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# AN INDEX THEOREM FOR TOEPLITZ OPERATORS ON TOTALLY ORDERED GROUPS

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ABSTRACT. We show that for every totally ordered group $\Gamma$ and invertible function $f \in C(\widehat{\Gamma})$ which does not have a logarithm, there is a representation in which the Toeplitz operator $T_f$ is a Breuer-Fredholm operator with nonzero index; this representation is the GNS-representation associated to a natural unbounded trace on the Toeplitz algebra $\mathcal{T}(\Gamma)$.

The Toeplitz algebra $\mathcal{T}(\Gamma)$ of a totally ordered abelian group $\Gamma$ is the $C^*$-algebra of operators on the Hardy space $H^2(\widehat{\Gamma})$ generated by the compressions $T_f$ of the multiplication operators $M_f$ for $f \in C(\widehat{\Gamma})$. When $\Gamma = \mathbf{Z}$, the Toeplitz operator $T_f$ is Fredholm if and only if $f \in C(\mathbf{T})^{-1}$, and then its Fredholm index is minus the winding number of $f$ about 0. For other $\Gamma, T_f$ is Fredholm if and only if it is invertible, and to get an interesting index theorem, one has to change one's concept of Fredholm operator.

Coburn, Douglas, Schaeffer and Singer [3] showed that, if $\Gamma$ is a subgroup of $\mathbf{R}$, there is a representation $\pi$ of $\mathcal{T}(\Gamma)$ such that $\pi(T_f)$ is a Breuer-Fredholm element of the $\mathrm{II}_\infty$-factor $\pi(\mathcal{T}(\Gamma))''$ whenever $f$ is invertible in $C(\widehat{\Gamma})$, and gave a formula for the index. Subsequently Murphy proved a version of this index theorem for more general ordered groups [8], but only a restricted class of $T_f$ with $f$ invertible are Fredholm in his representation. Here we extend Murphy's result in two directions. First of all, we prove that for each totally ordered abelian group $\Gamma$ and each invertible $f \in C(\widehat{\Gamma})$, there is a representation $\pi$ of $\mathcal{T}(\Gamma)$ in which $\pi(T_f)$ is Breuer-Fredholm. Secondly, we show that these representations are the GNS-representations of certain unbounded traces on the Toeplitz algebra $\mathcal{T}(\Gamma)$, thus explaining more clearly how the representations used in [3], [8] are canonically associated to the Toeplitz algebra.

Murphy has shown elsewhere [9] that a trace $\tau$ on a $C^*$-algebra $B$ naturally gives rise to an index theory of Fredholm elements of $B$. This is not obvious: since $C^*$-algebras need not contain projections, the obvious definitions of dimension as the trace of a projection and index as the difference of two dimensions are not available. His elegant solution, which deserves to be much better known, is to declare $b \in B$ to be Fredholm if it has a inverse $c$ modulo the ideal $\mathcal{M}_\tau$ of elements of finite trace, and then define the index of $b$ to be $\tau(bc - cb)$. We have couched our index theorem

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first in Murphy's $C^*$-algebraic context, and then apply the GNS-construction to convert to statements about the Breuer-Fredholm index.

We begin by constructing traces on the Toeplitz algebra $\mathcal{T}(\Gamma)$. We realise $\mathcal{T}(\Gamma)$ as a corner in a crossed product $B_\Gamma \times \Gamma$, construct invariant traces on $B_\Gamma$ from Archimedean subquotients of $\Gamma$, and then use a theorem of Zeller-Meier to extend them to traces on $B_\Gamma \times \Gamma$. Our construction is based on an embedding of an order ideal as a subgroup of $\mathbf{R}$; in retrospect, we have merely used the language of traces to abbreviate Murphy's construction of an invariant measure in [8, pp.102–108]. Our index theorem is in §2; the idea is to associate to each invertible function $f$ in $C(\widehat{\Gamma})$ an Archimedean subquotient of $\Gamma$, and then use the construction of §1 to produce traces on $\mathcal{T}(\Gamma)$ for which $T_f$ has nonzero $\tau$-index. The Breuer-Fredholm index version of our theorem is in §3, where we have also taken some care to relate our result to the original theorem of [3].

