1-1-2013

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Abstract

Inspired by earlier results on the quasilinear mean curvature flow, and recent investigations of fully nonlinear curvature flow of closed hypersurfaces which are not convex, we consider contraction of axially symmetric hypersurfaces by convex, degree-one homogeneous fully nonlinear functions of curvature. With a natural class of Neumann boundary conditions, we show that evolving hypersurfaces exist for a finite maximal time. The maximal time is characterised by a curvature singularity at either boundary. Some results continue to hold in the cases of mixed Neumann-Dirichlet boundary conditions and more general curvature-dependent speeds.

Keywords

axially, flow, curvature, symmetric, hypersurfaces, boundary, fully, conditions, nonlinear

Disciplines

Engineering | Science and Technology Studies

Publication Details

McCoy, J. A., Mofarreh, F. Y Y. & Williams, G. H. (2014). Fully nonlinear curvature flow of axially symmetric hypersurfaces with boundary conditions. Annali di Matematica Pura ed Applicata, 193 (5), 1443-1455.

1

FULLY NONLINEAR CURVATURE FLOW OF AXIALLY SYMMETRIC HYPERSURFACES WITH BOUNDARY CONDITIONS

JAMES A. MCCOY*, FATEMAH Y. Y. MOFARREH, AND GRAHAM H. WILLIAMS

Abstract. Inspired by earlier results on the quasilinear mean curvature flow, and recent investigations of fully nonlinear curvature flow of closed hypersurfaces which are not convex, we consider contraction of axially symmetric hypersurfaces by convex, degree-one homogeneous fully nonlinear functions of curvature. With a natural class of Neumann boundary conditions we show that evolving hypersurfaces exist for a finite maximal time. The maximal time is characterised by a curvature singularity at either boundary. Some results continue to hold in the cases of mixed Neumann-Dirichlet boundary conditions and more general curvature-dependent speeds.

1. Introduction

After Huisken's classical study of contraction of smooth convex hypersurfaces without boundary by their mean curvature [Hu1], and his subsequent study of the mean curvature flow with boundary conditions [Hu2], several papers have considered the formation of singularities in the mean curvature flow of axially symmetric surfaces and hypersurfaces [Hu3, AAG, DK, Ma, EM]. For the mean curvature flow, the position vector $X(x,t)$ of the evolving hypersurface $M_t = X(M,t)$ satisfies the system of quasilinear weakly parabolic partial differential equations

$$\frac{\partial X}{\partial t}(x,t) = -H(x,t)\,\nu(x,t)\,, \tag{1}$$

with initial condition

$$X(x,0) = X_0(x)$$

for some initial embedding X_0 of a given hypersurface M_0, possibly with boundary. Above, H is the mean curvature of M_t at the point $X(x,t)$ and $\nu(x,t)$ is a smooth choice of unit normal vector. In the case of boundary conditions which are not pure Neumann conditions, a tangential component is added to the right hand side of (1) such that the problem is well-posed, yielding 'non-parametric mean curvature flow'. Results obtained in this setting have been useful in classification of singularities and the extension beyond singularities of the mean curvature flow [Hu3, AAG, HS1, HS2, HS3]. Recently there has been interest in singularities of fully nonlinear curvature flows of closed nonconvex hypersurfaces [ALM1, ALM2, ALM3] so it is natural to consider such flows of axially symmetric hypersurfaces as model cases. There have also been some related studies of volume preserving mean curvature flow of axially symmetric surfaces [At1, At2, CRM1, CRM2, CRM3, AK]. In this paper we generalise results of [DK, Ma, EM] for the mean curvature flow of axially symmetric surfaces with monotone nondecreasing generating

[1]*Corresponding author. The research of the first author was supported by Discovery Project DP120100097 of the Australian Research Council. The research of the second author was supported by a postgraduate scholarship from Princess Nora bin Abdulrahman University.

1991 *Mathematics Subject Classification.* 53C44.

Key words and phrases. curvature flow, parabolic partial differential equation, hypersurface, initial-boundary value problem, Neumann boundary condition.

function and certain boundary conditions to the fully nonlinear case. Our flow equation is motivated by the normal flow

$$\frac{\partial X}{\partial t}(x,t) = -F(\mathcal{W}(x,t))\,\nu(x,t)\,, \tag{2}$$

where F is a suitably smooth function of the eigenvalues of the Weingarten map $\mathcal{W}$ of the evolving hypersurface M_t. Exact requirements on F will be given in the next section.

