Simple implementations of mutually orthogonal complementary sets of sequences

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SIMPLE IMPLEMENTATIONS OF MUTUALLY ORTHOGONAL COMPLEMENTARY SETS OF SEQUENCES

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ABSTRACT

This paper presents simple software and hardware implementations for a class of mutually orthogonal complementary sets of sequences based on its closed-form construction formula. Following a brief review of the Golay-paired Hadamard matrix concept, the flow graph for constructing mutually orthogonal Golay-paired Hadamard matrices, which represent the scalable complete complementary sets of sequences, is proposed. Then, their superb scalability and completeness are summarized. Finally, the C and Matlab functions and a logic schematic diagram are given to easily generate these complementary sequences.

1. INTRODUCTION

Complementary sequences and mutually orthogonal complementary sets of sequences [1-5] have been studied and found wide applications in digital signal processing and digital communications for decades [6-9,12]. In reviewing and implementing the methods available in the literature for constructing complementary sequences and mutually orthogonal complementary sets of sequences, it is noticed that there is a lack of efficient software or hardware implementation since almost all construction procedures use recursions based on pre-selected initial complementary sequences. To facilitate research on complementary sequences and make use of their unique properties in real-time applications, finding simple ways to generate complementary sequences is of great significance.

The scalable complete complementary sets of sequences [10,11] have been proposed to offer the superb scalability for a class of mutually orthogonal complementary sets of sequences and also provide a means for directly constructing these complementary sequences using a closed-form equation in addition to recursion. A revisit of the construction process reveals that this closed-form equation can be further exploited to derive more efficient complementary sequence implementation.

In this paper, two simple software functions and a hardware schematic diagram for easily generating any sequence in the scalable complete complementary sets of sequences are presented. Hence, it is proved that the scalable complete complementary sets of sequences not only provide ideal complementary property, but also have great implementation advantage. In the following sections, the fundamental definitions used to derive the scalable complete complementary sets of sequences are briefly reviewed. A flow graph, called construction pyramid, is then proposed to illustrate the construction process, followed by a summary of the scalability and completeness of the resulting complementary sequence sets. Finally, the simple software functions and a hardware circuit for generating these complementary sequences are given, and the conclusions are drawn.

2. SCALABLE COMPLETE COMPLEMENTARY SETS OF SEQUENCES

2.1 Golay-paired Hadamard Matrix

A pair of binary sequences with the same finite length is complementary if the sum of their respective auto-correlation functions is zero for all non-zero offsets [1]. We refer to such a pair of complementary sequences as a Golay sequence pair. This concept is extended to matrix form in [10], where a pair of matrices of the same dimension is defined as a *Golay matrix pair* if the sequence given by any row of one matrix in the matrix pair and the sequence given by the corresponding row of the other matrix in the matrix pair constitute a Golay sequence pair. Furthermore, denoting the Golay matrix pair as A and B, a *Golay-paired matrix* is defined as the matrix composed of the Golay matrix pair with one matrix of the pair being the upper half and the other the lower half, and denoted as $\begin{bmatrix} A \\ B \end{bmatrix}$. Similarly, $\begin{bmatrix} B \\ A \end{bmatrix}$ is also a Golay-paired matrix, called the commuted matrix of $\begin{bmatrix} A \\ B \end{bmatrix}$.

Golay sequence pair contains only two complementary sequences. A generalized complementary set of more than two sequences is defined in [2], where the sum of all auto-correlation functions of the sequences in the set is zero for all non-zero offsets. Apparently, Golay-paired matrix represents a special set of complementary sequences, since it is composed of a number of Golay sequence pairs.

