PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 142, Number 1, January 2014, Pages 213–225
S 0002-9939(2013)11755-7
Article electronically published on September 24, 2013

THE PATH SPACE OF A DIRECTED GRAPH

SAMUEL B. G. WEBSTER

(Communicated by Marius Junge)

ABSTRACT. We construct a locally compact Hausdorff topology on the path space of a directed graph E and identify its boundary-path space ∂E as the spectrum of a commutative C^*-subalgebra D_E of $C^*(E)$. We then show that ∂E is homeomorphic to a subset of the infinite-path space of any desingularisation F of E. Drinen and Tomforde showed that we can realise $C^*(E)$ as a full corner of $C^*(F)$, and we deduce that D_E is isomorphic to a corner of D_F. Lastly, we show that this isomorphism implements the homeomorphism between the boundary-path spaces.

INTRODUCTION

Cuntz and Krieger introduced and studied C^*-algebras associated to finite $(0,1)$-matrices in [3]. Within a year, Enomoto and Watatani showed in [6] how to interpret the Cuntz-Krieger relations and the hypotheses of Cuntz and Krieger's main theorems very naturally in terms of directed graphs. This opened many doors to operator algebraists: graph C^*-algebras have provided a rich supply of very tractable examples. In particular, the combinatorial properties of a graph are strongly tied to the algebraic properties of its C^*-algebra. Graph C^*-algebras include (up to Morita equivalence) all AF algebras [4] and all Kirchberg algebras with free abelian K_1 [17], as well many nonsimple examples of purely infinite nuclear C^*-algebras.

The original analyses of graph C^*-algebras utilised the powerful theory of groupoid C^*-algebras [15]. In [9], Kumjian, Pask, Raeburn and Renault built a groupoid $\mathcal{G}_E$ from each directed graph E. Then, using Renault's theory of groupoid C^*-algebras, they defined the graph C^*-algebra to be the groupoid C^*-algebra $C^*(\mathcal{G}_E)$. By interpreting Renault's hypotheses in terms of the graph E from which $\mathcal{G}_E$ was built, they were able to link properties of E to those of $C^*(\mathcal{G}_E)$. The analysis of [9] establishes among other things that $C^*(\mathcal{G}_E)$ is the universal C^*-algebra generated by a collection of partial isometries satisfying relations now known as the Cuntz-Krieger relations (Section 1.1).

The results of [9] were proved only for graphs in which each vertex emits and receives only finitely many edges. A significantly different way to construct $\mathcal{G}_E$ was introduced by Paterson in [11]. Paterson's construction proceeds via inverse semigroups and provides a framework for a groupoid-based analysis of the graph algebras of directed graphs which may contain infinite receiving vertices. Common

Received by the editors February 6, 2011 and, in revised form, March 1, 2012.
2010 *Mathematics Subject Classification.* Primary 46L05, 05C20.
Key words and phrases. Graph algebra, directed graph, path space, boundary paths.
This research was supported by the ARC Discovery Project DP0984360.

to both groupoid models is that the locally compact Hausdorff unit space $\mathcal{G}_E^0$ of the groupoid is a collection of paths in the graph: for a row-finite graph with no sources, $\mathcal{G}_E^0$ is the collection of right-infinite paths in E; but for more complicated graphs, the infinite paths are replaced with the *boundary paths*.

One can view graph C^*-algebras spatially by considering the image of the natural left-regular representation on the Hilbert space with an orthonormal basis indexed by an invariant subspace of the path space. Using the finite-path space yields the Topelitz-Cuntz-Krieger algebra associated to a graph, whereas using the boundary-path space yields the Cuntz-Krieger algebra.

Similar notions of boundary are key in other areas. For example, the boundary-quotient algebras of Crisp and Laca [2] are determined by analogues of the Cuntz-Krieger relations, and their idea of a minimal closed invariant set is exactly what our boundary captures. This boundary played a prominent role in Laca and Raeburn's analysis [10] of the KMS structure of the Cuntz-Li algebra $\mathcal{Q}_{\mathbb{N}}$ and its Toeplitz extension.

That the boundary-path space of a graph in any generality is locally compact and Hausdorff is more or less attributed to folklore. This result is stated for locally finite graphs without proof as in [9, Corollary 2.2] and has since appeared in more general settings, for example [7, 8, 11]. Paterson and Welch suggest a technique for a construction in [12], but overlook some subtleties which result in an erroneous statement of the basic open sets. We provide details of the construction in this paper and correct their result. Corollary 2.4 of [12] is incorrect as stated, and our resulting Theorem 2.1 is the correct replacement in the setting of directed graphs.

Drinen and Tomforde [5] construct from an arbitrary directed graph E a row-finite graph F such that $C^*(F)$ contains $C^*(E)$ as a full corner. Their construction adds an infinite path to each source and each infinite receiver in E. In the case of infinite receivers, the incoming edges are distributed along the appended infinite path. The resulting graph F is called a *Drinen-Tomforde desingularisation of* E. At an infinite receiver, there is a choice in the way in which edges are distributed along the appended path, and hence a Drinen-Tomforde desingularisation of E is not unique. Motivated by [5], Raeburn developed a 'collapsing' technique in [13, Section 5] which we use in this paper. He defined a desingularisation by identifying paths in a row-finite graph F with no sources, which we call *collapsible paths* (Definition 1.1), then 'collapsed' these paths to yield a graph E such that by applying Drinen and Tomforde's construction (and making the right choices along the way), we can recover F.

This paper is an in-depth analysis of the path space of a directed graph and how it is affected by desingularisation. We begin in Section 1 by recalling the standard definitions and notation for directed graphs, their C^*-algebras, and define a Drinen-Tomforde desingularisation.

In Section 2 we construct a topology on the path space of an arbitrary directed graph E and show that it is a locally compact Hausdorff topology. The construction we use follows an approach suggested by Paterson and Welch in [12]. We then construct a homeomorphism ϕ_∞, which identifies a subset of the infinite-path space of a desingularisation with the boundary-path space in the original graph.

In Section 3, we define the diagonal C^*-subalgebra of a graph C^*-algebra. We then build a homeomorphism h_E between the boundary-path space ∂E of an arbitrary graph E and the spectrum of its diagonal. We show that for a desingularisation F of E, the isomorphism which embeds $C^*(E)$ as a full corner in $C^*(F)$ implements the homeomorphism ϕ_∞ constructed in Section 2 via the homeomorphisms h_E and h_F.

1. Preliminaries

A *directed graph* $E = (E^0, E^1, r, s)$ consists of two countable sets E^0, E^1 and functions $r, s : E^1 \to E^0$. The elements of E^0 are called *vertices*, and the elements of E^1 are called *edges*. For each edge e, we call $s(e)$ the *source* of e and $r(e)$ the *range* of e. If $s(e) = v$ and $r(e) = w$, we say that v *emits* e and that w *receives* e or that e is an edge from v to w. Since all graphs in this paper are directed, we often just call a directed graph E a graph.

