Some new results of regular Hadamard matrices and SBIBD II

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Abstract

In this paper we prove that there exist $4-\{k^2; \frac{1}{2}k(k-1); k(k-2)\}$ SDS, regular Hadamard matrices of order $4k^2$, and SBIBD $(4k^2, 2k^2 + k, k^2 + k)$ for k = 47, 71, 151, 167, 199, 263, 359, 439, 599, 631, 727, 919, $5q_1$, $5q_2N$, $7q_3$, where q1, q2 and q3 are prime power such that $q_1 \equiv 1(mod\,4)$, $q_2 \equiv 5(mod\,8)$ and $q_3 \equiv 3(mod\,8)$, $N = 2^a3^bt^2$, a, b = 0 or 1, $t \neq 0$ is an arbitrary integer. We find new SBIBD$(4k^2, 2k^2 + k, k^2 + k)$ for 43 values of k less than 1000.

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Some new results of regular Hadamard matrices and SBIBD II [*]

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Abstract

In this paper we prove that there exist $4-\{k^2; \frac{1}{2}k(k-1); k(k-2)\}$ SDS, regular Hadamard matrices of order $4k^2$, and $SBIBD(4k^2, 2k^2+k, k^2+k)$ for $k = $ 47, 71, 151, 167, 199, 263, 359, 439, 599, 631, 727, 919, $5q_1$, $5q_2N$, $7q_3$, where q_1, q_2 and q_3 are prime power such that $q_1 \equiv 1(mod\ 4)$, $q_2 \equiv 5(mod\ 8)$ and $q_3 \equiv 3(mod\ 8)$, $N = 2^a3^bt^2$, $a, b = 0$ or 1, $t \neq 0$ is an arbitrary integer. We find new $SBIBD(4k^2, 2k^2+k, k^2+k)$ for 43 values of k less than 1000.

1 Preliminaries

An $n \times n$ matrix H is called an Hadamard matrix (or H-matrix) if every entry of the matrix is 1 or -1, and
$$HH^T = nI_n,$$
where I_n is an $n \times n$ identity matrix. In this paper we use H^T to denote the transpose of a matrix H.

We denote the excess of a Hadamard matrix $H = [a_{ij}]$ by $\sigma(H)$, where
$$\sigma(H) = \sum_{1 \leq i,j \leq n} a_{ij}.$$

Let $\sigma(n) = \max\{\sigma(H)\}$. The weight of a Hadamard matrix H, denoted by $W(H)$, is the number of ones in the H. We define $W(n) = max\{W(H)\}$. Note that the maxima are taken over all Hadamard matrices H of order n. It is obvious that $\sigma(H) = 2W(H) - n^2$ and $\sigma(n) = 2W(n) - n^2$ (see [3], [4], [5], [6] for details).

M. R. Best [1] proved that

$$\sigma(n) \leq n\sqrt{n}. \tag{1}$$

Definition 1 *(Regular Hadamard Matrix) A regular Hadamard matrix has the sum of each column of the matrix and the sum of each row of the matrix constant.*

Definition 2 *(SBIBD) A symmetric balanced incomplete block design, called as $SBIBD(v, k, \lambda)$, is defined by a $v \times v$ matrix M, which has every entry 0 or 1. The sum of each column and the sum of each row of the matrix is k. For any two columns c_i, c_j (and two rows r_i, r_j), $1 \leq i \neq j \leq v$, the inner product of c_i and c_j (r_i and r_j) is λ (see [8]).*

In 1989 J. Seberry proved the following theorem which is very useful for constructing $SBIBD(4k^2, 2k^2 + k, k^2 + k)$.

