2002

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Urzyczyn proved in [6] that for both these systems it is undecidable whether a given intersection type is empty. Kurata and Takahashi however have shown in [5] that this emptiness problem is decidable for the sytem including (η). but without ($\wedge I$).

The aim of this paper is to classify intersection type systems lacking some of ($\wedge I$), ($\wedge E$) and (η), into equivalence classes according to their strength in typing λ-terms and also according to their strength in possessing inhabitants.

This classification is used in a later paper to extend the above (un)decidability results to two of the five inhabitation-equivalence classes. This later paper also shows that the systems in two more of these classes have decidable inhabitation problems and develops algorithms to find such inhabitants.

Keywords

systems, type, classification, intersection

Disciplines

Engineering | Science and Technology Studies

Publication Details

Bunder, M. W. (2002). A classification of intersection type systems. Journal of Symbolic Logic, 67 (1), 353-368.

THE JOURNAL OF SYMBOLIC LOGIC
Volume 67, Number 1, March 2002

A CLASSIFICATION OF INTERSECTION TYPE SYSTEMS

M. W. BUNDER

In honour of Roger Hindley on his 60th birthday.

Abstract. The first system of intersection types, Coppo and Dezani [3], extended simple types to include intersections and added intersection introduction and elimination rules (($\wedge I$) and ($\wedge E$)) to the type assignment system. The major advantage of these new types was that they were invariant under β-equality, later work by Barendregt, Coppo and Dezani [1], extended this to include an (η) rule which gave types invariant under $\beta\eta$-reduction.

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§1. The system $\lambda\wedge$ and subsystems.

1.1. DEFINITION (Types).

(i) Type variables $a, b, c, \ldots,$ and ω, the universal type, are types.
(ii) If α and β are types so are $(\alpha \to \beta)$ and $(\alpha \wedge \beta)$.

1.2. DEFINITION (TA (type assignment) statements). If α is a type and M a λ-term, $M : \alpha$ is a TA-statement.

1.3. DEFINITION (TA-judgements). If $\Delta = \{x_1 : \alpha_1, \ldots, x_n : \alpha_n\}$ is a set of TA-statements and $M : \alpha$ is a TA-statement then $\Delta \vdash M : \alpha$ is a TA-judgement.

1.4. DEFINITION (The type assignment system $\mathrm{TA}_\lambda(\wedge, \omega)$ or $\lambda \wedge \omega$).

Axiom scheme (ω)	$\vdash M : \omega$
(Var)	if $x : \alpha \in \Delta$, $\quad \Delta \vdash x : \alpha$

Received January 11, 2000; revised October 24, 2000; accepted April 18, 2001.
This work was supported by Fondi Per Professori Visitatori dell Universitate di Torino. The author wishes to thank Mariangiola Dezani for her help with this project.

0022-4812/02/6701-0024/$2.60

$$(\to E) \qquad \frac{\Delta \vdash M : \alpha \to \beta \qquad \Delta \vdash N : \alpha}{\Delta \vdash MN : \beta}$$

$$(\to I) \qquad \frac{\Delta, x : \alpha \vdash N : \beta \qquad x \notin \mathrm{FV}(\Delta)}{\Delta \vdash \lambda x.N : \alpha \to \beta}$$

($\mathrm{FV}(\Delta)$ is the set of free variables of Δ.)

$$(\wedge I) \qquad \frac{\Delta \vdash M : \alpha \qquad \Delta \vdash M : \beta}{\Delta \vdash M : \alpha \wedge \beta}$$

$$(\wedge E) \qquad \frac{\Delta \vdash M : \alpha \wedge \beta}{\Delta \vdash M : \alpha} \qquad \frac{\Delta \vdash M : \alpha \wedge \beta}{\Delta \vdash M : \beta}$$

$$(\eta) \qquad \frac{\Delta \vdash \lambda x.Mx : \alpha \qquad x \notin \mathrm{FV}(M)}{\Delta \vdash M : \alpha}.$$

1.5. Definition (The systems $\lambda(\dots)$). The system $\lambda\wedge\omega$, without ω as a type and without rule (ω) is called $\lambda\wedge$.

$\lambda\wedge$ without rules $(\wedge E)$, (η) and $(\wedge I)$ we will call $\lambda(\ )$. $\lambda(\ )$ with $(\wedge I)$ is called $\lambda(\wedge I)$, with $(\wedge I)$ and (η), $\lambda(\wedge I, \eta)$, etc.

We will be, in this paper, dealing only with $\lambda\wedge$ and its subsystems that include $\lambda(\ )$. Van Bakel has shown in [7]:

1.6. Lemma. *If M is in normal form and Δ and α are ω-free then*

$$\Delta \vdash_{\lambda\wedge\omega} M : \alpha \Leftrightarrow \Delta \vdash_{\lambda\wedge} M : \alpha.$$

Similar results hold for the subsystems of $\lambda \wedge \omega$ and $\lambda\wedge$.

Some formulations of $\lambda\wedge$ use a partial order $\leq$ over types. This is defined as follows:

1.7. Definition ($\leq$).

(1) $\alpha \leq \alpha$

(2) $\alpha \wedge \beta \leq \alpha$

(3) $\alpha \wedge \beta \leq \beta$

(4) $(\alpha \to \beta) \wedge (\alpha \to \gamma) \leq \alpha \to \beta \wedge \gamma$.

(5) $\alpha \leq \beta \;\&\; \beta \leq \gamma \Rightarrow \alpha \leq \gamma$.

(6) $\alpha \leq \beta \;\&\; \alpha \leq \beta^1 \Rightarrow \alpha \leq \beta \wedge \beta^1$.

(7) $\alpha \leq \beta \;\&\; \alpha^1 \leq \beta^1 \Rightarrow \beta \to \alpha^1 \leq \alpha \to \beta^1$.

Note that the standard formulation of this has instead of (6):

(8) $\alpha \leq \alpha \wedge \alpha$ and

(9) $\alpha \leq \beta \;\&\; \alpha^1 \leq \beta^1 \Rightarrow \alpha \wedge \alpha^1 \leq \beta \wedge \beta^1$.

(9) can easily be derived using (2), (3), (5) and (6). (8) follows from (1) and (6). Conversely (6) follows from (9), (8) and (5).

The following rule is then valid in $\lambda\wedge$:

1.8. Rule ($\leq$).

$$\frac{\Delta \vdash M : \alpha \qquad \alpha \leq \beta}{\Delta \vdash M : \beta}.$$

In fact Hindley [4, Lemma 3.3.4] shows that (η) in $\lambda\wedge$ is equivalent to $(\leq)$. By 1.7(3) and (4), $(\wedge E)$ also follows from $(\leq)$.

For each of the weaker type systems A we will find a weaker form of $(\leq)$ (called $(\leq_A)$) replacing (η) and/or $(\wedge E)$. Each A determines a subset of the clauses (1) to (9).

