# Group divisible designs, GBRDSDS and generalized weighing matrices

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## Abstract

We give new constructions for regular group divisible designs, pairwise balanced designs, generalized Bhaskar Rao supplementary difference sets and generalized weighing matrices. In particular if p is a prime power and q divides p - 1 we show the following exist;

(i) GDD $(2(p^2+p+1), 2(p^2+p+1), rp^2, 2p^2, \lambda 1 = p^2\lambda, \lambda 2 = (p^2-p)r, m=p^2+p+1, n=2)$, r_+1,2;

(ii) GDD$(q(p+1), q(p+1), p(q-1), p(q-1), \lambda 1=(q-1)(q-2), \lambda_2=(p-1)(q-1)^2/q, m=q, n=p+1)$;

(iii) PBD(21,10;K),K={3,6,7} and PDB(78,38;K), K={6,9,45};

(iv) GW$(vk,k^2;EA(k))$ whenever a $(v,k,\lambda)$-difference set exists and k is a prime power;

(v) PBIBD$(vk^2,vk^2,k^2,k^2;\lambda_1=0,\lambda_2=\lambda,\lambda_3=k)$ whenever a $(v,k,\lambda)$-difference set exists and k is a prime power;

(vi) we give a GW$(21;9;Z_3)$.


## Disciplines

Physical Sciences and Mathematics


## Publication Details

Dinesh Sarvate and Jennifer Seberry, Group divisible designs, GBRDSDS and generalized weighing matrices, Utilitas Mathematica, 54, (1998), 157-174.

# GROUP DIVISIBLE DESIGNS, GBRSDS AND GENERALIZED WEIGHING MATRICES


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June 11, 1998


**Abstract**

We give new constructions for regular group divisible designs, pairwise balanced designs, generalized Bhaskar Rao supplementary difference sets and generalized weighing matrices. In particular if $p$ is a prime power and $q$ divides $p-1$ we show the following exist:

(i) $GDD(2(p^2+p+1), 2(p^2+p+1), rp^2, 2p^2, \lambda_1 = p^2\lambda, \lambda_2 = (p^2-p)r, m = p^2+p+1, n = 2)$, $r = 1, 2$;

(ii) $GDD(q(p+1), q(p+1), p(q-1), p(q-1), \lambda_1 = (q-1)(q-2), \lambda_2 = (p-1)(q-1)^2/q, m = q, n = p+1)$;

(iii) $PBD(21, 10; K)$, $K = \{3, 6, 7\}$ and $PBD(78, 38; K)$, $K = \{6, 9, 45\}$;

(iv) $GW(vk, k^2; EA(k))$ whenever a $(v, k, \lambda)$-difference set exists and $k$ is a prime power;

(v) $PBIBD(vk^2, vk^2, k^2, k^2; \lambda_1 = 0, \lambda_2 = \lambda, \lambda_3 = k)$ whenever a $(v, k, \lambda)$-difference set exists and k is a prime power;

(vi) we give a $GW(21; 9; Z_3)$.

The $GDD$ obtained are not found in W.H. Clatworthy, *Tables of Two-Associate-Class, Partially Balanced Designs*, NBS, US Department of Commerce, 1971.


## 1 INTRODUCTION

In this paper we set out to explore the usefulness of Bhaskar Rao designs and generalized matrices in the construction of $GDD$ and found them to be very rich indeed.

A design is a pair $(X, B)$ where $X$ is a finite set of elements and $B$ is a collection of (not necessarily distinct) subsets $B_i$ (called blocks) of $X$.

A *balanced incomplete block design*, $BIBD(v, b, r, k, \lambda)$, is an arrangement of $v$ elements into $b$ blocks such that:

(i) each element appears in exactly $r$ blocks;

(ii) each block contains exactly $k(< v)$ elements; and

(iii) each pair of distinct elements appear together in exactly $\lambda$ blocks.


*Research supported by Telecom grant 7027 and ARC grant A48830241

As $r(k-1)=\lambda(v-1)$ and $vr=bk$ are well known necessary conditions for the existence of a $BIBD(v,b,r,k,\lambda)$ we denote this design by $BIBD(v,k,\lambda)$.

Let $v$ and $\lambda$ be positive integers and $K$ a set of positive integers.

An arrangement of the elements of a set $X$ into blocks is a *pairwise balanced design*, $PBD(v;K;\lambda)$, if:

(i) $X$ contains exactly $v$ elements;

(ii) if a block contains $k$ elements then $k$ belongs to $K$;

(iii) each pair of distinct elements appear together in exactly $\lambda$ blocks.

A pairwise balanced design $PBD(v;\{k\};\lambda)$, that is where $K=\{k\}$ consists of exactly one integer, is a $BIBD(v,k,\lambda)$. It is well known that a $PBD(v-1;\{k,k-1\};\lambda)$ can be obtained from the $BIBD(v,b,r,k,\lambda)$.

For the definition of a partially balanced incomplete block design with $m$ associate classes ($PBIBD(m)$) see Raghavarao [34] or Street and Street [53].

A *generalized Bhaskar Rao design*, $W$ is defined as follows. Let $W$ be a $v\times b$ matrix with entries from $G\bigcup\{0\}$ where $G=\{h_1=e,h_2,\ldots,h_g\}$ is a finite group of order $g$. $W$ is then expressed as a sum $W=h_1A_1+\cdots+h_gA_g$, where $A_1,\ldots,A_g$ are $v\times b$ $(0,1)$ matrices such that the Hadamard product $A_i\star A_j=0$ for any $i\neq j$.

