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Distance geometry in quasihypermetric spaces. I

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Abstract

Let (X,d) be a compact metric space and let (X) denote the space of all finite signed Borel measures on X. Define I:(X)→ by ...

and set $M(X)$=sup $I(\mu)$, where μ ranges over the collection of signed measures in (X) of total mass 1.

The metric space (X,d) is *quasihypermetric* if for all n, all $\alpha_1,...,\alpha_n$ satisfying $\sum_{i=1}^{n}\alpha_i=0$ and all $x_1,...,x_nX$, the inequality $\sum_{i,j=1}^{n}\alpha_i\alpha_j d(x_i,x_j)\leq 0$ holds. Without the quasihypermetric property $M(X)$ is infinite, while with the property a natural semi-inner product structure becomes available on $_0(X)$, the subspace of (X) of all measures of total mass 0. This paper explores: operators and functionals which provide natural links between the metric structure of (X,d), the semi-inner product space structure of $_0(X)$ and the Banach space $C(X)$ of continuous real-valued functions on X; conditions equivalent to the quasihypermetric property; the topological properties of $_0(X)$ with the topology induced by the semi-inner product, and especially the relation of this topology to the weak-* topology and the measure-norm topology on $_0(X)$; and the functional-analytic properties of $_0(X)$ as a semi-inner product space, including the question of its completeness. A later paper [P. Nickolas and R. Wolf, Distance geometry in quasihypermetric spaces. II, *Math. Nachr.*, accepted] will apply the work of this paper to a detailed analysis of the constant $M(X)$.

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DISTANCE GEOMETRY IN QUASIHYPERMETRIC SPACES. I

PETER NICKOLAS AND REINHARD WOLF

Abstract. Let (X, d) be a compact metric space and let $\mathcal{M}(X)$ denote the space of all finite signed Borel measures on X. Define $I \colon \mathcal{M}(X) \to \mathbb{R}$ by

$$I(\mu) = \int_X \int_X d(x, y)\, d\mu(x) d\mu(y),$$

and set $M(X) = \sup I(\mu)$, where μ ranges over the collection of signed measures in $\mathcal{M}(X)$ of total mass 1.

The metric space (X, d) is *quasihypermetric* if for all $n \in \mathbb{N}$, all $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ satisfying $\sum_{i=1}^n \alpha_i = 0$ and all $x_1, \ldots, x_n \in X$, one has $\sum_{i,j=1}^n \alpha_i \alpha_j d(x_i, x_j) \le 0$. Without the quasihypermetric property $M(X)$ is infinite, while with the property a natural semi-inner product structure becomes available on $\mathcal{M}_0(X)$, the subspace of $\mathcal{M}(X)$ of all measures of total mass 0. This paper explores: operators and functionals which provide natural links between the metric structure of (X, d), the semi-inner product space structure of $\mathcal{M}_0(X)$ and the Banach space $C(X)$ of continuous real-valued functions on X; conditions equivalent to the quasihypermetric property; the topological properties of $\mathcal{M}_0(X)$ with the topology induced by the semi-inner product, and especially the relation of this topology to the weak-$*$ topology and the measure-norm topology on $\mathcal{M}_0(X)$; and the functional-analytic properties of $\mathcal{M}_0(X)$ as a semi-inner product space, including the question of its completeness. A later paper [Peter Nickolas and Reinhard Wolf, *Distance Geometry in Quasihypermetric Spaces. II*] will apply the work of this paper to a detailed analysis of the constant $M(X)$.

1. Introduction

Let (X, d) be a compact metric space and let $\mathcal{M}(X)$ denote the space of all finite signed Borel measures on X. Define $I \colon \mathcal{M}(X) \to \mathbb{R}$ by

$$I(\mu) = \int_X \int_X d(x, y)\, d\mu(x) d\mu(y),$$

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and set

$$M(X) = \sup I(\mu),$$

where μ ranges over $\mathcal{M}_1(X)$, the collection of signed measures in $\mathcal{M}(X)$ of total mass 1. Our main aim in this paper and its sequels [27] and [28] is to investigate the properties of the geometric constant $M(X)$.

The so-called quasihypermetric property (for the definition, see below) turns out to play an essential role in our analysis. Indeed, we show that if (X, d) does not have the quasihypermetric property, then $M(X)$ is infinite, and, with the exception of some general results, our attention is therefore mostly confined to quasihypermetric spaces. When (X, d) is a quasihypermetric space, we introduce a semi-inner product on $\mathcal{M}_0(X)$, the subspace of all measures in $\mathcal{M}(X)$ of total mass 0. The resulting semi-inner product space has interesting properties in its own right, and is our fundamental tool for studying the properties of $M(X)$.

In this paper, we focus largely on the analysis of this semi-inner product space, and then in [27] and [28] we use the framework that this provides for a comprehensive discussion of the properties of M. Specifically, we explore in this paper:

(1) the properties of several operators and functionals which provide natural links between the metric structure of (X, d), the semi-inner product space structure of $\mathcal{M}_0(X)$ and the Banach space $C(X)$ of continuous real-valued functions on X,
(2) conditions equivalent to the quasihypermetric property,
(3) the topological properties of $\mathcal{M}_0(X)$ with the topology induced by the semi-inner product, and especially the relation of this topology to the weak-$*$ topology and the measure-norm topology on $\mathcal{M}_0(X)$,
(4) the functional-analytic properties of $\mathcal{M}_0(X)$ as a semi-inner product space, especially under the condition that $M(X)$ is finite, and
(5) the question of the completeness of $\mathcal{M}_0(X)$ as a semi-inner product space.

These items describe respectively the contents of the five main sections of the paper.

As remarked above, the sequels [27] and [28] to this paper pursue in detail the applications of our work here to the study of the constant $M(X)$. Further papers are also planned, in which we will study a number of questions related to the issues raised in the first two papers. These include the behaviour of M in several specific classes of metric spaces and the relation of M to other constants appearing in distance geometry.

1.1. **Definitions and notation.** Let (X, d) (abbreviated when possible to X) be a compact metric space. The diameter of X is denoted by $D(X)$.

We denote by $C(X)$ the Banach space of all real-valued continuous functions on X equipped with the usual supremum norm. Further,

- $\mathcal{M}(X)$ denotes the space of all finite signed Borel measures on X,
- $\mathcal{M}_0(X)$ denotes the subspace of $\mathcal{M}(X)$ consisting of all measures of total mass 0,

- $\mathcal{M}_1(X)$ denotes the affine subspace of $\mathcal{M}(X)$ consisting of all measures of total mass 1,
- $\mathcal{M}^+(X)$ denotes the set of all positive measures in $\mathcal{M}(X)$, and
- $\mathcal{M}_1^+(X)$ denotes the intersection of $\mathcal{M}^+(X)$ and $\mathcal{M}_1(X)$, the set of all probability measures on X.

The support of $\mu \in \mathcal{M}(X)$ is denoted by $\operatorname{supp}(\mu)$. For $x \in X$, we denote by $\delta_x \in \mathcal{M}_1^+(X)$ the point measure at x.

Recall that the weak-$*$ topology on $\mathcal{M}(X)$ is characterized by the fact that a net $\{\mu_\alpha\}$ in $\mathcal{M}(X)$ converges to $\mu \in \mathcal{M}(X)$ if and only if $\int_X f\, d\mu_\alpha \to \int_X f\, d\mu$ for all $f \in C(X)$.

Each $\mu \in \mathcal{M}(X)$ has a Hahn–Jordan decomposition, allowing us to write either $\mu = \mu^+ - \mu^-$, where $\mu^+, \mu^- \in \mathcal{M}^+(X)$ and $\operatorname{supp}(\mu^+) \cap \operatorname{supp}(\mu^-) = \emptyset$, or, equivalently, $\mu = \alpha\mu_1 - \beta\mu_2$, where $\mu_1, \mu_2 \in \mathcal{M}_1^+(X)$, $\alpha, \beta \geq 0$ and $\operatorname{supp}(\mu_1) \cap \operatorname{supp}(\mu_2) = \emptyset$. We denote by $\|\cdot\|_{\mathcal{M}}$ the measure norm on $\mathcal{M}(X)$. Since our standing assumption will be that X is compact, we have the simple expression $\|\mu\|_{\mathcal{M}} = \mu^+(X) + \mu^-(X) = \alpha + \beta$, for μ as above.

The Riesz representation theorem tells us that $\mathcal{M}(X)$, equipped with the measure norm, is a Banach space isometrically isomorphic to the space $C(X)'$, the dual space of $C(X)$. In the following, we will freely identify signed Borel measures with continuous linear functionals, writing as convenient either $\mu(f)$ or $\int_X f\, d\mu$ when $f \in C(X)$ and $\mu \in \mathcal{M}(X)$.

Two functionals on measures will play a central role in this paper. If (X, d) is a compact metric space, then for $\mu, \nu \in \mathcal{M}(X)$, we set

$$I(\mu, \nu) = \int_X \int_X d(x, y)\, d\mu(x) d\nu(y),$$

and then we set

$$I(\mu) = I(\mu, \mu).$$

We also make use of the linear functionals $J(\mu)$ on $\mathcal{M}(X)$, defined for each $\mu \in \mathcal{M}(X)$ by $J(\mu)(\nu) = I(\mu, \nu)$ for all $\nu \in \mathcal{M}(X)$. The functional $I(\cdot, \cdot)$ is obviously bilinear on $\mathcal{M}(X) \times \mathcal{M}(X)$, and this immediately gives identities such as

$$I(\mu \pm \nu) = I(\mu) + I(\nu) \pm 2I(\mu, \nu),$$

which we will use frequently. It is useful to note that $I(\delta_x) = 0$.

For $\mu \in \mathcal{M}(X)$, we define the function d_μ by

$$d_\mu(x) = \int_X d(x, y)\, d\mu(y)$$

for $x \in X$. Of course, $d_\mu \in C(X)$ for all μ, and we define a linear map

$$T\colon \mathcal{M}(X) \to C(X)$$

by setting $T(\mu) = d_\mu$ for $\mu \in \mathcal{M}(X)$. Note that we may express the functional $I(\cdot,\cdot)$ in terms of the functions d_μ:

$$I(\mu,\nu) = \int_X d_\mu \, d\nu = \int_X d_\nu \, d\mu = I(\nu,\mu).$$

We also make use of the linear map T_0, which is the restriction of T to the subspace $\mathcal{M}_0(X)$.

