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Abstract

We give an overview on the existence of 4-NPAF (1,2w) sequences. We sketch some construction methods which give new 4-NPAF(1,2w) sequences and new orthogonal designs OD(4n;1,2w).

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On 4-NPAF(1, 2w) Sequences

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ABSTRACT. We give an overview on the existence of 4-NPAF(1, 2w) sequences. We sketch some construction methods which give new 4-NPAF(1, 2w) sequences and new orthogonal designs $OD(4n; 1, 2w)$.

1 Introduction

Definition 1 An orthogonal design A , of order n , and type (s_1, s_2, \dots, s_u) , denoted $OD(n; s_1, s_2, \dots, s_u)$, on the commuting variables $(\pm x_1, \pm x_2, \dots, \pm x_u, 0)$ is a square matrix of order n with entries $\pm x_k$, where each x_k occurs s_k times in each row and column such that the distinct rows are pairwise orthogonal.

In other words

$$AA^T = (s_1x_1^2 + \dots + s_ux_u^2)I_n$$

where I_n is the identity matrix. It is known that the maximum number of variables in an orthogonal design is $\rho(n)$, the Radon number, where for $n = 2^ab$, b odd, set $a = 4c + d$, $0 \leq d < 4$, then $\rho(n) = 8c + 2^d$.

Definition 2 A weighing matrix $W = W(n, k)$ is a square matrix with entries $0, \pm 1$ having k non-zero entries per row and column and inner product of distinct rows zero. Hence, W satisfies $WW^T = kI_n$. The number k is called the weight of W . A $W(n, n)$, for $n \equiv 0 \pmod{4}$, 1 or 2 , whose entries are ± 1 only is called an Hadamard matrix. A $W(n, n-1)$ for $n \equiv 0 \pmod{4}$ is equivalent to an $OD(n; 1, n-1)$ and a skew-Hadamard matrix of order n .

There are a number of conjectures concerning weighing matrices:

Conjecture 1 (Weighing Matrix Conjecture) *There exists a weighing matrix $W(4t, k)$ for $k \in \{1, \dots, 4t\}$.*

Conjecture 2 (Skew Weighing Matrix Conjecture) *When $n \equiv 4 \pmod{8}$, there exist a skew-weighing matrix (also written as an $OD(n; 1, k)$) when $k \leq n - 1$, $k = a^2 + b^2 + c^2$, a, b, c integers except that $n - 2$ must be the sum of two squares.*

Conjecture 3 *When $n \equiv 0 \pmod{8}$, there exists a skew-weighing matrix (also written as an $OD(n; 1, k)$) for all $k \leq n - 1$.*

Definition 3 (Nonperiodic Autocorrelation Function) *Let $X = \{\{x_{11}, \dots, x_{1n}\}, \{x_{21}, \dots, x_{2n}\}, \dots, \dots, \{x_{m1}, \dots, x_{mn}\}\}$ be a family of m sequences of elements 1, 0 and -1 and length n . The nonperiodic autocorrelation function of the family of sequences X , denoted by N_X , is a function defined by*

$$N_X(s) = \sum_{i=1}^{n-s} (x_{1i}x_{1,i+s} + x_{2i}x_{2,i+s} + \dots + x_{mi}x_{m,i+s}),$$

where s can range from 1 to $n - 1$.

Definition 4 *The weight $w(X_i)$ of a sequence X_i is defined as the total number of non-zero elements in X_i . The weight $w(X)$ of a family of sequences X is defined as $w(X) = \sum_{i=1}^m w(X_i)$.*

It is well known (see for example [8] or [7]) that the sum of the squares of the row sums of sequences with zero nonperiodic autocorrelation function must add to the total weight. That is,

$$\sum_{i=1}^m \left(\sum_{j=1}^n x_{ij} \right)^2 = w(X).$$

Notation 1 *Given binary or ternary sequences $S = \{s_1, s_2, \dots, s_n\}$ and $T = \{t_1, t_2, \dots, t_m\}$, we shall use \bar{S} for the sequence S negated, S^* for the sequence S reversed and ST for the sequence T appended onto S . That is,*

$$\begin{aligned} \bar{S} &= \{-s_1, -s_2, \dots, -s_n\}, \\ S^* &= \{s_n, \dots, s_2, s_1\}, \\ ST &= \{s_1, \dots, s_n, t_1, \dots, t_m\}. \end{aligned}$$

It can be shown that negating and/or reversing one or more sequences does not affect the (non)periodic autocorrelation function.

