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Abstract

We realize the Hecke C^*-algebra C_Q of Bost and Connes as a direct limit of Hecke C^*-algebras which are semigroup crossed products by N^F, for F a finite set of primes. For each approximating Hecke C^*-algebra we describe a composition series of ideals. In all cases there is a large type I ideal and a commutative quotient, and the intermediate subquotients are direct sums of simple C^*-algebras. We can describe the simple summands as ordinary crossed products by actions of Z^S for S a finite set of primes. When |S|=1, these actions are odometers and the crossed products are Bunce–Deddens algebras; when |S|>1, the actions are an apparently new class of higher-rank odometer actions, and the crossed products are an apparently new class of classifiable AT-algebras.

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Subquotients of Hecke C^*-algebras

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Abstract. We realize the Hecke C^*-algebra $\mathcal{C}_{\mathbb{Q}}$ of Bost and Connes as a direct limit of Hecke C^*-algebras which are semigroup crossed products by $\mathbb{N}^F$, for F a finite set of primes. For each approximating Hecke C^*-algebra we describe a composition series of ideals. In all cases there is a large type I ideal and a commutative quotient, and the intermediate subquotients are direct sums of simple C^*-algebras. We can describe the simple summands as ordinary crossed products by actions of $\mathbb{Z}^S$ for S a finite set of primes. When $|S| = 1$, these actions are odometers and the crossed products are Bunce–Deddens algebras; when $|S| > 1$, the actions are an apparently new class of higher-rank odometer actions, and the crossed products are an apparently new class of classifiable AT-algebras.

0. *Introduction*

In [**2**], Bost and Connes studied a particular Hecke C^*-algebra $\mathcal{C}_{\mathbb{Q}}$ arising in number theory. The algebra $\mathcal{C}_{\mathbb{Q}}$ can be realized as a semigroup crossed product $C^*(\mathbb{Q}/\mathbb{Z}) \rtimes_\alpha \mathbb{N}^*$ by an endomorphic action α of the multiplicative semigroup $\mathbb{N}^*$ on the group C^*-algebra $C^*(\mathbb{Q}/\mathbb{Z})$ (see [**8**]), and this realization has provided useful insight into the analysis of $\mathcal{C}_{\mathbb{Q}}$ (see [**6, 16**]). Since individual elements of $\mathbb{Q}/\mathbb{Z}$ and $\mathbb{N}^*$ involve only finitely many primes, $C^*(\mathbb{Q}/\mathbb{Z}) \rtimes_\alpha \mathbb{N}^*$ is the direct limit of subalgebras $C^*(G_F) \rtimes_\alpha \mathbb{N}^F$, where F is a finite set of

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primes, G_F is the subgroup of $\mathbb{Q}/\mathbb{Z}$ in which the denominators have all prime factors in F, and $\mathbb{N}^F$ acts through the embedding $(n_p) \mapsto \prod_{p\in F} p^{n_p}$ of $\mathbb{N}^F$ in $\mathbb{N}^*$ (see §1). One can therefore hope to understand the Hecke algebra $\mathcal{C}_{\mathbb{Q}}$ in terms of the finite-prime analogues $C^*(G_F) \rtimes_\alpha \mathbb{N}^F$.

Our goal is to analyse the structure of these finite-prime analogues of the Bost–Connes algebra. We started this analysis in [**11**], where we described a composition series for the two-prime analogue and identified the subquotients in familiar terms: there is a large type I ideal, a commutative quotient isomorphic to $C(\mathbb{T}^2)$, and the intermediate subquotient is isomorphic to a direct sum of Bunce–Deddens algebras. Here we describe a composition series for $C^*(G_F) \rtimes_\alpha \mathbb{N}^F$. Again there is a large type I ideal and a commutative quotient, and the intermediate subquotients are direct sums of simple C^*-algebras. We can describe the simple summands as ordinary crossed products by actions of $\mathbb{Z}^S$ for $S \subset F$. When $|S| = 1$, these actions are odometers and the crossed products are Bunce–Deddens algebras; when $|S| > 1$, the actions are an apparently new class of higher-rank odometer actions, and the crossed products are an apparently new class of classifiable AT-algebras.

We begin with a short section in which we describe the algebras we intend to study. In §2, we describe our composition series for the semigroup crossed product $C^*(G_F) \rtimes_\alpha \mathbb{N}^F$. It has $|F| + 1$ subquotients, and all but two of them are direct sums of algebras stably isomorphic to ordinary crossed products of the form $C(\mathcal{U}(\mathbb{Z}_{F\setminus S})) \rtimes \mathbb{Z}^S$, where $S \subset F$ and $\mathcal{U}(\mathbb{Z}_{F\setminus S})$ is the group of units in the ring $\prod_{p\in F\setminus S} \mathbb{Z}_p$. Our main tools are the analysis of invariant ideals in semigroup crossed products from [**10**] and some technical lemmas on sums and intersections of ideals in C^*-algebras. We also use the general results of [**19**] to see that the simple summands are classifiable.

In §3, we show that when $S = \{q\}$ is a singleton, $C(\mathcal{U}(\mathbb{Z}_{F\setminus S})) \rtimes \mathbb{Z}^S$ is a direct sum of finitely many Bunce–Deddens algebras; as in [**11**], the number of summands depends on the orders of q in the finite groups $\prod_{p\neq q} \mathcal{U}(\mathbb{Z}/p^l\mathbb{Z})$ for large $l \in \mathbb{N}$. We then consider the case where $S = \{q, r\}$. By computing the K-theory of $C(\mathcal{U}(\mathbb{Z}_{F\setminus S})) \rtimes \mathbb{Z}^S$, we can see that they are not Bunce–Deddens algebras, for example. We expect these summands to be even harder to recognize when $|S| > 2$.

In §4, we use techniques like those of §2 to identify subquotients of the Bost–Connes algebra $C^*(\mathbb{Q}/\mathbb{Z}) \rtimes_\alpha \mathbb{N}^*$. These include algebras stably isomorphic to $C(\mathcal{U}(\mathbb{Z}_{\mathcal{P}\setminus S})) \rtimes \mathbb{Z}^S$ when S is a cofinite subset of the set $\mathcal{P}$ of all primes; in this case, though, these crossed products are themselves simple, and even though the general theory of [**19**] no longer applies, we can see using results from [**1**] that they are classifiable AT-algebras. We finish with a purely number theoretic Appendix A in which we identify the orders of an odd integer in the groups $\mathcal{U}(\mathbb{Z}/p^l\mathbb{Z})$ and their products. As in [**11**, Theorem 3.1], these are needed when we want to identify the number of simple summands in the various subquotients.

