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On the complete pivoting conjecture for Hadamard matrices of small orders

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Dedicated to John Makepeace Bennett

Abstract

In this paper we study explicitly the pivot structure of Hadamard matrices of small orders $16, 20$ and 32. An algorithm computing the $(n-j)\times(n-j)$ minors of Hadamard matrices is presented and its implementation for $n = 12$ is described. Analytical tables summarizing the pivot patterns attained are given.

Key words and phrases: Gaussian elimination, pivot size, complete pivoting, minors, Hadamard matrices.

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1 Introduction

Let A be an $n \times n$ real matrix, and let $\underline{b}$ be a real n-vector. In his fundamental work on backward error analysis Wilkinson [13] proved that when the linear system $A \cdot \underline{x} = \underline{b}$ is solved in floating point arithmetic by Gaussian elimination (GE) with either partial or complete pivoting the computed solution $\hat{\underline{x}}$ satisfies

$$(A + E) \cdot \hat{\underline{x}} = \underline{b}$$

where the norm of the perturbation matrix E can be bounded from above as follows

$$||E||_\infty \leq g(n, A) \cdot f(n) \cdot u||A||_\infty \tag{1}$$

where u is the unit roundoff, f(n) is a cubic polynomial of n, and g(n,A) is the growth factor defined by

$$g(n, A) = \frac{\max_{i,j,k} |a_{ij}^{(k)}|}{|a_{11}^{(0)}|}$$

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where $a_{ij}^{(k)}$, $k = 1, 2, \ldots, n-1$ denotes the (i, j)th element that occurs at the k-th step of elimination. The elements $a_{ii}^{(n-1)}$ are called pivots.

Thus the growth factor is closely related to the stability of GE since the smaller its value the sharper the bound attained in (1) and consequently the more stable the method becomes.

We say that a matrix A is completely pivoted (CP) if the rows and columns have been permuted so that Gaussian elimination with no pivoting satisfies the requirements for complete pivoting.

Let $g(n, A)$ denote the growth associated with Gaussian elimination on a CP $n \times n$ matrix A and $g(n) = \sup\{ g(n, A) \}$. The problem of determining $g(n)$ for various values of n is called the *growth problem.*

The determination of $g(n)$ remains a challenging problem. Wilkinson in [13],[14] noted that there were no known examples of matrices for which $g(n) > n$. In [1] Cryer conjectured that "$g(n, A) \leq n$, with equality if and only if A is a Hadamard matrix". This was proved to be "untrue" in [6].

An Hadamard matrix H of order n is an $n \times n$ matrix with elements ± 1 and $HH^T = nI$.

Since Wilkinson's initial conjecture seems to be connected with Hadamard matrices the following, more refined, conjecture was posed (see [1],[2], [10]):

Conjecture (The growth conjecture for Hadamard matrices)

Let A be an $n \times n$ CP Hadamard matrix. Reduce A by GE. Then

(i) $g(n, A) = n$.

(ii) The four last pivots are equal to $\frac{n}{2}$ or $\frac{n}{4}, \frac{n}{2}, \frac{n}{2}, n$.

(iii) The fifth last pivot can take the values $\frac{n}{3}$ or $\frac{n}{2}$.

(iv) The sixth last pivot can take the values $\frac{n}{4}$, $\frac{n}{10/3}$, or $\frac{n}{8/3}$.

(v) Every pivot before the last has magnitude at most $\frac{n}{2}$.

(vi) The first six pivots are equal to $1, 2, 2, 4$, 2 or 3, $\frac{10}{3}$ or $\frac{8}{3}$ or 4.

The equality in (i) above has been proved for a certain class of $n \times n$ Hadamard matrices [2]. Cryer [1] has shown (ii) for the pivots $\frac{n}{2}$, $\frac{n}{2}$ and n. Day and Peterson [2] have shown that the values $\frac{n}{2}$ or $\frac{n}{4}$ appear in the fourth last pivot when Gaussian elimination, not necessarily with complete pivoting, is applied to a Hadamard matrix. They posed the conjecture that when Gaussian elimination with complete pivoting is applied to an Hadamard matrix the value of $\frac{n}{2}$ is impossible. In [3] a counter example of an Hadamard matrix of order 16 which has fourth last pivot $\frac{n}{2}$ is given. In [9] ten CP Hadamard matrices of order 16 are given having fourth last pivot 8, as well as a CP Hadamard matrix of order 32 with fourth last pivot 16. In each case this distinguished case with the fourth pivot $\frac{n}{2}$ arose in the equivalence class containing the Sylvester Hadamard matrix. Moreover, in [9] the first infinite family of Hadamard matrices of order n with fourth last pivot $\frac{n}{2}$ is discovered. The values in (vi) are proved in [2] for the first five values, 1, 2, 2, 4, 2 or 3, and experimental evidence in [10] and this paper strongly supports the next values.

Wilkinson's initial conjecture seems to be connected with Hadamard matrices. Interesting results in the size of pivots appears when GE is applied to CP skew-Hadamard and weighing matrices of order n and weight $n-1$. In these matrices, the growth is also large, and experimentally, we have been led to believe it equals $n-1$ and special structure appears for the first few and last few pivots [7].