We are looking for four ternary sequences A, B, C, C of lengths m, m, n, n ($m \geq n$) and weight w where A, B, C, C have zero nonperiodic autocorrelation function. That is,

$$N_A(s) + N_B(s) + 2N_C(s) = 0, \quad s = 1, \dots, m-1, \text{ and } c_i = 0, \quad i > n.$$

From these sequences we can form the sequences $X = AB, Y = A\bar{B}, Z = C0C^*$ and $W = Ca\bar{C}^*, a \neq 0$. X, Y, Z, W are called 4-NPAF(1, 2w) sequences.

Lemma 1 *If there are sequences A, B, C, C of length m, m, n, n ($m \geq n$) and weight w , then there is an orthogonal design $OD(4p; 1, 2w)$, where $p \geq \max(2m, 2n + 1)$.*

Proof. Given A, B and C take the 4-NPAF(1, 2w) sequences $AB, A\bar{B}, C0C^*, Ca\bar{C}^*$ as the first rows of length $p \geq \max(2m, 2n + 1)$ (if necessary append zeros) of four circulant matrices which can be used in the Goethals-Seidel array to form an $OD(4p; 1, 2w)$. \square

2 Existence of Sequences of Type A, B, C, C

We performed exhaustive searches on the computer for sequences of type A, B, C, C for particular weights and lengths. The results are presented in Table 1. In the table, 1 is replaced by '+' and -1 by '-'. Where the weight could not be written as $a^2 + b^2 + 2c^2$ (a, b, c being the row sums of A, B, C respectively), we write "n/e", meaning that sequences for this particular weight cannot exist. "n/f" means that there were no sequences found subjected to complete search for some lengths m and n . However, sequences may exist for longer lengths.

We also construct sequences of type A, B, C, C from Golay sequences and ternary complementary pairs (TCP's).

Lemma 2 *If there are Golay sequences or ternary complementary pairs of lengths ℓ and weight u , then there are sequences of type A, B, C, C of*

- (i) lengths $m = n = \ell$ and weight $w = 2u$; and
- (ii) lengths $m = \ell + 1, n = \ell$ and weight $w = 2u + 2$.

Proof. Given Golay sequences or a ternary complementary pair X, Y of length n and weight w take the four sequences

- (i) $A = X, B = X, C = Y$; or
- (ii) $A = \{X, 1\}, B = \{X, -1\}, C = Y$.

Now A, B, C, C have the desired properties. \square

m	n	Weight	Sequences
5	4	17	+++--+, ++0---, +-+--+
5	4	18	++++-, -++++-, +-+--+
5	5	19	+++---, +-0+-, ++++--+
5	5	20	+++---, +-++--+, ++++--+
7	5	21	++-+-+++, ++++0+---, ++0--+
7	6	22	+++--0-, +-+-0+, ++++0--+
7	6	23	++-0+--+, -++++--+, ++++0+--
8	6	24	++++-+--+, +-+-+---+, ++00--+
9	7	25	+0++0--0+, +-+0++0---, ++++0+--
9	9	25	++00+-+0+, +-+-+---00+, +0+00++--
7	6	26	+++--+-, +-++++-, -++++--
7	7	26	++++--+, ++++--+-, ++++0+--
9	7	27	++00+-+0+, +-+-++++00-, ++++--+-
9	8	27	+++0+0+---, +-000--+, ++-+-0++
7	7	28	+++--+-, +-+-+--+, ++++--+-
8	7	28	++-++-0+, -++++-0+, ++++--+-
		29	n/f
		30	n/e
9	8	31	+++--+-, -++++0--+-, +-++++0-
10	8	31	+++++0--+-, ++0--+-0--+, ++-+-0++
8	8	32	+++--+-, -++++--+, ++++--+-
9	8	33	+++--+-, ++++0+---, +++--+-
9	8	34	++++--+-, +-+-+---+, +++--+-
11	10	35	+++0+0+0+---, +-0+-0--+, ++0+-+-++

Table 1. Sequences of type A, B, C, C found via computer.

Corollary 1 *There are sequences of type A, B, C, C of*

(i) *lengths $m = n = 2^a 10^b 26^c$ and weight $w = 4 \times 2^a 10^b 26^c$; and*

(ii) *lengths $m = 2^a 10^b 26^c + 1, n = 2^a 10^b 26^c$ and weight $w = 4 \times 2^a 10^b 26^c + 2$,*

$a, b, c \geq 0$.

