}^{\Lambda^0}$ be the maps with matrices

$$\delta_1 = \begin{pmatrix} 1 - B^t & 1 - R^t \end{pmatrix} \quad \text{and} \quad \delta_2 = \begin{pmatrix} R^t - 1 \\ 1 - B^t \end{pmatrix}.$$

Then [4, Proposition 3.16] says that the K -groups are given by

$$\begin{aligned} K_0(C^*(\Lambda)) &\cong \text{coker } \delta_1 \oplus \text{ker } \delta_2 \\ K_1(C^*(\Lambda)) &\cong \text{ker } \delta_1 / \text{img } \delta_2. \end{aligned}$$

We were able to calculate the size of the K -groups for a large number of examples by implementing the following procedure in the Magma computational algebra system. Magma recognizes that we are dealing with integer matrices and so it performs calculations over the integers; for example, on being asked to find a basis for the column space of an integer matrix it returns an integer basis. When calculating $|K_0(C^*(\Lambda))|$, we noticed that $\text{ker } \delta_2 = 0$ in every example. To calculate $|\text{coker } \delta_1|$, we find a basis matrix M whose columns are an integer basis for the column space of the matrix of δ_1 . Then

$$|K_0(C^*(\Lambda))| = |\text{coker } \delta_1| = |\det M|.$$

To calculate $|K_1(C^*(\Lambda))|$, first we find a basis matrix H for $\text{ker } \delta_1$. Since the columns of H are linearly independent, for each column vector z of the matrix of δ_2 the equation $Hw = z$ has a unique solution w . Form the matrix whose columns are the solutions w ; then the i th column of W contains the coordinates of the basis vector z with respect to the basis for $\text{ker } \delta_1$. Thus

$$|K_1(C^*(\Lambda))| = |\text{ker } \delta_1 / \text{img } \delta_2| = |\det W|.$$

We give details of these calculations for the Ledrappier graph.

Example 7.1. Consider the sock tile \square with $q = 2, t = 0$ and rule $w \equiv 1$. Vertex matrices for the 2-graph Λ are

$$B = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \quad \text{and} \quad R = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix}.$$

The matrices $\delta_1 : \mathbb{Z}^8 \rightarrow \mathbb{Z}^4$ and $\delta_2 : \mathbb{Z}^4 \rightarrow \mathbb{Z}^8$ are

$$\delta_1 = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 \\ -1 & 1 & -1 & 0 & -1 & 1 & 0 & 1 \\ 0 & -1 & 1 & -1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & -1 & -1 & 1 \end{pmatrix} \quad \text{and}$$

$$\delta_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & -1 & 0 \\ -1 & 1 & -1 & 0 \\ 0 & -1 & 1 & -1 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

Here, δ_1 is onto so we can choose a basis for $\text{img } \delta_1$ such that M is the 4×4 identity matrix; hence $|\det M| = 1$, and $K_0(C^*(\Lambda)) = 0$. Magma gives us the matrices H and W below, which satisfy $HW = \delta_2$:

$$H = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 1 & -1 \\ 0 & -1 & 1 & -1 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \quad \text{and} \quad W = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 \end{pmatrix}.$$

Then since $|\det W| = 1$, we have $K_1(C^*(\Lambda)) = 0$.

Some of the results of our calculations are listed in Table 1. We begin with some explanatory comments.

- A tile T can be uniquely described by the lengths of its rows from longest to shortest. For example, in the table we write $[2, 1]$ for the sock.
- The tile obtained by reflecting T about the line $y = x$ is called the *conjugate tile* of T . For example, the conjugate of the tile $[3, 1]$ is $[2, 1, 1]$ and the sock tile is its own conjugate. A tile and its conjugate give C^* -algebras with the same K -theory since this amounts to swapping the roles of B and R in the K -theory formulas. So in the table we list only one out of each pair of conjugate tiles.
- The results in the table refer to basic data with $t = 0$ and $w \equiv 1$. We also performed calculations for other rules and traces, but we obtained the same values of $|K_0|$ and $|K_1|$. A partial explanation for this is in Remark 5.5.
- Blank spaces in the table would require calculations beyond sensible computation time. We were able to do more calculations when $q = 2$, but the results did not reveal any interesting new phenomena.

A more detailed look at the results of our calculations suggests the following.

TABLE 1. Table of K -theory calculations.