We follow the convention of [13] so that a *path of length* n in a directed graph E is a sequence $\mu = \mu_1 \dots \mu_n$ of edges in E such that $s(\mu_i) = r(\mu_{i+1})$ for $1 \le i \le n-1$. We write $|\mu| = n$ for the length of μ and regard vertices as paths of length 0; we denote by E^n the set of paths of length n, and define $E^* := \bigcup_{n\in\mathbb{N}} E^n$. We extend the range and source maps to E^* by setting $r(\mu) = r(\mu_1)$ and $s(\mu) = s(\mu_{|\mu|})$ for $|\mu| > 1$, and $r(v) = v = s(v)$ for $v \in E^0$. If μ and ν are paths with $s(\mu) = r(\nu)$, we write $\mu\nu$ for the path $\mu_1 \dots \mu_{|\mu|}\nu_1 \dots \nu_{|\nu|}$. For a set of vertices $V \subset E^0$ and a set of paths $F \subset E^*$ we define $VF := \{\mu \in F : r(\mu) \in V\}$ and $FV := \{\mu \in F : s(\mu) \in V\}$. If $V = \{v\}$ we drop the braces and write vF and Fv. We define the infinite paths E^∞ of E to be infinite strings $\mu_1 \dots \mu_n \dots$ such that $s(\mu_i) = r(\mu_{i+1})$ for all $i \ge 1$, we extend the range map to E^∞ by setting $r(\mu) = r(\mu_1)$, and for a set of vertices $V \subset E^0$, we define $VE^\infty := \{x \in E^\infty : r(x) \in V\}$.

If $r^{-1}(v)$ is finite for every $v \in E^0$, we say that E is *row-finite*. A vertex v is *singular* if $|r^{-1}(v)| \in \{0, \infty\}$. The *boundary paths* of E are defined by $\partial E := E^\infty \cup \{\alpha \in E^* : s(\alpha) \text{ is singular}\}$.

1.1. Graph C^*-algebras.

Let E be a directed graph. Define

$$E^{\le n} := \{\mu \in E^* : |\mu| = n, \text{ or } |\mu| < n \text{ and } s(\mu)E^1 = \emptyset\}.$$

A *Cuntz-Krieger E-family* consists of mutually orthogonal projections $\{s_v : v \in E^0\}$ and partial isometries $\{s_\mu : \mu \in E^*\}$ such that $\{s_\mu : \mu \in E^{\le n}\}$ have mutually orthogonal ranges for each $n \in \mathbb{N}$, and such that:

(CK1) $s_\mu^* s_\mu = s_{s(\mu)}$ for every $\mu \in E^*$;
(CK2) $s_\mu s_\mu^* \le s_{r(\mu)}$ for every $\mu \in E^*$; and
(CK3) $s_v = \sum_{\nu \in vE^{\le n}} s_\nu s_\nu^*$ for every $v \in E^0$ and $n \in \mathbb{N}$ such that $|vE^{\le n}| < \infty$.

The C^*-algebra of E is the universal C^*-algebra $C^*(E)$ generated by a Cuntz-Krieger E-family $\{s_\mu : \mu \in E^*\}$. The existence of such a C^*-algebra follows from an argument like that of [13, Proposition 1.21].

These relations are slightly different from the Cuntz-Krieger relations appearing elsewhere (for example in [1, 5, 13]), but straightforward calculations show that our definition is equivalent to the one usually stated. For details refer to [18, Section 2.3].

1.2. Desingularisation.

Let $\mu \in E^\infty$ and $e \in E^1$. We say that e *exits* μ if there exists $i \ge 1$ such that $s(e) = s(\mu_i)$ and $e \ne \mu_i$; note that edges with source $r(\mu)$ are not considered exits of μ. We say that e *enters* μ if there exists $i \ge 1$ such that $r(e) = r(\mu_i)$ and $e \ne \mu_i$.

Definition 1.1. Let E be a directed graph. We say that an infinite path $\mu \in E^\infty$ is *collapsible* if:

(C1) μ has no exits,
(C2) $r^{-1}(r(\mu_i))$ is finite for every i,
(C3) $r^{-1}(r(\mu)) = \{\mu_1\}$,
(C4) $\mu_i \neq \mu_j$ for all $i \neq j$, and
(C5) μ has either zero or infinitely many entries.

In [13, p. 42] only (C1)–(C3) are present. Condition (C4) was added after we realized that a cycle with no entrance could be collapsible under the original definition, and (C5) was added to ensure that we collapse only paths (a process described in Remark 1.2) which yield singular vertices, thus avoiding a complication in the proof of [13, Proposition 5.2],[1] the key result for this theory. These conditions are not all necessary to carry out the process of collapsing, but they ensure the simplest formulae and also that we collapse as few paths as possible.

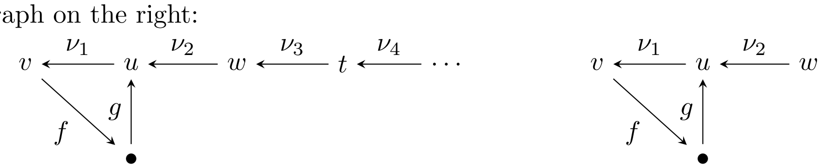
Remark 1.2. As the name suggests, we will collapse these paths to form a new graph. Suppose that μ is a collapsible path in a row-finite graph F. Define $s_\infty(\mu) := \{s(\mu_i) : i \geq 1\}$ and

$$F^*(\mu) := \{\nu \in F^* : |\nu| > 1, \nu = \mu_1\mu_2 \ \ldots \mu_{|\nu-1|}e \text{ for some } e \neq \mu_{|\nu|}\}.$$

Set $F^0_\mu := F^0 \setminus s_\infty(\mu)$ and $F^1_\mu := \big(F^1 \setminus (r^{-1}(s_\infty(\mu)) \cup \{\mu_1\})\big) \cup \{e_\nu : \nu \in F^*(\mu)\}$, and extend the range and source maps to F^1_μ by setting $r(e_\nu) := r(\nu) = r(\mu)$ and $s(e_\nu) := s(\nu)$. Then F_μ is the graph obtained by collapsing the path μ in F. Notice that for $\alpha \in F^*_\mu$, $s(\alpha)$ is singular if and only if $s(\alpha) = r(\mu)$.

Given a collection M of collapsible paths such that no two paths in M have any edge or vertex in common, we call the paths in M *disjoint*. We can carry out the process described in Remark 1.2 on all the paths in M simultaneously, yielding a graph F_M which may no longer be row-finite.

Example 1.3. Collapsing the path $\nu_3\nu_4\ldots$ in the graph on the left yields the graph on the right:



Notice that the path $(\nu_1 g f)^\infty := \nu_1 g f \nu_1 g f \ldots$ is not collapsible as it fails (C4), and $\nu_1\nu_2\ldots$ is not collapsible either as it has exactly one entry, failing (C5).

Definition 1.4. Let E be a directed graph. A *Drinen-Tomforde desingularisation* of E is a pair (F, M) consisting of a row-finite graph F with no sources and of a collection M of disjoint collapsible paths such that $F_M \cong E$.

2. Topology

For $\mu \in E^*$, we define the *cylinder set of* μ by $\mathcal{Z}(\mu) := \{\nu \in E^* \cup E^\infty : \nu = \mu\nu'\}$. Following Paterson and Welch's approach in [12], define $\alpha : E^* \cup E^\infty \to \{0,1\}^{E^*}$ by $\alpha(w)(y) = 1$ if $w \in \mathcal{Z}(y)$, and 0 otherwise. We endow $\{0,1\}^{E^*}$ with the topology

[1] The proof of [13, Proposition 5.2] contained an error when proving that the Cuntz-Krieger relation holds in F_μ at the vertex resulting from collapsing a path μ with finitely many entries.

of pointwise convergence, and W with the initial topology induced by $\{\alpha\}$. The following theorem is considered a folklore result, for which we provide a proof.