Notation 1. Write A for a matrix of order n, which is completely pivoted (CP). Write $A(j)$ for the absolute value of the determinant of the $j \times j$ principal submatrix in the upper lefthand corner of the matrix A and $A[j]$ for the absolute value of the determinant of the $(n-j) \times (n-j)$ principal submatrix in the bottom righthand corner of the matrix A. The magnitude of the pivots appearing after the application of GE operations is given by

$$p_j = A(j)/A(j-1). \tag{2}$$

Notation 2. We use $-$ for -1 in matrices in this paper. Also, when we are saying the determinants of a matrix we mean the absolute values of the determinants.

2 Minors of Hadamard matrices

Hadamard matrices of order n have determinant $n^{\frac{n}{2}}$. Sharpe [12] observed that the determinants of all the $(n-1) \times (n-1)$ minors of an Hadamard matrix of order n are zero or $n^{\frac{n}{2}-1}$, and that the determinants of all the $(n-2) \times (n-2)$ minors are zero or $2n^{\frac{n}{2}-2}$, and the determinants of all the $(n-3) \times (n-3)$ minors are zero or $4n^{\frac{n}{2}-3}$. We note that the maximum determinant corresponds to having the submatrix

$$\begin{bmatrix} 1 & 1 \\ 1 & - \end{bmatrix} \text{ or } \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & - \\ 1 & - & 1 \end{bmatrix}$$

in the upper lefthand corner of the Hadamard matrix for $n-2$ and $n-3$ respectively.

Theorem 1 *[8] The determinants of the $(n-3) \times (n-3)$ minors of a CP Hadamard matrix of order n are zero or $4n^{\frac{n}{2}-3}$.*

Theorem 2 *[8] The determinants of the $(n-4) \times (n-4)$ minors of a CP Hadamard matrix of order n are zero, $8n^{\frac{n}{2}-4}$ or $16n^{\frac{n}{2}-4}$.*

A direct consequence of the above theorems and of relation (2) is the following:

Theorem 3 *When Gaussian Elimination is applied on a CP Hadamard matrix H of order n the last four pivots are $n, \frac{n}{2}, \frac{n}{2}, \frac{n}{2}$ or $\frac{n}{4}$.*

We now outline the method to evaluate the $(n-5), (n-6), \ldots, (n-j)$ minors. The $(n-j) \times (n-j)$ minors are denoted by M_{n-j}.

Lemma 1 *(The Distribution Lemma) [8]*

Let H be any Hadamard matrix, of order $n > 2$. Then for every triple of rows of H there are precisely $\frac{n}{4}$ columns which are

(a) $(1,1,1)^T$ *or* $(-,-,-)^T$

(b) $(1,1,-)^T$ *or* $(-,-,1)^T$

(c) $(1,-,1)^T$ *or* $(-,1,-)^T$

(d) $(1,-,-)^T$ *or* $(-,1,1)^T$

If we are considering the $(n-j) \times (n-j)$ minors, then the first j rows, ignoring the upper lefthand $j \times j$ matrix, have 2^{j-1} potentially different first j elements in each column. Let $\underline{x}_{\beta+1}^T$ be the vectors containing the binary representation of each integer $\beta + 2^{j-1}$ for $\beta = 0, \ldots, 2^{j-1}-1$. Replace all zero entries of $\underline{x}_{\beta+1}^T$ by -1 and define the $j \times 1$ vectors

$$\underline{u}_k = \underline{x}_{2^{j-1}-k+1}, \quad k = 1, \ldots, 2^{j-1}$$

Let u_k indicate the number of columns beginning with the vectors $\underline{u}_k$, $\quad k = 1, \ldots, 2^{j-1}$. We note

$$\sum_{i=1}^{2^{j-1}} u_i = n - j. \tag{3}$$

Then it holds (see [8]) that

$$M_{n-j} = n^{n-2^{j-1}-j} det D \tag{4}$$

where D is the following $2^{j-1} \times 2^{j-1}$ matrix.

$$D = \begin{bmatrix} n - ju_1 & u_2m_{12} & u_3m_{13} & \cdots & u_zm_{1z} \\ u_1m_{21} & n - ju_2 & u_3m_{23} & \cdots & u_zm_{2z} \\ \vdots & \vdots & \vdots & & \vdots \\ u_1m_{z1} & u_2m_{z2} & u_3m_{z2} & \cdots & n - ju_z \end{bmatrix}$$

where $(m_{ik}) = (-\underline{u}_i \cdot \underline{u}_k)$, with $\cdot$ the inner product.

We note if $n = 4t$, the orthogonality of the rows of the Hadamard matrix gives

$$t - j \leq u_1 + u_2 + \ldots + u_{2^{j-3}} \leq t, \quad t - j \leq u_{2^{j-3}+1} + \ldots + u_{2^{j-2}} \leq t$$

$$t - j \leq u_{2^{j-2}+1} + \ldots + u_{2^{j-2}+2^{j-3}} \leq t, \quad t - j \leq u_{2^{j-2}+2^{j-3}+1} + \ldots + u_{2^{j-1}} \leq t$$

Each of these equations can be rewritten so the constraints become

$$\begin{gathered} 0 \leq t - u_1 - u_2 - \ldots - u_{2^{j-3}} \leq j, \quad 0 \leq t - u_{2^{j-3}+1} - \ldots - u_{2^{j-2}} \leq j \\ 0 \leq t - u_{2^{j-2}+1} - \ldots - u_{2^{j-2}+2^{j-3}} \leq j, \quad 0 \leq t - u_{2^{j-2}+2^{j-3}+1} - \ldots - u_{2^{j-1}} \leq j \end{gathered} \tag{5}$$

The Algorithm for $(n-j) \times (n-j)$ minors of an $n \times n$ Hadamard matrix H.