1. *Preliminaries*

We denote by $\mathbb{N}^*$ the semigroup of positive integers under multiplication, and by $\mathbb{N}$ the semigroup of non-negative integers under addition. It was shown in [**8**, Proposition 2.1] that there is an action α of $\mathbb{N}^*$ by endomorphisms of $C^*(\mathbb{Q}/\mathbb{Z})$ such that

$$\alpha_n(\delta_r) = \frac{1}{n}\sum_{ns=r}\delta_s \quad \text{for } r \in \mathbb{Q}/\mathbb{Z} \text{ and } n \in \mathbb{N}^*.$$

The corresponding semigroup crossed product $C^*(\mathbb{Q}/\mathbb{Z}) \rtimes_\alpha \mathbb{N}^*$ is isomorphic to the Hecke C^*-algebra $\mathcal{C}_\mathbb{Q}$ of Bost and Connes [**8**, Corollary 2.10]. We denote by $(i_A, i_{\mathbb{N}^*})$ the canonical covariant representation of $(C^*(\mathbb{Q}/\mathbb{Z}), \mathbb{N}^*, \alpha)$ in $C^*(\mathbb{Q}/\mathbb{Z}) \rtimes_\alpha \mathbb{N}^*$.

Let F be a set of prime numbers. The rational numbers of the form $k\big(\prod_{p\in F} p^{m_p}\big)^{-1}$ form a subgroup of $\mathbb{Q}$, whose image in $\mathbb{Q}/\mathbb{Z}$ we denote by G_F. The integrated form of the map $r \mapsto \delta_r : G_F \to UC^*(\mathbb{Q}/\mathbb{Z})$ is a homomorphism i_F of $C^*(G_F)$ into $C^*(\mathbb{Q}/\mathbb{Z})$; a standard duality argument shows that i_F is injective, so that we can identify $C^*(G_F)$ with the subalgebra $i_F(C^*(G_F))$ of $C^*(\mathbb{Q}/\mathbb{Z})$. When n has all of its prime factors in F, α_n leaves this subalgebra invariant, and hence composing α with the map $(m_p)_{p\in F} \mapsto \prod_{p\in F} p^{m_p}$ gives an action of $\mathbb{N}^F$ on $C^*(G_F)$, which we also denote by α. The pair $(i_F, i_{\mathbb{N}^*}|_{\mathbb{N}^F})$ is a covariant representation of $(C^*(G_F), \mathbb{N}^F, \alpha)$ in $C^*(\mathbb{Q}/\mathbb{Z}) \rtimes_\alpha \mathbb{N}^*$. Since i_F is injective, we can deduce from the main theorem of [**12**] (or by minor modifications to the argument in [**8**, §3]) that the corresponding homomorphism

$$i_F \times i_{\mathbb{N}^*}|_{\mathbb{N}^F} : C^*(G_F) \rtimes_\alpha \mathbb{N}^F \to C^*(\mathbb{Q}/\mathbb{Z}) \rtimes_\alpha \mathbb{N}^*$$

is also an injection. We use this injection to identify $C^*(G_F) \rtimes_\alpha \mathbb{N}^F$ with a subalgebra of $C^*(\mathbb{Q}/\mathbb{Z}) \rtimes_\alpha \mathbb{N}^*$.

The crossed product $C^*(\mathbb{Q}/\mathbb{Z}) \rtimes_\alpha \mathbb{N}^*$ is spanned by the elements of the form $i_A(\delta_r)i_{\mathbb{N}^*}(m)i_{\mathbb{N}^*}(n)^*$ [**8**, Lemma 3.2]. If F contains all of the prime factors of m, n and the denominator of r, then $i_A(\delta_r)i_{\mathbb{N}^*}(m)i_{\mathbb{N}^*}(n)^*$ lies in $C^*(G_F) \rtimes_\alpha \mathbb{N}^F$. Thus, $C^*(\mathbb{Q}/\mathbb{Z}) \rtimes_\alpha \mathbb{N}^*$ is the direct limit $\overline{\bigcup_F C^*(G_F) \rtimes_\alpha \mathbb{N}^F}$ over increasing finite subsets F of the set $\mathcal{P}$ of prime numbers.

In the next section, we describe a composition series for $C^*(G_F) \rtimes_\alpha \mathbb{N}^F$ when F is a finite subset of $\mathcal{P}$, and identify the subquotients in terms of ordinary crossed products $C(X_S) \rtimes \mathbb{Z}^S$ associated to subsets S of F. The underlying space X_S is the group of units $\mathcal{U}(\mathbb{Z}_{F\setminus S})$ in the ring $\mathbb{Z}_{F\setminus S} := \prod_{p\in F\setminus S}\mathbb{Z}_p$; as an additive group, $\mathbb{Z}_{F\setminus S}$ is the dual group of $G_{F\setminus S}$. The action of a prime $q \in S$ on

$$C(\mathcal{U}(\mathbb{Z}_{F\setminus S})) \subset C(\mathbb{Z}_{F\setminus S}) \cong C^*(G_{F\setminus S}) \subset C^*(\mathbb{Q}/\mathbb{Z})$$

induced by α_q is multiplication by q on $\mathcal{U}(\mathbb{Z}_{F\setminus S})$ (see [**11**, Lemma 1.1]), which is an automorphism because q is a unit in $\mathbb{Z}_{F\setminus S}$. Thus, the action of $\mathbb{N}^S$ on $C(\mathcal{U}(\mathbb{Z}_{F\setminus S}))$ extends to an action σ of $\mathbb{Z}^S$ such that

$$\sigma_{(m_p)}(f)(x) = f\left(\left(\prod_{p\in S} p^{m_p}\right)^{-1} x\right) \quad \text{for } (m_p) \in \mathbb{N}^S.$$

As a matter of notation, we view a crossed product $A \rtimes_\beta G$ by an action of a group as the universal C^*-algebra generated by a copy of A and a unitary representation $i_G : G \to U(A \rtimes_\beta G)$ satisfying the covariance relation $\beta_s(a) = i_G(s)a i_G(s)^*$.

2. *Finitely many primes*

The object of this section is to prove the following theorem. For the definitions of AT-algebra, real rank zero and stable rank one, see [**23**] and the references therein.