## 1. Traces on the Toeplitz algebra

Let $\Gamma$ be a discrete totally ordered abelian group. For $x \in \Gamma$, we define $1_x \in \ell^\infty(\Gamma)$ by

$$1_x(y) = \begin{cases} 1 & \text{if } y \geq x, \\ 0 & \text{if } y < x. \end{cases}$$

Since $1_x 1_y = 1_{\max(x,y)}$, the closed span $B_\Gamma := \overline{\operatorname{sp}}\{1_x : x \in \Gamma\}$ is a $C^*$-subalgebra of $\ell^\infty(\Gamma)$, and the action of $\Gamma$ by translation on $\ell^\infty(\Gamma)$ restricts to an action $\alpha : \Gamma \to \operatorname{Aut} B_\Gamma$ such that $\alpha_x(1_y) = 1_{x+y}$. If $\lambda : \Gamma \to U(\ell^2(\Gamma))$ is the left regular representation of $\Gamma$, and $M$ is the action of $\ell^\infty(\Gamma)$ by multiplication operators, then $M \times \lambda$ is a faithful representation of $B_\Gamma \times_\alpha \Gamma$ which carries the corner associated to the projection $i_{B_\Gamma}(1_0)$ onto $\mathcal{T}(\Gamma) = C^*(M(1_0)\lambda_x M(1_0))$. This is Theorem 3.14 of [6]; alternatively, one can realise $\mathcal{T}(\Gamma)$ as a semigroup crossed product $B_{\Gamma^+} \times_\alpha \Gamma^+$ as in [2], and deduce from the general theory of [1] that $B_{\Gamma^+} \times_\alpha \Gamma^+ = i_{B_\Gamma}(1_0)\,(B_\Gamma \times_\alpha \Gamma)\,i_{B_\Gamma}(1_0)$.

To obtain traces on $\mathcal{T}(\Gamma)$, we first need to construct invariant traces on $B_\Gamma$. When $\Gamma$ is a subgroup of $\mathbf{R}$, the linearisation of the map $1_x \mapsto \chi_{[x,\infty)}$ is an isometric isomorphism of $\operatorname{sp}\{1_x : x \in \Gamma\}$ onto a $^*$-subalgebra of $L^\infty(\mathbf{R})$, which therefore extends to an embedding $\Phi$ of $B_\Gamma$ as a $C^*$-subalgebra of $L^\infty(\mathbf{R})$. Thus we can define a trace $\sigma$ on $B_\Gamma$ by integrating with respect to Lebesgue measure: $\sigma(f) = \int_{\mathbf{R}} \Phi(f)\,dm$. More generally, we have:

**Proposition 1.** *Suppose $I$ is an Archimedean ordered ideal of $\Gamma$, so that there is an order isomorphism $\phi$ of $I$ into $\mathbf{R}$. For every such $\phi$, there is an invariant semifinite lower semicontinuous trace $\sigma$ on $B_\Gamma$ such that $\sigma(1_x - 1_y) = \phi(y - x)$ whenever $y - x \in I$.*

*Proof.* Let $B_I^+ = \overline{\operatorname{sp}}\{1, 1_x : x \in I\}$ be the $C^*$-algebra obtained by adjoining an identity to $B_I$. The formula $\Phi(1_x) = \chi_{[\phi(x),\infty)}$ extends by linearity to an isometric homomorphism of $\operatorname{sp}\{1, 1_x : x \in I\}$ onto a $*$-subalgebra of $L^\infty(\mathbf{R})$, and hence to a homomorphism $\Phi$ on all of $B_I^+$. Next, choose a complete set of coset representatives $\{x_r\}$ for $\Gamma/I$. For each $r$, define $\Phi_r : B_\Gamma \to L^\infty(\mathbf{R})$ by

$$\Phi_r(f) := \Phi\big(\alpha_{x_r}^{-1}(f|_{x_r+I})\big) \quad \text{for } f \in B_\Gamma;$$