Size	Tile	$q = 2$		$q = 3$		$q = 4$		$q = 5$	
		$ K_0 $	$ K_1 $	$ K_0 $	$ K_1 $	$ K_0 $	$ K_1 $	$ K_0 $	$ K_1 $
3	[3]	3	3	8	8	15	15	24	24
	[2, 1]	1	1	2	2	3	3	4	4
4	[4]	7	7	26	26	63	63	124	124
	[3, 1]	1	1	2	2	3	3	4	4
	[2, 2]	1	1	2	2	3	3	4	4
5	[5]	15	15	80	80	255	255	624	624
	[4, 1]	1	1	2	2	3	3	4	4
	[3, 2]	1	1	2	2	3	3	4	4
	[3, 1, 1]	3	3	8	8	15	15	24	24
6	[6]	31	31	242	242	1023	1023		
	[5, 1]	1	1	2	2	3	3		
	[4, 2]	1	1	2	2	3	3		
	[4, 1, 1]	1	1	2	2	3	3		
	[3, 3]	1	1	2	2	3	3		
	[3, 2, 1]	3	3	8	8	15	15		
7	[7]	63	63	728	728				
	[6, 1]	1	1	2	2				
	[5, 2]	1	1	2	2				
	[5, 1, 1]	3	3	8	8				
	[4, 3]	1	1	2	2				
	[4, 2, 1]	1	1	2	2				
	[4, 1, 1, 1]	7	7	26	26				
	[3, 3, 1]	3	3	8	8				

Conjectures.

- (1) $\ker \delta_2 = 0$.
- (2) $|K_0(C^*(\Lambda))| = |K_1(C^*(\Lambda))|$.
- (3) $|K_i(C^*(\Lambda))|$ always has the form $q^n - 1$. Our calculations are consistent with the formula $|K_i(C^*(\Lambda))| = (q^{c_2} - 1, q^{c_1} - 1)$.

We prove Conjectures (1) and (2) in Theorems 8.1 and 8.9 below. We do not know whether $K_0(C^*(\Lambda))$ is isomorphic to $K_1(C^*(\Lambda))$ in general, though calculations in Magma confirm that $K_i(C^*(\Lambda))$ is cyclic in all the examples listed in Table 1, and hence K_0 is isomorphic to K_1 for all these examples. However, this is automatically true in most cases because there is only one group of the given order, so the number of examples we have considered where there is something to prove (that is, the ones where $|K_i| = 4$ or 8) is fairly small. Certainly we have not yet identified a potential reason for the existence of such an isomorphism, and our proof of Conjecture (2) does not help. We have only numerical evidence for Conjecture (3).

7.1. *Implications for the classification.* The graphs whose K -theory is computed in Table 1 are of two types. For tiles with $c_2 = 0$, the blue graph consists of disjoint cycles, and the aperiodicity condition of [22] fails, so the C^* -algebras of these graphs are not

simple. (Their structure is nevertheless quite intricate and will be discussed in a future paper.) For all other graphs, the basic data satisfies the hypotheses of Theorem 6.1, and hence the C^* -algebras are simple and satisfy the hypotheses of the Kirchberg–Phillips theorem. The Kirchberg–Phillips theorem (as stated in [25, Theorem 8.4.1(iv)], for example) says that two suitable unital C^* -algebras A and B are isomorphic if and only if $K_1(A) \cong K_1(B)$ and there is an isomorphism of $K_0(A)$ onto $K_0(B)$ which takes the class $[1_A]$ of the identity to $[1_B]$. When $|K_0(C^*(\Lambda))| = |K_1(C^*(\Lambda))| = 1$, the last condition is trivially satisfied and $C^*(\Lambda)$ is isomorphic to the Cuntz algebra \mathcal{O}_2 . (Somewhat disappointingly, the Ledrappier graph is one of these graphs.) When $|K_i| > 1$, we computed the class of $[1] = \sum_{v \in \Lambda^0} [p_v]$ in $K_0(C^*(\Lambda)) = \text{coker } \delta_1$ (see [4, Corollary 5.1]), and found that it is always a generator for $K_0(C^*(\Lambda))$. So in all our examples, $K_0(C^*(\Lambda))$ is cyclic. We do not know whether this is always true. To sum up: if Λ_1 and Λ_2 are any graphs in the table with $c_2 \geq 1$, and if $K_0(C^*(\Lambda_1)) = K_0(C^*(\Lambda_2))$, then $C^*(\Lambda_1) \cong C^*(\Lambda_2)$.