Denote by $W^+$ the transpose of $h_1^{-1}A_1+\cdots+h_g^{-1}A_g$ and let $N=A_1+\cdots+A_g$. In this paper we are concerned with the special case where $W$, denoted by $GBRD(v,b,r,k,\lambda;G)$, satisfies

(i) $WW^+=\mathrm{re}I+\frac{\lambda}{g}(h_1+\cdots+h_g)(J-I)$, and

(ii) $NN^T=(r-\lambda)I+\lambda J$.

It can be seen that the second condition requires that $N$ be the incidence matrix of a $BIBD(v,b,r,k,\lambda)$ and thus we can use the shorter notation $GBRD(v,k,\lambda;G)$ for a generalized Bhaskar Rao design. A $GBRD(v,k,\lambda;Z_2)$ is also referred to as a $BRD(v,k,\lambda)$.

A $GBRD(v,k,\lambda;G)$ with $v=b$ is a symmetric $GBRD$ or generalized weighing matrix, but a generalized weighing matrix, $W=GW(v,k;G)$ is also used for any square matrix satisfying $WW^+=\mathrm{ke}I$ where $h_1+\cdots+h_g=0$ is used (as in the $g$th roots of unity). If $W$ has no 0 entries the $GBRD$ is also known as a generalized Hadamard matrix ($GH$).

A *group divisible design*, $GDD(v,b,r,k,\lambda_1,\lambda_2,m,n)$, on $v$ points is a triple $(X,S,A)$ where

(i) $X$ is a set (of points), where $|X|=v$,

(ii) $S$ is a class of non-empty subsets $X$ (called groups), of size $n$, which partitions $X$, and $|S|=n$,

(iii) $A$ is a class of subsets of $X$ (called blocks), each containing at least two points, and $|A|=b$,

(iv) each pair of distinct points $\{x,y\}$ where $x$ and $y$ are from the same group is contained in precisely $\lambda_1$ blocks.

(v) each pair of distinct points $\{x,y\}$ where $x$ and $y$ are not from the same group is contained in precisely $\lambda_2$ blocks.

In general, the number of elements in a group is denoted by $n$.

Bhaskar Rao designs with elements $0, \pm 1$ have been studied by a number of authors including Bhaskar Rao [3, 4], Seberry [42, 44], Singh [48], Sinha [49], Street [51], Street and Rodger [52] and Vyas [54]. Bhaskar Rao [3] used these designs to construct partially balanced designs and this was improved by Street and Rodger [52] and Seberry [44]. Another technique for studying partially balanced designs has involved looking at generalized orthogonal matrices which have elements from elementary abelian groups and the element 0. Matrices with group elements as entries have been studied by Berman [1, 2], Butson [5, 6], Delsarte and Goethals [13], Drake [16], Rajkundlia [35], Seberry [40, 41], Shrikhande [47] and Street [50].

Generalized Hadamard matrices has been studied by Street [50], Seberry [40, 41], Dawson [8], and de Launey [9, 10].

Bhaskar Rao designs over elementary abelian groups other than $Z_2$ have been studied by Lam and Seberry [26] and Seberry [45]. de Launey, Sarvate and Seberry [12] considered Bhaskar Rao designs over $Z_4$ which is an abelian (but not elementary) group. Some Bhaskar Rao designs over the non-abelian groups $S_3$ and $Q_4$ have been studied by Gibbons and Mathon [20].

Palmer and Seberry [33] study generalized Bhaskar Rao designs over the non-abelian groups $S_3$, $D_4$, $Q_4$, $D_6$ and over the small abelian group $Z_2 \times Z_4$. Seberry [46] completed the study of groups of order 8.

We use the following notation for initial blocks of a $GBRD$. We say $(a_\alpha, b_\beta, \ldots, c_\gamma)$ is an initial block, when the Latin letters are developed mod $v$ and the Greek subscripts are the elements of the group, which will be placed in the incidence matrix in the position indicated by the Latin letter. For example in the $(i, b-1+i)$th position we place $\beta$ and so on.

We form the difference table of an initial block $(a_\alpha, b_\beta, \ldots, c_\gamma)$ by placing in the position headed by $x_\delta$ and by row $y_\eta$ the element $(x-y)_{\delta\eta^{-1}}$ where $(x-y)$ is mod $v$ and $\delta\eta^{-1}$ is in the group.

By the term *totality* of elements we mean that repetitions remain: hence the set union of $\{1,2,3,4,5,6\} \bigcup \{3,4,7,8\} = \{1,2,3,4,5,6,7,8\}$ while the totality of elements in the two sets $\{1,2,3,4,5,6\}$ & $\{3,4,7,8\} = \{1,2,3,3,4,4,5,6,7,8\}$. The symbol & is sometimes written as $\uplus$.

A set of initial blocks will be said to form a $GBR$ difference set (if there is one initial block) or $GBR$ supplementary difference sets (if more than one) if in the totality of elements

$$(x-y)_{\delta\eta^{-1}} \pmod{v, G}$$

each non-zero element $a_g$, $a \pmod{v}$, $g \in G$, occurs $\lambda / \mid G \mid$ times.

Examples of the use of these $GBR$ supplementary difference sets ($GBRSDS$) are given in Seberry [42].

## 2 GROUP DIVISIBLE DESIGNS

Let $B$ be the incidence matrix of a $BIBD(v, b, r, k, \lambda)$. Let $A$ be the matrix formed from a $GBRD(V, B, R, j, tv; G)$, where $|G| = v$, by replacing each zero of the $GBRD$ by the $v \times v$ zero matrix and each group element of the $GBRD$ by the right regular permutation matrix representation from the group $EA(v)$.

Then $A$ is a $GDD(vV, vB, R, j,\ \lambda_1 = 0,\ \lambda_2 = t,\ m = V,\ n = v)$.

