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Abstract

Suppose a C * -algebra A acts by adjointable operators on a Hilbert A -module X. Pimsner constructed a C * -algebra ? $_X$ which includes, for particular choices of X , crossed products of A by Z , the Cuntz algebras ? $_n$, and the CuntzKrieger algebras ? $_B$. Here we analyse the representations of the corresponding Toeplitz algebra. One consequence is a uniqueness theorem for the ToeplitzCuntz-Krieger algebras of directed graphs, which includes Cuntz's uniqueness theorem for ? $_\infty$.

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The Toeplitz Algebra of a Hilbert Bimodule

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ABSTRACT. Suppose a C^*-algebra A acts by adjointable operators on a Hilbert A-module X. Pimsner constructed a C^*-algebra $\mathcal{O}_X$ which includes, for particular choices of X, crossed products of A by $\mathbb{Z}$, the Cuntz algebras $\mathcal{O}_n$, and the Cuntz-Krieger algebras $\mathcal{O}_B$. Here we analyse the representations of the corresponding Toeplitz algebra. One consequence is a uniqueness theorem for the Toeplitz-Cuntz-Krieger algebras of directed graphs, which includes Cuntz's uniqueness theorem for $\mathcal{O}_\infty$.

A Hilbert bimodule X over a C^*-algebra A is a right Hilbert A-module with a left action of A by adjointable operators. The motivating example comes from an automorphism α of A: take $X_A = A_A$, and define the left action of A by $a \cdot b := \alpha(a)b$. In [23], Pimsner constructed a C^*-algebra $\mathcal{O}_X$ from a Hilbert bimodule X in such a way that the $\mathcal{O}_X$ corresponding to an automorphism α is the crossed product $A \times_\alpha \mathbb{Z}$. He also produced interesting examples of bimodules which do not arise from automorphisms or endomorphisms, including bimodules over finite-dimensional commutative C^*-algebras for which the corresponding $\mathcal{O}_X$ are the Cuntz-Krieger algebras. The Cuntz algebra $\mathcal{O}_n$ is $\mathcal{O}_X$ when ${}_{\mathbb{C}}X_{\mathbb{C}}$ is a Hilbert space of dimension n and the left action of $\mathbb{C}$ is by multiples of the identity.

Here we use methods developed in [18, 9] for analysing semigroup crossed products to study Pimsner's algebras. These methods seem to apply more directly to Pimsner's analogue of the Toeplitz-Cuntz algebras rather than his analogue $\mathcal{O}_X$ of the Cuntz algebras. Nevertheless, our results yield new information about the Cuntz-Krieger algebras of some infinite graphs, giving a whole class of these algebras which behave like $\mathcal{O}_\infty$.

The uniqueness theorems for C^*-algebras generated by algebraic systems of isometries say, roughly speaking, that all examples of a given system in which the isometries are non-unitary generate isomorphic C^*-algebras. We can approach such a theorem by introducing a C^*-algebra which is universal for systems of the given type, and then characterising its faithful representations. Here the systems consist of representations ψ of X and π of A on the same Hilbert space

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which convert the module actions and the inner product to operator multiplication; we call these Toeplitz representations of X. (The partial isometries and isometries appearing in more conventional systems are obtained by applying ψ to the elements of a basis for X.) In Section 1, we discuss these Toeplitz representations, show that there is a universal C^*-algebra $\mathcal{T}_X$ generated by a Toeplitz representation, and prove some general results relating these representations to the induced representations of Rieffel.

Our first theorem is very much in the spirit of other theorems about C^*-algebras generated by systems of isometries: it gives a condition on a Toeplitz representation (ψ, π) which implies that the corresponding representation $\psi \times \pi$ of $\mathcal{T}_X$ is faithful (Theorem 2.1). In broad terms, this condition says that the ranges of all the operators $\psi(x)$ should leave enough room for A to act faithfully. The proof follows standard lines: we use a canonical gauge action γ to construct an expectation onto a core $\mathcal{T}_X^\gamma$, and then show that both the core and the expectation are implemented faithfully in the given Toeplitz representation.

When the bimodule ${}_{\mathbb{C}}X_{\mathbb{C}}$ is an infinite-dimensional Hilbert space, Theorem 2.1 says that a family $\{S_i : i \in \mathbb{N}\}$ of isometries on $\mathcal{H}$ with orthogonal ranges generates a faithful representation of $\mathcal{O}_\infty$ if the ranges $S_i\mathcal{H}$ do not span $\mathcal{H}$. However, more is true: Cuntz proved that *every* family of isometries with orthogonal ranges generates a faithful representation of $\mathcal{O}_\infty$. Our main theorem is an improvement of Theorem 2.1 which gives the full strength of Cuntz's result (Theorem 3.1): we assume that X has a direct-sum decomposition $X = \bigoplus_\lambda X^\lambda$, but only ask that A acts faithfully on $(\bigoplus_{\lambda \in F} \psi(X^\lambda)\mathcal{H})^\perp$ for every finite subset F of indices. For ${}_{\mathbb{C}}X_{\mathbb{C}}$, the decomposition is parametrised by a basis of X, and the hypothesis asks that $\sum_{i=1}^n S_i S_i^* < 1$ for all finite n, which is trivially true if there are infinitely many S_i. To prove Theorem 3.1, we use the direct-sum decomposition to go further into the core; we need the special case in Theorem 2.1 to construct the expectation which does this.

The new applications of our theorem involve the C^*-algebras of directed graphs. For a locally finite graph E, the C^*-algebra $C^*(E)$ is by definition universal for Cuntz-Krieger E-families: families $\{S_f\}$ of partial isometries, parametrised by the edge set E^1 of the graph, and satisfying in particular

$$S_e^* S_e = \sum_{\{f : s(f) = r(e)\}} S_f S_f^*,$$

where $r, s : E^1 \to E^0$ send edges to their range and source vertices [17, 16]. The graph algebra $C^*(E)$ can be realised in a very natural way as the Cuntz-Pimsner algebra $\mathcal{O}_X$ of a bimodule X over the algebra $A = c_0(E^0)$ (see [24, 14] and Example 1.2 below). For graphs in which vertices can emit infinitely many edges, the Cuntz-Krieger relations involve infinite sums which do not make sense in a C^*-algebra, and it is not clear how to best define a useful notion of graph C^*-algebra. We show that this problem disappears if all vertices emit infinitely many edges: all families satisfying $S_e^* S_e \geq \sum_{\{f:s(f)=r(e)\}} S_f S_f^*$ generate isomorphic C^*-algebras (Theorem 4.1). If the graph is also transitive, this C^*-algebra is simple (Corollary 4.5).

Since Hilbert bimodules are a relatively new field of study, and since they arise in so many different ways, the precise axioms are not yet standard. Thus different authors have assumed that $\varphi : A \to \mathcal{L}(X)$ is injective, that A acts by compact operators on X, that A acts nondegenerately on X, or that X is full. We have been careful to avoid such assumptions, and in our final section we illustrate using the bimodules of graphs why we believe this to be helpful. We also give a couple of new applications involving other classes of Hilbert bimodules.

1. Toeplitz representations and the Toeplitz algebra

By a *Hilbert bimodule* over a C^*-algebra A we shall mean a right Hilbert A-module X together with an action of A by adjointable operators on X. The left action gives a homomorphism of A into the C^*-algebra $\mathcal{L}(X)$ of adjointable operators, which we denote by φ.

A *Toeplitz representation* (ψ, π) of a Hilbert bimodule X in a C^*-algebra B consists of a linear map $\psi : X \to B$ and a homomorphism $\pi : A \to B$ such that

$$\psi(x \cdot a) = \psi(x)\pi(a), \tag{1.1}$$

$$\psi(x)^*\psi(y) = \pi(\langle x, y\rangle_A), \quad \text{and} \tag{1.2}$$

$$\psi(a \cdot x) = \pi(a)\psi(x) \tag{1.3}$$

for x, $y \in X$ and $a \in A$. When $B = B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$, we call (ψ, π) a Toeplitz representation of X on $\mathcal{H}$.

Remark 1.1. *In fact Condition* (1.2) *implies that ψ is linear, as in* [1, p.8]. *It also implies that ψ is bounded: for $x \in X$ we have*

$$\|\psi(x)\|^2 = \|\psi(x)^*\psi(x)\| = \|\pi(\langle x, x\rangle_A)\| \leq \|\langle x, x\rangle_A\| = \|x\|^2.$$

If π is injective, then we have equality throughout, and ψ is isometric.

While many important examples of Hilbert bimodules are given in [23, Section 1], [21, Example 22] and [20, Section 3], the examples of most interest to us are associated to an infinite directed graph. These are not entirely new: it is shown in [23, p.193] how to build a bimodule from a finite $\{0,1\}$-matrix A, and that bimodule can be obtained by applying the following construction to the finite graph with incidence matrix A. However, the simplicity of the formulas in the next Example suggests that it may be more natural to think in terms of graphs rather than $\{0,1\}$-matrices.

Example 1.2 (The Cuntz-Krieger bimodule). Suppose $E = (E^0, E^1, r, s)$ is a directed graph with vertex set E^0, edge set E^1, and r, $s : E^1 \to E^0$ describing the range and source of edges. Let $X = X(E)$ be the vector space of functions $x : E^1 \to \mathbb{C}$ for which the function

$$v \in E^0 \mapsto \sum_{\{f \in E^1 : r(f) = v\}} |x(f)|^2$$

belongs to $A := c_0(E^0)$. Then with the operations

$$\begin{aligned} (x \cdot a)(f) &:= x(f)a(r(f)) \quad \text{for } f \in E^1, \\ \langle x, y \rangle_A(v) &:= \sum_{\{f \in E^1 : r(f) = v\}} \overline{x(f)} y(f) \quad \text{for } v \in E^0, \text{ and} \\ (a \cdot x)(f) &:= a(s(f))x(f) \quad \text{for } f \in E^1, \end{aligned}$$

X is a Hilbert bimodule over A.

Both the module X and the algebra A are spanned in an appropriate sense by point masses δ_f, δ_v, and we have

$$\langle \delta_e, \delta_f \rangle_A = \begin{cases} \delta_{r(e)} & \text{if } e = f \\ 0 & \text{otherwise;} \end{cases}$$

the elements δ_v are a family of mutually orthogonal projections in the C^*-algebra A. If (ψ, π) is a Toeplitz representation of this Hilbert bimodule X on $\mathcal{H}$, then the operators $P_v := \pi(\delta_v)$ are mutually orthogonal projections on $\mathcal{H}$, and (1.2) implies that the operators $S_f := \psi(\delta_f)$ are partial isometries with initial projection $P_{r(f)}$ and mutually orthogonal range projections; (1.3) implies that these range projections satisfy

$$\sum_{\{f \in E^1 : s(f) = v\}} S_f S_f^* \leq P_v \quad \text{for } v \in E^0. \tag{1.4}$$

We say that $\{S_f, P_v\}$ is a *Toeplitz-Cuntz-Krieger family* for the graph E. Conversely, given any such family on $\mathcal{H}$, we can define a representation $\pi : A \to B(\mathcal{H})$ by $\pi(a) := \sum_v a(v)P_v$, and a linear map $\psi : C_c(E^1) \to B(\mathcal{H})$ by $\psi(x) := \sum_f x(f)S_f$; routine calculations show that ψ is isometric for the A-norm on $C_c(E^1) \subset X$ and hence extends to a linear map on all of X, and that (ψ, π) is a Toeplitz representation of X.