Theorem 2.1. *Let E be a directed graph. For $\mu \in E^*$ and a finite subset $G \subset s(\mu)E^1$, define $\mathcal{Z}(\mu \setminus G) := \mathcal{Z}(\mu) \setminus \bigcup_{e\in G} \mathcal{Z}(\mu e)$. Then the collection*

$$\{\mathcal{Z}(\mu \setminus G) : \mu \in E^*, G \subset s(\mu)E^1 \text{ is finite}\}$$

is a basis for the initial topology induced by $\{\alpha\}$. Moreover, it is a locally compact Hausdorff topology on $E^* \cup E^\infty$.

Proof. First we consider the topology on $\{0,1\}^{E^*}$. Given disjoint finite subsets $F, G \subset E^*$, define sets $U_\mu^{F,G}$ to be $\{1\}$ if $\mu \in F$, $\{0\}$ if $\mu \in G$ and $\{0,1\}$ otherwise. Then the sets $N(F,G) := \prod_{\mu\in E^*} U_\mu^{F,G}$, where F, G range over all finite, disjoint pairs of subsets of E^*, form a basis for the topology on $\{0,1\}^{E^*}$. Clearly, α is a homeomorphism onto its range; hence the sets $\alpha^{-1}(N(F,G))$ form a basis for a topology on $E^* \cup E^\infty$. Observe that

$$\alpha^{-1}(N(F,G)) = \lambda \in \left(\bigcap_{\mu\in F} \mathcal{Z}(\mu)\right) \setminus \left(\bigcup_{\nu\in G} \mathcal{Z}(\nu)\right).$$

Notice that if $\alpha^{-1}(N(F,G))$ is nonempty, then $\bigcap_{\mu\in F} \mathcal{Z}(\mu) \neq \emptyset$. This implies that for $\mu, \nu \in F$, we have either

$$\mu \in \mathcal{Z}(\nu) \text{ if } |\mu| \geq |\nu| \quad \text{or } \nu \in \mathcal{Z}(\mu) \text{ if } |\nu| > |\mu|.$$

By choosing μ such that $|\mu| = \max\{|\nu| : \nu \in F\}$ and appropriately adjusting G, we see that each $\alpha^{-1}(N(F,G))$ has the form $\mathcal{Z}(\mu \setminus G)$ for some $\mu \in E^*$ and finite $G \subset s(\mu)E^*$.

Claim 2.1.1. $\{\mathcal{Z}(\mu\setminus G) : \mu \in E^*, G \subset s(\mu)E^1$ is finite$\}$ and $\{\mathcal{Z}(\mu\setminus G) : \mu \in E^*, G \subset s(\mu)E^*$ is finite$\}$ are bases for the same topology.

Proof. Fix $\mu \in E^*$ and a finite subset $G \subset s(\mu)E^*$. Let $\lambda \in \mathcal{Z}(\mu \setminus G)$. We seek $\alpha \in E^*$ and a finite set $F \subset s(\alpha)E^1$ such that

$$\lambda \in \mathcal{Z}(\alpha \setminus F) \subset \mathcal{Z}(\mu \setminus G).$$

We consider two cases: λ is finite or λ is infinite. If $\lambda \in E^\infty$, let $N = \max\{|\mu\nu| : \nu \in G\}$, $\alpha = \lambda_1 \dots \lambda_N$, and $F = \emptyset$. Then $\mathcal{Z}(\alpha \setminus F) = \mathcal{Z}(\alpha)$ clearly contains λ. Since $|\alpha| \geq |\mu\nu|$ for all $\nu \in G$, we have $\mathcal{Z}(\alpha) \subset \mathcal{Z}(\mu \setminus G)$, as required.

Now suppose that $\lambda \in E^*$. Set $\alpha = \lambda$ and

$$F = \{(\mu\nu)_{|\lambda|+1} : \nu \in G \text{ satisfies } |\mu\nu| > |\lambda|\}.$$

Then $\mathcal{Z}(\alpha \setminus F) = \mathcal{Z}(\lambda \setminus F)$ clearly contains λ. To see that $\mathcal{Z}(\lambda \setminus F) \subset \mathcal{Z}(\mu \setminus G)$, fix $\beta \in \mathcal{Z}(\lambda \setminus F)$. Factor $\lambda = \mu\lambda'$; then we have $\beta = \lambda\beta' = \mu\lambda'\beta' \in \mathcal{Z}(\mu)$. We now show that $\lambda'\beta' \notin \bigcup_{\nu\in G} \mathcal{Z}(\nu)$. Fix $\nu \in G$. If $|\mu\nu| \leq |\lambda|$, then $|\nu| \leq |\lambda'|$. Since $\lambda' \notin \mathcal{Z}(\nu)$, we have $\lambda'\beta' \notin \mathcal{Z}(\nu)$. If $|\mu\nu| > |\lambda|$, then since $\beta'_1 \notin F$, we have $(\mu\lambda'\beta')_{|\lambda|+1} = \beta'_1 \neq (\mu\nu)_{|\lambda|+1}$. So $(\lambda'\beta')_{|\lambda|-|\mu|+1} \neq \nu_{|\lambda|-|\mu|+1}$. $\square_{\text{Claim}}$

So the collection $\{\mathcal{Z}(\mu \setminus G) : \mu \in E^*, G \subset s(\mu)^1$ is finite$\}$ is a basis for our topology on $E^* \cup E^\infty$.

To see that $E^* \cup E^\infty$ is a locally compact Hausdorff space, we follow the strategy of [12] to show that $\mathcal{Z}(v)$ is compact for each $v \in E^0$. Since α is a homeomorphism onto its range, it suffices to prove that $\alpha(\mathcal{Z}(v))$ is compact. Since $\{0,1\}^{E^*}$ is compact, we show that $\alpha(\mathcal{Z}(v))$ is closed.

Let $\{\omega^{(n)} \in \mathcal{Z}(v) : n \in \mathbb{N}\}$ be such that $\alpha(\omega^{(n)}) \to f \in \{0,1\}^{E^*}$. We seek $\omega \in \mathcal{Z}(v)$ such that $f = \alpha(\omega)$. Let $A := \{\mu \in E^* : \alpha(\omega^{(n)})(\mu) \to 1\}$. Then if $\mu, \nu \in A$, for large n, we have that $w^{(n)} \in \mathcal{Z}(\mu) \cap \mathcal{Z}(\nu)$. In particular, $\mathcal{Z}(\mu) \cap \mathcal{Z}(\nu) \neq \emptyset$; without loss of generality say $\mu = \nu\nu'$, and denote it $\beta_{\mu,\nu}$. Then for large n we have that $\omega^{(n)} \in \mathcal{Z}(\beta_{\mu,\nu})$, so $\beta_{\mu,\nu} \in A$.

Since A is countable, we can list $A = \{\nu^1, \nu^2, \dots, \nu^m, \dots\}$. Let $y^1 := \nu^1$, and iteratively define $y^n := \beta_{y^{n-1},\nu^n}$. Then $\{y^n : n \in \mathbb{N}\}$ satisfy $y^n_1 y^n_2 \dots y^n_{|y^{n-1}|} = y^{n-1}$, and hence they determine a unique path $\omega \in E^* \cup E^\infty$.

