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Keywords

aligned, finitely, graphs, cofinality, higher, aperiodicity, rank

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Aperiodicity and cofinality for finitely aligned higher-rank graphs†

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Abstract

We introduce new formulations of aperiodicity and cofinality for finitely aligned higher-rank graphs Λ, and prove that $C^*(\Lambda)$ is simple if and only if Λ is aperiodic and cofinal. The main advantage of our versions of aperiodicity and cofinality over existing ones is that ours are stated in terms of finite paths. To prove our main result, we first characterise each of aperiodicity and cofinality of Λ in terms of the ideal structure of $C^*(\Lambda)$. In an appendix we show how our new cofinality condition simplifies in a number of special cases which have been treated previously in the literature; even in these settings our results are new.

1. *Introduction*

From the groundbreaking work of Cuntz and Krieger [**4**] the theory of Cuntz–Krieger algebras has been generalised through the efforts of many authors to include C^*-algebras of finite directed graphs [**7**], infinite directed graphs [**12**, **18**], infinite $\{0, 1\}$-matrices [**9**], ultragraphs [**32**], topological graphs and quivers [**15**, **19**] and higher-rank graphs [**16**], to name a few. In this paper we focus on the C^*-algebras of finitely aligned higher-rank graphs [**11**, **22**]. In graph-based generalisations of Cuntz–Krieger algebras, simplicity is characterised by two conditions on the graph, now known as aperiodicity and cofinality. Cofinality is traditionally phrased in terms of infinite paths, and in the setting of higher-rank graphs, the same is true of aperiodicity. This is problematic because, especially in higher-rank graphs, the infinite paths in question can be difficult to identify and work with. In this paper we introduce new formulations of aperiodicity and cofinality — which involve only finite paths — for finitely aligned higher-rank graphs Λ, and prove that $C^*(\Lambda)$ is simple if and only if Λ is aperiodic and cofinal. This generalises the results of [**23**, **24**] to finitely aligned k-graphs.

A directed graph E is a quadruple (E^0, E^1, r, s) where E^0 and E^1 are countable sets and r and s are maps from E^1 to E^0. The elements of E^0 are called vertices and the elements of E^1 are called edges. For each edge $e \in E^1$ the vertex $r(e)$ is called the range of e and $s(e)$ is called the source of e. We visualise the vertices as dots and each edge e as an arrow from $s(e)$ to $r(e)$.

In 1980, Enomoto and Watatani [**7**] associated a C^*-algebra to each finite directed graph E with no sources as follows. Suppose H is a Hilbert space. Then a *Cuntz–Krieger E-family*

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on H consists of a set $\{P_v : v \in E^0\}$ of mutually orthogonal projections on H and a set $\{S_e : e \in E^1\}$ of partial isometries on H satisfying:

(i) $S_e^* S_e = P_{s(e)}$ for every $e \in E^1$; and
(ii) $P_v = \sum_{\{e \in E^1 : r(e) = v\}} S_e S_e^*$ for all $v \in E^0$.

The two relations above are now known as the Cuntz–Krieger relations. Enomoto and Watatani's definition was subsequently generalised by various authors to infinite graphs [**1**, **2**, **12**, **17**, **18**]. In all cases a key justification of the chosen Cuntz–Krieger relations is the so-called Cuntz–Krieger uniqueness theorem. For directed graphs this theorem states that if every loop in E has an entrance, then any two Cuntz–Krieger E-families consisting of nonzero partial isometries generate isomorphic C^*-algebras.

A higher-rank graph, or k-graph, is an analogue of a directed graph in which paths have a degree in $\mathbb{N}^k$ rather than a length in $\mathbb{N}$. These k-graphs and their C^*-algebras were introduced by Kumjian and Pask [**16**] as graph-based models for the higher-rank Cuntz–Krieger algebras studied by Robertson and Steger [**25**]. For technical reasons, Kumjian and Pask only considered k-graphs in which each vertex receives at least one and at most finitely many paths of any given degree; such k-graphs are said to be row-finite with no sources. Subsequently Raeburn, Sims and Yeend generalised the theory of k-graph C^*-algebras to finitely aligned k-graphs [**21**, **22**].

In recent years k-graph algebras have attracted a great deal of attention. Exel [**8**] realises higher-rank graph algebras as combinatorial algebras and recovers the underlying path space from the algebra. Farthing, Muhly and Yeend [**11**] provide inverse semigroup and groupoid models for higher-rank graph algebras while Katsoulis and Kribs [**14**] explore the relationship between Cuntz–Krieger algebras of higher-rank graphs and nonselfadjoint operator algebras. Many other authors have contributed both to the fundamental theory of k-graph algebras [**3**, **20**, **30**] and to its applications [**5**, **6**, **13**, **31**].

In their seminal paper, Kumjian and Pask proved a generalisation of the Cuntz–Krieger uniqueness theorem for row-finite k-graphs with no sources [**16**, theorem 4·6]. Informed by the original groupoid model for graph C^*-algebras [**17**], Kumjian and Pask observed that in a directed graph the hypothesis that every loop has an entrance ensures that each vertex receives at least one infinite path which is not equal to any left-shift of itself. This was the formulation which they generalised to the higher-rank graph setting, and has become known as the aperiodicity condition. In particular, the aperiodicity condition in Kumjian and Pask's theorem is phrased in terms of infinite paths. In [**11**, **22**] the Cuntz–Krieger uniqueness theorem was further generalised to k-graphs which admit sources as well as vertices which may receive an infinite number of edges of the same degree. Each new generalisation has necessitated a new notion of an infinite path and hence a new notion of aperiodicity, so several different notions of aperiodicity have now appeared in the literature. Moreover, as the class of k-graphs considered broadens, the associated collection of infinite paths becomes more complicated, so the corresponding aperiodicity condition becomes harder to verify.

The Cuntz–Krieger uniqueness theorem can be reinterpreted as the assertion that any nontrivial ideal must contain a vertex projection. From the Cuntz–Krieger uniqueness theorem, it is typically a short step to a sufficient condition for simplicity. One identifies a cofinality condition which implies that an ideal containing one vertex projection must contain all the others, and hence must be the whole C^*-algebra. For row-finite graphs with no sources, the appropriate condition is that for any vertex v and any infinite path x in the graph, there exists a path with range v whose source lies on x. This condition, like aperiodicity, becomes more and more complicated for more general versions of the theory because the appropriate notion of an infinite path becomes more involved. In particular, for finitely aligned higher-rank graphs, the appropriate notion of cofinality, recently identified by Shotwell [**26**], is potentially quite difficult to check in examples.

In this paper, we improve on previous formulations of both aperiodicity and cofinality with equivalent conditions which only involve finite paths. In particular, our new conditions are more easily verified in practice than their predecessors. Our main result is Theorem 3·4: for a finitely aligned k-graph Λ, $C^*(\Lambda)$ is simple if and only if Λ is aperiodic and cofinal. As well as involving only finite paths, this is an improvement on [**29**, proposition 8·5] where only a sufficient condition is established. Our presentation is as self-contained as possible, and we have largely chosen to present direct proofs rather than appeal to existing results elsewhere in the literature.

Our characterisation of simplicity is not entirely new: we discovered late in the course of this research that Shotwell has recently proved that $C^*(\Lambda)$ is simple if and only if Λ has no local periodicity and is cofinal in the sense of [**29**]. His work is available as a preprint [**26**]. However, even where our results converge with Shotwell's, our approach is quite different. For example, though we use the same notion of no local periodicity as Shotwell to prove that no local periodicity combined with cofinality in Λ implies that $C^*(\Lambda)$ is simple, our proof is completely different from his. Shotwell's approach is to show that no local periodicity is equivalent to Condition (A) of [**11**] and then appeal to existing results, while our approach is via a direct argument which does not appeal to any heavy machinery. A similar comparison applies to the two proofs (ours and Shotwell's) that cofinality is necessary for simplicity. Moreover, as mentioned above, our definitions of aperiodicity and cofinality involve only finite paths, which make them easier to work with.

We begin by setting up the background and notation needed for the rest of the paper in Section 2. In Section 3 we introduce our new definitions of aperiodicity and cofinality, and state our main result, Theorem 3·4. As a first step to proving this main theorem, we show that our aperiodicity condition is equivalent to Shotwell's no local periodicity condition, and hence to two another aperiodicity condition used elsewhere in the literature. In Section 4 we explore the C^*-algebraic consequences of aperiodicity of a k-graph. In Section 5 we show that our notion of cofinality is equivalent to the definition of cofinality in [**26**, **29**] and then explore the relationship between cofinality of Λ and the structure of $C^*(\Lambda)$; we conclude Section 5 with the proof of our main result. We have also included an appendix in which we indicate how our new cofinality condition simplifies in a number of special cases; it is new even in these contexts.

2. *Preliminaries*

2·1. *k-graphs*

We regard $\mathbb{N}^k$ as a semigroup under addition, and use e_i to denote the ith generator. For $m, n \in \mathbb{N}^k$ we write m_i, n_i for the ith coordinates of m and n and $m \vee n$ for their coordinatewise maximum and $m \wedge n$ for their coordinatewise minimum. We write $m \leqslant n$ if $m_i \leqslant n_i$ for all i.

A k-graph (Λ, d) is a category Λ endowed with a functor $d\colon \Lambda \to \mathbb{N}^k$ satisfying the *factorisation property*: for every $\lambda \in \Lambda$ and $m, n \in \mathbb{N}^k$ satisfying $d(\lambda) = m + n$ there exist unique $\mu, \nu \in \Lambda$ such that $d(\mu) = m$, $d(\nu) = n$ and $\lambda = \mu\nu$. We read $d(\lambda)$ as the *degree of* λ and think of d as a generalised length function. For $n \in \mathbb{N}^k$ we write Λ^n for $\{\lambda \in \Lambda : d(\lambda) = n\}$.

The factorisation property implies that for each $\lambda \in \Lambda$ there exist unique elements $r(\lambda), s(\lambda) \in \Lambda^0$ such that $\lambda = r(\lambda)\lambda = \lambda s(\lambda)$, and also that $\Lambda^0 = \{r(\lambda) : \lambda \in \Lambda\} = \{s(\lambda) : \lambda \in \Lambda\}$. We call elements of Λ^0 vertices, and the functions r, s the *range* and *source* maps. Given k-graphs Λ and Γ, a k-graph morphism ϕ from Λ to Γ is a functor which preserves the degree map.