For the compact metric space (X,d), we define

$$M^+(X) = \sup\bigl\{I(\mu) : \mu \in \mathcal{M}_1^+(X)\bigr\}$$

and

$$M(X) = \sup\bigl\{I(\mu) : \mu \in \mathcal{M}_1(X)\bigr\}.$$

The geometric constant $M(X)$ is our main focus in this paper, but use will be made from time to time of $M^+(X)$.

A metric space (X,d) is called *quasihypermetric* if for all $n \in \mathbb{N}$, all $\alpha_1,\ldots,\alpha_n \in \mathbb{R}$ satisfying $\sum_{i=1}^n \alpha_i = 0$ and all $x_1,\ldots,x_n \in X$, we have

$$\sum_{i,j=1}^n \alpha_i\alpha_j d(x_i,x_j) \le 0.$$

1.2. **Connections with other work.** The geometric constant $M(X)$ appeared for the first time in the work of Alexander and Stolarsky [5], who dealt with the case when X is a compact subset of euclidean space and d is the usual euclidean metric. They showed that in this case $M(X)$ is always finite, and that when the subset X itself is finite, the supremum $M(X)$ is achieved for some signed measure $\mu \in \mathcal{M}_1(X)$, allowing the explicit computation of $M(X)$. Further papers by Alexander, especially [3] and [1], carried the analysis of the euclidean case further. Because euclidean space is quasihypermetric, the references just cited do not give explicit emphasis to the role of the quasihypermetric property and have little need for the development of a general framework for the analysis.

Our interest is in the analysis of $M(X)$ in a general compact metric space X, and our primary aim in the present paper is to develop the framework mentioned and in particular to make explicit the role of the quasihypermetric property. Indeed, the constant $M(X)$, which is ultimately our main interest, is discussed in this paper only as far as is needed to do this, and a detailed analysis of $M(X)$ itself will be taken up in [27], [28] and later papers.

Some of the ideas developed here have obvious parallels with the ideas of potential theory. In modern accounts of classical potential theory (see Landkof [22], for example), one deals with a space X which is a suitable region in a euclidean space and a kernel $k(x,y)$ on $X \times X$ which is typically of the form $\|x-y\|^\alpha$ for certain values of $\alpha < 0$ which depend on the dimension of the euclidean space (here, $\|\cdot\|$ denotes the euclidean norm). Energy integrals $I_k(\mu) = \iint k(x,y)\, d\mu(x) d\mu(y)$ and potentials $d_{k,\mu}(x) = \int k(x,y)\, d\mu(y)$

are then defined for signed measures μ, paralleling our definitions above, and one seeks, for example, to find measures μ which minimize $I_k(\mu)$ or which yield a constant potential $d_{k,\mu}$.

The classical framework may be generalized in several ways (see Fuglede [16]): the space X may be replaced by a (locally) compact Hausdorff topological space and quite general classes of kernels k can be considered. As discovered already by Björck [6], the theory takes on a significantly different character even in the euclidean case if the kernel has the non-classical form $\|x-y\|^\alpha$ for $\alpha > 0$, since one then naturally seeks to maximize rather than to minimize the corresponding generalized energy integral. Moreover, if X is not a euclidean domain, then standard analytical techniques, especially that of the Fourier transform, are no longer available.

For these reasons and others (relating, for example, to the quasihypermetric constraint), one cannot expect to find precise parallels between our results and arguments and those of either classical or generalized potential theory, even though the theories have global features in common at many points.

Some of the ideas in this paper can be generalized straightforwardly along the lines suggested by Fuglede's work. The reader can verify easily, for example, that analogues of a number of our results hold in the case of a continuous, symmetric kernel k on a compact Hausdorff space X. Using Fuglede's work, Farkas and Revész [14, 15] recently carried out a generalized potential-theoretic analysis of the so-called rendezvous number, another constant appearing in distance geometry (see, for example, [32], [17], [26], [11], [37], [35], [36], [34] and [19]).

Despite the possibility of such generalization, however, our discussion here takes place exclusively in the setting of a compact metric space X and its metric d, because our motivation is essentially geometric: the analysis of the geometric properties of X and related structures, and especially the geometric constant $M(X)$.

2. Properties of the Mappings T and I

Recall from section 1.1 that when (X, d) is a compact metric space, $T\colon \mathcal{M}(X) \to C(X)$ is the linear map defined by $T(\mu) = d_\mu$ for $\mu \in \mathcal{M}(X)$. We denote the image of T by $\operatorname{im} T$.

Theorem 2.1. *Let (X, d) be a compact metric space. Then $\dim(\operatorname{im} T)$ is finite if and only if X is finite.*

For the proof of the theorem, we need the following lemma. If S is any subset of a linear space, we write $[S]$ (omitting set braces if appropriate) to denote the linear hull of S.

Lemma 2.2. *Let (X, d) be a compact metric space. Then we have the following.*

(1) *If $i\colon X \to C(X)$ is the function defined by $i(x) = d_{\delta_x}$ for $x \in X$, then $\|i(x) - i(y)\|_\infty = d(x, y)$ for all $x, y \in X$.*

(2) $\overline{\operatorname{im} T} = \overline{[i(x) : x \in X]}$.

Proof. Since $d_{\delta_x}(y) = d(x, y)$ for all $x, y \in X$, the first statement is an easy consequence of the triangle inequality.

To prove the second statement, assume that $\mu \in \mathcal{M}(X)$ is such that $d_\mu \notin \overline{[i(x) : x \in X]}$. Then by the Hahn–Banach theorem, there exists $\nu \in \mathcal{M}(X)$ such that $\nu(d_\mu) = 1$ and $\nu(i(x)) = 0$ for all $x \in X$. But then $d_\nu(x) = 0$ for all $x \in X$, while $\mu(d_\nu) = \nu(d_\mu) = 1$, a contradiction. Therefore, $\overline{\operatorname{im} T} \subseteq \overline{[i(x) : x \in X]}$, and since the reverse inclusion clearly holds, the proof is complete. ☐

Proof of Theorem 2.1. If X is finite, then of course $\dim(\operatorname{im} T)$ is finite.

Let us assume that $\dim(\operatorname{im} T) = n$ for some integer $n \geq 0$. It is easy to see that if $n = 0$, then X is a one-point space, so we can assume that $n \geq 1$. By Lemma 2.2, there are $x_1, \ldots, x_n \in X$ such that $\operatorname{im} T = [i(x_1), \ldots, i(x_n)]$, and so for every $x \in X$, there exists a unique $\lambda(x) = (\lambda_1(x), \ldots, \lambda_n(x)) \in \mathbb{R}^n$ such that $i(x) = \lambda_1(x)i(x_1) + \cdots + \lambda_n(x)i(x_n)$. It follows that $d(x, y) = \sum_{i=1}^n \lambda_i(x) d(x_i, y)$ for all $x, y \in X$, and so we have

$$d(x, y) = d(y, x) = \sum_{j=1}^n \lambda_j(y) d(x, x_j) = \sum_{i,j=1}^n \lambda_i(x)\lambda_j(y) d(x_i, x_j).$$

Define an $n \times n$ matrix $A = (a_{i,j})$ by setting $a_{i,j} = -(1/2)d(x_i, x_j)$ for all i and j, and view A as a bounded linear operator on the euclidean space $\mathbb{R}^n$. It follows that

$$d(x, y) = \big(A(\lambda(x) - \lambda(y)) \bigm| \lambda(x) - \lambda(y)\big).$$

Now by the Cauchy–Schwarz inequality, we have

$$d(x, y) \leq \big\|A(\lambda(x) - \lambda(y))\big\| \cdot \big\|\lambda(x) - \lambda(y)\big\| \leq \|A\| \cdot \big\|\lambda(x) - \lambda(y)\big\|^2.$$

To estimate $\|\lambda(x) - \lambda(y)\|$, define $\phi_j \colon \operatorname{im} T \to \mathbb{R}$ by setting

$$\phi_j\Big(\sum_{i=1}^n \beta_i i(x_i)\Big) := \beta_j$$

for $j = 1, \ldots, n$. Since ϕ_j is linear and $\dim(\operatorname{im} T) = n < \infty$, we know that ϕ_j is bounded. Hence, for all $x, y \in X$, we have

$$\begin{aligned} \big|\lambda_j(x) - \lambda_j(y)\big| &= \big|\phi_j(i(x)) - \phi_j(i(y))\big| \\ &= \big|\phi_j(i(x) - i(y))\big| \\ &\leq \|\phi_j\| \cdot \big\|i(x) - i(y)\big\|_\infty \\ &= \|\phi_j\| \cdot d(x, y), \end{aligned}$$

by Lemma 2.2. Hence for $K := \max_j \|\phi_j\|$, we have $|\lambda_j(x) - \lambda_j(y)| \leq K \cdot d(x, y)$ for all $x, y \in X$ and for all $j = 1, \ldots, n$, and therefore $\|\lambda(x) - \lambda(y)\|^2 \leq nK^2 d(x, y)^2$ for all $x, y \in X$. Combining our inequalities, we obtain $d(x, y) \leq \|A\| nK^2 d(x, y)^2$ for all

$x, y \in X$, and hence we have $d(x,y) \geq 1/(n\|A\|K^2)$ for all distinct $x, y \in X$. Since X is compact, we conclude that X is finite. □

Remark 2.3. We note that Theorem 2.1 does not in general hold if the metric property of d is weakened. For $n \geq 2$, let S^{n-1} denote the euclidean unit sphere in $\mathbb{R}^n$, let X be a compact subset of S^{n-1} and let $d(x,y) = \|x-y\|^2$ for all $x, y \in X$, where $\|\cdot\|$ is the euclidean norm. Also, for $k = 1, \ldots, n$, define $f_k \in C(X)$ by $f_k(x) := \|x - e_k\|^2$, where e_k denotes the kth canonical unit vector in $\mathbb{R}^n$. Then defining T and i formally as earlier (though d now may not be a metric), we see easily that for each $x = (x_1, \ldots, x_n) \in S^{n-1}$

$$i(x) = 2\Big(1 - \sum_{k=1}^{n} x_k\Big) \cdot \underline{1} + \sum_{k=1}^{n} x_k f_k,$$

where $\underline{1}$ denotes the constant function $\underline{1}(y) := 1$ for all $y \in X$. But it is clear that Lemma 2.2 part (2) still holds, and so we have $\operatorname{im} T \subseteq [\underline{1}, f_1, \ldots, f_n]$, and it follows that $\dim(\operatorname{im} T) \leq n + 1 < \infty$. While the function d is non-negative and symmetric, and $d(x,y) = 0$ if and only if $x = y$, however, it follows from a theorem of Danzer and Grünbaum [12] that d cannot satisfy the triangle inequality if X has more than 2^n elements. Thus the forward implication of Theorem 2.1 fails for every infinite choice of X.