To see that $\alpha(\omega^{(n)}) \to \alpha(\omega)$, we first show that $\nu \in A$ if and only if $\omega \in \mathcal{Z}(\nu)$. Clearly, $\omega \in \mathcal{Z}(y^m) \subset \mathcal{Z}(\nu^m)$ for each $\nu^m \in A$. Conversely, let $\omega \in \mathcal{Z}(\nu^m)$. Then $y^m \in \mathcal{Z}(\nu^m) \cap A$ implies that for large enough n we have $\omega^{(n)} \in \mathcal{Z}(y^m) \subset \mathcal{Z}(\nu^m)$, so $\nu^m \in A$. Now fix $\nu \in E^*$. We will show that $\alpha(\omega^{(n)})(\nu) \to \alpha(\omega)(\nu)$. If $\alpha(\omega)(\nu) = 1$, then $\omega \in \mathcal{Z}(\nu)$. So $\nu \in A$, and hence $\omega^{(n)}(\nu) \to 1$. If $\alpha(\omega)(\nu) = 0$, we have $\omega \notin \mathcal{Z}(\nu)$, forcing $\alpha(\omega^{(n)})(\nu) \to 0$. So $\alpha(\omega^{(n)}) \to \alpha(\omega)$. Hence $\alpha(\mathcal{Z}(v))$ is closed. □

Theorem 2.2. *Let E be a directed graph and F be a Drinen-Tomforde desingularisation of E. Then E^0F^∞ is homeomorphic to ∂E.*

Suppose E is a directed graph and (F, M) is a Drinen-Tomforde desingularisation of E. Define $F^*(M) := \bigcup_{\mu \in M} F^*(\mu)$. Define $\phi' : (F^1 \cap E^1) \cup F^*(M) \to E^1$ by $\phi'|_{F^1 \cap E^1} := \mathrm{id}_{F^1 \cap E^1}$ and $\phi'|_{F^*(M)} : \nu \mapsto e_\nu$. So ϕ' acts as the identity on unchanged edges and takes collapsible paths in F to the associated edges in E.

If $\beta \in F^*$ with $r(\beta), s(\beta) \in E^0$, then β has the form $\beta = b^1 b^2 \dots b^n$ where each $b^k \in (F^1 \cap E^1) \cup F^*(M)$. Define $E^0F^*E^0 := \{\beta \in F^* : r(\beta), s(\beta) \in E^0\}$. We extend the map ϕ' above to a map ϕ on finite paths: define $\phi : E^0F^*E^0 \to E^*$ by

$$\phi(\beta) := \phi(b^1 b^2 \dots b^n) = \phi'(b^1) \dots \phi'(b^n). \tag{2.1}$$

We will extend this map to E^0F^∞ and ultimately show that it is a homeomorphism from E^0F^∞ to ∂E. To do so precisely we use the following results.

Lemma 2.3. *Let E be a directed graph and (F, M) be a desingularisation of E. If $\lambda \in E^0F^\infty$, then either*

- *$\lambda = l^1 \dots l^k \mu$ for some $\mu \in M$ and $l^i \in (F^1 \cap E^1) \cup F^*(M)$ or*
- *$\lambda = l^1 l^2 \dots l^n \dots$, where $l^i \in (F^1 \cap E^1) \cup F^*(M)$.*

Proof. Fix $\lambda \in E^0F^\infty$. We construct the l^i inductively. Either $\lambda_1 \in F^1 \cap E^1$ or $\lambda_1 = \mu_1$ for some $\mu \in M$. If $\lambda_1 \in F^1 \cap E^1$, then let $l^1 = \lambda_1$. If $\lambda_1 = \mu_1$, then either

(i) $\lambda_i = \mu_i$ for all $i \in \mathbb{N}$, in which case $\lambda = \mu$ or
(ii) there exists k such that $\lambda_i = \mu_i$ for all $i < k$ and $\lambda_k \neq \mu_k$, in which case we set $l^1 = \mu_1 \dots \mu_{k-1}\lambda_k$. Since paths in M have no edges in common, we have $l^1 \in F^*(\mu)$.

In case (i), $\lambda = \mu$, in which case we are done. In case (ii), $\lambda = l^1\lambda'$ for some $\lambda' \in F^\infty$. Iterating will either terminate with $\lambda = l^1 \dots l^n \mu$, where $\mu \in M$, or continue ad infinitum, in which case $\lambda = l^1 \dots l^n \dots$. □

Define $\phi_\infty : E^0F^\infty \to \partial E$ by

$$\phi_\infty(\lambda) := \begin{cases} \phi(\lambda') & \text{if } \lambda = \lambda'\mu \text{ for some } \mu \in M, \\ \phi'(\lambda^1)\dots\phi'(\lambda^n)\dots & \text{if } \lambda = l^1\dots l^n\dots. \end{cases} \tag{2.2}$$

Proposition 2.4 ([5, Lemma 2.6a])**.** *Let E be a directed graph and (F, M) be a desingularisation of E. Then ϕ and ϕ_∞, defined in* (2.1) *and* (2.2) *respectively, are bijections and preserve range and source.*

Remark 2.5. When working with the topology on the infinite path space of a row-finite directed graph F with no sources, the finite complements are unnecessary [9, Corollary 2.2]. For a detailed proof of this statement, see the author's Ph.D. thesis [18, Proposition 2.1.2].

Proof of Theorem 2.2. It suffices to show that ϕ_∞ and ϕ_∞^{-1} are continuous.

To see that ϕ_∞ is continuous, fix a basic open set $\mathcal{Z}(\alpha\backslash G)\cap\partial E$. If $\mathcal{Z}(\alpha\backslash G)\cap\partial E = \emptyset$, then $\phi_\infty^{-1}(\mathcal{Z}(\alpha \setminus G) \cap \partial E) = \emptyset$ is open. Suppose that $\mathcal{Z}(\alpha \setminus G) \cap \partial E \neq \emptyset$ and fix $\lambda \in \phi_\infty^{-1}(\mathcal{Z}(\alpha \setminus G) \cap \partial E)$. We seek $\gamma \in F^*$ such that

$$\lambda \in \mathcal{Z}(\gamma) \cap E^0F^\infty \subset \phi_\infty^{-1}(\mathcal{Z}(\alpha \setminus G) \cap \partial E).$$

We consider two cases:

(i) Either $\lambda = l^1l^2\dots$ or $\lambda = l^1\dots l^k\mu$ with $k > |\alpha|$ and
(ii) $\lambda = l^1\dots l^{|\alpha|}\mu$,

where $\mu \in M$, and $l^i \in (F^1 \cap E^1) \cup F^*(M)$ for each i.

In case (i), let $\gamma = l^1\dots l^{|\alpha|+1}$. Clearly $\lambda \in \mathcal{Z}(\gamma) \cap E^0F^\infty$. Furthermore, for $y \in \mathcal{Z}(\gamma) \cap E^0F^\infty$, $\phi'(l^1)\dots\phi'(l^{|\alpha|}) = \alpha$ and $\phi'(l^{|\alpha|+1}) \notin G$. So $\phi_\infty(y) = \phi_\infty(l^1\dots l^{|\alpha|+1}y') \in \mathcal{Z}(\alpha \setminus G) \cap \partial E$.