For $\lambda \in \Lambda$ and $m \leqslant n \leqslant d(\lambda)$ we define $\lambda(m, n)$ to be the unique path in Λ^{n-m} such that $\lambda = \lambda'\big(\lambda(m, n)\big)\lambda''$ for some $\lambda' \in \Lambda^m$ and $\lambda'' \in \Lambda^{d(\lambda)-n}$ (the existence and uniqueness of $\lambda(m, n)$ follows from two applications of the factorisation property). Unlike in [**21**, **22**], we do **not** write $\lambda(0, n)$ to mean $\lambda\big(0, n \wedge d(\lambda)\big)$; so $\lambda(0, n)$ is undefined if $n \nleqslant d(\lambda)$, and $d\big(\lambda(p, q)\big)$ is always equal to $q - p$ when $\lambda(p, q)$ makes sense. For $v \in \Lambda^0$ and $X \subset \Lambda$, define $vX = \{\lambda \in X : r(\lambda) = v\}$. In particular, $v\Lambda^n = \{\lambda \in \Lambda^n : r(\lambda) = v\}$ for each $v \in \Lambda^0$ and $n \in \mathbb{N}^k$. For $n \in \mathbb{N}^k$ define

$$\Lambda^{\leqslant n} := \{\lambda \in \Lambda : d(\lambda) \leqslant n, \text{ and } d(\lambda)_i < n_i \Rightarrow s(\lambda)\Lambda^{e_i} = \emptyset\}.$$

2·2. *Cuntz–Krieger families and k-graph C^*-algebras*

In this subsection we indicate how to associate a C^*-algebra to a higher-rank graph.

Fix a k-graph Λ. For $\mu, \nu \in \Lambda$, we write

$$\mathrm{MCE}(\mu, \nu) := \{\lambda \in \Lambda : d(\lambda) = d(\mu) \vee d(\nu), \lambda\big(0, d(\mu)\big) = \mu, \lambda\big(0, d(\nu)\big) = \nu\} \qquad (2{\cdot}1)$$

for the collection of *minimal common extensions* of μ and ν. We say that Λ is *finitely aligned* if $|\mathrm{MCE}(\mu, \nu)| < \infty$ for all $\mu, \nu \in \Lambda$.

Let Λ be a k-graph and fix $v \in \Lambda^0$ and $E \subset v\Lambda$. We say that E is *exhaustive* if for each $\mu \in v\Lambda$ there exists $\lambda \in E$ such that $\mathrm{MCE}(\mu, \lambda) \neq \emptyset$. If $|E| < \infty$ we say E is *finite exhaustive*. Define $\mathrm{FE}(\Lambda)$ to be the set of finite exhaustive sets of Λ and for each $v \in \Lambda^0$, define $v\,\mathrm{FE}(\Lambda)$ to be the set of finite exhaustive sets whose elements all have range v.

Definition 2·1 ([**22**, definition 2·5]). Let Λ be a finitely aligned k-graph. A Cuntz–Krieger Λ-family is a set $\{t_\lambda : \lambda \in \Lambda\}$ of partial isometries satisfying:

(CK1) $\{t_v : v \in \Lambda^0\}$ is a collection of mutually orthogonal projections;
(CK2) $t_\mu t_\nu = t_{\mu\nu}$ whenever $s(\mu) = r(\nu)$;
(CK3) $t_\mu^* t_\nu = \sum_{\mu\alpha=\nu\beta\in\mathrm{MCE}(\mu,\nu)} t_\alpha t_\beta^*$ for every $\mu, \nu \in \Lambda$; and
(CK4) $\prod_{\lambda\in E}(t_v - t_\lambda t_\lambda^*) = 0$ for every $v \in \Lambda^0$ and $E \in v\,\mathrm{FE}(\Lambda)$.

Given a finitely aligned k-graph Λ there exists a C^*-algebra $C^*(\Lambda)$ generated by a Cuntz–Krieger Λ-family $\{s_\lambda : \lambda \in \Lambda\}$ which is universal in the following sense: given any other Cuntz–Krieger Λ-family $\{t_\lambda : \lambda \in \Lambda\}$ there exists a unique homomorphism π_t such that $\pi_t(s_\lambda) = t_\lambda$ for every $\lambda \in \Lambda$.

LEMMA 2·2 ([**22**, lemma 2·7]). *Let Λ be a finitely aligned k-graph and let $\{t_\lambda : \lambda \in \Lambda\}$ be a Cuntz–Krieger Λ-family. Then*:

(i) $t_\mu t_\mu^* t_\nu t_\nu^* = \sum_{\lambda\in\mathrm{MCE}(\mu,\nu)} t_\lambda t_\lambda^*$ *for all* $\mu, \nu \in \Lambda$. *In particular,* $\{t_\lambda t_\lambda^* : \lambda \in \Lambda\}$ *is a family of commuting projections*;
(ii) *for* $\mu, \nu \in \Lambda^{\leqslant n}$, *we have* $t_\mu^* t_\nu = \delta_{\mu,\nu} t_{s(\mu)}$;
(iii) *if* $E \subset v\Lambda^{\leqslant n}$ *is finite, then* $t_v \geqslant \sum_{\lambda\in E} t_\lambda t_\lambda^*$;
(iv) $C^*(\{t_\lambda : \lambda \in \Lambda\}) = \overline{\mathrm{span}}\{t_\mu t_\nu^* : \mu, \nu \in \Lambda\} = \overline{\mathrm{span}}\{t_\mu t_\nu^* : \mu, \nu \in \Lambda, s(\mu) = s(\nu)\}$.

We have written (CK3) and Lemma 2·2(i) in terms of $\mathrm{MCE}(\mu, \nu)$, whereas they are rendered in [**22**] in terms of a different set, denoted $\Lambda^{\min}(\mu, \nu)$. The two definitions are equivalent because, for fixed $\mu, \nu \in \Lambda$ the map $(\alpha, \beta) \mapsto \mu\alpha$ is a bijection between $\Lambda^{\min}(\mu, \nu)$ and $\mathrm{MCE}(\mu, \nu)$. We will avoid reference to $\Lambda^{\min}$ in this paper to reduce the amount of notation required as much as possible.

2·3. *The $\partial\Lambda$ representation*

We construct for each finitely aligned k-graph Λ a Cuntz–Krieger Λ-family consisting of nonzero partial isometries. We begin by making sense of paths of infinite degree in a k-graph.

Fix an integer $k \geqslant 1$, and fix $m \in (\mathbb{N} \cup \{\infty\})^k$. We define a k-graph $(\Omega_{k,m}, d)$ as follows: for each $p \in \mathbb{N}^k$, the morphisms of degree p are

$$\Omega_{k,m}^p = \{(q, q + p) : q \in \mathbb{N}^k, q + p \leqslant m\},$$

and we define $r(p, q) := (p, p)$, $s(p, q) := (q, q)$, $d(p, q) := q - p$, and $(p, q)(q, r) = (p, r)$. By convention, we denote a vertex (q, q) of $\Omega_{k,m}$ just by q.

For the next definition, recall that a k-graph morphism is a degree-preserving functor between k-graphs. Given a k-graph morphism $x : \Omega_{k,m} \to \Lambda$, we denote $x(0)$ by $r(x)$, and m by $d(x)$.

Definition 2·3 ([**11**, definition 5·10]). Fix a finitely aligned k-graph Λ. We denote by $\partial\Lambda$ the set

$$\bigcup_{m\in(\mathbb{N}\cup\{\infty\})^k} \big\{x : \Omega_{k,m} \to \Lambda \mid x \text{ is a } k\text{-graph morphism, and for all } n \in \mathbb{N}^k \text{ with } n \leqslant d(x) \text{ and all } E \in x(n)\,\mathrm{FE}(\Lambda), \text{ there exists } p \leqslant d(x) - n \text{ such that } x(n, n+p) \in E\big\}.$$

For $v \in \Lambda^0$, we write $v\partial\Lambda$ for $\{x \in \partial\Lambda : r(x) = v\}$.

In [**11**], the elements of $\partial\Lambda$ were referred to as *boundary paths*. However, the same term has been used elsewhere in the k-graph literature [**10**, **21**, **28**] with different meanings. In order to avoid confusion we refrain from using this term at all.

We will use the following lemma to construct a concrete Cuntz–Krieger Λ-family below. This lemma is not new — indeed it is proved in greater generality in the thesis [**27**, lemma 4·3·3] — but we nevertheless provide a short proof here for ease of reference. To state the lemma, recall that if $x\colon \Omega_{k,m} \to \Lambda$ is a graph morphism, then: (1) for each $n \in \mathbb{N}^k$ with $n \leqslant m$, there is a graph morphism $\sigma^n(x)\colon \Omega_{k,m-n} \to \Lambda$ determined by $\sigma^n(x)(p, q) := x(n + p, n + q)$; and (2) for each $\lambda \in \Lambda r(x)$, there is a unique graph morphism $\lambda x\colon \Omega_{k,m+d(\lambda)} \to \Lambda$ such that $(\lambda x)(0, d(\lambda)) = \lambda$ and $\sigma^{d(\lambda)}(\lambda x) = x$.

LEMMA 2·4. *Let (Λ, d) be a finitely aligned k-graph and let $x \in \partial\Lambda$.*
(i) *If $m \in \mathbb{N}^k$ and $m \leqslant d(x)$, then $\sigma^m(x) \in \partial\Lambda$.*
(ii) *If $\lambda \in \Lambda r(x)$, then $\lambda x \in \partial\Lambda$.*

To prove this lemma we must recall a definition and another lemma from [**22**]. We also use this definition and lemma again in Section 3. Suppose $\lambda \in \Lambda$ and $E \subseteq r(\lambda)\Lambda$, write $\mathrm{Ext}(\lambda, E)$ for the set

$$\bigcup_{\mu\in E} \{\nu\big(d(\lambda), d(\lambda) \vee d(\mu)\big) : \nu \in \mathrm{MCE}(\lambda, \mu)\}.$$

LEMMA 2·5 ([**22**, lemma C·5]). *Let (Λ, d) be a finitely aligned k-graph, let $v \in \Lambda^0$, $\lambda \in v\Lambda$ and suppose $E \in vFE(\Lambda)$. Then* $\mathrm{Ext}(\lambda, E) \in s(\lambda)FE(\Lambda)$.

Proof of Lemma 2·4. (i) Fix $n \in \mathbb{N}^k$ such that $n \leqslant d\big(\sigma^m(x)\big)$ and fix a set $E \in \sigma^m(x)(n)\,\mathrm{FE}(\Lambda)$. Then $m+n \leqslant d(x)$ and since $x \in \partial\Lambda$ there exists $p \leqslant d(x) - (m+n)$ such that $x(m+n, m+n+p) \in E$. That is, $p \leqslant d\big(\sigma(x)\big) - n$ and $\big(\sigma^m(x)\big)(n, n+p) \in E$.