1.9. DEFINITION. If $i_1, \dots, i_k \in \{1, 2, \dots, 7\}$, $\leq_{i_1,\dots,i_k}$ denotes a weaker form of $\leq$ defined by postulates $(i_1), \dots, (i_k)$.

$\leq_{-j_1,\dots,j_{7-k}}$ is $\leq_{i_1,\dots,i_k}$ if $\{i_1, \dots, i_k, j_1, \dots, j_{7-k}\} = \{1, 2, \dots, 7\}$.
For each of the $\leq_{i_1,\dots,i_k}$ we postulate:

1.10. RULE $(\leq_{i_1,\dots,i_k})$.

$$\frac{\Delta \vdash M : \alpha \qquad \alpha \leq_{i_1,\dots,i_k} \beta}{\Delta \vdash M : \beta}.$$

Each of the systems A will be associated with a $\leq \dots$ as follows.

1.11. DEFINITION $(\leq_A)$. If A is:

$\lambda()$	$\leq_A$ is $\equiv$	$\lambda(\wedge E)$	$\leq_A$ is $\leq_{-4,6,7}$
$\lambda(\wedge I, \wedge E)$	$\leq_A$ is $\leq_{-4,7}$	$\lambda(\eta)$	$\leq_A$ is $\leq_{1,5,7}$
$\lambda(\wedge I, \eta)$	$\leq_A$ is $\leq_{1,5,6,7}$	$\lambda(\wedge E, \eta)$	$\leq_A$ is $\leq_{-4,6}$
$\lambda(\wedge I)$	$\leq_A$ is $\leq_{1,5,6}$	$\lambda(\wedge I, \wedge E, \eta)$, or $\lambda(\wedge I, \leq)$	$\leq_A$ is $\leq$.

We need one lemma before we can show that each system A satisfies "Rule $(\leq_A)$".

1.12. LEMMA. *If A is a type system and $x \notin \mathrm{FV}(\Delta)$ then*

(a) $$\Delta \vdash_A M : \alpha$$

implies

$$\Delta, x : \beta \vdash_A M : \alpha.$$

PROOF. By an easy induction on the derivation of (a). ⊣

1.13. LEMMA $(\leq_A)$. *If A is type system, $\alpha \leq_A \beta$ and*

(b) $$\Delta \vdash_A M : \alpha$$

then

(c) $$\Delta \vdash_A M : \beta.$$

PROOF. By induction on the derivation of $\alpha \leq_A \beta$. The cases below are numbered according to the last clause used in this derivation.
Case (1). $\beta \equiv \alpha$. Obvious.
Cases (2) and (3). $\alpha \equiv \beta \wedge \gamma$ or $\gamma \wedge \beta$ for some γ. As (2) and (3) are included in the definition of $\leq_A$, A includes $(\wedge E)$ so (b) gives (c).
Case (4). In this case

$$\alpha \equiv (\alpha_1 \to \beta_1) \wedge (\alpha_1 \to \gamma_1)$$

and

$$\beta \equiv \alpha_1 \to \beta_1 \wedge \gamma_1.$$

As (4) is included in the definition of $\leq_A$, A has $(\leq)$ or $(\wedge E)$, (η) and $(\wedge I)$ from which $(\leq)$ is derivable, so in each case (c) can be derived from (b).
Case (5). If $\alpha \leq_A \beta$ comes from $\alpha \leq_A \gamma$ and $\gamma \leq_A \beta$, both of these have shorter derivations, so by the induction hypothesis first:

$$\Delta \vdash_A M : \gamma$$

and then (c).

Case (6). In this case $\alpha \leq_A \gamma$, $\alpha \leq_A \delta$, again with shorter derivations, and $\beta \equiv \gamma \wedge \delta$.

As (6) is included in the definition of $\leq_A$, A has $(\leq)$ or $(\wedge I)$. The former case is obvious, in the latter case we have, by the induction hypothesis:

$$\Delta \vdash_A M : \gamma$$

and

$$\Delta \vdash_A M : \delta$$

and (c) follows by $(\wedge I)$.

Case (7). In this case

$$\alpha \equiv \gamma \to \delta,$$
$$\beta \equiv \xi \to \zeta$$

where, by shorter derivations, $\xi \leq_A \gamma$ and $\delta \leq_A \zeta$.

As (7) is included in the definition of $\leq_A$, A has $(\leq)$ or (η). The former case is obvious, in the latter case we have using Lemma 1.12:

$$\Delta, x : \xi \vdash_A M : \gamma \to \delta$$

By $\xi \leq_A \gamma$ and the induction hypothesis:

$$\Delta, x : \xi \vdash_A x : \gamma,$$

so by $(\to E)$

$$\Delta, x : \xi \vdash_A Mx : \delta.$$

By $\delta \leq_A \zeta$ and the induction hypothesis

$$\Delta, x : \xi \vdash_A Mx : \zeta,$$

so by $(\to I)$ and (η) we get (c). ⊣

§2. Relations between subsystems of $\lambda\wedge$.

2.1. Definition. If A and B are type systems then

$$A \preceq_1 B \equiv (\forall \Delta, M, \alpha)(\Delta \vdash_A M : \alpha \to \Delta \vdash_B M : \alpha)$$
$$A \preceq_2 B \equiv (\forall \Delta, \alpha)\big((\exists M)\Delta \vdash_A M : \alpha \to (\exists M)\Delta \vdash_B M : \alpha\big)$$
$$A \approx_i B \equiv A \preceq_i B \,\&\, B \preceq_i A \qquad (i = 1, 2)$$
$$A \prec_i B \equiv A \preceq_i B \,\&\, \sim A \approx_i B \qquad (i = 1, 2)$$
$$A \prec_{1,2} B \equiv A \prec_1 B \,\&\, A \prec_2 B.$$

The $\approx_1$ equivalence is the standard one, and $\prec_1$ the weaker than relation between type systems. Systems are $\approx_2$ equivalent if they have equivalent emptiness problems, $\prec_2$ is the corresponding "weaker than".

The lemma below follows from Definition 2.1.

2.2. Lemma. *If A, B, and C are type systems*

(i) $A \preceq_1 B \to A \preceq_2 B$
(ii) $A \approx_1 B \to A \approx_2 B$
(iii) $A \prec_2 B \to \sim A \approx_1 B$
(iv) $A \preceq_1 B \,\&\, \sim A \approx_2 B \to A \prec_{1,2} B$
(v) $A \preceq_i B \,\&\, B \preceq_i C \to A \preceq_i C \quad (i = 1, 2)$.