Theorem 1 *(J. Seberry [6]) The following conditions are equivalent:*

(i) *There exists a Hadamard matrix of order $4k^2$ with maximum excess $8k^3$.*

(ii) *There exists a regular Hadamard matrix of order $4k^2$.*

(iii) *There exists an $SBIBD(4k^2, 2k^2 + k, k^2 + k)$.*

Many regular Hadamard matrices of order $4k^2$ and $SBIBD(4k^2, 2k^2 + k, k^2 + k)$ were given in [3, 7, 11]. In particular, there were 169 values of k less than 1000 for which the existence of $SBIBD(4k^2, 2k^2 + k, k^2 + k)$ was still undetermined (see the list of [11]).

2   Construct SBIBD from SDS

Definition 3 *(SDS) Let G be an Abelian group of order v. We denote the group operation by multiplication. Subsets $D_1, \cdots, D_r$ of G are called $r - \{v; \mid D_1 \mid, \cdots, \mid D_r \mid; \lambda\}$ supplementary difference sets (SDS), if for every nonidentity element g in G there are exactly λ ordered pairs (d, d') in $D_1 \times D_1$, or $D_2 \times D_2$, $\cdots$, or $D_r \times D_r$, such that $gd' = d$.*

It is convenient to use the group ring $Z[G]$ of the group G over the ring Z of rational integers with addition and multiplication. Here the elements of $Z[G]$ are of the form

$$a_1 g_1 + a_2 g_2 + \cdots + a_v g_v, \quad a_i \in Z, \quad g_i \in G.$$

In $Z[G]$ the addition $+$ is given by the rule

$$\sum_{g \in G} a(g)g + \sum_{g \in G} b(g)g = \sum_{g \in G} (a(g) + b(g))g.$$

The multiplication in $Z[G]$ is given by the rule

$$(\sum_g a(g)g)(\sum_h b(h)h) = \sum_k (\sum_{gh=k} a(g)b(h))k.$$

For any subset A of G we define

$$\sum_{g \in A} g \in Z[G],$$

and by abusing the notation we will denote it by A.

For any two subsets $A, B \subset G$, let t be an integer. We define

$$B^t = \sum_{b \in B} b^t \in Z[G], \quad AB^{-1} = \sum_{a \in A, \, b \in B} ab^{-1} \in Z[G],$$

and denote

$$\triangle A = AA^{-1}, \quad \triangle(A, B) = AB^{-1} + BA^{-1}.$$

If $A = \phi$, then we have

$$\triangle \phi = 0, \quad \triangle(\phi, B) = 0.$$

It is obviously that $\triangle(A, A) = 2\triangle A$.

With this convention $D_1, D_2, \cdots, D_r$ being $r - \{v; \mid D_1 \mid, \cdots, \mid D_r \mid; \lambda\}$ SDS are equivalent to

$$\sum_{i=1}^r \triangle D_i = (\sum_{r=1}^r \mid D_i \mid - \lambda) + \lambda G.$$

If $k_1 = \cdots = k_r = k$, we simplify $D_1, \cdots, D_r$ to $r - \{v; k; \lambda\}$ SDS. When $r = 1$, the single SDS becomes a difference set (DS) in the usual sense. When $r = 4$ and $\lambda = \sum_{i=1}^4 k_i - v$, we call D_1, D_2, D_3, D_4 type $H-$SDS.

In this paper special interest is devoted to the case: $v = q^2$, $k_1 = k_2 = k_3 = k_4 = \frac{1}{2}q(q-1)$ and $\lambda = q(q-2)$ (see [6]).

To construct SBIBD from SDS we need the following theorem.

Theorem 2 *(T. Xia, M.Y. Xia and J. Seberry [11]) If there exist $4 - \{q^2; \frac{1}{2}q(q-1); q(q-2)\}$ SDS on an Abelian group G of order q^2, then there exist an $SBIBD(4q^2, 2q^2 + q, q^2 + q)$.*

In the following we assume p is an odd prime, $r > 0$, and

$$q = p^r = 4m + 3.$$

Let g be a generator of $GF(q^2)^*$. Put

$$E_i = \{g^{8(m+1)j+i} : j = 0, \cdots, 2m\}, \quad i = 0, \cdots, 8m + 7. \tag{2}$$