(ii) Fix $E \in (\lambda x)(n)\,\mathrm{FE}(\Lambda)$. Let $\lambda' = (\lambda x)\big(n, n \vee d(\lambda)\big)$. By Lemma 2·5, $\mathrm{Ext}(\lambda', E) \in x\big((n \vee d(\lambda)) - n\big)\,\mathrm{FE}(\Lambda)$, so there exists $p \leqslant d(x) - \big((n \vee d(\lambda)) - d(\lambda)\big)$ such that $\alpha = (\lambda x)\big(n \vee d(\lambda), (n \vee d(\lambda)) + p\big) = x\big((n \vee d(\lambda)) - d(\lambda), (n \vee d(\lambda)) - d(\lambda) + p\big)$ belongs to $\mathrm{Ext}(\lambda', E)$, say $\lambda'\alpha = \mu\beta$ where $\mu \in E$. But now $q = d(\mu)$ satisfies

$$(\lambda x)(n, n+q) = (\lambda'\alpha)(0, q) = (\mu\beta)(0, q) = \mu \in E.$$

Definition 2·6 (The $\partial\Lambda$ Representation). Let (Λ, d) be a finitely aligned k-graph and let $\{\zeta_x : x \in \partial\Lambda\}$ denote the standard orthonormal basis for $\ell^2(\partial\Lambda)$. For $\lambda \in \Lambda$, define

$$S_\lambda \zeta_x = \begin{cases} \zeta_{\lambda x} & \text{if } s(\lambda) = r(x), \\ 0 & \text{otherwise.} \end{cases}$$

Then $\{S_\lambda : \lambda \in \Lambda\}$ is a Cuntz–Krieger Λ-family in $\mathcal{B}\big(\ell^2(\partial\Lambda)\big)$ called the $\partial\Lambda$ *representation* (see [**28**, lemma 4·6]), and for each $\lambda \in \Lambda$,

$$S_\lambda^* \zeta_x = \begin{cases} \zeta_{\sigma^{d(\lambda)}(x)} & \text{if } x(0, d(\lambda)) = \lambda \\ 0 & \text{otherwise.} \end{cases}$$

Remark 2·7. Fix a finitely aligned k-graph Λ. [**11**, lemma 5·15] implies that each $S_v \in \ell^2(\partial\Lambda)$ is nonzero. The universal property of $C^*(\Lambda)$ then implies that the universal generating partial isometries in $C^*(\Lambda)$ are all nonzero.

3. *Aperiodicity, cofinality and the main theorem*

In this section we introduce our new formulations of aperiodicity and cofinality, and state our main result, Theorem 3·4.

Definition 3·1. We say that a k-graph Λ is *aperiodic* if for every pair of distinct paths $\alpha, \beta \in \Lambda$ with $s(\alpha) = s(\beta)$ there exists $\tau \in s(\alpha)\Lambda$ such that $\mathrm{MCE}(\alpha\tau, \beta\tau) = \emptyset$.

Remark 3·2. To check that Λ is aperiodic, it suffices to show that for every distinct pair μ, ν such that $s(\mu) = s(\nu)$, $r(\mu) = r(\nu)$ and $d(\mu) \wedge d(\nu) = 0$, there exists $\tau \in s(\mu)\Lambda$ such that $\mathrm{MCE}(\mu\tau, \nu\tau) = \emptyset$. For suppose that this is indeed the case, and fix distinct $\alpha, \beta \in \Lambda$ with $s(\alpha) = s(\beta)$. Let $m := d(\alpha) \wedge d(\beta)$, let $\mu := \alpha\big(m, d(\alpha)\big)$, and let $\nu := \beta\big(m, d(\beta)\big)$. If $\alpha(0, m) \neq \beta(0, m)$, then $\tau = s(\alpha)$ satisfies $\mathrm{MCE}(\alpha\tau, \beta\tau) = \emptyset$. On the other hand, if $\alpha(0, m) = \beta(0, m)$, then that $\alpha \neq \beta$ forces $\mu \neq \nu$. Hence by assumption there exists $\tau \in s(\mu)\Lambda$ such that $\mathrm{MCE}(\mu\tau, \nu\tau) = \emptyset$. Thus $\mathrm{MCE}(\alpha\tau, \beta\tau) = \{\alpha(0, m)\rho : \rho \in \mathrm{MCE}(\mu\tau, \nu\tau)\} = \emptyset$ as required.

Definition 3·3. Suppose Λ is a k-graph. We say that Λ is *cofinal* if for every $v, w \in \Lambda^0$ there exists $E \in w\,\mathrm{FE}(\Lambda)$ such that $v\Lambda s(\alpha) \neq \emptyset$ for every $\alpha \in E$.

THEOREM 3·4. *Suppose Λ is a finitely aligned k-graph. Then $C^*(\Lambda)$ is simple if and only if Λ is aperiodic and cofinal.*

The proof of Theorem 3·4 occupies the rest of the paper. We begin by establishing the equivalence of aperiodicity with earlier conditions appearing in the literature. To this end we recall the notion of local periodicity for finitely aligned k-graphs. This was introduced by Shotwell in ([**26**, definition 3·1 and remarks 3·4]).

Definition 3·5. Let (Λ, d) be a finitely aligned k-graph and suppose $m, n \in \mathbb{N}^k$ are distinct. We say Λ has *local periodicity* m, n *at* v if for every $x \in v\partial\Lambda$ we have:

(LP1) $m \vee n \leqslant d(x)$; and

(LP2) $\sigma^m(x) = \sigma^n(x)$.

We say Λ has *no local periodicity* if Λ does not have local periodicity m, n at any $v \in \Lambda^0$ for any distinct $m, n \in \mathbb{N}^k$. That is, Λ has no local periodicity if for every $v \in \Lambda^0$, and every distinct $m, n \in \mathbb{N}^k$ either:

(NLP1) there exists $x \in v\partial\Lambda$ such that $d(x) \ngeqslant m \vee n$; or

(NLP2) $d(x) \geqslant m \vee n$ for every $x \in v\partial\Lambda$, and there exists $y \in v\partial\Lambda$ such that $\sigma^m(y) \neq \sigma^n(y)$.

Recall from [**11**] that Λ is said to satisfy condition (A) if for each $v \in \Lambda^0$ there exists $x \in v\partial\Lambda$ such that $\sigma^m(x) = \sigma^n(x)$ implies $m = n$ for all $m, n \leqslant d(x)$.

PROPOSITION 3·6. *Suppose* Λ *is a finitely aligned* k*-graph. Then the following are equivalent*:

(i) Λ *is aperiodic*;

(ii) Λ *has no local periodicity*;

(iii) Λ *satisfies condition* (A) *of* [**11**, theorem 7·1]

Proof. Shotwell uses condition (A) as his definition of aperiodicity and establishes (ii) $\Leftrightarrow$ (iii) in [**26**, proposition 3·10], so it suffices to show (i) $\Leftrightarrow$ (ii).

(i) $\Rightarrow$ (ii). Suppose Λ is aperiodic. Fix $v \in \Lambda^0$ and distinct $m, n \in \mathbb{N}^k$. If there exists $x \in v\partial\Lambda$ such that $d(x) \ngeqslant m \vee n$ then (NLP1) holds and we are done. So we may suppose that $d(x) \geqslant m \vee n$ for every $x \in v\partial\Lambda$. Since $v\partial\Lambda$ is nonempty we have $v\Lambda^{m\vee n} \neq \emptyset$. Fix $\lambda \in v\Lambda^{m\vee n}$ and write $\lambda = \mu\alpha = \nu\beta$ where $d(\mu) = m$ and $d(\nu) = n$. Note that this implies $d(\mu) + d(\alpha) = d(\nu) + d(\beta)$ so $d(\alpha) - d(\beta) = d(\nu) - d(\mu)$ and in particular $d(\alpha) \neq d(\beta)$. Since Λ is aperiodic, there exists τ such that $\operatorname{MCE}(\alpha\tau, \beta\tau) = \emptyset$. Fix $z \in s(\tau)\partial\Lambda$ and let $y = \lambda\tau z \in v\partial\Lambda$. Then $\sigma^m(y) = \alpha\tau z$ and $\sigma^n(y) = \beta\tau z$. If these were equal then $\big(\sigma^m(y)\big)\big(0, d(\alpha\tau) \vee d(\beta\tau)\big)$ would belong to $\operatorname{MCE}(\alpha\tau, \beta\tau)$, contradicting our choice of τ. So we must have $\sigma^n(y) \neq \sigma^m(y)$.

(ii) $\Rightarrow$ (i). Suppose Λ has no local periodicity and fix distinct $\alpha, \beta \in \Lambda$ with $s(\alpha) = s(\beta) = v$ and $d(\alpha) \neq d(\beta)$. By Remark 3·2 we may suppose that $r(\alpha) = r(\beta)$ and $d(\alpha) \wedge d(\beta) = 0$. Let $D := d(\alpha) \vee d(\beta) = d(\alpha) + d(\beta)$. Then $D - d(\alpha) = d(\beta)$ and $D - d(\beta) = d(\alpha)$. We consider two cases: for $m = d(\beta)$ and $n = d(\alpha)$ either:

(i) (NLP1) holds; or

(ii) (NLP2) holds.

We consider case (ii) first because it is simpler.

Case (ii). Suppose (NLP2) holds. That is, for every $x \in v\partial\Lambda$ we have $d(x) \geqslant D$ and there exists $y \in v\partial\Lambda$ such that $\sigma^{d(\beta)}(y) \neq \sigma^{d(\alpha)}(y)$. Then there exists $M_{\alpha,\beta} \leqslant d(y) - (d(\beta) \vee d(\alpha))$ such that

$$\big(\sigma^{d(\beta)}(y)\big)(0, M_{\alpha,\beta}) \neq \big(\sigma^{d(\alpha)}(y)\big)(0, M_{\alpha,\beta})$$

or equivalently,

$$y(d(\beta), d(\beta) + M_{\alpha,\beta}) \neq y(d(\alpha), d(\alpha) + M_{\alpha,\beta}). \tag{3·1}$$

Let $\tau = y\big(0, D + M_{\alpha,\beta}\big)$. Then $d(\beta) + M_{\alpha,\beta} \leqslant d(\tau)$ and $d(\alpha) + M_{\alpha,\beta} \leqslant d(\tau)$. Moreover,

$$(\alpha\tau)(D, D + M_{\alpha,\beta}) = \tau\big(D - d(\alpha), D - d(\alpha) + M_{\alpha,\beta}\big) = y\big(d(\beta), d(\beta) + M_{\alpha,\beta}\big)$$

and similarly,

$$(\beta\tau)(D, D + M_{\alpha,\beta}) = y\big(d(\alpha), d(\alpha) + M_{\alpha,\beta}\big).$$

It follows from (3·1) that $\mathrm{MCE}(\alpha\tau, \beta\tau) = \varnothing$.

Case (i). Now suppose (NLP1) holds. That is, there exists $x \in v\partial\Lambda$ such that $d(x) \ngeqslant D$. We consider two subcases:

(a) for every $\lambda \in v\Lambda$ there exists $\lambda' \in s(\lambda)\Lambda$ such that $d(\lambda\lambda') \geqslant D$; or
(b) there exists $\lambda \in \Lambda$ such that $s(\lambda)\Lambda^{D-d(\lambda)} = \varnothing$.

Case (i)(a). We claim there exist $\tau_f \in v\Lambda$ and $i \in \{1, \ldots, k\}$ such that $d(\alpha)_i \neq d(\beta)_i$ and $|s(\tau_f)\Lambda^{e_i}| = \infty$. Let $p = D \wedge d(x)$, $\lambda = x(0, p)$ and $G = s(\lambda)\Lambda^{D-p}$. Then (a) implies that G is exhaustive. Moreover, since $d(x) \ngeqslant D$, there is no $q \leqslant d(x) - n$ such that $x(p, p+q) \in G$. Since $x \in \partial\Lambda$ it follows that $G \notin \mathrm{FE}(\Lambda)$ and hence that $|G| = \infty$.