Theorem 2.4. *Let (X,d) be a compact metric space. Then T is injective if and only if $\operatorname{im} T$ is dense in $C(X)$.*

Proof. Assume that $\operatorname{im} T$ is not dense in $C(X)$. Then by the Hahn–Banach theorem, there exists $\mu \neq 0$ in $\mathcal{M}(X)$ such that $\mu(d_\nu) = 0$ for all $\nu \in \mathcal{M}(X)$. Therefore $0 = \mu(d_\nu) = \nu(d_\mu)$ for all $\nu \in \mathcal{M}(X)$, and so $d_\mu = 0$. Hence T is not injective. On the other hand, assume that $\operatorname{im} T$ is dense in $C(X)$ and that $d_\mu = 0$ for some $\mu \in \mathcal{M}(X)$. Now $d_\mu = 0$ implies $\nu(d_\mu) = 0$ for all $\nu \in \mathcal{M}(X)$, and therefore $0 = \nu(d_\mu) = \mu(d_\nu)$ for all $\nu \in \mathcal{M}(X)$. Then, since $\operatorname{im} T$ is dense in $C(X)$, we have $\mu = 0$, and T is injective. □

We now discuss the continuity of the functionals $I(\cdot)$ and $I(\cdot,\cdot)$ on $\mathcal{M}(X)$ and $\mathcal{M}(X) \times \mathcal{M}(X)$, and on various subsets. We omit the straightforward proofs of the first two results, the second of which generalizes parts of the statement and proof of Lemma 1 of [35].

Theorem 2.5. *If (X,d) is a compact metric space and $\mathcal{M}(X)$ is given the weak-$*$ topology, then the functional $I(\cdot,\cdot)$ on $\mathcal{M}(X) \times \mathcal{M}(X)$ is separately continuous in each variable.*

Theorem 2.6. *Let (X,d) be a compact metric space and let $\mathcal{M}(X)$ be given the weak-$*$ topology. Then the functional $I(\cdot)$ is continuous on any subset of $\mathcal{M}(X)$ which is $\|\cdot\|_{\mathcal{M}}$-bounded.*

Corollary 2.7. *The functional $I(\cdot)$ is weak-$*$ sequentially continuous on $\mathcal{M}(X)$.*

Proof. Suppose that $\mu_n \to \mu$ is a weak-$*$ convergent sequence in $\mathcal{M}(X)$. Then since $\mu_n(f) \to \mu(f)$ for each $f \in C(X)$, the set $\{|\mu_n(f)| : n \in \mathbb{N}\}$ is bounded for each $f \in C(X)$, and it follows from the Banach–Steinhaus (or uniform boundedness) theorem that the set $\{\|\mu_n\|_{\mathcal{M}} : n \in \mathbb{N}\}$ is also bounded. It now follows from Theorem 2.6 that $I(\mu_n) \to I(\mu)$, and so $I(\cdot)$ is sequentially continuous. ☐

Corollary 2.8.

(1) *The functional $I(\cdot)$ is weak-$*$ continuous on $\mathcal{M}^+(X)$ (and hence in particular on $\mathcal{M}_1^+(X)$).*

(2) *When X is finite, the functional $I(\cdot)$ is weak-$*$ continuous on $\mathcal{M}(X)$.*

Proof. Both parts follow from Corollary 2.7, using for part (1) the fact that the subset $\mathcal{M}^+(X)$ of positive measures in $\mathcal{M}(X)$ is metrizable (see [10, Theorem 12.10]) and for part (2) the obvious fact that when X is finite $\mathcal{M}(X)$ is metrizable (see also [10, Theorem 16.9]). ☐

Part (1) in the case of $\mathcal{M}_1^+(X)$ was observed earlier in [35].

Remark 2.9. It is useful to note that the identity

$$I(\mu,\nu) = \tfrac{1}{2}\big(I(\mu+\nu) - I(\mu) - I(\nu)\big)$$

allows information about the continuity of $I(\cdot,\cdot)$ to be deduced from information about the continuity of $I(\cdot)$ (this was pointed out to the first author by Ben Chad). Hence Theorem 2.6 and Corollaries 2.7 and 2.8 extend in an obvious way to the functional $I(\cdot,\cdot)$.

We now establish a negative result about the continuity of the functionals I, which shows in particular that significantly stronger positive results than those above are impossible.

Theorem 2.10. *Let (X,d) be an infinite compact metric space. Then the functionals $I(\cdot)$ and $I(\cdot,\cdot)$ are weak-$*$ discontinuous everywhere.*

Proof. We use here some ideas from exercise 2, §4, Chap. 3 of [8]. We define a net of pairs of measures in $\mathcal{M}(X) \times \mathcal{M}(X)$. For index set, we take the set A of all finite subsets of $C(X)$, directed by set inclusion. Consider a fixed collection $\{f_1,\dots,f_n\} \in A$, where $f_1,\dots,f_n$ are distinct. Then by Theorem 2.1, there exists $\mu \in \mathcal{M}(X)$ such that d_μ is not in the linear span of $\{f_1,\dots,f_n\}$. We may clearly assume that $\|\mu\|_{\mathcal{M}} = 1$. By the Hahn–Banach theorem, there exists $\nu \in \mathcal{M}(X)$ such that $\nu(f_i) = 0$ for $i = 1,\dots,n$ but $\nu(d_\mu) \neq 0$; that is, in our usual notation, $I(\mu,\nu) \neq 0$. We may clearly rescale ν so that $I(\mu,\nu)$ has any desired non-zero value, and it is convenient here to assume that $I(\mu,\nu) = n$. Writing $\alpha = \{f_1,\dots,f_n\}$, let us denote the measures μ and ν just found by μ_α and ν_α, respectively.

We claim that the net $\{\nu_\alpha\}$ converges weak-$*$ to 0 in $\mathcal{M}(X)$. Indeed, given $f \in C(X)$, we have $\{f\} \in A$, and our choice of $\nu_{\{f\}}$ means that $\nu_{\{f\}}(f) = 0$. Also, if $\alpha \in A$ is such that $\{f\} \subseteq \alpha$, then $\nu_\alpha(f) = 0$, so $\nu_\alpha(f) \to 0$ in $\mathbb{R}$, as required for weak-$*$ convergence.

We chose the measures $\{\mu_\alpha\}$ so that $\|\mu_\alpha\|_{\mathcal{M}} = 1$ for all α, so the μ_α all lie in the unit ball of $\mathcal{M}(X)$, which by the Banach–Alaoglu theorem (see also Corollary 12.7 of [10]) is weak-$*$ compact. Hence there exists a weak-$*$ convergent subnet, say $\mu_{\alpha(\beta)} \to \mu$, of the μ_α. Thus we have $(\mu_{\alpha(\beta)}, \nu_{\alpha(\beta)}) \to (\mu, 0)$, where the convergence is weak-$*$ in each coordinate. But μ_α and ν_α were chosen in such a way that $I(\mu_\alpha, \nu_\alpha) = |\alpha|$ (the cardinality of α), so it follows that the net $I(\mu_{\alpha(\beta)}, \nu_{\alpha(\beta)})$ diverges in $\mathbb{R}$. That is, the functional $I(\cdot, \cdot)$ is discontinuous at $(\mu, 0) \in \mathcal{M}(X) \times \mathcal{M}(X)$.

A straightforward argument now shows that $I(\cdot, \cdot)$ is discontinuous at all points in $\mathcal{M}(X) \times \mathcal{M}(X)$, and the observation in Remark 2.9 then implies that $I(\cdot)$ is discontinuous everywhere. □

We note the following result for later application.

Corollary 2.11. *Let (X, d) be an infinite compact metric space. Then the functional $I(\cdot)$, when restricted to the domain $\mathcal{M}_0(X)$, is weak-$*$ discontinuous at all points.*

Proof. It is easy to show that $I(\cdot)$ is continuous at all points of $\mathcal{M}_0(X)$ if and only if it is continuous at one, so it suffices to show that $I(\cdot)$ is discontinuous at $0 \in \mathcal{M}_0(X)$.

Assume that $I(\cdot)$ is continuous at $0 \in \mathcal{M}_0(X)$, and suppose that $\mu_\alpha \to 0$ for some net $\{\mu_\alpha\}$ in $\mathcal{M}(X)$. Let δ be any fixed atomic probability measure, and let $m_\alpha = \mu_\alpha(X)$. Then $\mu_\alpha(X) = \int_X d\mu_\alpha \to 0$, so $m_\alpha \to 0$. Put $\nu_\alpha = \mu_\alpha - m_\alpha \delta$, so that $\nu_\alpha \in \mathcal{M}_0(X)$. Now $\nu_\alpha \to 0$ weak-$*$, since for any $f \in C(X)$ we have $\int_X f\, d\nu_\alpha = \int_X f\, d\mu_\alpha - m_\alpha \int_X f\, d\delta \to 0$. Hence, by assumption, $I(\nu_\alpha) \to 0$. But $I(\nu_\alpha) = I(\mu_\alpha - m_\alpha\delta) = I(\mu_\alpha) - 2m_\alpha I(\mu_\alpha, \delta)$, and $I(\mu_\alpha, \delta) \to 0$ by separate continuity of $I(\cdot, \cdot)$, so $\left|m_\alpha I(\mu_\alpha, \delta)\right| = |m_\alpha| \cdot |I(\mu_\alpha, \delta)| \to 0$, and hence $I(\mu_\alpha) \to 0$. Therefore, $I(\cdot)$ is continuous at $0 \in \mathcal{M}(X)$, contradicting Theorem 2.10, and this completes the proof. □

3. The Quasihypermetric Property

The quasihypermetric property is the most important metric property considered in this paper and in [27] and [28]. In view of the fact that our ultimate interest is the study of the geometric constant M, the following simple result explains why our focus is almost exclusively on these spaces.