since every $f \in B_\Gamma$ has restriction $f|_I$ in $B_I^+$, and $\alpha_{x_r}^{-1}(f\big|_{x_{r+I}}) = \alpha_{x_r}^{-1}(f)|_I$, this definition of $\Phi_r$ makes sense. Now define $\sigma$ on $B_\Gamma$ by $\sigma(f) = \sum_r \int_{\mathbf{R}} \Phi_r(f)dm$. Note that, although $\Phi_r(f)$ depends on the choice of coset representatives $\{x_r\}$, the expression $\int_{\mathbf{R}} \Phi_r(f)\,dm$ does not, because Lebesgue measure is translation invariant. Similarly, the function $\sigma$ is translation invariant: translating $\sigma$ by $x \in \Gamma$ may move the cosets around, but $\{x_r + x\}$ is still a complete set of coset representatives of $I$ in $\Gamma$. Because each $f \mapsto \int_{\mathbf{R}} \Phi_r(f)dm$ is semifinite, so is $\sigma$.

To check that $\sigma$ is lower semicontinuous, let $\{f_n\}$ be an increasing sequence of positive functions in $B_\Gamma$ converging to $f$. Then the sequence $s_n := \sum_r \Phi_r(f_n)$ is increasing with $s_n \to \sum_r \Phi_r(f)$. Thus by the Monotone Convergence Theorem, $\int s_n dm \to \int \sum_r \Phi_r(f)dm$, and hence $\sigma(f_n) \to \sigma(f)$ by two applications of Tonelli's Theorem.

Finally, if $y - x \in I$, then $x$ and $y$ are in the same coset $x_s + I$ for some $s \in \Gamma/I$. Therefore $\Phi_r(1_x - 1_y) = 0$ for $r \neq s$, and

$$\begin{aligned}\sigma(1_x - 1_y) &= \int_{\mathbf{R}} \Phi_s(1_x - 1_y)\,dm = \int_{\mathbf{R}} \Phi(1_{x-x_s} - 1_{y-x_s})\,dm \\ &= \int_{\mathbf{R}} \chi_{[\phi(x-x_s),\phi(y-x_s))}\,dm \\ &= \phi\left((y-x_s)-(x-x_s)\right) = \phi(y-x),\end{aligned}$$

as required. □

**Corollary 2.** *If $I$ is an Archimedean order ideal in $\Gamma$ and $\phi$ is an isomorphism of $I$ into $\mathbf{R}$, then there is a semifinite lower semicontinuous trace $\tau$ on $\mathcal{T}(\Gamma)$ such that $\tau(T_xT_x^* - T_yT_y^*) = \phi(y-x)$ whenever $x, y \in \Gamma$ satisfy $y - x \in I$.*

*Proof.* Proposition 1 gives a semifinite lower semicontinuous trace $\sigma$ on $B_\Gamma$. By [11, 9.3], the composition $\sigma \circ E$ of $\sigma$ with the conditional expectation $E : B_\Gamma \times_\alpha \Gamma \to B_\Gamma$ is a lower semicontinuous trace on $B_\Gamma \times_\alpha \Gamma$. Let $\tau$ be the restriction of $\sigma \circ E$ to the corner $i_{B_\Gamma}(1_0)(B_\Gamma \times \Gamma)i_{B_\Gamma}(1_0)$. Then $\tau$ is a semifinite lower semicontinuous trace on $\mathcal{T}(\Gamma)$, and whenever $y - x \in I$, we have

$$\tau(T_xT_x^* - T_yT_y^*) = \sigma \circ E\big(i_{B_\Gamma}(1_x - 1_y)\big) = \sigma(1_x - 1_y) = \phi(y-x).$$

□

## 2. A $C^*$-algebraic index theorem

Suppose $\tau$ is a trace on a unital $C^*$-algebra $B$, and

$$\mathcal{M}_\tau := \operatorname{sp}\{b \in B^+ : \tau(b) < \infty\}$$

is the (non-closed) $^*$-ideal of elements of finite trace. An element $b$ of $B$ is *Fredholm relative to* $\tau$ if there is an element $c \in B$ such that $1 - bc$ and $1 - cb$ belong to $\mathcal{M}_\tau$, and the index is then $\tau\text{-ind}(b) := \tau(bc - cb)$. Murphy has shown that most of the usual index theory of Fredholm operators carries over to this setting [9, §3]. (The possible exception is the vanishing of the index for normal elements of $B$, which is open.)