None of the C^* -algebras of graphs in Table 1 with non-zero K -theory can be isomorphic to the C^* -algebra of an ordinary directed graph E , because $K_1(C^*(E))$ is always free (being a subgroup of the free group \mathbb{Z}^{E^0}).

8. *K-theory results*

Let T be a tile, and again write $(c_1, c_2) = \sqrt{\{j \mid j \in T\}}$. For $0 \leq i \leq c_1$, we let h_i denote the second coordinate of the top box (i, h_i) in each column of T ; for $0 \leq i \leq c_2$, w_i is the first coordinate of the right-hand box (w_i, i) in each row.

In this section we prove Conjectures (1) and (2) about $K_*(C^*(\Lambda(T, q, t, w)))$, under some mild hypotheses on the shape of the tile.

THEOREM 8.1. *Suppose we have basic data (T, q, t, w) in which w has invertible corners and $c_1, c_2 \geq 1$. Suppose further that either $h_0 > h_1$ or $w_0 > w_1$. If B and R are the vertex matrices associated to $\Lambda = \Lambda(T, q, t, w)$, then the map*

$$\delta_2 = \begin{pmatrix} R^t - 1 \\ 1 - B^t \end{pmatrix} : \mathbb{Z}^{\Lambda^0} \rightarrow \mathbb{Z}^{\Lambda^0} \oplus \mathbb{Z}^{\Lambda^0}$$

has trivial kernel and $K_0(C^(\Lambda)) = \text{coker } \delta_1$.*

Reference [4, Proposition 3.16] says that the bijection of $\text{coker } \delta_1$ onto $K_0(C^*(\Lambda))$ carries the generator δ_v of \mathbb{Z}^{Λ^0} to $[p_v]$, and therefore Theorem 8.1 says that these generate $K_0(C^*(\Lambda))$. The image of δ_1 is then generated by the images of the elements $(1 - B^t)\delta_v$ and $(1 - R^t)\delta_v$. Thus Theorem 8.1 says that, for one of our 2-graphs Λ , $K_0(C^*(\Lambda))$ is generated by $\{[p_v] \mid v \in \Lambda^0\}$ modulo the relations

$$[p_v] = \sum_{r(e)=v, d(e)=e_1} [p_{s(e)}], \quad [p_v] = \sum_{r(e)=v, d(e)=e_2} [p_{s(e)}]$$

imposed by the blue and red Cuntz–Krieger relations.

Theorem 8.1 will follow immediately from the following proposition. For the rest of the section, we fix a set of basic data (T, q, t, w) in which w has invertible corners and $c_1, c_2 \geq 1$.

PROPOSITION 8.2. Let B and R be the vertex matrices of $\Lambda = \Lambda(T, q, t, w)$.

- (1) If $h_0 > h_1$ then the map $1 - B^t : \mathbb{Z}^{\Lambda^0} \rightarrow \mathbb{Z}^{\Lambda^0}$ has trivial kernel.
- (2) If $w_0 > w_1$ then the map $1 - R^t : \mathbb{Z}^{\Lambda^0} \rightarrow \mathbb{Z}^{\Lambda^0}$ has trivial kernel.

To prove this we use the special structure of the vertex matrices of Λ . By symmetry it suffices to prove part (1). From Proposition 3.5 we know that B is a $\{0, 1\}$ -matrix; that the number of 1s in each row/column is q^{e_2} ; and that any two rows/columns are either equal or orthogonal. The crucial observation is that the matrices with these properties are the ones which arise as the vertex matrices of dual graphs. Recall from [20, p. 17], for example, that the dual graph of a directed graph E is the directed graph \widehat{E} with $\widehat{E}^0 := E^1$, $\widehat{E}^1 := \{(e, f) \in E^1 \times E^1 \mid r(f) = s(e)\}$, and range and source maps given by $r(e, f) = e$ and $s(e, f) = f$.

To describe the graphs whose duals arise we need some notation. Let S be the tile $S := T \cap (T - e_1)$ and let S^+ be the tile $S^+ := S \cup \{(h_1 + 1)e_2\}$. In the visual model, S is the tile obtained from T by deleting the first column and shifting one unit to the left, and S^+ is obtained from S by adding one box to the top of its first column. For a directed graph F and an integer $n \geq 1$, the directed graph nF has vertex set $(nF)^0 = F^0$, edge set

$$(nF)^1 = F^1 \times \{1, \dots, n\} = \{(f, i) \mid f \in F^1, 1 \leq i \leq n\}$$

and range and source maps given by $r(f, i) = r(f)$ and $s(f, i) = s(f)$. Then the vertex matrix of nF is n times the vertex matrix of F . (Note if $n = 1$ then $1F \cong F$.)