Since $D - p < \infty$ there exists $i \in \{1, \ldots, k\}$ and a smallest $a < (D - p)_i \in \mathbb{N}$ such that $|\{\mu(ae_i, (a+1)e_i) : \mu \in G\}| = \infty$. Since a is the smallest element of $\mathbb{N}$ with the given property we have $|\{\mu(0, ae_i) : \mu \in G\}| < \infty$ and there exists $\lambda' \in G$ such that $|\lambda'(ae_i)\Lambda^{e_i}| = \infty$. Let $\tau_f = \lambda\lambda'(0, ae_i)$.

We now show that $d(\alpha)_i \neq d(\beta)_i$. Since $\lambda' \in G$ we have the inequality $0 < (a+1) \leqslant d(\lambda')_i = (D-p)_i$, which implies $D_i > 0$. Since $D = d(\alpha) \vee d(\beta)$ and since $d(\alpha) \wedge d(\beta) = 0$ by assumption, it follows that $d(\alpha)_i \neq d(\beta)_i$. This establishes the claim.

Since Λ is finitely aligned, $|\mathrm{MCE}(\alpha\tau_f, \beta\tau_f)| < \infty$ and since $d(\alpha\tau_f)_i \neq d(\beta\tau_f)_i$ we deduce that $d(\psi)_i > \min\{d(\alpha\tau_f)_i, d(\beta\tau_f)_i\}$ for every $\psi \in \mathrm{MCE}(\alpha\tau_f, \beta\tau_f)$. However $|s(\tau_f)\Lambda^{e_i}| = \infty$, so there exists $\tau_i \in s(\tau_f)\Lambda^{e_i}$ such that $\psi\big(d(\alpha\tau_f), d(\alpha\tau_f\tau_i)\big) \neq \tau_i$ for all $\psi \in \mathrm{MCE}(\alpha\tau_f, \beta\tau_f)$.

Let $\tau = \tau_f\tau_i$. We now must argue that $\mathrm{MCE}(\alpha\tau, \beta\tau) = \varnothing$. Suppose for contradiction that $\psi \in \mathrm{MCE}(\alpha\tau, \beta\tau)$. Then in particular $\psi\big(0, d(\alpha\tau) \vee d(\beta\tau)\big) \in \mathrm{MCE}(\alpha\tau_f, \beta\tau_f)$. However $\psi\big(d(\alpha\tau_f), d(\alpha\tau_f\tau_i)\big) = \tau_i$ which contradicts our choice of τ_i.

Case (i)(b). Let λ be as in Case (i)(b). Fix $\lambda' \in s(\lambda)\Lambda^{\leqslant D-p} \subseteq s(\lambda)\Lambda$. Then $d(\lambda\lambda') \ngeqslant D$ and there exists $i \in \{1, \ldots, k\}$ such that $d(\lambda\lambda')_i < D_i$. For each i with $d(\lambda\lambda')_i < D_i$, we have $d(\lambda')_i < (D - p)_i$ so by definition of $\Lambda^{\leqslant D-p}$ we have $s(\lambda')\Lambda^{e_i} = \varnothing$. We now claim that there exists i such that $d(\lambda\lambda')_i < D_i$ and $d(\alpha)_i \neq d(\beta)_i$. Suppose for contradiction that $d(\alpha)_i = d(\beta)_i$ for each $i \in \{1, \ldots, k\}$ such that $d(\lambda\lambda') < D_i$. By assumption we have $d(\alpha) \wedge d(\beta) = 0$ which implies that whenever $d(\alpha)_i = d(\beta)_i$ we must have $d(\alpha)_i = d(\beta)_i = 0$. Fix $i \in \{1, \ldots, k\}$ such that $d(\lambda\lambda')_i < D_i$. Then $d(\alpha)_i = d(\beta)_i = 0$ and we have the contradiction

$$0 < d(\lambda\lambda')_i < D_i = (d(\alpha) \vee d(\beta))_i = 0.$$

We now let $\tau = \lambda\lambda'$. Then $d(\alpha\tau)_i \neq d(\beta\tau)_i$ but $s(\tau)\Lambda^{e_i} = s(\lambda')\Lambda^{e_i} = \varnothing$ and hence $\mathrm{MCE}(\alpha\tau, \beta\tau) = \varnothing$ as required.

4. *Consequences of aperiodicity*

We now characterise aperiodicity of Λ in terms of the ideal structure of $C^*(\Lambda)$ and prove a version of the Cuntz–Krieger uniqueness theorem.

THEOREM 4·1. *Let (Λ, d) be a finitely aligned k-graph. Then the following are equivalent*:

(i) *Λ is aperiodic*;

(ii) *every non-zero ideal of $C^*(\Lambda)$ contains a vertex projection*;

(iii) *the $\partial\Lambda$ representation π_S is faithful.*

The bulk of the work goes into (i) $\Rightarrow$ (ii). This is the Cuntz–Krieger uniqueness theorem and we prove it in the next subsection. The implication (ii) $\Rightarrow$ (iii) follows from Remark 2·7. We therefore begin by proving (iii) $\Rightarrow$ (i). We first establish two preliminary results.

LEMMA 4·2. *Let Λ be a finitely aligned k-graph and fix $v \in \Lambda^0$. Suppose that Λ has local periodicity m, n at v. If $x \in v\partial\Lambda$ and $i \in \{1, \ldots, k\}$ satisfy $d(x)_i < \infty$, then $m_i = n_i$.*

Proof. Suppose $x \in v\partial\Lambda$ and $i \in \{1, \ldots, k\}$ satisfy $d(x)_i < \infty$. Since $\sigma^m(x) = \sigma^n(x)$ we have $d(x) - m = d(x) - n$ and in particular $\big(d(x) - m\big)_i = \big(d(x) - n\big)_i$ which implies $m_i = n_i$ since $d(x)_i < \infty$.

LEMMA 4·3. *Let Λ be a finitely aligned k-graph. Suppose Λ has local periodicity m, n at v. Then there exist $\mu, \nu, \alpha \in \Lambda$ such that $r(\mu) = r(\nu) = v$, $s(\mu) = s(\nu) = r(\alpha)$, $d(\mu) = m$, $d(\nu) = n$ and $\mu\alpha z = \nu\alpha z$ for all $z \in s(\alpha)\partial\Lambda$.*

Proof. Fix $x \in v\partial\Lambda$. Since Λ has local periodicity m, n at v we know that $m \vee n \leqslant d(x)$. Let $\mu = x(0, m)$, $\nu = x(0, n)$ and $\alpha = x(m, m \vee n)$. Then $s(\mu) = r\big(\sigma^m(x)\big) = r\big(\sigma^n(x)\big) = s(\nu)$. Fix $z \in s(\alpha)\partial\Lambda$. Since $(\mu\alpha z)(0, n) = \nu$ and $d(\mu) = m$ we have

$$\mu\alpha z = \nu\sigma^n(\mu\alpha z) = \nu\sigma^m(\mu\alpha z) = \nu\alpha z.$$

Recall that given a finitely aligned k-graph Λ, there is a strongly continuous *gauge action* $\gamma\colon \mathbb{T} \to \operatorname{Aut}\big(C^*(\Lambda)\big)$ given using multi-index notation by $\gamma_z(s_\lambda) = z^{d(\lambda)} s_\lambda$.

Proof of (iii) $\Rightarrow$ (i) *in Theorem* 4·1. We prove the contrapositive statement. Suppose Λ is not aperiodic. Then Proposition 3·6 implies that there exist $v \in \Lambda^0$ and distinct $m, n \in \mathbb{N}^k$ such that Λ has local periodicity m, n at v. By Lemma 4·3 there exist $\mu, \nu, \alpha \in \Lambda$ such that $r(\mu) = r(\nu) = v$, $s(\mu) = s(\nu) = r(\alpha)$, $d(\mu) = m$, $d(\nu) = n$ and $\mu\alpha y = \nu\alpha y$ for every $y \in s(\alpha)\partial\Lambda$.

We claim that $a := s_{\mu\alpha}s^*_{\mu\alpha} - s_{\nu\alpha}s^*_{\mu\alpha} \in \ker(\pi_S)\backslash\{0\}$. To see that $a \neq 0$ we check that for $\omega \in \mathbb{T}^k$ with $\omega^{d(\nu)-d(\mu)} = -1$ we have $(\mathrm{id} + \gamma_\omega)(a) = 2s_{\mu\alpha}s^*_{\mu\alpha} \neq 0$. To see that $\pi_S(a) = 0$ we check directly using our choice of μ, ν, α that $\pi_S(a)\zeta_x = 0$ for all $x \in \partial\Lambda$. The details are the same as [**23**, proposition 3·5].

4·1. *The Cuntz–Krieger Uniqueness Theorem*

We now use our definition of aperiodicity to prove a version of the Cuntz–Krieger uniqueness theorem. We start with a technical lemma.

LEMMA 4·4. *Suppose (Λ, d) is an aperiodic, finitely aligned k-graph. Fix $v \in \Lambda^0$ and let H be a finite subset of Λv. Then there exists $\tau \in v\Lambda$ such that* $\mathrm{MCE}(\alpha\tau, \beta\tau) = \varnothing$ *for every pair of distinct paths $\alpha, \beta \in H$.*

Proof. We proceed by induction on $|H|$. If $|H| = 2$ then Lemma 4·4 reduces to the definition of aperiodicity.

Now suppose the result is true whenever $|H| = k$. Fix $H \subseteq \Lambda v$ with $|H| = k+1$. Fix $\alpha \in H$ and list $H \backslash \{\alpha\} = \{\beta_1, \ldots, \beta_k\}$. By the inductive hypothesis there exists $\tau_\alpha \in v\Lambda$ such that $\mathrm{MCE}(\beta\tau_\alpha, \beta'\tau_\alpha) = \emptyset$ for all distinct $\beta, \beta' \in H\backslash\{\alpha\}$. Inductively applying aperiodicity, we obtain paths $\tau_1, \ldots, \tau_k$ such that for each $j \leqslant k$,

$$\mathrm{MCE}(\alpha\tau_\alpha\tau_1 \cdots \tau_j,\ \beta_j\tau_\alpha\tau_1 \cdots \tau_j) = \emptyset.$$

We claim that $\tau = \tau_\alpha\tau_1 \ldots \tau_k$ satisfies $\mathrm{MCE}(\mu\tau, \nu\tau) = \emptyset$ for all distinct $\mu, \nu \in H$. Fix distinct $\mu, \nu \in H$. First suppose that $\mu, \nu \in H \setminus \{\alpha\}$. Then $\mathrm{MCE}(\mu\tau_\alpha, \nu\tau_\alpha) = \emptyset$ and since τ_α is an initial segment of τ it follows that $\mathrm{MCE}(\mu\tau, \nu\tau) = \emptyset$. Now suppose that one of μ, ν is equal to α; without loss of generality suppose $\mu = \alpha$. It remains to show that $\mathrm{MCE}(\alpha\tau, \nu\tau) = \emptyset$. Since $\nu \neq \alpha$ we have $\nu = \beta_p$ for some $1 \leqslant p \leqslant k$. By construction $\mathrm{MCE}(\alpha\tau_\alpha\tau_1 \cdots \tau_p,\ \beta_p\tau_\alpha\tau_1 \cdots \tau_p) = \emptyset$. Since $\tau_1 \cdots \tau_p$ is an initial segment of τ it follows that $\mathrm{MCE}(\alpha\tau, \nu\tau) = \emptyset$.