The structure of this article is as follows. After setting up notation and preliminaries in Section 2, in Section 3 we observe some fundamental behaviour of the flow, while in Section 4 we characterise the finite maximal time of existence of the flow in terms of a curvature singularity. Finally, in Section 5 we consider a more general case where the speed F in (2) is replaced by F^k for constant $k > 0$.

In a later paper we intend to examine the nature of curvature singularities in the more general setting where the generating function of the surface is no longer monotone nondecreasing.

The authors would like to thank Prof Dong-Ho Tsai, Dr Glen Wheeler and Dr Valentina Wheeler for useful discussions.

2. Preliminaries

We will use similar notation to earlier work. In particular, $g = \{g_{ij}\}$, $A = \{h_{ij}\}$ and $\mathcal{W} = \left\{h^i_{\ j}\right\}$ denote respectively the metric, second fundamental form and Weingarten map of M_t. The mean curvature of M_t is

$$H = g^{ij}h_{ij} = h^i_{\ i}$$

and the norm of the second fundamental form is

$$|A|^2 = g^{ij}g^{lm}h_{il}h_{jm} = h^j_{\ l}h_j^{\ l}$$

where g^{ij} is the (i,j)-entry of the inverse of the matrix (g_{ij}). Throughout this paper we sum over repeated indices from 1 to n unless otherwise indicated. Raised indices indicate contraction with the metric.

We will denote by $\left(\dot{F}^{kl}\right)$ the matrix of first partial derivatives of F with respect to the components of its argument:

$$\left.\frac{\partial}{\partial s}F(A+sB)\right|_{s=0} = \dot{F}^{kl}(A)\,B_{kl}.$$

Similarly for the second partial derivatives of F we write

$$\left.\frac{\partial^2}{\partial s^2}F(A+sB)\right|_{s=0} = \ddot{F}^{kl,rs}(A)\,B_{kl}B_{rs}.$$

We will also use the notation

$$\dot{f}_i(\kappa) = \frac{\partial f}{\partial \kappa_i}(\kappa) \text{ and } \ddot{f}_{ij}(\kappa) = \frac{\partial^2 f}{\partial \kappa_i \kappa_j}(\kappa)\,.$$

Unless otherwise indicated, throughout this paper we will always evaluate partial derivatives of F at $\mathcal{W}$ and partial derivatives of f at $\kappa(\mathcal{W})$. The nonlinear speed functions F should have the following properties:

Conditions 2.1.

i) $F(\mathcal{W}) = f(\kappa(\mathcal{W}))$ *where* $\kappa(\mathcal{W})$ *gives the eigenvalues of* $\mathcal{W}$ *and* f *is a smooth, symmetric function defined on an open convex cone* Γ *containing the positive cone*

$$\Gamma_+ = \{\kappa = (\kappa_1, \ldots, \kappa_n) \in \mathbb{R}^n : \kappa_i > 0 \textit{ for all } i = 1, 2, \ldots, n\}\,.$$

ii) *f is strictly increasing in each argument: $\frac{\partial f}{\partial \kappa_i} > 0$ for each $i = 1, \ldots, n$ at every point of Γ.*
iii) *f is homogeneous of degree one: $f(k\kappa) = kf(\kappa)$ for any $k > 0$.*
iv) *f is normalised, $f(1, \ldots, 1) = 1$.*
v) *f is convex.*

These conditions, sometimes with some adjustments, have been used before in curvature contraction flows of convex hypersurfaces [An1, Han, An4, An5, AM, CKK, AMZ] and recently in flows of closed hypersurfaces not necessarily convex [Mc, ALM1, ALM2, ALM3]. Some example functions F are given in those papers; in particular many examples satisfying the above properties including positivity on a cone larger than the positive cone can be built from appropriate operations of the elementary symmetric functions of the principal curvatures. One particular example is $F = H + \eta |A|$ for any constant $\eta \in [0, 1)$.

Conditions 2.1, ii) ensures existence at least for a short time of a solution to (2); we will state a precise short time existence result in the next section (Theorem 3.1).