THEOREM 2.1. *Let F be a finite set of primes. Then there is a composition series $\{I_k \mid 1 \le k \le |F|\}$ of ideals in $C^*(G_F) \rtimes_\alpha \mathbb{N}^F$ such that:*
(a) $I_1 \cong C(\mathcal{U}(\mathbb{Z}_F), \mathcal{K}(l^2(\mathbb{N}^F)))$;
(b) $I_{k+1}/I_k \cong \bigoplus_{S\subset F,\,|S|=k}(C(\mathcal{U}(\mathbb{Z}_{F\setminus S})) \rtimes_\sigma \mathbb{Z}^S) \otimes \mathcal{K}(l^2(\mathbb{N}^{F\setminus S}))$ *if* $|F| \ge 2$;
(c) $(C^*(G_F) \rtimes \mathbb{N}^F)/I_{|F|} \cong C(\mathbb{T}^F)$.
If $|F| \ge 2$, each $C(\mathcal{U}(\mathbb{Z}_{F\setminus S})) \rtimes_\sigma \mathbb{Z}^S$ is a finite direct sum of simple AT-algebras with real rank zero and a unique tracial state.

The proof of the theorem will occupy the rest of the section. We need some notation and a number of preliminary results.

Under the Fourier transform $C^*(G_F) \cong C(\mathbb{Z}_F)$ the action α becomes

$$\alpha_{(n_p)}(f)(x) = \begin{cases} f\left(\left(\prod_{p\in F} p^{n_p}\right)^{-1} x\right) & \text{if } x \in \left(\prod_{p\in F} p^{n_p}\right)\mathbb{Z}_F \\ 0 & \text{otherwise} \end{cases}$$

(see [**11**, Lemma 1.1]). For $S \subset F$, we set $\mathcal{Z}_S := \{a \in \mathbb{Z}_F \mid a_p = 0 \text{ for } p \in S\}$, and we write $\mathcal{Z}_p$ for $\mathcal{Z}_{\{p\}}$. The next lemma identifies $C_0(\mathbb{Z}_F \setminus \mathcal{Z}_S)$ as the kind of ideal for which taking crossed products behaves well (see [**10**]).

LEMMA 2.2. *For $S \subset F$, $C_0(\mathbb{Z}_F \setminus \mathcal{Z}_S)$ is an extendibly invariant ideal in $C(\mathbb{Z}_F)$.*

Proof. It suffices by [**10**, Theorem 4.3] to show that for each $n \in \mathbb{N}^F$, the endomorphism $x \mapsto \left(\prod_{p\in F} p^{n_p}\right)x$ of $\mathbb{Z}_F$ leaves both $\mathcal{Z}_S$ and $\mathbb{Z}_F \setminus \mathcal{Z}_S$ invariant. Certainly $\left(\prod_{p\in F} p^{n_p}\right)\mathcal{Z}_S$ is contained in $\mathcal{Z}_S$. If $x \notin \mathcal{Z}_S$, then $x_r \ne 0$ for some $r \in S$, $\prod_{p\in F} p^{n_p} x_r \ne 0$ for this r, and $\left(\prod_{p\in F} p^{n_p}\right)x \notin \mathcal{Z}_S$. □

Theorem 1.7 of [**10**] now allows us to identify $C_0(\mathbb{Z}_F \setminus \mathcal{Z}_S) \rtimes \mathbb{N}^F$ with an ideal J_S in $C(\mathbb{Z}_F) \rtimes_\alpha \mathbb{N}^F$ such that $(C(\mathbb{Z}_F) \rtimes_\alpha \mathbb{N}^F)/J_S = C(\mathcal{Z}_S) \rtimes \mathbb{N}^F$; we write J_p for $J_{\{p\}}$.

LEMMA 2.3. *We have $J_S = \sum_{p\in S} J_p$.*

Proof. Since $\mathcal{Z}_S = \bigcap_{p\in S} \mathcal{Z}_p$, we have $\mathbb{Z}_F \setminus \mathcal{Z}_S = \bigcup_{p\in S} \mathbb{Z}_F \setminus \mathcal{Z}_p$, and $C_0(\mathbb{Z}_F \setminus \mathcal{Z}_S) = \sum_{p\in S} C_0(\mathbb{Z}_F \setminus \mathcal{Z}_p)$. It follows from [**10**, Lemma 1.3] that if I, J and $I + J$ are extendibly invariant ideals in (A, P), then $(I + J) \rtimes P = (I \rtimes P) + (J \rtimes P)$. Thus, the result follows from Lemma 2.2. □

For $1 \le k \le |F|$, we define

$$I_k := \prod_{S\subset F,\,|S|=k} J_S = \bigcap_{S\subset F,\,|S|=k} J_S. \tag{2.1}$$

It follows from [**10**, Lemma 1.3] that if I and J are extendibly invariant ideals in (A, P), then

$$(I \rtimes P)(J \rtimes P) = (IJ) \rtimes P,$$

and hence $I_k = C_0\big(\bigcap_{S\subset F,\,|S|=k}(\mathbb{Z}_F \setminus \mathcal{Z}_S)\big) \rtimes \mathbb{N}^F$. Therefore,

$$I_1 = C_0\bigg(\bigcap_{p\in F}(\mathbb{Z}_F \setminus \mathcal{Z}_p)\bigg) \rtimes \mathbb{N}^F = C_0\bigg(\prod_{p\in F}(\mathbb{Z}_p \setminus \{0\})\bigg) \rtimes \mathbb{N}^F;$$

since $\mathbb{Z}_p \setminus \{0\}$ is homeomorphic to $\mathcal{U}(\mathbb{Z}_p) \times \mathbb{N}$ by [**11**, Lemma 2.3], Theorem 2.1(a) follows from an argument similar to that in the last paragraph of [**11**, p. 176]. Similarly, we can prove Theorem 2.1(c) by following the proof of (2.4) of [**11**], because $(C^*(G_F) \rtimes_\alpha \mathbb{N}^F)/I_{|F|} = \mathbb{C} \rtimes \mathbb{N}^F$.

To prove Theorem 2.1(b), we need some lemmas. The first contains some general facts about families of ideals in C^*-algebras.

