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On sufficient conditions for some orthogonal designs and sequences with zero autocorrelation function

Abstract

We give new sets of sequences with entries from {0, ±a, ±b, ±c, ±d} on the commuting variables a, b, c, d and zero autocorrelation function. Then we use these sequences to construct some new orthogonal designs. This means that for order 28 only the existence of the following five cases, none of which is ruled out by known theoretical results, remain in doubt: OD(28; 1, 4, 9, 9), OD(28; 1, 8, 8, 9), OD(28; 2, 8, 9, 9), OD(28; 3, 6, 8, 9), OD(28; 4, 4, 4, 9). We consider 4 - N PAF(S_l, S_2, S_3, S_4) sequences or four sequences of commuting variables from the set {0, ±a, ±b, ±c, ±d} with zero nonperiodic autocorrelation function where ±a occurs S_l times, ±b occurs S_2 times, etc. We show the necessary conditions for the existence of an 0D(4n; S_1,S_2, S_3,S_4) constructed using four circulant matrices are sufficient conditions for the existence of 4 - NPAF(S_1, S_2, S_3, S_4) sequences for all lengths ≥ n,

i) for n = 3, with the extra condition (S_1,S_2,S_3,S_4) ≠ (1,1,1,9),

ii) for n = 5, provided there is an integer matrix P satisfying PPT = diag (S_1,S_2,S_3,S_4),

iii) for n = 7, with the extra condition that (S_1,S_2,S_3,S_4) ≠ (1,1,1,25), and possibly (S_1,S_2,S_3,S_4) =I- (1,1,1,16), (1,1,8,18), (1,1,13,13), (1,4,4,9), (1,4,9,9), (1,4,10,10), (1,8,8,9), (1,9,9,9), (2,4,4,18), (2,8,9,9), (3,4,6,8), (3,6,8,9); (4,4,4,9), (4,4,9,9), (4,5,5,9), (5,5,9,9).

We show the necessary conditions for the existence of an OD(4n; S_1,S_2) constructed using four circulant matrices are sufficient conditions for the existence of 4 - NPAF(S_1,S_2) sequences for all lengths ≥ n, where n = 3 or 5.

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On sufficient conditions for some orthogonal designs and sequences with zero autocorrelation function

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Dedicated to Albert Leon Whiteman on his 80th birthday

Abstract

We give new sets of sequences with entries from $\{0, \pm a, \pm b, \pm c, \pm d\}$ on the commuting variables a, b, c, d and zero autocorrelation function. Then we use these sequences to construct some new orthogonal designs.

This means that for order 28 only the existence of the following five cases, none of which is ruled out by known theoretical results, remain in doubt:

$OD(28;\ 1,\ 4,\ 9,\ 9)$, $OD(28;\ 1,\ 8,\ 8,\ 9)$, $OD(28;\ 2,\ 8,\ 9,\ 9)$, $OD(28;\ 3,\ 6,\ 8,\ 9)$, $OD(28;\ 4,\ 4,\ 4,\ 9)$.

We consider $4 - NPAF(s_1, s_2, s_3, s_4)$ *sequences* or four sequences of commuting variables from the set $\{0, \pm a, \pm b, \pm c, \pm d\}$ with zero non-periodic autocorrelation function where $\pm a$ occurs s_1 times, $\pm b$ occurs s_2 times, etc.

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We show the necessary conditions for the existence of an $OD(4n; s_1, s_2, s_3, s_4)$ constructed using four circulant matrices are sufficient conditions for the existence of $4-NPAF(s_1, s_2, s_3, s_4)$ sequences for all lengths $\geq n$,

i) for $n = 3$, with the extra condition $(s_1, s_2, s_3, s_4) \neq (1, 1, 1, 9)$,

ii) for $n = 5$, provided there is an integer matrix P satisfying $PP^T = diag\,(s_1, s_2, s_3, s_4)$,

iii) for $n = 7$, with the extra condition that $(s_1, s_2, s_3, s_4) \neq (1, 1, 1, 25)$, and possibly $(s_1, s_2, s_3, s_4) \neq (1, 1, 1, 16)$, $(1, 1, 8, 18)$, $(1, 1, 13, 13)$, $(1, 4, 4, 9)$, $(1, 4, 9, 9)$, $(1, 4, 10, 10)$, $(1, 8, 8, 9)$, $(1, 9, 9, 9)$, $(2, 4, 4, 18)$, $(2, 8, 9, 9)$, $(3, 4, 6, 8)$, $(3, 6, 8, 9)$, $(4, 4, 4, 9)$, $(4, 4, 9, 9)$, $(4, 5, 5, 9)$, $(5, 5, 9, 9)$.

We show the necessary conditions for the existence of an $OD(4n; s_1, s_2)$ constructed using four circulant matrices are sufficient conditions for the existence of $4 - NPAF(s_1, s_2)$ sequences for all lengths $\geq n$, where $n =$ 3 or 5.

1 Introduction

An *orthogonal design* of order n and type $(s_1, s_2, \ldots, s_u)$ $(s_i > 0)$, denoted $OD(n; s_1, s_2, \ldots, s_u)$, on the commuting variables $x_1, x_2, \ldots, x_u$ is an $n \times n$ matrix A with entries from $\{0, \pm x_1, \pm x_2, \ldots, \pm x_u\}$ such that

$$AA^T = (\sum_{i=1}^{u} s_i x_i^2) I_n.$$

Alternatively, the rows of A are formally orthogonal and each row has precisely s_i entries of the type $\pm x_i$. In [2], where this was first defined, it was mentioned that

$$A^T A = (\sum_{i=1}^{u} s_i x_i^2) I_n.$$

and so our alternative description of A applies equally well to the columns of A. It was also shown in [2] that $u \leq \rho(n)$, where $\rho(n)$ (Radon's function) is defined by $\rho(n) = 8c + 2^d$, when $n = 2^a b$, b odd, $a = 4c + d$, $0 \leq d < 4$.

A weighing matrix $W = W(n, k)$ is a square matrix with entries $0, \pm 1$ having k non-zero entries per row and column and inner product of distinct rows zero. Hence W satisfies $WW^T = kI_n$, and W is equivalent to an orthogonal design $OD(n; k)$. The number k is called the *weight* of W.

Given the sequence $A = \{a_1, a_2, \ldots, a_n\}$ of length n the *non-periodic autocorrelation function* $N_A(s)$ is defined as

$$N_A(s) = \sum_{i=1}^{n-s} a_i a_{i+s}, \quad s = 0, 1, \ldots, n-1, \tag{1}$$

If $A(z) = a_1 + a_2 z + \ldots + a_n z^{n-1}$ is the associated polynomial of the sequence A, then

$$A(z)A(z^{-1}) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j z^{i-j} = N_A(0) + \sum_{s=1}^{n-1} N_A(s)(z^s + z^{-s}), z \neq 0. \qquad (2)$$

Given A as above of length n the *periodic autocorrelation function* $P_A(s)$ is defined, reducing $i+s$ modulo n, as

$$P_A(s) = \sum_{i=1}^{n} a_i a_{i+s}, \quad s = 0, 1, \ldots, n-1. \qquad (3)$$

2 Preliminary results

We make extensive use of the book of Geramita and Seberry [6]. We quote the following theorems, giving their reference from the aforementioned book, that we use:

Lemma 1 **[6, Lemma 4.11]** *If there exists an orthogonal design $OD(n; s_1, s_2, \ldots, s_u)$ then there exists an orthogonal design $OD(2n; s_1, s_1, es_2, \ldots, es_u)$ where $e = 1$ or 2.* □

Lemma 2 **[6, Lemma 4.4]** *If A is an orthogonal design $OD(n; s_1, s_2, \ldots, s_u)$ on the commuting variables $\{0, \pm x_1, \pm x_2, \ldots, \pm x_u\}$ then there is an orthogonal design $OD(n; s_1, s_2, \ldots, s_i + s_j, \ldots, s_u)$ and $OD(n; s_1, s_2, \ldots, s_{j-1}, s_{j+1}, \ldots, s_u)$ on the $u-1$ commuting variables $\{0, \pm x_1, \pm x_2, \ldots, \pm x_{j-1}, \pm x_{j+1}, \ldots, \pm x_u\}$.* □

Theorem 1 **[6, Theorems 2.19 and 2.20]** *Suppose $n \equiv 0 (mod\ 4)$. Then the existence of a $W(n, n-1)$ implies the existence of a skew-symmetric $W(n, n-1)$. The existence of a skew-symmetric $W(n,k)$ is equivalent to the existence of an $OD(n; 1, k)$.* □

Theorem 2 **[6, Proposition 3.54 and Theorem 2.20]** *An orthogonal design $OD(n; 1, k)$ can only exist in order $n \equiv 4(mod\ 8)$ if k is the sum of three squares. An orthogonal design $OD(n; 1, n-2)$ can only exist in order $n \equiv 4(mod\ 8)$ if $n-2$ is the sum of two squares.* □

Theorem 3 **[6, Theorem 4.49]** *Suppose there exist four circulant matrices A, B, C, D of order n satisfying*

$$AA^T + BB^T + CC^T + DD^T = fI_n.$$

Let R be the back diagonal matrix. Then

$$GS = \begin{pmatrix} A & BR & CR & DR \\ -BR & A & D^TR & -C^TR \\ -CR & -D^TR & A & B^TR \\ -DR & C^TR & -B^TR & A \end{pmatrix}.$$

is a $W(4n, f)$ *when* A, B, C, D *are* $(0, 1, -1)$ *matrices, and an orthogonal design* $OD(4n; s_1, s_2, \ldots, s_u)$ *on* $x_1, x_2, \ldots, x_u$ *when* A, B, C, D *have entries from* $\{0, \pm x_1, \ldots, \pm x_u\}$ *and* $f = \sum_{j=1}^{u}(s_j x_j^2)$. □