Theorem 3.1. *If (X, d) is a compact non-quasihypermetric space, then $M(X) = \infty$.*

Proof. If X is non-quasihypermetric, then there exist $n \in \mathbb{N}$, $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ with $\sum_{i=1}^n \alpha_i = 0$ and $x_1, \ldots, x_n \in X$ such that $\sum_{i,j=1}^n \alpha_i \alpha_j d(x_i, x_j) > 0$. Writing $\mu = \sum_{i=1}^n \alpha_i \delta_{x_i}$, we therefore have $\mu \in \mathcal{M}_0(X)$ and $I(\mu) > 0$ (see also condition (3) in Theorem 3.2 below). Now choose any $x \in X$, and define $\mu_n \in \mathcal{M}_1(X)$ by setting $\mu_n = n\mu + \delta_x$ for each $n \in \mathbb{N}$. Then $I(\mu_n) = n^2 I(\mu) + 2n d_\mu(x) \to \infty$ as $n \to \infty$, giving the result. □

We record a list of conditions which are equivalent to the quasihypermetric condition.

Theorem 3.2. *Let (X, d) be a compact metric space. Then the following conditions are equivalent.*

(1) (X, d) *is quasihypermetric.*
(2) $\sum_{i,j=1}^{n} d(x_i, x_j) + \sum_{i,j=1}^{n} d(y_i, y_j) \leq 2 \sum_{i,j=1}^{n} d(x_i, y_j)$ *for all* $n \in \mathbb{N}$ *and for all* $x_1, \ldots, x_n, y_1, \ldots, y_n \in X$.
(3) $I(\mu) \leq 0$ *for all* $\mu \in \mathcal{M}_0(X)$.
(4) $I(\mu, \nu)^2 \leq I(\mu) I(\nu)$ *for all* $\mu, \nu \in \mathcal{M}_0(X)$.
(5) $I(\mu) + I(\nu) \leq 2I(\mu, \nu)$ *for all* $\mu, \nu \in \mathcal{M}_1(X)$.
(6) $I(\mu) + I(\nu) \leq 2I(\mu, \nu)$ *for all* $\mu, \nu \in \mathcal{M}_1^+(X)$.
(7) $\frac{1}{2}\big(I(\mu) + I(\nu)\big) \leq I\big(\frac{1}{2}(\mu + \nu)\big)$ *for all* $\mu, \nu \in \mathcal{M}_1(X)$.
(8) $\frac{1}{2}\big(I(\mu) + I(\nu)\big) \leq I\big(\frac{1}{2}(\mu + \nu)\big)$ *for all* $\mu, \nu \in \mathcal{M}_1^+(X)$.

To these, for completeness, we add the following variants of the last two conditions.

(7′) $\alpha I(\mu) + \beta I(\nu) \leq I(\alpha\mu + \beta\nu)$ *for all* $\mu, \nu \in \mathcal{M}_1(X)$ *and all* $\alpha, \beta \in \mathbb{R}$ *such that* $\alpha, \beta \geq 0$ *and* $\alpha + \beta = 1$.
(8′) $\alpha I(\mu) + \beta I(\nu) \leq I(\alpha\mu + \beta\nu)$ *for all* $\mu, \nu \in \mathcal{M}_1^+(X)$ *and all* $\alpha, \beta \in \mathbb{R}$ *such that* $\alpha, \beta \geq 0$ *and* $\alpha + \beta = 1$.

Proof. The proofs are for the most part straightforward, and we show only the equivalence of (3) and (4) (and note also that the equivalence of (1) and (2) is outlined on page 2049 of [25]).

Assuming (3), we define a semi-inner product $(\cdot \mid \cdot)$ on the space $\mathcal{M}_0(X)$ by the formula $(\mu \mid \nu) = -I(\mu, \nu)$ for $\mu, \nu \in \mathcal{M}_0(X)$; that the semi-inner product axioms are satisfied is clear (we will study and use this semi-inner product extensively below). It is clear that the Cauchy–Schwarz inequality for the semi-inner product gives (4). Conversely, assume (4). If X is singleton, then (3) is immediate. Otherwise, let ν be any element of $\mathcal{M}_0(X)$ such that $I(\nu) < 0$; we may take $\nu = \delta_x - \delta_y$ for any pair of distinct elements $x, y \in X$, for example. Then (4) implies that $I(\mu) \leq 0$ for all $\mu \in \mathcal{M}_0(X)$, giving (3). □

An important and much less elementary equivalence is given by Schoenberg [31]: a separable metric space (X, d) is quasihypermetric if and only if the metric space $(X, d^{\frac{1}{2}})$ is isometrically embeddable in the Hilbert space ℓ^2.

The quasihypermetric property has been discovered several times; it appears independently, for example, in Lévy [24], Schoenberg [31], Björck [6] and Kelly [20], in each case as part of a study involving more general geometric inequalities. The term 'quasihypermetric' was introduced by Kelly [20]; elsewhere, quasihypermetric spaces, or their metrics, have been referred to as *of negative type* (see [9], for example).

There are several important classes of quasihypermetric spaces.

(1) The euclidean spaces $\mathbb{R}^n$ for all $n \geq 1$.

(2) More generally, the space $\mathbb{R}^n$ for all n, equipped with the usual p-norm for $1 \leq p \leq 2$.
(3) All two-dimensional real normed spaces.
(4) All metric spaces with four or fewer points.
(5) The n-dimensional sphere S^n in $\mathbb{R}^{n+1}$ for $n \geq 1$, equipped with the great-circle metric.

When $n \geq 3$, the space $\mathbb{R}^n$ equipped with the p-norm for $2 < p \leq \infty$ is not quasihypermetric.

The first three classes of examples above are essentially given by classical results from the theory of L^1-embeddability (see [24, 13, 18]), as is the negative statement; the case of a metric space with four points is part of Blumenthal's 'four-point theorem' (see [7, Theorem 52.1]); and the case of the sphere with the great-circle metric is given in [21]. The cases of $\mathbb{R}^n$ with the p-norm for $1 \leq p \leq 2$ and of S^n with the great-circle metric are also given by a general construction of Alexander [3] using the methods of integral geometry.

Definition 3.3. A compact quasihypermetric space (X, d) is said to be *strictly quasihypermetric* if $I(\mu) = 0$ only when $\mu = 0$, for $\mu \in \mathcal{M}_0(X)$.

Lemma 1 of [6], in this terminology, yields the following statement.

Theorem 3.4. *Every compact subset of $\mathbb{R}^n$ is strictly quasihypermetric.*

This fact has also been discovered independently more than once; it is noted in [4] that it is equivalent to the uniqueness theorem for the Radon transform. A theorem implying the weaker statement that finite subsets of $\mathbb{R}^n$ are strictly quasihypermetric was proved in [30].

Example 3.5. Let X be the circle S^1 of radius 1, given the arc-length metric d. Since d is the one-dimensional form of the great-circle metric, X is quasihypermetric, as noted above. We claim that X is not strictly quasihypermetric. Indeed, let x_1 and y_1 be diametrically opposite points in X. Then, if we set $\mu_1 = \delta_{x_1} + \delta_{y_1}$, it is easy to see that the integral $\int_X d(x, y)\, d\mu_1(x)$ has the constant value $\pi = D(X)$ for all y. Hence, if a second measure μ_2 is similarly defined for a different pair of points x_2, y_2, and if we write $\mu = \mu_1 - \mu_2$, then we have $0 \neq \mu \in \mathcal{M}_0(X)$, while $I(\mu) = I(\mu_1) + I(\mu_2) - 2I(\mu_1, \mu_2) = \pi + \pi - 2\pi = 0$.

The same argument shows that the sphere S^{n-1} with the great-circle metric fails to be strictly quasihypermetric for all $n > 1$. The argument shows moreover that the subspace $\{x_1, y_1, x_2, y_2\}$ of S^1 is a 4-element metric space which is quasihypermetric but not strictly quasihypermetric.

Theorem 3.6. *Let (X, d) be a non-trivial compact strictly quasihypermetric space. Then*
(1) *T is injective, and*
(2) *$\operatorname{im} T$ is dense in $C(X)$.*

Proof. By Theorem 2.4, it suffices to show that T is injective. Suppose that $d_\mu = 0$ for some $\mu \in \mathcal{M}(X)$. If $\mu(X) \neq 0$, define $\overline{\mu} \in \mathcal{M}_1(X)$ by setting $\overline{\mu} = \mu/\mu(X)$, and choose $\nu \in \mathcal{M}_1^+(X)$ with $I(\nu) > 0$ (we may take $\nu = \frac{1}{2}(\delta_x + \delta_y)$ for any pair of distinct elements $x, y \in X$, for example). Then since $d_{\overline{\mu}} = 0$ and $2I(\overline{\mu}, \nu) \geq I(\overline{\mu}) + I(\nu)$ (Theorem 3.2), we have $I(\nu) \leq 0$, a contradiction. Therefore, $\mu(X) = 0$. But since $d_\mu = 0$ implies $I(\mu) = 0$, the strictly quasihypermetric assumption now implies that $\mu = 0$, as required. □

We note that the assumption that X is non-trivial is necessary: if X is singleton, it is easy to see that $\operatorname{im} T$ is not dense in $C(X)$.

Example 3.7. Consider again the quasihypermetric, non-strictly quasihypermetric space (X, d), where X is the circle of radius 1 and d is the arc-length metric. It is obvious that whenever x and x' are diametrically opposite points in X, we have $d(x, y) + d(x', y) = \pi$ for all $y \in X$, and integration with respect to an arbitrary measure $\mu \in \mathcal{M}(X)$ then yields $d_\mu(x) + d_\mu(x') = \pi\mu(X)$. But the collection of functions $f \in C(X)$ such that $f(x) + f(x')$ is constant for all diametrically opposite pairs of points x and x' is clearly a proper closed subspace of $C(X)$, and since it contains $\operatorname{im} T$, the latter is not dense in $C(X)$.