Proposition 1.3. *Let X be a Hilbert bimodule over A. Then there is a C^*-algebra $\mathcal{T}_X$ and a Toeplitz representation $(i_X, i_A) : X \to \mathcal{T}_X$ such that*

(a) *for every Toeplitz representation (ψ, π) of X, there is a homomorphism $\psi \times \pi$ of $\mathcal{T}_X$ such that $(\psi \times \pi) \circ i_X = \psi$ and $(\psi \times \pi) \circ i_A = \pi$; and*
(b) *$\mathcal{T}_X$ is generated as a C^*-algebra by $i_X(X) \cup i_A(A)$.*

The triple $(\mathcal{T}_X, i_X, i_A)$ is unique: if (B, i'_X, i'_A) has similar properties, there is an isomorphism $\theta : \mathcal{T}_X \to B$ such that $\theta \circ i_X = i'_X$ and $\theta \circ i_A = i'_A$. Both maps i_X and i_A are injective. There is a strongly continuous action $\gamma : \mathbb{T} \to \operatorname{Aut} \mathcal{T}_X$ such that $\gamma_z(i_A(a)) = i_A(a)$ and $\gamma_z(i_X(x)) = z i_X(x)$ for $a \in A$, $x \in X$.

We call $\mathcal{T}_X$ the *Toeplitz algebra* of X and γ the *gauge action*. To prove the existence of $\mathcal{T}_X$, we need to know that the bimodule has lots of nontrivial representations. Here the fundamental example is a modification of Fock space, due essentially to Pimsner [23].

Example 1.4 (The Fock representation). For $n \geq 1$, the n-fold internal tensor product $X^{\otimes n} := X \otimes_A \cdots \otimes_A X$ is naturally a right Hilbert A-module, and A acts on the left by

$$a \cdot \big(x_1 \otimes_A \cdots \otimes_A x_n\big) := (a \cdot x_1) \otimes_A \cdots \otimes_A x_n;$$

if we need a name for the operator we call it $\varphi(a) \otimes_A 1^{n-1}$, and we continue to write x for a typical element of $X^{\otimes n}$. For $n = 0$, we take $X^{\otimes 0}$ to be the Hilbert module A with left action $a \cdot b := ab$. Then the Hilbert-module direct sum $F(X) := \bigoplus_{n=0}^{\infty} X^{\otimes n}$ carries a diagonal left action of A in which $a \cdot (x_n) := (a \cdot x_n)$. We can induce a representation $\pi_0 : A \to B(\mathcal{H})$ to a representation $F(X)\text{-}\mathrm{Ind}_A^{\mathcal{L}(F(X))} \pi_0$ of $\mathcal{L}(F(X))$ on $F(X) \otimes_A \mathcal{H}$, which restricts to a representation $\pi := F(X)\text{-}\mathrm{Ind}_A^A \pi_0$ of A.

For each $x \in X$, we can define a *creation operator* $T(x)$ on $F(X)$ by

$$T(x)y = \begin{cases} x \cdot y & \text{if } y \in X^{\otimes 0} = A \\ x \otimes_A y & \text{if } y \in X^{\otimes n} \text{ for some } n \geq 1; \end{cases}$$

routine calculations show that $T(x)$ is adjointable with

$$T(x)^* z = \begin{cases} 0 & \text{if } z \in X^{\otimes 0} = A \\ \langle x, x_1 \rangle_A \cdot y & \text{if } z = x_1 \otimes_A y \in X \otimes_A X^{\otimes n-1} = X^{\otimes n}. \end{cases}$$

If we now define $\psi : X \to B(F(X) \otimes_A \mathcal{H})$ by

$$\psi(x) := F(X)\text{-}\mathrm{Ind}_A^{\mathcal{L}(F(X))} \pi_0(T(x)),$$

then (ψ, π) is a Toeplitz representation of X, called the *Fock representation induced from* π_0. Note that, since A acts faithfully on $X^{\otimes 0} = A$ and the representation $F(X)\text{-}\mathrm{Ind}_A^{\mathcal{L}(F(X))} \pi_0$ is faithful whenever π_0 is, the representation π is faithful whenever π_0 is; by Remark 1.1, so is ψ.

***Remark* 1.5.** If we denote by φ_∞ the diagonal embedding of A in $\mathcal{L}(F(X))$, then (T, φ_∞) is a Toeplitz representation of X in the C^*-algebra $\mathcal{L}(F(X))$. Pimsner's Toeplitz algebra of X is by definition the C^*-subalgebra of $\mathcal{L}(F(X))$ generated by $T(X) \cup \varphi_\infty(A)$ [23, Definition 1.1], which is precisely the image of $\mathcal{T}_X$ under $T \times \varphi_\infty$. In Corollary 2.2, we will show that our Toeplitz algebra is isomorphic to his by proving that $T \times \varphi_\infty$ is faithful.

***Proof of Proposition* 1.3.** Say that a (Toeplitz) representation (ψ, π) of X on a Hilbert space $\mathcal{H}$ is *nondegenerate* (resp. *cyclic*) if the C^*-algebra $C^*(\psi, \pi)$ generated by $\psi(X) \cup \pi(A)$ acts nondegenerately (resp. cyclically). For an arbitrary representation (ψ, π) of X, let P be the orthogonal projection onto the essential subspace $\mathcal{K} := \overline{C^*(\psi, \pi)\mathcal{H}}$; then $(P\psi, P\pi)$ is a nondegenerate representation of X on $P\mathcal{H}$, and $((I-P)\psi, (I-P)\pi)$ is the zero representation. By the usual Zorn's lemma argument, $\mathcal{K}$ decomposes as a direct sum of subspaces on which $C^*(\psi, \pi)$ acts cyclically. Hence every representation is the direct sum of a zero representation and a collection of cyclic representations.

Let S be a set of cyclic representations of X such that every cyclic representation of X is unitarily equivalent to an element of S. (It can be shown that such a set S exists by fixing a Hilbert space $\mathcal{H}$ of sufficiently large dimension, and considering only cyclic representations on subspaces of $\mathcal{H}$. The set S is nonempty because the Fock representations must have nonzero cyclic summands.) Let

$$\mathcal{H} := \bigoplus_{(\psi,\pi)\in S} \mathcal{H}_{\psi,\pi}, \quad i_X := \bigoplus_{(\psi,\pi)\in S} \psi, \quad \text{and} \quad i_A := \bigoplus_{(\psi,\pi)\in S} \pi$$

(the direct sum defining i_X makes sense because every ψ is contractive). Then (i_X, i_A) is a representation of X in $\mathcal{T}_X := C^*(i_X, i_A)$; (b) is satisfied by definition, and (a) can be routinely verified.

The uniqueness follows by a standard argument, and the maps i_X and i_A are injective because the Fock representations factor through (i_X, i_A) by (a). To establish the existence of the gauge automorphism γ_z, just note that $(\mathcal{T}_X, zi_X, i_A)$ is also universal, and invoke the uniqueness. The continuity of the gauge action follows from a straightforward $\varepsilon/3$-argument. □

Whenever a C^*-algebra C acts by adjointable operators on a Hilbert A-module, one can use the module to induce representations of A to representations of C. If the representation π of A is half of a Toeplitz representation, we can realise the induced representation on the Hilbert space of π:

Proposition 1.6. *Let X be a right Hilbert A-module, and suppose (ψ, π) is a representation of X on $\mathcal{H}$; that is, $\psi : X \to B(\mathcal{H})$ is linear, $\pi : A \to B(\mathcal{H})$ is a representation, and* (1.1) *and* (1.2) *hold.*

(1) *There is a unique representation* $\rho = \rho^{\psi,\pi}$ *of* $\mathcal{L}(X)$ *on* $\mathcal{H}$ *with essential subspace* $\overline{\psi(X)\mathcal{H}} := \overline{\operatorname{span}}\{\psi(x)h : x \in X, h \in \mathcal{H}\}$ *such that*

$$\rho^{\psi,\pi}(S)(\psi(x)h) = \psi(Sx)h \quad \text{for } S \in \mathcal{L}(X),\ x \in X,\ \text{and } h \in \mathcal{H},$$

and we then have $\rho(\Theta_{x,y}) = \psi(x)\psi(y)^*$.

(2) *If* $\mathcal{K}$ *is a subspace of* $\mathcal{H}$ *which is invariant for* π, *then the subspace* $\mathcal{M} = \overline{\psi(X)\mathcal{K}}$ *is invariant for* ρ. *If* $\pi|_{\mathcal{K}}$ *is faithful, so is* $\rho|_{\mathcal{M}}$.

Proof.

(1) The map $(x, h) \mapsto \psi(x)h$ is bilinear, and hence there is a linear map $U : X \odot \mathcal{H} \to \mathcal{H}$ such that $U(x \otimes h) = \psi(x)h$. Since

$$\begin{aligned}(U(x \otimes h) \mid U(y \otimes k)) &= (\psi(x)h \mid \psi(y)k) = (h \mid \psi(x)^*\psi(y)k) \\ &= (h \mid \pi(\langle x, y\rangle_A)k) = (x \otimes h \mid y \otimes k),\end{aligned}$$

U extends to an isometry from $X \otimes_A \mathcal{H}$ to $\mathcal{H}$ such that $U(x \otimes_A h) = \psi(x)h$. For $S \in \mathcal{L}(X)$ we have

$$U \operatorname{Ind}\pi(S)U^*(\psi(x)h) = U \operatorname{Ind}\pi(S)\big(x \otimes_A h\big) = U\big(Sx \otimes_A h\big) = \psi(Sx)h,$$

so we can define $\rho := \operatorname{Ad} U \circ \operatorname{Ind}\pi$.

If x, y, $z \in X$, then

$$\rho(\Theta_{x,y})\psi(z) = \psi(x \cdot \langle y, z\rangle_A) = \psi(x)\pi(\langle y, z\rangle_A) = \psi(x)\psi(y)^*\psi(z),$$

so $\rho(\Theta_{x,y})$ and $\psi(x)\psi(y)^*$ agree on $\overline{\psi(X)\mathcal{H}}$. If k is orthogonal to $\psi(X)\mathcal{H}$, then $\rho(\Theta_{x,y})k = 0$, so we must show that $\psi(x)\psi(y)^*k = 0$. But this follows from $(\psi(x)\psi(y)^*k \mid h) = (k \mid \psi(y)\psi(x)^*h) = 0$.

(2) The subspace $\mathcal{M}$ is invariant for ρ because $\rho(S)(\psi(x)k) = \psi(Sx)k$. The restriction of U to $X \otimes_A \mathcal{K}$ implements a unitary equivalence between $\operatorname{Ind}\pi|_{X\otimes_A\mathcal{K}}$ and $\rho|_{\mathcal{M}}$; since the first of these is equivalent to $\operatorname{Ind}(\pi|_{\mathcal{K}})$, it is faithful if $\pi|_{\mathcal{K}}$ is, and hence so is $\rho|_{\mathcal{M}}$. □

Remark 1.7. The formula $\rho(\Theta_{x,y}) = \psi(x)\psi(y)^*$ implies that the representation ρ is the canonical extension to $M(\mathcal{K}(X)) = \mathcal{L}(X)$ of the map Pimsner would call $\pi^{(1)}$; see [23, page 202]. (We have avoided the notation $\pi^{(1)}$ because the map depends on both ψ and π.) For a representation (ψ, π) of X in a C^*-algebra B, we can represent B on a Hilbert space and apply the Proposition to obtain a homomorphism $\rho^{\psi,\pi} : \mathcal{K}(X) \to B$, but it need not extend canonically to $\mathcal{L}(X)$.

Proposition 1.8. *Let* A *and* B *be* C^**-algebras, let* X *and* Y *be Hilbert bimodules over* A, *and suppose that* $\pi : A \to B$ *is a homomorphism which forms part of Toeplitz representations* (ψ, π) *and* (μ, π) *of* X *and* Y *in* B.