Corollary 1 *If there are four sequences* A, B, C, D *of length* n *with entries from* $\{0, \pm x_1, \pm x_2, \pm x_3, \pm x_4\}$ *with zero periodic or non-periodic autocorrelation function, then these sequences can be used as the first rows of circulant matrices which can be used in the Goethals-Seidel array to form an* $OD(4n; s_1, s_2, s_3, s_4)$. *We note that if there are sequences of length* n *with zero non-periodic autocorrelation function, then there are sequences of length* $n + m$ *for all* $m \geq 0$. □

3 Constructing orthogonal designs by using circulant matrices

Suppose A is an $OD\,(4n; s_1, s_2, s_3, s_4)$ on the commuting variables x_1, x_2, x_3, x_4, constructed using four circulant matrices A_1, A_2, A_3, A_4 of order n, with entries from $\{0, \pm x_1, \pm x_2, \pm x_3, \pm x_4\}$, in the Goethals-Seidel array. Suppose the row and column sum of A_i is

$$r_i = p_{1i}x_1 + p_{2i}x_2 + p_{3i}x_3 + p_{4i}x_4, \quad i = 1, 2, 3, 4.$$

Let e^T be the $1 \times n$ vector of 1's, then $e^T A_i = r_i e^T$. Now since A is an orthogonal design,

$$\sum_{i=1}^{4} A_i A_i^T = \left(\sum_{i=1}^{4} s_i x_i^2\right) I_n. \tag{4}$$

Multiplying on the left by e^T and the right by e we have

$$\sum_{i=1}^{4}(e^T A_i)(e^T A_i)^T = n \sum_{i=1}^{4} s_i x_i^2.$$

or

$$\sum_{i=1}^{4}(r_i e^T)(r_i e^T)^T = n \sum_{i=1}^{4} r_i^2 = n \sum_{i=1}^{4} s_i x_i^2.$$

Thus we have

$$\begin{aligned} s_1x_1^2 + s_2x_2^2 + s_3x_3^2 + s_4x_4^2 = \;& x_1^2 \sum_{i=1}^{4} p_{1i}^2 + x_2^2 \sum_{i=1}^{4} p_{2i}^2 + x_3^2 \sum_{i=1}^{4} p_{3i}^2 \\ & + x_4^2 \sum_{i=1}^{4} p_{4i}^2 + 2x_1x_2 \sum_{i=1}^{4} p_{1i}p_{2i} \end{aligned}$$

$$+2x_1x_3\sum_{i=1}^{4}p_{1i}p_{3i}+2x_1x_4\sum_{i=1}^{4}p_{1i}p_{4i}$$

$$+2x_2x_3\sum_{i=1}^{4}p_{2i}p_{3i}+2x_2x_4\sum_{i=1}^{4}p_{2i}p_{4i}$$

$$+2x_3x_4\sum_{i=1}^{4}p_{3i}p_{4i}.$$

Hence we have four integer vectors $p_1^T=(p_{11},p_{12},p_{13},p_{14})$, $p_2^T=(p_{21},p_{22},p_{23},p_{24})$, $p_3^T=(p_{31},p_{32},p_{33},p_{34})$, $p_4^T=(p_{41},p_{42},p_{43},p_{44})$, which are pairwise orthogonal. Also $|p_1^T|^2=s_1$, $|p_2^T|^2=s_2$, $|p_3^T|^2=s_3$, $|p_4^T|^2=s_4$.

Form these vectors into an orthogonal integer matrix P with $P^T=(p_1,p_2,p_3,p_4)$. Then $PP^T=diag\,(s_1,s_2,s_3,s_4)$ and $det\,P=\sqrt{s_1s_2s_3s_4}$. But P is integer so $s_1s_2s_3s_4$ is a square. Thus we have

Lemma 3 *The Goethals-Seidel construction for an orthogonal design $OD\,(4n;s_1,s_2,s_3,s_4)$ can only be used if*

(i) there is an integer matrix P satisfying $PP^T=diag\,(s_1,s_2,s_3,s_4)$ and hence

(ii) $s_1s_2s_3s_4$ is a square.

□

Since the row sum of A_j is $\sum_{i=1}^{4}p_{ij}x_i$ for $1\le j\le 4$, the 4×4 matrix $P=(p_{ij})$ is called the *sum matrix* of A_1, A_2, A_3, A_4.

Theorem 4 [1, Eades and Seberry Wallis] *The sum matrix P of a solution of (4) satisfies*

$$PP^T=diag\,(s_1,s_2,s_3,s_4). \tag{5}$$

□

If the first row of A_j has m_{ij} entries of the kind $\pm x_i$, $i,j=1,2,3,4$, then the 4×4 matrix $M=(m_{ij})$ is called the *entry matrix* of A_1,A_2,A_3,A_4. The *fill matrix* of A_1,A_2,A_3,A_4 is $Q=M-abs(P)$. It is clear that

$$\sum_{j=1}^{4}m_{ij}=s_i \quad \text{for} \quad 1\le i\le 4. \tag{6}$$

In most cases, if the s_i are small, then the solution of (5) is essentially unique and can be found easily by hand.

It is clear that if Q is the fill matrix of a solution of (4), then the entries of Q are even non-negative integers, and if $M=(m_{ij})=abs(P)+Q$, then M satisfies (6) and the sum of a column of M is at most n. Thus the content of A_i is determined by the i-th columns of the sum and entry matrices.

The above remarks help in the search for circulant matrices to fit into the Goethals-Seidel array.

4 New orthogonal designs

The method outlined in the previous section has been used successfully to compute the following fourteen new four variable orthogonal designs:

Theorem 5 *There exist orthogonal designs*

$$\begin{array}{lll} OD\,(28;1,2,4,18) & OD\,(28;1,2,6,12) & OD\,(28;1,3,6,18) \\ OD\,(28;1,4,4,16) & OD\,(28;1,4,8,8) & OD\,(28;1,4,10,10) \\ OD\,(28;1,9,9,9) & OD\,(28;2,2,4,9) & OD\,(28;2,3,6,9) \\ OD\,(28;2,4,4,18) & OD\,(28;2,4,8,9) & OD\,(28;4,5,5,9) \\ OD\,(28;5,5,8,8) & OD\,(28;5,5,9,9), & \end{array}$$

constructed using four circulant matrices in the Goethals-Seidel array.

Proof. We use the sequences given in Appendix D, which have zero periodic and non-periodic autocorrelation function, as the first rows of the corresponding circulant matrices in the Goethals-Seidel array to obtain the required orthogonal designs. □

Remark 1 This means, using the results of [6, p352-357] that only the existence of the following five cases, none of which is ruled out by known theoretical results, remains in doubt:

$$\begin{array}{lll} OD(28;\ 1,\ 4,\ 9,\ 9), & OD(28;\ 1,\ 8,\ 8,\ 9), & OD(28;\ 2,\ 8,\ 9,\ 9), \\ OD(28;\ 3,\ 6,\ 8,\ 9), & OD(28;\ 4,\ 4,\ 4,\ 9). & \end{array}$$

4.1 Using sequences with NPAF = 0

We note that sequences of commuting variables with zero non-periodic autocorrelation function can also be used. The following theorems summarize the results, many of which are given here for the first time, for lengths 3, 5 and 7.

Notation 1 We write $(a_1, a_2)_p$ for the *Hilbert Norm Residue symbol.*

We note from Hall [8, p114] that the Hilbert Norm Residue symbol $(a_1, a_2)_p = 1$ for all primes p, including zero, if and only if the diophantine equation

$$a_1x^2 + a_2y^2 = z^2$$

has a solution in the integers not all zero.

The properties of the Hilbert Norm residue symbol are given in Hall [8, p115]. We note that the primes $p = 2$ and ∞ can be ignored as $(a_1, a_2)_p = 1$ for these primes in the context of this paper.

Notation 2 We call four sequences of j commuting variables $\pm a_1, \pm a_2, \cdots, \pm a_j$ which occur $s_1,\ s_2,\ \cdots,\ s_j$ times each, respectively, over the whole four sequences and which have zero non-periodic autocorrelation function $4 - NPAF(s_1, s_2, \cdots, s_j)$ *sequences.*

Theorem 6 (Geramita and Seberry and Lemma 3 [1, 6]) *The following conditions are necessary for the existence of an $OD(4n; s_1, s_2, \cdots, s_\ell)$ in orders with n odd:*

i) for $\ell = 2$; $(-1, s_1 s_2)_p (s_1, s_2)_p = 1$ for all primes p;

ii) for $\ell = 3$; $(s_1, s_2)_p (s_1, s_3)_p (s_2, s_3)_p (-1, s_1 s_2 s_3)_p = 1$ for all primes p; if $s_1 + s_2 + s_3 = n - 1$ then $s_1 s_2 s_3$ is a square and $(s_1, s_2)_p (s_1, s_3)_p (s_2, s_3)_p = 1$ for all primes p;

iii) for $\ell = 4$; $s_1 s_2 s_3 s_4$ is a square, $\prod_{1 \le i < j \le 4} (s_i, s_j)_p = 1$ for all primes p, where (s_i, s_j) is the Hilbert norm residue symbol, and $s_1 + s_2 + s_3 + s_4 \neq n - 1$.