**Theorem 3.** *Suppose $\Gamma$ is a totally ordered group, and $f \in C(\widehat{\Gamma})^{-1}$ does not have a logarithm in $C(\widehat{\Gamma})$. Then there is a semifinite lower semicontinuous trace $\tau$ on the Toeplitz algebra $\mathcal{T}(\Gamma)$ such that $T_f$ is Fredholm relative to $\tau$ and $\tau\text{-}\operatorname{ind} T_f \neq 0$.*

*Proof.* Since $\Gamma$ is totally ordered, it is torsion-free, and hence has connected dual $\widehat{\Gamma}$. Thus by a theorem of van Kampen (e.g. [7, Theorem 1.1]), there exist $x \in \Gamma$ and $g \in C(\widehat{\Gamma})$ such that $f = \epsilon_x e^g$ and $x \neq 0$. The set

$$I_x := \{y \in \Gamma : ny \leq |x| \text{ for all } n \in \mathbf{Z}\}$$

is an order ideal of $\Gamma$. The quotient $\Gamma/I_x$ is itself a totally ordered group in which

$$z + I_x \geq 0 \iff z + y \geq 0 \text{ for some } y \in I_x;$$

let $p : \Gamma \to \Gamma/I_x$ be the quotient map. The image $p(x)$ of $x$ is almost by definition a finite element of $\Gamma/I_x$, and is nonzero because $x \notin I_x$; in particular, the order ideal $F(\Gamma/I_x)$ of finite elements is nonzero. Since $F(\Gamma/I_x)$ is always Archimedean, there is an embedding $\phi$ of $F(\Gamma/I_x)$ in $\mathbf{R}$, and hence by Corollary 2 there is a trace $\tau_1$ on $\mathcal{T}(\Gamma/I_x)$ such that $\tau_1(T_{p(x)}T_{p(x)}^* - 1) = -\phi(p(x))$. From the universal property of $\mathcal{T}(\Gamma)$, we deduce that there is a canonical map $q : \mathcal{T}(\Gamma) \to \mathcal{T}(\Gamma/I_x)$ satisfying $q(T_x) = T_{p(x)}$, and then $\tau := \tau_1 \circ q$ is a lower semicontinuous trace on $\mathcal{T}(\Gamma)$ satisfying $\tau(T_x T_x^* - 1) = -\phi(p(x))$.

To see that $T_f$ is $\tau$-Fredholm, suppose first that $x \geq 0$. Then $T_f = T_{e^g} T_x$. It is quite easy to see that $T_{e^g}$ is invertible (for example, [7, p.4]), and hence it is trivially $\tau$-Fredholm. Since $T_x^* T_x = 1$, the equation $\tau(1 - T_x T_x^*) = \phi(p(x))$ implies that $T_x^*$ is a partial inverse for $T_x$ relative to $\tau$, and hence $T_x$ is $\tau$-Fredholm. We deduce that the product $T_f = T_{e^g} T_x$ is $\tau$-Fredholm, with

$$(*) \qquad \tau\text{-ind}\, T_f = \tau\text{-ind}\, T_{e^g} + \tau\text{-ind}\, T_x = 0 - \phi(p(x)) = -\phi(p(x)),$$

which is nonzero because $x \notin I_x$ and $\phi$ is injective on $\Gamma/I_x$. If $x \leq 0$, we can apply the previous case to $T_f^* = T_{\overline{f}}$, and the result follows. □

*Remark 4.* Since the traces we constructed in §1 depend on the choice of an isomorphism $\phi$ of an abstract Archimedean ideal $I$ into $\mathbf{R}$, and multiplying $\phi$ by any $c \in (0, \infty)$ gives another such isomorphism, the numerical value of the index is not significant. If, however, we start with a subgroup $\Gamma$ of $\mathbf{R}$, then in some sense this choice has already been made, and our construction gives the trace $\sigma$ on $B_\Gamma$ corresponding to Lebesgue integration on $L^\infty(\mathbf{R}) \supset B_\Gamma$. Composing with the dual embedding of $\mathbf{R}$ in $\widehat{\Gamma}$ converts functions in $C(\widehat{\Gamma})$ to almost periodic functions on $\mathbf{R}$; if $f = \epsilon_x e^g$ for some $x \in \Gamma$, then the corresponding function $F$ on $\mathbf{R}$ has the form $t \mapsto e^{ixt} e^{G(t)}$, where $G$ is the almost periodic function $g|_{\mathbf{R}}$, and one can recover $x$ as the mean motion