Step1: Generate all ± 1 matrices M, of order j.
Step2: Form the general matrix, $N = [M \ \ U_j]$, of size $j \times n$ for the first j rows of an $n \times n$ Hadamard matrix H, where

$$U_j = \begin{matrix} \overbrace{1...1}^{u_1} & \overbrace{1...1}^{u_2} & \dots & \overbrace{1...1}^{u_{2^{j-1}-1}} & \overbrace{1...1}^{u_{2^{j-1}}} \\ 1...1 & 1...1 & \dots & 1...1 & 1...1 \\ 1...1 & 1...1 & \dots & -...- & -...- \\ . & . & \dots & . & . \\ . & . & \dots & . & . \\ 1...1 & 1...1 & \dots & 1...1 & -...- \\ 1...1 & -...- & \dots & -...- & -...- \end{matrix}$$

Step3: For each M consider all $\binom{j}{3}$ subsets of three rows of N and use the Distribution Lemma with $\sum_{i=1}^{2^{j-1}} u_i = n - j$ to form 4 equations in the variables $u_1, \dots, u_{2^{j-1}}$ for each subset.
Step 4: For each M search for all feasible solutions to the different equations generated at Step 3.
Step 5 For each M and for each feasible solution found in Step 4 use the matrix D to find all possible values of the $(n-j) \times (n-j)$ minors.

Implementation of the algorithm

Let $n = 12$ and $j = 5$. We demonstrate analytically the implementation of the algorithm.

Step1-Step2:

For each 5×5 matrix M with elements ± 1 and first row and column all 1 we form the matrix $N = [M, U_5]$. There are 2^{16} different matrices M. The first five rows of matrix N are:

	v_1				v_2				v_3				v_4			
	z_1		z_2		z_3		z_4		z_5		z_6		z_7		z_8	
	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8	u_9	u_{10}	u_{11}	u_{12}	u_{13}	u_{14}	u_{15}	u_{16}
M	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	1	1	1	1	1	1	1	1	−	−	−	−	−	−	−	−
	1	1	1	1	−	−	−	−	1	1	1	1	−	−	−	−
	1	1	−	−	1	1	−	−	1	1	−	−	1	1	−	−
	1	−	1	−	1	−	1	−	1	−	1	−	1	−	1	−

Recalling that $u_1, u_2, \dots, u_{16}$ are the number of columns of the kind indicated. From the order we have

$$\sum_{i=1}^{16} u_i = 7. \tag{6}$$

We first see by the Distribution Lemma that

$$0 \le u_i \le 3. \tag{7}$$

Step 3-Step 4:

(I) For every triple of rows (i,k,l) of N let v_1,v_2,v_3,v_4 be the number of columns starting correspodingly with:

(a) $(1,1,1)^T$ or $(-,-,-)^T$

(b) $(1,1,-)^T$ or $(-,-,1)^T$

(c) $(1,-,1)^T$ or $(-,1,-)^T$

(d) $(1,-,-)^T$ or $(-,1,1)^T$

By the orthogonality of these three rows and the order we form the equations:

$$\begin{aligned} v_1+v_2-v_3-v_4 &= -m_{ik} \\ v_1-v_2+v_3-v_4 &= -m_{il} \\ v_1-v_2-v_3+v_4 &= -m_{kl} \\ v_1+v_2+v_3+v_4 &= 7 \end{aligned}$$

This system can be solved uniquely

$$\begin{aligned} v_1 &= \frac{1}{4}(7-m_{ik}-m_{il}-m_{kl}) \\ v_2 &= \frac{1}{4}(7-m_{ik}+m_{il}+m_{kl}) \\ v_3 &= \frac{1}{4}(7-m_{ik}-m_{il}+m_{kl}) \quad (8) \\ v_4 &= \frac{1}{4}(7+m_{ik}+m_{il}-m_{kl}) \quad (9) \end{aligned}$$

The distribution lemma gives that $0 \le v_i \le 3, \;\; i=1,2,3,4$ and v_i must be integer. Thus for each M and for all possible triplets (there are $\binom{5}{3}=10$ triplets) of rows if there is at least one triplet and at least one m: $v_m \ge 4$ or $v_m < 0$ or v_m not integer, then we reject this matrix M. Finally the number of possible matrices M is reduced to 40860.

We call the vector

$$ip = (m_{12}, m_{13}, m_{14}, m_{15}, m_{23}, \ldots, m_{25}, \ldots, m_{45}),$$

for a given M the *inner product (IP) profile vector*. We call two matrices inner product equivalent if they have the same (IP). Hereinafter we can work with the (IP) instead of M. We found 1975 inequivalent (IP) profile vectors for the 40860 M found previously.

For each (IP) and for the first three lines we calculate v_4.