(1) *There is a linear map* $\psi \otimes_A \mu$ *of the internal tensor product* $X \otimes_A Y$ *into* B *which satisfies*

$$\psi \otimes_A \mu(x \otimes_A y) = \psi(x)\mu(y), \quad x \in X,\ y \in Y, \tag{1.5}$$

and $(\psi \otimes_A \mu, \pi)$ *is a Toeplitz representation of* $X \otimes_A Y$.

(2) *Suppose* $B = B(\mathcal{H})$. *Denote by* $S \mapsto S \otimes_A 1$ *the canonical homomorphism of* $\mathcal{L}(X)$ *into* $\mathcal{L}(X \otimes_A Y)$ *given by the left action of* $\mathcal{L}(X)$ *on* X, *and let* $P_{\psi\otimes_A\mu}$ *be the projection of* $\mathcal{H}$ *onto* $\overline{\psi \otimes_A \mu(X \otimes_A Y)(\mathcal{H})}$. *Then the representations* $\rho^{\psi,\pi}$ *and* $\rho^{\psi\otimes_A\mu,\pi}$ *of Proposition 1.6 are related by*

$$\rho^{\psi\otimes_A\mu,\pi}(S \otimes_A 1) = \rho^{\psi,\pi}(S)P_{\psi\otimes_A\mu} \quad \text{for } S \in \mathcal{L}(X).$$

Proof. Since $(x, y) \mapsto \psi(x)\mu(y)$ is bilinear, it induces a linear map $\psi \odot \mu$ on the algebraic tensor product $X \odot Y$. For any x, $z \in X$ and y, $w \in Y$ we have

$$\begin{aligned} \mu(y)^*\psi(x)^*\psi(z)\mu(w) &= \mu(y)^*\pi(\langle x, z\rangle_A)\mu(w) \\ &= \mu(y)^*\mu(\langle x, z\rangle_A \cdot w) \\ &= \pi(\langle y, \langle x, z\rangle_A \cdot w\rangle_A) \\ &= \pi\big(\langle x \otimes_A y, z \otimes_A w\rangle_A\big). \end{aligned} \tag{1.6}$$

Thus for $v = \sum_i x_i \otimes y_i \in X \odot Y$ we have

$$\begin{aligned} \big\|\psi \odot \mu(v)\big\|^2 &= \big\|\psi \odot \mu(v)^*\psi \odot \mu(v)\big\| = \Big\|\sum_{i,j} \mu(y_i)^*\psi(x_i)^*\psi(x_j)\mu(y_j)\Big\| \\ &= \Big\|\pi\Big(\sum_{i,j}\Big\langle x_i \otimes_A y_i, x_j \otimes_A y_j\Big\rangle_A\Big)\Big\| \\ &\le \Big\|\sum_{i,j}\Big\langle x_i \otimes_A y_i, x_j \otimes_A y_j\Big\rangle_A\Big\| = \big\|v\big\|^2, \end{aligned}$$

so $\psi\odot\mu$ induces a contractive linear map $\psi\otimes_A\mu$ on $X\otimes_A Y$. Routine calculations on elementary tensors show that $(\psi \otimes_A \mu, \pi)$ is a Toeplitz representation of $X \otimes_A Y$.

For part (2), note that the vectors $\psi \otimes_A \mu(x \otimes_A y)h = \psi(x)\mu(y)h$ span a dense subspace of the essential subspace $\overline{\psi \otimes_A \mu(X \otimes_A Y)\mathcal{H}}$ of $\rho^{\psi\otimes_A\mu,\pi}$. Thus the calculation

$$\begin{aligned} \rho^{\psi\otimes_A\mu,\pi}(S \otimes_A 1)(\psi(x)\mu(y)h) &= \psi \otimes_A \mu\big((S \otimes_A 1)(x \otimes_A y)\big)h \\ &= \psi(Sx)\mu(y)h \\ &= \rho^{\psi,\pi}(S)(\psi(x)\mu(y)h) \end{aligned}$$

implies the result. □

2. Faithful representations

If (ψ, π) is a Toeplitz representation of a Hilbert bimodule X over A on a Hilbert space $\mathcal{H}$, then the subspace

$$\overline{\psi(X)\mathcal{H}} := \overline{\operatorname{span}}\{\psi(x)h : x \in X,\ h \in \mathcal{H}\}$$

is invariant for π: $\pi(a)(\psi(x)h) = \psi(a \cdot x)h$. Thus the complement $(\psi(X)\mathcal{H})^{\perp}$ is also invariant for π. Our first main theorem says that if π is faithful on this complement, then $\psi \times \pi$ is faithful.

Theorem 2.1. *Let X be a Hilbert bimodule over a C^*-algebra A, and let (ψ, π) be a Toeplitz representation of X on a Hilbert space $\mathcal{H}$. If A acts faithfully on $(\psi(X)\mathcal{H})^{\perp}$, then $\psi \times \pi$ is a faithful representation of $\mathcal{T}_X$. If the homomorphism $\varphi : A \to \mathcal{L}(X)$ describing the left action of A on X has range in $\mathcal{K}(X)$ and if $\psi \times \pi$ is faithful, then A acts faithfully on $(\psi(X)\mathcal{H})^{\perp}$.*

Before we prove this theorem we deduce from it that our Toeplitz algebra is isomorphic to Pimsner's. This implies in particular that his algebra is universal for Toeplitz representations [23, Theorem 3.4].

Corollary 2.2. *The Fock representation $T \times \varphi_\infty$ of $\mathcal{T}_X$ is faithful.*

Proof. Let π_0 be a faithful representation of A on $\mathcal{H}$, and consider

$$(\psi, \pi) := \big((F(X)\text{-}\mathrm{Ind}_A^{\mathcal{L}(F(X))}\pi_0) \circ T\,,\ (F(X)\text{-}\mathrm{Ind}_A^{\mathcal{L}(F(X))}\pi_0) \circ \varphi_\infty\big),$$

which is a Toeplitz representation because (T, φ_∞) is. For each $n \geq 0$ and $y \in X^{\otimes n}$, we have $\psi(x)(y \otimes_A h) = (x \otimes_A y) \otimes_A h$; thus

$$\overline{\psi(X)\big(F(X) \otimes_A \mathcal{H}\big)} = \Big(\bigoplus_{n=1}^{\infty} X^{\otimes n}\Big) \otimes_A \mathcal{H} \cong \bigoplus_{n=1}^{\infty}(X^{\otimes n} \otimes_A \mathcal{H})$$

has complement $X^{\otimes 0} \otimes_A \mathcal{H} = A \otimes_A \mathcal{H} = \mathcal{H}$. The restriction of π to this subspace is just $A\text{-}\mathrm{Ind}_A^A \pi_0 = \pi_0$, which is faithful. Thus Theorem 2.1 says that $\psi \times \pi = \big(F(X)\text{-}\mathrm{Ind}_A^{\mathcal{L}(F(X))}\pi_0\big) \circ (T \times \varphi_\infty)$ is faithful, and hence $T \times \varphi_\infty$ is too. □

Averaging over the gauge action gives an expectation E of $\mathcal{T}_X$ onto the fixed-point algebra $\mathcal{T}_X^\gamma$:

$$E(b) := \int_{\mathbb{T}} \gamma_w(b)\, dw \quad \text{for } b \in \mathcal{T}_X.$$

The map E is a positive linear idempotent of norm one, and is faithful on positive elements in the sense that $E(b^*b) = 0 \implies b = 0$. The main step in the proof of Theorem 2.1 is to show that the expectation E is spatially implemented: there is a compatible expectation $E_{\psi,\pi}$ of $\psi \times \pi(\mathcal{T}_X)$ onto $\psi \times \pi(\mathcal{T}_X^\gamma)$.

Proposition 2.3. *Let (ψ, π) be a Toeplitz representation of X such that π is faithful on $(\psi(X)\mathcal{H})^\perp$.*

(1) *There is a norm-decreasing map $E_{\psi,\pi}$ on $\psi \times \pi(\mathcal{T}_X)$ such that*

$$E_{\psi,\pi} \circ (\psi \times \pi) = (\psi \times \pi) \circ E;$$

(2) *$\psi \times \pi$ is faithful on the fixed-point algebra $\mathcal{T}_X^\gamma$.*

Before we try to construct $E_{\psi,\pi}$ we need to understand what E does, and for this we need a description of a dense subalgebra of $\mathcal{T}_X$.

Suppose (ψ, π) is a Toeplitz representation of X in a C^*-algebra B. For $n \geq 1$, Proposition 1.8 gives us a representation $(\psi^{\otimes n}, \pi)$ of the tensor power $X^{\otimes n} := X \otimes_A \cdots \otimes_A X$ such that $\psi^{\otimes n}(x_1 \otimes_A \cdots \otimes_A x_n) = \psi(x_1) \cdots \psi(x_n)$. We define $\psi^{\otimes 0} := \pi$. When $m \geq 1$, $X^{\otimes m} \otimes_A X^{\otimes n} = X^{\otimes(m+n)}$ for every $n \geq 0$, and $\psi^{\otimes m} \otimes_A \psi^{\otimes n} = \psi^{\otimes(m+n)}$. There is a slight subtlety for $m = 0$: the natural map $a \otimes_A x \mapsto a \cdot x$ identifies $X^{\otimes 0} \otimes_A X^{\otimes n} = A \otimes_A X^{\otimes n}$ with the essential submodule $A \cdot X^{\otimes n}$ of $X^{\otimes n}$, and then $\psi^{\otimes 0} \otimes_A \psi^{\otimes n}$ is the restriction of $\psi^{\otimes n}$ to this submodule.

Lemma 2.4. *With the above notation, we have*

$$\mathcal{T}_X = \overline{\operatorname{span}}\{i_X^{\otimes m}(x) i_X^{\otimes n}(y)^* : m, n \geq 0,\ x \in X^{\otimes m},\ y \in X^{\otimes n}\}.$$

The expectation E is given by

$$E(i_X^{\otimes m}(x) i_X^{\otimes n}(y)^*) = \begin{cases} i_X^{\otimes m}(x) i_X^{\otimes n}(y)^* & \text{if } m = n, \\ 0 & \text{if } m \neq n. \end{cases}$$

Proof. The algebra $\mathcal{T}_X$ is spanned by products of elements $i_X(x)$, $i_A(a)$ and $i_X(y)^*$; given a word in these generators, we can usually absorb $i_A(a)$'s into $i_X(x)$'s, and use $i_X(y)^* i_X(x) = i_A(\langle y, x\rangle_A)$ to cancel any $i_X(y)^*$ appearing to the left of an $i_X(x)$. (This is [23, Lemma 3.1].) Since $\gamma_z(i_X^{\otimes m}(x) i_X^{\otimes n}(y)^*) = z^{m-n} i_X^{\otimes m}(x) i_X^{\otimes n}(y)^*$, the second assertion is easy. □

Lemma 2.4 implies that the image $\psi \times \pi(\mathcal{T}_X)$ is spanned by elements $\psi^{\otimes m}(x)\psi^{\otimes n}(y)^*$ and that $E_{\psi,\pi}$ must satisfy

$$E_{\psi,\pi}(\psi^{\otimes m}(x)\psi^{\otimes n}(y)^*) = \begin{cases} \psi^{\otimes m}(x)\psi^{\otimes n}(y)^* & \text{if } m = n, \\ 0 & \text{if } m \neq n. \end{cases} \tag{2.1}$$

We shall show that the formal linear extension $E_{\psi,\pi}$ of the map defined by (2.1) is norm-decreasing, and hence extends to a well-defined norm-decreasing map on $\psi \times \pi(\mathcal{T}_X)$. We analyse the norm of an element $E_{\psi,\pi}(S)$ by showing that the subspaces $\overline{\psi^{\otimes n}(X^{\otimes n})\mathcal{H}}$ form a decreasing chain of reducing subspaces, in which the differences are large enough to see operators in each $\mathcal{L}(X^{\otimes n})$ faithfully.