PROPOSITION 8.3. Suppose that $h_0 > h_1$. Set $r_B = q^{h_0 - h_1 - 1}$, and let $B\Lambda(S^+, q, 0, 1)$ be the blue graph of the tile S^+ with alphabet q , trace 0 and rule which is identically 1. Then the blue graph $B\Lambda$ of $\Lambda(T, q, t, w)$ is isomorphic to the dual of $r_B B\Lambda(S^+, q, 0, 1)$.

To prove this we need the following lemma.

LEMMA 8.4. Let $v_1, v_2 \in \Lambda^0$. Then the set

$$\{u \in \Lambda^0 : u|_{S+e_1} = v_2|_S \text{ and } u|_S = v_1|_S\} \tag{8.1}$$

contains $r_B = q^{h_0 - h_1 - 1}$ vertices if v_1, v_2 satisfy

$$v_1(i) = v_2(i - e_1) \quad \text{for } i \in S \cap (S + e_1), \tag{8.2}$$

and is empty otherwise.

Proof. If (8.2) holds, then define a function $u : T \rightarrow \mathbb{Z}/q\mathbb{Z}$ by $u|_{S+e_1} = v_2|_S$ and $u|_S = v_1|_S$. Since $|T \setminus (S \cup (S + e_1))| = h_0 - h_1 > 0$, there are r_B such functions u which define vertices in Λ^0 . □

Define a relation \sim on Λ^0 by

$$v_1 \sim v_2 \iff v_1|_S = v_2|_S.$$

It is straightforward to check that \sim is an equivalence relation. Let $[v]$ denote the equivalence class of $v \in \Lambda^0$ under \sim . By definition of \sim the set in Lemma 8.4 does not change if we replace v_1 and v_2 by other elements of $[v_1]$ and $[v_2]$. So for v_1, v_2 satisfying (8.2), we can list the vertices in the set (8.1) as $u_i([v_1], [v_2])$ for $1 \leq i \leq r_B$.

Proof of Proposition 8.3. Let F be the directed graph with vertices $F^0 = \Lambda^0 / \sim$ and edges

$$F^1 = \{([v_1], [v_2]) \in F^0 \times F^0 \mid v_1, v_2 \text{ satisfy (8.2)}\}.$$

We prove first that $B\Lambda$ is isomorphic to the dual of the directed graph $r_B F$, and then that F is isomorphic to $B\Lambda(S^+, q, 0, 1)$.

By definition, $r_B F$ has vertices F^0 and edges

$$(r_B F)^1 = \{([v_1], [v_2], i) \mid ([v_1], [v_2]) \in F^1, 1 \leq i \leq r_B\}.$$

So the dual $\widehat{r_B F}$ has vertices $(\widehat{r_B F})^0 = (r_B F)^1$ and there is an edge from $([v_3], [v_4], j)$ to $([v_1], [v_2], i)$ if and only if $[v_2] = [v_3]$.

Define a map $\phi^0 : (\widehat{r_B F})^0 \rightarrow \Lambda^0$ by $\phi^0([v_1], [v_2], i) = u_i([v_1], [v_2])$. Then ϕ^0 is a bijection since $u_i([v_1], [v_2])$ is uniquely determined by $([v_1], [v_2])$ and i , and every $v \in \Lambda^0$ belongs to a set in (8.1) for some v_1 and v_2 (for example, take $v_1 = v$ and v_2 to be any vertex adjacent to v).

Suppose there is an edge in $\widehat{r_B F}$ from $([v_3], [v_4], j)$ to $([v_1], [v_2], i)$ —that is, suppose $[v_2] = [v_3]$. Then Proposition 3.5 says there is a unique edge in $B\Lambda$ from $\phi([v_3], [v_4], j) = u_j([v_3], [v_4])$ to $\phi([v_1], [v_2], i) = u_i([v_1], [v_2])$ since