LEMMA 2.4. *Suppose that $I_1, \ldots, I_n$ are ideals in a C^*-algebra B.*

(a) *With $F_n = \{1, \ldots, n\}$, we have*

$$\prod_{S\subset F_n,\,|S|=k}\bigg(\sum_{i\in S} I_i\bigg) = \sum_{R\subset F_n,\,|R|=n-k+1}\bigg(\prod_{j\in R} I_j\bigg) \quad \textit{for } 1 \le k \le n. \tag{2.2}$$

(b) *Suppose that K is an ideal such that $I_iI_j \subset K$ for all i, j. Then $\big(\sum_{i=1}^n I_i\big)/K$ is naturally isomorphic to $\bigoplus_{i=1}^n (I_i/I_i \cap K)$.*

Proof. We prove (a) by induction on n. The statement is trivial for $n = 1, 2$. Suppose that it holds for $n - 1$. When $k = 1$, both sides of (2.2) are $\prod_{i=1}^n I_i$, so we assume $k \ge 2$. Writing the left-hand side (LHS) of (2.2) as $\big(\prod_{n\in S}\big)\big(\prod_{n\notin S}\big)$ and applying the inductive hypothesis to F_{n-1} shows that

$$\text{LHS} = \bigg(\prod_{|S|=k,n\in S}\bigg(I_n + \sum_{i\in S\setminus\{n\}} I_i\bigg)\bigg)\bigg(\sum_{R\subset F_{n-1},\,|R|=n-k}\bigg(\prod_{j\in R} I_j\bigg)\bigg). \tag{2.3}$$

As I_n is an ideal and $I_n^2 = I_n$, the first term of (2.3) simplifies to give

$$\text{LHS} = \bigg(I_n + \prod_{S'\subset F_{n-1},\,|S'|=k-1}\bigg(\sum_{i\in S'} I_i\bigg)\bigg)\bigg(\sum_{R\subset F_{n-1},\,|R|=n-k}\bigg(\prod_{j\in R} I_j\bigg)\bigg).$$

We can use the inductive hypothesis on F_{n-1} with k replaced by $k - 1$ to get

$$\text{LHS} = \bigg(I_n + \sum_{R'\subset F_{n-1},\,|R'|=n-k+1}\bigg(\prod_{j\in R'} I_j\bigg)\bigg)\bigg(\sum_{R\subset F_{n-1},\,|R|=n-k}\bigg(\prod_{j\in R} I_j\bigg)\bigg), \tag{2.4}$$

which is contained in

$$\sum_{R\subset F_{n-1},\,|R|=n-k}\bigg(\prod_{j\in R\cup\{n\}} I_j\bigg) + \sum_{R'\subset F_{n-1},\,|R'|=n-k+1}\bigg(\prod_{j\in R'} I_j\bigg). \tag{2.5}$$

Since (2.5) is the same as the right-hand side (RHS) of (2.2), LHS $\subset$ RHS. On the other hand, every element of every $\prod_{j\in R'} I_j$ arises in (2.4) because we can pick $R \subset R'$, so RHS $\subset$ LHS.

To prove (b), note that the map $\phi_i : a + I_i \cap K \mapsto a + K$ is an injection of $I_i/(I_i \cap K)$ into $\big(\sum_{i=1}^n I_i\big)/K$, and

$$\phi_i(a + I_i \cap K)\phi_j(b + I_j \cap K) = ab + K = 0 \quad \text{for } i \ne j$$

because $ab \in I_iI_j \subset K$. So the ϕ_j combine to give an injection ϕ of $\bigoplus(I_i/I_i \cap K)$ into $\big(\sum_{i=1}^n I_i\big)/K$, which is clearly surjective. □

LEMMA 2.5. *The ideals I_k of $C^*(G_F) \rtimes_\alpha \mathbb{N}^F$ defined in (2.1) satisfy*

$$I_{k+1}/I_k = \bigoplus_{S\subset F,\,|S|=k} \Big(\bigcap_{p\notin S} J_{S\cup\{p\}}\Big)\Big/ J_S.$$

Proof. Lemma 2.4(a) gives $I_{k+1} = \sum_{R\subset F,\,|R|=n-k}(\prod_{p\in R} J_p)$. The product of any two ideals $\prod_{p\in R} J_p$ with $|R| = n-k$ has at least $n-k+1$ factors J_p, and, hence, is contained in $I_k = \sum_{R\subset F,\,|R|=n-k+1}(\prod_{p\in R} J_p)$. Thus, Lemma 2.4(b) gives

$$I_{k+1}/I_k = \bigoplus_{R\subset F,\,|R|=n-k} \frac{\prod_{p\in R} J_p}{I_k \cap \big(\prod_{p\in R} J_p\big)}. \tag{2.6}$$

Now

$$I_k \cap \Big(\prod_{p\in R} J_p\Big) = \sum_{|T|=n-k+1} \Big(\prod_{q\in T} J_q\Big)\Big(\prod_{p\in R} J_p\Big);$$

each of these summands has at least one factor J_q for $q \notin R$, and is then contained in $J_q\big(\prod_{p\in R} J_p\big)$. Using $I \cap J = IJ$ again gives

$$I_k \cap \Big(\prod_{p\in R} J_p\Big) = \sum_{q\notin R} J_q\Big(\prod_{p\in R} J_p\Big) = \Big(\sum_{q\notin R} J_q\Big)\Big(\prod_{p\in R} J_p\Big),$$

and using the isomorphism $(I+J)/I = J/(I\cap J)$ and Lemma 2.3 gives

$$\frac{\prod_{p\in R} J_p}{I_k \cap \big(\prod_{p\in R} J_p\big)} = \frac{J_{F\setminus R} + \big(\prod_{p\in R} J_p\big)}{J_{F\setminus R}}.$$

Finally we observe that

$$J_{F\setminus R} + \Big(\prod_{p\in R} J_p\Big) = \prod_{p\in R} (J_{F\setminus R} + J_p) = \prod_{p\in R} J_{(F\setminus R)\cup\{p\}}$$

and write $S = F \setminus R$ to deduce the result. □

LEMMA 2.6. *The ideals J_S in $C^*(G_F) \rtimes_\alpha \mathbb{N}^F$ satisfy*

$$\Big(\bigcap_{p\in F\setminus S} J_{S\cup\{p\}}\Big)\Big/ J_S \cong (C(\mathcal{U}(\mathbb{Z}_{F\setminus S})) \rtimes_\sigma \mathbb{Z}^S) \otimes \mathcal{K}(l^2(\mathbb{N}^{F\setminus S})).$$