The Goethals-Seidel construction can only be used if

iv) there exists an integer matrix P, with all entries of modulus $\le n$ which satisfies $PP^T = diag(s_1, s_2, s_3, s_4)$.

Theorem 7 *There are no $4 - NPAF(1, 1, 1, 9)$ sequences of length $n = 3$.*
There are no $4 - NPAF(1, 1, 1, 25)$ sequences of length $n = 7$.

Proof. We first note that $9 = 3^2 + 0^2 + 0^2 + 0^2 = 2^2 + 2^2 + 1^2 + 0^2$. For the second set of sums of squares it is not possible to incorporate the three variables which occur once. For the first sum of squares, one of the sequences has the same variable 3 times with a positive sign. The only way to incorporate a single variable in such an $NPAF$ sequence is for it to be of the form $x, y, -x$, where x and y are variables. However this is not possible for three single variables when another must occur three times if the $NPAF = 0$. Hence this case is not possible.

For the $4 - NPAF(1, 1, 1, 25)$ we proceed similarly, first noting that $25 = 5^2 + 0^2 + 0^2 + 0^2 = 4^2 + 3^2 + 0^2 + 0^2$. To accommodate the 3 single variables three of the sequences must be skew or of the form $x, y, 0, -y, -x$ before the single variable is included. However this means the row sum is zero so only the first sum of squares could be possible. A little consideration of the $NPAF$ of the skew sequences $x, -x, 0, x, -x$ and $x, x, 0, -x, -x$ will show that they cannot be combined with a sequence with 5 equal variables and obtain zero $NPAF$. Hence this case is also impossible. □

Theorem 8 *The necessary conditions for the existence of an $OD(12; s_1, s_2, s_3, s_4)$ plus the extra condition that $(s_1, s_2, s_3, s_4) \neq (1, 1, 1, 9)$ are sufficient for the existence of $4 - NPAF(s_1, s_2, s_3, s_4)$ sequences for every length $n \ge 3$.*

Proof. The required sequences can be found in Appendix A. □

Remark 2 Sequences of length 3 with the variables occurring s_1, s_2, s_3, or s_4, times and with zero non-periodic autocorrelation function are given in Appendix A for the first time for the following 4-tuples:

$$(1,1,1,4),\quad (1,1,2,8),\quad (1,1,4,4),\quad (1,1,5,5),\quad (1,2,2,4),$$
$$(1,2,3,6),\quad (2,2,4,4).$$

Corollary 2 *Let $n \geq 3$. Then the necessary conditions for the existence of an $OD(4n; s_1, s_2, s_3, s_4)$, with $s_1 + s_2 + s_3 + s_4 \leq 12$, $(s_1, s_2, s_3, s_4) \neq (1,1,1,9)$, constructed using four circulant matrices in the Goethals-Seidel array are sufficient.*

Proof. The results for the 4-tuples

$$(1,1,1,1),\quad (1,1,2,2),\quad (2,2,2,2),\quad (3,3,3,3),$$

are given in [6, p348-351]. The remaining sequences are given in Appendix A and have zero non-periodic autocorrelation function. Since these sequences have zero non-periodic autocorrelation function, the sequences can be first padded with sufficient zeros added to the end to make their length $n \geq 3$. □

Corollary 3 *Let $n \geq 3$. Then the necessary conditions for the existence of an $OD(4n; s_1, s_2, s_3, s_4)$, with $s_1 + s_2 + s_3 + s_4 \leq 12$, $(s_1, s_2, s_3, s_4) \neq (1,1,1,9)$, constructed using four circulant matrices in the Goethals-Seidel array are sufficient for the existence of $4 - NPAF(s_1, s_2, s_3, s_4)$ sequences for every length n.*

Noting that sequences with zero non-periodic autocorrelation function and length 3 are not possible for the 4-tuple (1,1,1,9) we consider the derivative 3-tuples (1,1,1), (1,1,2), (1,1,9), (1,1,10) and (1,2,9) and find

Lemma 4 *The necessary conditions for the existence of an $OD(12; s_1, s_2, s_3)$ are sufficient for the existence of $4 - NPAF(s_1, s_2, s_3)$ sequences.*

Proof. The required sequences can be found in Appendix A. □

Theorem 9 *The necessary conditions for the existence of an $OD(20; s_1, s_2, s_3, s_4)$ plus the extra condition that $(s_1, s_2, s_3, s_4) \neq (1,3,6,8)$, $(1,4,4,9)$, $(2,2,5,5)$ are sufficient for the existence of $4 - NPAF(s_1, s_2, s_3, s_4)$ sequences.*

Proof. The required sequences can be found in Appendix B. □

Remark 3 Is is not known if the 4-tuples (1,3,6,8), (1,4,4,9) and (2,2,5,5) are the types of orthogonal designs of order 20.

Sequences of length 5 with zero non-periodic autocorrelation function and with the variables occurring s_1, s_2, s_3, s_4, times are given in Appendix B for the first time for the following 3-tuples and 4-tuples (we do not include sequences here with $s_1 + s_2 + s_3 + s_4 \leq 12$, except (1,1,1,9), as these are covered by Appendix A):

$$\begin{array}{lllll}
(1,1,1,9), & (1,1,4,9), & (1,1,8,8), & (1,1,9,9), & (1,2,2,9), \\
(1,2,4,8), & (1,2,8,9), & (1,4,4,4), & (1,4,5,5), & (1,5,5,9), \\
(2,2,4,9), & (2,3,4,6), & (2,3,6,9), & (2,4,4,8), & (2,5,5,8), \\
(3,3,6,6), & (4,4,5,5), & (1,1,13), & (1,2,17), & (1,2,11), \\
(3,6,8), & (4,4,10), & (4,6,8), & (7,7), & (7,10).
\end{array}$$

Corollary 4 *Let $n \geq 5$. Then the necessary conditions for the existence of an $OD(4n; s_1, s_2, s_3, s_4)$, with $s_1 + s_2 + s_3 + s_4 \leq 20$, constructed using four circulant matrices in the Goethals-Seidel array are sufficient.*

Proof. $(1,3,6,8)$, $(1,4,4,9)$, $(2,2,5,5)$ cannot be the types of orthogonal designs in order 20 constructed using circulant matrices. The results for the 4-tuples

$$\begin{array}{lllll} (1,1,1,1), & (1,1,2,2), & (2,2,2,2), & (2,2,2,8), & (2,2,4,4), \\ (2,2,8,8), & (3,3,3,3), & (4,4,4,4), & (5,5,5,5), & \end{array}$$

are given in [6, p348-351]. The remaining sequences are given in Appendices A and B and have zero non-periodic autocorrelation function. Since these sequences have zero non-periodic autocorrelation function, the sequences can be first padded with sufficient zeros added to the end to make their length $n \geq 5$. □

Lemma 5 *If an $OD(20; 3, 7, 8)$ exists it cannot be constructed from four circulant matrices.*

Proof. An exhaustive computer search gave the result.

Lemma 6 *The necessary conditions for the existence of an $OD(20; s_1, s_2, s_3)$ plus the extra condition that $(s_1, s_2, s_3) \neq (3, 7, 8)$ are sufficient for the existence of $4-NPAF(s_1, s_2, s_3)$ sequences with the possible exception of*

$$(\ 1,\ 3,\ 14), \quad (\ 1,\ 4,\ 13), \quad (\ 1,\ 6,\ 11), \quad (\ 1,\ 8,\ 11), \quad (\ 2,\ 5,\ 7).$$

Proof. We note that the sequences exist for $n \geq 5$ for $(1,1,13)$, $(1,2,17)$, $(1,2,11)$, $(3,6,8)$, $(4,4,10)$, and $(4,6,8)$ which are given for the first time in Appendix B. The other required sequences may be constructed by equating variables from those for four variables found in Appendix B. □

Lemma 7 *The necessary conditions for the existence of an $OD(20; s_1, s_2)$ are sufficient for the existence of $4-NPAF(s_1, s_2)$ sequences.*

Proof. The required sequences may be constructed by equating variables from those found in Appendix B. □

Theorem 10 *Let $n \geq 7$. Then the necessary conditions for the existence of an $OD(4n; s_1, s_2, s_3, s_4)$, with $20 < s_1 + s_2 + s_3 + s_4 \leq 28$, constructed using four circulant matrices in the Goethals-Seidel array plus the extra condition that $(s_1, s_2, s_3, s_4) \neq (1,1,1,25)$, are sufficient for the existence of $4-NPAF(s_1, s_2, s_3, s_4)$ sequences except possibly $(s_1, s_2, s_3, s_4) \neq$*

$$\begin{array}{lllll} (1,1,1,16), & (1,1,8,18), & (1,1,13,13), & (1,3,6,18), & (1,4,4,9), \\ (1,4,9,9), & (1,8,8,9), & (1,9,9,9), & (2,4,4,18), & (2,4,8,9), \\ (2,8,9,9), & (3,3,3,12), & (3,4,6,8), & (3,6,8,9), & (4,4,4,9), \\ (4,4,9,9), & (4,5,5,9), & (5,5,9,9). & & \end{array}$$

Proof. The required sequences can be found in Appendix C. □

Corollary 5 *Let $n \geq 7$. Then the necessary conditions for the existence of an $OD(4n; s_1, s_2, s_3, s_4)$, with $20 < s_1 + s_2 + s_3 + s_4 \leq 28$, constructed using four circulant matrices in the Goethals-Seidel array are sufficient except possibly for the 4-tuples*

$$(1,4,9,9), \quad (1,8,8,9), \quad (2,8,9,9), \quad (3,6,8,9), \quad (4,4,4,9).$$