$$u_i([v_1], [v_2])|_{S+e_1} = v_2|_S = v_3|_S = u_j([v_3], [v_4])|_S.$$

Define the map $\phi^1 : (\widehat{r_B F})^1 \rightarrow \Lambda^{e_1}$ by taking $\phi^1(([v_1], [v_2], i), ([v_3], [v_4], j))$ to be the unique edge in $B\Lambda$ with source $u_j([v_3], [v_4])$ and range $u_i([v_1], [v_2])$. Then ϕ^1 is a bijection since ϕ^0 is. We have $r \circ \phi^1 = \phi^0 \circ r$ since

$$\begin{aligned} r(\phi^1((([v_1], [v_2], i), ([v_3], [v_4], j)))) &= u_i([v_1], [v_2]) \\ &= \phi^0([v_1], [v_2], i) \\ &= \phi^0(r((([v_1], [v_2], i), ([v_3], [v_4], j)))) \end{aligned}$$

and similarly $s \circ \phi^1 = \phi^0 \circ s$. Thus $\phi = (\phi^0, \phi^1)$ is a graph isomorphism from $\widehat{r_B F}$ to $B\Lambda$.

It remains to show that F is isomorphic to $B\Lambda(S^+, q, 0, 1)$. Let $[v] \in F^0$. Define $v^+ : S^+ \rightarrow \mathbb{Z}/q\mathbb{Z}$ by

$$v^+|_S = v|_S \quad \text{and} \quad v^+(0, h_1 + 1) = - \sum_{j \in S} v(j) \pmod{q}.$$

This is well-defined since $S^+ \setminus S = \{(0, h_1 + 1)\}$ and each element in $[v]$ takes the same values on S . We also have that v^+ is uniquely determined by $[v]$, and v^+ is clearly a vertex in $B\Lambda(S^+, q, 0, 1)$. Define $\psi^0 : F^0 \rightarrow B\Lambda(S^+, q, 0, 1)^0$ by $\psi^0([v]) = v^+$. Then we claim that ψ^0 is a bijection. It is one-to-one because v^+ is uniquely determined by $[v]$. To see that ψ^0 is onto, let u be a vertex in $B\Lambda(S^+, q, 0, 1)$ and suppose $u^- \in \Lambda^0$ with $u^-|_{S^+} = u$. Then $(u^-)^+|_S = u^-|_S = u|_S$ which implies $(u^-)^+|_{S^+} = u|_{S^+}$; this says that $\psi([u^-]) = (u^-)^+ = u$, so ψ is onto.

Suppose $([v_1], [v_2]) \in F^1$. Then (8.2) and $S^+ \cap (S^+ + e_1) = S \cap (S + e_1)$ imply

$$v_1^+(i) = v_2^+(i - e_1) \quad \text{for } i \in S^+ \cap (S^+ + e_1).$$

Now Proposition 3.5 implies that there is a unique edge in $B\Lambda(S^+, q, 0, 1)$ from $\psi^0([v_2]) = v_2^+$ to $\psi^0([v_1]) = v_1^+$. Define a map ψ^1 from F^1 to the edge set of $B\Lambda(S^+, q, 0, 1)$ by taking $\psi^1([v_1], [v_2])$ to be the unique edge in $B\Lambda(S^+, q, 0, 1)$ with source v_2^+ and range v_1^+ . Then ψ^1 is a bijection since ψ^0 is. We have $r \circ \psi^1 = \psi^0 \circ r$ since

$$r(\psi([v_1], [v_2])) = \psi^0([v_1]) = \psi^0(r([v_1], [v_2]))$$

and similarly $s \circ \psi^1 = \psi^0 \circ s$. Thus $\psi = (\psi^0, \psi^1)$ is a graph isomorphism from F to $B\Lambda(S^+, q, 0, 1)$. \square

So the blue graph of T is related to the blue graph of S^+ , which is a tile with one fewer column than T . In fact we can repeatedly apply Proposition 8.3 since the new tile S^+ satisfies the hypotheses of that proposition. The tile S_i^+ in the next corollary is obtained from T by deleting the first i columns, shifting to the origin and adding one box to the new first column.

COROLLARY 8.5. *Suppose that $h_0 > h_1$. For $1 \leq i \leq c_1$, let S_i^+ be the tile*

$$S_i^+ = (S_{i-1}^+ \cap (S_{i-1}^+ - e_1)) \cup \{(h_i + 1)e_2\},$$

define r_{B_i} by

$$r_{B_i} = \begin{cases} q^{h_0-h_1-1} & \text{if } i = 1, \\ q^{h_{i-1}-h_i} & \text{if } i > 1, \end{cases}$$

and let H_i be the directed graph $B\Lambda(S_i^+, q, 0, 1)$. Then $B\Lambda(T, q, t, w) \cong \widehat{(r_{B_1}H_1)}$ and $H_i \cong \widehat{(r_{B_{i+1}}H_{i+1})}$ for $1 \leq i \leq c_1 - 1$.