**Lemma 1** *Suppose there exists a* $BIBD(v, b, r, k, \lambda)$, $Y$, *and a* $GBRD(V, B, R, j, tv; G)$, $A$, *with* $|G| = v$. *Then there exists a* $GDD(vV, bB, rR, jk,\ \lambda_1 = R\lambda,\ \lambda_2 = trk,\ m = V,\ n = v)$.

**Proof.** Let $C = A \times Y$, where the group element $g_i$ of $G$ with matrix representation $G_i$ is replaced by $G_iY$ and zero by the $v \times b$ zero matrix. Then all the parameters of $C$ except $\lambda_1$ and $\lambda_2$ are immediate. The inner product of any two rows of $Y$ is $\lambda$ and $G_iY$ also has inner product of rows $\lambda$. $G_iY$'s occur $R$ times in each row of $C$ so $\lambda_1 = R\lambda$.

The inner product of rows of the $GBRD$ gives t copies of the group so the contribution to the inner product of rows of different $GDD$ groups is

$$\underset{g_i, g_j \in G}{\&} \quad tG_iYY^TG_j^{-1} = \underset{g_i, g_j \in G}{\&} \quad tG_i(r-\lambda)I + \lambda J)G_j^{-1} = t(r - \lambda + \lambda v)J.$$

Hence $\lambda_2 = trk$. Another way to check that $\lambda_2 = trk$ is to observe that we will have as inner product

$$t(G_1 + \cdots + G_v)YY^T = t(J)((r-\lambda)I + \lambda J) = t(r-\lambda)J + t(\lambda v)J,$$

Now use $\lambda(v-1) = r(k-1)$ to get the result. □

**Example 1** Let the $Y = SBIBD(3,2,1)$ and $A$ be the $GBRD(6,6,6;Z_3)$. Then the Lemma gives us $GDD(18,18,12,12,\ \lambda_1 = 6,\ \lambda_2 = 8,\ m = 6,\ n = 3)$ which is given below and which is not found in Clatworthy's tables [7].

**Example 2** Let A be formed from the $GBRD(7,21,9,3,3;Z_3) = D$ which may be written as:

$$D = \left[\begin{array}{ccccccc|ccccccc|ccccccc}
0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & \omega & 0 & \omega^2 & 0 & 0 & 0 & 1 & \omega^2 & 0 & w & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & \omega & 0 & \omega^2 & 0 & 0 & 0 & 1 & \omega^2 & 0 & w & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & \omega & 0 & \omega^2 & 0 & 0 & 0 & 1 & \omega^2 & 0 & w \\
1 & 0 & 0 & 0 & 1 & 1 & 0 & \omega^2 & 0 & 0 & 0 & 1 & \omega & 0 & w & 0 & 0 & 0 & 1 & \omega^2 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & \omega^2 & 0 & 0 & 0 & 1 & \omega & 0 & w & 0 & 0 & 0 & 1 & \omega^2 \\
1 & 0 & 1 & 0 & 0 & 0 & 1 & \omega & 0 & \omega^2 & 0 & 0 & 0 & 1 & \omega^2 & 0 & w & 0 & 0 & 0 & 1 \\
1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & \omega & 0 & \omega^2 & 0 & 0 & 0 & 1 & \omega^2 & 0 & w & 0 & 0 & 0
\end{array}\right]$$

Let $B = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$, $\quad T = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$, $\quad$ and $\quad \mathbf{O} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

We form $C$ from $A$ by replacing 0 by $\mathbf{O}$ and $\omega^i$ by $T^iB$. Then $C$ is a $GDD(21,\ 63,\ 18,\ 6,\ \lambda_1 = 9,\ \lambda_2 = 4,\ m = 7,\ n = 3)$.

**Corollary 2** *A* $GBRD(p^2+p+1,\ p^2+p+1,\ p^2,\ p^2,\ p^2-p;\ Z_2)$ *always exists for* $p$ *a prime power so a* $GDD(2(p^2+p+1),\ 2(p^2+p+1),\ rp^2,\ kp^2,\ \lambda_1 = p^2\lambda,\ \lambda_2 = (p^2-p)r)$ *exists where* $(k,r,\lambda) = (2,2,2)$ *or* $(1,1,0)$.

**Corollary 3** *A* $GBRD(p+1,p,p-1;\ Z_q)$ *exists for every prime power* $p$, *if* $q$ *divides* $p-1$. *An* $SBIBD(q,q-1,q-2)$ *exists. Hence there exists a* $GDD(q(p+1),\ q(p+1),\ p(q-1),\ p(q-1),\ \lambda_1 = (q-1)(q-2),\ \lambda_2 = (p-1)(q-1)^2/q,\ m = p+1, n = q)$.

**Corollary 4** *If a* $GBRD(V,B,R,j,tv;\ EA(v))$ *exists then a* $GDD(vV, vB, R(v-1), j(v-1),\ \lambda_1 = R(v-2),\ \lambda_2 = t(v-1)^2,\ m = V,\ n = v)$ *exists.*

**Proof.** An $SBIBD(v,v-1,v-2)$ always exists. □

**Example 2** (continued). There exists a $GH(3t, Z_3) = H, 3t > 7$, and write $SBIBD(7,3,1) = S$. Take seven rows of $H$ and replace its elements by 0 and $T^i$ to give,

$$H' = GDD(21, 9t, 3t, 7,\ \lambda_1 = 0,\ \lambda_2 = t,\ m = 7,\ n = 3).$$

Replace the zeros and ones of $S$ by the $3 \times 1$ matrix of zeros and ones respectively to form,

$$S' = GDD(21, 7, 3, 9,\ \lambda_1 = 3,\ \lambda_2 = 1,\ m = 7,\ n = 3).$$

We note $L = I_7 \times J_{3,1}$ is a $GDD(21, 7, 1, 3,\ \lambda_1 = 1,\ \lambda_3 = 0,\ m = 7,\ n = 3)$.
Then

$$C_1 = [C : H'(t = 3) : H'(t = 4) : S']$$

is a $PBD(21, 12; K)$, where $K = \{6, 7, 9\}$, and

$$C_2 = [C : H'(t = 6) : L]$$

is a $PBD(21, 10; K)$, where $K = \{6, 7, 3\}$.