Lemma 1 *(M.Y. Xia nd G. Liu [9]) In $GF(q^2)$ the following equations hold:*

(i) $\triangle E_i = (2m+1) + m(E_i + E_i^{-1})$,

(ii) $\triangle(E_i, E_i^{-1}) = (2m+1)(E_i + E_i^{-1})$,

(iii) $\triangle(E_i, E_j + E_j^{-1}) = GF(q^2)^* - (E_i + E_i^{-1} + E_j + E_j^{-1})$,

where $0 \le i \ne j \le 8m+7$, $GF(q^2)^*$ is the set of all nonzero elements of $GF(q^2)$.

Let

$$U = \{a_i : \ 0 \le a_i \le 8m+7, \ i = 0, \cdots, 2t\},$$
$$V = \{b_j : \ 0 \le b_j < 4m+4, \ j = 1, \cdots, 2m+1-t\},$$

where $0 \le t \le 2m+1$, such that

$$| \ \{a(mod\ 4m+4) : \ a \in U\} \cup V \ | = 2m+2+t. \tag{3}$$

The equation (3) means that $a_i \not\equiv a_j (mod\ 4m+4)$ for $i \ne j$ and $a_i \not\equiv b_j (mod\ 4m+4)$ for any i, j.

Write

$$A = \bigcup_{a \in U} E_a, \ \ B = \bigcup_{b \in V}(E_b \cup E_{b+4m+4}) \tag{4}$$

and set

$$D = A \cup B. \tag{5}$$

Lemma 2 *Under the condition (3) we have*

$$\begin{aligned}\triangle D = \ &2(2m+1-t)(2m+1) + ((2m+1)^2 - t^2)GF(q^2)^* \\ &-(2m+1-t)(A + A^{-1}) + \triangle A.\end{aligned} \tag{6}$$

Proof. (6) follows from Lemma 1 by direct calculation. $\qquad\square$

From (6) we can see that $\triangle D$ only depends on A and does not depend on the particular choice of B. This is a very useful property in searching SDS.

Put

$$D_i = g^{(m+1)i}D, \ \ i = 0, 1, 2, 3. \tag{7}$$

We investigate when D_0, D_1, D_2 and D_3 defined as above can form $4 - \{q^2; \frac{1}{2}q(q-1); \ q(q-2)\}$ SDS for some appropriate set A, i.e.,

$$\sum_{i=1}^{3}\triangle D_i = q^2 + q(q-2)GF(q^2). \tag{8}$$

When $q \equiv 3(mod\ 8)$, many subsets in $GF(q^2)$ can be taken as the A that makes (8) true [9, 10]. In other cases we know of no general answer so far. Fortunately, when $q \equiv 7(mod\ 16)$, we find many positive results.

Lemma 3 *For* $q = 71, 151, 167, 199, 263, 359, 439, 599, 631, 727$ *and* 919, *there exist subsets* A *of* $GF(q^2)$ *that make (8) true.*

Concrete constructions for A in the above cases are given in the Appendix of the paper. Using Lemma 1 and Lemma 2 one can check they satisfy (8). We refer verification to the reader.

Lemma 3 is an attempt to search for $4 - \{q^2; \frac{1}{2}q(q-1); q(q-2)\}$ SDS when $q \equiv 7(mod\ 8)$. From Lemma 3 it follows

Corollary 1 *There exist* $SBIBD(4k^2; 2k^2+k; k^2+k)$ *for* $k = 71, 151, 167, 199,$ $263, 359, 439, 599, 631, 727$ *and* 919, *respectively.*

For $q \equiv 7(mod\ 16)$ the first three gaps of k are 103, 311 and 487. The regular Hadamard matrices of corresponding orders are unknown as yet. This means that we can not use the method given to find solutions for all cases.