Remark 4 Sequences of length 7 with zero non-periodic autocorrelation function and with the variables occurring s_1, s_2, s_3, s_4, times are given in Appendix C for the first time for the following 4-tuples (we do not include sequences here with $s_1 + s_2 + s_3 + s_4 \leq 20$, as these are covered by Appendices A and B):

$$\begin{array}{lllll} (1,1,2,18), & (1,1,10,10), & (1,2,4,18), & (1,2,6,12), & (1,3,6,8), \\ (1,3,6,18), & (1,4,4,16), & (1,4,8,8), & (1,4,10,10), & (2,2,2,18), \\ (2,2,9,9), & (5,5,8,8). \end{array}$$

Theorem 11 *Let $n = 7$. Then the necessary conditions for the existence of an $OD(28; s_1, s_2, s_3, s_4)$ are sufficient for existence except possibly for*

$$(1,4,9,9), \quad (1,8,8,9), \quad (2,8,9,9), \quad (3,6,8,9), \quad (4,4,4,9).$$

Proof. Use [6] and Appendix D. □

Also, although an $OD(28; 4,4,9,9)$ is known, it is not constructed using four circulant matrices. Hence we consider the derivative 3-tuples, $(4,4,9)$, $(4,4,18)$, $(4,9,9)$, $(4,9,13)$, $(8,9,9)$ and together with the types for which orthogonal designs of order 28 are not yet known we have:

Lemma 8 *The necessary conditions for the existence of an $OD(28; s_1, s_2, s_3)$ are sufficient for the existence of $4-NPAF(s_1, s_2, s_3)$ sequences with the possible exception of the 76 cases indicated in Table 1.*

Proof. The sequences for $(1,1,13)$, $(1,2,17)$, $(1,2,11)$ and $(1,4,13)$ with $n \geq 7$ are given in Appendices B and C. The other required sequences may be constructed by equating variables from those found in Appendix C. □

Remark 5 The 236 3-tuples given in Table 1 are possible types of orthogonal designs in order 28. There are 41 cases as yet unresolved. We use

N if they are made from 4-sequences with zero NPAF;
P if they are made from 4-sequences with zero PAF;
F if they are made from the $OD(28; 4,4,9,9)$.

(1, 1, 1)	N	(1, 4, 16)	N	(2, 2, 10)	N	(2, 7, 19)	P	(3, 8, 10)	P	(5, 5, 10)	N
(1, 1, 2)	N	(1, 4, 17)	N	(2, 2, 13)	N	(2, 8, 8)	N	(3, 8, 15)		(5, 5, 13)	P
(1, 1, 4)	N	(1, 4, 18)	N	(2, 2, 16)	N	(2, 8, 9)	N	(3, 9, 14)		(5, 5, 16)	N
(1, 1, 5)	N	(1, 4, 20)	N	(2, 2, 17)	N	(2, 8, 10)	N	(3, 10, 15)		(5, 5, 18)	
(1, 1, 8)	N	(1, 5, 5)	N	(2, 2, 18)	N	(2, 8, 13)	N	(3, 11, 14)		(5, 6, 9)	N
(1, 1, 9)	N	(1, 5, 6)	N	(2, 2, 20)	N	(2, 8, 16)	N	(4, 4, 4)	N	(5, 6, 14)	
(1, 1, 10)	N	(1, 5, 9)	N	(2, 3, 4)	N	(2, 8, 18)	P	(4, 4, 5)	N	(5, 6, 15)	
(1, 1, 13)	N	(1, 5, 14)	N	(2, 3, 6)	N	(2, 9, 9)	N	(4, 4, 8)	N	(5, 7, 8)	N
(1, 1, 16)	N	(1, 5, 16)	N	(2, 3, 7)	N	(2, 9, 11)	N	(4, 4, 9)	N	(5, 7, 10)	
(1, 1, 17)	P	(1, 5, 19)		(2, 3, 9)	N	(2, 9, 12)	N	(4, 4, 10)	N	(5, 7, 14)	
(1, 1, 18)	N	(1, 5, 20)		(2, 3, 10)	N	(2, 9, 17)		(4, 4, 13)		(5, 8, 8)	N
(1, 1, 20)	N	(1, 6, 8)	N	(2, 3, 15)	N	(2, 10, 10)	N	(4, 4, 16)	N	(5, 8, 13)	N
(1, 1, 25)	P	(1, 6, 11)	P	(2, 3, 16)	N	(2, 10, 12)	N	(4, 4, 17)	N	(5, 9, 9)	P
(1, 1, 26)	P	(1, 6, 12)	N	(2, 4, 4)	N	(2, 11, 11)		(4, 4, 18)	P	(5, 9, 10)	P
(1, 2, 2)	N	(1, 6, 14)	N	(2, 4, 6)	N	(2, 11, 13)		(4, 4, 20)	N	(5, 9, 14)	
(1, 2, 3)	N	(1, 6, 18)	N	(2, 4, 8)	N	(2, 11, 15)		(4, 5, 5)	N	(5, 10, 10)	P
(1, 2, 4)	N	(1, 6, 21)	N	(2, 4, 9)	N	(2, 13, 13)	P	(4, 5, 6)	N	(6, 6, 6)	N
(1, 2, 6)	N	(1, 8, 8)	N	(2, 4, 11)	N	(3, 3, 3)	N	(4, 5, 9)	N	(6, 6, 12)	N
(1, 2, 8)	N	(1, 8, 9)	N	(2, 4, 12)	N	(3, 3, 6)	N	(4, 5, 14)	P	(6, 7, 8)	P
(1, 2, 9)	N	(1, 8, 11)	N	(2, 4, 16)	N	(3, 3, 12)	N	(4, 5, 16)	N	(6, 8, 9)	N
(1, 2, 11)	N	(1, 8, 12)	N	(2, 4, 17)	N	(3, 3, 15)	N	(4, 5, 19)		(6, 8, 11)	
(1, 2, 12)	N	(1, 8, 16)		(2, 4, 18)	N	(3, 4, 6)	N	(4, 6, 8)	N	(6, 8, 12)	N
(1, 2, 16)	P	(1, 8, 17)		(2, 4, 19)	N	(3, 4, 8)	N	(4, 6, 11)	P	(6, 9, 11)	
(1, 2, 17)	N	(1, 8, 18)	P	(2, 4, 22)		(3, 4, 14)	P	(4, 6, 12)	N	(7, 7, 7)	N
(1, 2, 18)	N	(1, 8, 19)	P	(2, 5, 5)	N	(3, 4, 18)	N	(4, 6, 14)	N	(7, 7, 14)	N
(1, 2, 19)	N	(1, 9, 9)	N	(2, 5, 7)	N	(3, 6, 6)	N	(4, 6, 18)	N	(7, 8, 10)	
(1, 2, 22)	N	(1, 9, 10)	N	(2, 5, 8)	N	(3, 6, 8)	N	(4, 8, 8)	N	(7, 8, 13)	
(1, 2, 25)	P	(1, 9, 13)	P	(2, 5, 13)	N	(3, 6, 9)	N	(4, 8, 9)	N	(7, 10, 11)	
(1, 3, 6)	N	(1, 9, 16)		(2, 5, 15)		(3, 6, 11)	N	(4, 8, 11)	N	(8, 8, 8)	N
(1, 3, 8)	N	(1, 9, 18)	N	(2, 5, 18)	N	(3, 6, 12)	N	(4, 8, 12)	N	(8, 8, 9)	
(1, 3, 14)	P	(1, 10, 10)	N	(2, 6, 7)	N	(3, 6, 16)		(4, 8, 16)	N	(8, 8, 10)	N
(1, 3, 18)	N	(1, 10, 11)	N	(2, 6, 9)	N	(3, 6, 17)		(4, 9, 9)	N	(8, 9, 9)	F
(1, 3, 22)		(1, 10, 14)	P	(2, 6, 11)		(3, 6, 18)	N	(4, 9, 10)	N	(8, 9, 11)	
(1, 3, 24)	N	(1, 13, 13)	P	(2, 6, 12)	N	(3, 6, 19)	N	(4, 9, 13)	F	(8, 10, 10)	N
(1, 4, 4)	N	(1, 13, 14)	P	(2, 6, 13)	N	(3, 7, 8)	N	(4, 10, 10)	N	(9, 9, 9)	P
(1, 4, 5)	N	(2, 2, 2)	N	(2, 6, 16)	N	(3, 7, 10)		(4, 10, 11)		(9, 9, 10)	P
(1, 4, 8)	N	(2, 2, 4)	N	(2, 6, 17)		(3, 7, 11)		(4, 10, 14)	N		
(1, 4, 9)	N	(2, 2, 5)	N	(2, 7, 10)		(3, 7, 15)		(5, 5, 5)	N		
(1, 4, 10)	N	(2, 2, 8)	N	(2, 7, 12)	N	(3, 7, 18)	N	(5, 5, 8)	N		
(1, 4, 13)	N	(2, 2, 9)	N	(2, 7, 13)		(3, 8, 9)	N	(5, 5, 9)	N		

Table 1: Known 3-variable designs in order 28

Lemma 9 *The necessary conditions for the existence of an $OD(28; s_1, s_2)$ are sufficient for the existence of $4 - NPAF(s_1, s_2)$ sequences with the possible exception of*

$$\begin{array}{ccccccc} (1,25) & (1,26) & (2,25) & (3,23) & (4,22) & (5,23) & (7,15) \\ (7,19) & (8,19) & (9,17) & (10,15) & (11,14) & (11,15) & (11,17) \\ (13,14) & (13,15). & & & & & \end{array}$$

Proof. The same 27 2-tuples indicated in Table 2 cannot be the types of four sequences of commuting variables of length $n \geq 7$. Appendices B and C and [6, p403] give designs for (1,27), (2,26), (6,20), (7,7), (7,10), (11,11), and (13,13), all others are obtained by equating variables in 4 variable designs given in [6, p401-403] and Appendix C. These all exist for all $n \geq 7$. □

Lemma 10 [6, Geramita and Seberry] *The necessary conditions for the existence of an $OD(28; s_1, s_2)$ are sufficient for existence using four circulant matrices.*

Remark 6 The 2 variable orthogonal designs indicated in Table 2 do not exist in order 28.