Our initial n-dimensional hypersurface M_0 is rotationally symmetric about the x-axis, ‘axially symmetric’, so there is a corresponding strictly positive and suitably smooth function $u_0 : [0, a] \to \mathbb{R}$ such that M_0 is parametrised by $X_0 : [0, a] \times \mathbb{S}^{n-1} \to \mathbb{R}^{n+1}$, where

$$X_0(x, \omega) = (x, u_0(x)\,\omega).$$

We will assume that u_0 is at least twice differentiable on $[0, a]$. Throughout the paper, derivatives at the endpoints $x = 0$ and $x = a$ are interpreted naturally as one-sided derivatives. We will denote by κ_1 the curvature of the generating curve $(x, u(x))$ of the surface of revolution, that is, κ_1 is the ‘axial curvature’. We denote by $\kappa_2 = \kappa_3 = \ldots = \kappa_n$ the ‘rotational curvatures’. It is straightforward to compute that these are given in terms of u_0 by

$$\kappa_1 = \frac{-(u_0)_{xx}}{\left(1 + (u_0)_x^2\right)^{\frac{3}{2}}}, \;\; \kappa_j = \frac{1}{u_0\sqrt{1 + (u_0)_x^2}}, j = 2, \ldots, n.$$

In the same process as for the mean curvature flow, we add an explicit diffeomorphism to the right hand side of (2) such that the parametrisation $X(x,t) = (x, u(x,t)\,\omega)$ is preserved and the evolution problem with boundary conditions is well-posed. Then the equivalent scalar evolution equation for the graph function is

$$\frac{\partial u}{\partial t} = -\sqrt{1 + u_x^2}\, F(\mathcal{W}), \tag{3}$$

and the Weingarten map $\mathcal{W}$ has everywhere the useful diagonal structure

$$\mathcal{W} = \begin{bmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 I \end{bmatrix} = \begin{bmatrix} \frac{-u_{xx}}{(1+u_x^2)^{\frac{3}{2}}} & 0 \\ 0 & \frac{1}{u\sqrt{1+u_x^2}} I \end{bmatrix} = \frac{1}{\sqrt{1 + u_x^2}} \begin{bmatrix} -(\arctan(u_x))_x & 0 \\ 0 & \frac{1}{u} I \end{bmatrix},$$

where I is the $(n-1) \times (n-1)$ identity matrix. Here and throughout the article we will write $u_x^2 = \left(\frac{\partial u}{\partial x}\right)^2$. Using the degree one homogeneity of F, equation (3) can be rewritten as

$$\frac{\partial u}{\partial t} = \dot{F}^{11} \frac{u_{xx}}{1 + u_x^2} - \sum_{j=2}^{n} \dot{F}^{jj} \frac{1}{u} = \dot{F}^{11} (\arctan(u_x))_x - \sum_{j=2}^{n} \dot{F}^{jj} \frac{1}{u}.$$

Since the matrix of the Weingarten map is everywhere diagonal, the matrix of $\dot{F}$ is everywhere diagonal and $\dot{F}^{kk} = \dot{f}^k$ for each k (see, for example, [An4]), and the above evolution equation for u becomes

$$\frac{\partial u}{\partial t} = \dot{f}^1 \left(\arctan\left(u_x\right)\right)_x - \sum_{j=2}^{n} \dot{f}^j \frac{1}{u}. \tag{4}$$

Our analysis will be performed by working directly with (3) and (4) and other evolution equations obtained from it. We will also need the following flow independent estimates.

Lemma 2.2. *Any function F satisfying Conditions 2.1 also satisfies*

i) $f \geq \frac{1}{n} H$,

ii) $\sum_{k=1}^{n} \dot{f}_k = \operatorname{trace}\left(\dot{F}^{kl}\right) \leq 1$.

Proof: Parts i) and ii) are proved in exactly the same way as in [U], Lemma 3.3 and Lemma 3.2, where the signs are opposite in that paper due to the concavity of f there. Here the proof works similarly, even when Γ is a larger convex cone than the positive cone. □

3. Behaviour of the flow

In this section we are interested in solutions of (3) with the boundary conditions

$$u_x(0,t) = 0, \quad u_x(a,t) = g(t), \tag{5}$$

where g is a suitably smooth function. Although not necessary for the short time existence theorem, Theorem 3.1, we will for for the subsequent results assume g is smooth, non-negative and non-increasing.

Our short time existence result for (3) is a special case of Theorem 8.5.4 from [Lu] whose proof uses semigroup theory. A similar result is presented for the case of mean curvature flow in [EM].