(II) For the first 4 lines of N we notice that there are 8 different columns $z_1, z_2, \ldots, z_8$. Using the orthogonality and the order and solving the system we have:

$$
\begin{aligned}
z_1 &= 1/4 \cdot (7 - m_{23} - m_{24} - m_{34}) - z_8 \\
z_2 &= 1/4 \cdot (m_{34} - m_{13} + m_{24} - m_{12}) + z_8 \\
z_3 &= 1/4 \cdot (+m_{34} - m_{14} + m_{23} - m_{12}) + z_8 \\
z_4 &= -1/4 \cdot (7 - m_{34} + m_{13} + m_{14}) - z_8 \qquad (10) \\
z_5 &= 1/4 \cdot (m_{24} - m_{14} + m_{23} - m_{13}) + z_8 \\
z_6 &= 1/4 \cdot (7 - m_{24} + m_{12} + m_{14}) - z_8 \\
z_7 &= 1/4 \cdot (7 - m_{23} + m_{12} + m_{13}) - z_8 \\
z_8 &= z_8 \qquad (11)
\end{aligned}
$$

For a given (IP) and its v_4 as found from (8) the possible values of the parameter z_8 are from 0 to v_4 because $z_7 + z_8 = v_4$ and thus $z_8 = v_4 - z_7 \leq v_4$ (z_7 is always a non negative number). For all the possible values of z_8 we compute the z_i checking also if $0 \leq z_i \leq 3$ and z_i integer. From all the possible solutions we need for the next step the vector containing the (IP) and the coordinates z_4, z_6, z_7, z_8 of the solutions denoted by $[IP, z_4, z_6, z_7, z_8]$.

(III) By the orthogonality of the five rows of N, the order and by solving the system of the equations produced we get:

$$
\begin{aligned}
u_1 &= 1/2 \cdot 7 - 1/4 \cdot (m_{45} + m_{25} + m_{35} - m_{15} - m_{12} + m_{23} - m_{13} \\
&\quad + m_{24} + m_{34} - m_{14}) - u_8 - u_{12} - u_{14} - u_{15} - 3 \cdot u_{16} \\
u_2 &= -1/4 \cdot (7 - m_{45} - m_{25} - m_{35} + m_{15} + m_{12} + m_{13} + m_{14}) \\
&\quad + u_8 + u_{12} + u_{14} + 2 \cdot u_{16} \\
u_3 &= -1/4 \cdot (7 - m_{45} + m_{15} + m_{12} + m_{13} - m_{24} - m_{34} + m_{14}) \\
&\quad + u_8 + u_{12} + u_{15} + 2 \cdot u_{16} \\
u_4 &= 1/4 \cdot (7 - m_{45} + m_{15} + m_{14}) - u_8 - u_{12} - u_{16} \\
u_5 &= -1/4 \cdot (7 - m_{35} + m_{15} + m_{12} - m_{23} + m_{13} - m_{34} + m_{14}) \\
&\quad + u_8 + u_{14} + u_{15} + 2 \cdot u_{16} \\
u_6 &= 1/4 \cdot (7 - m_{35} + m_{15} + m_{13}) - u_8 - u_{14} - u_{16} \qquad (12) \\
u_7 &= 1/4 \cdot (7 - m_{34} + m_{13} + m_{14}) - u_8 - u_{15} - u_{16} \\
u_8 &= u_8 \\
u_9 &= -1/4 \cdot (7 - m_{25} + m_{15} + m_{12} + m_{13} - m_{23} - m_{24} + m_{14}) \\
&\quad + u_{12} + u_{14} + u_{15} + 2 \cdot u_{16} \\
u_{10} &= 1/4 \cdot (7 + m_{15} - m_{25} + m_{12}) - u_{12} - u_{14} - u_{15} \\
u_{11} &= 1/4 \cdot (7 + m_{14} - m_{24} + m_{12}) - u_{12} - u_{15} - u_{16} \\
u_{12} &= u_{12} \\
u_{13} &= 1/4 \cdot (7 + m_{13} - m_{23} + m_{12}) - u_{14} - u_{15} - u_{16}
\end{aligned}
$$

$$
\begin{aligned}
u_{14} &= u_{14} \\
u_{15} &= u_{15} \\
u_{16} &= u_{16}
\end{aligned} \tag{13}
$$

For each $[IP, z_4, z_6, z_7, z_8]$ found from (10) we specify the possible values of the parameters $u_8, u_{12}, u_{14}, u_{16}$. These are $0 \leq u_8 \leq z_4$, $0 \leq u_{12} \leq z_6$, $0 \leq u_{14} \leq z_7$ and $0 \leq u_{16} \leq z_8$. u_{15} is not a "real" parameter because $u_{15} + u_{16} = z_8$ and thus $u_{15} = z_8 - u_{16}$.

Step 5: For each $[u_1, u_2, \ldots, u_{16}]$ and with the use of matrix D we find the possible values of the $(12-5) \times (12-5)$ minors. Thus the 7×7 minors can take the values $\{0, 192, 384, 576\}$.

Expanding the above algorithm for $j = 6$ we find as possible values for the 6×6 minors the following: $\{0, 32, 64, 96, 128, 160\}$.