$$\lim_{t \to \infty} \frac{1}{2t} \big( \arg F(t) - \arg F(-t) \big) = \lim_{t \to \infty} \frac{1}{2t} \big( xt - iG(t) - (-xt - iG(-t)) \big)$$

of the almost periodic function $F$.

The Toeplitz algebra of a subgroup $\Gamma$ of $\mathbf{R}$ has a faithful representation as Wiener-Hopf operators on $L^2[0, \infty)$; the Toeplitz operator on $H^2(\widehat{\Gamma})$ with symbol $f \in C(\widehat{\Gamma})$ is carried into the Wiener-Hopf operator $W_F$ with symbol $F := f|_{\mathbf{R}} \in AP(\mathbf{R})$. (It follows from a theorem of Douglas [5] that this is a faithful representation of $\mathcal{T}(\Gamma)$; alternatively, note that $B_\Gamma^+$ acts faithfully on $L^2[0, \infty)$ and apply the main theorem of [2].) Thus we can deduce from $(*)$ that there is a natural trace on the Wiener-Hopf $C^*$-algebra on $L^2[0, \infty)$ for which the index of $W_F$ is minus the mean motion of $F$.

## 3. The Breuer-Fredholm index theorem

**Proposition 5.** *Suppose $\sigma : B \to [0, \infty]$ is a semifinite lower semicontinuous trace on a $C^*$-algebra $B$, and $\pi_\sigma$ is the associated GNS-representation of $B$ on $H_\sigma$. Then there is a faithful semifinite normal trace $\widetilde{\sigma}$ on $\pi_\sigma(B)'' \subset B(H_\sigma)$ such that $\widetilde{\sigma}(\pi_\sigma(b)) = \sigma(b)$ for $b \in B^+$.*

*Proof.* Let $\mathcal{N}_\sigma := \{b \in B : \sigma(b^*b) < \infty\}$. Then $\sigma$ extends uniquely to a functional $\sigma'$ on $\mathcal{M}_\sigma := \mathcal{N}_\sigma^2$ which agrees with $\sigma$ on $\mathcal{M}_\sigma^+ = \mathcal{M}_\sigma \cap B^+$ [4, 6.1.2]. It follows from [4, §6.4 and 6.2] that if $N_\sigma := \{b \in B : \sigma(b^*b) = 0\}$, then $A := \mathcal{N}_\sigma / N_\sigma$ is a Hilbert algebra with respect to the inner product $(a + N_\sigma | b + N_\sigma) := \sigma'(b^*a)$. The Hilbert space completion $H$ used in [4] is precisely $H_\sigma$, the operator $\pi_\sigma(b)$ is the operator $U_b$ on $H$, and $U(A) = \pi_\sigma(B)''$. Thus [4, A60] says that there is a faithful semifinite normal trace $\widetilde{\sigma}$ on $\pi_\sigma(B)''$, and for each $b \in \mathcal{N}_\sigma$ we have

$$\widetilde{\sigma}(\pi_\sigma(b^*b)) = \widetilde{\sigma}(\pi_\sigma(b)^*\pi_\sigma(b)) = (b + N_\sigma | b + N_\sigma) = \sigma(b^*b) < \infty;$$

in particular this implies that $\pi_\sigma(\mathcal{N}_\sigma) \subset \mathcal{N}_{\widetilde{\sigma}}$. One further deduces from [4, A60] that the unique extension $\widetilde{\sigma}'$ of $\widetilde{\sigma}$ to $\mathcal{M}_{\widetilde{\sigma}}$ satisfies

$$\widetilde{\sigma}'(\pi_\sigma(b)^*\pi_\sigma(a)) = (a + N_\sigma | b + N_\sigma) = \sigma'(b^*a),$$

so that for every $x \in \mathcal{M}_\sigma$ we have $\sigma'(x) = \widetilde{\sigma}'(\pi_\sigma(x))$. It follows from [4, §6.6] that $\sigma = \widetilde{\sigma}' \circ \pi_\sigma$. □