The following technical lemma allows us to replace the use of condition (B) in the proof of the Cuntz–Krieger uniqueness theorem in [**22**, theorem 4·5] with our aperiodicity hypothesis.

We first introduce some notation. Given a finite set X, we write $M_X(\mathbb{C})$ for the matrix algebra with matrix units indexed by X; that is, the unique C^*-algebra generated by nonzero partial isometries $\{\theta_{x,y} : x, y \in X\}$ satisfying $\theta^*_{x,y} = \theta_{y,x}$, and $\theta_{x,y}\theta_{w,z} = \delta_{y,w}\theta_{x,z}$. Of course, for each enumeration $X = \{x_1, \ldots, x_{|X|}\}$ of X, there is a corresponding isomorphism of $M_X(\mathbb{C})$ with the standard matrix algebra $M_{|X|}(\mathbb{C})$ determined by $\theta_{x_i,y_i} \mapsto \theta_{i,j}$; but choosing an enumeration is unnecessary and complicates notation, so we avoid doing so here.

LEMMA 4·5. *Let $\{t_\lambda : \lambda \in \Lambda\}$ be a Cuntz–Krieger Λ-family with $t_v \neq 0$ for each $v \in \Lambda^0$. Fix $v \in \Lambda^0$ and let H be a finite subset of Λv. Fix $N \in \mathbb{N}^k$ and a linear combination $a = \sum_{\mu,\nu\in H} a_{\mu,\nu}t_\mu t^*_\nu$ such that whenever $a_{\mu,\nu} \neq 0$ we have $\mu \in \Lambda^{\leqslant N}$. Also let $a_0 = \sum_{\mu,\nu\in H, d(\mu)=d(\nu)} a_{\mu,\nu}t_\mu t^*_\nu$. Then there exists a norm-decreasing linear map Q: $C^*(\Lambda) \to C^*(\Lambda)$ such that $\|Q(a_0)\| = \|a_0\|$ and $Q(a_0) = Q(a)$. In particular, $\|a_0\| \leqslant \|a\|$.*

Remark. [**22**, lemma 4·11] claims that Q maps $\pi\big(C^*(\Lambda)\big)$ to $\pi\big(C^*(\Lambda)^\gamma\big)$. However this assertion is not proved and is not obviously true. Fortunately it is also not needed.

For the proof of Lemma 4·5 the following remark will prove useful.

Remark 4·6. If $\alpha, \beta \in \Lambda$ satisfy $s(\alpha) = s(\beta)$, $d(\alpha) = d(\beta)$ and $\alpha \in \Lambda^{\leqslant n}$ then $\beta \in \Lambda^{\leqslant n}$ also. To see this suppose $d(\beta)_i < n_i$. Then $d(\alpha)_i = d(\beta)_i < n_i$, and $\alpha \in \Lambda^{\leqslant n}$ forces $s(\beta)\Lambda^{e_i} = s(\alpha)\Lambda^{e_i} = \emptyset$.

Proof of Lemma 4·5. Since $H \subseteq \Lambda v$ is finite, Lemma 4·4 implies that there exists $\tau \in v\Lambda$ such that $\mathrm{MCE}(\alpha\tau, \beta\tau) = \emptyset$ for every pair of distinct paths $\alpha, \beta \in H$. For each $n \leqslant N$ let

$$Q_n := \sum_{\rho\in H\cap\Lambda^{\leqslant N}\cap\Lambda^n} t_{\rho\tau}t^*_{\rho\tau}$$

and define $Q : C^*(\Lambda) \to C^*(\Lambda)$ by

$$Q(b) := \sum_{n\leqslant N} Q_n b Q_n.$$

Lemma 2·2 implies that the Q_n are mutually orthogonal projections, so $Q\big(C^*(\Lambda)\big) \cong \oplus_{n\leqslant N} Q_n C^*(\Lambda) Q_n$. Hence $\|Q(a)\| = \max_{n\leqslant N} \|Q_n a Q_n\| \leqslant \|a\|$, and so Q is norm-decreasing; it is clearly linear.

We will now show that $\|Q(a_0)\| = \|a_0\|$. Consider

$$G = \operatorname{span}\{t_\alpha t_\beta^* : \alpha, \beta \in H, d(\alpha) = d(\beta), \alpha \in \Lambda^{\leqslant N}\}.$$

We will show that G is a finite-dimensional C^*-subalgebra of $C^*(\Lambda)$ and use this to see that Q restricts to an isomorphism of G onto $Q(G)$. It then follows that $\|Q(a_0)\| = \|a_0\|$.

For each $n \leqslant N$ let $F_n := \{t_\alpha t_\beta^* : \alpha, \beta \in H, d(\alpha) = n = d(\beta) \text{ and } \alpha \in \Lambda^{\leqslant N}\}$. Fix $n \leqslant N$ and elements $t_\alpha t_\beta^*$ and $t_\mu t_\nu^*$ of F_n. Then $\beta, \mu \in \Lambda^n$ and (CK3) forces

$$(t_\alpha t_\beta^*)(t_\mu t_\nu^*) = t_\alpha(\delta_{\beta,\mu} t_\upsilon) t_\nu^* = \delta_{\beta,\mu} t_\alpha t_\nu^*.$$

A standard argument using (CK3) and that $t_\upsilon \neq 0$ shows that each $t_\alpha t_\beta^*$ is nonzero. Hence $t_\alpha t_\beta^* \mapsto \theta_{\alpha,\beta}$ determines an isomorphism $G_n := \operatorname{span}(F_n) \cong M_{H\cap\Lambda^{\leqslant N}\cap\Lambda^n}(\mathbb{C})$.

Fix distinct $m, n \leqslant N$. Let $t_\alpha t_\beta^* \in F_m$ and $t_\mu t_\nu^* \in F_n$. Since $\alpha \in \Lambda^{\leqslant N}$ and $d(\alpha) = d(\beta)$, Remark 4·6 implies $\beta \in \Lambda^{\leqslant N}$. Since $d(\beta) \neq d(\mu)$, we have $\beta \neq \mu$, so Lemma 2·2 (ii) implies that $(t_\alpha t_\beta^*)(t_\mu t_\nu^*) = 0$. Hence $G_m \perp G_n$. Thus $G = \oplus_{n\leqslant N} G_n \cong \oplus_{n\leqslant N} M_{H\cap\Lambda^{\leqslant N}\cap\Lambda^n}(\mathbb{C})$ via $t_\alpha t_\beta^* \mapsto \theta_{\alpha,\beta}$.

For $\alpha, \beta \in H$ such that $a_{\alpha,\beta} \neq \emptyset$, we have $\alpha \in \Lambda^{\leqslant N}$ and we calculate

$$\begin{aligned}
Q(t_\alpha t_\beta^*) &= \sum_{n\leqslant N} Q_n (t_\alpha t_\beta^*) Q_n \\
&= \sum_{n\leqslant N} \Big(\sum_{\rho\in H\cap\Lambda^{\leqslant N}\cap\Lambda^n} t_{\rho\tau} t_{\rho\tau}^*\Big)(t_\alpha t_\beta^*)\Big(\sum_{\rho'\in H\cap\Lambda^{\leqslant N}\cap\Lambda^n} t_{\rho'\tau} t_{\rho'\tau}^*\Big) \\
&= \sum_{n\leqslant N} \Big(\sum_{\rho,\rho'\in H\cap\Lambda^{\leqslant N}\cap\Lambda^n} t_{\rho\tau} t_\tau^* (t_\rho^* t_\alpha)(t_\beta^* t_{\rho'}) t_\tau t_{\rho'\tau}^*\Big) \\
&= \sum_{\rho'\in H\cap\Lambda^{\leqslant N}\cap\Lambda^{d(\alpha)}} t_{\alpha\tau}(t_{\beta\tau}^* t_{\rho'\tau}) t_{\rho'\tau}^* \quad \text{by Lemma 2·2(ii).}
\end{aligned} \tag{4·1}$$

If, in addition, $d(\alpha) = d(\beta)$, then $\beta \in \Lambda^{\leqslant N}$ by Remark 4·6, so Lemma 2·2(ii) implies $t_\beta^* t_{\rho'} = 0$ unless $\rho' = \beta$. So continuing our calculation from above, with $\alpha, \beta \in H, \alpha \in \Lambda^{\leqslant N}$ and $d(\alpha) = d(\beta)$, we have

$$Q(t_\alpha t_\beta^*) = t_{\alpha\tau} t_{\alpha\tau}^* t_\alpha t_\beta^* t_{\beta\tau} t_{\beta\tau}^* = t_{\alpha\tau} t_{\beta\tau}^*$$

In particular, for $t_\alpha t_\beta^*, t_\mu t_\nu^* \in \bigcup_{n\leqslant N} F_n$,

$$Q(t_\alpha t_\beta^*) Q(t_\mu t_\nu^*) = t_{\alpha\tau} t_{\beta\tau}^* t_{\mu\tau} t_{\nu\tau}^* = t_{\alpha\tau} t_\tau^* (t_\beta^* t_\mu) t_\tau t_{\nu\tau}^* = \delta_{\beta,\mu} t_{\alpha\tau} t_{\nu\tau}^* = \delta_{\beta,\mu} Q(t_\alpha t_\nu^*).$$

So the set $\{Q(t_\alpha t_\beta^*) : \alpha, \beta \in H, d(\alpha) = d(\beta), \alpha \in \Lambda^{\leqslant N}\}$ is a system of non-zero matrix units for an isomorphic copy of $\oplus_{n\leqslant N} M_{\{\alpha\tau:\alpha\in H\cap\Lambda^{\leqslant N}\cap\Lambda^n\}}(\mathbb{C})$. In particular $t_\alpha t_\beta^* \mapsto t_{\alpha\tau} t_{\beta\tau}^*$ determines an isomorphism of F_n. Thus Q restricts to an isomorphism of G onto $Q(G)$ forcing $\|Q(a_0)\| = \|a_0\|$.