Table 1 gives some GDD on 21 varieties which exist.

| $v$ | $k$ | $m$ | $n$ | $\lambda_1$ | $\lambda_2$ | Comment |
|---|---|---|---|---|---|---|
| 21 | 12 | 7 | 3 | 4 | 2 | from $SBIBD(7,4,2)$ |
| 21 | 7 | 7 | 3 | 0 | $t$ | (a $GH(3t, Z_3)$ exists, $3t > 7$) |
| 21 | 18 | 7 | 3 | 6 | 5 | $(J - I)_7 \times J_{3,1}$ |
| 21 | 3 | 7 | 3 | 1 | 0 | $I_7 \times J_{3,1}$ |
| 21 | 9 | 7 | 3 | 3 | 1 | from $SBIBD(7,3,1)$ |
| 21 | $k$ | 7 | 3 | $r$ | $\lambda$ | from $BIBD(7, b, r, k, \lambda)$ |
| 21 | $K$ | 7 | 3 | 0 | $\Delta$ | from $GBRD(7, B, R, K, 3\Delta; Z_3)$ |

Table 1

**Remark 1** The $GDD$s with $\lambda_1 = r$ in Table 1 and in Table 2 can be constructed from a $BIBD$ and are singular but we have listed these parameters for easy reference so as to be able to apply them in the following Lemma and the Table 2 parameters in Lemma 6.

Clatworthy's tables [7] give $R188$ with $v = b = 21$, $k = r = 8$, $m = 7$, $n = 3$, $\lambda_1 = 7$, $\lambda_2 = 1$ but no $GDD$ with $v = b = 21$, and the designs of Table 1 appear to be new.

**Lemma 5** *Combinations from Table 1 can be used to give $PBD(21, \mu; K)$ for many $\mu$ and $k_i \in K$.*

Glynn [21] has found a $GW(13, 9, 6; S_3)$ which is circulant with the following first row: $[o\ e\ a\ d\ o\ a\ f\ e\ f\ o\ d]$ where $a = (1)$, $b = (123)(456)$, $c = (132)(465)$, $d = (14)(26)(35)$, $e = (15)$.

**Example 7** Using Lemma 1 with the $SBIBD(6,5,4)$ we get

$$C = GDD(78, 78, 45, 45,\ \lambda_1 = 36,\ \lambda_2 = 20,\ m = 13,\ n = 6).$$

| No. | $v$ | $k$ | $m$ | $n$ | $\lambda_1$ | $\lambda_2$ | Comment |
|---|---|---|---|---|---|---|---|
| $B_1$ | 78 | 54 | 13 | 6 | 9 | 6 | from $SBIBD(13,9,6)$ |
| $B_2$ | 78 | 45 | 13 | 6 | 36 | 20 | above |
| $B_3$ | 78 | 13 | 13 | 6 | 0 | $t$ | (if a $GH(6t,G)$ exists $\|G\|=b$, $6t>13:$ none are known), |
| $B_4$ | 78 | 72 | 13 | 6 | 12 | 11 | $(J-I)13\times J_{6,1}$ |
| $B_5$ | 78 | 6 | 13 | 6 | 1 | 0 | $I_{13}\times J_{6,1}$ |
| $B_6$ | 78 | 24 | 13 | 6 | 4 | 1 | from $SBIBD(13,4,1)$ |
| $B_7$ | 78 | $k$ | 13 | 6 | $r$ | $\lambda$ | from $BIBD(13,b,r,k,\lambda)$ |
| $B_8$ | 78 | $K$ | 13 | 6 | 0 | $\Delta$ | from $GBRD(13,B,R,K,6\Delta;S_3)$ |
| $B_9$ | 78 | 9 | 13 | 6 | 0 | 6 | from $GW(13,9,6;S_3)$ |

Table 2

Table 2 gives some $GDD$ on 78 varieties which exist.

We note that a $PBD(78,38,K)$ with $K=\{6,9,45\}$ can be formed by taking

$$[B2:2 \text{ copies } B5:3 \text{ copies } B9].$$

Clatworthy's tables list $R201$ which has $v=b=78$, $r=k=9$, $m=13$, $n=6$ but all the other designs in Table 2 appear to be new.

**Lemma 6** *Combinations from Table 2 can be used to give $PBD(78,\mu;K)$ for many $\mu$ and $K$.*

# 3 GENERALIZED SUPPLEMENTARY DIFFERENCE SETS

We slightly extend a Lemma of de Launey and Seberry [11, Lemma 6.1.1] to get a new result.