Proof. Applying Proposition 8.3 to $B\Lambda(T, q, t, w)$ gives the result for $i = 1$. Let $1 \leq i \leq c_1 - 1$. Each tile S_i^+ has columns $h_i + 1, h_{i+1}, \dots, h_{c_1}$. Since T is hereditary, $h_i \geq h_{i+1}$. Then S_i^+ satisfies $(h_i + 1) - h_{i+1} > 0$ and so we can apply Proposition 8.3 to H_i to get the result for $i > 1$. \square

Example 8.6. (1) Suppose T is the tile with $c_1 = c_2 = 3$ and columns $h_0 = 3, h_1 = h_2 = 1, h_3 = 0$. Let $q = 2, t \in \mathbb{Z}/2\mathbb{Z}$ and w is a rule with invertible corners. The tiles in Corollary 8.5 are

$$T = \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array} \quad S_1^+ = \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} \quad S_2^+ = \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array} \quad S_3^+ = \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}$$

and the constants are $r_{B_1} = 2, r_{B_2} = 1, r_{B_3} = 2$.

(2) Suppose T has columns h_0, \dots, h_{c_1} satisfying $h_0 = h_1 + 1$ and $h_1 = h_2 = \dots = h_{c_1}$. Since $S_{c_1}^+$ is the tile with one column with $h_{c_1} + 1$ boxes, we have $S_{c_1} \cap (S_{c_1} + e_1) = \emptyset$. So in H_{c_1} there is a directed edge between every pair of vertices, that is, H_{c_1} is the complete graph $K_{q^{h_{c_1}}}$ with $q^{h_{c_1}}$ vertices. Corollary 8.5 implies that $r_{B_1} = r_{B_2} = \dots = r_{B_{c_1}} = 1$ and so $B\Lambda(T, q, t, w)$ is obtained by c_1 times taking the dual of $K_{q^{h_{c_1}}}$. For example, the sock tile has $c_1 = 1$ and columns $h_0 = 1$ and $h_1 = 0$, and so the blue graph of the Ledrappier graph is isomorphic to $\widehat{K_2}$.

We need two more lemmas for the proof of Proposition 8.2.

LEMMA 8.7. *If $n \in \mathbb{Z}$ with $n > 1$ and B is an integer matrix, then*

$$\ker(1 - nB) = \{0\}.$$

Proof. Suppose $v \in \ker(1 - nB)$, that is, $nBv = v$. We claim that $n^p|v$ for all $p \geq 1$. To see this, in the $p = 1$ case we have $v = nBv$ and so v is n times some vector $Bv \in \mathbb{Z}^{\Lambda^0}$. For the inductive step suppose $n^p|v$. Then there exists $u \in \mathbb{Z}^{\Lambda^0}$ such that $v = n^p u$. Then

$$v = nBv = nB(n^p u) = n^{p+1} Bu$$

and so $n^{p+1}|v$. Hence $n^p|v$ for all $p \geq 1$, which is only possible if $v = 0$. □

LEMMA 8.8. *Let $n > 1$ be an integer. If K is the $n \times n$ matrix of all 1s, then*

$$\ker(1 - K^t) = \{0\}.$$

Proof. The matrix $1 - K^t$ is the circulant matrix $\dagger \text{Circ}(v)$ with $v = (0, 1, \dots, 1) \in \mathbb{Z}^n$. If ω is a primitive n th root of unity then using the formula for determinant of a circulant given in [5] we have

$$\begin{aligned} \det(1 - K^t) &= \det \text{Circ}(v) = \prod_{j=0}^{n-1} \sum_{i=0}^{n-1} \omega^{ij} v_i = \prod_{j=0}^{n-1} \sum_{i=1}^{n-1} \omega^{ij} \\ &= \prod_{j=1}^{n-1} \sum_{i=1}^{n-1} \omega^{ij} \times \sum_{i=1}^{n-1} \omega^{i0} = \prod_{j=1}^{n-1} (-1) \times \sum_{i=1}^{n-1} 1 \\ &= (-1)^{n-1} (n - 1). \end{aligned}$$