$(1,7)$	$(1,15)$	$(1,23)$	$(2,14)$	$(3,5)$	$(3,13)$	$(3,20)$
$(3,21)$	$(4,7)$	$(4,15)$	$(4,23)$	$(5,11)$	$(5,12)$	$(5,19)$
$(5,22)$	$(6,10)$	$(7,9)$	$(7,16)$	$(7,17)$	$(7,20)$	$(8,14)$
$(9,15)$	$(10,17)$	$(11,13)$	$(11,16)$	$(12,13)$	$(12,15)$	

Table 2: 2-tuples which cannot be the types of orthogonal designs in order 28

5 Asymptotic Results

Theorem 12 *The known asymptotic results are summarized in Table 3.*

Proof. All the results quoted except those improved here, may be found in Geramita and Seberry [6, p168].

The 4-sequences with $NPAF = 0$ which may be used in Corollary 1 of Theorem 3 to give $OD(4n; s_1, s_2, s_3, s_4)$ for (s_1, s_2, s_3, s_4) = (1,2,8,9), (1,5,5,9), (2,2,4,9) and (2,3,6,9), for all $n \geq 5$ are given in Appendix B.

The 4-sequences with $NPAF = 0$ which may be used in Corollary 1 of Theorem 3 to give $OD(4n; 1, 3, 6, 8)$ for $n \geq 7$ are given in Appendix C.

From Appendix A there exists an $OD(12; 1, 2, 3, 6)$ and hence an $OD(24; 1, 2, 3, 6, 6)$ and thus an $OD(24; 1, 3, 6, 8)$. Hence we have an $OD(4n; 1, 3, 6, 8)$ for all $n \geq 6$.

The existence of $OD(4n; s_1, s_2, s_3, s_4)$ for (s_1, s_2, s_3, s_4) = (1,1,2,18), (1,1,10, 10), (2,2,4,16), (2,2,10,10), and (2,4,6,12) for all $n \geq 6$ is obtained by using the sequences in Appendix C in Corollary 1 of Theorem 3.

The sequences corresponding to the existence of $OD(4n; s_1, s_2, s_3, s_4)$ for (s_1, s_2, s_3, s_4) = (1,4,8,8), and (2,2,2,18), and (3,3,3,12), for all $n \geq 7$ can be found in Appendix C. These designs for $4n = 24$ can be found on [6, p373].

Appendix C establishes the existence of $OD(4n; s_1, s_2, s_3, s_4)$ for (s_1, s_2, s_3, s_4) = (1,2,4,18), (1,3,6,18), (1,4,4,16), (4,4,4,16), (4,4,10,10), (5,5,8,8), and (7,7,7,7) for all $n \geq 7$.

There is an $OD(28; 4, 5, 5, 9)$ given in Appendix D. An $OD(24; 4, 5, 5, 9)$ and an $OD(32; 4, 5, 5, 9)$ are given in [6]. Also in [6] an $OD(20; 4, 5, 9)$ is given ensuring the existence of an $OD(40; 4, 5, 5, 9)$. Hence by taking direct sums of the designs we have the existence of an $OD(4n; 4, 5, 5, 9)$ for n = 6, 7, 8 and 10, and hence for all $n \geq 12$. □

$12 \leq \sum_{i=1}^{4} s_i \leq 16$	N	$16 < \sum_{i=1}^{4} s_i \leq 20$	N	$20 < \sum_{i=1}^{4} s_i \leq 24$	N	$24 < \sum_{i=1}^{4} s_i \leq 28$	N
(1,1,4,9)	16	(1,1,1,16)	24	(1,1,2,18)	24	(1,1,1,25)	56
(1,2,2,9)	16	(1,1,8,8)	20	(1,1,4,16)	24	(1,1,5,20)	144
(1,2,4,8)	16	(1,1,9,9)	20	(1,1,10,10)	24	(1,1,8,18)	56
(1,4,4,4)	16	(1,2,8,9)	20	(1,2,2,16)	48	(1,1,9,16)	312
(1,4,5,5)	16	(1,3,6,8)	24	(1,2,6,12)	24	(1,1,13,13)	48
(2,2,2,8)	16	(1,4,4,9)	48	(1,4,8,8)	24	(1,2,4,18)	28
(2,2,5,5)	24	(1,5,5,9)	20	(1,4,9,9)	72	(1,3,6,18)	28
(2,3,4,6)	16	(2,2,4,9)	20	(2,2,2,18)	24	(1,4,4,16)	28
(4,4,4,4)	16	(2,2,8,8)	20	(2,2,4,16)	24	(1,4,10,10)	40
		(2,3,6,9)	20	(2,2,9,9)	24	(1,8,8,9)	80
		(2,4,4,8)	20	(2,2,10,10)	24	(1,9,9,9)	56
		(2,5,5,8)	20	(2,4,6,12)	24	(2,4,4,18)	28
		(3,3,6,6)	20	(2,4,8,9)	160	(2,8,8,8)	28
		(4,4,5,5)	20	(3,3,3,12)	24	(2,8,9,9)	80
		(5,5,5,5)	20	(3,4,6,8)	28	(3,6,8,9)	952
				(4,4,4,9)	112	(4,4,4,16)	28
				(4,4,8,8)	24	(4,4,9,9)	48
				(4,5,5,9)	48	(4,4,10,10)	28
				(6,6,6,6)	24	(5,5,8,8)	28
						(5,5,9,9)	80
						(7,7,7,7)	28

Table 3: N is the order such that the indicated design exists in every order $4t \geq N$.

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Appendix A: Order 12 (Sequences with zero nonperiodic autocorrelation function)

Design	A_1			A_2			A_3			A_4		
(1, 1, 1, 1)	a	0	0	b	0	0	c	0	0	d	0	0
(1, 1, 1, 4)	d	a	$-d$	d	0	d	b	0	0	c	0	0
(1, 1, 1, 9)	do not exist											
(1, 1, 2, 2)	a	0	0	b	0	0	c	d	0	c	$-d$	0
(1, 1, 2, 8)	d	a	$-d$	d	b	$-d$	d	c	d	d	$-c$	d
(1, 1, 4, 4)	c	a	$-c$	c	0	c	d	b	$-d$	d	0	d
(1, 1, 5, 5)	c	a	$-c$	c	d	c	d	b	$-d$	d	$-c$	d
(1, 2, 2, 4)	d	a	$-d$	d	0	d	b	c	0	b	$-c$	0
(1, 2, 3, 6)	d	a	$-d$	d	c	d	d	$-c$	b	d	$-c$	$-b$
(2, 2, 2, 2)	a	b	0	a	$-b$	0	c	d	0	c	$-d$	0
(2, 2, 4, 4)	a	$-b$	c	a	$-b$	$-c$	a	b	d	a	b	$-d$
(3, 3, 3, 3)	a	b	c	$-b$	a	d	$-c$	$-d$	a	$-d$	c	$-b$
(1, 2, 9)	a	b	$-a$	a	$-a$	c	a	$-a$	$-c$	a	a	a
(1, 1, 10)	a	b	$-a$	c	d	$-c$	a	c	a	c	$-a$	c

Appendix B: Order 20 (Sequences with zero nonperiodic autocorrelation function)