In case (ii), we have that $s(\alpha)$ is singular in E. Since $G \subset s(\alpha)E^1$, (C3) implies that $G \subset \phi(F^*(M))$. Let $N = \max_{\nu\in\phi^{-1}(G)} |\nu|$. Each $\nu \in G \cap E^N$ has the form $\mu_1\dots\mu_{N-1}e$, where $e \neq \mu_N$. Set $\gamma = \phi^{-1}(\alpha)\mu_1\dots\mu_N$. Then $\lambda = \phi^{-1}(\alpha)\mu \in \mathcal{Z}(\gamma) \cap E^0F^\infty$ and $\mathcal{Z}(\gamma) \cap E^0F^\infty \subset \phi_\infty^{-1}(\mathcal{Z}(\alpha \setminus G) \cap \partial E$.

To see that ϕ_∞^{-1} is continuous, a basic open set $\mathcal{Z}(\gamma) \cap E^0F^\infty$ is in E^0F^∞. If $\mathcal{Z}(\gamma) \cap E^0F^\infty = \emptyset$, then $\phi_\infty(\mathcal{Z}(\gamma) \cap E^0F^\infty) = \emptyset$ is open, so suppose that $\mathcal{Z}(\gamma) \cap E^0F^\infty \neq \emptyset$. Let $x \in \phi_\infty(\mathcal{Z}(\gamma) \cap E^0F^\infty)$. We seek $\alpha \in E^*$ and a finite subset $G \subset s(\alpha)E^1$ such that

$$x \in \mathcal{Z}(\alpha \setminus G) \cap \partial E \subset \phi_\infty(\mathcal{Z}(\gamma) \cap E^0F^\infty).$$

Let $\lambda = \phi_\infty^{-1}(x) = \gamma\lambda'$, where $\lambda' \in F^\infty$. We consider two cases:

(i) $x \in E^\infty$ or
(ii) $x \in E^*$ and $s(x)$ is singular.

In case (i), λ does not 'start' with a collapsible path, so by Lemma 2.3 $\lambda = l^1l^2\dots$ for some $l^i \in (E^1 \cap F^1) \cup F^*(M)$. Let $j = \min\{i \in \mathbb{N} : |l^1\dots l^i| \geq |\gamma|\}$ and set $\alpha = \phi(l^1\dots l^j)$ and $G = \emptyset$. It follows that $x \in \mathcal{Z}(\alpha) \cap \partial E \subset \phi_\infty(\mathcal{Z}(\gamma) \cap E^0F^\infty)$.

In case (ii), we have $\lambda = \gamma\lambda' = \omega\mu$ for some $\omega \in F^*$ and $\mu \in M$. Let $\alpha := x$. Our choice of G depends on $|\gamma|$, so we argue in cases:

(1) If $|\gamma| \leq |\omega|$, let $G = \emptyset$.
(2) If $|\gamma| > |\omega|$, then $\gamma = \omega\mu_1\dots\mu_j$ for some $j \in \mathbb{N}$; let

$$G = \{e_\nu : \nu = \mu_1\dots\mu_k\nu_{k+1} \in F^*(\mu) \text{ and } k < j\}.$$

Since $x \in \mathcal{Z}(\alpha \setminus G) \cap \partial E$ by definition, we just need to show that

$$\mathcal{Z}(x \setminus G) \cap \partial E \subset \phi_\infty(\mathcal{Z}(\gamma) \cap E^0F^\infty).$$

Fix $y = xy' \in \mathcal{Z}(x \setminus G) \cap \partial E$. Since $x = \phi_\infty(\omega\mu) = \phi(\omega)$, we have $\phi_\infty^{-1}(y) = \omega\phi_\infty^{-1}(y')$. In case (1), $|\gamma| \leq |\omega|$ implies that $\omega = \gamma\omega'$ for some $\omega' \in F^*$, so

$$\phi_\infty^{-1}(y) = \gamma\omega'\phi_\infty^{-1}(y') \in \mathcal{Z}(\gamma) \cap E^0F^\infty.$$

For case (2), observe that if $y' \in E^0$, then $y = x \in \phi_\infty(\mathcal{Z}(\gamma) \cap E^0F^\infty)$ by assumption. Suppose that $|y'| \geq 1$. Then $y'_1 = e_\nu$ for some $\nu \in F^*(\mu)$. Since $y \in \mathcal{Z}(x \setminus G)$, $y'_1 \notin G$, so $\nu = \mu_1 \dots \mu_k\nu_{k+1}$ for some $k \geq j$, and thus

$$\phi_\infty^{-1}(y) = \phi_\infty^{-1}(xy') = \omega\nu\phi_\infty^{-1}(y'_2 \dots) = \gamma\mu_{j+1} \dots \mu_k\nu_{k+1}\phi_\infty^{-1}(y'_2 \dots)$$

is an element of $\mathcal{Z}(\gamma) \cap E^0F^\infty$. So $y \in \phi_\infty(\mathcal{Z}(\gamma) \cap E^0F^\infty)$, and hence $\phi_\infty : E^0F^\infty \to \partial E$ is a homeomorphism. □

3. The diagonal and the spectrum

For a directed graph E, we call $C^*(\{s_\mu s_\mu^* : \mu \in E\}) \subset C^*(E)$ the diagonal C^*-algebra of E and denote it by D_E, dropping the subscript when confusion is unlikely. We denote the spectrum of a commutative C^*-algebra B by $\Delta(B)$. Given a homomorphism $\pi : A \to B$ of commutative C^*-algebras, we denote by π^* the induced map from $\Delta(B)$ to $\Delta(A)$ such that $\pi^*(f)(y) = f(\pi(y))$ for all $f \in \Delta(B)$ and $y \in A$.

Remark 3.1. Suppose E is a directed graph and that (F, M) is a Drinen-Tomforde desingularisation of E. Let $\{s_\mu : \mu \in E^*\}$ and $\{t_\mu : \mu \in F^*\}$ be the Cuntz-Krieger families generating $C^*(E)$ and $C^*(F)$. Then it follows from [13, Proposition 5.2] that there exists a projection p such that $pC^*(F)p$ is a full corner in $C^*(F)$ and that there is an isomorphism $\pi : C^*(E) \cong pC^*(F)p$ such that $\pi(s_v) = t_v$ for each $v \in E^0$, $\pi(s_\mu) = t_{\phi^{-1}(\mu)}$ for each $\mu \in E^*$.

The goal for this section is the following theorem.

Theorem 3.2. *Let E be a directed graph and (F, M) be a Drinen-Tomforde desingularisation of E. Let $\phi_\infty : E^0F^\infty \to \partial E$ be the homeomorphism from Theorem* 2.2, *and let p and π be as in Remark* 3.1. *Then $\pi(D_E) = pD_Fp$, and there exist homeomorphisms $h_E : \partial E \to \Delta(D_E)$ and $h : E^0F^\infty \to \Delta(pD_Fp)$ such that the following diagram commutes:*

$$\begin{array}{ccc} E^0F^\infty & \xrightarrow{\phi_\infty} & \partial E \\ h\downarrow & & \downarrow h_E \\ \Delta(pD_Fp) & \xrightarrow[\pi^*]{} & \Delta(D_E) \end{array}$$

We prove Theorem 3.2 later in this section. First, we establish some technical results.