Lemma 2.5. *Suppose that (ψ, π) is a Toeplitz representation of X on $\mathcal{H}$. For $n \geq 1$, let P_n denote the projection of $\mathcal{H}$ onto $\overline{\psi^{\otimes n}(X^{\otimes n})\mathcal{H}}$, and let $P_0 = 1$. Write ρ_n for the representation $\rho^{\psi^{\otimes n},\pi}$ of $\mathcal{L}(X^{\otimes n})$ (so that ρ_0 is the extension of π on its essential subspace).*

(1) *We have $P_n \geq P_{n+1}$ for all $n \geq 0$, so $Q_n := P_n - P_{n+1}$ is also a projection for $n \geq 0$.*

(2) *For every $n \geq 0$, $k \geq 0$, and x, $y \in X^{\otimes n}$ we have*

$$\psi^{\otimes n}(x)P_k = P_{n+k}\psi^{\otimes n}(x), \quad \text{and} \tag{2.2}$$

$$P_k\psi^{\otimes n}(x)\psi^{\otimes n}(y)^* = \psi^{\otimes n}(x)\psi^{\otimes n}(y)^* P_k. \tag{2.3}$$

(3) *If π is faithful on $(\psi(X)\mathcal{H})^\perp$, then each ρ_n restricts to a faithful representation of $\mathcal{L}(X^{\otimes n})$ on $Q_n(\mathcal{H})$.*

Proof. For part (1), observe that the vectors $\psi^{\otimes n}(z)\psi(w)h = \psi^{\otimes(n+1)}(z \otimes_A w)h$ span the range of P_{n+1}, and are clearly in the range of P_n.

Equation (2.2) is trivially true for $k = 0$ and/or $n = 0$. If $k \geq 1$, $n \geq 1$, and $w \in X^{\otimes k}$, then

$$\psi^{\otimes n}(x)P_k\psi^{\otimes k}(w) = \psi^{\otimes n}(x)\psi^{\otimes k}(w) = P_{n+k}\psi^{\otimes n}(x)\psi^{\otimes k}(w),$$

so $\psi^{\otimes n}(x)P_k$ and $P_{n+k}\psi^{\otimes n}(x)$ agree on $P_k(\mathcal{H})$. If $f \in P_k(\mathcal{H})^\perp$, then for any $z \in X^{\otimes n}$, $w \in X^{\otimes k}$, and $h \in \mathcal{H}$ we have

$$\begin{aligned}(\psi^{\otimes n}(x)f \mid \psi^{\otimes n}(z)\psi^{\otimes k}(w)h) &= (f \mid \pi(\langle x, z\rangle_A)\psi^{\otimes k}(w)h) \\ &= (f \mid \psi^{\otimes k}(\langle x, z\rangle_A \cdot w)h) = 0,\end{aligned}$$

which implies $P_{n+k}\psi^{\otimes n}(x)f = 0$ because the vectors

$$\psi^{\otimes n}(z)\psi^{\otimes k}(w)h = \psi^{\otimes(n+k)}(z \otimes_A w)h$$

span the range of P_{n+k}. This gives (2.2). When $k < n$, both sides of (2.3) reduce to $\psi^{\otimes n}(x)\psi^{\otimes n}(y)^*$; for $k \geq n$, (2.3) follows from two applications of (2.2).

Part (3) is trivial for $n = 0$. For $n \geq 1$, we apply Proposition 1.6(2): since $\pi|_{(I-P_1)\mathcal{H}}$ is faithful, ρ_n is a faithful representation of $\mathcal{L}(X^{\otimes n})$ on

$$\overline{\psi^{\otimes n}(X^{\otimes n})(1 - P_1)\mathcal{H}}.$$

But this space is precisely $Q_n(\mathcal{H})$, because (2.2) implies that $\psi^{\otimes n}(x)(1 - P_1) = (P_n - P_{n+1})\psi^{\otimes n}(x)$. □

Proof of Proposition 2.3.

(1) We have to prove that for every finite sum

$$S := \sum_j i_X^{\otimes m_j}(x_j) i_X^{\otimes n_j}(y_j)^*$$

we have $\|\psi \times \pi(E(S))\| \le \|\psi \times \pi(S)\|$; equivalently, we have to prove

$$\Big\| \sum_{\{j:m_j=n_j\}} \psi^{\otimes n_j}(x_j)\psi^{\otimes n_j}(y_j)^* \Big\| \le \Big\| \sum_j \psi^{\otimes m_j}(x_j)\psi^{\otimes n_j}(y_j)^* \Big\|.$$

We know from (2.3) that the projections Q_k commute with every summand in $\psi \times \pi(E(S))$. If $m > k$, we have $Q_k \psi^{\otimes m}(x) = Q_k P_m \psi^{\otimes m}(x) = 0$, and if $m \le k$ and $n \le k$, (2.2) gives

$$Q_k \psi^{\otimes m}(x)\psi^{\otimes n}(y)^* Q_k = \psi^{\otimes m}(x) Q_{k-m} Q_{k-n} \psi^{\otimes n}(y)^*,$$

which is 0 unless $m = n$. Let $K := \max n_j$. Then $\rho_K(T \otimes_A 1^{K-n}) = \rho_n(T)P_K$ by Proposition 1.8(2), so we have

$$P_K(\psi \times \pi(E(S))) = P_K \rho_K\Big(\sum_{\{j:m_j=n_j\}} \Theta_{x_j,y_j} \otimes_A 1^{K-n_j} \Big); \tag{2.4}$$

because $Q_K \rho_K$ is faithful on $\mathcal{L}(X^{\otimes K})$ by the previous lemma, it follows that

$$\|P_K(\psi \times \pi(E(S)))\| = \|Q_K(\psi \times \pi(E(S)))\|.$$

Since $Q_0 + \cdots + Q_{K-1} + P_K = 1$, this gives

$$\begin{aligned} \|\psi \times \pi(E(S))\| &= \sup\{\|Q_k(\psi \times \pi(E(S)))\| : 0 \le k \le K\} \\ &= \sup\{\|Q_k(\psi \times \pi(E(S)))Q_k\| : 0 \le k \le K\} \\ &= \sup\{\|Q_k(\psi \times \pi(S))Q_k\| : 0 \le k \le K\} \\ &\le \|\psi \times \pi(S)\|. \end{aligned}$$

Thus $E_{\psi,\pi}$ extends to a norm-decreasing map on $\psi \times \pi(\mathcal{T}_X)$, giving (1).

Next let $R := \sum_j i_X^{\otimes n_j}(x_j) i_X^{\otimes n_j}(y_j)^*$ be a typical finite sum in the core $\mathcal{T}_X^\gamma$; such sums are dense because E is continuous and maps finite sums to finite sums. For $k < K := \max n_j$, Proposition 1.8(2) implies that

$$Q_k(\psi \times \pi(R)) = Q_k \rho_k\Big(\sum_{\{j:n_j \le k\}} \Theta_{x_j,y_j} \otimes_A 1^{k-n_j} \Big),$$

and hence

$$\|Q_k(\psi \times \pi(R))\| \leq \Big\| \sum_{\{j:n_j \leq k\}} \Theta_{x_j,y_j} \otimes_A 1^{k-n_j} \Big\|.$$

There is a similar formula for $\|P_K(\psi \times \pi(R)\|$ (see (2.4)), so

$$\begin{aligned} \|\psi \times \pi(R)\| &= \max\big\{\|P_K(\psi \times \pi(R))\|; \|Q_k(\psi \times \pi(R))\| : 0 \leq k < K\big\} \\ &\leq \max\Big\{\Big\| \sum_{\{j:n_j \leq k\}} \Theta_{x_j,y_j} \otimes_A 1^{k-n_j} \Big\| : 0 \leq k \leq K\Big\} \end{aligned} \tag{2.5}$$

for every Toeplitz representation (ψ, π). Applying this to a faithful representation shows that (2.5) is an upper bound for $\|R\|$.

When π is faithful on $(\psi(X)\mathcal{H})^\perp$, the representations $Q_k\rho_k$ and ρ_K are faithful too, so we actually have

$$\|\psi \times \pi(R)\| = \max\Big\{\Big\| \sum_{\{j:n_j \leq k\}} \Theta_{x_j,y_j} \otimes_A 1^{k-n_j} \Big\| : 0 \leq k \leq K\Big\}. \tag{2.6}$$

In particular, this implies that $\|R\|$ is at least (2.5); since we have already seen that $\|R\|$ is at most (2.5), we deduce that $\|R\| =$ (2.5), and (2.6) implies that $\psi \times \pi$ is isometric on the core. $\square$

Proof of Theorem 2.1. Suppose π is faithful on $(\psi(X)\mathcal{H})^\perp$ and $S \in \ker \psi \times \pi$. Then by Proposition 2.3(1) we have $\psi \times \pi(E(S^*S)) = E_{\psi,\pi}(\psi \times \pi(S^*S)) = 0$, which by Proposition 2.3(2) implies that $E(S^*S) = 0$. Because E is faithful, this forces $S^*S = 0$ and $S = 0$.

Now suppose that $\varphi(A) \subset \mathcal{K}(X)$. Proposition 1.6 gives a homomorphism $\rho^{i_X,i_A} : \mathcal{K}(X) \to \mathcal{T}_X$ (see Remark 1.7), and we claim that, for any Toeplitz representation (ψ, π),

$$\psi \times \pi(i_A(a) - \rho^{i_X,i_A}(\varphi(a))) = \pi(a)(1 - P_1) = \pi(a)|_{(\psi(X)\mathcal{H})^\perp}. \tag{2.7}$$

For any rank-one operator $\Theta_{x,y}$ we have

$$\psi \times \pi(\rho^{i_X,i_A}(\Theta_{x,y})) = \psi \times \pi(i_X(x)i_X(y)^*) = \psi(x)\psi(y)^* = \rho^{\psi,\pi}(\Theta_{x,y}),$$

and hence $(\psi \times \pi) \circ \rho^{i_X,i_A} = \rho^{\psi,\pi}$ on $\mathcal{K}(X)$. On the other hand, since $\rho^{\psi,\pi}(\varphi(a))$ agrees with $\pi(a)$ on $\psi(X)\mathcal{H}$, we have $\rho^{\psi,\pi}(\varphi(a)) = \pi(a)P_1$. These two observations give the claim (2.7).