In particular $\det(1 - K^t) \neq 0$, so we have $\ker(1 - K^t) = \{0\}$. □

Proof of Proposition 8.2. We can deduce (2) by applying part (1) to the conjugate tile, so it suffices to prove (1). Choose $r_{B_1}, \dots, r_{B_{c_1}}$ and $B\Lambda, H_1, \dots, H_{c_1}$ as in Corollary 8.5. Let B, B_1, \dots, B_{c_1} be the vertex matrices of $B\Lambda, H_1, \dots, H_{c_1}$. Since the vertex matrix of a dual graph \widehat{E} is the edge matrix of E , [14, Proposition 4.1] gives isomorphisms

$$\ker(1 - B^t) \cong \ker(1 - r_{B_1} B_1^t) \quad \text{and} \quad \ker(1 - B_i^t) \cong \ker(1 - r_{B_{i+1}} B_{i+1}^t), \quad (8.3)$$

for $1 \leq i \leq c_1 - 1$. If $r_{B_i} = 1$ for all i , then since the tile $S_{c_1}^+$ has only one column, Proposition 3.5(b) implies that every entry in the matrix B_{c_1} is 1, $\ker(1 - B_{c_1}^t) = \{0\}$ by Lemma 8.8, and all the kernels in (8.3) are trivial. If there exists r_{B_j} which is bigger than 1, then there is a first such j ; then Lemma 8.7 implies $\ker(1 - r_{B_j} B_j^t) = \{0\}$, and $\ker(1 - B_i^t) = \{0\}$ for $i < j$. Hence $\ker(1 - B^t) = \{0\}$. □

This completes the proof of Theorem 8.1, and hence settles Conjecture (1). The next theorem settles Conjecture (2).

\dagger A circulant matrix is a square matrix in which each row vector is obtained from the previous row vector by rotating one element to the right. Then an $n \times n$ circulant matrix $\text{Circ}(v)$ can be fully specified by the first row vector $v = (v_0, v_1, \dots, v_{n-1}) \in \mathbb{Z}^n$. (See [3]).

THEOREM 8.9. *Suppose we have basic data (T, q, t, w) in which w has invertible corners and $c_1, c_2 \geq 1$. Suppose further that either $h_0 > h_1$ and $w_0 > w_1$. Then the C^* -algebra of the 2-graph $\Lambda = \Lambda(T, q, t, w)$ has*

$$|K_0(C^*(\Lambda))| = |K_1(C^*(\Lambda))|.$$

Proof. Since $h_0 > h_1$ and $w_0 > w_1$ we know from Proposition 8.2 that $\ker(1 - B^t)$ and $\ker(1 - R^t)$ are trivial. Hence $1 - B^t$ and $1 - R^t$ are invertible over \mathbb{Q} , and both K_0 and K_1 are finite. Let $C := 1 - B^t$ and $D := 1 - R^t$. Then C and D commute because B and R do, and

$$\delta_1 = \begin{pmatrix} C & D \end{pmatrix} : \mathbb{Z}^{\Lambda^0} \oplus \mathbb{Z}^{\Lambda^0} \rightarrow \mathbb{Z}^{\Lambda^0} \quad \text{and} \quad \delta_2 = \begin{pmatrix} -D \\ C \end{pmatrix} : \mathbb{Z}^{\Lambda^0} \rightarrow \mathbb{Z}^{\Lambda^0} \oplus \mathbb{Z}^{\Lambda^0}.$$

By Theorem 8.1 we have $\ker \delta_2 = \{0\}$ and $K_0(C^*(\Lambda)) = \text{coker } \delta_1$, that is,

$$K_0(C^*(\Lambda)) = \mathbb{Z}^{\Lambda^0} / (C\mathbb{Z}^{\Lambda^0} + D\mathbb{Z}^{\Lambda^0}).$$

On the other hand we have

$$\begin{aligned} K_1(C^*(\Lambda)) &= \ker \delta_1 / \text{img } \delta_2 \\ &= \{(u, v) \mid u, v \in \mathbb{Z}^{\Lambda^0}, Cu + Dv = 0\} / \{(-Dw, Cw) \mid w \in \mathbb{Z}^{\Lambda^0}\}. \end{aligned}$$