Testing the above algorithm till the $(n-6)$ case we can summarize our results as follows:

order	Values of Minors
n	$n^{\frac{n}{2}}$
n-1	0, $n^{\frac{n}{2}-1}$
n-2	0, $2n^{\frac{n}{2}-2}$
n-3	0, $4n^{\frac{n}{2}-3}$
n-4	0, $8n^{\frac{n}{2}-4}$, $16n^{\frac{n}{2}-4}$
n-5	0, $16n^{\frac{n}{2}-5}$, $32n^{\frac{n}{2}-5}$, $48n^{\frac{n}{2}-5}$
n-6	0, $32n^{\frac{n}{2}-6}$, $64n^{\frac{n}{2}-6}$, $96n^{\frac{n}{2}-6}$, $128n^{\frac{n}{2}-6}$, $160n^{\frac{n}{2}-6}$

Table 1

Remark 1 We see that as the value of $n-j$ decreases the range of values of the corresponding minors M_{n-j} increases. We notice that the number of values at each step is the double number of values at the previous step.

3 Pivot structure of the 16×16 Hadamard matrices

Hall [4] (see also [11]) proved that there are 5 equivalence classes for Hadamard matrices of order 16 and gave examples of each.

In our subsequent experiments we took 40000 cases from each of the five equivalence classes and applied GECP to each. We found 9 different pivot patterns for class I. For class II we found 18 different pivot patterns. for class III we found 21 different pivot patterns. Since classes IV and V are one another's transpose they are identical for the purpose of GECP [2]. We found classes IV and V gave 12 different pivot patterns but each of these patterns had already occurred in classes I,II or III. The following tables summarize the different pivot structures attained for each class and highlight the difference between the classes. We note that the fourth last pivot 8 only occurs in class I which is the the equivalence class of Sylvester Hadamard matrices.

	growth	Class I- Pivot Pattern
1	16	$(1,2,2,4,2,4,4,\ 8,\ 2,4,\ 4,\ 8,4,8,8,16)$
2	16	$(1,2,2,4,2,4,4,\frac{16}{8/3},\frac{8}{3},4,\frac{16}{8/3},\frac{16}{3},4,8,8,16)$
3	16	$(1,2,2,4,2,4,4,\frac{16}{8/3},\frac{8}{3},4,\ 4,\ 8,4,8,8,16)$
4	16	$(1,2,2,4,3,\frac{8}{3},4,\frac{16}{8/3},\frac{8}{3},4,\ 4,\ 8,4,8,8,16)$
5	16	$(1,2,2,4,3,\frac{8}{3},4,\frac{16}{8/3},\frac{8}{3},4,\frac{16}{8/3},\frac{16}{3},4,8,8,16)$
6	16	$(1,2,2,4,3,\frac{8}{3},2,\ 4,\ 4,8,\frac{16}{8/3},\frac{16}{3},4,8,8,16)$
7	16	$(1,2,2,4,3,\frac{8}{3},2,\ 4,\ 4,8,\ 4,\ 8,4,8,8,16)$
8	16	$(1,2,2,4,3,\frac{8}{3},2,\ 4,\ 4,4,\ 8,\ 8,4,8,8,16)$
9	16	$(1,2,2,4,3,\frac{8}{3},2,\ 4,\ 4,4,\ 4,\ 8,8,8,8,16)$

Table 2

	growth	Class II- Pivot Pattern
1	16	$(1,2,2,4,2,4,4,\frac{16}{16/5},\frac{16}{5},\frac{16}{16/5},\frac{16}{10/3},\frac{16}{3},4,8,8,16)$
2	16	$(1,2,2,4,2,4,4,\frac{16}{16/5},\frac{16}{5},\ 4\ ,\ \frac{16}{8/3},\frac{16}{3},4,8,8,16)$
3	16	$(1,2,2,4,2,4,4,\ 4\ ,\ 4\ ,\frac{16}{16/5},\frac{16}{10/3},\frac{16}{3},4,8,8,16)$
4	16	$(1,2,2,4,2,4,4,\ 4,\ 4,\ 4,\ \frac{16}{8/3},\ \frac{16}{3},4,8,8,16)$
5	16	$(1,2,2,4,2,4,4,\ 4,\ 4,\ 4,\ 4,\ 8,\ 4,8,8,16)$
6	16	$(1,2,2,4,2,4,4,\frac{16}{16/5},\frac{16}{5},\ 4,\ 4,\ 8,4,8,8,16)$
7	16	$(1,2,2,4,3,\frac{10}{3},\ \frac{8}{10/3},\ 4,\ \frac{16}{3},\frac{16}{16/5},\frac{16}{10/3},\ \frac{16}{3},4,8,8,16)$
8	16	$(1,2,2,4,3,\frac{10}{3},\ \frac{16}{5},4,4,4,\frac{16}{8/3},\frac{16}{3},4,8,8,16)$
9	16	$(1,2,2,4,3,\frac{10}{3},\frac{8}{10/3},4,\frac{16}{3},4,4,8,4,8,8,16)$
10	16	$(1,2,2,4,3,\frac{10}{3},\frac{16}{5},4,4,\frac{16}{16/5},\frac{16}{10/3},\frac{16}{3},4,8,8,16)$
11	16	$(1,2,2,4,3,\frac{10}{3},\frac{16}{5},4,4,4,4,8,4,8,8,16)$
12	16	$(1,2,2,4,3,\frac{10}{3},\frac{16}{5},\frac{16}{16/5},\frac{16}{5},4,\frac{16}{8/3},\frac{16}{3},4,8,8,16)$
13	16	$(1,2,2,4,3,\frac{10}{3},\frac{16}{5},\frac{16}{16/5},\frac{16}{5},\frac{16}{16/5},\frac{16}{10/3},\frac{16}{3},4,8,8,16)$
14	16	$(1,2,2,4,3,\frac{10}{3},\ \frac{16}{5},\frac{16}{16/5},\frac{16}{5},\ 4,\ 4,\ 8,4,8,8,16)$
15	16	$(1,2,2,4,3,\frac{10}{3},\frac{8}{10/3},4,\frac{16}{3},4,\frac{16}{8/3},\frac{16}{3},4,8,8,16)$