**Theorem 7** *Suppose there exist $n$-$\{v;k;\lambda\}$ supplementary difference sets and a square $GBRD$ $(k,j,tg;G)$, $Y=(y_{su})$, where $|G|=g$. Then there exist $nk-\{v;j;t\lambda g;G\}-GBRSDS$.*

**Proof.** Let the $n-\{v;k;\lambda\}$ $SDS$, $D_i, i=1,\dots,n$, have elements $d^i_1,d^i_2,\dots,d^i_k$.
Using the $GBRD(y_{su})$ we form $nk$ $GBRSDS$ by choosing the initial blocks

$$d^i_{1_{y_{1u}}}, d^i_{2_{y_{2u}}},\dots,d^i_{k_{y_{ku}}}, i=1,2,\dots,n; u=1,2,\dots,k,$$

where if $y_{su}$ is 0, then we remove $d^i_{s_{y_{su}}}$ from the block (see Example 8).
These blocks are developed modulo $v$ so that in a block, developed from an initial block with $y_{au}$ in position $(1,a)$, position $(1+b,a+b)$ is also $y_{au}$. Note that $1+b$ and $a+b$ are both reduced modulo $v$.
Because the initial sets, $D_i$, had each element $1,\ 2,\ \dots,\ v-1$ occurring as the solution of the equation

$$d^i_a - d^i_b,\ i\in\{1,\ \dots,\ n\},\ a,\ b\in\{1,\ \dots,\ k\}$$

exactly $\lambda$ times, the new design will have

$$g_{aj}g^{-1}_{bj},\ j=1,\ \dots,\ k,\ \ a,\ b\in\{1,\ \dots,\ k\}$$

occurring $\lambda tg$ times. Hence we have the starting blocks of an $nk-\{v;j;t\lambda g;G\}-GBRSDS$.
□

**Corollary 8** *Let $p \equiv 1 \pmod 4$ be a prime power. Then there exist $2 - \{p; \frac{1}{2}(p-1); \frac{1}{2}(p-3)\}$ SDS. Suppose there exists a $GBRD((p-1)/2, k, tg; G)$ where $|G| = g$. Then there exist $(p-1) - \{p; k; tg(p-3)/2; G\} - GBRSDS$.*

**Example 8** We use the $GBRD(5, 5, 4, 4, 3; Z_3)$

$$Y = (y_{iu}) = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & \omega & \omega^2 \\ 1 & 1 & 0 & \omega^2 & \omega \\ 1 & \omega & \omega^2 & 0 & 1 \\ 1 & \omega^2 & \omega & 1 & 0 \end{bmatrix} \text{ written as } \begin{bmatrix} \star & 0 & 0 & 0 & 0 \\ 0 & \star & 0 & 1 & 2 \\ 0 & 0 & \star & 2 & 1 \\ 0 & 1 & 2 & \star & 0 \\ 0 & 2 & 1 & 0 & \star \end{bmatrix}.$$

Attaching $\star$ to an element is the same as multiplying it by zero in multiplicative notation and so removes that element from the starting block.

Now there are $3 - \{7; 5; 10\}$ $SDS$ namely $\{0,1,2,3,4\}$, $\{0,1,2,4,5\}$ and $\{0,1,2,3,5\}$. So we make 15 starting blocks

$$\begin{array}{lllll} \{1_0, 2_0, 3_0, 4_0\}, & \{0_0, 2_0, 3_1, 4_2\}, & \{0_0, 1_0, 3_2, 4_1\}, & \{0_0, 1_1, 2_2, 4_0\}, & \{0_0, 1_2, 2_1, 3_0\}, \\ \{1_0, 2_0, 4_0, 5_0\}, & \{0_0, 2_0, 4_1, 5_2\}, & \{0_0, 1_0, 4_2, 5_1\}, & \{0_0, 1_1, 2_2, 5_0\}, & \{0_0, 1_2, 2_1, 4_0\}, \\ \{1_0, 2_0, 3_0, 5_0\}, & \{0_0, 2_0, 3_1, 5_2\}, & \{0_0, 1_0, 3_2, 5_1\}, & \{0_0, 1_1, 2_2, 5_0\}, & \{0_0, 1_2, 2_1, 3_0\}, \end{array}$$

which give a $15 - \{7; 4; 30; Z_3\} - GBRSDS$.

Applying the same method to the $(11, 5, 2)$-difference set $\{1, 3, 4, 5, 9\}$ gives 5 starting blocks, $D_i^1, i = 1, \ldots, 5$, namely $\{3_0, 4_0, 5_0, 9_0\}$, $\{1_0, 4_0, 5_1, 9_2\}$, $\{1_0, 3_0, 5_2, 9_1\}$, $\{1_0, 3_1, 4_2, 9_0\}$ and $\{1_0, 3_2, 4_1, 5_0\}$ which give a $5 - \{11; 4; 6; Z_3\} - GBRSDS$. (Note: superscript 1 in $D_i^1$ is not necessary in this example.)

**Corollary 9** *Let $p \equiv 3 \pmod 4$ be a prime power. Suppose there exists a $GBRD(v, b, r, k, tp; G)$, $|G| = p$. Then there exist a $GDD(vp, bp, \frac{1}{2}r(p-1),\ \frac{1}{2}k(p-1),\ \lambda_1 = \frac{1}{4}r(p-3),\ \lambda_2 = \frac{1}{4}t(p-1)^2,\ m = v,\ n = p)$ and a $GDD(vp, bp, \frac{1}{2}r(p+1), \frac{1}{2}k(p+1),\ \lambda_1 = \frac{1}{4}r(p+1),\ \lambda_2 = \frac{1}{4}(p+1)^2,\ m = v,\ n = p)$.*

**Proof.** Use the $(p, \frac{1}{2}(p-1), \frac{1}{4}(p-3))$-difference set in the theorem or the $(p, \frac{1}{2}(p+1), \frac{1}{4}(p+1))$ difference set. □

**Corollary 10** *Let $p \equiv 3 \pmod 4$ and $p+1$ both be prime powers. Then there exist a $GDD(p(p+2),\ p(p+2),\ \frac{1}{2}(p^2-1),\ \frac{1}{2}(p^2-1),\ \lambda_1 = \frac{1}{4}(p+1)(p-3),\ \lambda_2 = \frac{1}{4}(p-1)^2,\ m = p+2,\ n = p)$ and an $SBIBD(p(p+2),\ \frac{1}{2}(p+1)^2,\ \frac{1}{4}(p+1)^2)$.*