3   Construct SBIBD from SDS and T-matrices

Definition 4 *(T−matrix) Let* T_1, T_2, T_3, T_4 *be* $n \times n$ *matrices with entries* $(0, \pm1)$. *Then we call* T_1, T_2, T_3, T_4 $T-$*matrices if*

(i) $T_i T_j = T_j T_i$, $1 \le i, j \le 4$,

(ii) *there exists an* $n \times n$ *monomial matrix* R *with* $R^T = R$, $R^2 = I_n$, *such that* $(T_i R)^T = T_i R$, $i = 1, 2, 3, 4$,

(iii) *if* $T_i = \left(t_{jk}^{(i)}\right)$, $1 \le j, k \le n$, $i = 1, 2, 3, 4$, *then* $\sum_{i=1}^{4} \mid t_{jk}^{(i)} \mid = 1$, $i \le j, k \le n$,

(iv) $\sum_{i=1}^{4} T_i T_i^T = nI_n$.

We use condition (i) and (ii) to replace the condition of circulant $T-$matrices.

More details of *T-matrices* are discussed in [2]. In this paper we refer to the paper [10]. The following theorem will be useful for constructing SBIBD.

Theorem 3 *([11]) If there exist* $4 - \{q^2; \frac{1}{2}q(q-1); q(q-2)\}$ *SDS* D_1, D_2, D_3, D_4 *of order* q^2 *in an Abelian group* G *of order* q^2, *and every element of* G *appears an even number of times in* D_1, D_2, D_3, D_4, *then there exist T-matrices* T_1, T_2, T_3, T_4 *that satisfy*

$$\sigma(T_1) = q^3, \quad \sigma(T_2) = \sigma(T_3) = \sigma(T_4) = 0. \tag{9}$$

For convenience we write the prime power $q \equiv 4m + 3 = 16n + 7$. Since the polynomial $x^2 + 1$ is irreducible over $GF(q)$, the set of all elements $\alpha x + \beta$, α, $\beta \in GF(q)$ modulo $x^2 + 1$ is a finite field $GF(q^2)$. In the following we will employ this concrete representation of $GF(q^2)$.

We know that there exist $T-$matrices T_1, T_2, T_3 and T_4 of order q^2 which satisfy (9) when $q \equiv 3(mod\ 8)$ [10]. Another example of $T-$matrices satisfying (9) is as follows:

Example 1 *Let $g = x + 1$ in $GF(5^2)$ and*

$$C_i = \{g^{8j+i}(mod\ x^2 - 3,\ mod\ 5) : \ j = 0, 1, 2\}, \ \ i = 0, 1, \cdots, 7.$$

Take

$$D_1 = \{0\} \cup C_0 \cup C_1 \cup C_2, \ \ D_2 = \{0\} \cup C_0 \cup C_1 \cup C_3,$$
$$D_3 = \{0\} \cup C_0 \cup C_2 \cup C_6, \ \ D_4 = \{0\} \cup C_0 \cup C_3 \cup C_6.$$

It is easy to verify that D_1, D_2, D_3 and D_4 defined above are $4 - \{25; 10; 15\}$ SDS and Theorem 3 holds for $q = 5$. Consequently, there exist $T-$matrices of order 25 that satisfy (9).

Theorem 4 *Suppose D_1, D_2, D_3 and D_4 are $4 - \{k^2; \frac{1}{2}k(k-1); k(k-2)\}$ SDS such that*

$$D_i = D_i^{-1}, \ \ i = 1, 2, 3, 4. \tag{10}$$

Then there exist $SBIBD(4(kt)^2, 2(kt)^2 + kt, (kt)^2 + kt)$ where t^2 is an order of $T-$matrices satisfying (9).