Theorem 3.1. *Given an initial function $u_0 \in C^2([0,a])$ $(C^{2,\alpha}[0,a])$, compatible with the boundary conditions* (5)*, there exists a $\delta > 0$ such that there is a unique solution $u \in C^2([0,a] \times [0,\delta))$ $(C^{2,\alpha}([0,a] \times [0,\delta))$ to* (3)*, with initial condition $u(\cdot,0) = u_0$ and satisfying the boundary conditions* (5)*.*

Remarks:

1. Uniform parabolicity of f is not required for the above result; Condition 2.1, ii) suffices.
2. Above we are using the standard notation for parabolic Hölder spaces, as in [Li], for example.
3. A similar short time existence result holds for Dirichlet or more general Robin boundary conditions, provided the inital data u_0 is compatible. Such a result is relevant for our later remarks concerning a mixed boundary value problem.
4. We will not pursue the optimal smoothing affect of the nonlinear operator F here, except to note that the case of $u_0 \in C^2([0,a])$ above gives that the curvatures of the hypersurface M_t are continuous, so (4) is uniformly parabolic on a possibly shorter time interval $\left[0, \tilde{\delta}\right)$. The short time existence result in Chapter 14 of [Li] then implies that $u \in C^{2,1}\left([0,a] \times \left(0,\tilde{\delta}\right)\right)$, moreover, classical Schauder estimates then provide higher short-time regularity provided F is sufficiently smooth. We will assume f is at least smooth enough for our maximum principle arguments to be valid. Importantly, we will use the C^2 version of Theorem 3.1 in characterising the maximal time T of existence (Theorem 4.2).

We now turn our attention to initial hypersurfaces for which the generating function u_0 is non-decreasing. The next Lemma does not require f to be convex.

Lemma 3.2. *Consider* (3) *under the boundary conditions* (5)*, with F satisfying Conditions 2.1, i) to iv). Let u_0 be at least $C^2([0,a])$.*

i) *If*

$$(u_0)_x \geq 0, \tag{6}$$

then $u_x(x,t) \geq 0$ for all $x \in [0,a]$, $t \in [0,T)$, that is, as long as a solution to (4) *exists.*

ii) *Suppose that $f \geq 0$ everywhere on the initial hypersurface M_0, that is, u_0 satisfies*

$$f\left(\frac{-(u_0)_{xx}}{\left(1+(u_0)_x^2\right)^{\frac{3}{2}}}, \frac{1}{u_0\sqrt{1+(u_0)_x^2}}, \dots, \frac{1}{u_0\sqrt{1+(u_0)_x^2}}\right) \geq 0. \tag{7}$$

Then $u_t \leq 0$ for all $x \in [0,a]$, $t \in [0,T)$.

Proof: To prove i) we differentiate (3) with respect to x to find that $v = u_x$ satisfies

$$\frac{\partial}{\partial t} v = \frac{\dot{f}^1}{1+v^2} v_{xx} - \frac{2\dot{f}^1}{(1+v^2)^2} v v_x^2 + \sum_{j=2}^{n} \frac{v}{u^2} \tag{8}$$

From (5) we see that $v \geq 0$ for $x = 0$ and $x = a$ and (6) implies $v \geq 0$ for $t = 0$. If $v(x,0) = 0$ at any $x \in (0,a)$ then this is a local minimum and by (8), v does not decrease. Moreover, from (8), if v attains an interior zero minimum then v does not decrease. Hence $v \geq 0$ remains true under (3).

The proof of ii) is similar; we instead differentiate (3) with respect to t to find that $v = u_t$ satisfies the equation

$$\frac{\partial}{\partial t} v = \frac{\dot{f}^1}{1+u_x^2} v_{xx} - \frac{2\dot{f}^1}{(1+u_x^2)^2} u_x u_{xx} v_x + \sum_{j=2}^{n} \dot{f}^j \frac{v}{u^2}. \tag{9}$$

In view of (7), v is initially nonpositive, so it remains so by short time existence and we have at least for a short time

$$\frac{\partial}{\partial t} v \leq \frac{\dot{f}^1}{1+u_x^2} v_{xx} - \frac{2\dot{f}^1}{(1+u_x^2)^2} u_x u_{xx} v_x. \tag{10}$$