Since there are Toeplitz representations (ψ, π) in which π is faithful on $(\psi(X)\mathcal{H})^\perp$ (for example, the Fock representation induced from a faithful representation of A) and $\psi \times \pi$ is then faithful, (2.7) implies that $\alpha : a \mapsto i_A(a) - \rho^{i_X,i_A}(\varphi(a))$ is an injective homomorphism of A into $\mathcal{T}_X$. (Warning: it is crucial here that $\varphi(A) \subset \mathcal{K}(X)$.) Thus if $\psi \times \pi$ is faithful, so is the composition with α, and (2.7) gives the result. □

3. Direct sums of Hilbert bimodules

If $\{X^\lambda : \lambda \in \Lambda\}$ is a family of Hilbert bimodules over the same C^*-algebra A, then the algebraic direct sum X_0 is a pre-Hilbert A-module with $(x_\lambda)\cdot a := (x_\lambda \cdot a)$ and $\langle (x_\lambda), (y_\lambda)\rangle_A := \sum_\lambda \langle x_\lambda, y_\lambda\rangle_A$. We can therefore complete X_0 to obtain a Hilbert A-module X, which we denote by $\bigoplus_{\lambda\in\Lambda} X^\lambda$ (see [25, Lemma 2.16]). There is a left action of A on X_0 defined by $a \cdot (x_\lambda) := (a \cdot x_\lambda)$, which we claim extends to an action of A by adjointable operators on $\bigoplus X^\lambda$. To see this, note that the left action of A on each X^λ satisfies $\langle a \cdot x_\lambda, a \cdot x_\lambda\rangle_A \leq \|a\|^2 \langle x_\lambda, x_\lambda\rangle_A$, and since the sum of positive elements is positive, we deduce that

$$\langle a \cdot (x_\lambda), a \cdot (x_\lambda)\rangle_A \leq \|a\|^2 \Big(\sum_\lambda \langle x_\lambda, x_\lambda\rangle_A\Big) = \|a\|^2 \langle (x_\lambda), (y_\lambda)\rangle_A.$$

Thus the map $(x_\lambda) \mapsto a \cdot (x_\lambda)$ is bounded for the norm on X_0 induced by $\langle \cdot, \cdot \rangle_A$, and extends to a map on all of X, which is adjointable with adjoint $(x_\lambda) \mapsto a^* \cdot (x_\lambda)$, as claimed. We have now shown that $X = \bigoplus_{\lambda\in\Lambda} X^\lambda$ is itself a Hilbert bimodule over A, which we call the *direct sum* of the Hilbert bimodules X^λ.

Theorem 3.1. *Let $\{X^\lambda : \lambda \in \Lambda\}$ be a family of Hilbert bimodules over a C^*-algebra A, let $X := \bigoplus_{\lambda\in\Lambda} X^\lambda$, and let (ψ, π) be a Toeplitz representation of X on a Hilbert space $\mathcal{H}$. If A acts faithfully on $(\psi(\bigoplus_{\lambda\in F} X^\lambda)\mathcal{H})^\perp$ for every finite subset F of Λ, then $\psi \times \pi$ is faithful on $\mathcal{T}_X$. If A acts by compact operators on the left of each X^λ and if $\psi \times \pi$ is faithful, then π acts faithfully on every $(\psi(\bigoplus_{\lambda\in F} X^\lambda)\mathcal{H})^\perp$.*

The proof of this Theorem exploits a grading of $\mathcal{T}_X$ by the free group $\mathbb{F}_\Lambda$ on Λ: picking off the e-graded piece gives an expectation E^Λ which goes further into the core $\mathcal{T}_X^\gamma$ than the expectation E used in Section 2. Such gradings are usually formalised in terms of a coaction of $\mathbb{F}_\Lambda$ on $\mathcal{T}_X$, but because $\mathbb{F}_\Lambda$ is not amenable, it would not be obvious from such a formalisation that the associated expectation E^Λ is faithful (see, for example, [18, Section 4]). Here we shall construct the expectation directly using the Fock representation of $\mathcal{T}_X$, which we know is faithful by Corollary 2.2.

First we need some notation. Let $\mathbb{F}_\Lambda^+$ be the subsemigroup of $\mathbb{F}_\Lambda$ generated by Λ and the identity e. For s, $t \in \mathbb{F}_\Lambda^+$, we write $s \leq t$ if t has the form sr for some $r \in \mathbb{F}_\Lambda^+$, and we define

$$s \vee t := \begin{cases} t & \text{if } s \leq t, \\ s & \text{if } t \leq s, \text{ and} \\ \infty & \text{otherwise.} \end{cases}$$

(The pair $(\mathbb{F}_\Lambda, \mathbb{F}_\Lambda^+)$ is an example of a *quasi-lattice ordered group* ([22, 18]): the subsemigroup defines a left-invariant partial order on $\mathbb{F}_\Lambda$ in which $s \leq t$ if and only if $s^{-1}t \in \mathbb{F}_\Lambda^+$, and, loosely speaking, every finite bounded subset has a least upper bound.)

For a reduced word $s = \lambda_1 \cdots \lambda_n$ in $\mathbb{F}_\Lambda^+ \setminus \{e\}$, we write $|s| := n$. We can identify the tensor power $X^s := X^{\lambda_1} \otimes_A \cdots \otimes_A X^{\lambda_n}$ with a submodule of $X^{\otimes n}$. If (ψ, π) is a Toeplitz representation of X, we can define $\psi^\lambda := \psi|_{X^\lambda}$ and $\psi^s := \psi^{\lambda_1} \otimes_A \cdots \otimes_A \psi^{\lambda_n}$, and then (ψ^s, π) is a Toeplitz representation of X^s by Proposition 1.8. The associativity of $\bigotimes_A$ gives an isomorphism of $X^s \otimes_A X^t$ onto X^{st} which carries $\psi^s \otimes_A \psi^t$ into ψ^{st}, and that ψ^s agrees with the restriction of $\psi^{\otimes |s|}$ to $X^s \subseteq X^{\otimes |s|}$.

Proposition 3.2. *Let (ψ, π) be a Toeplitz representation of X in a C^*-algebra.*

(1) *Suppose $s,\ t \in \mathbb{F}_\Lambda^+$ and $s \leq t$. Then for every $x,\ y_1 \in X^s$ and $y_2 \in X^{s^{-1}t}$ we have $\psi^s(x)^* \psi^t(y_1 \otimes_A y_2) = \psi^{s^{-1}t}(\langle x, y_1 \rangle_A \cdot y_2)$.*
(2) *Suppose $s,\ t \in \mathbb{F}_\Lambda^+$ and $s \vee t = \infty$. Then for every $x \in X^s$ and $y \in X^t$ we have $\psi^s(x)^* \psi^t(y) = 0$.*
(3) $\psi \times \pi(\mathcal{T}_X) = \overline{\operatorname{span}}\{\psi^s(x)\psi^t(y)^* : x \in X^s,\ y \in X^t,\ s,\ t \in \mathbb{F}_\Lambda^+\}$.
(4) *There is a norm-decreasing linear map E^Λ on $\mathcal{T}_X$ which satisfies*

$$E^\Lambda(i_X^s(x) i_X^t(y)^*) = \begin{cases} i_X^s(x) i_X^t(y)^* & \text{if } s = t \text{ in } \mathbb{F}_\Lambda^+, \\ 0 & \text{otherwise,} \end{cases}$$

and which is faithful on positive elements.

Proof.

Part (1) is a straightforward computation.

For (2), let r be the longest common initial segment in s and t, so that $s = r\lambda s_1$ and $t = r\mu t_1$ for $r,\ s_1,\ t_1 \in \mathbb{F}_\Lambda^+$ and $\lambda \neq \mu \in \Lambda$. Then $X^{r\lambda}$ and $X^{r\mu}$ are orthogonal submodules of $X^{\otimes(|r|+1)}$. Since vectors of the form $x \otimes_A y \in X^{r\lambda} \otimes_A X^{s_1}$ span X^s and similarly for X^t, the calculation

$$\begin{aligned} \psi^s(x \otimes y)^* \psi^t(w \otimes z) &= \psi^{s_1}(y)^* \psi^{\otimes(|r|+1)}(x)^* \psi^{\otimes(|r|+1)}(w) \psi^{t_1}(z) \\ &= \psi^{s_1}(y)^* \pi(\langle x, w \rangle_A) \psi^{t_1}(z) = 0 \end{aligned}$$

implies (2).

For (3), we show that $C := \overline{\operatorname{span}}\{\psi^s(x)\psi^t(y)^*\}$ is a C^*-subalgebra of $\psi \times \pi(\mathcal{T}_X)$ which contains $\psi(X)$ and $\pi(A)$. It is clearly closed under taking adjoints. To see that it is a subalgebra, consider $\psi^s(x)\psi^t(y)^*$ and $\psi^u(z)\psi^v(w)^*$. Part (2) implies that $\psi^t(y)^*\psi^u(z) = 0$ if $t \vee u = \infty$. Otherwise, (1) implies that $\psi^t(y)^*\psi^u(z)$ has the form $\psi^{t^{-1}u}(z')$ (if $t \leq u$) or $\psi^{u^{-1}t}(y')^*$ (if $u \leq t$). Absorbing this element into either $\psi^s(x)$ or $\psi^v(w)^*$ shows that the product $\psi^s(x)\psi^t(y)^*\psi^u(z)\psi^v(w)^*$ belongs to C.

Since X is essential as a right A-module, every element has the form $y \cdot a$ for $y \in X$ and $a \in A$. Approximating y by a finite sum of the form $\sum y_\lambda$ shows that $\psi(y \cdot a) = \psi(y)\pi(a) \sim \sum_\lambda \psi^\lambda(y_\lambda)\psi^e(a^*)^*$ belongs to C. Similarly, writing an arbitary element of A as bc^* shows that $\pi(bc^*) = \psi^e(b)\psi^e(c)^* \in C$.

(4) Part (2) implies that the subspaces X^s of $X^{\otimes n}$ corresponding to different words of length n are orthogonal; thus the natural map is an isomorphism of the Fock bimodule $\bigoplus_{s\in\mathbb{F}_\Lambda^+} X^s$ onto $F(X)$. For $r \in \mathbb{F}_\Lambda^+$, let R_r be the orthogonal projection of $F(X)$ onto X^r. Then for each $S \in \mathcal{L}(F(X))$, the sum $\sum_{r\in\mathbb{F}_\Lambda^+} R_r S R_r$ converges $*$-strongly to an adjointable operator $\Phi(S)$; the resulting linear mapping Φ on $\mathcal{L}(F(X))$ is idempotent, norm-decreasing, and faithful on positive operators. Let $T \times \varphi_\infty$ be the Fock representation of $\mathcal{T}_X$, which is faithful by Corollary 2.2. We want to define $E^\Lambda := (T \times \varphi_\infty)^{-1} \circ \Phi \circ (T \times \varphi_\infty)$; before we can do this, we need to know that Φ leaves the range of $T \times \varphi_\infty$ invariant. Both this and the formula in (4) will follow if we can show that

$$\Phi(T^s(x)T^t(y)^*) = \begin{cases} T^s(x)T^t(y)^* & \text{if } s = t \text{ in } \mathbb{F}_\Lambda^+, \\ 0 & \text{otherwise.} \end{cases} \tag{3.1}$$

Let $x \in X^s$ and $y \in X^t$, and note that X^r is spanned by vectors of the form $T^r(z)a$, where $a \in A = X^{\otimes 0}$. If $t \vee r = \infty$, then (2) gives $T^s(x)T^t(y)^*T^r(z)a = 0$. If $r < t$, then (1) implies that $T^t(y)^*T^r(z)a = T^{r^{-1}t}(y')^*a$ for some $y' \in X^{r^{-1}t}$, and this vanishes because $T^{r^{-1}t}(y')^*$ kills $A = X^{\otimes 0} \subset F(X)$. If $t \leq r$, then $T^t(y)^*T^r(z) = T^{t^{-1}r}(z')$ for some $z' \in X^{t^{-1}r}$, and $T^s(x)T^t(y)^*T^r(z)a = T^{st^{-1}r}(x \otimes z')a \in X^{st^{-1}r}$. Thus

$$R_r T^s(x)T^t(y)^* R_r = \begin{cases} T^s(x)T^t(y)^* R_r & \text{if } s = t \text{ in } \mathbb{F}_\Lambda^+, \\ 0 & \text{otherwise,} \end{cases}$$

and summing over $r \in \mathbb{F}_\Lambda^+$ gives (3.1). □

Now suppose that (ψ, π) is a Toeplitz representation of X on $\mathcal{H}$. As in the previous section, we aim to show that if π satisfies the hypothesis of Theorem 3.1, then the expectation E^Λ is spatially implemented. The analogues of the projections P_n are the projections P_s onto the subspaces $\overline{\psi^s(X^s)\mathcal{H}}$, and in the next Lemma we write down some of their properties. The analogues of the projections $Q_n = P_n - P_{n+1}$ are the projections Q_s^F described in Lemma 3.4, which is based on [18, Lemma 1.4]; in Lemma 3.5 we show that Q_s^F is large enough to see $\mathcal{L}(X^s)$ faithfully.