Remark 3.3. Let E be a directed graph, and let $\mu, \nu \in E^*$. Then

$$(s_\mu s_\mu^*)(s_\nu s_\nu^*) = \begin{cases} s_\mu s_\mu^* \text{ if } \mu = \nu\nu', \\ s_\nu s_\nu^* \text{ if } \nu = \mu\mu', \\ 0 \text{ otherwise.} \end{cases} \tag{3.1}$$

This result is proved for row-finite directed graphs in [13, Corollary 1.14(b)]. The proof is only marginally different for arbitrary directed graphs; for a detailed argument see [18, Lemma 2.4.4].

Lemma 3.4. *Let E be a directed graph, and let $F \subset E^*$ be finite. For $\mu \in F$, define*

$$q_\mu^F := s_\mu s_\mu^* \prod_{\mu\mu' \in F \setminus \{\mu\}} (s_\mu s_\mu^* - s_{\mu\mu'} s_{\mu\mu'}^*).$$

Then the q_μ^F are mutually orthogonal projections in $\operatorname{span}\{s_\mu s_\mu^* : \mu \in F\}$*, and for each $\nu \in F$, we have*

$$s_\nu s_\nu^* = \sum_{\nu\nu' \in F} q_{\nu\nu'}^F. \tag{3.2}$$

Proof. By Remark 3.3, $p : \lambda \to s_\lambda s_\lambda^*$ is a Boolean representation of E in the sense of [16, Definition 3.1]. The result then follows from [16, Lemma 3.1]. □

Remark 3.5. Let A be a C^*-algebra, let p be a projection in A, let Q be a finite set of commuting subprojections of p and let q_0 be a nonzero subprojection of p. Then $\prod_{q\in Q}(p-q)$ is a projection. If q_0 is orthogonal to each $q \in Q$, then $q_0 \prod_{q\in Q}(p-q) = q_0$, so, in particular, $\prod_{q\in Q}(p-q) \neq 0$. The proof of this is relatively simple; details can be found in [18, Lemma A.0.7].

Remark 3.6. Let E be a directed graph, and let $F \subset E^*$ be finite. For $\mu \in F$, let $F_\mu = \{\mu' \in s(\mu)E \setminus \{s(\mu)\} : \mu\mu' \in F\}$. It follows from an induction on $|F_\mu|$ that

$$q_\mu^F = s_\mu \Big(\prod_{\mu' \in F_\mu} (s_{s(\mu)} - s_{\mu'} s_{\mu'}^*) \Big) s_\mu^*.$$

We say that $\mu, \nu \in E^*$ have a *common extension* if either $\mu = \nu\nu'$ or $\nu = \mu\mu'$, and we call the longer path the *minimal common extension* of μ and ν. A set $F \subset E^*$ is *exhaustive* if for every $\mu \in E^*$ there exists $\nu \in F$ such that μ and ν have a common extension. We denote the set of finite exhaustive sets by $\mathcal{FE}(E)$, and for a vertex v we define $v\mathcal{FE}(E) := \{F \in \mathcal{FE}(E) : F \subset vE^*\}$.

Theorem 3.7. *Let E be a directed graph. Then $D = \overline{\operatorname{span}}\{s_\mu s_\mu^* : \mu \in E\}$, and for each $x \in \partial E$ there exists a unique $h_E(x) \in \Delta(D)$ such that*

$$h_E(x)(s_\mu s_\mu^*) = \begin{cases} 1 & \text{if } x \in \mathcal{Z}(\mu), \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, $x \mapsto h_E(x)$ is a homeomorphism of ∂E onto $\Delta(D)$.

Proof. That $D = \overline{\operatorname{span}}\{s_\mu s_\mu^* : \mu \in E^*\}$ follows from equation (3.1).

Fix $x \in \partial E$ and $\sum_{\mu \in F} b_\mu s_\mu s_\mu^* \in \operatorname{span}\{s_\mu s_\mu^* : \mu \in E^*\}$. Let $n = \max\{p \in \mathbb{N} : x_1 \dots x_p \in F\}$, and define $F_x := \{\mu' \in x(n)E \setminus \{x(n)\} : x(0,n)\mu' \in F\}$.

Claim 3.7.1. The projection $q_{x_1 \dots x_n}^F \neq 0$.

Proof. If $s(x_n)E^* = \emptyset$, then $F_x = \emptyset$, and hence $q_{x_1\dots x_n}^F = s_{x_1\dots x_n} s_{x_1\dots x_n}^* \neq 0$. Now suppose that $s(x_n)E^* \neq \emptyset$. We first show that there exists $\nu \in s(x_n)E^*$ such that for each $\mu' \in F_x$, ν and μ' have no common extension. We argue in cases:

(i) If $s(x)$ is a source in E and $|x| > n$, let $\nu = x_{n+1} \dots x_{|x|}$. Then by choice of n, ν has no common extension with any μ' in F_x.

(ii) If $s(x)$ is an infinite receiver, such a ν exists since $|F_x| \leq |F| < |s(x)E^*| = \infty$.

(iii) If $x \in E^\infty$, let $k = \max\{|\mu'| : \mu' \in F_x\}$. Then it follows from our choice of n that $\nu = x_{n+1} \dots x_{n+k}$ is not a common extension of any μ' in F_x.

By Remark 3.3, we have $s_\nu s_\nu^* s_{\mu'} s_{\mu'}^* = 0$ for all $\mu' \in F_x$. Applying Remark 3.5 with $p = s_{s(x_n)}$, $q_0 = s_\nu s_\nu^*$, $Q = F_x$, we have $\prod_{\mu' \in F_x}(s_{s(x_n)} - s_{\mu'} s_{\mu'}^*) \neq 0$. So

$$q^F_{x_1 \dots x_n} = s_{x_1 \dots x_n} \prod_{\mu' \in F_x} (s_{s(x_n)} - s_{\mu'} s_{\mu'}^*) s^*_{x_1 \dots x_n} \neq 0. \qquad \square_{\text{Claim}}$$

By the above claim,

$$\Big\| \sum_{\nu \in F} b_\mu s_\mu s_\mu^* \Big\| = \Big\| \sum_{\nu \in F} \Big(\sum_{\substack{\mu \in F \\ \nu \in \mathcal{Z}(\mu)}} b_\mu \Big) q_\nu^F \Big\| = \max_{\substack{\nu \in F \\ q_\nu^F \neq 0}} \Big\{ \Big| \sum_{\substack{\mu \in F \\ \nu \in \mathcal{Z}(\mu)}} b_\mu \Big| \Big\} \geq \Big| \sum_{\substack{\mu \in F \\ x_1 \dots x_n \in \mathcal{Z}(\mu)}} b_\mu \Big|.$$

Hence the formula

$$h_E(x)\Big(\sum_{\mu \in F} b_\mu s_\mu s_\mu^* \Big) = \sum_{\substack{\mu \in F \\ x \in \mathcal{Z}(\mu)}} b_\mu \tag{3.3}$$

determines a well-defined, norm-decreasing linear map $h_E(x)$ on $\operatorname{span}\{s_\mu s_\mu^* : \mu \in E\}$.