Design	A_1					A_2					A_3					A_4				
(1, 1, 1, 1)	a	0	0	0	0	b	0	0	0	0	c	0	0	0	0	d	0	0	0	0
(1, 1, 1, 4)	d	a	$-d$	0	0	d	0	d	0	0	b	0	0	0	0	c	0	0	0	0
(1, 1, 1, 9)	d	a	$-d$	0	0	d	b	$-d$	0	0	d	0	c	0	$-d$	d	0	d	0	d
(1, 1, 2, 2)	a	0	0	0	0	b	0	0	0	0	c	d	0	0	0	d	$-c$	0	0	0
(1, 1, 2, 8)	d	a	$-d$	0	0	d	b	$-d$	0	0	d	c	d	0	0	d	$-c$	d	0	0
(1, 1, 4, 4)	c	a	$-c$	0	0	c	0	c	0	0	d	b	$-d$	0	0	d	0	d	0	0
(1, 1, 4, 9)	d	a	$-d$	0	0	d	b	$-d$	0	0	d	c	0	c	$-d$	d	c	d	$-c$	d
(1, 1, 5, 5)	c	a	$-c$	0	0	c	d	c	0	0	d	b	$-d$	0	0	d	$-c$	d	0	0
(1, 1, 8, 8)	c	$-d$	a	d	$-c$	d	c	b	$-c$	$-d$	d	c	0	c	d	c	$-d$	0	$-d$	c
(1, 1, 9, 9)	d	c	a	$-c$	$-d$	c	$-d$	b	d	$-c$	c	$-c$	c	c	c	d	$-d$	d	d	d
(1, 2, 2, 4)	d	a	$-d$	0	0	d	0	d	0	0	b	c	0	0	0	b	$-c$	0	0	0
(1, 2, 2, 9)	d	0	a	0	$-d$	d	0	d	0	d	b	d	c	$-d$	0	b	$-d$	$-c$	d	0
(1, 2, 3, 6)	d	a	$-d$	0	0	d	c	d	0	0	d	$-c$	b	0	0	d	$-c$	$-b$	0	0
(1, 2, 4, 8)	d	c	b	$-c$	d	d	0	$-b$	0	d	d	0	a	0	$-d$	d	c	0	c	$-d$
(1, 2, 8, 9)	d	c	a	$-c$	$-d$	d	c	d	$-c$	d	b	c	$-d$	c	d	b	$-c$	d	$-c$	$-d$
(1, 3, 6, 8)	if it exists it cannot be constructed from circulants																			
(1, 4, 4, 4)	b	a	$-b$	0	0	b	0	b	0	0	c	d	$-d$	c	0	c	d	d	$-c$	0
(1, 4, 4, 9)	if it exists it cannot be constructed from circulants																			
(1, 4, 5, 5)	c	a	$-c$	0	0	c	d	c	0	0	b	d	0	$-d$	b	b	d	$-c$	d	$-b$
(1, 5, 5, 9)	d	$-c$	a	c	$-d$	b	d	$-c$	$-d$	$-c$	b	$-d$	b	d	c	d	b	d	$-b$	d
(2, 2, 2, 2)	a	b	0	0	0	b	$-a$	0	0	0	c	d	0	0	0	d	$-c$	0	0	0
(2, 2, 2, 8)	a	d	b	$-d$	0	a	$-d$	$-b$	d	0	c	0	d	0	d	c	0	$-d$	0	$-d$
(2, 2, 4, 4)	a	b	0	0	0	b	$-a$	0	0	0	0	c	d	$-d$	c	0	c	d	d	$-c$
(2, 2, 4, 9)	a	0	$-d$	b	d	a	0	d	$-b$	$-d$	d	c	0	c	$-d$	d	c	d	$-c$	d

Appendix B (continued): Order 20 (Sequences with zero nonperiodic autocorrelation function)

Design	A_1					A_2					A_3					A_4				
(2, 2, 5, 5)	if it exists it cannot be constructed from circulants																			
(2, 2, 8, 8)	a	d	c	d	$-c$	a	$-d$	$-c$	$-d$	c	b	c	$-d$	c	d	b	$-c$	d	$-c$	$-d$
(2, 3, 4, 6)	a	d	0	$-d$	a	a	d	c	d	$-a$	b	d	$-c$	0	0	b	$-d$	c	0	0
(2, 3, 6, 9)	b	a	$-d$	c	d	d	a	$-b$	$-c$	$-d$	d	c	$-d$	c	$-b$	d	$-c$	d	c	d
(2, 4, 4, 8)	d	c	a	$-c$	d	d	b	$-a$	$-b$	d	d	b	0	b	$-d$	d	c	0	c	$-d$
(2, 5, 5, 8)	b	c	a	$-c$	b	b	c	d	c	$-b$	b	d	$-a$	$-d$	b	b	d	$-c$	d	$-b$
(3, 3, 3, 3)	a	b	c	0	0	a	$-b$	0	d	0	a	0	$-c$	$-d$	0	b	$-c$	d	0	0
(3, 3, 6, 6)	a	d	b	$-d$	a	a	d	c	d	$-a$	c	$-d$	a	0	$-b$	c	$-d$	$-a$	0	b
(4, 4, 4, 4)	0	a	b	$-b$	a	0	a	b	b	$-a$	0	c	d	$-d$	c	0	c	d	d	$-c$
(4, 4, 5, 5)	a	c	0	$-c$	a	a	c	d	c	$-a$	b	d	0	$-d$	b	b	d	$-c$	d	$-b$
(5, 5, 5, 5)	a	b	b	d	$-d$	$-b$	a	a	c	$-c$	d	c	c	$-a$	a	$-c$	d	d	$-b$	b
(1, 1, 13)	a	a	b	$-a$	$-a$	a	$-a$	c	a	$-a$	a	0	a	0	a	a	0	0	0	a
(1, 2, 17)	c	$-c$	a	c	$-c$	b	c	$-c$	$-c$	$-c$	b	$-c$	c	c	c	c	$-c$	c	c	c
(1, 2, 11)	c	0	a	0	$-c$	c	c	$-c$	b	0	c	c	$-c$	$-b$	0	c	0	c	0	c
(1, 3, 14)																				
(1, 4, 13)																				
(1, 6, 11)																				
(1, 8, 11)																				
(2, 5, 7)																				
(3, 6, 8)	a	c	b	$-c$	b	a	c	0	c	$-b$	b	c	$-a$	c	0	b	c	$-b$	$-c$	0
(3, 7, 8)	if it exists it cannot be constructed from circulants																			
(4, 4, 10)	a	a	$-a$	b	c	a	a	$-a$	$-b$	$-c$	a	0	a	b	$-c$	a	0	a	$-b$	c
(4, 6, 8)	a	b	$-a$	b	$-c$	a	b	$-a$	$-b$	$-c$	a	0	a	b	$-c$	a	0	a	$-b$	c
(7, 7)	a	a	$-a$	b	0	a	0	a	b	0	b	b	$-b$	$-a$	0	b	0	b	$-a$	0
(7, 10)	a	0	b	b	$-b$	a	0	$-b$	b	b	a	b	a	b	$-a$	a	$-b$	0	$-b$	a

Appendix C: Order 28 (Sequences with zero nonperiodic autocorrelation function)

Design	A_1							A_2						
(1, 1, 1, 1)	a	0	0	0	0	0	0	b	0	0	0	0	0	0
(1, 1, 1, 4)	a	0	0	0	0	0	0	b	0	0	0	0	0	0
(1, 1, 1, 9)	d	a	$-d$	0	0	0	0	d	b	$-d$	0	0	0	0
(1, 1, 1, 16)														
(1, 1, 1, 25)	do not exist													
(1, 1, 2, 2)	a	0	0	0	0	0	0	b	0	0	0	0	0	0
(1, 1, 2, 8)	d	a	$-d$	0	0	0	0	d	b	$-d$	0	0	0	0
(1, 1, 2, 18)	a	a	c	$-a$	$-a$	0	0	a	$-a$	d	a	$-a$	0	0
(1, 1, 4, 4)	a	0	0	0	0	0	0	b	0	0	0	0	0	0
(1, 1, 4, 9)	d	a	$-d$	0	0	0	0	d	b	$-d$	0	0	0	0
(1, 1, 4, 16)	b	0	b	0	b	0	b	b	0	b	a	$-b$	0	$-b$
(1, 1, 5, 5)	a	0	0	0	0	0	0	b	0	0	0	0	0	0
(1, 1, 8, 8)	c	d	a	$-d$	$-c$	0	0	$-c$	d	b	$-d$	c	0	0
(1, 1, 8, 18)														
(1, 1, 9, 9)	d	c	a	$-c$	$-d$	0	0	c	$-d$	b	d	$-c$	0	0
(1, 1, 10, 10)	d	c	a	$-c$	$-d$	0	0	c	$-d$	b	d	$-c$	0	0
(1, 1, 13, 13)														
(1, 2, 2, 4)	b	a	$-b$	0	0	0	0	b	0	b	0	0	0	0
(1, 2, 2, 9)	d	0	a	0	$-d$	0	0	d	0	d	0	d	0	0
(1, 2, 2, 16)	d	a	$-d$	d	b	$-d$	0	d	a	$-d$	$-d$	$-b$	d	0
(1, 2, 3, 6)	a	b	c	0	0	0	0	a	b	$-c$	0	0	0	0
(1, 2, 4, 8)	c	a	$-c$	0	0	0	0	c	0	c	0	0	0	0
(1, 2, 4, 18)	d	c	$-d$	a	d	$-c$	$-d$	0	d	$-d$	d	b	d	d
(1, 2, 6, 12)	$-d$	c	0	a	0	$-c$	d	d	c	d	0	d	b	$-d$
(1, 2, 8, 9)	d	c	a	$-c$	$-d$	0	0	d	c	d	$-c$	d	0	0
(1, 3, 6, 8)	d	c	a	$-c$	$-d$	0	0	b	c	$-d$	c	d	0	0
(1, 3, 6, 18)	d	c	$-d$	a	d	$-c$	$-d$	$-c$	d	d	b	$-c$	$-d$	$-d$
(1, 4, 4, 4)	b	a	$-b$	0	0	0	0	b	0	b	0	0	0	0
(1, 4, 4, 9)														
(1, 4, 4, 16)	$-d$	$-b$	d	a	$-d$	b	d	d	b	$-d$	0	$-d$	b	d
(1, 4, 5, 5)	b	a	$-b$	0	0	0	0	b	0	b	0	0	0	0