Lemma 3.3. *Let (ψ, π) be a Toeplitz representation of X on $\mathcal{H}$. For $s \in \mathbb{F}_\Lambda^+$, denote by ρ_s the representation $\rho^{\psi^s,\pi} : \mathcal{L}(X^s) \to B(\mathcal{H})$, and let P_s be the projection of $\mathcal{H}$ onto $\overline{\psi^s(X^s)\mathcal{H}}$; take $P_e = 1$ and $P_\infty = 0$.*

(1) *We have $P_sP_t = P_{s\vee t}$ for $s,\ t \in \mathbb{F}_\Lambda^+$.*

(2) *For $s,\ t \in \mathbb{F}_\Lambda^+$ and $x,\ y \in X^s$, we have*

$$\psi^s(x)P_t = P_{st}\psi^s(x), \quad \text{and} \tag{3.2}$$

$$P_t\psi^s(x)\psi^s(y)^* = \psi^s(x)\psi^s(y)^*P_t. \tag{3.3}$$

The proofs are like those of Part (2) of Lemma 2.5; the orthogonality of P_s and P_t when $s \vee t = \infty$ follows from Proposition 3.2(2).

Lemma 3.4. *Let F be a finite subset of $\mathbb{F}_\Lambda^+$ such that $e \in F$. For $s \in F$, let*

$$Q_s^F := P_s\Big(\prod_{\{t\in F:s<t\}}(1-P_t)\Big).$$

Then $1 = \sum_{s\in F} Q_s^F$.

Proof. We proceed by induction on $|F|$. If $|F| = 1$, then $F = \{e\}$, and $Q_e^F = P_e = 1$. If $|F| \geq 2$, we remove a maximal element c from F, and apply the inductive hypothesis to $G := F \setminus \{c\}$. There is a unique longest word $b \in G$ such that $b < c$. We claim that only the summand Q_b^G in the decomposition $1 = \sum_{s\in G} Q_s^G$ is changed by adding c to G; in other words, we claim that $Q_s^F = Q_s^G$ for $s \neq b$. Suppose $s \in G \setminus \{b\}$. Then Q_s^F and Q_s^G have the same factors except for an extra $1 - P_c$ in Q_s^F when $s < c$. But $s < c$ implies $s < b$, because b is the longest word in G with $b < c$, and $P_bP_c = P_c$ by Lemma 3.3(1); thus $1 - P_b = (1 - P_b)(1 - P_c)$ and $Q_s^F = Q_s^G$, as claimed.

We now have $\sum_{s\in F} Q_s^F = \sum_{s\in G\setminus\{b\}} Q_s^G + Q_b^F + Q_c^F$, and it suffices to show that $Q_b^G = Q_b^F + Q_c^F$. If $t \in G$ and $b < t$, the maximality of b implies that $c \vee t = \infty$, and hence $P_cP_t = 0$ by Lemma 3.3(1). Thus

$$\begin{aligned}
Q_b^G &= P_b(1-P_c)\Big(\prod_{\{t\in G:b<t\}}(1-P_t)\Big) + P_bP_c\Big(\prod_{\{t\in G:b<t\}}(1-P_t)\Big)\\
&= P_b\Big(\prod_{\{t\in F:b<t\}}(1-P_t)\Big) + P_bP_c\\
&= Q_b^F + P_c = Q_b^F + Q_c^F,
\end{aligned}$$

as required. □

Lemma 3.5. *Suppose (ψ,π) is a Toeplitz representation such that A acts faithfully on $(\psi(\bigoplus_{\lambda\in F} X^\lambda)\mathcal{H})^\perp$ for every finite subset F of Λ.*

(1) *Let G be a finite subset of $\mathbb{F}_\Lambda^+ \setminus \{e\}$, and let $s \in \mathbb{F}_\Lambda^+$. Then $\rho_s := \rho^{\psi^s,\pi}$ is a faithful representation of $\mathcal{L}(X^s)$ on $P_s \prod_{t\in G}(1-P_{st})\mathcal{H}$.*

(2) *If F is a finite subset of $\mathbb{F}_\Lambda^+$ with $e \in F$, then for each $s \in F$, ρ_s is a faithful representation of $\mathcal{L}(X^s)$ on $Q_s^F\mathcal{H}$.*

Proof.

(1) Each $t \in G$ has a unique decomposition $t = \lambda_t r$ with $\lambda_t \in \Lambda$ and $r \in \mathbb{F}_\Lambda^+$; write $G' := \{\lambda_t : t \in G\}$. Lemma 3.3(1) implies that the projections P_λ for $\lambda \in \Lambda$ are mutually orthogonal, so $\overline{\psi(\bigoplus_{\lambda\in G'} X^\lambda)\mathcal{H}} = \bigoplus_{\lambda\in G'} P_\lambda\mathcal{H}$, and our hypothesis says that π is faithful on the range of $1 - \sum_{\lambda\in G'} P_\lambda = \prod_{\lambda\in G'}(1-P_\lambda)$. But $P_t \le P_{\lambda_t}$ for each t, so $\prod_{t\in G}(1-P_t) \ge \prod_{\lambda\in G'}(1-P_\lambda)$, and π is also faithful on $\prod_{t\in G}(1-P_t)\mathcal{H}$. Now Proposition 1.6(2) implies that ρ_s is faithful on

$$\mathcal{M}_s := \overline{\operatorname{span}}\Big\{\psi^s(x)\Big(\prod_{t\in G}(1-P_t)h\Big) : x \in X^s,\ h \in \mathcal{H}\Big\},$$

which by (3.2) is precisely $\prod_{t\in G}(1-P_{st})P_s\mathcal{H}$, at least for $s \neq e$. When $s = e$, $\mathcal{M}_e$ is a subspace of $\prod_{t\in G}(1-P_t)\mathcal{H}$, and the result follows.

(2) Apply (1) with $G := \{s^{-1}t : t \in F,\ s < t\}$. □

We can now construct our spatial implementation of the expectation E^Λ.

Proposition 3.6. *Suppose (ψ,π) is a Toeplitz representation of $\bigoplus X^\lambda$ such that A acts faithfully on $(\psi(\bigoplus_{\lambda\in F} X^\lambda)\mathcal{H})^\perp$ for every finite subset F of Λ.*

(1) *There is a norm-decreasing linear map $E_{\psi,\pi}^\Lambda$ on $\psi\times\pi(\mathcal{T}_X)$ such that*

$$E_{\psi,\pi}^\Lambda \circ (\psi\times\pi) = (\psi\times\pi)\circ E^\Lambda;$$

(2) *$\psi\times\pi$ is faithful on $E^\Lambda(\mathcal{T}_X)$.*

Proof. (1) We show that for each finite sum $S := \sum_j i_X^{s_j}(x_j)i_X^{t_j}(y_j)^*$, we have

$$\Big\|\sum_{\{j:s_j=t_j\}} \psi^{s_j}(x_j)\psi^{s_j}(y_j)^*\Big\| \le \Big\|\sum_j \psi^{s_j}(x_j)\psi^{t_j}(y_j)^*\Big\|;$$

then the map $E_{\psi,\pi}^\Lambda : \psi\times\pi(S) \mapsto \psi\times\pi(E^\Lambda(S))$ extends to a well-defined norm-decreasing map on $\psi\times\pi(\mathcal{T}_X)$ with the required properties.

Let $F := \{e\} \cup \{s_j\} \cup \{t_j\}$. Equation (3.3) implies that the projections P_s and Q_s^F commute with every summand in $\psi \times \pi(E^\Lambda(S))$; it follows from Lemma 3.4 that there exists $c \in F$ such that

$$\|\psi \times \pi(E^\Lambda(S))\| = \|Q_c^F(\psi \times \pi(E^\Lambda(S)))\|.$$

If $t \in F$ with $c < t$, then $Q_c^F\psi^t(x) = Q_c^F(1 - P_t)P_t\psi^t(x) = 0$, and if $c \vee t = \infty$, then $Q_c^F\psi^t(x) = Q_c^F P_c P_t \psi^t(x) = 0$; thus compressing by Q_c^F kills all summands in $\psi \times \pi(S)$ except possibly those for which $s_j \leq c$ and $t_j \leq c$. As in the proof of Proposition 2.3, it follows from Proposition 1.8(2) that

$$Q_c^F(\psi \times \pi(E^\Lambda(S))) = Q_c^F\rho_c\Big(\sum_{\{j:s_j=t_j\leq c\}} \Theta_{x_j,y_j} \otimes_A 1^{s_j^{-1}c}\Big),$$

and from Lemma 3.5(2) that

$$\|\psi \times \pi(E^\Lambda(S))\| = \Big\|\sum_{\{j:s_j=t_j\leq c\}} \Theta_{x_j,y_j} \otimes_A 1^{s_j^{-1}c}\Big\|.$$

The idea now is to replace Q_c^F by a smaller projection Q, in such a way that compressing by Q kills the remaining off-diagonal terms of $Q_c^F(\psi \times \pi(S))Q_c^F$ but still preserves the norm of $\psi \times \pi(E^\Lambda(S))$.

For each $s,\ t \in F$ such that $s \neq t$, $s,\ t \leq c$ and $s^{-1}c \vee t^{-1}c < \infty$, we define $d_{s,t} \in \mathbb{F}_\Lambda^+$ as in [18, Lemma 3.2]:

$$d_{s,t} = \begin{cases} (s^{-1}c)^{-1}(t^{-1}c) & \text{if } s^{-1}c < t^{-1}c \\ (t^{-1}c)^{-1}(s^{-1}c) & \text{if } t^{-1}c < s^{-1}c, \end{cases}$$

noting in particular that $d_{s,t}$ is never the identity in $\mathbb{F}_\Lambda^+$. Let

$$G := \{c^{-1}t : t \in F,\ c < t\} \cup \{d_{s,t}\},$$

and define $Q := P_c \prod_{t\in G}(I - P_{ct})$. Notice that we have added factors to the formula for Q_c^F, so $Q \leq Q_c^F$.