Suppose there is a first time when $v = 0$. Applying the maximum principle to (9), this cannot occur at an interior point. At a boundary point, in view of (5), we have $v_x(0,t) = 0$ and $v_x(a,t) = g'(t) \leq 0$. By the Hopf Lemma (see, for example, [PW]) a boundary maximum would have $v_x(0,t) < 0$ or $v_x(a,t) > 0$, so there can be no boundary maximum. We conclude that $v \leq 0$ is preserved. □

Remarks:

1. Since the cone of definition of f is larger than the positive cone, $f > 0$ does not immediately follow from Conditions 2.1 ii) and iii) via the Euler identity as in the case of convex hypersurfaces. Consequently, the above result Lemma 3.2, iii) is useful because it implies that f does not become negative under the flow.
2. In the case of pure Neumann conditions ($g \equiv 0$) we can construct an entire C^2 solution of (3) by reflecting $u : [0,a] \times [0,T)$ in the x-axis to create a spatially even function on $[-a,a] \times [0,T)$ and then extending this periodically in space to $\mathbb{R} \times [0,T)$. Such a construction is done in [Hu3, At1]. We can then apply the maximum principle considering only interior extrema. In particular, the speed F evolves according to

$$\frac{\partial}{\partial t} F = \mathcal{L} F + \dot{F}^{kl} h_k^m h_{ml} F, \tag{11}$$

where $\mathcal{L} = \dot{F}^{kl}\nabla_k\nabla_l$ and ∇ denotes the covariant derivative on $M_t = X(\cdot, t)$. Applying the maximum principle to (11) we observe that F remains bounded below by its initial minimum.

Further, under (2), as in [An1], for example, the mean curvature evolves according to

$$\frac{\partial}{\partial t}H = \mathcal{L}H + \ddot{F}^{kl,rs}\nabla^i h_{kl}\nabla_i h_{rs} + \dot{F}^{kl}h_k^m h_{ml}H.$$

In the case that F is convex, that $H \geq 0$ remains true under (3) now follows directly by the maximum principle.

We refer the reader to Section 5 where we will use similar properties for more general speeds.

We may specify in a similar way as in [Ma] a condition on g which ensures that the solution u exists for a finite maximal time $T > 0$. Here we need F convex, so we can use Lemma 2.2, i). Of course, in the special case that $g \equiv 0$ we know as in [DK] that the maximal existence time is finite by comparing the solution of (3), with initial data u_0, with an enclosing cylinder; such a comparison does not require F convex nor F homogeneous.

Lemma 3.3. *Suppose in addition to* (6) *and* (7) *that*

$$\arctan g(0) < \frac{(n-1)\,a^2}{\int_0^a u_0(x)\,dx}. \tag{12}$$

Then the maximal existence time T of solution u to (3) *is finite.*

Proof: Otherwise a solution u exists and is positive for every finite time. We define the function $E : [0, \infty) \to \mathbb{R}_+$ by $E(t) = \int_0^a u(x,t)\,dx$. We have

$$E'(t) = \int_0^a \frac{\partial 1}{\partial} t u(x,t)\,dx = -\int_0^a \sqrt{1+u_x^2}F\left(\mathcal{W}\right)\,dx.$$

Using Lemma 2.2, i), we obtain

$$E'(t) \leq -\frac{1}{n}\int_0^a \sqrt{1+u_x^2}H\,dx \leq \frac{1}{n}\int_0^a \left(\arctan u_x\right)_x\,dx - \frac{n-1}{n}\int_0^a \frac{1}{u}\,dx.$$

By Hölder's inequality,

$$-\int_0^a \frac{1}{u}\,dx \int_0^a u\,dx \leq -a^2$$

so

$$\begin{aligned}E'(t) &\leq \frac{1}{n}\arctan u_x(a,t) - \frac{1}{n}\arctan u_x(0,t) - \frac{n-1}{n}\frac{a^2}{\int_0^a u\,dx}\\ &= \frac{1}{n}\arctan g(t) - \frac{n-1}{n}\frac{a^2}{\int_0^a u\,dx} \leq \frac{1}{n}\arctan g(0) - \frac{n-1}{n}\frac{a^2}{\int_0^a u_0\,dx} < 0.\end{aligned}$$

Integrating, we obtain

$$E(t) \leq E(0) + \frac{1}{n}\left[\arctan g(0) - \frac{(n-1)\,a^2}{\int_0^a u_0\,dx}\right]t$$

which implies that E becomes negative in finite time, a contradiction. □

Remarks:

1. If the boundary condition (5) is replaced by the mixed condition

$$u_x(0,t) = 0,\ u(a,t) = h(t) \tag{13}$$

for a positive function h which is bounded by $\frac{2(n-1)a}{\pi}$, then similar arguments show that again the flow speed remains nonpositive and, using the energy $E(t) = \int_0^a u^2 dx$, the maximal existence time is finite.
2. The second spatial derivative u_{xx} also satisfies a parabolic equation. The fully nonlinear version of the Sturmian theorem gives that the number of zeros of u_{xx} does not increase during the evolution. This tells us that the number of sign-changes of the axial curvature does not increase under the evolution, a property that could be of interest in applications. We refer the reader to [G, AK] for details of the nonlinear Sturmian theorem and its applications.

4. Singularity

Now we characterise the maximal existence time T as the time of a curvature singularity, that is, when the norm $|A|$ of the second fundamental form becomes unbounded. More specifically, we show that if the axial curvature κ_1 does not blow up at $x = a$ as $t \to T$, then the rotational curvatures blows up at $x = 0$ and in view of the formula for κ_j, $j = 2, \dots, n$, we must have $u(0,t) \to 0$ as $t \to T$. This result is also analogous to the corresponding result for the mean curvature flow in [Ma]. Critical to the argument in [Ma] was that the mean curvature remains positive under the evolution. We do not have this, but an extra, reasonable structure condition on f, permits a similar deduction.

Condition 4.1. *Suppose f satisfies $\lim_{z\to-\infty} f(z,1,\dots,1) < 0$, where we allow the case that the limit is equal to $-\infty$.*

Remark: For the above condition to be satisfied the cone Γ of definition of f must allow the above limit to be taken.

In view of Lemma 3.2, ii), $F \geq 0$ is preserved under the evolution. Condition 4.1 then implies that z does not become too negative. Two examples of f satisfying Condition 4.1 are

i) The mean curvature, $F = H$, so $f(z,1,\dots,1) = z + (n-1)$;
ii) Our earlier fully nonlinear example, $F = H + \eta|A|$ for any $\eta \in [0,1)$. In this case $f(z,1,\dots,1) = z + (n-1) + \eta\sqrt{z^2 + (n-1)}$.

Theorem 4.2. *Suppose that F satisfies Conditions 2.1 and 4.1. Suppose in addition to* (6), (7) *and* (12) *that $\lim_{t\to T} \kappa_1^2(a,t) < \infty$. Then $\lim_{t\to T} \kappa_j^2(0,t) = \infty$ for $j = 2,\dots,n$.*

Proof: Suppose for the sake of establishing a contradiction that $\lim_{t\to T} u(0,t) = \delta > 0$. Then for $j = 2,\dots,n$ we have that for all $(x,t) \in [0,a] \times [0,T)$,

$$\kappa_j^2(x,t) = \frac{1}{u^2(x,t)(1+u_x^2(x,t))} \leq \frac{1}{u^2(x,t)} \leq \frac{1}{u^2(0,t)} \leq \frac{1}{\delta^2}. \tag{14}$$

It follows from Lemma 3.2, ii) that under the evolution

$$f(\kappa_1,\kappa_2,\dots,\kappa_2) = \kappa_2 f\left(\frac{\kappa_1}{\kappa_2},1,\dots,1\right) \geq 0.$$

Since the rotational curvatures $\kappa_2 > 0$, this means that for $z = \frac{\kappa_1}{\kappa_2}$,

$$f(z,1,\dots,1) \geq 0$$

as long as the solution exists. Condition 4.1 on F implies therefore that

$$z = \frac{\kappa_1}{\kappa_2} \geq -c_0,$$

for some $c_0 > 0$, which, in terms of derivatives of u means

$$\frac{-u\, u_{xx}}{1+u_x^2} \geq -c_0,$$

that is, in view of our assumption, we have on $[0,a]\times[0,T)$ that

$$\frac{u_{xx}}{1+u_x^2} \leq \frac{c_0}{u} \leq \frac{c_0}{\delta}. \tag{15}$$

Multiplying equation (9) by $-\mathrm{e}^{-\lambda t}$ and setting $w = -u_t\, \mathrm{e}^{-\lambda t}$, the function w satisfies the equation

$$\frac{\partial}{\partial t} w = \frac{\dot f^1}{1+u_x^2} w_{xx} - \frac{2\dot f^1}{(1+u_x^2)^2} u_x u_{xx} w_x + \left(\sum_{j=2}^{n} \dot f^j \frac{1}{u^2} - \lambda \right) w.$$