Design	A_3							A_4						
(1, 1, 1, 1)	c	0	0	0	0	0	0	d	0	0	0	0	0	0
(1, 1, 1, 4)	d	c	$-d$	0	0	0	0	d	0	d	0	0	0	0
(1, 1, 1, 9)	d	0	c	0	$-d$	0	0	d	0	d	0	d	0	0
(1, 1, 1, 16)														
(1, 1, 1, 25)														
(1, 1, 2, 2)	c	d	0	0	0	0	0	c	$-d$	0	0	0	0	0
(1, 1, 2, 8)	d	c	d	0	0	0	0	d	$-c$	d	0	0	0	0
(1, 1, 2, 18)	b	a	a	a	$-a$	a	0	$-b$	a	a	a	$-a$	a	0
(1, 1, 4, 4)	c	c	d	$-d$	0	0	0	d	d	$-c$	c	0	0	0
(1, 1, 4, 9)	d	c	0	c	$-d$	0	0	d	c	d	$-c$	d	0	0
(1, 1, 4, 16)	b	c	$-b$	0	$-b$	c	b	b	c	$-b$	d	b	$-c$	$-b$
(1, 1, 5, 5)	c	c	$-c$	d	0	d	0	d	d	$-d$	$-c$	0	$-c$	0
(1, 1, 8, 8)	c	d	0	d	c	0	0	c	$-d$	0	$-d$	c	0	0
(1, 1, 8, 18)														
(1, 1, 9, 9)	c	$-c$	c	c	c	0	0	d	$-d$	d	d	d	0	0
(1, 1, 10, 10)	c	$-c$	c	c	c	d	0	d	$-d$	d	d	d	$-c$	0
(1, 1, 13, 13)														
(1, 2, 2, 4)	c	d	0	0	0	0	0	c	$-d$	0	0	0	0	0
(1, 2, 2, 9)	b	d	c	$-d$	0	0	0	b	$-d$	$-c$	d	0	0	0
(1, 2, 2, 16)	d	0	d	c	$-d$	0	$-d$	d	0	d	0	d	0	d
(1, 2, 3, 6)	b	$-a$	b	0	0	0	0	b	d	$-b$	0	0	0	0
(1, 2, 4, 8)	d	b	$-d$	d	0	d	0	d	b	$-d$	$-d$	0	$-d$	0
(1, 2, 4, 18)	d	d	d	$-b$	d	$-d$	0	d	c	$-d$	0	$-d$	c	d
(1, 2, 6, 12)	d	c	d	0	$-d$	$-b$	d	d	$-c$	0	0	0	$-c$	d
(1, 2, 8, 9)	b	c	$-d$	c	d	0	0	b	$-c$	d	$-c$	$-d$	0	0
(1, 3, 6, 8)	d	0	b	0	d	$-c$	0	$-d$	0	$-d$	0	b	$-c$	0
(1, 3, 6, 18)	c	$-d$	$-d$	b	$-d$	d	$-d$	d	d	c	b	d	$-d$	d
(1, 4, 4, 4)	c	c	d	$-d$	0	0	0	d	d	$-c$	c	0	0	0
(1, 4, 4, 9)														
(1, 4, 4, 16)	$-d$	d	$-d$	d	0	c	c	d	d	d	d	0	$-c$	c
(1, 4, 5, 5)	c	c	$-c$	d	0	d	0	d	d	$-d$	$-c$	0	$-c$	0

Appendix C (continued): Order 28 (Sequences with zero nonperiodic autocorrelation function)

Design	A_1							A_2							A_3							A_4						
(1, 4, 8, 8)	0	$-d$	c	a	$-c$	d	0	b	d	c	$-c$	$-d$	b	0	b	$-d$	$-c$	$-c$	$-d$	$-b$	0	d	$-c$	0	$-c$	d	0	0
(1, 4, 9, 9)																												
(1, 4, 10, 10)																												
(1, 5, 5, 9)	d	$-c$	a	c	$-d$	0	0	b	d	$-c$	$-d$	$-c$	0	0	b	$-d$	b	d	c	0	0	d	b	d	$-b$	d	0	0
(1, 8, 8, 9)																												
(1, 9, 9, 9)																												
(2, 2, 2, 2)	a	b	0	0	0	0	0	a	$-b$	0	0	0	0	0	c	d	0	0	0	0	0	c	$-d$	0	0	0	0	0
(2, 2, 2, 8)	a	b	0	0	0	0	0	a	$-b$	0	0	0	0	0	c	d	$-c$	c	0	c	0	c	d	$-c$	$-c$	0	$-c$	0
(2, 2, 2, 18)	$-d$	b	d	0	$-d$	a	d	d	$-b$	$-d$	0	$-d$	a	d	0	$-d$	d	c	d	d	d	d	$-d$	d	$-c$	d	d	0
(2, 2, 4, 4)	a	$-b$	c	0	0	0	0	a	$-b$	$-c$	0	0	0	0	a	b	d	0	0	0	0	a	b	$-d$	0	0	0	0
(2, 2, 4, 9)	a	0	$-d$	b	d	0	0	a	0	d	$-b$	$-d$	0	0	d	c	0	c	$-d$	0	0	d	c	d	$-c$	d	0	0
(2, 2, 4, 16)	a	b	$-a$	a	c	a	0	a	b	$-a$	a	$-c$	a	0	a	b	$-a$	$-a$	d	$-a$	0	a	b	$-a$	$-a$	$-d$	$-a$	0
(2, 2, 5, 5)	a	b	0	0	0	0	0	a	$-b$	0	0	0	0	0	c	c	$-c$	d	0	d	0	d	d	$-d$	$-c$	0	$-c$	0
(2, 2, 8, 8)	a	d	c	d	$-c$	0	0	a	$-d$	$-c$	$-d$	c	0	0	b	c	$-d$	c	d	0	0	b	$-c$	d	$-c$	$-d$	0	0
(2, 2, 9, 9)	d	b	$-d$	c	a	$-c$	0	d	b	$-d$	$-c$	$-a$	c	0	d	c	0	c	$-d$	c	0	$-c$	d	0	d	c	d	0
(2, 2, 10, 10)	c	c	$-c$	d	a	d	0	c	c	$-c$	d	$-a$	d	0	d	d	$-d$	$-c$	b	$-c$	0	d	d	$-d$	$-c$	$-b$	$-c$	0
(2, 3, 4, 6)	a	d	0	$-d$	a	0	0	a	d	c	d	$-a$	0	0	b	c	$-d$	0	0	0	0	b	$-c$	d	0	0	0	0
(2, 3, 6, 9)	b	a	$-d$	c	d	0	0	d	a	$-b$	$-c$	$-d$	0	0	d	c	$-d$	c	$-b$	0	0	d	$-c$	d	c	d	0	0
(2, 4, 4, 8)	d	c	a	$-c$	d	0	0	d	b	$-a$	$-b$	d	0	0	d	b	0	b	$-d$	0	0	d	c	0	c	$-d$	0	0
(2, 4, 4, 18)																												
(2, 4, 6, 12)	a	b	c	a	b	$-c$	0	a	b	c	$-a$	$-b$	c	0	b	$-a$	b	b	d	$-b$	0	b	$-a$	b	$-b$	$-d$	b	0
(2, 4, 8, 9)	c	$-d$	0	$-a$	0	d	c	c	$-b$	0	a	0	b	c	c	b	$-d$	0	d	b	$-c$	c	$-d$	d	$-d$	$-d$	$-d$	$-c$
(2, 5, 5, 8)	a	d	$-a$	a	0	a	0	a	d	$-a$	$-a$	0	$-a$	0	c	c	$-c$	b	0	b	0	b	b	$-b$	$-c$	0	$-c$	0
(2, 8, 8, 8)	a	a	b	$-b$	c	d	$-c$	a	a	b	$-b$	$-c$	$-d$	c	a	$-a$	b	b	c	0	c	a	$-a$	b	b	$-c$	0	$-c$
(2, 8, 9, 9)																												
(3, 3, 3, 3)	a	b	$-c$	0	0	0	0	a	$-b$	0	d	0	0	0	a	0	c	$-d$	0	0	0	b	c	d	0	0	0	0
(3, 3, 3, 12)	c	b	c	$-c$	d	c	0	c	$-b$	c	0	0	0	a	c	0	c	c	$-d$	$-c$	$-a$	0	b	0	c	$-d$	$-c$	a
(3, 3, 6, 6)	a	d	b	$-d$	a	0	0	a	d	c	d	$-a$	0	0	c	$-d$	a	0	$-b$	0	0	c	$-d$	$-a$	0	b	0	0
(3, 4, 6, 8)																												
(3, 6, 8, 9)																												
(4, 4, 4, 4)	a	b	c	d	0	0	0	a	$-b$	c	$-d$	0	0	0	a	b	$-c$	$-d$	0	0	0	a	$-b$	$-c$	d	0	0	0

Appendix C (continued): Order 28 (Sequences with zero nonperiodic autocorrelation function)