We now show that $h_E(x)$ is a homomorphism. Since $h_E(x)$ is linear and norm-decreasing, it suffices to calculate

$$\begin{aligned} h_E(x)(s_\mu s_\mu^* s_\alpha s_\alpha^*) &= \begin{cases} 1 & \text{if } \alpha \in \mathcal{Z}(\mu) \text{ and } x \in \mathcal{Z}(\alpha), \\ & \text{or } \mu \in \mathcal{Z}(\alpha) \text{ and } x \in \mathcal{Z}(\mu), \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} 1 & \text{if } x \in \mathcal{Z}(\alpha) \cap \mathcal{Z}(\mu), \\ 0 & \text{otherwise} \end{cases} \\ &= h_E(x)(s_\mu s_\mu^*) h_E(x)(s_\alpha s_\alpha^*). \end{aligned}$$

Now $h(x)$ is a nonzero bounded homomorphism on a dense subspace of D and hence extends uniquely to a nonzero homomorphism $h(x) : D \to \mathbb{C}$. It remains to show that $h_E : \partial E \to \Delta(D)$ is a homeomorphism. The trickiest part is to show that h_E is onto.

Claim 3.7.2. The map h_E is surjective.

Proof. Fix $\phi \in \Delta(D)$. For each $n \in \mathbb{N}$, $\{s_\mu s_\mu^* : |\mu| = n\}$ are mutually orthogonal projections; thus there exists at most one $\nu^n \in E^n$ such that $\phi(s_{\nu^n} s_{\nu^n}^*) = 1$. Let

$$S := \{n \in \mathbb{N} : \text{ there exists } \nu^n \in E^n \text{ such that } \phi(s_{\nu^n} s_{\nu^n}) = 1\}.$$

Since ϕ is nonzero, S is nonempty. If $\nu = \mu\nu'$ and $\phi(s_\nu s_\nu^*) = 1$, then

$$1 = \phi(s_\nu s_\nu^*) = \phi(s_\nu s_\nu^* s_\mu s_\mu^*) = \phi(s_\nu s_\nu^*) \phi(s_\mu s_\mu^*),$$

so $\phi(s_\mu s_\mu^*) = 1$. It follows that either $S = \mathbb{N}$ or to $\{1, \dots, N\}$ for some N.

If $S = \mathbb{N}$, define $x \in E^\infty$ by $x(0, n) = \nu^n$ for all n. If $S = \{1, \dots, N\}$, define $x := \nu^N$. We see that $x \in \partial E$ is trivial if $S = \mathbb{N}$ and follows from (CK3) otherwise.

To see that $h_E(x) = \phi$, notice that for each $\mu \in E^*$ we have

$$\begin{aligned}\phi(s_\mu s_\mu^*) = 1 &\iff |\mu| \in S \text{ and } \nu^{|\mu|} = \mu \\ &\iff x(0, |\mu|) = \mu \\ &\iff h_E(x)(s_\mu s_\mu^*) = 1.\end{aligned}$$

Since both $\phi(s_\mu s_\mu^*)$ and $h_E(x)(s_\mu s_\mu^*)$ only take values in $\{0, 1\}$, it follows that $h_E(x) = \phi$. $\square_{\text{Claim}}$

To see h is injective, suppose that $h_E(x) = h_E(y)$. Then for each $n \in \mathbb{N}$, let $n_x = \min\{n, |x|\}$. Then we have

$$h_E(y)(s_{x(0,n_x)} s^*_{x(0,n_x)}) = h_E(x)(s_{x(0,n_x)} s^*_{x(0,n_x)}) = 1.$$

Hence $y(0, n \wedge |x|) = x(0, n \wedge |x|)$ for all $n \in \mathbb{N}$. By symmetry, we also have that $y(0, n \wedge |y|) = x(0, n \wedge |y|)$ for all n. In particular, $|x| = |y|$ and $y(0, n) = x(0, n)$ for all $n \leq |x|$. Thus $x = y$.

Recall that $\Delta(D)$ carries the topology of pointwise convergence. For openness, it suffices to check that h_E^{-1} is continuous. Suppose that $h(x^n) \to h(x)$. Fix a basic open set $\mathcal{Z}(\mu)$ containing x, so $h(x)(s_\mu s_\mu^*) = 1$. Since $h(x^n)(s_\mu s_\mu^*) \in \{0, 1\}$ for all n, for large enough n, we have $h(x^n)(s_\mu s_\mu^*) = 1$. So $x^n \in \mathcal{Z}(\mu)$. For continuity, a similarly straightforward argument shows that if $x^n \to x$, then $h(x^n)(s_\mu s_\mu^*) \to h(x)(s_\mu s_\mu^*)$. This convergence extends to $\operatorname{span}\{s_\mu s_\mu^* : \mu \in E^*\}$ by linearity and to D by an $\varepsilon/3$ argument. $\square$

We can now prove our main result.

Proof of Theorem 3.2. The projection p from Remark 3.1 satisfies

$$(3.4) \qquad p t_\mu t_\mu^* p = \begin{cases} t_\mu t_\mu^* & \text{if } r(\mu) \in E^0, \\ 0 & \text{otherwise.} \end{cases}$$

We will show that π maps D_E onto pD_Fp. It follows from (3.4) that $\pi(D_E) \subset pD_Fp$. To see the reverse inclusion, fix $\mu \in F^*$. If $r(\mu) \notin E^0$, then $pt_\mu t_\mu^* p = 0 \in \pi(D_E)$, so suppose that $r(\mu) \in E^0$. If $s(\mu) \in E^0$, then $pt_\mu t_\mu^* p = t_\mu t_\mu^* = \pi(s_{\phi(\mu)} s^*_{\phi(\mu)}) \in \pi(D_E)$. Now suppose that $s(\mu) \notin E^0$; then $s(\mu) = s(\nu_n)$ for some collapsible path $\nu \in F^\infty$ and $n \in \mathbb{N}$. Since ν has no exits except at $r(\nu)$, we have $\mu = \mu'\nu_n$ for $\mu' = \mu(0, |\mu| - 1)$. Furthermore, $s(\mu')F^1$ is finite; thus (CK3) implies that

$$(3.5) \qquad ps_\mu s_\mu^* p = ps_{\mu'} s_{\nu_n} s_{\nu_n}^* s_{\mu'}^* p = ps_{\mu'} s_{\mu'}^* p - \sum_{f \in s(\mu')F^1 \setminus \{\nu_n\}} ps_{\mu'} s_f s_f^* s_{\mu'}^* p.$$

An induction on n gives $ps_{\mu'} s_{\mu'}^* p \in \pi(D_E)$. It then follows from (3.5) that $ps_\mu s_\mu^* p \in \pi(D_E)$, and hence $\pi(D_E) = pD_Fp$.

We now construct the homeomorphism h. Since p commutes with D_F, the space pD_Fp is an ideal of D_F. Then Propositions A26(a) and A27(b) in [14] imply that the map $k : \phi \mapsto \phi|_{pD_Fp}$ is a homeomorphism of $\{\phi \in \Delta(D_F) : \phi|_{pD_Fp} \neq 0\}$ onto $\Delta(pD_Fp)$. Since F has no singular vertices, $\partial F = F^\infty$. Let $h_F : F^\infty \to \Delta(D_F)$ be the homeomorphism obtained from Theorem 3.7. Then $h_F(x) \in \operatorname{dom}(k)$ for all $x \in E^0F^\infty$. Define $h := k \circ h_F|_{E^0F^\infty} : E^0F^\infty \to \Delta(pD_Fp)$.