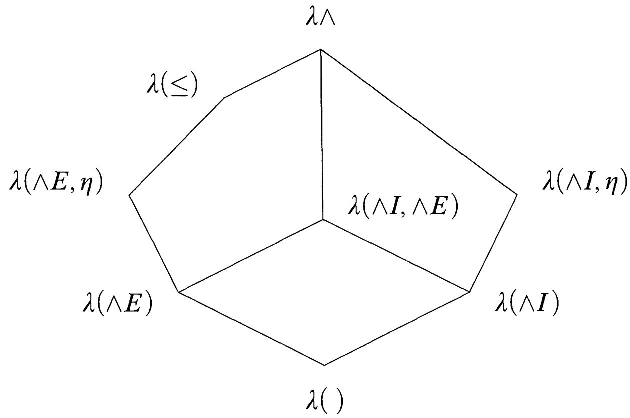

FIGURE 1.

We will see below that it is possible to have $A \prec_1 B$ and $A \approx_2 B$ (i.e., $\sim A \prec_2 B$) as well as $A \prec_2 B$ and $\sim A \preceq_1 B$.

In Section 6 below we will prove the following equivalences:

2.3. THEOREM.

(i) $\lambda\wedge \equiv \lambda(\wedge I, \wedge E, \eta) \approx_1 \lambda(\wedge I, \leq)$
(ii) $\lambda(\leq) \approx_1 \lambda(\wedge E, \eta, \leq)$
(iii) $\lambda(\ ) \approx_1 \lambda(\eta)$
(iv) $\lambda\wedge \approx_2 \lambda(\wedge I, \wedge E)$
(v) $\lambda(\wedge E, \eta) \approx_2 \lambda(\wedge E)$
(vi) $\lambda(\wedge I) \approx_2 \lambda(\wedge I, \eta)$.

Note that by Lemma 2.2(ii) there are further $\approx_2$ equivalences.

Theorems 7.1 and 7.2 below imply the inequivalence results given in the next two theorems.

2.4. THEOREM. *Type systems are related as given in Figure* 1 *where a downward line from a system B to a system A represents $A \prec_1 B$. Unconnected systems are independent.*

2.5. THEOREM. *Type systems are related as in Figure* 2 *where a downward line represents $A \prec_2 B$. Unconnected systems are independent.*

A later paper will show that the inhabitation problem for systems $\approx_2$ equivalent to $\lambda\wedge$ is undecidable and decidable for systems $\approx_2$ to $\lambda(\leq)$. This follows from the $\approx_2$ equivalences in Theorem 2.3 and the work of Urzyczyn [6] and Kurata and Takahashi [5]. New work will show that the systems $\approx_2$ equivalent to $\lambda(\ )$ and $\lambda(\wedge E)$ also have a decidable inhabitation problem. Algorithms will be supplied to find such inhabitants. We suspect, but have not yet proved, that the inhabitation problem for systems $\approx_2$ equivalent to $\lambda(\wedge I)$ is also decidable.

2.6. NOTE. In $\lambda\wedge$ we have:

$$\Delta \vdash M : \alpha \wedge \beta \Leftrightarrow \Delta \vdash M : \beta \wedge \alpha$$

and

$$\Delta \vdash M : \alpha \wedge (\beta \wedge \gamma) \Leftrightarrow \Delta \vdash M : (\alpha \wedge \beta) \wedge \gamma,$$

so that all bracketings and orderings of the α_i's in $\alpha_1 \wedge \cdots \wedge \alpha_\eta$ are equal.

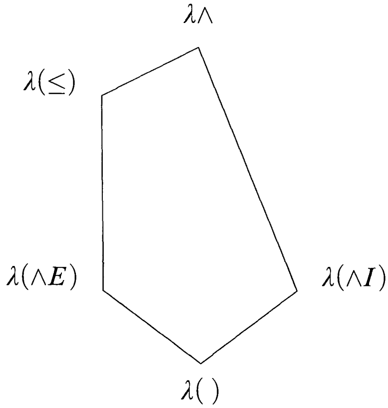

FIGURE 2.

In systems that do not have both $(\wedge I)$ and $(\wedge E)$ the commutative and associative laws do not hold. In these systems expressions such as $\Delta \vdash M : \alpha_1 \wedge \cdots \wedge \alpha_n, M : \alpha_1 \wedge \cdots \wedge \alpha_n \in \Delta$, or $\gamma \equiv \alpha_1 \wedge \cdots \wedge \alpha_n$ will be interpreted as having a particular bracketing and ordering of $\alpha_1, \ldots, \alpha_n$, which will be clear from the context.

Before we can examine the relations between the various systems we need a large number of preliminary results.

§3. Condensing and replacement lemmas.

3.1. CONDENSING LEMMA. *If A is any type system*

$$\Delta, x : \alpha \vdash_A M : \beta \tag{d}$$

and $x \notin \mathrm{FV}(M)$, *then, by a derivation no longer than that of* (d)

$$\Delta \vdash_A M : \beta.$$

PROOF. By an easy induction on the derivation of (d). ⊣

3.2. LEMMA. *If A is any type theory and*

$$\Delta \vdash_A N : \beta \tag{e}$$

then

(i) $y \in \mathrm{FV}(N) \rightarrow y \in \mathrm{FV}(\Delta)$
(ii) $y \in \mathrm{BV}(N) \rightarrow y \notin \mathrm{FV}(\Delta)$

($\mathrm{BV}(N)$ *is the set of bound variables of* N).

PROOF. By induction on the derivation of (e). ⊣

3.3. THE REPLACEMENT LEMMA. *If A is a type system and*

$$\Delta, x : \alpha \vdash_A M : \beta \tag{f}$$

and

$$\Delta \vdash_A N : \alpha$$

then

$$\Delta \vdash_A [N/x]M : \beta. \tag{g}$$

PROOF. By induction on the derivation of (f). Most cases are simple (the (Var) case uses Lemma 3.1). We will consider the cases where (f) is obtained by $(\to I)$ and (η).
Case $(\to I)$. (f) comes by $(\to I)$ from:

$$\Delta, x : \alpha, y : \gamma \vdash_A P : \delta$$

where $\beta \equiv \gamma \to \delta$ and $M \equiv \lambda y.P$.

By the induction hypothesis

(h) $$\Delta, y : \gamma \vdash_A [N/x]P : \delta.$$

By Lemma 3.2 applied to (f), $y \notin \mathrm{FV}(\Delta)$ and so $y \notin \mathrm{FV}(N)$ and $\lambda y.[N/x]P \equiv [N/x]\lambda y.P$.

By $(\to I)$, therefore, (g) follows from (h).
Case (η). (f) comes by (η) from

$$\Delta, x : \alpha \vdash_A \lambda y.My : \beta$$

where $y \notin \mathrm{FV}(M)$.