In view of Lemma 2.2, ii) and our assumption, taking $\lambda > \frac{1}{\delta^2}$ ensures the coefficient of w is negative and so w cannot obtain an interior maximum. Further,

$$w\,(x,0) = \sqrt{1+u_x^2\,(x,0)}F \leq C\,(M_0) \tag{16}$$

Let us now show that w is bounded on the sides $x=0$ and $x=a$. We have

$$w(0,t) = \left. \left(-\frac{\dot f^1 u_{xx}}{1+u_x^2} + \sum_{j=2}^{n} \dot f^j \frac{1}{u} \right) \right|_{(0,t)} \mathrm{e}^{-\lambda t}$$

and at $x=0$, $u_{xx} \geq 0$ in view of our assumption and Lemma 3.2, i), so using also Lemma 2.2, ii) we have

$$w(0,t) \leq \frac{1}{\delta}.$$

Similarly, using our assumption and that $g\,(t)$ is nonincreasing, there is a non-positive constant $\alpha \leq u_{xx}\,(a,t)$ for all $t \in [0,T)$ and

$$w(a,t) \leq \left. \left(-\dot f^1 \alpha + \sum_{j=2}^{n} \dot f^j \frac{1}{u} \right) \mathrm{e}^{-\lambda t} \right|_{(a,t)}$$

from which it follows using Lemma 2.2, ii) that on $[0,T)$

$$w(a,t) \leq \left(\frac{1}{\delta} - \alpha \right) \mathrm{e}^{-\lambda t} < \frac{1}{\delta} - \alpha.$$

Therefore, together with (16) we have an upper bound for w, that is

$$w = -u_t e^{-\lambda t} \leq \overline{C}\,(M_0, \delta, T)$$

and so on $[0,a]\times[0,T)$,

$$-u_t = \sqrt{1+u_x^2}F \leq \overline{C} e^{\lambda T}.$$

Using now Lemma 2.2, i), we obtain

$$\kappa_1 = \frac{-u_{xx}}{1+u_x^2} \leq n\overline{C} e^{\lambda T}.$$

Together with (15) we have on $[0,a]\times[0,T)$ that

$$\kappa_1^2 \leq \max\left(\frac{c_0^2}{\delta^2}, 4\overline{C}^2 e^{2\lambda T} \right). \tag{17}$$

Now the assumption together with Lemma 3.2, implies $u_T\,(x) = \lim_{t\to T} u\,(x,t) \geq 0$ exists, and (14) and (17) imply u_T generates a C^2 axially-symmetric hypersurface which could be used as an initial hypersurface in the short time existence result, Theorem 3.1, contradicting the maximality of T. Thus our assumption is false and the theorem is proved. □

5. Extension

In this section we are interested in generalising our earlier results to the case where the flow speed is homogeneous of degree $k > 0$, that is,

$$\frac{\partial X}{\partial t}(x,t) = -F^k\left(\mathcal{W}(x,t)\right)\nu(x,t),$$

where F continues to satisfy Conditions 2.1. Such flows of hypersurfaces have been considered before, particularly flows by Gauss curvature and powers of the mean curvature and often for surfaces, usually in the context of convex initial data or translating solutions [I, Han, An3, An2, An5, Schu1, Schu1, Schu2, Schn, JJ, AS, AM, AMZ, CKK].

The corresponding evolution equation of the graph function u is now

$$\frac{\partial u}{\partial t} = -\sqrt{1+u_x^2}\,F^k \tag{18}$$

Under the flow (18), we have the following evolution equations.