**Proof.** Use the previous corollary and the $GBRD(p+2, p+1, p; Z_p)$. □

**Example 9** Over $GF(2^3)$ with the primitive equation $\gamma^3 = \gamma + 1$ we have

$$\gamma, \gamma^2, \gamma^3 = \gamma + 1, \gamma^4 = \gamma^2 + \gamma, \gamma^5 = \gamma^3 = \gamma + 1, \gamma^6 = \gamma^2 + 1, \gamma^7 = 1$$

and choosing $m_{00} = m_{ii} = 0$, $m_{0i} = m_{i0} = 1$, $i = 1, \ldots, 8$ and $m_{ij} = a^k$ if $\gamma^k = \gamma^j + \gamma^i$

| | | $0$ | $1$ | $\gamma$ | $\gamma^2$ | $\gamma^3$ | $\gamma^4$ | $\gamma^5$ | $\gamma^6$ |
|---|---|---|---|---|---|---|---|---|---|
| | $0$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ |
| $0$ | $1$ | $0$ | $1$ | $a$ | $a^2$ | $a^3$ | $a^4$ | $a^5$ | $a^6$ |
| $1$ | $1$ | $1$ | $0$ | $a^3$ | $a^2$ | $a$ | $a^5$ | $a^4$ | $a^2$ |
| $\gamma$ | $1$ | $a$ | $a^3$ | $0$ | $a^4$ | $1$ | $a^2$ | $a^6$ | $a^5$ |
| $\gamma^2$ | $1$ | $a^2$ | $a^2$ | $a^4$ | $0$ | $a^5$ | $a$ | $a^3$ | $1$ |
| $\gamma^3$ | $1$ | $a^3$ | $a$ | $1$ | $a^5$ | $0$ | $a^6$ | $a^2$ | $a^4$ |
| $\gamma^4$ | $1$ | $a^4$ | $a^2$ | $a^2$ | $a$ | $a^6$ | $0$ | $1$ | $a^3$ |
| $\gamma^5$ | $1$ | $a^5$ | $a^4$ | $a^6$ | $a^3$ | $a^2$ | $1$ | $0$ | $a$ |
| $\gamma^6$ | $1$ | $a^6$ | $a^2$ | $a^5$ | $1$ | $a^4$ | $a^3$ | $a$ | $0$ |

We map $0 \to 0_7$, $a^i \to T^iB$. If $B$ is an $SBIBD(7,k,\lambda)$ the new matrix has order 63 , row and column sum $8k$, $\lambda_1 = 8\lambda$ and $\lambda_2 = k^2$. So we have an $SBIBD(63,63,32,32,16)$ or a $GDD(63,63,24,24;\ \lambda_1 = 8,\ \lambda_2 = 9,\ m = 9,\ n = 7)$.

In general this construction takes a $GBRD(p+1,p,p-1)$ where $p$ is a prime power and an $SBIBD(p-1,k,\lambda)$ and makes a $GDD(p^2-1,\ p^2-1,\ pk,\ pk,\ \lambda_1 = p\lambda,\ \lambda_2 = k^2)$.

**Corollary 11** *Let $p \equiv 3 \pmod 4$ and $q = p-1$ both be prime powers. Then there exists a $GDD(p(q^2+q+1),\ p^2(q^2+q+1),\ \frac{1}{2}p^2(p-1),\ \frac{1}{2}p(p-1);\ \lambda_1 = \frac{1}{4}p^2(p-3),\ \lambda_2 = \frac{1}{4}(p-1)^2)$ and a $GDD(p(q^2+q+1),\ p^2(q^2+q+1),\ \frac{1}{2}p^2(p+1),\ \frac{1}{2}p(p-1);\ \lambda_1 = \frac{1}{4}p^2(p+1),\ \lambda_2 = \frac{1}{4}(p+1)^2,\ m = q^2+q+1,\ n = p)$.*

**Proof.** Use the $p-\{q^2+q+1;\ q+1;\ q+1;\ G\}-GBRSDS$, $|G| = p$, to make a $GBRD(q^2+q+1,\ p(q^2+q+1),\ p^2,\ p,\ p;\ G)$. Then use the $SBIBD(p,\ \frac{1}{2}(p-1),\ \frac{1}{4}(p-3))$ and $SBIBD(p,\ \frac{1}{2}(p+1),\ \frac{1}{4}(p+1))$ to obtain the second GDD of the enunciation. □

For example, we know that there exist a difference set for $SBIBD(7,3,1)$ and a $GBRD(3,3,3;Z_3)$. We apply Theorem 7 to get $3-\{7,3,3;Z_3\}-GBRSDS$.

Now we use Lemma 1 and the trivial $BIBD(3,1,0) = I_3$ to obtain a $GDD(21,63,9,3;0,1)$ and Lemma 1 and the $BIBD(3,2,1)$ to obtain a $GDD(21,63,9,3;9,4)$.

# 4 GENERALIZED WEIGHING MATRICES

Write $G = (g_{ij})$ for a symmetric $GH(k,G)$, $|G| = k$, where $G$ comprises the $k$th roots of unity, $1,\ \gamma, \ldots,\ \gamma^{k-1}$ with the relation $1+\gamma+\gamma^2+\ldots+\gamma^{k-1} = 0$. $G$ is in normalized form so $g_{0i} = g_{i0} = 1$, $i = 0, \ldots, k-1$ and $g_{ij} = \gamma^{ij}$.

Let $D = \{d_1, \ldots, d_k\}$ be a $(v,k,\lambda)$-difference set. Form the $k-\{v;k;k\lambda\}-GBRSDS$, $D_i = \{g_{i1}d_1,\ g_{i2}d_2, \ldots, g_{ik}d_k\}$, $i = 1, \ldots, k$. Call the matrices developed from $D_i$, $A_i$. Now we form a matrix, $W$, of order $k^2$ by choosing the circulant matrix with first now

$$[A_1:\ A_2:\ \ldots:\ A_k].$$

We claim $W$ is a generalized weighing matrix.