Again by Lemma 3.2, $y \notin \mathrm{FV}(N)$ and so

$$\Delta \vdash_A [N/x]\lambda y.My : \beta,$$

obtained by the induction hypothesis, is

$$\Delta \vdash_A \lambda y.([N/x]M)y : \beta$$

which gives (g) by (η). ⊣

§4. **Lemmas concerning $\leq_{\dots}$.** Lemma 13 of [2] gives us:

4.1. LEMMA. *$\alpha \leq \beta \Leftrightarrow \alpha$ is some conjunction of $a_1, \dots, a_n, (\alpha_1 \to \gamma_1), \dots, (\alpha_m \to \gamma_m)$, β is some conjunction of $b_1, \dots, b_k, (\beta_1 \to \delta_1), \dots, (\beta_\ell \to \delta_\ell)$ where*:

$$\{b_1, \dots, b_k\} \subseteq \{a_1, \dots, a_n\}$$

and

$$(\forall i)\big(1 \leq i \leq \ell \to (\exists j_1, \dots, j_r \in \{1, \dots, m\})\ \alpha_{j_1} \wedge \dots \wedge \alpha_{j_r} \geq \beta_i \ \&\ \gamma_{j_1} \wedge \dots \wedge \gamma_{j_r} \leq \delta_i\big).$$

Here we need some special cases of this.

4.2. LEMMA.

(i) $$\alpha \wedge \beta \leq \delta \to \gamma \Leftrightarrow \alpha = a_1 \wedge \dots \wedge a_p \wedge (\alpha_1 \to \gamma_1) \wedge \dots \wedge (\alpha_q \to \gamma_q)$$
$$\& \ \beta = a_{p+1} \wedge \dots \wedge a_n \wedge (\alpha_{q+1} \to \gamma_{q+1}) \wedge \dots \wedge (\alpha_m \to \gamma_m)$$
$$\& \ (\exists j_1, \dots, j_r \in \{1, \dots, m\})\alpha_{j1} \wedge \dots \wedge \alpha_{j_r} \geq \delta \ \&\ \gamma_{j_1} \wedge \dots \wedge \gamma_{j_r} \leq \gamma.$$

(ii) $\alpha \to \beta \leq \delta \to \gamma \Leftrightarrow \delta \leq \alpha \ \&\ \beta \leq \gamma$.

We require some similar results for weaker forms of $\leq$.

4.3. LEMMA.

(i) $\alpha \leq_{1,5,7} \beta \Leftrightarrow \alpha \equiv \beta$.
(ii) $\alpha \wedge \beta \leq_{1,5,6,7} \gamma \Leftrightarrow \gamma \equiv (\alpha \wedge \beta) \wedge \dots \wedge (\alpha \wedge \beta)$.

(iii) $\alpha \leq_{1,5,6,7} \beta \rightarrow \gamma \Leftrightarrow \alpha \equiv \sigma \rightarrow \tau \;\&\; \tau \leq_{1,5,6,7} \gamma \;\&\; \beta \leq_{1,5,6,7} \sigma$.
(iv) $\alpha \leq_{1,5,6} \beta \Leftrightarrow \beta \equiv \alpha \wedge \cdots \wedge \alpha$.
(v) $\alpha \leq_{-4,6,7} \beta \Leftrightarrow (\exists \gamma_1 \ldots \gamma_n)\alpha \equiv \gamma_1 \wedge \cdots \wedge \beta \wedge \cdots \wedge \gamma_n$.
(vi) $\alpha \rightarrow \beta \leq_{-4,7} \gamma \rightarrow \delta \Leftrightarrow \alpha \equiv \gamma \;\&\; \beta \equiv \delta$.
(vii) $\alpha \leq_{-4,7} \gamma \rightarrow \delta \Leftrightarrow \alpha \equiv \alpha_1 \wedge \cdots \wedge (\gamma \rightarrow \delta) \wedge \cdots \wedge \alpha_n$.

PROOF. [$\Rightarrow$] By induction on the derivation of $\alpha \leq_{\ldots} \beta$. [$\Leftarrow$] Obvious. $\dashv$

§5. Cut elimination theorems and generation lemmas.

5.1. DEFINITION (Cuts and cut formulas). A **cut** is a sequence of proof steps consisting of a $(\wedge I)$ or $(\rightarrow I)$ step, zero or more $(\leq_A)$ or (η) steps and a $(\wedge E)$ or $(\rightarrow E)$ step. The type to which the elimination rule is applied is called the **cut formula**. Cuts are called $\rightarrow$ or $\wedge$ cuts if they have no $(\leq_A)$ or (η) steps otherwise they are $\wedge{\leq}\wedge$, $\wedge{\leq}{\rightarrow}$, ${\rightarrow}{\leq}\wedge$, ${\rightarrow}{\leq}{\rightarrow}$, ${\rightarrow}\eta{\rightarrow}$, or $\wedge\eta\wedge$ cuts, etc., depending on the rules involved.

The longest cut formula in a derivation is called a **maximal cut formula** and the cut that contains it a **maximal cut**.

5.2. CUT ELIMINATION THEOREM. *If A is a type theory with* (Var), $(\rightarrow E)$, $(\rightarrow I)$ *any of* $(\wedge I)$, $(\wedge E)$ *and* (η) *and*

(i) $$\Delta \vdash_A M : \alpha$$

then there is a derivation with no cut formulas of

$$\Delta \vdash_A M^1 : \alpha$$

where $M =_{\beta\eta} M^1$.

PROOF. Consider the derivations of (i) that have the shortest maximal cut formulas. Of these consider the ones that have the fewest maximal cut formulas and of these consider the shortest derivations.

Now consider the first maximal cut (if any) in a shortest derivation.
Case 1. The derivation has a maximal $\rightarrow$ cut of the form:

$$(\rightarrow E)\ \dfrac{\begin{array}{c} D_2 \\ \Delta \vdash_A N : \beta \end{array} \qquad (\rightarrow I)\ \dfrac{\begin{array}{c} D_1 \\ \Delta, x : \beta \vdash_A P : \gamma \end{array}}{\Delta \vdash_A \lambda x.P : \beta \rightarrow \gamma}}{\Delta \vdash_A (\lambda x.P)N : \gamma}\ .$$

This can be replaced, by the Replacement Lemma (3.3) by a derivation which replaces each use of (Var) of the form:

$$\Delta, x : \beta \vdash_A x : \beta$$

by

$$\begin{array}{c} D_2 \\ \Delta \vdash_A N : \beta, \end{array}$$

removes all $x : \beta$'s from the left of the $\vdash_A$ and replaces each x to the right of the $\vdash_A$, by N giving a new D_1', replacing D_1 and:

$$\begin{array}{c} D_1' \\ \Delta \vdash_A [N/x]P : \gamma. \end{array}$$

Note that γ may now have become a cut formula and also β above, but these are shorter than the eliminated maximal cut formula $\beta \to \alpha$.

Thus either the only maximal cut formula is reduced in length or the number of cut formulas of maximal length has been reduced by one. This is impossible, so none of the maximal cuts can be $\to$ cuts.