Recall from [10] that if $\tau$ is a semifinite normal trace on a von Neumann algebra $N$, then an operator $T \in N$ is *Fredholm relative to* $\tau$ if the projection $N_T$ on $\ker T$ has $\tau(N_T) < \infty$, and there is a projection $E \in N$ such that $\tau(E) < \infty$ and the range of $(1 - E)$ is contained in the range of $T$; the *Breuer-Fredholm $\tau$-index* of $T$ is then $\tau\text{-ind}\, T := \tau(N_T) - \tau(N_{T^*})$.

**Theorem 6.** *Let $\Gamma$ be a totally ordered abelian group, and suppose that $f \in C(\widehat{\Gamma})$ does not have a logarithm in $C(\widehat{\Gamma})$. Then there are a representation $\pi$ of the Toeplitz algebra $\mathcal{T}(\Gamma)$ and a trace $\mu$ on $\pi(\mathcal{T}(\Gamma))''$ such that $\pi(T_f) \in \pi(\mathcal{T}(\Gamma))''$ is Fredholm relative to $\mu$, with Breuer-Fredholm index $\mu\text{-}\operatorname{ind}(\pi(T_f)) \neq 0$.*

*Proof.* We proceed as in Theorem 3, writing $f = \epsilon_x e^g$, etc. We then take $\pi$ to be the GNS-representation $\pi_\tau$ associated to the trace $\tau$ in that theorem, and $\mu := \widetilde{\tau}$. Then $\pi_\tau(T_x)$ is an isometry with range projection $\pi_\tau(T_x T_x^*)$ satisfying

$$\widetilde{\tau}(1 - \pi_\tau(T_x T_x^*)) = \tau(1 - T_x T_x^*) = \phi(p(x));$$

since $T_{e^g}$ is invertible, we deduce that $\pi_\tau(T_f)$ is Breuer-Fredholm with $\mu\text{-ind}\, \pi_\tau(T_f) = -\phi(p(x)) \neq 0$, as required. □

*Remark 7.* If $\Gamma$ is a subgroup of $\mathbf{R}$, we can view $B_\Gamma$ as a $C^*$-subalgebra of $L^\infty(\mathbf{R})$, and the trace $\sigma$ on $B_\Gamma$ is then given by $\sigma(f) := \int f\, dm$ (see Remark 4). Thus the trace $\tau := \sigma \circ E$ of Corollary 2 is given on positive elements of $C_c(\Gamma, B_\Gamma)$ by

$$\begin{aligned}\sigma \circ E(f^*f) &= \sigma(f^*f(e)) = \sigma\Big(\sum_s f^*(s)\alpha_s(f(s^{-1}))\Big)\\ &= \sum_s \sigma\big(\alpha_s\big(\overline{f(s^{-1})}f(s^{-1})\big)\big) = \sum_t \sigma\big(|f(t)|^2\big)\\ &= \sum_t \int_{\mathbf{R}} |f(t)|^2\, dm;\end{aligned}$$

we deduce that $H_{\sigma\circ E} = \ell^2(\Gamma, L^2(\mathbf{R}))$. Since the representation $\pi_{\sigma\circ E}$ of $C_c(\Gamma, B_\Gamma) \subset B_\Gamma \times \Gamma$ extends the action by left multiplication on $C_c(\Gamma, B_\Gamma) \subset \ell^2(\Gamma, L^2(\mathbf{R}))$, we can see by inspection that $\pi_{\sigma\circ E}$ is the integrated form of the covariant representation $(1 \otimes M, \lambda \otimes \lambda)$ of $(B_\Gamma, \Gamma, \alpha)$ on $\ell^2(\Gamma) \otimes L^2(\mathbf{R}) = \ell^2(\Gamma, L^2(\mathbf{R}))$. This is precisely the representation used in [3], and in view of Remark 4, our index theorem reduces in this case to that of [3].

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