Remark 3.8. We note that Theorem 3.6 gives a very simple proof of Theorem 2.1 in the case of a strictly quasihypermetric space X. Indeed, if $\operatorname{im} T$ were finite-dimensional for such a space, then $\operatorname{im} T$ would be both closed and dense in $C(X)$, and hence equal to $C(X)$, and the finite-dimensionality of $C(X)$ would then imply that X was finite.

4. Topologies on $\mathcal{M}(X)$ and its Subspaces

Let (X, d) be a compact quasihypermetric space. In the proof of Theorem 3.2, we noted in passing that a semi-inner product $(\cdot \mid \cdot)$ can be defined on the subspace $\mathcal{M}_0(X)$ of $\mathcal{M}(X)$ of measures of total mass 0 by the formula

$$(\mu \mid \nu) = -I(\mu, \nu)$$

for $\mu, \nu \in \mathcal{M}_0(X)$. When $\mathcal{M}_0(X)$ is equipped with this semi-inner product, we will denote the resulting semi-inner product space by $E_0(X)$. We note that the associated seminorm $\|\cdot\|$ on $E_0(X)$ is given by

$$\|\mu\| = \left[-I(\mu)\right]^{\frac{1}{2}}$$

for $\mu \in \mathcal{M}_0(X)$. In referring to the topology of $E_0(X)$, we will from here on always mean the topology induced by this seminorm; other topologies on $\mathcal{M}_0(X)$—specifically, the topologies induced on $\mathcal{M}_0(X)$ by the weak-$*$ topology and the measure-norm topology on $\mathcal{M}(X)$—will be named explicitly.

It is clear that the above semi-inner product becomes an inner product, and $E_0(X)$ an inner product space, precisely when (X, d) is strictly quasihypermetric.

The use of functionals such as $I(\cdot,\cdot)$ to define a (semi-)inner product structure is a standard procedure in potential theory (see [22] and [16], for example) and has also been explored in somewhat different settings such as the study of irregularity of distribution ([2] and [4]) and distance geometry (see [23]).

Recall the definition of the constant $M(X)$:

$$M(X) = \sup I(\mu),$$

where μ ranges over $\mathcal{M}_1(X)$. In the case when $M(X)$ is finite, there is a natural extension of the semi-inner product on $E_0(X) = \mathcal{M}_0(X)$ to a semi-inner product on the collection $\mathcal{M}(X)$ of all signed Borel measures on X. Specifically, we define

$$(\mu \mid \nu) = (M(X) + 1)\mu(X)\nu(X) - I(\mu, \nu)$$

for $\mu, \nu \in \mathcal{M}(X)$, and note that the semi-inner product space axioms are straightforward to check. Further, the new semi-inner product is once again an inner product precisely when X is a strictly quasihypermetric space. It is easy to see that the new semi-inner product is indeed an extension of the earlier one. When $\mathcal{M}(X)$ is equipped with the extended semi-inner product, we will denote the resulting semi-inner product space by $E(X)$.

Remark 4.1. We note that if the term $M(X) + 1$ in the definition is replaced by $M(X) + \epsilon$, for any $\epsilon > 0$, then the expression still defines an extension of the earlier semi-inner product, though working with the initially given form will suffice for our purposes here.

It is straightforward to show that the induced norms are equivalent for all ϵ, so that in particular the metric and topological properties of $E(X)$ are independent of ϵ. Further, the identity mapping on $\mathcal{M}_0(X)$ can be extended to an isomorphism between the corresponding semi-inner product spaces if and only if there exists a measure $\mu_0 \in \mathcal{M}_1(X)$ such that d_{μ_0} is a constant function. (The existence of measures of this type will play an important role in [27] and [28] in our analysis of $M(X)$.)

We will later make extensive use of the semi-inner product space $E_0(X)$. We begin this in the next section of this paper, and continue it in [27] and [28], where we will relate the structure of $E_0(X)$ in a detailed way to the properties of the constant $M(X)$. In this section, however, we wish to study some of the properties of $E_0(X)$ as a topological space, especially the question of the relation between the topology of $E_0(X)$ and other topologies induced on $\mathcal{M}_0(X)$ as a subspace of $\mathcal{M}(X)$. The other topologies that we discuss are the topology induced on $\mathcal{M}(X)$ and its subspaces by the measure norm $\|\cdot\|_{\mathcal{M}}$, and the weak-$*$ topology. The question of the completeness of $E_0(X)$ will be discussed later, in section 6.

Theorem 4.2. *Let (X, d) be a compact quasihypermetric space. Then for $\mu \in \mathcal{M}_0(X)$, we have $\|\mu\| \leq (D(X)/2)^{\frac{1}{2}} \|\mu\|_{\mathcal{M}}$.*

Proof. Suppose that $\mu \in \mathcal{M}_0(X)$ has Hahn–Jordan decomposition $\mu = \mu^+ - \mu^-$. Then, since μ^+ and μ^- are positive measures, we have

$$\begin{aligned}
\|\mu\|^2 &= -I(\mu) \\
&= -I(\mu^+ - \mu^-) \\
&= -I(\mu^+) - I(\mu^-) + 2I(\mu^+, \mu^-) \\
&\leq 2I(\mu^+, \mu^-) \\
&= 2\int_X \int_X d(x, y)\, d\mu^+(x) d\mu^-(x) \\
&\leq 2D(X)\mu^+(X)\mu^-(X) \\
&\leq (D(X)/2)\left(\mu^+(X) + \mu^-(X)\right)^2 \\
&= (D(X)/2)\,\|\mu\|^2_{\mathcal{M}},
\end{aligned}$$

giving the result. ☐

Corollary 4.3. *The topology of $E_0(X)$ is contained in the topology induced on $\mathcal{M}_0(X)$ by the measure norm on $\mathcal{M}(X)$.*

Remark 4.4. We note that no better constant than $(D(X)/2)^{\frac{1}{2}}$ is in general possible in the inequality above. In any space (X, d), let x and y be two points in X such that $d(x, y) = D(X)$, and set $\mu = \delta_x - \delta_y \in \mathcal{M}_0(X)$. Then it is easy to see that $\|\mu\|^2 = 2d(x, y) = 2D(X)$ and $\|\mu\|_{\mathcal{M}} = 2$, so that equality holds.

The argument above also shows, with minimal changes, that if the support of μ lies in a closed sphere of radius r, then $\|\mu\| \leq r^{1/2}\|\mu\|_{\mathcal{M}}$.

We will prove that if (X, d) is a compact quasihypermetric space, then the norm topology on $E_0(X)$ is incomparable with the topology induced on $\mathcal{M}_0(X)$ by the weak-$*$ topology on $\mathcal{M}(X)$ unless X is finite. One half of what we require is given by the following result.

Theorem 4.5. *Let (X, d) be an infinite compact quasihypermetric space. Then there exists a sequence in $\mathcal{M}_0(X)$ which converges to 0 in $E_0(X)$ but does not converge in the weak-$*$ topology or in the measure-norm topology.*

Proof. Since X is infinite and compact, it contains a non-trivial convergent sequence. Fix such a sequence, say $x_n \to x$, in which the points x_n are all distinct from x. Write $c_n = d(x, x_n)$ for all $n \in \mathbb{N}$.

Define $\mu_n \in \mathcal{M}_0(X)$ by setting $\mu_n = c_n^{-1/3}(\delta_x - \delta_{x_n})$. Then $\mu_n \to 0$ in $E_0(X)$, since

$$\|\mu_n\|^2 = -I(\mu_n) = 2c_n^{-2/3} I(\delta_x, \delta_{x_n}) = 2c_n^{-2/3} d(x, x_n) = 2c_n^{1/3} \to 0.$$

But the argument in the proof of Corollary 2.7 shows that if $\{\mu_n\}$ converges weak-$*$ then the sequence of measure norms $\|\mu_n\|_{\mathcal{M}}$ must be bounded, and since clearly $\|\mu_n\|_{\mathcal{M}} =$

$2c_n^{-1/3} \to \infty$, we conclude as required that $\{\mu_n\}$ converges neither weak-$*$ nor in norm. □

Corollary 4.6. *The topology of $E_0(X)$ does not contain the weak-$*$ topology on $\mathcal{M}_0(X)$, and is strictly weaker than the measure norm topology on $\mathcal{M}_0(X)$.*

By Corollary 2.11, there exists a weak-$*$ convergent net $\mu_\alpha \to 0$ in $\mathcal{M}_0(X)$ such that $I(\mu_\alpha) \not\to 0$ in $\mathbb{R}$; but $\|\mu_\alpha\| = \left[-I(\mu_\alpha)\right]^{\frac{1}{2}}$ by definition, so we have $\mu_\alpha \not\to 0$ in $E_0(X)$. Thus, we have:

Corollary 4.7. *The topology of $E_0(X)$ is not contained in the weak-$*$ topology on $\mathcal{M}_0(X)$.*

We now have the result claimed earlier.

Theorem 4.8. *If (X, d) is an infinite compact quasihypermetric space, then the topology of $E_0(X)$ and the weak-$*$ topology on $\mathcal{M}_0(X)$ are incomparable.*

Remark 4.9. As the discussion above shows, the convergence of a net weak-$*$ in $\mathcal{M}_0(X)$ does not imply the convergence of the net with respect to the semi-inner product space topology of $E_0(X)$. It is therefore worth noting that if $\mu_n \to \mu$ is a weak-$*$ convergent sequence in $\mathcal{M}_0(X)$, then we also have $\mu_n \to \mu$ in $E_0(X)$. Further, if $M(X) < \infty$, then weak-$*$ convergence of an arbitrary sequence in $\mathcal{M}(X)$ implies its convergence with respect to the topology of the semi-inner product space $E(X)$. These statements can be proved straightforwardly using Corollary 2.7 and Theorem 2.5.

5. $M(X)$ and the Properties of $E_0(X)$

Let (X, d) be a compact quasihypermetric space. As noted in section 4, we can define the following semi-inner product and seminorm on $\mathcal{M}_0(X)$:

$$(\mu \mid \nu) := -I(\mu, \nu), \quad \|\mu\| := (\mu \mid \mu)^{\frac{1}{2}}$$

for $\mu, \nu \in \mathcal{M}_0(X)$. Recall also from section 4 that $E_0(X)$ denotes $\mathcal{M}_0(X)$ equipped with this semi-inner product, and that $E_0(X)$ is an inner product space if and only if X is strictly quasihypermetric. We begin by collecting some elementary properties of $E_0(X)$.