**Theorem 12** *If $k$ is a prime power and there exists a $(v,k,\lambda)$-difference set then there exists a $GW(vk,k^2;\ EA(k))$.*

**Proof.** We use the normalized $GH(k,\ EA(k))$, $G = (g_{ij})$ whose elements are the $k$th roots of unity as above. We form $W$ as above.

There are three products to check: the inner product of row $x$ and row $x+yk$, $y \neq 0$; the inner product of row $x$ and row $y$ where $x, y \in S_i = \{ik,\ ik+1,\ \ldots,\ ik+k-1,\ i = 1,\ \ldots,\ k\}$; the inner product of row $x$ and row $y$ where $x \in S_i$, $y \in S_j$, $i \neq j$.

The first row of $A_i$ has $a_{1,d_j} = g_{ij}$, $a_{1,n} = 0$ otherwise. Hence the $x$th row of $A_i$ has $a_{x,j} = a_{1,j-x+1} = g_{1m}$ if $j - x + 1 = d_m$ and $a_{xj} = 0$ otherwise.

**Case 1:** The inner product of the $x$th row and the $x + yk$th row of $W$ is

$$\sum_{z=0}^{k-1} \sum_{j=1}^{v} a_{x,j+zk} a^{-1}_{x+yk,j+zk}, \quad y \neq 0$$

$$\begin{aligned}
&= \sum_{z=0}^{k-1} \sum_{j=1}^{v} a_{1,j+zk-x+1} a^{-1}_{1,j+zk-x-yk+1}, \quad y \neq 0 \\
&= \sum_{z=0}^{k-1} \sum_{d_m \in D} g_{zm} g^{-1}_{(z-y),m}, \quad y \neq 0, \text{ if } d_m = j - x + 1,\ d_m \in D \\
&= 0
\end{aligned}$$

since $\sum_{d \in D} g_{zm} g^{-1}_{(z-y),m}$ is the inner product of two rows of the $GH$, $G$, for which $1 + \gamma + \gamma^2 + \ldots + \gamma^{k-1} = 0$.

**Case 2:** The inner product of the $x$th row and the $y$th row, $x,\ y \in S_i$ is

$$\sum_{z=0}^{k-1} \sum_{j=1}^{v} a_{x,j+zk} a^{-1}_{y,j+zk}, \qquad y \neq x$$

$$\begin{aligned}
&= \sum_{z=0}^{k-1} \sum_{d_m, d_n \in D} g_{zm} g^{-1}_{zn}, \quad d_m = j - x + 1, \quad d_n = j - y + 1, \quad d_m,\ d_n \in D \\
&= 0
\end{aligned}$$

since $\sum_{z=0}^{k-1} g_{zm} g^{-1}_{zn}$ is the inner product of two rows of the $GH$.

**Case 3:** The inner product of $x$th and $y$th rows, $x \in S_i$, $y \in S_j$, $i \neq j$ is

$$\sum_{z=0}^{k-1} \sum_{j=1}^{v} a_{x,j+zk} a^{-1}_{y,j+zk}$$

$$= \sum_{z=0}^{k-1} \sum_{d_m, d_n \in D} g_{zm} g^{-1}_{z+w,n}, \quad \text{some} \quad w \neq 0,$$

$$d_m = j - x + 1, \quad d_n = j - y + 1, \quad d_m,\ d_n \in D.$$

( The $w$ reflects that where $x, y$ come from different $S_i$, the elements of row $y$ have all been incremented by the same fixed constant ($w$) due to the block cyclic structure of $W$ )

$$\begin{aligned}
&= \sum_{z=0}^{k-1} \sum_{d_m, d_n \in D} \gamma^{zm} \gamma^{-zn-wn} \\
&= \sum_{z=0}^{k-1} \sum_{d_m, d_n \in D} \gamma^{z(m-n)-wn}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{d_m, d_n \in D} \sum_{z=0}^{k-1} \gamma^{z(m-n)-wn} \\
&= 0 \text{ as } \sum_{z=0}^{k-1} \gamma^{z(m-n)-wn} = 0.
\end{aligned}
$$