Proof. From Theorem 3 of [11] the theorem holds. □

Example 2 *In $GF(7^2)$ let $g = x + 2$ and set*

$$S_i = \{g^{12j+i}(mod\ x^2 + 1,\ mod\ 7) : \ j = 0, 1, 2, 3\}, \ \ i = 0, 1, \cdots, 11.$$

Take

$$D_i = \{0\} \cup S_{3+3i} \cup S_{5+3i} \cup S_{6+3i} \cup S_{7+3i} \cup S_{9+3i}, \ \ i = 0, 1, 2, 3.$$

It is easy to verify that D_0, D_1, D_2 and D_3 are $4 - \{49; 21; 35\}$ SDS and $D_i^{-1} = D_i$, $i = 0, 1, 2, 3$.

From Theorem 3, Theorem 4, Example 1, Example 2 we have the following corollaries.

Corollary 2 *There exist $SBIBD(4k^2, 2k^2 + k, k^2 + k)$ for $k = 35$, $5q_1$ and $5q_2N$, where q_1, q_2 are prime power, such that $q_1 \equiv 1(mod\ 4)$, $q_2 \equiv 5(mod\ 8)$, $N = 2^a 3^b t^2$, where $a, b = 0$ or 1 and $t \neq 0$ any integer.*

Corollary 3 *There exist $SBIBD(4k^2, 2k^2 + k, k^2 + k)$ for $k = 7q_3$ where $q_3 \equiv 3(mod\ 8)$ is a prime power.*

6

4 Summary

4.1 Numerical results

Example 3 *In $GF(47^2)$, let $g = x + 2$ and set*

$$
\begin{aligned}
C_i &= \{g^{32j+i} : j = 0, \cdots, 68\}, \ i = 0, \cdots, 31, \\
E_i &= \{g^{96j+i} : j = 0, \cdots, 22\}, \ i = 0, \cdots, 95.
\end{aligned}
$$

Write $I_1 = \{0, 1, 3, 6, 8, 13, 15, 18, 28\}$, $I_2 = \{3, 5, 11, 12, 14, 15, 24, 25, 26\}$ and put

$$
A_i = \bigcup_{j \in I_i} C_j, \quad B_i = \bigcup_j (E_j \cup E_{j+48}),
$$

such that

$$
A_i \cap B_i = \phi \ and \ \mid A_i \mid + \mid B_i \mid = 1081, \ \ i = 1, 2.
$$

Take $D_i = A_i \cup B_i$ and $D_{i+2} = g^8 D_i$, $i = 1, 2$. Then D_1, D_2, D_3 and D_4 are $4 - \{2209; 1081; 2115\}$ SDS. Thus we can construct a regular Hadamard matrix of order 8836.

From Example 3, Corollary 1, 2 and 3 one can assert that there are 43 new values of $k < 1000$ for which there exist $SBIBD(4k^2, 2k^2 + k, k^2 + k)$. They are 47, 71, 151, 167, 199, 263, 359, 439, 599, 631, 727, 919 (Corollary 1);
$77(7 \cdot 11)$, $133(7 \cdot 19)$, $301(7 \cdot 43)$, $413(7 \cdot 59)$, $469(7 \cdot 67)$, $581(7 \cdot 83)$, $749(7 \cdot 107)$, $917(7 \cdot 131)$, $973(7 \cdot 139)$ (Corollary 3);
$265(5 \cdot 53)$, $305(5 \cdot 61)$, $365(5 \cdot 73)$, $435(5 \cdot 29 \cdot 3)$, $445(5 \cdot 89)$, $485(5 \cdot 97)$, $505(5 \cdot 101)$, $545(5 \cdot 109)$, $555(5 \cdot 37 \cdot 3)$, $565(5 \cdot 113)$, $585(5 \cdot 13 \cdot 9)$, $685(5 \cdot 137)$, $745(5 \cdot 149)$, $785(5 \cdot 157)$, $795(5 \cdot 53 \cdot 3)$, $905(5 \cdot 181)$, $915(5 \cdot 61 \cdot 3)$, $965(5 \cdot 193)$, $985(5 \cdot 197)$ (Corollary 2);
$459(3^3 \cdot 17)$ (Theorem 3 of [11]); $681(227 \cdot 3)$, $825(11 \cdot 3 \cdot 5^2)$ (Proposition 6 of [11]). The last three values of k were missed from the list of [11].