Design	A_1							A_2							A_3							A_4						
(4, 4, 4, 9)																												
(4, 4, 4, 16)	$-c$	$-d$	c	$-c$	$-a$	$-c$	$-b$	c	d	$-c$	c	$-a$	c	$-b$	c	d	$-c$	$-c$	a	$-c$	$-b$	$-c$	$-d$	c	c	a	c	$-b$
(4, 4, 5, 5)	a	a	b	$-b$	0	0	0	b	b	$-a$	a	0	0	0	c	0	c	d	d	$-d$	0	d	0	d	$-c$	$-c$	c	0
(4, 4, 8, 8)	a	a	b	$-b$	c	d	0	a	a	b	$-b$	$-c$	$-d$	0	$-a$	a	$-b$	$-b$	$-c$	d	0	a	$-a$	b	b	$-c$	d	0
(4, 4, 9, 9)																												
(4, 4, 10, 10)	b	c	a	c	d	d	$-d$	b	$-c$	a	$-c$	$-d$	$-d$	d	b	d	$-a$	d	$-c$	$-c$	c	b	$-d$	$-a$	$-d$	c	c	$-c$
(4, 5, 5, 9)																												
(5, 5, 5, 5)	a	a	$-a$	b	0	b	0	$-b$	$-b$	b	a	0	a	0	$-c$	$-c$	c	$-d$	0	$-d$	0	$-d$	$-d$	d	c	0	c	0
(5, 5, 8, 8)	$-c$	d	b	a	b	$-d$	c	$-c$	$-d$	a	$-b$	a	d	c	c	d	$-a$	0	a	d	c	$-c$	d	$-b$	0	b	d	$-c$
(5, 5, 9, 9)																												
(6, 6, 6, 6)	a	a	b	$-b$	c	d	0	$-b$	$-b$	a	$-a$	d	$-c$	0	$-c$	$-c$	$-d$	d	a	b	0	$-d$	$-d$	c	$-c$	$-b$	a	0
(7, 7, 7, 7)	a	a	$-a$	b	c	b	d	$-b$	$-b$	b	a	d	a	$-c$	$-c$	$-c$	c	$-d$	a	$-d$	b	$-d$	$-d$	d	c	$-b$	c	a
(1, 4, 13)	a	$-a$	c	a	$-a$	0	0	a	0	0	0	a	0	0	b	a	a	0	$-a$	$-a$	b	b	a	0	a	0	a	$-b$
(3, 6, 8)	a	c	b	$-c$	b	0	0	a	c	0	c	$-b$	0	0	b	c	$-a$	c	0	0	0	b	c	$-b$	$-c$	0	0	0
(4, 4, 10)	a	a	$-a$	b	c	0	0	a	a	$-a$	$-b$	$-c$	0	0	a	0	a	b	$-c$	0	0	a	0	a	$-b$	c	0	0
(4, 6, 8)	a	b	$-a$	b	$-c$	0	0	a	b	$-a$	$-b$	$-c$	0	0	a	0	a	b	$-c$	0	0	a	0	a	$-b$	c	0	0
(6, 8, 12)	a	d	0	$-d$	a	c	$-d$	a	d	0	$-d$	a	$-c$	d	a	d	c	d	$-a$	c	$-d$	a	d	c	d	$-a$	$-c$	d
(1, 27)	a	a	$-a$	b	a	$-a$	$-a$	a	a	$-a$	a	$-a$	a	$-a$	a	a	$-a$	$-a$	$-a$	$-a$	a	a	a	$-a$	a	a	a	a
(2, 26)	a	b	b	$-b$	b	$-b$	$-b$	$-a$	b	b	$-b$	b	$-b$	$-b$	b	b	b	$-b$	b	b	b	b	b	$-b$	$-b$	$-b$	b	$-b$
(7, 7)	a	a	$-a$	b	0	0	0	a	0	a	b	0	0	0	b	b	$-b$	$-a$	0	0	0	b	0	b	$-a$	0	0	0
(7, 10)	a	0	b	b	$-b$	0	0	a	0	$-b$	b	b	0	0	a	b	a	b	$-a$	0	0	a	$-b$	0	$-b$	a	0	0
(11, 11)	a	a	a	b	$-b$	b	0	a	$-a$	a	$-b$	$-b$	$-b$	0	a	a	$-a$	b	0	$-b$	0	a	0	$-a$	$-b$	b	b	0
(13, 13)	a	a	a	$-a$	b	0	b	b	b	b	$-b$	$-a$	0	$-a$	a	a	$-a$	a	b	b	$-b$	b	b	$-b$	b	$-a$	$-a$	a

Appendix D: Order 28 (Sequences with zero periodic autocorrelation function)

Design	A_1							A_2							A_3							A_4						
(1, 1, 1, 16)	a	d	d	$-d$	d	$-d$	$-d$	b	0	d	0	0	$-d$	0	c	0	d	0	0	$-d$	0	0	d	d	$-d$	d	d	d
(1, 1, 1, 25)	a	d	d	$-d$	d	$-d$	$-d$	b	d	d	$-d$	d	$-d$	$-d$	c	d	d	$-d$	d	$-d$	$-d$	$-d$	d	d	d	d	d	d
(1, 1, 8, 18)	$-a$	b	b	a	b	a	a	a	b	b	$-a$	b	$-a$	$-a$	c	b	b	$-b$	b	$-b$	$-b$	d	b	b	$-b$	b	$-b$	$-b$
(1, 1, 13, 13)	$-a$	b	b	a	b	a	a	$-b$	$-a$	$-a$	b	$-a$	b	b	c	a	a	$-a$	a	$-a$	$-a$	d	b	b	$-b$	b	$-b$	$-b$
(1, 4, 4, 9)	a	d	d	$-d$	d	$-d$	$-d$	0	d	d	0	d	0	0	c	c	b	$-b$	0	0	0	b	b	$-c$	c	0	0	0
(1, 4, 10, 10)	a	$-d$	$-d$	d	$-d$	d	d	c	b	b	$-c$	$-c$	c	0	$-b$	$-c$	$-c$	$-d$	0	$-c$	b	0	c	$-c$	$-d$	$-d$	c	$-d$
(1, 9, 9, 9)	$-b$	a	b	$-d$	c	$-c$	d	$-b$	c	c	d	d	b	b	d	$-c$	$-d$	d	b	$-c$	b	$-d$	$-c$	$-d$	b	c	b	c
(2, 4, 4, 18)	$-b$	d	c	d	$-d$	d	d	b	a	c	$-d$	$-d$	d	d	b	$-a$	c	d	d	$-d$	$-d$	b	d	$-c$	d	$-d$	d	d
(3, 4, 6, 8)	$-c$	c	c	0	c	a	b	c	$-c$	$-c$	0	$-c$	a	b	d	b	0	$-b$	d	0	0	$-d$	b	$-a$	b	d	0	0
(4, 4, 9, 9)	known but not constructed from circulants																											
(4, 5, 5, 9)	a	$-b$	c	$-c$	0	a	b	a	0	$-a$	b	$-c$	$-c$	0	$-b$	d	d	$-d$	$-d$	$-b$	$-c$	d	$-d$	$-d$	0	$-d$	0	$-d$
(5, 5, 9, 9)	d	$-d$	a	b	$-c$	b	c	c	$-d$	a	d	a	$-b$	$-c$	a	$-d$	$-a$	c	d	c	c	b	d	d	$-c$	d	$-b$	c
(1, 3, 24)	a	c	$-c$	c	$-c$	c	$-c$	b	c	$-c$	$-c$	$-c$	$-c$	$-c$	b	$-c$	$-c$	c	c	c	c	b	$-c$	c	c	$-c$	c	c
(1, 6, 21)	c	b	$-b$	b	$-b$	b	$-b$	a	b	b	b	b	$-b$	$-b$	a	$-a$	b	b	$-a$	b	b	a	a	b	$-b$	$-b$	b	b
(1, 9, 13)	a	b	b	$-b$	b	$-b$	$-b$	0	b	b	0	b	0	0	0	c	$-c$	c	c	$-c$	c	c	c	c	$-c$	$-c$	c	c
(1, 10, 14)	c	$-b$	$-b$	b	$-b$	b	b	0	$-b$	a	a	a	$-a$	a	0	b	$-a$	b	a	b	a	0	b	b	a	$-b$	$-a$	b
(2, 7, 19)	c	a	a	a	$-a$	b	$-a$	$-c$	a	b	a	$-a$	a	$-a$	b	$-b$	a	a	$-b$	a	a	b	b	a	$-a$	$-a$	a	a
(2, 8, 13)	$-b$	b	b	c	b	0	0	$-b$	b	b	$-c$	b	0	0	0	a	$-a$	a	a	$-a$	a	a	a	a	$-a$	$-a$	a	a
(3, 23)	$-a$	0	b	b	b	$-b$	b	a	$-b$	b	b	b	$-b$	b	a	b	b	$-b$	b	$-b$	0	b	b	b	b	b	$-b$	$-b$
(5, 21)	b	$-a$	a	b	$-b$	a	a	b	$-b$	a	$-a$	a	$-a$	a	a	a	$-a$	$-a$	$-a$	a	0	a	a	a	a	$-a$	a	0
(5, 23)	a	a	$-a$	$-b$	b	b	b	$-a$	b	$-a$	b	$-b$	b	$-b$	b	b	b	b	b	$-b$	$-b$	b	b	b	$-b$	b	b	$-b$
(6, 20)	b	a	b	a	$-a$	0	a	b	a	$-b$	a	0	$-a$	$-a$	b	a	a	$-a$	$-a$	a	$-a$	b	$-a$	$-a$	$-a$	$-a$	a	$-a$
(7, 15)	b	a	b	a	$-b$	0	a	b	$-a$	0	b	0	0	0	b	$-a$	a	$-a$	0	a	a	b	$-a$	$-a$	$-a$	a	a	$-a$
(9, 17)	a	$-a$	a	$-a$	$-a$	a	0	a	a	$-a$	$-b$	$-a$	$-a$	b	a	a	$-b$	b	a	a	0	a	b	$-a$	b	b	b	$-b$
(11, 12)	a	$-a$	a	a	0	0	0	$-b$	a	a	a	$-a$	$-a$	a	a	b	a	b	b	$-b$	$-b$	b	b	$-b$	b	0	b	0
(11, 15)	b	b	$-b$	b	0	b	$-a$	b	b	$-b$	$-b$	0	$-b$	$-a$	b	a	a	$-a$	$-a$	a	a	a	a	a	a	$-a$	a	$-a$
(11, 17)	b	$-b$	b	b	b	a	$-a$	b	$-b$	$-b$	$-b$	b	$-a$	$-a$	b	a	$-a$	$-a$	$-a$	$-a$	a	a	a	a	$-a$	a	a	$-a$
(12, 14)	b	$-a$	b	b	$-b$	a	$-a$	b	a	b	a	b	$-a$	$-a$	b	b	$-a$	$-a$	a	0	$-a$	a	a	b	$-a$	$-b$	$-a$	0

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