Proof. We first realize $\big(\bigcap_{p\in F\setminus S} J_{S\cup\{p\}}\big)/J_S$ as a semigroup crossed product:

$$\begin{aligned}\bigcap_{p\in F\setminus S} J_{S\cup\{p\}} &= C_0\Big(\bigcap_{p\in F\setminus S} (\mathbb{Z}_F \setminus \mathcal{Z}_{S\cup\{p\}})\Big) \rtimes \mathbb{N}^F \\ &= C_0\Big(\mathbb{Z}_F \Big\backslash \Big(\bigcup_{p\in F\setminus S} \mathcal{Z}_{S\cup\{p\}}\Big)\Big) \rtimes \mathbb{N}^F.\end{aligned}$$

Thus,

$$\begin{aligned}\Big(\bigcap_{p\in F\setminus S} J_{S\cup\{p\}}\Big)\Big/ J_S &= C_0\Big(\mathcal{Z}_S \Big\backslash \Big(\bigcup_{p\in F\setminus S} \mathcal{Z}_{S\cup\{p\}}\Big)\Big) \rtimes \mathbb{N}^F \\ &= C_0\Big(\Big(\prod_{p\in F\setminus S} \mathbb{Z}_p \setminus \{0\}\Big) \times \Big(\prod_{p\in S} \{0\}\Big)\Big) \rtimes \mathbb{N}^F.\end{aligned}$$

The arguments of Corollary 2.4 and Lemma 2.5 of [**11**] show that this last crossed product is isomorphic to $(C(\mathcal{U}(\mathbb{Z}_{F\setminus S})) \rtimes_\sigma \mathbb{Z}^S) \otimes \mathcal{K}(l^2(\mathbb{N}^{F\setminus S}))$. □

Theorem 2.1(b) follows immediately from Lemmas 2.5 and 2.6.

To finish the proof of Theorem 2.1, it remains to prove the statements about the structure of $C(\mathcal{U}(\mathbb{Z}_{F\setminus S})) \rtimes_\sigma \mathbb{Z}^S$. Corollary A.6 implies that $H := \overline{\mathbb{Z}^S}$ has finite index in $\mathcal{U}(\mathbb{Z}_{F\setminus S})$. The argument at the end of the proof of [**11**, Theorem 3.1] shows that $C(\mathcal{U}(\mathbb{Z}_{F\setminus S})) \rtimes_\sigma \mathbb{Z}^S$ is a finite direct sum of algebras isomorphic to $C(H) \rtimes_\sigma \mathbb{Z}^S$, which is simple because $\mathbb{Z}^S$ acts minimally and freely on H. Since H is an open and closed subset of $\mathcal{U}(\mathbb{Z}_{F\setminus S})$, it is totally disconnected and it follows from [**19**, Theorem 6.11] that $C(H) \rtimes_\sigma \mathbb{Z}^S$ has real rank zero and stable rank one.

The space $\mathcal{U}(\mathbb{Z}_{F\setminus S})$ is the inverse limit of the finite groups $\mathcal{U}\big(\mathbb{Z}/\big(\prod_{p\in F\setminus S} p^{l_p}\big)\mathbb{Z}\big)$ over $l = (l_p) \in \mathbb{N}^{F\setminus S}$. The diagonally embedded copy of $\mathbb{N}$ is cofinal in $\mathbb{N}^{F\setminus S}$ and, hence,

$$\mathcal{U}(\mathbb{Z}_{F\setminus S}) = \varprojlim \mathcal{U}\bigg(\mathbb{Z}\Big/\Big(\prod_{p\in F\setminus S} p^n\Big)\mathbb{Z}\bigg). \tag{2.7}$$

Let π_n denote the canonical surjection of $\mathcal{U}(\mathbb{Z}_{F\setminus S})$ onto $\mathcal{U}\big(\mathbb{Z}/\big(\prod_{p\in F\setminus S} p^n\big)\mathbb{Z}\big)$.

LEMMA 2.7. *Let $H_n := \pi_n(H) \subset \mathcal{U}\big(\mathbb{Z}/\big(\prod_{p\in F\setminus S} p^n\big)\mathbb{Z}\big)$ and let $\mathbb{Z}^S$ act on H_n via the embedding $(n_q) \mapsto \prod_{q\in S} q^{n_q}$ of $\mathbb{Z}^S$ in $\mathbb{Z}$. Then there are C^*-subalgebras A_n of $C(H) \rtimes_\sigma \mathbb{Z}^S$ such that $A_n \cong C(H_n) \rtimes \mathbb{Z}^S$ and $C(H) \rtimes_\sigma \mathbb{Z}^S = \overline{\bigcup A_n}$.*

Proof. The homomorphism π_n induces an injection π_n^* of $C(H_n)$ into $C(H)$, and then $C(H) = \overline{\bigcup_{n\in\mathbb{N}} \pi_n^*(C(H_n))}$. On $\mathbb{Z}^S \subset H$, π_n is reduction modulo $\prod_{p\in F\setminus S} p^n$, so π_n^* converts the action σ into the canonical action of $\mathbb{Z}^S$ by multiplication on H_n. Thus, π_n^* induces a homomorphism $\pi_n^* \rtimes \mathrm{id}$ of $C(H_n) \rtimes \mathbb{Z}^S$ into $C(H) \rtimes_\sigma \mathbb{Z}^S$. The homomorphism π_n^* is faithful on $C(H_n)$ and intertwines the dual actions and, hence, a standard argument shows that $\pi_n^* \rtimes \mathrm{id}$ is faithful on $C(H_n) \rtimes \mathbb{Z}^S$ (see, for example, [**11**, Lemma 4.2]). Since $\bigcup_n \pi_n^*(C(H_n))$ is dense in $C(H)$, we therefore have

$$C(H) \rtimes_\sigma \mathbb{Z}^S = \overline{\bigcup_{n\in\mathbb{N}} \pi_n^* \rtimes \mathrm{id}(C(H_n) \rtimes \mathbb{Z}^S)},$$

as claimed. □

We can identify the subalgebras A_n explicitly.