It remains to show that $Q(a_0) = Q(a)$. Fix $\alpha, \beta \in H$ such that $d(\alpha) \neq d(\beta)$ and $a_{\alpha,\beta} \neq 0$. It suffices to show that $Q(t_\alpha t_\beta^*) = 0$. Using (4·1) we have

$$Q(t_\alpha t_\beta^*) = \sum_{\rho'\in H\cap\Lambda^{\leqslant N}\cap\Lambda^n} t_{\alpha\tau} t_{\beta\tau}^* t_{\rho'\tau} t_{\rho'\tau}^*.$$

Since $d(\beta) \neq d(\alpha)$ we have $\beta \neq \rho'$ for all $\rho' \in H \cap \Lambda^{\leqslant N} \cap \Lambda^{d(\alpha)}$. Thus $\mathrm{MCE}(\beta\tau, \rho'\tau) = \emptyset$ for every $\rho' \in H \cap \Lambda^{\leqslant N} \cap \Lambda^{d(\alpha)}$ by choice of τ. Thus (CK3) forces $Q(t_\alpha t_\beta^*) = 0$.

We can now prove our version of the Cuntz–Krieger Uniqueness Theorem.

THEOREM 4·7 (The Cuntz–Krieger uniqueness theorem). *Suppose (Λ, d) is an aperiodic, finitely aligned k-graph and suppose π is a representation of $C^*(\Lambda)$ such that $\pi(s_v) \neq 0$ for every $v \in \Lambda^0$. Then π is faithful.*

Proof. The opening paragraph of [**22**, section 4] together with [**22**, proposition 4·1] show that it suffices to fix a finite subset $H \subseteq \Lambda$ and scalars $\{a_{\lambda,\mu} : \lambda, \mu \in H\}$ and show that

$$\Big\| \sum_{\lambda,\mu\in H, d(\lambda)=d(\mu)} a_{\lambda,\mu}\pi(s_\lambda s_\mu^*)\Big\| \leqslant \Big\| \sum_{\lambda,\mu\in H} a_{\lambda,\mu}\pi(s_\lambda s_\mu^*)\Big\|.$$

Proposition 4·10 and equation (4·4) of [**22**] show that we may assume that there exist $v \in \Lambda^0$ and $N \in \mathbb{N}^k$ such that $H \subseteq v\Lambda$ and also that $a_{\lambda,\mu} \neq \emptyset$ implies $\lambda \in \Lambda^{\leqslant N}$. We are now in the situation of Lemma 4·5 with $t_\lambda = \pi(s_\lambda)$ for every $\lambda \in \Lambda$, $a = \sum_{\lambda,\mu\in H} a_{\lambda,\mu}t_\lambda t_\mu^*$ and $a_0 = \sum_{\lambda,\mu\in H, d(\lambda)=d(\mu)} a_{\lambda,\mu}t_\lambda t_\mu^*$. Thus Lemma 4·5 gives

$$\|a_0\| = \|Q(a_0)\| = \|Q(a)\| \leqslant \|a\|.$$

as required.

Remark. We have really just recycled the proof of [**22**, theorem 4·5], which occupies all of [**22**, section 4·2], replacing [**22**, lemma 4·11] (in which condition (B) was invoked) with our Lemma 4·5 which uses aperiodicity instead.

Proof of Theorem 4·1. (i) $\Rightarrow$ (ii) follows from Theorem 4·7; (ii) $\Rightarrow$ (iii) follows from Remark 2·7; and we proved (iii) $\Rightarrow$ (ii) on page 341.

5. *Consequences of cofinality*

In this section we characterise cofinality in Λ in terms of the ideal structure in $C^*(\Lambda)$, and conclude by proving our main result, Theorem 3·4. The meat of Theorem 5·1, namely (ii) $\Rightarrow$ (iii), is identical to [**26**, proposition 4·3] and we have only included the argument for completeness. The key points of difference are our formulation of cofinality in terms of finite paths, and the fact that our proof is direct: our arguments do not appeal to the classification of gauge-invariant ideals in $C^*(\Lambda)$ of [**29**].

THEOREM 5·1. *Suppose Λ is a finitely aligned k-graph. Then the following are equivalent*;

(i) *Λ is cofinal*;
(ii) *for each $v \in \Lambda^0$ and each $x \in \partial\Lambda$ there exists $n \leqslant d(x)$ such that $v\Lambda x(n) \neq \emptyset$*;
(iii) *the only ideal of $C^*(\Lambda)$ which contains s_v for some $v \in \Lambda^0$, is $C^*(\Lambda)$*; *and*
(iv) *the only ideal of $C^*(\Lambda)$ which nontrivially intersects $C^*(\Lambda)^\gamma$ is $C^*(\Lambda)$.*

We begin with a technical lemma. Conditions (i) and (ii) of the following Lemma are precisely conditions (MT1) and (MT2) of [**27**, proposition 5·5·3].

LEMMA 5·2. *Let $K \subseteq \Lambda^0$ be a nonempty set such that*:
(i) *if $u \in K$ and $E \in u\,\mathrm{FE}(\Lambda)$ then there exists $\alpha \in E$ such that $s(\alpha) \in K$*; *and*
(ii) *if $u \in K$ and $v\Lambda u \neq \emptyset$ then $v \in K$.*
Then there exists $x \in \partial\Lambda$ such that $x(n) \in K$ for every $n \leqslant d(x)$.

Proof. We draw heavily on the techniques used in the proof of [**28**, lemma 4·7]. Define $P : (\mathbb{N} \setminus \{0\})^2 \to \mathbb{N} \setminus \{0\}$ by

$$P(m,n) := \frac{(m+n-1)(m+n-2)}{2} + m.$$

Then P is the position function for the diagonal listing

$$\{(1,1),(1,2),(2,1),(1,3),(2,2),(3,1),(1,4),\ldots\} \text{ of } (\mathbb{N} \setminus \{0\})^2.$$

That is, if $P(m,n) = l$ then (m,n) is the lth term in the above listing. For each $l \in \mathbb{N} \setminus \{0\}$ let $(i_l, j_l) \in (\mathbb{N} \setminus \{0\})^2$ be the unique pair such that $P(i_l, j_l) = l$.

Fix $v \in K$. We claim there exists a sequence $(\lambda_l)_{l=1}^\infty \subset v\Lambda$ and for each l an enumeration $s(\lambda_l)\,\mathrm{FE}(\Lambda) = \{E_{l,j} : j \geqslant 1\}$ which satisfy:

(a) $\lambda_{l+1}\big(0, d(\lambda_l)\big) = \lambda_l$ for every $l \geqslant 1$; and
(b) $\lambda_{l+1}\big(d(\lambda_{i_l}), d(\lambda_{l+1})\big) \in E_{i_l, j_l}\Lambda K$ for every $l \geqslant 1$.

We proceed by induction on l. For $l = 0$ define $\lambda_{l+1} = \lambda_1 = v$. Since Λ is countable, for each $w \in \Lambda^0$ the collection of finite subsets of $w\Lambda$ is countable. In particular, $w\,\mathrm{FE}(\Lambda)$ is countable. Let $\{E_{1,j} : j \in \mathbb{N} \setminus \{0\}\}$ be a listing of $v\,\mathrm{FE}(\Lambda)$. Then (a) and (b) are trivially satisfied because $l = 0 < 1$.

Now suppose $l \geqslant 1$ and that λ_n and $\{E_{n,j} : j \geqslant 1\}$ satisfy (a) and (b) for $1 \leqslant n \leqslant l$. Recall that $P(i_l, j_l) = l$ so i_l is the horizontal coordinate of the lth term in the diagonal listing. In particular $i_l < l$ and by assumption the listing $\{E_{i_l,j} : j \geqslant 1\}$ of $s(\lambda i_l)\,\mathrm{FE}(\Lambda)$ satisfies (a) and (b). In particular the set E_{i_l,j_l} has already been fixed. We now must find λ_{l+1} satisfying (a) and (b).

Let $\mu := \lambda_l\big(d(\lambda_{i_l}), d(\lambda_l)\big)$ and $E := \mathrm{Ext}(\mu, E_{i_l,j_l})$. Lemma 2·5 implies that $E \in s(\mu)\,\mathrm{FE}(\Lambda)$. Condition (b) implies that $s(\mu) \in K$, so hypothesis (i) implies that there exists $\alpha \in E$ such that $s(\alpha) \in K$. By definition of E there exist $\nu \in E_{i_l,j_l}$ and $\beta \in \Lambda$ such that $\mu\alpha = \nu\beta \in \mathrm{MCE}(\mu,\nu)$. Then $\lambda_{l+1} := \lambda_l\alpha$ satisfies (b) and trivially satisfies (a). This proves the claim.

Define $m \in (\mathbb{N} \cup \{\infty\})^k$ by $m_i = \sup\{d(\lambda_l)_i : l \geqslant 1\}$, and define $x : \Omega_{k,m} \to \Lambda$ by $x\big(0, d(\lambda_l)\big) = \lambda_l$ for all l. Once we show that $x \in v\partial\Lambda$, hypothesis (ii) will force $x(n) \in K$ for each $n \leqslant d(x)$ as required.

Fix $n \leqslant d(x)$ and $F \in x(n)\,\mathrm{FE}(\Lambda)$. Fix l such that $d(\lambda_{i_l}) \geqslant n$ and let $E := \mathrm{Ext}\,\big(x(n, d(\lambda_{i_l})), F\big)$. Then Lemma 2·5 implies that $E \in s(\lambda_{i_l})\,\mathrm{FE}(\Lambda)$. By definition of the position function P, there exists $k \geqslant l$ such that $i_k = l$ and $E = E_{l,j_k}$. Then condition (b) implies

$$x\big(d(\lambda_l), d(\lambda_{k+1})\big) = x\big(d(\lambda_{i_k}), d(\lambda_{k+1})\big) \in E.$$

On the other hand, since $E = \mathrm{Ext}\,\big(x(n, d(\lambda_{i_l})), F\big)$ the definition of Ext (see page 337) implies that there exists $\mu \in F$ and $\alpha \in \Lambda$ such that

$$x\big(n, d(\lambda_l)\big)x\big(d(\lambda_l), d(\lambda_{k+1})\big) = \mu\alpha \in \mathrm{MCE}\,\big(x(n, d(\lambda_l)), \mu\big).$$

In particular

$$x(n, n + d(\mu)) = (x(n, d(\lambda_{k+1})))(0, d(\mu)) = (\mu\alpha)(0, d(\mu)) = \mu.$$

Hence $x \in \partial\Lambda$ as required.

Proof of Theorem 5·1. (i) $\Rightarrow$ (ii). Suppose Λ is cofinal. Fix $v \in \Lambda^0$ and $x \in \partial\Lambda$. Since Λ is cofinal there exists $E \in x(0)\,\mathrm{FE}(\Lambda)$ such that $v\Lambda s(\alpha) \neq \emptyset$ for all $\alpha \in E$. Since $x \in \partial\Lambda$ there exists $\alpha \in E$ such that $x\big(0, d(\alpha)\big) = \alpha$. In particular $n = d(\alpha)$ satisfies

$$x(n) = s\big(x\big(0, d(\alpha)\big)\big) = s(\alpha)$$

so $v\Lambda x(n) \neq \emptyset$.