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A Golay-paired matrix can be constructed in many different ways. However, a class of square Golay-paired matrix can be constructed recursively as follows

$$H_N = \begin{bmatrix} H_{N/2} & \tilde{H}_{N/2} \\ H_{N/2} & -\tilde{H}_{N/2} \end{bmatrix} \qquad (1)$$

where H_N denotes the Golay-paired matrix of dimension $N \times N$ ($N = 2^n$, $n > 0$) and $\tilde{H}_N$ the commuted matrix of H_N. It is easily verified that the matrix H_N also demonstrates the property

$$H_N H_N^T = H_N^T H_N = N I_N \qquad (2)$$

where H_N^T denotes the transpose of H_N, and I_N is the identity matrix of order N. Equation (2) implies that all sequences given by rows or columns of H_N are orthogonal with each other. We refer to this orthogonal square Golay-paired matrix as a *Golay-paired Hadamard matrix.*

From (1) and the commuted matrix definition, $\tilde{H}_N$ can be expressed as

$$\tilde{H}_N = \begin{bmatrix} H_{N/2} & -\tilde{H}_{N/2} \\ H_{N/2} & \tilde{H}_{N/2} \end{bmatrix}. \qquad (3)$$

It is also a Golay-paired Hadamard matrix.

As an example, the Golay-paired Hadamard matrix of order 8 is constructed using recursion (1) as

$$H_8 = \begin{bmatrix} +1 & +1 & +1 & -1 & +1 & +1 & -1 & +1 \\ +1 & -1 & +1 & +1 & +1 & -1 & -1 & -1 \\ +1 & +1 & -1 & +1 & +1 & +1 & +1 & -1 \\ +1 & -1 & -1 & -1 & +1 & -1 & +1 & +1 \\ +1 & +1 & +1 & -1 & -1 & -1 & +1 & -1 \\ +1 & -1 & +1 & +1 & -1 & +1 & +1 & +1 \\ +1 & +1 & -1 & +1 & -1 & -1 & -1 & +1 \\ +1 & -1 & -1 & -1 & -1 & +1 & -1 & -1 \end{bmatrix}. \qquad (4)$$

It can be easily verified that any sequence given by a row in the upper half of H_8 and the sequence given by the corresponding row in the lower half constitute a Golay sequence pair and thus all the sequences in H_8 constitute a complementary sequence set. These sequences are also orthogonal with each other.

2.2 Mutually Orthogonal Golay-paired Hadamard Matrices

Alternately using recursion (1) and (3) to construct multiple Golay-paired Hadamard matrices, a set of mutually orthogonal Golay-paired Hadamard matrices can be formed. The term "mutually orthogonal" means that every two matrices in the set are mates of each other, and a matrix is said to be a mate of another matrix of the same order if the sum of all cross-correlation functions between the corresponding sequences given by the two matrices is always zero for any offset according to the definition in [2]. It can be easily shown that a Golay-paired Hadamard matrix and its commuted version are mates of each other.

Starting from $H_1 = \tilde{H}_1 = [+1]$, and using the solid-lined arrow to represent the recursion (1) and the dashed-lined arrow to represent the recursion (3), a set of N mutually orthogonal Golay-paired Hadamard matrices of order N, $H_N^{(0)}$, $H_N^{(1)}$, ..., and $H_N^{(N-1)}$, can be constructed after n step iterations following a flow graph, called *construction pyramid*, as shown in Figure 1. Compared with the original description in [10], this construction pyramid not only provides a vivid illustration of the construction process, but also can be used to prove the orthogonality and derive other useful properties for the constructed matrices.

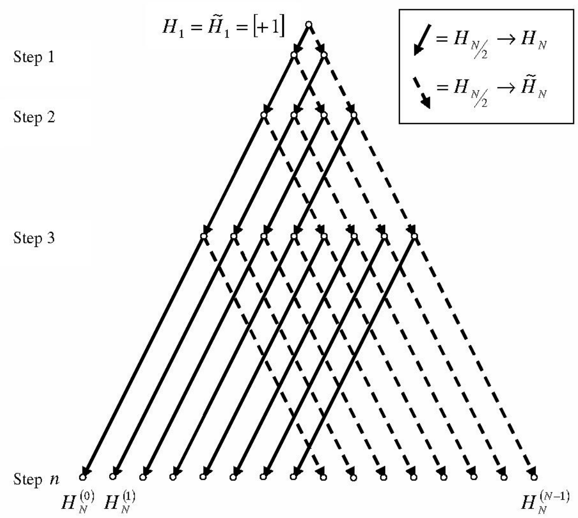

Figure 1. Construction pyramid of mutually orthogonal Golay-paired Hadamard matrices.