PROPOSITION 2.8. *Let F be a finite quotient of $\mathbb{Z}^k$. Then $C(F) \rtimes \mathbb{Z}^k$ is isomorphic to $C(\mathbb{T}^k, M_{|F|}(\mathbb{C}))$.*

Proof. Let H be the subgroup of $\mathbb{Z}^k$ with $F = \mathbb{Z}^k/H$. Then H is itself a free abelian group of rank k, and hence has the form $A\mathbb{Z}^k$ for some $A \in M_k(\mathbb{Z}) \cap GL_k(\mathbb{Q})$. The matrix A has a Smith normal form: there are matrices $P, Q \in GL_k(\mathbb{Z})$ such that $B := P^{-1}AQ^{-1}$ is diagonal [**15**, §3.22]. Then $H = A\mathbb{Z}^k = PBQ\mathbb{Z}^k = PB\mathbb{Z}^k \cong B\mathbb{Z}^k = b_{11}\mathbb{Z} \oplus \cdots \oplus b_{kk}\mathbb{Z}$. In other words, multiplying by P^{-1} gives an automorphism of $\mathbb{Z}^k$ which carries H into $\bigoplus b_{ii}\mathbb{Z}$. Thus,

$$C(F) \rtimes \mathbb{Z}^k \cong C\bigg(\prod_{i=1}^{k} (\mathbb{Z}/b_{ii}\mathbb{Z})\bigg) \rtimes \mathbb{Z}^k \cong \bigotimes_{i=1}^{k} \Big(C(\mathbb{Z}/b_{ii}\mathbb{Z}) \rtimes_\tau \mathbb{Z}\Big),$$

where τ is the action of $\mathbb{Z}$ by translation.

By [**17**, Corollary 2.5], $C(\mathbb{Z}/b\mathbb{Z}) \rtimes_\tau \mathbb{Z}$ is isomorphic to the induced algebra

$$\mathrm{Ind}_{(b\mathbb{Z})^\perp}^{\mathbb{T}}(C(\mathbb{Z}/b\mathbb{Z}) \rtimes_\tau (\mathbb{Z}/b\mathbb{Z}), \widehat{\tau}),$$

which is described in terms of a generator β of the dual action $\widehat{\tau}$ as the mapping torus

$$MT(\beta) = \{f : [0,1] \to C(\mathbb{Z}/b\mathbb{Z}) \rtimes_\tau (\mathbb{Z}/b\mathbb{Z}) \mid f(1) = \beta(f(0))\}. \tag{2.8}$$

Since $C(\mathbb{Z}/b\mathbb{Z}) \rtimes_\tau (\mathbb{Z}/b\mathbb{Z}) \cong B(l^2(\mathbb{Z}/b\mathbb{Z})) = M_{|b|}(\mathbb{C})$, the automorphism β is inner and there is a continuous path β_t in $\mathrm{Aut}\, M_{|b|}(\mathbb{C})$ such that $\beta_0 = \mathrm{id}$ and $\beta_1 = \beta$. Now $\phi(f)(t) = \beta_t^{-1}(f(t))$ defines an isomorphism ϕ of (2.8) onto $C(\mathbb{T}, M_{|b|}(\mathbb{C}))$. We therefore have

$$C(F) \rtimes \mathbb{Z}^k \cong \bigotimes_{i=1}^{k} C(\mathbb{T}, M_{|b_{ii}|}(\mathbb{C})) \cong C(\mathbb{T}^k, M_{\prod_i |b_{ii}|}(\mathbb{C})),$$

and the result follows on observing that $\prod_i |b_{ii}| = |\det B| = |\det A| = |F|$. □

It follows from Proposition 2.8 and the decomposition $C(H) \rtimes_\sigma \mathbb{Z}^S = \overline{\bigcup A_n}$ that $C(H) \rtimes_\sigma \mathbb{Z}^S$ is an AH-algebra†, see [**23**]. The K-theory of $C(H_n) \rtimes \mathbb{Z}^S$ is torsion-free and this property is preserved under inductive limits, so $C(H) \rtimes_\sigma \mathbb{Z}^S$ has torsion-free K-theory. Since $C(H) \rtimes_\sigma \mathbb{Z}^S$ is a simple AH-algebra with real rank zero and no dimension growth, it follows from [**20**, Lemma 7.5], using the results of Elliott [**4**, Theorems 8.3 and 4.3] and Lin [**13**, Proposition 2.6, **14**, Theorem 5.2], that it is an AT-algebra.

We also use the decomposition $C(H) \rtimes_\sigma \mathbb{Z}^S = \overline{\bigcup A_n}$ to prove that $C(H) \rtimes_\sigma \mathbb{Z}^S$ has a unique tracial state. Let μ denote the Haar measure on $H \subset \mathcal{U}(\mathbb{Z}_{F\setminus S})$. The action σ permutes the cylinder sets $\{\pi_n^{-1}(m) \mid m \in H_n\}$, so every invariant probability measure agrees with μ on cylinder sets. Since the characteristic functions of such sets span a dense subspace of $C(H)$, it follows that μ is the only invariant probability measure and $C(H) \rtimes \mathbb{Z}^S$ has a unique tracial state by [**3**, Corollary VIII.3.8].

This completes the proof of Theorem 2.1.

3. *The structure of* $C(\mathcal{U}(\mathbb{Z}_{F\setminus S})) \rtimes_\sigma \mathbb{Z}^S$

3.1. *When S contains just one prime.* We consider $C(\mathcal{U}(\mathbb{Z}_{F\setminus S})) \rtimes_\sigma \mathbb{Z}^S$ when $|F| \geq 2$ and $S = \{q\}$. To simplify the notation, we relabel $F \setminus \{q\}$ as F. The following result generalizes [**11**, Theorem 3.1] in two directions: to sets F with $|F| > 1$ and to sets F containing the even prime 2. If $\mathbf{l} = (l_p) \in \mathbb{N}^F$ is a multi-index, we write $o_{\mathbf{l}}(q)$ for the order of q in $\prod_{p\in F} \mathcal{U}(\mathbb{Z}/p^{l_p}\mathbb{Z})$.

THEOREM 3.1. *Suppose that F is a finite set of primes and q is a prime which does not belong to F. Then there are a multi-index $\mathbf{K} = (K_p) \in \mathbb{N}^F$ and $d \in \mathbb{N}$ such that*

$$o_{\mathbf{K}+\mathbf{l}}(q) = d\left(\prod_{p\in F} p^{l_p}\right) \quad \textit{for every } \mathbf{l} \in \mathbb{N}^F, \tag{3.1}$$

† To see that an inductive limit $\overline{\bigcup A_n}$ is an AH-algebra, it suffices to show that each A_n is a corner in a matrix algebra $M_N(C(X))$, or, equivalently, that A_n is a homogeneous algebra with vanishing Dixmier–Douady class. Since the Dixmier–Douady class $\delta(A)$ of an m-homogeneous algebra satisfies $m\delta(A) = 0$ and $H^3(\mathbb{T}^k, \mathbb{Z})$ has no torsion, it suffices to prove that each A_n is a homogeneous algebra with spectrum $\mathbb{T}^k$. In our situation we could prove this in several ways. However, Proposition 2.8 makes the stronger statement that A_n is isomorphic to $M_m(C(\mathbb{T}^k))$.

and $C(\mathcal{U}(\mathbb{Z}_F)) \rtimes_\sigma \mathbb{Z}$ is the direct sum of $\big(\prod_{p\in F}(p-1)p^{K_p-1}\big)/d$ copies of a Bunce–Deddens algebra with supernatural number $d\big(\prod_{p\in F} p^\infty\big)$.