Case 2. The derivation has a maximal $\to\eta\to$ cut of the form:

$$(\to E)\,\dfrac{\Delta \vdash_A N:\beta \qquad (\eta)\,\dfrac{(\to I)\,\dfrac{\begin{array}{c}D_1\\ \Delta, x:\beta \vdash_A (\lambda x_1.(\lambda x_2.(\cdots(\lambda x_n.Qx_n)\cdots)x_1))x:\gamma\end{array}}{\Delta \vdash_A \lambda x.(\lambda x_1.(\lambda x_1.(\lambda x_2.(\cdots(\lambda x_n.Qx_n)\cdots)x_1))x:\beta\to\gamma}}{\Delta \vdash_A Q:\beta\to\gamma}}{\Delta \vdash_A QN:\gamma}$$

where $x, x_1, \dots, x_n \notin \mathrm{FV}(Q)$.

This can be replaced exactly as in Case 1 giving

$$\begin{array}{c}D_1'\\ \Delta \vdash_A \lambda x_1.(\lambda x_2.(\cdots(\lambda x_n.Qx_n)\cdots)x_1))N:\gamma\end{array}$$

where $\lambda x_1.(\lambda x_2.(\cdots(\lambda x_n.Qx_n)\cdots)x_1))N =_{\beta\eta} QN$.

Again there can be no such maximal cut.

Case 3. The derivation has a maximal $\wedge\eta\wedge-$ (or if $n = 0$, a $\wedge-$) cut of the form:

$$(\wedge E)\,\dfrac{(\eta)\,\dfrac{(\wedge I)\,\dfrac{\Delta \vdash_A \lambda x_1\dots x_n.Qx_1\dots x_n:\alpha \qquad \Delta \vdash_A \lambda x_1\dots x_n.Qx_1\dots x_n:\beta}{\Delta \vdash_A \lambda x_1\dots x_n.Qx_1\dots x_n:\alpha\wedge\beta}}{\Delta \vdash_A Q:\alpha\wedge\beta}}{\Delta \vdash_A Q:\alpha}$$

(or $\Delta \vdash Q:\beta$). This can be replaced by:

$$(\eta)\,\dfrac{\Delta \vdash_A \lambda x_1\dots x_n.Qx_1\dots x_n:\alpha}{\Delta \vdash_A Q:\alpha}$$

(or β). α may now be a cut formula, but a larger maximal cut has been eliminated, which, as is shown above, is impossible.

Hence the derivation described has no cuts at all. ⊣

5.3. Generation lemma for systems without $\wedge I$. *If A is a type system that has at least* (Var), $(\to I)$ *and* $(\to E)$ *but not* $(\wedge I)$ *and*

(j) $$\Delta \vdash_A M:\alpha$$

then one of:

(i) $M \equiv x, (\exists\beta)x:\beta\in\Delta$ & $\beta \leq_A \alpha$.

(ii) $M \equiv PQ, (\exists\beta,\gamma)\Delta \vdash_A P:\gamma\to\beta, \Delta\vdash_A Q:\gamma$ & $\beta\leq_A\alpha$ *where the derivations together are shorter than that of* (j).

(iii) $M \equiv \lambda x.N, (\exists\beta,\gamma)\Delta, x:\beta\vdash_A N:\gamma$ & $\beta\to\gamma\leq_A\alpha$ *where the derivation is shorter than that of* (j).

Proof. By induction on the derivation of (j).

Case 1. (j) comes by (Var). Then $\alpha\equiv\beta$, so by (1) $\beta\leq_A\alpha$, i.e., (i) holds.

Case 2. (j) comes by $(\to E)$ from $\Delta\vdash_A P:\gamma\to\alpha$ and $\Delta\vdash_A Q:\gamma$, so (ii) holds.

Case 3. (j) comes by $(\to I)$ from $\Delta, x : \beta \vdash_A N : \gamma$ where $\alpha \equiv \beta \to \gamma$, then (iii) holds.
Case 4. (j) comes by $(\wedge E)$ from $\Delta \vdash_A M : \alpha \wedge \sigma$ or $\Delta \vdash_A M : \sigma \wedge \alpha$, for some σ. By Definition 1.11, $\leq_A$ includes (2) and (3) so $\sigma \wedge \alpha \leq_A \alpha$ and $\alpha \wedge \sigma \leq_A \alpha$. (i), (ii) or (iii) now holds, depending on the form of M by the induction hypothesis and (5).
Case 5. (j) comes by (η) from $\Delta \vdash_A \lambda x.Mx : \alpha$ and $x \notin \mathrm{FV}(M)$. Then by the induction hypothesis (iii),

$$\Delta, x : \xi \vdash_A Mx : \zeta$$

where $\xi \to \zeta \leq_A \alpha$, and by induction hypothesis (ii) and (i):

$$\Delta, x : \xi \vdash_A M : \sigma \to \tau \tag{k}$$

and

$$\Delta, x : \xi \vdash_A x : \sigma,$$

where the derivations, together, are shorter than that of (g), $\xi \leq_A \sigma$ and $\tau \leq_A \zeta$. By Definition 1.11 as A has (η), $\leq_A$ has (5) and (7) so $\sigma \to \tau \leq_A \xi \to \zeta \leq_A \alpha$. As $x \notin \mathrm{FV}(M)$ by (k) we have

$$\Delta \vdash_A M : \sigma \to \tau \tag{l}$$

by a derivation no longer than that of (k). By the induction hypothesis applied to (l), (i), (ii) or (iii) now holds depending on the form of M. ⊣

Note that by Lemma 1.13 (Rule $(\leq_A)$) (i), (ii) and (iii) also imply (j).

5.4. Corollary. *If A is a type system that has* (Var), $(\to I)$, $(\to E)$ *and* (η), *but not* $(\wedge I)$ *and*

$$\Delta \vdash_A \lambda x.Mx : \alpha$$

where $x \notin \mathrm{FV}(M)$, then by a deduction that is shorter,

$$\Delta \vdash_A M : \sigma \to \tau,$$

where $\sigma \to \tau \leq_A \alpha$.

Also $\Delta \vdash_A M : \alpha$.

Proof. Case 5 of the proof of Lemma 5.3 and Lemma 1.13. ⊣

5.5. Corollary. $\Delta \vdash_A M : \alpha \Leftrightarrow \Delta \vdash_{\lambda(\leq_A)} M : \alpha$.

Proof. [$\Leftarrow$] By Lemma 1.13. [$\Rightarrow$] By induction on the derivation of (j).

Each case is trivial except the one where the last step in the derivation of (j) is by (η). In this case we have by a derivation shorter than that of (j):

$$\Delta \vdash_A \lambda x.Mx : \alpha$$

and so by Corollary 5.4 (as in the proof of Case 5 of Lemma 5.3) by a derivation that is not longer we have:

$$\Delta \vdash_A M : \sigma \to \tau$$

where $\sigma \to \tau \leq_A \alpha$. Now by the induction hypothesis we have

$$\Delta \vdash_{\lambda(\leq_A)} M : \sigma \to \tau$$

and so by $(\leq_A)$

$$\Delta \vdash_{\lambda(\leq_A)} M : \alpha.$$ ⊣

5.6. GENERATION LEMMA FOR SYSTEMS CONTAINING $\lambda(\wedge I)$. *If A is a type system that has at least* (Var), $(\rightarrow I)$, $(\rightarrow E)$ *and* $(\wedge I)$ *and*

(m) $$\Delta \vdash_A M : \alpha$$

then one of:

(i) $M \equiv x \;\&\; (\exists \beta) x : \beta \in \Delta \;\&\; \beta \leq_A \alpha$.