Table 3

	growth	**Class III- Pivot Pattern**
1	16	$(1,2,2,4,2,4,\ \ 4,\ 4,\frac{9}{2},\frac{16}{18/5},\frac{16}{10/3},\frac{16}{3},4,8,8,16)$
2	16	$(1,2,2,4,2,4,\ \ 4,\ \frac{9}{2},4,\frac{16}{18/5},\frac{16}{10/3},\frac{16}{3},4,8,8,16)$
3	16	$(1,2,2,4,3,\frac{10}{3},\frac{18}{5},4,4,\frac{16}{18/5},\frac{16}{10/3},\frac{16}{3},4,8,8,16)$
4	16	$(1,2,2,4,3,\frac{8}{3},4,4,4,\frac{16}{16/5},\frac{16}{10/3},\frac{16}{3},4,8,8,16)$
5	16	$(1,2,2,4,3,\frac{8}{3},4,4,4,4,\frac{16}{8/3},\frac{16}{3},4,8,8,16)$
6	16	$(1,2,2,4,3,\frac{8}{3},4,4,4,4,4,8,4,8,8,16)$
7	16	$(1,2,2,4,3,\frac{8}{3},4,\frac{16}{16/5},\frac{16}{5},4,\frac{16}{8/3},\frac{16}{3},4,8,8,16)$
8	16	$(1,2,2,4,3,\frac{8}{3},4,\frac{16}{16/5},\frac{16}{5},\frac{16}{16/5},\frac{16}{10/3},\frac{16}{3},4,8,8,16)$
9	16	$(1,2,2,4,3,\frac{8}{3},4,\frac{16}{16/5},\frac{16}{5},4,4,8,4,8,8,16)$
10	16	$(1,2,2,4,3,\frac{8}{3},4,4,\frac{9}{2},\frac{16}{18/5},\frac{16}{10/3},\frac{16}{3},4,8,8,16)$

Table 4

Summarizing the above Tables we have the following Table for the values appearing in each class.

Pivot	1st Class	2nd Class	3rd Class	4th Class
1	1	1	1	1
2	2	2	2	2
3	2	2	2	2
4	4	4	4	4
5	2,3	2,3	2,3	2,3
6	$4,\frac{8}{3}$	$4,\frac{10}{3}$	$4,\frac{8}{3},\frac{10}{3}$	$4,\frac{10}{3}$
7	2,4	$4,\frac{8}{10/3},\frac{16}{5}$	$4,\frac{18}{5}$	$4,\frac{18}{5}$
8	4,6,8	4,5,6,8	$4,\frac{9}{2}$,5,6,8	4,5,6,8
9	$2,4,\ \frac{8}{3}$	$2,4,\frac{8}{3},\frac{16}{3},\frac{16}{5}$	$2,4,\frac{9}{2},\frac{8}{3}\ \frac{16}{5}$	$2,4,\frac{9}{2},\frac{8}{3}\ \frac{16}{5}$
10	4,8	4,5	$4,5,\frac{16}{18/5}$	$4,5,\frac{16}{18/5}$
11	4,6,8	$4,6,\frac{16}{10/3}$	$4,6,\frac{16}{10/3}$	$4,6,\frac{16}{10/3}$
12	$8,\frac{16}{3}$	$8,\frac{16}{3}$	$8,\frac{16}{3}$	$8,\frac{16}{3}$
13	4,8	4	4	4
14	8	8	8	8
15	8	8	8	8
16	16	16	16	16

Table 5

4 Pivot structure of the 20×20 Hadamard matrices

Hall [5] (see also [11]) found 3 equivalence classes of Hadamard matrices of order 20. After testing 3000000 equivalent matrices we found totally 1015 different pivot patterns. The following Table presents the 12 different pivot patterns attained concerning the first seven and and the last seven pivots. The intermediate pivots are varying and according to their possible values the 1015 pivot patterns were computed. The frequency of each pivot pattern concerns all the possible values of the intermediate pivots that are appearing i.e. for the first pivot pattern of the next Table were found 125 different pivot patterns starting with the specified seven first pivots and ending with the specified seven last pivots.