The existence of $\mathbf{K}$ and d satisfying (3.1) is established in Proposition A.5. We saw in §2 that $C(\mathcal{U}(\mathbb{Z}_F)) \rtimes_\sigma \mathbb{Z}$ is the direct sum of copies of the simple algebra $C(H) \rtimes_\sigma \mathbb{Z}$, where H is the closure of $q^{\mathbb{Z}}$ in $\mathcal{U}(\mathbb{Z}_F)$. It remains to prove that $C(H) \rtimes_\sigma \mathbb{Z}$ is a Bunce–Deddens algebra and to calculate the index $|\mathcal{U}(\mathbb{Z}_F) : H|$, which is the number of simple direct summands.

Let $\{n_k\}$ be integers with $n_k \geq 2$, and let $X_k = \{0, 1, \dots, n_k - 1\}$. Addition by 1 with carry over is a homeomorphism of the totally disconnected space $X := \prod_{k\geq 0} X_k$ called an odometer action, and the resulting crossed product $C(X) \rtimes_\tau \mathbb{Z}$ is a *Bunce–Deddens algebra with supernatural number* $\mathbf{n} := \prod_{k\geq 0} n_k$ (see [**3**, Ch. VIII.4]).

Our claim that $C(H) \rtimes_\sigma \mathbb{Z}$ is a Bunce–Deddens algebra will follow from the next proposition, which generalizes [**11**, Proposition 3.6].

PROPOSITION 3.2. *Suppose $\{G_l \mid \in \mathbb{N}\}$ are finite groups and $G = \varprojlim(G_l, \pi_l)$. Fix $g \in G$ and let L denote the closed subgroup of G generated by g. Consider the action $\sigma : \mathbb{Z} \to$ $\operatorname{Aut} C(G)$ such that $\sigma_n(f)(x) = f(g^{-n}x)$. Let $o_l(g)$ denote the order of $\pi_l(g)$ in G_l and let*

$$d_l := \begin{cases} o_0(g) & \text{if } l = 0 \\ o_l(g)/o_{l-1}(g) & \text{if } l \geq 1. \end{cases} \tag{3.2}$$

Then $C(L) \rtimes_\sigma \mathbb{Z}$ is a Bunce–Deddens algebra with supernatural number $\prod_{l\geq 0} d_l$.

Proof. Let $X := \prod_{l\geq 0}\{0, 1, \dots, d_l - 1\}$. The argument in the proof of [**11**, Proposition 3.6] shows that the continuous maps $h_l : X \to G_l$ given by

$$h_l(\{a_n\}) = \pi_l(g^{a_0 + a_1 d_0 + \cdots + a_l d_0 d_1 \dots d_{l-1}}) \tag{3.3}$$

combine to give an equivariant homeomorphism $h : X \to L$, which induces the required isomorphism. □

Our subgroup H of $\mathcal{U}(\mathbb{Z}_F)$ is the inverse limit $\varprojlim \pi_l(H)$, where $\pi_l : \mathcal{U}(\mathbb{Z}_F) \to \mathcal{U}\big(\mathbb{Z}/\big(\prod_{p\in F} p^{K_p+l}\big)\mathbb{Z}\big)$ is the canonical surjection. Then Proposition 3.2 and (3.1) imply that $C(H) \rtimes_\sigma \mathbb{Z}$ is a Bunce–Deddens algebra with supernatural number $d(\prod_{p\in F} p)^\infty$ for $d = o_{\mathbf{K}}(q)$. By Corollary A.6, we have that

$$|\mathcal{U}(\mathbb{Z}_F) : H| = \left(\prod_{p\in F}(p-1)p^{K_p-1}\right)\Big/ d, \tag{3.4}$$

which finishes the proof of Theorem 3.1.

3.2. *When S consists of two primes.* We now analyse $C(\mathcal{U}(\mathbb{Z}_{F\setminus S})) \rtimes \mathbb{Z}^S$ when $S = \{q, r\}$. For simplicity, we consider only the case $F = \{p, q, r\}$, so that we are interested in the crossed product $C(\mathcal{U}(\mathbb{Z}_p)) \rtimes_\sigma \mathbb{Z}^2$, where

$$\sigma_{m,n}(f)(x) = f(q^{-m} r^{-n} x).$$

THEOREM 3.3. *The C^*-algebra $C(\mathcal{U}(\mathbb{Z}_p)) \rtimes_\sigma \mathbb{Z}^2$ is a finite direct sum of copies of a simple AT-algebra A which has real rank zero, a unique tracial state and K-theory satisfying two short exact sequences:*

$$\begin{aligned} 0 \longrightarrow \mathbb{Z}[p^{-1}] \longrightarrow K_0(A) \longrightarrow \mathbb{Z} \longrightarrow 0 \\ 0 \longrightarrow \mathbb{Z} \longrightarrow K_1(A) \longrightarrow \mathbb{Z}[p^{-1}] \longrightarrow 0. \end{aligned} \tag{3.5}$$

Everything except the assertion about K-theory was proved in Theorem 2.1; the simple C^*-algebra A is $C(H) \rtimes_\sigma \mathbb{Z}^2$, where H is the closure of $q^{\mathbb{Z}} r^{\mathbb{Z}}$ in $\mathcal{U}(\mathbb{Z}_p)$. We aim to analyse $C(H) \rtimes_\sigma \mathbb{Z}^2$ by writing it as an iterated crossed product $(C(H) \rtimes_{\sigma^q} \mathbb{Z}) \rtimes \mathbb{Z}$. The inside crossed product is not simple unless $q^{\mathbb{Z}}$ is dense in H and it is helpful to reduce to this case using the following lemma.