(ii) $M \equiv PQ \;\&\; (\exists \beta_1, \gamma_1, \dots, \beta_n, \gamma_n)(\forall i)(1 \leq i \leq n \rightarrow \Delta \vdash_A P : \beta_i \rightarrow \gamma_i$, $\Delta \vdash_A Q : \beta_i)$ *by shorter derivations than that of* (m), *and*

$$\gamma_1 \wedge \cdots \wedge \gamma_n \leq_A \alpha.$$

(iii) $M \equiv \lambda x.N \;\&\; (\exists \beta_1, \gamma_1, \dots, \beta_n, \gamma_n)(\forall i)(1 \leq i \leq n \rightarrow \Delta, x : \beta_i \vdash_A N : \gamma_i)$ *by a shorter derivation than that of* (m) *and*

$$(\beta_1 \rightarrow \gamma_1) \wedge \cdots \wedge (\beta_n \rightarrow \gamma_n) \leq_A \alpha.$$

PROOF. As for Lemma 5.3 except for [$\Rightarrow$] Cases 5 and an additional Case 7 for $\wedge I$. $\leq_A$ is $\leq_{1,5,6}$, $\leq_{-4,7}$, $\leq_{1,5,6,7}$ or $\leq$.

Case 5. (m) comes by (η) from $y \notin \mathrm{FV}(M)$ and

$$\Delta \vdash_A \lambda y.My : \alpha.$$

In this case we note that $\leq_A$ is $\leq_{1,5,6,7}$ or $\leq$, so each $\leq_A$ has at least (1), (5), (6) and (7).

Then, by the induction hypothesis (iii) by a shorter derivation,

(n) $$(\exists \xi_1, \zeta_1, \dots, \xi_k, \zeta_k)(\forall i)\big(1 \leq i \leq k \rightarrow \Delta, y : \xi_i \vdash_A My : \zeta_i \;\&\; (\xi_1 \rightarrow \zeta_1) \wedge \cdots \wedge (\xi_k \rightarrow \zeta_k) \leq_A \alpha\big).$$

By (n) and the induction hypothesis (ii) and (i) we have for each i $(1 \leq i \leq n)$:

(o) $$(\exists \sigma_{i1}, \tau_{i1}, \dots, \sigma_{im_i} \tau_{im_i})(\forall j)(1 \leq j \leq m_i \rightarrow \Delta, y : \xi_i \vdash_A M : \sigma_{ij} \rightarrow \tau_{ij} \quad \Delta, y : \xi_i \vdash_A y : \sigma_{ij})$$

by shorter deductions than that of (n), where $\tau_{i1} \wedge \cdots \wedge \tau_{im_i} \leq_A \zeta_i$ and for each j, $\xi_i \leq_A \sigma_{ij}$.

By Lemma 3.1 we have, by a deduction no longer than that of (o):

(p) $$\Delta \vdash_A M : \sigma_{ij} \rightarrow \tau_{ij}.$$

Case 5(i). If $M \equiv x$, there is, by the induction hypothesis (i), a β such that $x : \beta \in \Delta$ and $\beta \leq_A \sigma_{ij} \rightarrow \tau_{ij}$.

If $\leq_A$ is $\leq_{1,5,6,7}$ by Lemma 4.3(iii) we have $\beta \equiv \beta_1 \rightarrow \beta_2$, $\beta_2 \leq_A \tau_{ij}$ and $\sigma_{ij} \leq_A \beta_1$, for each i and j. By (6) we obtain $\beta_2 \leq_A \tau_{i1} \wedge \cdots \wedge \tau_{im_i} \leq_A \zeta_i$ and by (5) $\xi_i \leq_A \beta_1$. So by (7) $\beta_1 \rightarrow \beta_2 \leq_A \xi_i \rightarrow \zeta_i$ and by (6) $\beta \equiv \beta_1 \rightarrow \beta_2 \leq_A (\xi_1 \rightarrow \zeta_1) \wedge \cdots \wedge (\xi_k \rightarrow \zeta_k) \leq_A \alpha$.

If $\leq_A$ is $\leq$, we have by (7) $\beta \leq_A \sigma_{ij} \rightarrow \tau_{ij} \leq_A \xi_i \rightarrow \tau_{ij}$. By (6) $\beta \leq_A (\xi_i \rightarrow \tau_{i1}) \wedge \cdots \wedge (\xi_i \rightarrow \tau_{im_i})$ and by (4), (5) and (9) $\beta \leq_A \xi_i \rightarrow \tau_{i1} \wedge \cdots \wedge \tau_{im_i}$. Now by (7) $\beta \leq_A \xi_i \rightarrow \zeta_i$ and by (6) $\beta \leq_A (\xi_i \rightarrow \zeta_1) \wedge \cdots \wedge (\xi_k \rightarrow \zeta_k) \leq \alpha$.

Case 5(ii). If $M \equiv PQ$, there are, by the induction hypothesis (ii) for each i and j $\beta_1, \gamma_1, \ldots \beta_n, \gamma_n$ such that

$$\Delta \vdash_A P : \beta_p \to \gamma_p$$
$$\Delta \vdash_A Q : \beta_p$$

by derivations shorter than that of (p) and

$$\gamma_1 \wedge \cdots \wedge \gamma_n \leq_A \sigma_{ij} \to \tau_{i1}.$$

If $\leq_A$ is $\leq_{1,5,6,7}$ by Lemma 4.3(ii) and (iii) this requires $n = 1$ and $\gamma_1 \equiv \beta_1 \to \beta_2$ and we obtain, as in Case 5(i),

$$\gamma_1 \wedge \cdots \wedge \gamma_n \equiv \gamma_1 \equiv \beta_1 \to \beta_2 \leq_A \alpha.$$

Similarly for $\leq$.

Case 5(iii). If $M \equiv \lambda x.N$, there are, by the induction hypothesis (iii) $\beta_1, \gamma_1, \ldots,$ β_n, γ_n such that for each i and j and each p, $1 \leq p \leq n$,

$$\Delta, x : \beta_p \vdash_A N : \gamma_p$$

where $(\beta_1 \to \gamma_1) \wedge \cdots \wedge (\beta_n \to \gamma_n) \leq_A \sigma_{ij} \to \tau_{ij}$. As before if $\leq_A$ is $\leq_{1,5,6,7}$ this requires $n = 1$ and

$$(\beta_1 \to \gamma_1) \wedge \cdots \wedge (\beta_n \to \gamma_n) = \beta_1 \to \gamma_1 \leq_A \alpha.$$

Similarly for $\leq$.