To see that Q has the required properties, fix $s,\ t \in F$ satisfying $s \neq t$, $s \leq c$, and $t \leq c$. Then from (3.2) we have

$$Q\psi^s(x)\psi^t(y)^*Q = QP_c\psi^s(x)\psi^t(y)^*P_cQ = Q\psi^s(x)P_{s^{-1}c}P_{t^{-1}c}\psi^t(y)^*Q,$$

which certainly vanishes if $s^{-1}c \vee t^{-1}c = \infty$. But if $s^{-1}c \vee t^{-1}c < \infty$, then $Q \leq P_c - P_{cd_{s,t}}$, so

$$\begin{aligned} Q\psi^s(x)\psi^t(y)^*Q &= Q(P_c - P_{cd_{s,t}})\psi^s(x)\psi^t(y)^*(P_c - P_{cd_{s,t}})Q \\ &= Q\psi^s(x)(P_{s^{-1}c} - P_{s^{-1}cd_{s,t}})(P_{t^{-1}c} - P_{t^{-1}cd_{s,t}})\psi^t(y)^*Q, \end{aligned}$$

which vanishes because either $s^{-1}cd_{s,t} = t^{-1}c$ or $t^{-1}cd_{s,t} = s^{-1}c$. We deduce that

$$Q(\psi \times \pi(S))Q = Q\rho_c\Big(\sum_{\{j:s_j=t_j\le c\}} \Theta_{x_j,y_j} \otimes_A 1^{s_j^{-1}c}\Big).$$

Since $Q\rho_c$ is faithful by Lemma 3.5(1), we have

$$\begin{aligned}\|\psi \times \pi(E^\Lambda(S))\| &= \Big\| \sum_{\{j:s_j=t_j\le c\}} \Theta_{x_j,y_j} \otimes_A 1^{s_j^{-1}c}\Big\| \\ &= \Big\|Q\rho_c\Big(\sum_{\{j:s_j=t_j\le c\}} \Theta_{x_j,y_j} \otimes_A 1^{s_j^{-1}c}\Big)\Big\| \\ &= \|Q(\psi \times \pi(S))Q\| \\ &\le \|\psi \times \pi(S)\|,\end{aligned}$$

giving (1).

Applying the argument of Proposition 2.3(2) to the partition $\{Q_s^F\}$ of 1 gives (2). □

Proof of Theorem 3.1. The first part follows from Proposition 3.6 just as Theorem 2.1 follows from Proposition 2.3. Suppose A acts by compact operators on each summand X^λ. Then A acts by compact operators on $\bigoplus_{\lambda\in F} X^\lambda$ for any finite set F of indices, giving maps $\varphi_F : A \to \mathcal{K}(X)$. An argument like that in the proof of Theorem 2.1 shows that

$$\psi \times \pi(i_A(a) - \rho^{i_X,i_A}(\varphi_F(a))) = \pi(a)\big|_{(1-\sum_{\lambda\in F} P_\lambda)\mathcal{H}}.$$

Applying this with (ψ, π) satisfying the hypothesis of the first part implies that $\alpha_F : a \mapsto i_A(a) - \rho^{i_X,i_A}(\varphi_F(a))$ is an injection of A in $\mathcal{T}_X$. If now (ψ, π) is a Toeplitz representation for which $\psi \times \pi$ is faithful, then composing with α_F shows that the hypothesis is necessary. □

4. THE TOEPLITZ ALGEBRA OF A DIRECTED GRAPH

Let $E = (E^0, E^1, r, s)$ be a directed graph and $X(E)$ the Hilbert bimodule over $A = c_0(E^0)$ discussed in Example 1.2. Recall that $X(E)$ consists of functions on the edge set E^1, and that $X(E)$ and A are spanned by point masses $\{\delta_f : f \in E^1\}$ and $\{\delta_v : v \in E^0\}$, respectively.

Theorem 4.1. *The Toeplitz algebra $\mathcal{T}_{X(E)}$ is generated by a Toeplitz-Cuntz-Krieger E-family $\{i_X(\delta_f),\ i_A(\delta_v) : f \in E^1,\ v \in E^0\}$. It is universal for such families: if $\{S_f, P_v\}$ is a Toeplitz-Cuntz-Krieger E-family on a Hilbert space $\mathcal{H}$, there is a representation $\pi^{S,P} : \mathcal{T}_{X(E)} \to B(\mathcal{H})$ such that $\pi^{S,P}(i_X(\delta_f)) = S_f$ and*

$\pi^{S,P}(i_A(\delta_v)) = P_v$. *The representation* $\pi^{S,P}$ *is faithful if and only if every* P_v *is nonzero (and hence every* S_f *is nonzero), and*

$$P_v > \sum_{\{f\in E^1 : s(f)=v\}} S_f S_f^*$$

for every vertex v *which emits at most finitely many edges.*

Proof. Write $X := X(E)$. We proved in Example 1.2 that $\{\psi(\delta_f), \pi(\delta_v)\}$ is a Toeplitz-Cuntz-Krieger E-family for any Toeplitz representation (ψ, π), and this applies in particular to the canonical representation (i_X, i_A) in $\mathcal{T}_X$. The family generates $\mathcal{T}_X$ because $i_X(X)$ and $i_A(A)$ do, because δ_f and δ_v span dense subspaces of X and A, and because i_X and i_A are isometric. We saw in Example 1.2 how the family $\{S_f, P_v\}$ generates a Toeplitz representation (ψ, π) with $\psi(\delta_f) = S_f$ and $\pi(\delta_v) = P_v$, so $\pi^{S,P} := \psi \times \pi$ has the required property.

For the final statement, we apply Theorem 3.1. For each $f \in E^1$, we let X_f be the bimodule $\mathbb{C}$ in which $a \cdot z = a(s(f))z$, $z \cdot a = za(r(f))$ and $\langle z, w\rangle_A = \bar{z}w\delta_{r(f)}$, and note that $(z_f) \mapsto \sum_f z_f \delta_f$ induces an isomorphism of $\bigoplus_{f\in E^1} X_f$ onto X. (It is easy to check on the algebraic direct sum that the map is a bimodule homomorphism which preserves the inner products.) Since $\mathcal{K}(X_f) = \mathcal{L}(X_f)$ for each f, A acts by compact operators on each X_f, and Theorem 3.1 says that $\pi^{S,P}$ is faithful if and only if A acts faithfully on each $(\bigoplus_{f\in F} \mathcal{H}_f)^\perp$, where $\mathcal{H}_f = \pi^{S,P}(i_X(\delta_f)\mathcal{H}) = S_f\mathcal{H}$. The action of $A = c_0(E^0)$ on any space is faithful iff every δ_v acts nontrivially, so A acts faithfully on $(\bigoplus_{f\in F} \mathcal{H}_f)^\perp$ if and only if

$$0 \neq P_v\Big(1 - \sum_{f\in F} S_f S_f^*\Big) = P_v - \sum_{\{f\in F : s(f)=v\}} S_f S_f^*.$$

If each P_w is nonzero and v emits infinitely many edges, this holds since $P_v \geq \sum_{\{f\in E^1 : s(f)=v\}} S_f S_f^*$, so the result follows. □

Corollary 4.2. *Let* E *be a directed graph, and suppose that* $\{S_f, P_v\}$ *and* $\{T_f, Q_v\}$ *are Toeplitz-Cuntz-Krieger* E*-families such that each* P_v *and* Q_v *is nonzero, and such that*

$$P_v > \sum_{\{f\in E^1 : s(f)=v\}} S_f S_f^* \quad \text{and} \quad Q_v > \sum_{\{f\in E^1 : s(f)=v\}} T_f T_f^*$$

for every vertex v *which emits at most finitely many edges. Then there is an isomorphism* θ *of* $C^*(S_f, P_v)$ *onto* $C^*(T_f, Q_v)$ *such that* $\theta(S_f) = T_f$ *for all* $f \in E^1$ *and* $\theta(P_v) = Q_v$ *for all* $v \in E^0$.

Proof. Take $\theta := \pi^{T,Q} \circ (\pi^{S,P})^{-1}$. □

Corollary 4.3. *Let E be a directed graph with at least one edge. Then $\mathcal{T}_{X(E)}$ is simple if and only if every vertex emits infinitely many edges and every pair of vertices are joined by a finite path.*

Proof. First we show that the hypotheses imply simplicity. Suppose θ is a representation of $\mathcal{T}_{X(E)}$ with a nontrivial kernel, and let $S_f := \theta(s_f)$ and $P_v := \theta(p_v)$. Since each vertex emits infinitely many edges, Theorem 4.1 implies that $P_v = 0$ for some v. If $s(f) = v$, then $S_f = P_v S_f = 0$, and hence $P_{r(f)} = S_f^* S_f = 0$ as well. Since every pair of vertices are joined by a finite path, it follows that $P_w = 0$ for every $w \in E^0$. But then $S_f = S_f S_f^* S_f = S_f P_{r(f)} = 0$ for every $f \in E^1$, and $\theta = 0$.

Conversely, suppose $\mathcal{T}_{X(E)}$ is simple. We prove that we can reach every vertex from a given vertex v by considering the ideal $\langle p_v \rangle$ generated by p_v, which is all of $\mathcal{T}_{X(E)}$ by simplicity. As usual, we write $s_\mu := s_{f_1} \cdots s_{f_n}$ for a finite path $\mu = f_1 \cdots f_n$, define $s_w := p_w$ for each vertex w, and verify that $\mathcal{T}_{X(E)} = \overline{\operatorname{span}}\{s_\mu s_\nu^*\}$. The ideal $\langle p_v \rangle$ is spanned by products of the form $s_\mu s_\nu^* p_v s_\sigma s_\tau^*$, which satisfy

$$s_\mu s_\nu^* p_v s_\sigma s_\tau^* = \begin{cases} s_\mu s_{\sigma'} s_\tau^* & \text{if } s(\nu) = s(\sigma) = v \text{ and } \sigma = \nu\sigma', \\ s_\mu s_{\nu'}^* s_\tau^* & \text{if } s(\nu) = s(\sigma) = v \text{ and } \nu = \sigma\nu', \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

On the other hand, if $r(\mu) = r(\tau)$ can be reached from v, say by α, then $s_\mu s_\tau^* = s_\mu s_\alpha^* s_\alpha s_\tau^* = s_\mu s_\alpha^* p_v s_\alpha s_\tau^*$ belongs to $\langle p_v \rangle$. Thus

$$\langle p_v \rangle = \overline{\operatorname{span}}\{s_\mu s_\tau^* : r(\mu) = r(\tau) \in H(v)\},$$

where $H(v)$ is the set of vertices w for which there is a path from v to w.

We want to prove that $H(v)$ is all of E^0. Suppose there exists $w \in E^0 \setminus H(v)$. We shall show that $\|p_w - b\| \geq 1$ for all $b \in \langle p_v \rangle$, which contradicts $\langle p_v \rangle = \mathcal{T}_{X(E)}$. Suppose $b = \sum \lambda_i s_{\mu_i} s_{\tau_i}^*$ is a typical finite sum in $\langle p_v \rangle$. Let F be the (finite) set of edges which start at w and are the initial edge of some μ_i. Theorem 4.1 implies that the projection $q := p_w - \sum_{f \in F} s_f s_f^*$ is nonzero. But $p_w s_{\mu_i} = 0$ unless $s(\mu_i) = w$, and then $s_f s_f^* s_{\mu_i} = s_{\mu_i}$ for the one f which starts μ_i. Thus

$$qb = \sum_i \lambda_i p_w s_{\mu_i} s_{\tau_i}^* - \sum_i \lambda_i \Big(\sum_{f \in F} s_f s_f^* \Big) s_{\mu_i} s_{\tau_i}^* = 0,$$

and $\|p_w - b\| \geq \|q(p_w - b)\| = \|q\| = 1$, as required.

The transitivity we have just proved implies that each vertex v emits at least one edge. If v emits only finitely many edges, then $q := p_v - \sum_{\{f : s(f) = v\}} s_f s_f^*$ is nonzero by Theorem 4.1. However, one can easily construct Toeplitz-Cuntz-Krieger E-families on Hilbert space such that $P_v = \sum_{\{f : s(f) = v\}} S_f S_f^*$, and then

q would be in the kernel of the corresponding representation of $\mathcal{T}_{X(E)}$. Thus each vertex must emit infinitely many edges. □

In passing from the Toeplitz algebra $\mathcal{T}_X$ to the Cuntz-Pimsner algebra $\mathcal{O}_X$, an important role is played by the ideal $J := \varphi^{-1}(\mathcal{K}(X))$; the theory simplifies when this ideal is either $\{0\}$ or A, and authors have often imposed hypotheses which force $J = A$. (This is done, for example, in [20] and [13].) For the bimodules of graphs, one can identify the ideal J explicitly.