Figure 2 (at the end of this paper) shows the 8 mutually orthogonal Golay-paired Hadamard matrices of order 8 obtained using the construction pyramid after 3 iterations. The orthogonality can be also easily verified.

2.3 Scalability and Completeness

The above mutually orthogonal Golay-paired Hadamard matrices represent a collection of complete complementary sequence sets as defined in [3], where "complete" means that the number of the complementary sequence sets in the collection is equal to the number of the sequences in each set. Moreover, these complete complementary sequence sets can be successively decomposed into constitutional subsets, which are also complete complementary sequence sets but of lower order, while the sequence length in all subsets remains the same. A lower order complete complementary sequence subset can be represented by a non-square Golay-paired matrix. We refer to these complete complementary sequence sets with such scalability as the scalable complete complementary sequence sets [10].

The scalability and completeness can be illustrated using Figure 2 for the complete complementary sequence sets of order 8, where the first subset of the complete complementary sets of order 4 is enclosed in the two solid-lined boxes and the first sub-subset of the complete complementary sets of order 2 within the first subset is enclosed in the two dashed-lined boxes. In general, the scalable complete complementary sequence sets of order N consists of $\left(\frac{N}{K}\right)^2$ complete complementary sequence subsets of order K (K is the number of sequences in a subset) for $K = N$, $\frac{N}{2}$, …, 4, 2. However, only $\frac{N}{K}$ of them are composed of different Golay-paired matrices of dimension $K \times N$.

3. SIMPLE SOFTWARE AND HARDWARE IMPLEMENTATIONS

3.1 Closed-form Expression

The mutually orthogonal Golay-paired Hadamard matrices can be also constructed directly using closed-form formula instead of recursion. Denoting an element at the ith row and the jth column of $H_N^{(k)}$ as $h_N^{(k)}(i,j)$, and representing i, j and k in the radix-2 forms respectively as

$$i = \sum_{r=0}^{n-1} 2^r i_r \tag{5}$$

$$j = \sum_{r=0}^{n-1} 2^r j_r \tag{6}$$

$$k = \sum_{r=0}^{n-1} 2^r k_r \tag{7}$$

where i_r, j_r and k_r, $r = 0,1,\ldots,n-1$, are binary bits, taking on value 0 or 1, we have

$$h_N^{(k)}(i,j) = (-1)^{\sum_{r=0}^{n-2}(j_{r+1}\oplus i_r \oplus k_r)j_r + (i_{n-1}\oplus k_{n-1})j_{n-1}} \tag{8}$$

where $\oplus$ denotes the modulo-2 sum.

The $\frac{N}{K}$ different complete complementary sequence subsets of order K can be also formulated accordingly. Each of them is represented by K Golay-paired matrices of dimension $K \times N$. For the lth subset, $l = 0,1,\ldots,\frac{N}{K}-1$, these K Golay-paired matrices are

$$\begin{bmatrix} h_N^{(k+lK)}(0,j) \\ h_N^{(k+lK)}(1,j) \\ \vdots \\ h_N^{(k+lK)}\left(\frac{K}{2}-1,j\right) \\ h_N^{(k+lK)}\left(\frac{N}{2},j\right) \\ h_N^{(k+lK)}\left(\frac{N}{2}+1,j\right) \\ \vdots \\ h_N^{(k+lK)}\left(\frac{N}{2}+\frac{K}{2}-1,j\right) \end{bmatrix}, \quad \text{for } k = 0,1,\ldots,K-1. \tag{9}$$

3.2 Software Functions and Hardware Schematic

Since an integer is represented in radix-2 form in any digital computer or digital signal processing system, the closed-form expression (8) provides us with an extremely simple way to generate any element $h_N^{(k)}(i,j)$ of $H_N^{(k)}$ by either software or hardware. A software implementation is given by the following function, written in the standard C language, which consists of only three sentences.

```
char hn(int i,int j,int k,int n)
{
    j=((j>>1)^i^k)&j;
    for(i=0;i<n-1;i++)j=(j>>1)^(j&1);
    return(j?-1:1);
}
```