(ii) $\Rightarrow$ (i). We prove the contrapositive statement. Suppose there exist $v, w \in \Lambda^0$ such that for every $E \in w\,\mathrm{FE}(\Lambda)$ there exists $\alpha \in E$ such that $v\Lambda s(\alpha) = \emptyset$. We aim to apply Lemma 5·2. To this end we let

$$K := \{u \in \Lambda^0 : \text{ for each } E \in u\,\mathrm{FE}(\Lambda) \text{ there exists } \alpha \in E \text{ such that } v\Lambda s(\alpha) = \emptyset\}.$$

We claim that K satisfies the hypothesis of Lemma 5·2. We have $K \neq \emptyset$ since $w \in K$. Fix $u \in K$ and $E \in u\,\mathrm{FE}(\Lambda)$. To show that K satisfies hypothesis (i) of Lemma 5·2 we claim there exists $\alpha \in E$ such that $s(\alpha) \in K$. Indeed, if for every $\alpha \in E$ we have $s(\alpha) \notin K$ then for every $\alpha \in E$ there exists $F_\alpha \in s(\alpha)\,\mathrm{FE}(\Lambda)$ such that $v\Lambda s(\eta) \neq \emptyset$ for every $\eta \in F_\alpha$. Let $G := \{\alpha\eta : \alpha \in E, \eta \in F_\alpha\}$ and $F := \{\alpha \in E : F_\alpha \text{ does not contain } s(\alpha)\}$. Then since $F_\alpha \in \mathrm{FE}(\Lambda)$ for all $\alpha \in E$, [**28**, definition 5·2] combined with [**28**, lemma 5·3] implies $G \in \mathrm{FE}(\Lambda)$. By construction of G we have $v\Lambda s(\lambda) \neq \emptyset$ for each $\lambda \in G$ and since $G \in u\,\mathrm{FE}(\Lambda)$ this contradicts $u \in K$. Hence K satisfies hypothesis (i) of Lemma 5·2.

For hypothesis (ii) fix $u \in K$ and suppose $v\Lambda u \neq \emptyset$, say $\lambda \in v\Lambda u$. We must show that $v \in K$. Fix $E \in v\,\mathrm{FE}(\Lambda)$ and let $F = \mathrm{Ext}(\lambda, E)$. It follows from Lemma 2·5 that $F \in u\,\mathrm{FE}(\Lambda)$. Since $u \in K$ there exists $\alpha \in F$ such that $v\Lambda s(\alpha) = \emptyset$. By definition of F there exists $\mu \in E$ and $\nu \in \mathrm{MCE}(\lambda, \mu)$ such that $\alpha = \nu\big(d(\lambda), d(\lambda) \vee d(\mu)\big)$. In particular

$$\beta := \nu\big(d(\mu), d(\lambda) \vee d(\mu)\big) \in s(\mu)\Lambda s(\alpha)$$

and since $v\Lambda s(\alpha) = \emptyset$ we have $v\Lambda s(\mu) = \emptyset$. That is, $\mu \in E$ satisfies $v\Lambda s(\mu) = \emptyset$. Hence $v \in K$ and thus K satisfies the hypotheses of Lemma 5·2 as claimed. By Lemma 5·2 there exists $x \in \partial\Lambda$ with $x(n) \in K$ for every $n \leqslant d(x)$. Since $\{x(n)\} \in x(n)\,\mathrm{FE}(\Lambda)$ for all $n \leqslant d(x)$, the definition of K then implies that $v\Lambda x(n) = \emptyset$ for all n.

(ii) $\Rightarrow$ (iii). Fix an ideal I of $C^*(\Lambda)$ with $s_w \in I$ for some $w \in \Lambda^0$. Let $H = \{w : s_w \in I\}$. We will show $\Lambda^0 \setminus H$ satisfies the hypothesis of Lemma 5·2 and then deduce that $H = \Lambda^0$.

Recall from [**22**, definition 3·3] that ΠE was defined as the smallest set containing E such that

> for all $\lambda, \mu, \nu, \rho \in \Pi E$ with $d(\lambda) = d(\mu), d(\nu) = d(\rho), s(\lambda) = s(\mu)$ and $s(\nu) = s(\rho)$ and each $\mu\alpha = \nu\beta \in \mathrm{MCE}(\mu, \nu)$, we have $\lambda\alpha, \rho\beta \in \Pi E$.

In particular, the construction of the set F in the proof of [**22**, lemma 3·2] combines with [**22**, definition 3·3] to show that each element μ of ΠE has the form $\mu = \nu\nu'$ for some $\nu \in E$.

We claim that if $v \in \Lambda^0 \setminus H$ then for every finite exhaustive set $F \subset v\Lambda$ there exists $\lambda \in F$ such that $s(\lambda) \in \Lambda^0 \setminus H$. Suppose for contradiction that $s(\nu) \in H$ for each $\nu \in F$. Fix $\mu \in \Pi F$ then $\mu = \nu\nu'$ for some $\nu \in F$. Since $s_{s(\nu)} \in I$ we have $s_{s(\mu)} = s_{\nu'}^* s_{s(\nu)} s_{\nu'} \in I$ for every $\mu \in \Pi F$. Then $s_\mu s_\mu^* = s_\mu s_{s(\mu)} s_\mu^* \in I$. [**22**, proposition 3·5] implies that

$$s_v = \prod_{\mu' \in v\Pi F} (s_v - s_{\mu'} s_{\mu'}^*) + \sum_{\mu \in v\Pi F} \left(s_\mu s_\mu^* \prod_{\nu\nu' \in \Pi F, d(\nu') > 0} (s_\nu s_\nu^* - s_{\nu\nu'} s_{\nu\nu'}^*) \right).$$

Since F is exhaustive, ΠF is also exhaustive and [**22**, lemma 3·2] implies that ΠF is finite.

Hence (CK4) implies that $\prod_{\mu'\in v\Pi F}(s_v - s_{\mu'}s_{\mu'}^*) = 0$ and thus

$$s_v = \sum_{\mu\in v\Pi F}\left(s_\mu s_\mu^* \prod_{\nu\nu'\in\Pi F, d(\nu')>0}(s_\nu s_\nu^* - s_{\nu\nu'}s_{\nu\nu'}^*)\right) \in I.$$

This contradicts $v \in \Lambda^0 \setminus H$. Thus $\Lambda^0 \setminus H$ satisfies hypothesis (i) of Lemma 5·2. For hypothesis (ii), suppose $u \in \Lambda^0 \setminus H$ and $\lambda \in v\Lambda u$. Suppose for contradiction that $v \in H$. Then $s_u = s_{s(\lambda)} = s_\lambda^* s_v s_\lambda \in H$ contradicting the definition of u. So $\Lambda^0 \setminus H$ satisfies the hypothesis of Lemma 5·2.

By Lemma 5·2 there exists $x \in \partial\Lambda$ with $x(n) \in \Lambda^0\setminus H$ for every $n \leqslant d(x)$. By hypothesis there exists $n \leqslant d(x)$ such that $w\Lambda x(n) \neq \emptyset$. Let $\mu \in w\Lambda x(n)$. Since $s_w \in I$ we have $s_{x(n)} = s_\mu^* s_w s_\mu \in I$ forcing $x(n) \in H$. This contradicts the definition of x. Hence $H = \Lambda^0$. For each $\mu \in \Lambda$ we now have $s(\mu) \in H$, so $s_{s(\mu)} \in I$. Thus $s_\mu = s_\mu s_\mu^* s_\mu = s_\mu s_{s(\mu)} \in I$. So I contains all the generators of $C^*(\Lambda)$, forcing $I = C^*(\Lambda)$.

(iii) $\Rightarrow$ (ii). We prove the contrapositive statement. Suppose statement (ii) does not hold. Fix $x \in \partial\Lambda$ and $v \in \Lambda^0$ such that $v\Lambda x(n) = \emptyset$ for all $n \in \mathbb{N}^k$. Define $T \subseteq \partial\Lambda$ by

$$T = \{y \in \partial\Lambda : \sigma^m(y) = \sigma^n(x) \text{ for some } m, n \in \mathbb{N}^k\}.$$

Let $W = \overline{\operatorname{span}}\{\zeta_y : y \in T\} \subseteq \ell^2(\partial\Lambda)$ and fix $y \in T$. Since $\lambda y, \sigma^n(y) \in T$ for all $\lambda \in \Lambda r(y)$ and $n \leqslant d(y)$, we have $S_\lambda W \subseteq W$ and $S_\lambda^* W \subseteq W$ for every $\lambda \in \Lambda$. Since the S_λ satisfy the Cuntz–Krieger relations, so do the $S_\lambda|_W$.

By the universal property of $C^*(\Lambda)$ there is a representation $\pi : C^*(\Lambda) \to \mathcal{B}(W)$ such that $\pi(s_\lambda) = S_\lambda$ for every $\lambda \in \Lambda$. We claim that $I := \ker(\pi)$ contains a vertex projection but is not equal to $C^*(\Lambda)$. First observe that $\pi(s_{r(x)})\zeta_x = S_{r(x)}\zeta_x = \zeta_x$, so $s_{r(x)} \notin I$ and hence $I \neq C^*(\Lambda)$. We will show that $s_v \in I$. To see this, fix a basis element $\zeta_y \in W$. Since $y \in T$ we have $\sigma^m(y) = \sigma^n(x)$ for some $m, n \in \mathbb{N}^k$. In particular $y(0, m) \in r(y)\Lambda x(n)$ but $v\Lambda x(n) = \emptyset$ which forces $r(y) \neq v$. Hence $\pi(s_v)\zeta_y = 0$. Hence (iii) does not hold.

(iii) $\Rightarrow$ (iv). For $E \subseteq \Lambda$ define $M_{\Pi E}^s := \overline{\operatorname{span}}\{s_\mu s_\nu^* : \mu, \nu \in \Pi E, d(\mu) = d(\nu)\}$. Then the proof of [**22**, theorem 3·1] shows that $C^*(\Lambda)^\gamma = \overline{\bigcup_{E\subset\Lambda \text{ finite}} M_{\Pi E}^s}$, and [**22**, lemma 3·2] implies that each $M_{\Pi E}^s$ is a finite-dimensional C^*-algebra. Let I be an ideal of $C^*(\Lambda)$ such that $I \cap C^*(\Lambda)^\gamma \neq \{0\}$. Then there exists a finite set $E \subseteq \Lambda$ such that $I \cap M_{\Pi E}^s \neq \{0\}$. Since $M_{\Pi E}^s = \oplus_{v\in s(\Pi E)} M_{(\Pi E)v}^s$ and $M_{(\Pi E)v}^s$ is simple for each $v \in s(\Pi E)$, there exists $v \in s(\Pi E)$ such that $I \cap M_{(\Pi E)v}^s = M_{(\Pi E)v}^s$. Since $(\Pi E)v \neq \emptyset$ there exists λ such that $s_\lambda s_\lambda^* \in M_{(\Pi E)v}^s \subset I$. Hence $s_{s(\lambda)} = s_\lambda^*(s_\lambda s_\lambda^*)s_\lambda \in I$ and (iii) implies $I = C^*(\Lambda)$.