growth	Pivot Pattern	Frequency
20	$(1,2,2,4,3,\frac{10}{3},\frac{18}{5},\ldots,\frac{20}{18/5},\frac{20}{10/3},\frac{20}{3},\frac{20}{4},\frac{20}{2},\frac{20}{2},20)$	125
20	$(1,2,2,4,3,\frac{10}{3},\frac{18}{5},\ldots,\frac{20}{4},\frac{20}{8/3},\frac{20}{3},\frac{20}{4},\frac{20}{2},\frac{20}{2},20)$	195
20	$(1,2,2,4,3,\frac{10}{3},\frac{18}{5},\ldots,\frac{20}{16/5},\frac{20}{10/3},\frac{20}{3},\frac{20}{4},\frac{20}{2},\frac{20}{2},20)$	123
20	$(1,2,2,4,3,\frac{10}{3},\frac{18}{5},\ldots,\frac{20}{4},\frac{20}{4},\frac{20}{2},\frac{20}{4},\frac{20}{2},\frac{20}{2},20)$	214
20	$(1,2,2,4,3,\frac{10}{3},\frac{16}{5},\ldots,\frac{20}{4},\frac{20}{4},\frac{20}{2},\frac{20}{4},\frac{20}{2},\frac{20}{2},20)$	75
20	$(1,2,2,4,3,\frac{10}{3},\frac{16}{5},\ldots,\frac{20}{16/5},\frac{20}{10/3},\frac{20}{3},\frac{20}{4},\frac{20}{2},\frac{20}{2},20)$	41
20	$(1,2,2,4,3,\frac{10}{3},\frac{16}{5},\ldots,\frac{20}{18/5},\frac{20}{10/3},\frac{20}{3},\frac{20}{4},\frac{20}{2},\frac{20}{2},20)$	17
20	$(1,2,2,4,3,\frac{10}{3},\frac{16}{5},\ldots,\frac{20}{4},\frac{20}{8/3},\frac{20}{3},\frac{20}{4},\frac{20}{2},\frac{20}{2},20)$	64
20	$(1,2,2,4,2,4,4,\ldots,\frac{20}{16/5},\frac{20}{10/3},\frac{20}{3},\frac{20}{4},\frac{20}{2},\frac{20}{2},20)$	31
20	$(1,2,2,4,2,4,4,\ldots,\frac{20}{4},\frac{20}{8/3},\frac{20}{3},\frac{20}{4},\frac{20}{2},\frac{20}{2},20)$	55
20	$(1,2,2,4,2,4,4,\ldots,\frac{20}{4},\frac{20}{4},\frac{20}{2},\frac{20}{4},\frac{20}{2},\frac{20}{2},20)$	56
20	$(1,2,2,4,2,4,4,\ldots,\frac{20}{18/5},\frac{20}{10/3},\frac{20}{3},\frac{20}{4},\frac{20}{2},\frac{20}{2},20)$	19

Table 6

In none of the cases we examined was the fourth last pivot $\frac{20}{2}$. The following Table summarizes the first seven and the last seven values that appear as pivots.

Pivot	1	2	3	4	5	6	7
Values	1	2	2	4	2,3	$4, \frac{10}{3}$	$4, \frac{18}{5}, \frac{16}{5}$

Table 7

Pivot	14	15	16	17	18	19	20
Values	$\frac{20}{4},\frac{20}{18/5},\frac{20}{16/5}$	$\frac{20}{4},\frac{20}{10/3},\frac{20}{8/3}$	$\frac{20}{3},\frac{20}{2}$	$\frac{20}{4}$	$\frac{20}{2}$	$\frac{20}{2}$	20

Table 8

5 Pivot structure of the 32×32 Hadamard matrices

We tested 1700000 Hadamard matrices of order 32. We found totally 414380 different pivot patterns. The following Table presents the 22 different pivot patterns attained concerning the seven first and and the seven last pivots. The intermediate pivots are varying and according to their possible values the 414380 pivot patterns were computed. The frequency of each pivot pattern concerns all the possible values of the intermediate pivots that are appearing i.e. for the first pivot pattern of the next Table were found 171244 different pivot patterns starting and ending with the specified seven first and seven last pivots. We did find one case, which was equivalent to the Sylvester Hadamard matrix, which had fourth last pivot $\frac{32}{2}$.