The map $C\mathbb{Z}^{\Lambda^0} \cap D\mathbb{Z}^{\Lambda^0} \rightarrow \ker \delta_1$ defined by $w \mapsto (-C^{-1}w, D^{-1}w)$ carries $CD\mathbb{Z}^{\Lambda^0}$ onto $\text{img } \delta_2$. This induces an isomorphism of $(C\mathbb{Z}^{\Lambda^0} \cap D\mathbb{Z}^{\Lambda^0}) / CD\mathbb{Z}^{\Lambda^0}$ onto $\ker \delta_1 / \text{img } \delta_2$, hence

$$K_1(C^*(\Lambda)) = (C\mathbb{Z}^{\Lambda^0} \cap D\mathbb{Z}^{\Lambda^0}) / CD\mathbb{Z}^{\Lambda^0}.$$

We have

$$C\mathbb{Z}^{\Lambda^0} \leq (C\mathbb{Z}^{\Lambda^0} + D\mathbb{Z}^{\Lambda^0}) \leq \mathbb{Z}^{\Lambda^0} \quad \text{and} \quad C(D\mathbb{Z}^{\Lambda^0}) \leq (C\mathbb{Z}^{\Lambda^0} \cap D\mathbb{Z}^{\Lambda^0}) \leq D\mathbb{Z}^{\Lambda^0}.$$

Since D is an isomorphism of \mathbb{Z}^{Λ^0} onto $D\mathbb{Z}^{\Lambda^0}$ which carries $C\mathbb{Z}^{\Lambda^0}$ onto $CD\mathbb{Z}^{\Lambda^0}$, we have $|D\mathbb{Z}^{\Lambda^0} : CD\mathbb{Z}^{\Lambda^0}| = |\mathbb{Z}^{\Lambda^0} : C\mathbb{Z}^{\Lambda^0}|$. Then

$$\begin{aligned} &|D\mathbb{Z}^{\Lambda^0} : C\mathbb{Z}^{\Lambda^0} \cap D\mathbb{Z}^{\Lambda^0}| |C\mathbb{Z}^{\Lambda^0} \cap D\mathbb{Z}^{\Lambda^0} : CD\mathbb{Z}^{\Lambda^0}| \\ &= |D\mathbb{Z}^{\Lambda^0} : CD\mathbb{Z}^{\Lambda^0}| \\ &= |\mathbb{Z}^{\Lambda^0} : C\mathbb{Z}^{\Lambda^0}| \\ &= |\mathbb{Z}^{\Lambda^0} : (C\mathbb{Z}^{\Lambda^0} + D\mathbb{Z}^{\Lambda^0})| |(C\mathbb{Z}^{\Lambda^0} + D\mathbb{Z}^{\Lambda^0}) : \mathbb{Z}^{\Lambda^0}|. \end{aligned} \tag{8.4}$$

The inclusion of $D\mathbb{Z}^{\Lambda^0}$ in $C\mathbb{Z}^{\Lambda^0} + D\mathbb{Z}^{\Lambda^0}$ induces an isomorphism of $D\mathbb{Z}^{\Lambda^0} / (C\mathbb{Z}^{\Lambda^0} \cap D\mathbb{Z}^{\Lambda^0})$ onto $(C\mathbb{Z}^{\Lambda^0} + D\mathbb{Z}^{\Lambda^0}) / C\mathbb{Z}^{\Lambda^0}$, and hence

$$|D\mathbb{Z}^{\Lambda^0} : C\mathbb{Z}^{\Lambda^0} \cap D\mathbb{Z}^{\Lambda^0}| = |(C\mathbb{Z}^{\Lambda^0} + D\mathbb{Z}^{\Lambda^0}) : C\mathbb{Z}^{\Lambda^0}|. \tag{8.5}$$

Equation (8.5) allows us to cancel in (8.4) and obtain

$$|C\mathbb{Z}^{\Lambda^0} \cap D\mathbb{Z}^{\Lambda^0} : CD\mathbb{Z}^{\Lambda^0}| = |\mathbb{Z}^{\Lambda^0} : (C\mathbb{Z}^{\Lambda^0} + D\mathbb{Z}^{\Lambda^0})|,$$

which gives the result. □

Remark 8.10. Notice that our proof does not give an explicit isomorphism between $C\mathbb{Z}^{\Lambda^0} \cap D\mathbb{Z}^{\Lambda^0} / CD\mathbb{Z}^{\Lambda^0}$ and $\mathbb{Z}^{\Lambda^0} / (C\mathbb{Z}^{\Lambda^0} + D\mathbb{Z}^{\Lambda^0})$, so we cannot deduce that $K_0 \cong K_1$, only that they have the same number of elements.

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