Lemma 5.1. *Let (X, d) be a compact quasihypermetric space. Then we have the following.*

1. $|I(\mu, \nu)| = |(\mu \mid \nu)| \leq \|\mu\| \cdot \|\nu\|$ *for all* $\mu, \nu \in E_0(X)$.
2. $F = \{\mu \in E_0(X) : \|\mu\| = 0\}$ *is a linear subspace of* $E_0(X)$.
3. $F = \{\mu \in E_0(X) : (\mu \mid \nu) = 0$ *for all* $\nu \in E_0(X)\}$.
4. *With* $(\nu_1 + F \mid \nu_2 + F) := (\nu_1 \mid \nu_2)$, *the quotient space* $E_0(X)/F$ *becomes an inner product space.*
5. $F = \{\mu \in E_0(X) : d_\mu$ *is a constant function*$\}$.

(6) *If there exist* $\varphi \in \mathcal{M}_1(X)$ *and* $c \geq 0$ *such that* $|I(\varphi, \nu)| \leq c\|\nu\|$ *for all* $\nu \in E_0(X)$, *then for each* $\mu \in \mathcal{M}(X)$ *there exists* $c_\mu \geq 0$ *such that* $|I(\mu, \nu)| \leq c_\mu\|\nu\|$ *for all* $\nu \in E_0(X)$.

(7) *If there exist* $\mu_0 \in \mathcal{M}_1^+(X)$ *and* $c \geq 0$ *such that* $|I(\mu_0, \nu)| \leq c\|\nu\|$ *for all* $\nu \in E_0(X)$, *then there exists* $K \geq 0$ *such that* $|I(\mu, \nu)| \leq K\|\nu\|$ *for all* $\nu \in E_0(X)$ *and for all* $\mu \in \mathcal{M}_1^+(X)$.

Proof. It is well known that (1), (2), (3) and (4) hold in all semi-inner product spaces.

(5) Let μ be in F. Part (3) implies that $I(\mu, \delta_x - \delta_y) = 0$ for all $x, y \in X$, and hence that $d_\mu(x) = d_\mu(y)$ for all $x, y \in X$. Conversely, if d_μ is a constant function, then it is easy to check that $I(\mu) = \mu(X) = 0$, giving $\|\mu\| = 0$.

(6) Consider $\varphi \in \mathcal{M}_1(X)$ and $c \geq 0$ such that $|I(\varphi, \nu)| \leq c\|\nu\|$ for all $\nu \in E_0(X)$, and let $\mu \in \mathcal{M}(X)$. If $\mu(X) = 0$, then the assertion follows by (1). If $\mu(X) \neq 0$, then we have

$$\begin{aligned}\left|I\left(\frac{\mu}{\mu(X)}, \nu\right)\right| &\leq \left|I\left(\frac{\mu}{\mu(X)} - \varphi, \nu\right)\right| + \left|I(\varphi, \nu)\right| \\ &\leq \left(\left\|\frac{\mu}{\mu(X)} - \varphi\right\| + c\right) \cdot \|\nu\|,\end{aligned}$$

and hence $\left|I(\mu, \nu)\right| \leq \big(\|\mu - \mu(X)\varphi\| + c|\mu(X)|\big)\|\nu\|$ for all $\nu \in E_0(X)$.

(7) Consider $\mu_0 \in \mathcal{M}_1^+(X)$ and $c \geq 0$ such that $\left|I(\mu_0, \nu)\right| \leq c\|\nu\|$ for all $\nu \in E_0(X)$. Then for $\mu \in \mathcal{M}_1^+(X)$, we have

$$\begin{aligned}\left|I(\mu, \nu)\right| &\leq \left|I(\mu - \mu_0, \nu)\right| + \left|I(\mu_0, \nu)\right| \\ &\leq \|\mu - \mu_0\| \cdot \|v\| + c\|v\| \\ &= \Big[\big(2I(\mu, \mu_0) - I(\mu) - I(\mu_0)\big)^{\frac{1}{2}} + c\Big] \cdot \|\nu\| \\ &\leq \Big[\big(2I(\mu, \mu_0) - I(\mu_0)\big)^{\frac{1}{2}} + c\Big] \cdot \|\nu\| \\ &\leq \|\nu\| \cdot \Big[\big(2D(X) - I(\mu_0)\big)^{\frac{1}{2}} + c\Big],\end{aligned}$$

for all $\nu \in E_0(X)$. □

Theorem 5.2. *Let* X *be a compact quasihypermetric space. If there exist* $\mu \in \mathcal{M}_0(X)$ *and* $c \neq 0$ *such that* $d_\mu(x) = c$ *for all* $x \in X$, *then*

(1) X *is not strictly quasihypermetric, and*

(2) $M(X) = \infty$.

Proof. Let $\mu \in \mathcal{M}_0(X)$ and $c \neq 0$ be such that $d_\mu(x) = c$ for all $x \in X$.

(1) Clearly, $\mu \neq 0$ and $I(\mu) = 0$, and so X is not strictly quasihypermetric.

(2) Write $\mu = \alpha\mu_1 - \beta\mu_2$, where $\alpha, \beta \geq 0$ and $\mu_1, \mu_2 \in \mathcal{M}_1^+(X)$. Since $\mu \neq 0$ and $\mu(X) = 0$, we have $\alpha = \beta \neq 0$. Now $c = d_\mu(x) = \alpha d_{\mu_1 - \mu_2}(x)$ for all $x \in X$. Hence

$d_{\mu_1-\mu_2}(x) = K$ for all $x \in X$, where $K := c/\alpha \neq 0$. For each $n \geq 1$, define $\nu_n \in \mathcal{M}_1(X)$ by setting $\nu_n = n \operatorname{sign} K(\mu_1 - \mu_2) + \mu_2$. Then, since $I(\mu_1 - \mu_2) = (1/\alpha^2)I(\mu) = 0$, we have

$$\begin{aligned} I(\nu_n) &= n^2 I(\mu_1 - \mu_2) + 2n \operatorname{sign} K I(\mu_1 - \mu_2, \mu_2) + I(\mu_2) \\ &= 2n|K| + I(\mu_2) \\ &\to \infty \end{aligned}$$

as $n \to \infty$, and so $M(X) = \infty$. □

Recall from section 4 that in the case when $M(X)$ is finite, there is a natural extension of the semi-inner product on $E_0(X)$ to a semi-inner product on the whole of $\mathcal{M}(X)$, which we then denote by $E(X)$, given by

$$(\mu \mid \nu) = (M(X) + 1)\mu(X)\nu(X) - I(\mu, \nu)$$

for $\mu, \nu \in \mathcal{M}(X)$. In the following results, we find that a great deal of extra information about the spaces and operators under consideration becomes available under the assumption that $M(X)$ is finite.

Theorem 5.3. *Let (X, d) be a compact quasihypermetric space. Then the following conditions are equivalent.*

1. $M(X) < \infty$.
2. *There exist $\mu \in \mathcal{M}_1(X)$ and $c \geq 0$ such that $|I(\mu, \nu)| \leq c \|\nu\|$ for all $\nu \in E_0(X)$.*
3. *For all $\mu \in \mathcal{M}(X)$, there exists $c_\mu \geq 0$ such that $|I(\mu, \nu)| \leq c_\mu \|\nu\|$ for all $\nu \in E_0(X)$.*
4. *There exists $K \geq 0$ such that $|I(\mu, \nu)| \leq K \|\nu\|$ for all $\nu \in E_0(X)$ and for all $\mu \in \mathcal{M}_1^+(X)$.*
5. *There exists $c \geq 0$ such that $\|d_\nu\|_\infty \leq c \|\nu\|$ for all $\nu \in E_0(X)$.*
6. *There exists $c \geq 0$ such that $|I(\mu_1) - I(\mu_2)| \leq c \|\mu_1 - \mu_2\|$ for all $\mu_1, \mu_2 \in \mathcal{M}_1^+(X)$.*

Theorem 5.4. *Let (X, d) be a compact quasihypermetric space, and assume that $M(X) < \infty$. Then*

1. *$|\mu(X)| \leq \|\mu\|$ for all $\mu \in E(X)$, and*
2. *there exists $c \geq 0$ such that $\|d_\mu\|_\infty \leq c \|\mu\|$ for all $\mu \in E(X)$.*

Before proving these two theorems, we note a useful corollary and remark.

Corollary 5.5. *Let (X, d) be a compact quasihypermetric space, and assume that $M(X) < \infty$. Then $E_0(X)$ is closed in $E(X)$.*

Remark 5.6. If additionally X is strictly quasihypermetric, then we can reformulate Theorem 5.3 in the usual language of normed linear spaces. Recall that each $\mu \in \mathcal{M}(X)$ defines a linear functional $J(\mu)$ on $\mathcal{M}(X)$ by $J(\mu)(\nu) = I(\mu, \nu)$ for $\nu \in \mathcal{M}(X)$. Then for X strictly quasihypermetric, Theorem 5.3 tells us that the following conditions are equivalent.

1. $M(X) < \infty$.

(2) $J(\mu)\colon E_0(X) \to \mathbb{R}$ is bounded for some $\mu \in \mathcal{M}_1(X)$.
(3) $J(\mu)\colon E_0(X) \to \mathbb{R}$ is bounded for all $\mu \in \mathcal{M}(X)$.
(4) $\sup \|J(\mu)\| < \infty$, where μ ranges over $\mathcal{M}_1^+(X)$.
(5) The mapping $T_0\colon E_0(X) \to C(X)$ defined by $T_0(\mu) = d_\mu$ for $\mu \in E_0(X)$ is a bounded linear operator.
(6) The concave functional I is Lipschitz-continuous on $\mathcal{M}_1^+(X)$ with respect to the norm-induced metric.

We now turn to proofs of the theorems.

Proof of Theorem 5.3. Parts (6) and (7) of Lemma 5.1 imply the equivalence of conditions (2), (3) and (4).