The key for this simple implementation is the use of bit-wise logic operations provided by C. A Matlab version of `hn()` is also provided as follows by using the bit-wise logic functions.

```
function h=hn(i,j,k,n)
j=bitand(bitxor(bitshift(j,1),bitxor(i,k)),j);
for i=1:n-1
    j=bitxor(bitshift(j,-1),bitand(j,1));
end
h=(-1)^j;
```

If we use digital circuit to implement the bit-wise logic operation, any sequence in the scalable complete complementary sequence sets can be easily generated by digital hardware. Figure 3 shows a schematic diagram of this hardware implementation, which includes three registers I, J, K, and a number of logic XOR and AND gates. The I register and K register are pre-loaded to specify the integers i and k. The J register with an input clock serves as an "add-one" counter. It is set to 0 at the beginning and reset after $N = 2^n$ clock ticks. The output is the sequence given by the i th row in $H_N^{(k)}$, where +1 is represented by logic 1 and -1 by logic 0.

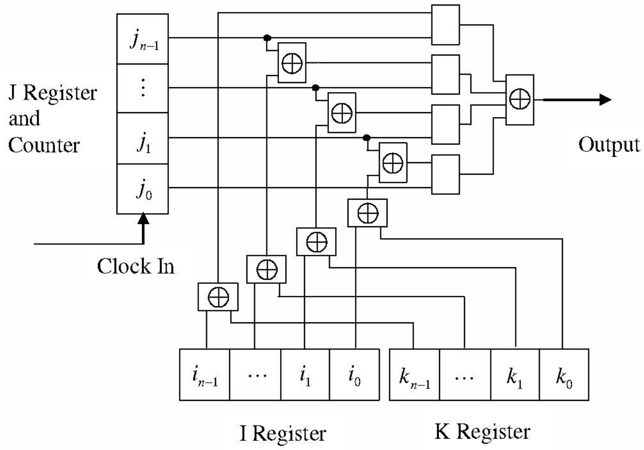

Figure 3. Schematic diagram for generating any sequence in the scalable complete complementary sets of sequences by digital hardware.

4. CONCLUSIONS

A Golay-paired Hadamard matrix represents a special complementary sequence set in which the sequences can be grouped into Golay sequence pairs and they are orthogonal with each other. The mutually orthogonal Golay-paired Hadamard matrices, constructed using the construction pyramid, represent a class of mutually orthogonal complementary sets of sequences, i.e., the scalable complete complementary sets of sequences. With the simple software functions and hardware circuit presented in this paper, great implementation advantage has been demonstrated for the scalable complete complementary sets of sequences, in addition to their desired complementary property, preferable orthogonality, and superb scalability and completeness.

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$H_8^{(0)}$	$H_8^{(1)}$	$H_8^{(2)}$	$H_8^{(3)}$	$H_8^{(4)}$	$H_8^{(5)}$	$H_8^{(6)}$	$H_8^{(7)}$
+++−++−+	+−+++−−−	++−++++−	+−−−+−++	+++−−−+−	+−++−+++	++−+−−−+	+−−−−+−−
+−+++−−−	+++−++−+	+−−−+−++	++−++++−	+−++−+++	+++−−−+−	+−−−−+−−	++−+−−−+
++−++++−	+−−−+−++	+++−++−+	+−+++−−−	++−+−−−+	+−−−−+−−	+++−−−+−	+−++−+++
+−−−+−++	++−++++−	+−+++−−−	+++−++−+	+−−−−+−−	++−+−−−+	+−++−+++	+++−−−+−
+++−−−+−	+−++−+++	++−+−−−+	+−−−−+−−	+++−++−+	+−+++−−−	++−++++−	+−−−+−++
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+−−−−+−−	++−+−−−+	+−++−+++	+++−−−+−	+−−−+−++	++−++++−	+−+++−−−	+++−++−+

Figure 2. Scalable complete complementary sequence sets of order 8 (+ and − denote the elements +1 and −1 respectively).