4.2 Unknown cases

There are at most 126 values of $k < 1000$ for which the existence of $SBIBD(4k^2, 2k^2 + k, k^2 + k)$ is undetermined. These values are:
79, 103, 127, 141, 191, 209, 213, 217, 223, 231, 237, 239, 253, 255, 271, 279, 309, 311, 329, 341, 355, 357, 367, 369, 377, 381, 383, 385, 395, 399, 403, 423, 425, 431, 437, 453, 455, 463, 465, 473, 479, 481, 483, 487, 493, 497, 501, 503, 515, 517, 527, 553, 561, 573, 589, 595, 597, 607, 611, 615, 627, 629, 635, 639, 647, 649, 651, 657, 663, 665, 669, 689, 693, 697, 705, 711, 713, 715, 717, 719, 721, 737, 743, 751, 755, 759, 765, 775, 781, 789, 793, 799, 801, 805, 813, 817, 823, 833, 835, 837, 839, 861, 863, 869, 873, 887, 889, 893, 899, 901, 903, 911, 913, 923, 927, 933, 935, 949, 955, 967, 969, 983, 987, 989, 991, 995.

Appendix

Though D in (5) contains A and B, the expression of $\triangle D$ only depends on A and does not depend on the particular choice of B. So at first we can ignore B and just search A satisfying (8), then we can take B as in (4) satisfying (3).

Now put

$$C_i = \{g^{16j+i} : j = 0, 1, \cdots, (2n+1)(8n+3) - 1\}, \quad i = 0, 1, \cdots, 15.$$

where g is a generator of $GF(q^2)$. It is clear that

$$C_i = \bigcup_{j=0}^{2n} E_{16j+i}, \quad i = 0, \cdots, 15,$$

where E_i is defined in (2). The list of A is as follows:

$$q = 71, \quad A = C_0 \cup C_1 \cup C_2 \cup C_3 \cup C_7, \text{ where } g = x + 8.$$
$$q = 151, \quad A = C_0 \cup C_1 \cup C_2 \cup C_6 \cup C_{13}, \text{ where } g = x + 9.$$
$$q = 167, \quad A = C_0 \cup C_1 \cup C_3 \cup C_4 \cup C_7, \text{ where } g = x + 2.$$
$$q = 199, \quad A = C_0 \cup C_1 \cup C_5 \cup C_6 \cup C_{11}, \text{ where } g = x + 13.$$
$$q = 263, \quad A = C_0 \cup C_1 \cup C_2 \cup C_3 \cup C_{12}, \text{ where } g = x + 2.$$
$$q = 359, \quad A = C_0 \cup C_1 \cup C_3 \cup C_6 \cup C_{13}, \text{ where } g = x + 11.$$
$$q = 439, \quad A = C_0 \cup C_1 \cup C_2 \cup C_3 \cup C_4 \cup C_6 \cup C_7, \text{ where } g = x + 9.$$
$$q = 599, \quad A = C_0 \cup C_1 \cup C_2 \cup C_3 \cup C_7, \text{ where } g = x + 11.$$
$$q = 631, \quad A = C_0 \cup C_1 \cup C_3 \cup C_6 \cup C_{13}, \text{ where } g = x + 5.$$
$$q = 727, \quad A = C_0 \cup C_1 \cup C_2 \cup C_4 \cup C_7 \cup C_{13} \cup C_{14}, \text{ where } g = x + 2.$$
$$q = 919, \quad A = C_0 \cup C_1 \cup C_2 \cup C_3 \cup C_{12}, \text{ where } g = x + 6.$$

Introducing cyclotomic classes C_i, $0 \le i < 16$ simplifies the procedure of searching for A in (5) satisfying (8) considerably. Further research will show this method is suitable to obtain more values of q which would lead to more new results.

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