(4) $\Rightarrow$ (1): Let $\mu \in \mathcal{M}_1(X)$. Write μ as $\mu = \alpha\mu_1 - \beta\mu_2$, with $\alpha, \beta \geq 0$, $\alpha - \beta = 1$ and $\mu_1, \mu_2 \in \mathcal{M}_1^+(X)$. Then we have

$$\begin{aligned} I(\mu) &= I(\alpha(\mu_1 - \mu_2) + \mu_2) \\ &= \alpha^2 I(\mu_1 - \mu_2) + 2\alpha I(\mu_2, \mu_1 - \mu_2) + I(\mu_2). \end{aligned}$$

If $I(\mu_1 - \mu_2) = 0$, then by assumption we have $|I(\mu_2, \mu_1 - \mu_2)| \leq K\,\|\mu_1 - \mu_2\| = 0$, and so $I(\mu) = I(\mu_2) \leq M^+(X)$. If $I(\mu_1 - \mu_2) < 0$, then we find

$$\begin{aligned} I(\mu) &= -\|\mu_1 - \mu_2\|^2\alpha^2 + 2\alpha I(\mu_2, \mu_1 - \mu_2) + I(\mu_2) \\ &= -\|\mu_1 - \mu_2\|^2 \left(\alpha - \frac{I(\mu_2, \mu_1 - \mu_2)}{\|\mu_1 - \mu_2\|^2}\right)^2 \\ &\qquad + \frac{I(\mu_2, \mu_1 - \mu_2)^2}{\|\mu_1 - \mu_2\|^2} + I(\mu_2). \end{aligned}$$

Hence, in both cases, we have

$$I(\mu) \leq I\left(\mu_2, \frac{\mu_1 - \mu_2}{\|\mu_1 - \mu_2\|}\right)^2 + M^+(X) \leq K^2 + M^+(X).$$

Therefore, we have $M(X) \leq K^2 + M^+(X) < \infty$.

(1) $\Rightarrow$ (2): Fix any $x \in X$, and assume that for all $n \in \mathbb{N}$ there exists $\nu_n \in E_0(X)$ with $|I(\delta_x, \nu_n)| > n\,\|\nu_n\|$. Suppose that $\|\nu_n\| = 0$ for some n. By Lemma 5.1 part (5) there exists $c \in \mathbb{R}$ with $d_{\nu_n}(y) = c$ for all $y \in X$. Since $M(X) < \infty$ by assumption, Theorem 5.2 implies that $c = 0$. Hence $I(\delta_x, \nu_n) = d_{\nu_n}(x) = 0$, a contradiction. Therefore, we have $\|\nu_n\| > 0$ for all $n \in \mathbb{N}$. Now, defining $\mu_n \in \mathcal{M}_1(X)$ by

$$\mu_n = \delta_x + \frac{n \operatorname{sign} I(\delta_x, \nu_n)}{\|\nu_n\|}\,\nu_n,$$

we have

$$I(\mu_n) = \frac{2n}{\|\nu_n\|}\big|I(\delta_x, \nu_n)\big| - n^2 > 2n^2 - n^2 = n^2 \to \infty$$

as $n \to \infty$, contradicting the fact that $M(X) < \infty$. Thus we have $|I(\delta_x, \nu)| \le c\|\nu\|$, for some $c \ge 0$ and for all $\nu \in E_0(X)$.

(4) $\Rightarrow$ (5): By assumption, we have $|I(\delta_x, \nu)| \le K\,\|\nu\|$, for all $\nu \in E_0(X)$ and for all $x \in X$. Since $d_\nu(x) = I(\delta_x, \nu)$ for $\nu \in E_0(X)$ and $x \in X$, we are done.

(5) $\Rightarrow$ (2): Fixing any $x \in X$, we have $|I(\delta_x, \nu)| = |d_\nu(x)| \le \|d_\nu\|_\infty \le c\,\|\nu\|$, for all $\nu \in E_0(X)$.

(4) $\Rightarrow$ (6): Let $\mu_1, \mu_2 \in \mathcal{M}_1^+(X)$. Then by assumption, we have

$$\big|I(\mu_1) - I(\mu_1, \mu_2)\big| = \big|I(\mu_1, \mu_1 - \mu_2)\big| \le K\,\|\mu_1 - \mu_2\|$$

and

$$\big|I(\mu_1, \mu_2) - I(\mu_2)\big| = \big|I(\mu_2, \mu_1 - \mu_2)\big| \le K\,\|\mu_1 - \mu_2\|,$$

and hence $|I(\mu_1) - I(\mu_2)| \le 2K\,\|\mu_1 - \mu_2\|$.

(6) $\Rightarrow$ (1): Let $\mu \in \mathcal{M}_1(X)$, and write $\mu = \alpha\mu_1 - \beta\mu_2$, where $\alpha, \beta \ge 0$, $\alpha - \beta = 1$ and $\mu_1, \mu_2 \in \mathcal{M}_1^+(X)$. Now

$$\begin{aligned} I(\mu) &= I\left(\alpha(\mu_1 - \mu_2) + \mu_2\right) \\ &= I\left(\beta(\mu_1 - \mu_2) + \mu_1\right) \\ &= -\|\mu_1 - \mu_2\|^2\alpha^2 + 2\alpha I(\mu_2, \mu_1 - \mu_2) + I(\mu_2) \\ &= -\|\mu_1 - \mu_2\|^2\beta^2 + 2\beta I(\mu_1, \mu_1 - \mu_2) + I(\mu_1). \end{aligned}$$

Therefore, $I(\mu_2, \mu_1 - \mu_2) \le 0$ implies $I(\mu) \le I(\mu_2) \le M^+(X)$ and $I(\mu_1, \mu_1 - \mu_2) \le 0$ implies $I(\mu) \le I(\mu_1) \le M^+(X)$.

Now suppose that $I(\mu_2, \mu_1 - \mu_2) > 0$ and $I(\mu_1, \mu_1 - \mu_2) > 0$. It follows that $I(\mu_1) > I(\mu_1, \mu_2) > I(\mu_2)$. Suppose that $\|\mu_1 - \mu_2\| = 0$. Now Lemma 5.1 part (5) implies the existence of some $\gamma \in \mathbb{R}$ such that $d_{\mu_1}(x) - d_{\mu_2}(x) = \gamma$ for all $x \in X$. Therefore, integrating, we have $I(\mu_1) - I(\mu_1, \mu_2) = \gamma = I(\mu_1, \mu_2) - I(\mu_2)$, which gives $I(\mu_1) - I(\mu_2) = 2\gamma$. But by assumption we have $|I(\mu_1) - I(\mu_2)| \le c\|\mu_1 - \mu_2\| = 0$, which gives $|2\gamma| \le 0$, and hence $\gamma = 0$, and it follows that $I(\mu_1) = I(\mu_1, \mu_2) = I(\mu_2)$, a contradiction.

Hence we can assume that $I(\mu_1) > I(\mu_1, \mu_2) > I(\mu_2)$ and that $\|\mu_1 - \mu_2\| > 0$. Now, as in the proof of the case (4) $\Rightarrow$ (1), we find

$$\begin{aligned} I(\mu) &\le \frac{I(\mu_2, \mu_1 - \mu_2)^2}{\|\mu_1 - \mu_2\|^2} + I(\mu_2) \\ &\le \frac{\big(I(\mu_1) - I(\mu_2)\big)^2}{\|\mu_1 - \mu_2\|^2} + I(\mu_2), \end{aligned}$$

so by assumption we have $I(\mu) \le c^2 + I(\mu_2) \le c^2 + M^+(X)$

Therefore, in either case, we have $M(X) \le c^2 + M^+(X) < \infty$. □

Proof of Theorem 5.4. (1) Consider $\mu \in E(X)$ with $\mu(X) \neq 0$. Then

$$\|\mu\|^2 = \big(M(X) + 1\big)\mu(X)^2 - I(\mu)$$

$$\begin{aligned} &= \mu(X)^2\big(M(X)+1-I(\mu/\mu(X))\big) \\ &\geq \mu(X)^2. \end{aligned}$$

Hence $|\mu(X)| \leq \|\mu\|$ for all $\mu \in E(X)$.

(2) Let $x \in X$ and $\mu \in E(X)$. Since $(\mu \mid \delta_x) = (M(X)+1)\mu(X) - I(\mu, \delta_x)$, we have

$$\begin{aligned} |d_\mu(x)| &= \big|(M(X)+1)\mu(X) - (\mu \mid \delta_x)\big| \\ &\leq \big(M(X)+1\big)\|\mu\| + \|\delta_x\| \cdot \|\mu\| \\ &= \|\mu\| \cdot \big(M(X)+1+(M(X)+1)^{\frac{1}{2}}\big), \end{aligned}$$

and so $\|d_\mu\|_\infty \leq \|\mu\| \cdot \big(M(X)+1+(M(X)+1)^{\frac{1}{2}}\big)$. □

Remark 5.7. The constant c in part (6) of Theorem 5.3 can be taken to be non-zero. This is clear if X is singleton. For non-trivial X, suppose that $c = 0$. Then $I(\mu_1) = I(\mu_2)$ for all $\mu_1, \mu_2 \in \mathcal{M}_1^+(X)$, and since $I(\delta_x) = 0$ for all $x \in X$, we have $I(\mu) = 0$ for all $\mu \in \mathcal{M}_1^+(X)$. But for any distinct $x, y \in X$, we have $\frac{1}{2}(\delta_x + \delta_y) \in \mathcal{M}_1^+(X)$, and then we have $I\big(\frac{1}{2}(\delta_x + \delta_y)\big) = d(x, y) = 0$, a contradiction. Thus we can assume that $c \neq 0$.

We can therefore interpret part (6) of the theorem as saying that $M(X) < \infty$ if and only if the following strengthened quasihypermetric property holds: there exists $L > 0$ such that

$$I(\mu_1 - \mu_2) + L \cdot \big|I(\mu_1) - I(\mu_2)\big|^{\frac{1}{2}} \leq 0$$

for all $\mu_1, \mu_2 \in \mathcal{M}_1^+(X)$. (Note that by condition (5) of Theorem 3.2, the quasihypermetric property is equivalent to the statement that $I(\mu_1 - \mu_2) \leq 0$ for all $\mu_1, \mu_2 \in \mathcal{M}_1^+(X)$.)

It turns out that with the imposition of the condition that $M(X) < \infty$, the assertion of Theorem 3.6 leads to a characterization of the strictly quasihypermetric property.