growth	Pivot Pattern	Frequency
32	$(1,2,2,4,3,\frac{10}{3},\frac{18}{5},\ldots,\frac{32}{4},\frac{32}{4},\frac{32}{2},\frac{32}{4},\frac{32}{2},\frac{32}{2},32)$	171244
32	$(1,2,2,4,3,\frac{10}{3},\frac{18}{5},\ldots,\frac{32}{18/5},\frac{32}{10/3},\frac{32}{3},\frac{32}{4},\frac{32}{2},\frac{32}{2},32)$	50646
32	$(1,2,2,4,3,\frac{10}{3},\frac{16}{5},\ldots,\frac{32}{4},\frac{32}{4},\frac{32}{2},\frac{32}{4},\frac{32}{2},\frac{32}{2},32)$	5612
32	$(1,2,2,4,3,\frac{10}{3},\frac{18}{5},\ldots,\frac{32}{4},\frac{32}{8/3},\frac{32}{3},\frac{32}{4},\frac{32}{4},\frac{32}{2},32)$	73245
32	$(1,2,2,4,2,4,4,\ldots,\frac{32}{4},\frac{32}{4},\frac{32}{2},\frac{32}{4},\frac{32}{2},\frac{32}{2},32)$	45681
32	$(1,2,2,4,3,\frac{10}{3},\frac{18}{5},\ldots,10,\frac{32}{10/3},\frac{32}{3},\frac{32}{4},\frac{32}{2},\frac{32}{2},32)$	33798
32	$(1,2,2,4,3,\frac{10}{3},\frac{16}{5},\ldots,\frac{32}{16/5},\frac{32}{10/3},\frac{32}{3},\frac{32}{4},\frac{32}{2},\frac{32}{2},32)$	960
32	$(1,2,2,4,3,\frac{10}{3},\frac{16}{5},\ldots,\frac{32}{4},\frac{32}{8/3},\frac{32}{3},\frac{32}{4},\frac{32}{2},\frac{32}{2},32)$	2042
32	$(1,2,2,4,2,4,4,\ldots,\frac{32}{18/5},\frac{32}{10/3},\frac{32}{3},\frac{32}{4},\frac{32}{2},\frac{32}{2},32)$	13247
32	$(1,2,2,4,2,4,4,\ldots,\frac{32}{4},\frac{32}{8/3},\frac{32}{3},\frac{32}{4},\frac{32}{2},\frac{32}{2},32)$	12641
32	$(1,2,2,4,2,4,4,\ldots,\frac{32}{16/5},\frac{32}{10/3},\frac{32}{3},\frac{32}{4},\frac{32}{2},\frac{32}{2},32)$	4407
32	$(1,2,2,4,2,4,4,\ldots,\frac{32}{2},\frac{32}{8/3},\frac{32}{3},\frac{32}{4},\frac{32}{2},\frac{32}{2},32)$,	102
32	$(1,2,2,4,3,\frac{10}{3},\frac{16}{5},\ldots,\frac{32}{18/5},\frac{32}{10/3},\frac{32}{3},\frac{32}{4},\frac{32}{2},\frac{32}{2},32)$	888
32	$(1,2,2,4,2,4,4,\ldots,\frac{32}{4},\frac{32}{2},\frac{32}{2},\frac{32}{4},\frac{32}{2},\frac{32}{2},32)$	21
32	$(1,2,2,4,2,4,4,\ldots,\frac{32}{2},\frac{32}{4},\frac{32}{2},\frac{32}{4},\frac{32}{2},\frac{32}{2},32)$	139
32	$(1,2,2,4,3,\frac{10}{3},\frac{16}{5},\ldots,\frac{32}{2},\frac{32}{4},\frac{32}{2},\frac{32}{4},\frac{32}{2},\frac{32}{2},32)$	34
32	$(1,2,2,4,3,\frac{10}{3},\frac{16}{5},\ldots,\frac{32}{2},\frac{32}{8/3},\frac{32}{3},\frac{32}{4},\frac{32}{2},\frac{32}{2},32)$	32,
32	$(1,2,2,4,3,\frac{10}{3},\frac{18}{5},\ldots,\frac{32}{2},\frac{32}{4},\frac{32}{2},\frac{32}{4},\frac{32}{2},\frac{32}{2},32)$	37
32	$(1,2,2,4,3,\frac{10}{3},\frac{18}{5},\ldots,\frac{32}{2},\frac{32}{8/3},\frac{32}{3},\frac{32}{4},\frac{32}{2},\frac{32}{2},32)$	37
32	$(1,2,2,4,3,\frac{10}{3},\frac{18}{5},\ldots,\frac{32}{4},\frac{32}{2},\frac{32}{2},\frac{32}{4},\frac{32}{2},\frac{32}{2},32)$	1
32	$(1,2,2,4,3,\frac{10}{3},\frac{16}{5},\ldots,\frac{32}{4},\frac{32}{2},\frac{32}{2},\frac{32}{4},\frac{32}{2},\frac{32}{2},32)$	13
32	$(1,2,2,4,3,\frac{10}{3},\frac{16}{5},\ldots,\frac{32}{3},\frac{32}{8/3},\frac{32}{3},\frac{32}{4},\frac{32}{2},\frac{32}{2},32)$	3

Table 9

The following Table summarizes the first seven and the last seven values which appear as the pivots.

Pivot	1	2	3	4	5	6	7
Values	1	2	2	4	2,3	$4, \frac{10}{3}$	$4, \frac{18}{5}, \frac{16}{5}$

Table 10

Pivot	26	27	28	29	30	31	32
Values	$\frac{32}{2}, \frac{32}{3}, \frac{32}{4}, \frac{32}{18/5}, \frac{32}{16/5}$	$\frac{32}{4}, \frac{32}{10/3}, \frac{32}{8/3}$	$\frac{32}{3}, \frac{32}{2}$	$\frac{32}{4}$	$\frac{32}{2}$	$\frac{32}{2}$	32

Table 11

6 Conclusions

The above results show that the magnitudes of a few of the first and last few pivot elements are determined. However, the sizes of the intermediate pivots vary and we believe only the last half of the pivots are directly dependent on n.

We conjecture that the fifth from last pivot can only have magnitude $\frac{n}{2}$ or $\frac{n}{3}$ and that the sixth last pivot can take the values $\frac{n}{4}$, $\frac{n}{10/3}$, or $\frac{n}{8/3}$. We also believe that the sixth pivot can have magnitude $\frac{10}{3}$ or $\frac{8}{3}$ or 4.

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