Case 7. If (m) comes by $(\wedge I)$ from

(q) $$\Delta \vdash_A M : \xi$$

and

(r) $$\Delta \vdash_A M : \zeta$$

where $\alpha \equiv \xi \wedge \zeta$.

Case 7(i). If $M \equiv x$, $x : \beta \in \Delta$, $\beta \leq_A \xi$ and $\beta \leq_A \zeta$ so by (6) (which is in each A), $\beta \leq_A \xi \wedge \zeta \equiv \alpha$.

Case 7(ii). If $M = PQ$ we have $\beta_1, \gamma_1, \ldots, \beta_m, \gamma_m, \ldots, \beta_p, \gamma_p$ such that for $1 \leq i \leq p$

$$\Delta \vdash_A P : \beta_i \to \gamma_i$$

and

$$\Delta \vdash_A Q : \beta_i$$

by derivations shorter than that of (q) and/or (r) $\gamma_1 \wedge \cdots \wedge \gamma_m \leq_A \xi$, and $\gamma_{m+1} \wedge \cdots \wedge \gamma_p \leq_A \zeta$.

If A includes $(\wedge E)$, $\leq_A$ has (6) and (9) and we have

$$\gamma_1 \wedge \cdots \wedge \gamma_p \leq_A \xi \wedge \zeta \equiv \alpha.$$

If A includes (η), but not $(\wedge E)$, $\leq_A$ is $\leq_{1,5,6,7}$ then by Lemma 4.3(ii), if $m > 1$

$$\xi \equiv (\gamma_1 \wedge \cdots \wedge \gamma_m) \wedge \cdots \wedge (\gamma_1 \wedge \cdots \wedge \gamma_m).$$

If $p > m + 1$

$$\zeta \equiv (\gamma_{m+1} \wedge \cdots \wedge \gamma_p) \wedge \cdots \wedge (\gamma_{m+1} \wedge \cdots \wedge \gamma_p).$$

Thus $(\gamma_1 \wedge \cdots \wedge \gamma_m) \wedge \cdots \wedge (\gamma_1 \wedge \cdots \wedge \gamma_m) \wedge (\gamma_{m+1} \wedge \cdots \wedge \gamma_p) \wedge \cdots \wedge (\gamma_{m+1} \wedge \cdots \wedge \gamma_p) \equiv$ $\alpha \leq_A \alpha$. If there are r $(\gamma_1 \wedge \cdots \wedge \gamma_m)$'s and s $(\gamma_{m+1} \wedge \cdots \wedge \gamma_p)$'s, the n in part (ii) of the lemma is $rm + sp$.

If $m = 1$ we have $\gamma_1 \leq_A \xi$, and so by (7) $\beta_1 \to \gamma_1 \leq_A \beta_1 \to \xi$, and by Lemma 1.13,

$$\Delta \vdash_A P : \beta_1 \to \xi.$$

Similarly if $m + 1 = p$ we have:

$$\Delta \vdash_A P : \beta_n \to \zeta,$$

so in these cases we have (ii), but with ξ replacing γ_1, ζ replacing γ_n or both.

If A has neither (η) nor $(\wedge E)$, $\leq_A$ is $\leq_{1,5,6}$ and we have (even if $m = 1$ or $m + 1 = p$), by Lemma 4.3(iv), ξ and ζ as in the $m > 1$ case for A with (η). We again have (ii) with $n = rm + sp$.

Case 7(iii). If $M \equiv \lambda x.N$ we have $\beta_1, \gamma_1, \dots, \beta_m, \gamma_m, \dots, \beta_n, \gamma_n$ such that for $1 \leq i \leq n$

$$\Delta, x : \beta_i \vdash N : \gamma_i$$

by derivations shorter than that of (q) and/or (r),

$$(\beta_1 \to \gamma_1) \wedge \cdots \wedge (\beta_m \to \gamma_m) \leq_A \zeta$$
$$(\beta_{m+1} \to \gamma_{m+1}) \wedge \cdots \wedge (\beta_n \to \gamma_n) \leq_A \zeta.$$

We have as above:

$$(\beta_1 \to \gamma_1) \wedge \cdots \wedge (\beta_n \to \gamma_n) \leq_A \alpha. \qquad \dashv$$

5.7. Corollary. $\Delta \vdash_A M : \alpha \Leftrightarrow \Delta \vdash \lambda(\wedge I, \leq_A) M : \alpha.$

Proof. As for Corollary 5.5. $\dashv$

§6. Proof of the equivalence theorems (Theorem 2.3). Each of (i) to (iii) follows from Definition 1.11 and either Corollary 5.5 or 5.7 and in the case of (i) Lemma 4.3(i). ($\lambda(\wedge I, \wedge E, \eta) \approx_1 \lambda(\wedge I, \wedge E, \leq)$ was first proved in Hindley [4, Lemma 3.3.4].)

(v) and (vi). We show that for an arbitrary system A, $A \approx_2 A + (\eta)$.

We prove, by induction on the proof of:

(s) $$\Delta \vdash_{A+(\eta)} N : \alpha$$

that

(t) $$(\exists M) \Delta \vdash_A M : \alpha$$

and $M \rhd_\eta N$.

Each case is simple. We show the $(\to E)$ and (η) cases.

If (s) is obtained by $(\to E)$ from

$$\Delta \vdash_{A+(\eta)} P : \beta \to \alpha$$

and

$$\Delta \vdash_{A+(\eta)} Q : \beta$$

where $N \equiv PQ$, we have by the induction hypothesis a P^1 and Q^1, such that:

$$\Delta \vdash_A P^1 : \beta \to \alpha$$

and

$$\Delta \vdash_A Q^1 : \beta$$

and $P'Q' \rhd_\eta PQ$.

We then have (t) with $M \equiv P^1Q^1$.

If (s) is obtained by (η) from:

$$\Delta \vdash_{A+(\eta)} \lambda x.Nx : \alpha \quad \text{and} \quad x \notin \mathrm{FV}(N)$$

we have (t) by the induction hypothesis, where $M \rhd_\eta \lambda x.Nx \rhd_\eta N$.
(iv). By (iii) and Lemma 2.2(ii) we have $\lambda(\wedge I, \wedge E, \eta) \approx_2 \lambda(\wedge I, \leq)$. The result then follows from (v).

§7. Inequivalences of systems.