Theorem 5.8. *Let X be a non-trivial compact quasihypermetric space with $M(X) < \infty$. Then the following conditions are equivalent.*

(1) *X is strictly quasihypermetric.*
(2) *T is injective.*
(3) $\operatorname{im} T$ *is dense in $C(X)$.*

Proof. (1) ⇒ (2) follows by Theorem 3.6 and (2) ⇔ (3) by Theorem 2.4, so it remains to show that (2) ⇒ (1). Let $I(\mu) = 0$ for some $\mu \in \mathcal{M}_0(X)$. Then part (5) of Lemma 5.1 gives us $c \in \mathbb{R}$ such that $d_\mu(x) = c$ for all $x \in X$. But by Theorem 5.2 we get $c = 0$, since $M(X) < \infty$. Hence $d_\mu = 0$, and therefore, using the injectivity of T, we have $\mu = 0$. □

Remark 5.9. The condition $M(X) < \infty$ is necessary in Theorem 5.8. In Theorem 5.4 of [27], we shall construct a space X which is quasihypermetric but not strictly quasihypermetric and has $M(X) = \infty$, but for which it is easy to check that T is injective.

Consider the interval $[a,b]$ in $\mathbb{R}$, with its usual metric. For each $c \in [a,b]$, we clearly have $d_{\delta_c}(x) = |x-c|$ for all $x \in [a,b]$. It is straightforward to confirm that the linear span of these functions in $C([a,b])$ is exactly the subspace of piecewise linear continuous functions, which is dense in $C([a,b])$, and it follows that $\operatorname{im} T$ is dense in $C([a,b])$. Since $M([a,b]) = (b-a)/2 < \infty$ (see Lemma 3.5 of [5] or Corollary 3.2 of [27]), we have the following:

Corollary 5.10. *Every compact subset of $\mathbb{R}$ with the usual metric is strictly quasihypermetric.*

We noted earlier (see Theorem 3.4) the fact that each compact subset X of $\mathbb{R}^n$ is strictly quasihypermetric for all n. By Theorem 5.8, this is equivalent to the fact that $\operatorname{im} T$ is dense in $C(X)$ for each such X. The fact that the latter statement holds is a fundamental result in the theory of radial basis functions; see [29, Theorem B.1].

6. Completeness

We now address the question of the completeness of the spaces $E_0(X)$ and $E(X)$, under the assumption that $M(X)$ is finite. Recall that the semi-norms on $E_0(X)$ and $E(X)$ become norms precisely when X is strictly quasihypermetric.

Our main result is the following (cf. [22, Theorem 1.19]).

Theorem 6.1. *Let (X,d) be a compact quasihypermetric space with $M(X) < \infty$. Then the semi-inner product space $E_0(X)$ is complete if and only if X is finite.*

For the proof, we need the following result. (Recall that $T_0 \colon \mathcal{M}_0(X) \to C(X)$ is the restriction of the linear map T to the subspace $\mathcal{M}_0(X)$. Also recall that for $\mu \in \mathcal{M}(X)$, the functional $J(\mu)$ is defined by $J(\mu)(\nu) = I(\mu,\nu)$ for $\nu \in E_0(X)$.)

Lemma 6.2. *Let (X,d) be a compact quasihypermetric space with $M(X) < \infty$. Then we have the following.*

1. *The operator $\widetilde{T}_0 \colon E_0(X)/F \to C(X)$ defined by $\widetilde{T}_0(\mu+F) = T_0(\mu)$ for $\mu \in E_0(X)$ is well defined and compact.*
2. *The adjoint operator $\widetilde{T}_0' \colon \mathcal{M}(X) \to (E_0(X)/F)'$ is given by $\widetilde{T}_0'(\mu)(\nu+F) = -(\mu \mid \nu)$ for all μ in $\mathcal{M}(X)$ and $\nu \in E_0(X)$.*
3. $\dim E_0(X)/F < \infty$ *if and only if X is finite.*
4. $E_0(X)/F$ *is complete if and only if* $\dim E_0(X)/F < \infty$.

Proof. (1) Suppose that $\mu_1 + F = \mu_2 + F$ for some $\mu_1, \mu_2 \in E_0(X)$. Then $\mu_1 - \mu_2 \in F$, and by Lemma 5.1 part (5) and Theorem 5.2, we conclude that $d_{\mu_1-\mu_2} \equiv 0$, and so $\widetilde{T}_0(\mu_1+F) = \widetilde{T}_0(\mu_2+F)$.

Let $B = \big\{\nu + F \in E_0(X)/F : \|\nu+F\| \le 1\big\}$. For $\nu + F \in B$, we have

$$\big|\widetilde{T}_0(\nu+F)(x) - \widetilde{T}_0(\nu+F)(y)\big| \;=\; \big|d_\nu(x) - d_\nu(y)\big|$$

$$\begin{aligned} &= \left|(\nu \mid \delta_x - \delta_y)\right| \\ &\leq \|\nu\| \cdot \|\delta_x - \delta_y\| \\ &= \|\nu + F\| \cdot \big(2d(x,y)\big)^{\frac{1}{2}} \\ &\leq \big(2d(x,y)\big)^{\frac{1}{2}}, \end{aligned}$$

for all $x, y \in X$.

By Theorem 5.3 part (5), we have, for each $\nu + F \in B$,

$$\|\widetilde{T}_0(\nu + F)\|_\infty = \|T_0(\nu)\|_\infty = \|d_\nu\|_\infty \leq c\|\nu\| = c\|\nu + F\| \leq c,$$

for some constant c. The Arzelà-Ascoli Theorem now implies that $\widetilde{T}_0(B)$ is relatively compact in $C(X)$, and therefore $\widetilde{T}_0$ is compact.

(2) By definition,we have $\widetilde{T}_0'(\mu)(\nu+F) = \mu(\widetilde{T}_0(\nu+F)) = \mu(T_0(\nu)) = \mu(d_\nu) = I(\mu,\nu) = -(\mu \mid \nu)$ for all $\mu \in \mathcal{M}(X)$ and $\nu \in E_0(X)$.

(3) Of course, if X is finite, then $\dim E_0(X)/F < \infty$, so let us assume that $\dim E_0(X)/F = n$ for some natural number n (note that $n = 0$ obviously implies that X is a one-point space). Thus there are $\mu_1, \ldots, \mu_n \in E_0(X)$ such that

$$E_0(X)/F = [\mu_1 + F, \ldots, \mu_n + F].$$

Now consider $\mu \in E_0(X)$. Then there exist $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ such that

$$\mu + F = \alpha_1(\mu_1 + F) + \cdots + \alpha_n(\mu_n + F),$$

and we have $\mu - \sum_{i=1}^n \alpha_i\mu_i \in F$. By Lemma 5.1 part (5) and Theorem 5.2, it follows that $d_{\mu - \sum_{i=1}^n \alpha_i\mu_i} = 0$. Therefore, $d_\mu \in [d_{\mu_1}, \ldots, d_{\mu_n}]$, and we conclude that $\operatorname{im} T_0 = [d_{\mu_1}, \ldots, d_{\mu_n}]$. But $\operatorname{im} T = [\operatorname{im} T_0, d_{\delta_x}]$ for each fixed $x \in X$, since $d_\nu = d_{\nu - \nu(X)\delta_x} + \nu(X) \cdot d_{\delta_x}$ for each $\nu \in \mathcal{M}(X)$, and so $\dim(\operatorname{im} T) < \infty$. Therefore, X is finite, by Theorem 2.1.

(4) Clearly $E_0(X)/F$ is complete if $\dim E_0(X)/F < \infty$, so let us assume that $E_0(X)/F$ is complete. The Riesz representation theorem, with Lemma 6.2 part (2), implies that $(E_0(X)/F)' = \widetilde{T}_0'(\mathcal{M}_0(X))$, since $\widetilde{T}_0'(\mu)(\nu + F) = -(\mu \mid \nu) = (-\mu + F \mid \nu + F)$ for all $\mu, \nu \in E_0(X)$. Therefore, $\widetilde{T}_0'\colon \mathcal{M}(X) \to (E_0(X)/F)'$ is compact, since $\widetilde{T}_0$ is compact by part (1), and $\operatorname{im} \widetilde{T}_0' = (E_0(X)/F)'$, which is by assumption complete.

But it is well known (see for example Theorem 7.4 in [33]) that this situation implies that $\operatorname{im} \widetilde{T}_0'$ is of finite dimension, and hence $(E_0(X)/F)'$ is of finite dimension. Therefore, $E_0(X)/F$ is of finite dimension, and so X is finite, by part (3). □

Corollary 6.3. *With the hypotheses of the lemma, $E_0(X)/F$ is complete if and only if X is finite.*

Proof of Theorem 6.1. If X is finite, the required conclusion is trivial, so let us assume that $E_0(X)$ is complete. Let $(\mu_n + F)_{n \geq 1}$ be a Cauchy sequence in $E_0(X)/F$, where $\mu_n \in E_0(X)$ for all n. Since $\|\mu_n - \mu_m\| = \|(\mu_n + F) - (\mu_m + F)\|$ for all n and m, we conclude that $(\mu_n)_{n\geq 1}$ is a Cauchy sequence in $E_0(X)$, and hence, by assumption, there exists $\mu \in E_0(X)$ (not necessarily unique) such that $\|\mu_n - \mu\| \to 0$ as $n \to \infty$. Hence $\|(\mu_n + F) - (\mu + F)\| \to 0$ as $n \to \infty$. Therefore, $E_0(X)/F$ is complete, and hence, by Corollary 6.3, X is finite. □

Corollary 6.4. *Let (X, d) be a compact strictly quasihypermetric space with $M(X) < \infty$. Then the inner product space $E_0(X)$ is a Hilbert space if and only if X is finite.*

Finally, we apply an earlier result to extend Theorem 6.1 to the space $E(X)$. Indeed, by Corollary 5.5, $E_0(X)$ is closed in $E(X)$ when $M(X) < \infty$, so the completeness of $E(X)$ would imply the completeness of $E_0(X)$, and we therefore have the following.

Corollary 6.5. *Let (X, d) be a compact quasihypermetric space with $M(X) < \infty$. Then the semi-inner product space $E(X)$ is complete if and only if X is finite.*

There is also of course a result paralleling Corollary 6.4 for $E(X)$ in the strictly quasihypermetric case.

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