Proposition 4.4. *Let $X(E)$ be the Hilbert bimodule of a directed graph E, and let $\varphi : A \to \mathcal{L}(X(E))$ be the homomorphism describing the left action of $A = c_0(E^0)$. Then*

$$\varphi^{-1}(\mathcal{K}(X(E))) = \overline{\operatorname{span}}\{\delta_v : v \text{ emits at most finitely many edges}\}.$$

Proof. Write $X := X(E)$. Since $\mathcal{K}(X)$ is an ideal in $\mathcal{L}(X)$, $J := \varphi^{-1}(\mathcal{K}(X))$ is an ideal in $A = c_0(E^0)$, and hence has the form

$$\{a \in A : a(w) = 0 \quad \text{for } w \notin F\} = \overline{\operatorname{span}}\{\delta_v : v \in F\}$$

for some subset F of the discrete space E^0. So it suffices to see that $\varphi(\delta_v)$ belongs to $\mathcal{K}(X)$ iff v emits finitely many edges. If v emits finitely many edges, then $\varphi(\delta_v) = \sum_{\{f:s(f)=v\}} \Theta_{\delta_f,\delta_f}$ is compact.

Suppose now that v emits infinitely many edges. Since $\operatorname{span}\{\delta_f\}$ is dense in X and $(x,y) \mapsto \Theta_{x,y}$ is continuous, we can approximate any compact operator on X by a finite linear combination of the form $K := \sum_{e,f\in F} \lambda_{e,f}\Theta_{\delta_e,\delta_f}$. But for any such combination K, we can find an edge $g \notin F$ such that $s(g) = v$, and then $\Theta_{\delta_e,\delta_f}(\delta_g) = \delta_e \cdot \langle \delta_f, \delta_g\rangle_A = 0$ for all $e, f \in F$. Thus

$$\begin{aligned} \|\varphi(\delta_v) - K\| &= \sup\{\|(\varphi(\delta_v) - K)(x)\| : \|x\|_A \le 1\} \\ &\ge \|\varphi(\delta_v)(\delta_g) - K(\delta_g)\| \\ &= \|\delta_g - 0\| = 1, \end{aligned}$$

and hence $\varphi(\delta_v)$ is not compact. □

Corollary 4.5. *If E is a directed graph in which every vertex emits infinitely many edges, then the Cuntz-Pimsner algebra $\mathcal{O}_{X(E)}$ coincides with the Toeplitz algebra $\mathcal{T}_{X(E)}$, and is simple if and only if E is transitive.*

Remark 4.6. Since (at least in the absence of sources and sinks) the Cuntz-Pimsner algebra $\mathcal{O}_{X(E)}$ is generated by a Cuntz-Krieger family for the edge matrix B of E, one might guess that $\mathcal{O}_{X(E)}$ is isomorphic to the Cuntz-Krieger algebra $\mathcal{O}_B$ of [8], and that this last Corollary follows from [8, Theorem 14.1]. This guess is correct, but the connection is nontrivial; since it concerns Cuntz-Pimsner algebras rather than Toeplitz algebras, we shall present the details elsewhere. We note also that our Toeplitz algebra $\mathcal{T}_{X(E)}$ is not the Toeplitz-Cuntz-Krieger algebra $\mathcal{TO}_B$ discussed in [8]: their relations do not imply that the initial projections P_v are mutually orthogonal.

5. Concluding remarks

To see why we have avoided placing additional hypotheses on our bimodules, consider the Cuntz-Krieger bimodules of graphs. We want to allow graphs with infinite valency, so Proposition 4.4 shows that A will not always act by compact operators. We also want to consider graphs with sinks (vertices which emit no edges) and sources (vertices which receive no edges). Since $v \in E^0$ is a sink iff $\delta_v \in c_0(E^0)$ acts trivially on the left of $X(E)$, $\varphi : A \to \mathcal{L}(X)$ may not be injective; since v is a source iff δ_v is not in the ideal $\overline{\operatorname{span}}\{\langle x, y\rangle_A\}$, X need not be full as a right Hilbert module.

Every Cuntz-Krieger bimodule $X = X(E)$ is essential, in the sense that $\overline{\operatorname{span}}\, A \cdot X = X$, because $\delta_{s(f)} \cdot \delta_f = \delta_f$ for every $f \in E^1$. However, the following non-essential submodules arise in analysing the ideal structure of $\mathcal{T}_{X(E)}$. Suppose $V \subset E^0$ is hereditary in the sense that $r(f) \in V$ whenever $s(f) \in V$. Then $I := c_0(V)$ is an ideal in $c_0(E^0)$ such that $I \cdot X(E) \subseteq X(E) \cdot I$, so $X(E) \cdot I$ is a Hilbert I-bimodule. However, if there is an edge f such that $s(f) \notin V$ and $r(f) \in V$, then $\delta_f \in X(E) \cdot I$ but $a \cdot \delta_f = 0$ for all $a \in I$.

Because our modules may not be essential, we cannot require that the representations π in our Toeplitz representations (ψ, π) are nondegenerate: in the Fock representation induced from a nondegenerate representation of A, π is nondegenerate if and only if X is essential. Moreover, the essential subspace of π need not be invariant under ψ, so it is not in general possible to reduce to the nondegenerate case as one typically does when dealing with representations of a $*$-algebra. The following Corollary illustrates an extreme case: when the left action is trivial, ψ and π have orthogonal ranges. In general, we believe the correct notion of nondegeneracy for a Toeplitz representation (ψ, π) is that the C^*-algebra generated by $\psi(X) \cup \pi(A)$ acts nondegenerately; see the proof of Proposition 1.3.

Corollary 5.1. *Suppose the left action of A on X is trivial.*

(1) *$\psi \times \pi$ is faithful if and only if π is faithful. If A is simple, so is $\mathcal{T}_X$.*
(2) *$\mathcal{T}_X$ is canonically isomorphic to the algebra*

$$L(X) := \mathcal{K}(X \oplus A) = \begin{pmatrix} \mathcal{K}(X) & X \\ \widetilde{X} & A \end{pmatrix};$$

if X_A is full, $L(X)$ is the linking algebra of the imprimitivity bimodule ${}_{\mathcal{K}(X)}X_A$ (see [25, Section 3.2]).

Proof.

(1) If $\psi \times \pi$ is faithful, so is $(\psi \times \pi) \circ i_A = \pi$. On the other hand, for $a \in A$ and $x \in X$ we have $\pi(a)\psi(x) = \psi(a \cdot x) = 0$, so π acts trivially on $\overline{\psi(X)\mathcal{H}}$. Thus if π is faithful it must be faithful on $(\psi(X)\mathcal{H})^{\perp}$, and $\psi \times \pi$ is faithful by the Theorem.

(2) The formulas $\psi(x) := \left(\begin{smallmatrix} 0 & x \\ 0 & 0 \end{smallmatrix}\right)$ and $\pi(a) := \left(\begin{smallmatrix} 0 & 0 \\ 0 & a \end{smallmatrix}\right)$ define a Toeplitz representation of X in $L(X)$ such that π is faithful and $\psi(X) \cup \pi(A)$ generates $L(X)$. Now use (1). □

Our next application is a different extension of Cuntz's result on the simplicity of $\mathcal{O}_\infty$: to recover it, take each $X^\lambda = {}_{\mathbb{C}}\mathbb{C}_{\mathbb{C}}$.

Corollary 5.2. *Let X be a Hilbert bimodule over a simple C^*-algebra A. If $X = \bigoplus_{\lambda \in \Lambda} X^\lambda$ and the left action of A is nontrivial on infinitely many summands, then the Toeplitz algebra $\mathcal{T}_X$ is simple.*

Proof. If $\psi \times \pi$ is a nonzero representation of $\mathcal{T}_X$ on $\mathcal{H}$, then the simplicity of A implies that π and ψ are faithful. Since the summands in X are mutually orthogonal, this implies that the action of π in each $(\psi(\bigoplus_{\lambda \in F} X^\lambda)\mathcal{H})^{\perp}$ is nonzero and hence faithful. Thus the result follows from Theorem 3.1. □

Our final application is motivated by Pimsner's realisation of crossed products by endomorphisms as $\mathcal{O}_X$ for suitable X. Let τ denote the forward-shift endomorphism on the C^*-algebra c of bounded sequences, and let $X := \tau(1)c$ be the Hilbert bimodule over c in which $x \cdot a := xa$, $\langle x, y\rangle_c := x^* y$ and $a \cdot x := \tau(a)x$. Since the identity operator on X is compact, Theorem 2.1 applies, and we recover a theorem of Conway, Duncan and Paterson [2] (see also [11, Theorem 1.3]). Recall that an element v in a C^*-algebra is a *power partial isometry* if v^n is a partial isometry for every $n \geq 1$.

Proposition 5.3. *$\mathcal{T}_X$ is unital, $v := i_X(\tau(1))^*$ is a power partial isometry, and $\mathcal{T}_X = C^*(1, v)$. The pair $(\mathcal{T}_X, v)$ has the following universal property: if B is a unital C^*-algebra and $V \in B$ is a power partial isometry, there is a unital homomorphism $\mathcal{T}_X \to B$ which maps v to V.*

Proof. $i_c(1)$ is an identity for $\mathcal{T}_X$, and the calculation

$$i_c(\tau(a)) = i_X(\tau(1))^* i_X(\tau(a)) = v i_X(a \cdot \tau(1)) = v i_c(a) v^*$$

shows that $v^n v^{*n} = i_c(\tau^n(1))$ is a projection. These projections and the identity generate $i_c(c)$; this and $i_X(x) = v^* i_c(x)$ show that $\mathcal{T}_X = C^*(1, v)$.

Suppose $V \in B$ is a power partial isometry. Since $V^n V^{*n} \geq V^{n+1} V^{*(n+1)}$, there is a unital homomorphism $\pi_V : c \to B$ which satisfies $\pi_V(\tau^n(1)) = V^n V^{*n}$. Define $\psi_V(x) := V^* \pi_V(x)$. We claim that (ψ_V, π_V) is a Toeplitz representation. Conditions (1.1) and (1.2) for a Toeplitz representation are easy. For (1.3) notice that $\pi_V(\tau(a)) = V\pi_V(a)V^*$, and recall from [10] that the initial and range projections of the powers of V form a commuting family, so that $V^*V \in \pi(c)'$; thus

$$\begin{aligned}\psi_V(a \cdot x) = \psi_V(\tau(a)x) = V^*\pi_V(\tau(a))\pi_V(x) = V^*V\pi_V(a)V^*\pi_V(x)\\ = \pi_V(a)V^*VV^*\pi_V(x) = \pi_V(a)V^*\pi_V(x) = \pi_V(a)\psi_V(x),\end{aligned}$$

as required. Since $\psi_V \times \pi_V(i_c(1)) = \pi_V(1) = 1$ and $\psi_V \times \pi_V(v) = \psi_V(\tau(1))^* = \pi_V(\tau(1))V = VV^*V = V$, $\psi_V \times \pi_V$ is the desired map. □

Corollary 5.4. *Let J_n denote the truncated shift on $\mathbb{C}^n$ (with $J_1 := 0$). Then $C^*(1, \oplus J_n)$ is the universal unital C^*-algebra generated by a power partial isometry.*

Proof. If $V := \bigoplus J_n$, then Theorem 2.1 implies that $\psi_V \times \pi_V$ is faithful. □

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