Proof. Take Golay sequences in the above constructions. □

[1] and [4] present many new TCP's of length n and weight w . Hence, we get many new sequences of type A, B, C, C , many new 4-NPAF(1, $4w$) and 4-NPAF(1, $4w + 4$) sequences and many new $OD(8n + 4; 1, 4w)$ and

$OD(8n + 8; 1; 4w + 4)$. These results are summarised in Table 2. The table indicates the two constructions from Lemma 2. [3] multiplies (or concatenates) the length and the weight of a TCP (where the two sequences have their zeros in the same positions) by certain small numbers which again leads to new TCP's and new sequences of type A, B, C, C . In particular multiplication is possible for the following parameters

$$(m, f) = (3, 4), (3, 5), (4, 5), (5, 8), (6, 8), (6, 10), (7, 8), (7, 10),$$

where m indicates the multiplier of the length and $\frac{1}{2}f$ the multiplier of the weight. For example, $(m, f) = (3, 5)$ means that we can multiply the length of a given TCP (or Golay sequences) by 3 and the weight by $2\frac{1}{2}$.

The multiplication $(m, f) = (3, 5)$ leads to new results.

Lemma 3 *If there is a TCP of length ℓ and weight w (where the two sequences have their zeros in the same positions), then there are sequences of type A, B, C, C of*

(i) *lengths $m = n = 3\ell$ and weight $5w$; and*

(ii) *lengths $m = 3\ell + 1, n = 3\ell$ and weight $5w + 2$.*

Proof. Given a TCP S, T of length ℓ (where S and T have their zeros in the same positions), let $P = \frac{1}{2}(S + T)$, $Q = \frac{1}{2}(S - T)$ and let

$$\begin{aligned} X &= SQB, \\ Y &= \bar{S}PB. \end{aligned}$$

Now X, Y is a TCP of length 3ℓ and weight $2\frac{1}{2}w$ (see [3]) and we can apply Lemma 2 to get the desired sequences of type A, B, C, C . \square

There are TCP's with the parameters in Table 2 with zeros in the same positions. Hence, we get many new sequences of type A, B, C, C and weights $5w$ and $5w + 2$ (where w is the initial weight of the TCP) and many 4-NPAF(1, $10w$) and 4-NPAF(1, $10w + 4$) sequences. In the last step, we get new $OD(24n + 4; 1, 10w)$ and $OD(24n + 8; 1, 10w + 4)$ (where n is the initial length of of the TCP).

Finally, in Table 3, we present some other sequences of type A, B, C, C which fill in some of the gaps for weights $w \leq 50$.

3 Numerical Consequences

Table 4 shows the parameters of the orthogonal designs $OD(4p; 1, 2w)$ obtained by the above constructions via sequences of type A, B, C, C .

TCP length	TCP weight	A,B,C,C lengths	A,B,C,C weight
5	8	5,5/6,5	16/18
6	10	6,6/7,6	20/22
7	10	7,7/8,7	20/22
8	16	8,8/9,8	32/34
9	16	9,9/10,9	32/34
10	20	10,10/11,10	40/42
11	16	11,11/12,11	32/34
12	20	12,12/13,12	40/42
13	20	13,13/14,13	40/42
14	26	14,14/15,14	52/54
15	20	15,15/16,15	40/42
16	32	16,16/17,16	64/66
17	32	17,17/18,17	64/66
18	32	18,18/19,18	64/66
19	32	19,19/20,19	64/66
20	40	20,20/21,20	80/82

Table 2. From TCP's to sequences of type A,B,C,C.

m	n	Weight	Sequences
11	11	37	++----+++--+, +-+++0++++-, +0+00--+-+--+
11	11	44	+--+--+-+--+-+--+, +---+++-+++++, ++++-+-+--++-

Table 3. Some other sequences of type A,B,C,C.

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p	w	How
11	16	Table 3
10	17	Table 2
10	18	Table 2
11	19	Table 2
11	20	Table 2
14	21	Table 2
14	22	Table 2 and 3
14	23	Table 2
16	24	Table 2
18	25	Table 2
14	26	Table 2
18	27	Table 2
15	28	Table 2
18	31	Table 2
17	32	Table 2 and 3
18	33	Table 2
18	34	Table 2 and 3
22	35	Table 2
23	37	Table 3
21	40	Table 3
22	42	Table 3
23	44	Table 3
37	50	Table 3 and multiplication by $(m, f) = (3, 5)$
29	52	Table 3
30	54	Table 3
33	64	Table 3
34	66	Table 3
41	80	Table 3
42	82	Table 3
61	100	Table 3 and multiplication by $(m, f) = (3, 5)$
62	102	Table 3 and multiplication by $(m, f) = (3, 5)$

Table 4. An orthogonal design $OD(4p; 1, 2w)$ exists.