Thus we have the result. □

**Example 10** The $GW(21, 9; Z_3)$ is given. Similarly one can construct a $GW(55, 25; Z_5)$.

$$
\left[\begin{array}{ccccccccccccccccccccc}
0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & \omega & 0 & \omega^2 & 0 & 0 & 0 & 1 & \omega^2 & 0 & \omega & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & \omega & 0 & \omega^2 & 0 & 0 & 0 & 1 & \omega^2 & 0 & \omega & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & \omega & 0 & \omega^2 & 0 & 0 & 0 & 1 & \omega^2 & 0 & \omega \\
1 & 0 & 0 & 0 & 1 & 1 & 0 & \omega^2 & 0 & 0 & 0 & 1 & \omega & 0 & \omega & 0 & 0 & 0 & 1 & \omega^2 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & \omega^2 & 0 & 0 & 0 & 1 & \omega & 0 & \omega & 0 & 0 & 0 & 1 & \omega^2 \\
1 & 0 & 1 & 0 & 0 & 0 & 1 & \omega & 0 & \omega^2 & 0 & 0 & 0 & 1 & \omega^2 & 0 & \omega & 0 & 0 & 0 & 1 \\
1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & \omega & 0 & \omega^2 & 0 & 0 & 0 & 1 & \omega^2 & 0 & \omega & 0 & 0 & 0 \\
 & & & & & & & & & & & & & & & & & & & & \\
0 & 1 & \omega^2 & 0 & \omega & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & \omega & 0 & \omega^2 & 0 & 0 \\
0 & 0 & 1 & \omega^2 & 0 & \omega & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & \omega & 0 & \omega^2 & 0 \\
0 & 0 & 0 & 1 & \omega^2 & 0 & \omega & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & \omega & 0 & \omega^2 \\
\omega & 0 & 0 & 0 & 1 & \omega^2 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & \omega^2 & 0 & 0 & 0 & 1 & \omega & 0 \\
0 & \omega & 0 & 0 & 0 & 1 & \omega^2 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & \omega^2 & 0 & 0 & 0 & 1 & \omega \\
\omega^2 & 0 & \omega & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & \omega & 0 & \omega^2 & 0 & 0 & 0 & 1 \\
1 & \omega^2 & 0 & \omega & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & \omega & 0 & \omega^2 & 0 & 0 & 0 \\
 & & & & & & & & & & & & & & & & & & & & \\
0 & 1 & \omega & 0 & \omega^2 & 0 & 0 & 0 & 1 & \omega^2 & 0 & \omega & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & \omega & 0 & \omega^2 & 0 & 0 & 0 & 1 & \omega^2 & 0 & \omega & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & \omega & 0 & \omega^2 & 0 & 0 & 0 & 1 & \omega^2 & 0 & \omega & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\
\omega^2 & 0 & 0 & 0 & 1 & \omega & 0 & \omega & 0 & 0 & 0 & 1 & \omega^2 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & \omega^2 & 0 & 0 & 0 & 1 & \omega & 0 & \omega & 0 & 0 & 0 & 1 & \omega^2 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\
\omega & 0 & \omega^2 & 0 & 0 & 0 & 1 & \omega^2 & 0 & \omega & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\
1 & \omega & 0 & \omega^2 & 0 & 0 & 0 & 1 & \omega^2 & 0 & \omega & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0
\end{array}\right]
$$

**Lemma 13** *If $k$ is a prime power and there exists a $(v, k, \lambda)$-difference set then there exists a $PBIBD(vk^2, vk^2, k^2, k^2;\ \lambda_1 = 0,\ \lambda_2 = \lambda,\ \lambda_3 = k)$.*

**Proof.** We replace the elements of the $GW(vk, k^2;\ EA(k))$ by their matrix representation as before. This gives $\lambda_1 = 0$.

The set of $x$th and $(x + yk)$th rows, $y = 0, \ldots, k-1$ of the $GW$ give the third association which has $\lambda_3 = k$.

The set of rows corresponding to the product of the $x$th rows and the $y$th rows, $x \in S_i$, $y \in S_j$, $i \neq j$ give the second association class with $\lambda_2 = \lambda$. □

Table 3 gives some of the generalized weighing matrices and $PBIBD$s parameters obtained by using Theorem 12 and Lemma 13.

**Example 11** From the $GW(21, 9; Z_3)$ with $\omega^i$ replaced by $T^i$ we have the classes comprising rows $3j+1$, $3j+2$, $3j+3$, $j = 0, 1, \ldots, 20$ with inner product zero.

Rows $3j+1$, $3j+2$, $3j+3$ with any of $21+3j+1$, $21+3j+2$, $21+3j+3$ (and vice versa) and with any of $42+3j+1$, $42+3j+2$, $42+3j+3$, $j = 0, 1, \ldots, 7$ (and vice versa) have inner product 3.

All other pairs of rows have inner product 1.

| Difference set $(v,k,\lambda)$ | $GW$ $(vk,k^2;EA(k))$ | $PBIBD$ $(vk^2,vk^2,k^2,k^2;0,\lambda,k)$ |
|---|---|---|
| $(4,3,2)$ | $(12,9;EA(3))$ | $(36,36,9,9;0,2,3)$ |
| $(7,3,1)$ | $(21,9;EA(3))$ | $(63,63,9,9;0,1,3)$ |
| $(5,4,3)$ | $(20,16;EA(4))$ | $(80,80,16,16;0,3,4)$ |
| $(7,4,2)$ | $(28,16;EA(4))$ | $(112,112,16,16;0,2,4)$ |
| $(13,4,1)$ | $(52,16;EA(4))$ | $(208,208,16,16;0,1,4)$ |
| $(6,5,4)$ | $(30,25;EA(5))$ | $(150,150,25,25;0,4,5)$ |
| $(11,5,2)$ | $(55,25;EA(5))$ | $(275,275,25,25;0,2,5)$ |
| $(21,5,1)$ | $(105,25;EA(5))$ | $(525,525,25,25;0,1,5)$ |
| $(8,7,6)$ | $(56,49;EA(7))$ | $(392,392,49,49;0,6,7)$ |
| $(15,7,3)$ | $(105,49;EA(7))$ | $(735,735,49,49;0,3,7)$ |
| $(9,8,7)$ | $(72,64;EA(8))$ | $(576,576,64,64;0,7,8)$ |
| $(15,8,4)$ | $(120,64;EA(8))$ | $(960,960,64,64;0,4,8)$ |
| $(57,8,1)$ | $(456,64;EA(64))$ | $(3648,3648,64,64;0,1,8)$ |

Table 3

So the $PBIBD$, $X$, satisfies $XJ = JX = 9$,

$$XX^T = 9I_3 \times I_7 \times I_3 + (J-I)_3 \times J_7 \times J_3 + (J-I)_3 \times I_7 \times 2J.$$

Hence we have a $PBIBD(63,63,9,9;\ \lambda_1 = 0,\ \lambda_2 = 1,\ \lambda_3 = 3)$.

**Acknowledgment:** We wish to thank Dr. Kishore Sinha for his helpful advice and comments.

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