We aim to show that $h_E \circ \phi_\infty = \pi^* \circ h$. Let $x \in E^0F^\infty$, and fix $\mu \in E^*$. Since $(h_E \circ \phi_\infty)(x)$ and $h(x)$ are homomorphisms, and since π is an isomorphism, it suffices to show that

$$(h_E \circ \phi_\infty)(x)(s_\mu s_\mu^*) = (\pi^* \circ h)(x)(s_\mu s_\mu^*). \tag{3.6}$$

Since $\mu \in E^*$, we have $t_{\phi^{-1}(\mu)} t^*_{\phi^{-1}(\mu)} \in pD_F p$. Then since $r(x) \in E^0$, the right-hand side of (3.6) becomes

$$\begin{aligned}\pi^*(h(x))(s_\mu s_\mu^*) = h(x)(t_{\phi^{-1}(\mu)} t^*_{\phi^{-1}(\mu)}) &= h_F(x)|_{pD_Fp}(t_{\phi^{-1}(\mu)} t^*_{\phi^{-1}(\mu)}) \\ &= \begin{cases} 1 & \text{if } x \in \mathcal{Z}(\phi^{-1}(\mu)), \\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

We break the left-hand side of (3.6) into cases: (i) $\phi_\infty(x) \in E^\infty$ or (ii) $\phi_\infty(x) \in E^*$. In case (i), since $\phi_\infty(x) \in \mathcal{Z}(\mu)$ if and only if $x = \phi^{-1}(\mu)\phi_\infty^{-1}(\mu')$ for some $\mu' \in E^\infty$, the left-hand side of (3.6) becomes

$$h_E(\phi_\infty(x))(s_\mu s_\mu^*) = \begin{cases} 1 & \text{if } \phi_\infty(x) \in \mathcal{Z}(\mu), \\ 0 & \text{otherwise}\end{cases} = \begin{cases} 1 & \text{if } x \in \mathcal{Z}(\phi^{-1}(\mu)), \\ 0 & \text{otherwise.}\end{cases}$$

In case (ii), $\phi_\infty(x) = \phi(x')$, where $x = x'\nu$ for some collapsible path $\nu \in M$. The left-hand side of (3.6) then becomes

$$h_E(\phi(x'))(s_\mu s_\mu^*) = \begin{cases} 1 & \text{if } \phi(x') \in \mathcal{Z}(\mu), \\ 0 & \text{otherwise.}\end{cases}$$

Since ϕ is a bijection, $x' = \phi^{-1}(\mu)x''$ if and only if $\phi(x') = \mu\phi(x'')$, so equation (3.6) is satisfied, and thus $h_E \circ \phi_\infty(x) = \pi^* \circ h(x)$. □

Acknowledgements

The work contained in this paper is from the author's Ph.D. thesis. As such the author extends many thanks to his Ph.D. supervisors, Iain Raeburn and Aidan Sims, for their support and willingness to proofread and guide his work.

References

[1] Teresa Bates, Jeong Hee Hong, Iain Raeburn, and Wojciech Szymański, *The ideal structure of the C^*-algebras of infinite graphs*, Illinois J. Math. **46** (2002), no. 4, 1159–1176. MR1988256 (2004i:46105)

[2] John Crisp and Marcelo Laca, *Boundary quotients and ideals of Toeplitz C^*-algebras of Artin groups*, J. Funct. Anal. **242** (2007), no. 1, 127–156, DOI 10.1016/j.jfa.2006.08.001. MR2274018 (2007k:46117)

[3] Joachim Cuntz and Wolfgang Krieger, *A class of C^*-algebras and topological Markov chains*, Invent. Math. **56** (1980), no. 3, 251–268, DOI 10.1007/BF01390048. MR561974 (82f:46073a)

[4] D. Drinen, *Viewing AF-algebras as graph algebras*, Proc. Amer. Math. Soc. **128** (2000), no. 7, 1991–2000, DOI 10.1090/S0002-9939-99-05286-7. MR1657723 (2000m:46114)

[5] D. Drinen and M. Tomforde, *The C^*-algebras of arbitrary graphs*, Rocky Mountain J. Math. **35** (2005), no. 1, 105–135, DOI 10.1216/rmjm/1181069770. MR2117597 (2006h:46051)

[6] Masatoshi Enomoto and Yasuo Watatani, *A graph theory for C^*-algebras*, Math. Japon. **25** (1980), no. 4, 435–442. MR594544 (83d:46069a)

[7] Cynthia Farthing, Paul S. Muhly, and Trent Yeend, *Higher-rank graph C^*-algebras: an inverse semigroup and groupoid approach*, Semigroup Forum **71** (2005), no. 2, 159–187, DOI 10.1007/s00233-005-0512-2. MR2184052 (2006h:46052)

[8] Alex Kumjian and David Pask, *Higher rank graph C^*-algebras*, New York J. Math. **6** (2000), 1–20. MR1745529 (2001b:46102)

[9] Alex Kumjian, David Pask, Iain Raeburn, and Jean Renault, *Graphs, groupoids, and Cuntz-Krieger algebras*, J. Funct. Anal. **144** (1997), no. 2, 505–541, DOI 10.1006/jfan.1996.3001. MR1432596 (98g:46083)

[10] Marcelo Laca and Iain Raeburn, *Phase transition on the Toeplitz algebra of the affine semigroup over the natural numbers*, Adv. Math. **225** (2010), no. 2, 643–688, DOI 10.1016/j.aim.2010.03.007. MR2671177 (2012g:46102)

[11] Alan L. T. Paterson, *Graph inverse semigroups, groupoids and their C^*-algebras*, J. Operator Theory **48** (2002), no. 3, suppl., 645–662. MR1962477 (2004h:46066)

[12] Alan L. T. Paterson and Amy E. Welch, *Tychonoff's theorem for locally compact spaces and an elementary approach to the topology of path spaces*, Proc. Amer. Math. Soc. **133** (2005), no. 9, 2761–2770 (electronic), DOI 10.1090/S0002-9939-05-08030-5. MR2146225 (2006c:54008)

[13] Iain Raeburn, *Graph algebras*, CBMS Regional Conference Series in Mathematics, vol. 103, published for the Conference Board of the Mathematical Sciences, Washington, DC, by the Amer. Math. Soc., Providence, RI, 2005. MR2135030 (2005k:46141)

[14] Iain Raeburn and Dana P. Williams, *Morita equivalence and continuous-trace C^*-algebras*, Mathematical Surveys and Monographs, vol. 60, American Mathematical Society, Providence, RI, 1998. MR1634408 (2000c:46108)

[15] Jean Renault, *A groupoid approach to C^*-algebras*, Lecture Notes in Mathematics, vol. 793, Springer, Berlin, 1980. MR584266 (82h:46075)

[16] Aidan Sims and Samuel B. G. Webster, *A direct approach to co-universal algebras associated to directed graphs*, Bull. Malays. Math. Sci. Soc. (2) **33** (2010), no. 2, 211–220. MR2666424 (2011d:46114)

[17] Wojciech Szymański, *The range of K-invariants for C^*-algebras of infinite graphs*, Indiana Univ. Math. J. **51** (2002), no. 1, 239–249, DOI 10.1512/iumj.2002.51.1920. MR1896162 (2003b:46077)

[18] S. B. G. Webster, Directed Graphs and k-graphs: Topology of the Path Space and How It Manifests in the Associated C^*-algebra, Ph.D. Thesis, University of Wollongong, 2010.

School of Mathematics and Applied Statistics, University of Wollongong, NSW 2522, Australia

E-mail address: sbgwebster@gmail.com