(iv) $\Rightarrow$ (iii): Trivial, since each s_v belongs to $C^*(\Lambda)^\gamma$.

Proof of Theorem 3·4. ($\Rightarrow$) Suppose $C^*(\Lambda)$ is simple. Then (iii) $\Rightarrow$ (i) of Theorem 5·1 implies that Λ is cofinal and (iii) $\Rightarrow$ (i) of Theorem 4·1 implies that Λ is aperiodic.

($\Leftarrow$) Suppose Λ is aperiodic and cofinal. Fix a nonzero ideal I of $C^*(\Lambda)$. Then (i) $\Rightarrow$ (ii) of Theorem 4·1 implies that I contains a vertex projection, and (i) $\Rightarrow$ (iii) of Theorem 5·1 then implies $I = C^*(\Lambda)$. Thus $C^*(\Lambda)$ is simple.

Appendix A. Cofinality

In the appendix we will show that our cofinality condition for finitely aligned k-graphs is equivalent to other, simpler conditions for less general classes of k-graphs.

A·1. *Row-finite locally convex k-graphs*

Recall from [**21**] that a k-graph Λ is called *locally convex* if, for all distinct $i, j \in \{1, \ldots, k\}$, and all $\mu \in \Lambda^{e_i}$ and $\nu \in \Lambda^{e_j}$ such that $r(\mu) = r(\nu)$, each of $s(\mu)\Lambda^{e_j}$ and $s(\nu)\Lambda^{e_i}$ is nonempty.

We begin with some notation: for $n \in \mathbb{N}^k$ we write $|n| := \sum_{i=1}^k n_i \in \mathbb{N}$. To show the "if" direction for row-finite locally convex k-graphs we will need to use the following technical lemma.

LEMMA A·1. *Let Λ be a locally convex k-graph and fix $\lambda \in \Lambda^{\leqslant m+n}$. Then $\mu = \lambda\big(0, m \wedge d(\lambda)\big)$ and $\nu = \lambda(m \wedge d(\lambda), d(\lambda))$ are the unique paths $\mu \in \Lambda^{\leqslant m}$ and $\nu \in \Lambda^{\leqslant n}$ such that $\lambda = \mu\nu$.*

Proof. We first establish the existence of paths $\mu \in \Lambda^{\leqslant m}$ and $\nu \in \Lambda^{\leqslant n}$ such that $\lambda = \mu\nu$ and then show that they must be as defined above. We proceed by induction on $|n|$. If $|n| = 1$ then this is precisely [**21**, lemma 3·12]. Suppose the statement is true for $|n| = l \geqslant 1$ and suppose $|n| = l + 1$. Fix $j \in \{1, \ldots, k\}$ such that $n_j \geqslant 1$. By [**21**, lemma 3·12] we may factorise λ as $\lambda'\lambda''$ where $\lambda' \in \Lambda^{(m+n)-e_j}$ and $\lambda'' \in \Lambda^{\leqslant e_j}$. The inductive hypothesis implies that $\lambda' = \mu\lambda'''$ where $\mu \in \Lambda^{\leqslant m}$ and $\lambda''' \in \Lambda^{\leqslant n-e_j}$. By [**21**, lemma 3·6] we have $\nu := \lambda'''\lambda'' \in \Lambda^{\leqslant (n-e_j)+e_j} = \Lambda^{\leqslant n}$ and $\lambda = \mu\nu$ as desired.

For uniqueness first observe that if $p \leqslant d(\lambda)$ satisfies $\lambda(0, p) \in \Lambda^{\leqslant m}$, then $p \leqslant m \wedge d(\lambda)$. Now suppose for contradiction that $p_i < (m \wedge d(\lambda))_i$. Then $p_i < m_i$ so $p + e_i \leqslant d(\lambda)$; hence $\lambda(p, p + e_i) \in \lambda(p)\Lambda^{e_i}$ contradicting $\lambda(0, p) \in \Lambda^{\leqslant m}$. It follows that if $\lambda = \mu\nu$ with $\mu \in \Lambda^{\leqslant m}$, we must have $\mu = \lambda(0, m \wedge d(\lambda))$, and then $\nu = \lambda(d(\mu), d(\lambda))$ by the factorisation property.

PROPOSITION A·2. *Suppose Λ is a row-finite locally convex k-graph. Then Λ is cofinal if and only if for every $v, w \in \Lambda^0$ there exists $n \in \mathbb{N}^k$ such that $v\Lambda s(\alpha) \neq \emptyset$ for each $\alpha \in w\Lambda^{\leqslant n}$.*

Proof. ($\Rightarrow$) Suppose Λ is cofinal. Fix $v, w \in \Lambda^0$. Then there exists $E \in w\,\mathrm{FE}(\Lambda)$ such that $v\Lambda s(\alpha) \neq \emptyset$ for each $\alpha \in E$. We must show there exists $n \in \mathbb{N}^k$ such that $v\Lambda s(\beta) \neq \emptyset$ for each $\beta \in w\Lambda^{\leqslant n}$. Let $n = \bigvee_{\alpha \in E} d(\alpha)$ and fix $\beta \in w\Lambda^{\leqslant n}$. Since E is exhaustive there exists $\alpha \in E$ such that $\mathrm{MCE}(\alpha, \beta) = \alpha\mu = \beta\nu$ for some $\mu \in s(\alpha)\Lambda^{(d(\alpha)\vee d(\beta))-d(\beta)}$ and $\nu \in s(\alpha)\Lambda^{(d(\alpha)\vee d(\beta))-d(\alpha)}$. However since $\beta \in \Lambda^{\leqslant n}$ we must have $\nu = s(\beta)$. Hence $v\Lambda s(\alpha) \neq \emptyset$ and thus $v\Lambda s(\beta) \supseteq v\Lambda\mu \neq \emptyset$.

($\Leftarrow$) It suffices to show that each $w\Lambda^{\leqslant n} \in w\,\mathrm{FE}(\Lambda)$. Since Λ is row-finite we have $|w\Lambda^{\leqslant n}| < \infty$. Fix $\alpha \in w\Lambda$ and $\beta \in s(\alpha)\Lambda^{\leqslant (n\vee d(\alpha))-d(\alpha)}$. By [**21**, Lemma 3.6] $\alpha\beta \in w\Lambda^{\leqslant n\vee d(\alpha)} = w\Lambda^{\leqslant n+((n\vee d(\alpha))-n)}$. Then by Lemma A·1 there exists $\mu \in \Lambda^{\leqslant n}$ and $\nu \in w\Lambda^{\leqslant (n\vee d(\alpha))-n}$ such that $\mu\nu = \alpha\beta$. Hence $\mu \in \Lambda^{\leqslant n}$ satisfies $\mathrm{MCE}(\mu, \alpha) \neq \emptyset$.

Remark A·3. If Λ has no sources, then each $\Lambda^{\leqslant n} = \Lambda^n$, so Λ is cofinal if and only if for every $v, w \in \Lambda^0$ there exists $n \in \mathbb{N}^k$ such that $v\Lambda s(\alpha) \neq \emptyset$ for every $\alpha \in w\Lambda^n$.

A·2. *Arbitrary directed graphs*

We will use the following notation to characterise cofinality in an arbitrary directed graph.

NOTATION. *Suppose E is an arbitrary directed graph. For $n \in \mathbb{N}$ we define*

$$X_n := \big\{\lambda \in E^* : d(\lambda) \leqslant n,\ |\lambda(m)E^1| < \infty \textit{ for every } m < d(\lambda) \\ \textit{and if } d(\lambda) < n \textit{ then } |s(\lambda)E^1| \in \{0, \infty\}\big\}.$$

PROPOSITION A·4. *Suppose E is an arbitrary directed graph. Then Λ is cofinal if and only if for every $v, w \in E^0$ there exists $n \in \mathbb{N}$ such that $vE^*s(\alpha) \neq \emptyset$ for each $\alpha \in wX_n$.*

Proof. ($\Rightarrow$) Suppose Λ is cofinal. Fix $v, w \in \Lambda^0$. Then there exists $F \in w\,\mathrm{FE}(E)$ such that $vE^*s(\alpha) \neq \emptyset$ for every $\alpha \in F$. Let $n = \max\{|\alpha| : \alpha \in F\}$. Consider wX_n. We show that $vE^*s(\mu) \neq \emptyset$ for every $\mu \in wX_n$. Fix $\mu \in wX_n$. Then there exists a smallest $\nu \in F$ such that $\mathrm{MCE}(\mu, \nu) \neq \emptyset$.

We claim that $|\mu| \geqslant |\nu|$. Suppose for contradiction that $|\mu| < |\nu|$. Then $|\nu(m)E^1| < \infty$ for every $m < |\nu|$; for if $m \in \mathbb{N}$ satisfies $m < |\nu|$ and $|\nu(m)E^1| = \infty$ then there exists $\lambda \in \nu(m)E^1$ such that $\nu(0, |m|)\lambda \notin F$, which is impossible because ν is minimal in F with $\mathrm{MCE}(\mu, \nu) \neq \emptyset$. In particular $|s(\mu)E^1| < \infty$. However, because $|\mu| < |\nu| \leqslant n$, by definition of X_n the set $s(\mu)E^1$ is either empty or infinite. Clearly $|s(\mu)E^1| \neq 0$ so $|s(\mu)E^1| = \infty$ giving a contradiction. Now since $|\mu| \geqslant |\nu|$ we have $\mathrm{MCE}(\mu, \nu) = \mu = \nu\mu(|\nu|, |\mu|)$ and it follows from $vE^*s(\nu) \neq \emptyset$ that $vE^*s(\mu) \neq \emptyset$.

($\Leftarrow$) Fix $v, w \in E^0$. Then there exists $n \in \mathbb{N}$ such that $vE^*s(\alpha) \neq \emptyset$ for each $\alpha \in wX_n$. By definition of X_n we have $|X_n| < \infty$. We claim that wX_n is exhaustive. Suppose for contradiction that wX_n is not exhaustive. Then there exists $\lambda \in wE^*$ such that $\mathrm{MCE}(\beta, \lambda) = \emptyset$ for every $\beta \in w(X_n)$. Since $r(\lambda) = w$ there exists a smallest $m \leqslant n$ such that $\lambda(0, m) = \beta(0, m)$ for some $\beta \in wX_n$ but $\lambda(m, m+1) \neq \mu\big(m, (m+1) \wedge d(\mu)\big)$ for every $\mu \in w(X_n)$. We consider two cases: when $m = n$ and when $m < n$.

We first consider the case when $m = n$. Then $\beta(m) = s(\beta)$ and since $\beta(m)E^*s(\lambda) \neq \emptyset$ it follows that $\mathrm{MCE}(\beta, \lambda) = \lambda$. Now suppose $m < n$. Then $m + 1 \leqslant n$ and hence by definition of X_n we have $|\beta(m)E^1| = \infty$. Thus $|\beta| = m$ and again it follows that $\mathrm{MCE}(\beta, \lambda) = \lambda$. Hence wX_n is exhaustive.

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