7.1. Theorem.

(i) $\lambda(\wedge I) \prec_1 \lambda(\wedge I, \eta)$
(ii) $\lambda(\wedge I, \wedge E) \prec_1 \lambda(\wedge I, \leq)$
(iii) $\lambda(\wedge E) \prec_1 \lambda(\wedge E, \eta)$
(iv) $\lambda(\wedge I) \prec_{1,2} \lambda(\wedge I, \wedge E)$
(v) $\lambda(\,) \prec_{1,2} \lambda(\wedge E)$
(vi) $\lambda(\wedge I, \eta) \prec_{1,2} \lambda(\wedge I, \leq)$
(vii) $\lambda(\wedge E) \prec_{1,2} \lambda(\wedge I, \wedge E)$
(viii) $\lambda(\wedge E, \eta) \prec_{1,2} \lambda(\leq)$
(ix) $\lambda(\,) \prec_{1,2} \lambda(\wedge I)$
(x) $\lambda(\leq) \prec_{1,2} \lambda(\wedge I, \leq)$.

Proof. In each of (i) to (iii) $A \preceq_1 B$ is obvious. We show each $A \not\approx_1 B$ by giving a judgement that clearly holds for B but not for A by Generation Lemma 5.3 or 5.6 and the appropriate part of Lemma 4.3.

(i) $x : a \wedge a \to b \vdash x : a \to b$.
(ii) $x : (a \to b) \wedge (a \to c) \vdash x : a \to b \wedge c$.
(iii) $x : b \to c \wedge d \vdash x : a \wedge b \to c$.

In each case of (iv) to (x) $A \preceq_{1,2} B$ is obvious. By Lemma 2.2(iv) it is enough to prove $\sim(A \approx_2 B)$ and for this it is enough to give an example of an M such that $\Delta \vdash_B M : \alpha$ and to prove $\sim(\exists M)\Delta \vdash_A M : \alpha$.
(iv)–(v).

$$x : a \wedge b \vdash_B x : a$$

where B is $\lambda(\wedge I, \wedge E)$, or $\lambda(\wedge E)$.

Consider a derivation of

$$x : a \wedge b \vdash_A M : a,$$

where A is $\lambda(\wedge I)$ or $\lambda(\,)$, that has no cuts and that is of minimal length. The last steps in the derivation must be in the form:

$$(\to E)\ \frac{x : a \wedge b \vdash_A P_1 : \alpha_1 \qquad x : a \wedge b \vdash_A N : \alpha_1 \to \cdots \to \alpha_n \to a}{x : a \wedge b \vdash_A NP_1 : \alpha_2 \to \cdots \to \alpha_n \to a}$$

$$(\to E)\ \frac{x : a \wedge b \vdash_A P_n : \alpha_n \qquad \vdots}{x : a \wedge b \vdash_A NP_1 \cdots P_n : a}$$

where $NP_1 \cdots P_n \equiv M$ and where $x : a \wedge b \vdash N : \alpha$ $(\alpha = \alpha_1 \to \cdots \to \alpha_n \to a)$ is not obtained by $(\to E)$.

If this is obtained by (Var), $a \wedge b \equiv \alpha_1 \to \cdots \to \alpha_m \to a$, which is impossible.

If this is obtained by $(\wedge I)$ from $x : a \wedge b \vdash_A N : \beta$ and $x : a \wedge b \vdash_A N : \gamma$, we have $\alpha \equiv \beta \wedge \gamma$, which is also impossible, so $x : a \wedge b \vdash_A M : a$ cannot be derived.
(vi)–(viii).

$$x : a \wedge b \vdash_B x : b \wedge a$$

where B is $\lambda(\wedge I, \leq)$, $\lambda(\leq)$, or $\lambda(\wedge I, \wedge E)$.

As above

$$x : a \wedge b \vdash_A M : b \wedge a$$

must have a cut free proof where A is $\lambda(\wedge I, \eta)$, $\lambda(\wedge E)$ or $\lambda(\wedge E, \eta)$. This means $(\rightarrow E)$ and $(\wedge E)$ steps from a judgement:

$$x : a \wedge b \vdash_A N : \alpha$$

where this comes by (Var).

This requires $a \wedge b \equiv \alpha$ which is impossible.
(ix)–(x).

$$\vdash_B \lambda x.x : (a \rightarrow a) \wedge (b \rightarrow b),$$

where B is $\lambda(\wedge I)$, or $\lambda(\wedge I, \leq)$. If

$$\vdash_A \lambda x.x : (a \rightarrow a) \wedge (b \rightarrow b),$$

where A is $\lambda(\,)$ or $\lambda(\leq)$, by Generation Lemma 5.3(iii) we have a β and γ such that:

$$x : \beta \vdash_A x : \gamma$$

and

$$\beta \rightarrow \gamma \leq_A (a \rightarrow a) \wedge (b \rightarrow b).$$

If A is $\lambda(\,)$, $\leq_A$ is $\equiv$, so this is impossible.

If A is $\lambda(\leq)$, $\leq_A$ is $\leq$, so we have:

$$\beta \rightarrow \gamma \leq a \rightarrow a$$

and

$$\beta \rightarrow \gamma \leq b \rightarrow b.$$

Now by Lemma 4.2(ii), $a \leq \beta, \gamma \leq a, b \leq \beta$ and $\gamma \leq b$. As we also have $\beta \leq \gamma$ by Lemma 5.3(i), we obtain $a \leq b$ which is impossible by Lemma 4.1. ⊣

7.2. Theorem.

(i) $\lambda(\leq), \lambda(\wedge E, \eta) \not\preceq_1 \lambda(\wedge I, \wedge E), \lambda(\wedge I, \eta), \lambda(\wedge I)$.
(ii) $\lambda(\wedge I, \eta) \not\preceq_1 \lambda(\wedge I, \wedge E)$.
(iii) $\lambda(\wedge I), \lambda(\wedge I, \leq) \not\preceq_2 \lambda(\leq), \lambda(\wedge E)$.
(iv) $\lambda(\wedge E), \lambda(\leq), \lambda(\wedge I, \leq) \not\preceq_2 \lambda(\wedge I)$.
(v) $\lambda(\wedge I) \not\preceq_2 \lambda(\leq)$.
(vi) $\lambda(\wedge E), \lambda(\leq) \not\preceq_2 \lambda(\wedge I)$.

Proof. As for Theorem 7.1.

(i) $x : a \rightarrow c \vdash x : a \wedge b \rightarrow c$ holds for $\lambda(\leq)$ and $\lambda(\wedge E, \eta)$, but not for $\lambda(\wedge I, \wedge E)$ or $\lambda(\wedge I, \eta)$, (and hence not for the other weaker systems) by Generation Lemma 5.6 and Lemma 4.3(iii) and (vi).

(ii) $x : a \rightarrow b \vdash x : a \wedge b \rightarrow b$ holds for $\lambda(\wedge I, \eta)$, but not for $\lambda(\wedge I, \wedge E)$ by Lemma 4.3(vi).

(iii) As for Theorem 7.1(ix)–(x).

(iv) As for Theorem 7.1(iv)–(v).

(v) As for Theorem 7.1(ix)–(x).

(vi) As for Theorem 7.1(iv)–(v). ⊣

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