Lemma 5.1.
i) $\frac{\partial}{\partial t}u_x = \frac{kF^{k-1}}{1+u_x^2}\dot{f}^1\left(u_x\right)_{xx} + (1-3k)\frac{F^{k-1}\dot{f}^1 u_x}{(1+u_x^2)^2}\left(\left(u_x\right)_x\right)^2 + (k-1)\frac{F^{k-1}\dot{f}^2 u_x}{u(1+u_x^2)}\left(u_x\right)_x + \frac{kf^{k-1}\dot{f}^2}{u^2}u_x$,
ii) $\frac{\partial}{\partial t}u_t = \frac{kF^{k-1}}{1+u_x^2}\dot{f}^1\left(u_t\right)_{xx} + (1-3k)\frac{F^{k-1}\dot{f}^1 u_x u_{xx}}{(1+u_x^2)^2}\left(u_t\right)_x + (k-1)\frac{F^{k-1}\dot{f}^2 u_x (u_t)_x}{u(1+u_x^2)} + \frac{kf^{k-1}\dot{f}^2}{u^2}u_t$,
iii) $\frac{\partial}{\partial t}F^k = \mathcal{L}F^k + k\dot{F}^{ij}h_i{}^m h_{mj}F^k$,

where we have used the notation $\mathcal{L} = kF^{k-1}\dot{F}^{ij}\nabla_i\nabla_j$.

Using these equations and similar arguments as in the previous section we have the following consequences.

Corollary 5.2. *Suppose the initial hypersurface M_0 has $f > 0$ everywhere and consider the flow* (18).
i) *With the boundary conditions* (5), $u_x \geq 0$ *and* $u_t \leq 0$ *continue to hold under the flow.*
ii) *With the boundary conditions* (13), $u_t < 0$ *continues to hold under the flow.*
iii) *In the case of pure Neumann boundary conditions* ((5) *with* $g \equiv 0$), *the minimum of* F *does not decrease under the flow.*

We have specified M_0 to have $f > 0$ strictly now, so our result Corollary 5.2, ii) is also a strict inequality. This is to ensure equation (18) is strictly parabolic for any $k > 0$, at least for a short time. Therefore a similar argument as in Section 2 gives an equivalent local existence result to Theorem 3.1 in the case that the initial hypersurface satisfies $\min_{M_0} F > 0$.

In the case of pure Neumann boundary conditions we can compare the surface M_t evolving via (18) with an enclosing cylinder. The cylinder shrinks to a line segment in finite time, providing a sharp bound on the time T by which the solution hypersurface M_t must have ceased to exist. This argument needs only that the flow (18) is parabolic; the speed need not be homogeneous nor F convex.

Finally in the case that $k \leq 1$, pure Neumann boundary conditions and F satisfies Conditions 2.1 and 4.1 we classify the singularity at time T, using a similar argument as in the proof of Theorem 4.2.

Theorem 5.3. *Let M_0 be such that $(u_0)_x \geq 0$ and $\min_{M_0} f > 0$. Consider the flow (18) with $k \leq 1$ and pure Neumann boundary conditions. Suppose $\lim_{t\to T} \kappa_1^2(a,t) < \infty$. Then $\lim_{t\to T} \kappa_j^2(0,t) = \infty$ for $j = 2, \ldots, n$.*

Proof: Suppose $\lim_{t\to T} u(0,t) = \delta > 0$. Then, as in the proof of Theorem 4.2,

$$\kappa_j^2(x,t) \leq \frac{1}{\delta^2}$$

for all $(x,t) \in [0,a] \times [0,T)$. It follows using Condition 4.1 in the same way as in the proof of Theorem 4.2 that on $[0,a] \times [0,T)$,

$$\frac{u_{xx}}{1+u_x^2} \leq \frac{c_0}{\delta}$$

for some finite $c_0 > 0$. For a constant λ to be chosen, the evolution equation for $w = -u_t e^{-\lambda t}$ is

$$\frac{\partial}{\partial t} w = \frac{kF^{k-1}}{n(1+u_x^2)} w_{xx} + (1-3k)\frac{F^{k-1}}{n(1+u_x^2)^2} u_x u_{xx} w_x + (k-1)\frac{n-1}{n}\frac{F^{k-1}}{u(1+u_x^2)} u_x w_x + \left(\frac{n-1}{n}\frac{kF^{k-1}}{u^2} - \lambda\right) w.$$

Since $k \leq 1$, $F^{k-1} \leq \min_{M_0} F^{k-1}$ holds under the flow, by Corollary 5.2, iii) and we can again choose λ such that the coefficient of w is negative and so w cannot obtain an interior maximum. The remainder of the proof is the same as for Theorem 4.2, where we now use our generalisation to the short time existence result of this section. □

The authors are grateful to Andrew Holder and the Soaring Eagle project of the Faculty of Informatics at the University of Wollongong for production assistance for this article.

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