LEMMA 3.4. *Let H_q denote the closure of $q^{\mathbb{Z}}$ in $\mathcal{U}(\mathbb{Z}_p)$. Then H_q has finite index $I(q)$ in H and, hence, is an open and closed subset of H. The inclusion of $C(H_q)$ in $C(H)$ induces an isomorphism of $C(H_q) \rtimes_\sigma (\mathbb{Z} \times I(q)\mathbb{Z})$ onto the corner $\chi_{H_q}(C(H) \rtimes_\sigma \mathbb{Z}^2)\chi_{H_q}$.*

Proof. Corollary A.6 implies that H_q has finite index in $\mathcal{U}(\mathbb{Z}_p)$, so it certainly has finite index in H. The inclusion of $C(H_q)$ in $C(H)$ and the map

$$(m, I(q)n) \mapsto \chi_{H_q} i_{\mathbb{Z}^2}(m, I(q)n)\chi_{H_q}$$

form a covariant representation of $(C(H_q), \mathbb{Z} \times I(q)\mathbb{Z}, \sigma)$ in $\chi_{H_q}(C(H) \rtimes_\sigma \mathbb{Z}^2)\chi_{H_q}$ and, hence, give a homomorphism

$$\phi : C(H_q) \rtimes_\sigma (\mathbb{Z} \times I(q)\mathbb{Z}) \to \chi_{H_q}(C(H) \rtimes_\sigma \mathbb{Z}^2)\chi_{H_q}.$$

We can identify $(\mathbb{Z} \times I(q)\mathbb{Z})^\wedge$ with $\mathbb{T}^2/(\mathbb{Z} \times I(q)\mathbb{Z})^\perp = \mathbb{T}^2/(1 \times C_{I(q)})$, where C_n denotes the group of nth roots of unity, and then ϕ carries the dual action $\hat{\sigma}_{[w,z]}$ into $\hat{\sigma}_{w,z}$; now a standard argument implies that ϕ is injective (or we could apply [**22**, Corollary 4.3], for example). We have

$$\chi_{H_q}(f i_{\mathbb{Z}^2}(m, n))\chi_{H_q} = (f\chi_{H_q}) i_{\mathbb{Z}^2}(m, n)\chi_{H_q} = i_{\mathbb{Z}^2}(m, n)\sigma_{m,n}^{-1}(f\chi_{H_q})\chi_{H_q}.$$

Since the support of $\sigma_{m,n}^{-1}(f\chi_{H_q})$ is contained in $q^{-m}r^{-n}H_q = r^{-n}H_q$, we have

$$\sigma_{m,n}^{-1}(f\chi_{H_q})\chi_{H_q} = \begin{cases} \sigma_{m,n}^{-1}(f\chi_{H_q}) & \text{if } I(q) \text{ divides } n \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\begin{aligned} \chi_{H_q}(f i_{\mathbb{Z}^2}(m, n))\chi_{H_q} &= \begin{cases} i_{\mathbb{Z}^2}(m, n)\sigma_{m,n}^{-1}(f\chi_{H_q}) & \text{if } I(q) \text{ divides } n \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} \phi((f\chi_{H_q}) i_{\mathbb{Z}^2}(m, n)) & \text{if } I(q) \text{ divides } n \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Thus, every $\chi_{H_q}(f i_{\mathbb{Z}^2}(m, n))\chi_{H_q}$ is in the range of ϕ and ϕ is surjective. □

COROLLARY 3.5. *Define $\gamma : \mathbb{Z} \to \mathrm{Aut}(C(H_q) \rtimes_{\sigma^q} \mathbb{Z})$ by*

$$\gamma_m(f i_{\mathbb{Z}}(n)) = \sigma^r_{I(q)m}(f) i_{\mathbb{Z}}(n). \tag{3.6}$$

Then $(C(H_q) \rtimes_{\sigma^q} \mathbb{Z}) \rtimes_\gamma \mathbb{Z}$ is isomorphic to a full corner in $C(H) \rtimes_\sigma \mathbb{Z}^2$.

Proof. Theorem 4.1 of [**18**] gives $C(H_q) \rtimes_\sigma (\mathbb{Z} \times I(q)\mathbb{Z}) \cong (C(H_q) \rtimes_{\sigma^q} \mathbb{Z}) \rtimes I(q)\mathbb{Z}$, so the result follows from Lemma 3.4 on replacing $I(q)\mathbb{Z}$ by the isomorphic group $\mathbb{Z}$. □

The analysis in §3.1 shows that $C(H_q) \rtimes_{\sigma^q} \mathbb{Z}$ is a Bunce–Deddens algebra. The K-theory of Bunce–Deddens algebras is well known. To state the version we need, recall that if $\mathbf{n} = (n_k)_{k\geq 0}$ is a sequence with $n_k \geq 2$, then $\mathbb{Z}[\mathbf{n}^{-1}]$ denotes the set of rational numbers with denominator $N_k = \prod_{i=0}^{k} n_i$ for some $k \geq 0$.

PROPOSITION 3.6. *Suppose that $\mathbf{n} = (n_k)_{k\geq 0}$, $X_k = \{0, \ldots, n_k - 1\}$, $X = \prod X_k$ and $\tau : \mathbb{Z} \to \operatorname{Aut} C(X)$ is the associated odometer. Then there are isomorphisms $\phi_0 : K_0(C(X) \rtimes_\tau \mathbb{Z}) \to \mathbb{Z}[\mathbf{n}^{-1}]$ such that $\phi_0([\chi_{J(a_0,\ldots,a_k)}]) = N_k^{-1}$ for each cylinder set $J(a_0, \ldots, a_k)$ and $\phi_1 : K_1(C(X) \rtimes_\tau \mathbb{Z}) \to \mathbb{Z}$ such that $\phi_1(i_{\mathbb{Z}}(1)) = 1$.*

Proof. As $K_1(C(X)) = 0$, the Pimsner–Voiculescu sequence for the system $(C(X), \mathbb{Z}, \tau)$ reduces to

$$0 \longrightarrow K_1(C(X) \rtimes_\tau \mathbb{Z}) \xrightarrow{\delta} K_0(C(X)) \xrightarrow{\mathrm{id} - \tau_*} K_0(C(X)) \xrightarrow{\mathrm{id}_*} K_0(C(X) \rtimes_\tau \mathbb{Z}) \longrightarrow 0.$$

Now let $C_k = \{J(a_0, \ldots, a_k)\}$ be the set of cylinder sets of length $k+1$ and note that $C(X) = \overline{\bigcup_{k=1} A_k}$, where $A_k = \operatorname{span}\{\chi_J \mid J \in C_k\}$. Each χ_J for $J \in C_k$ is the sum of n_{k+1} basis elements of A_{k+1}, so the maps $[\chi_{J(a_0,\ldots,a_k)}] \mapsto {N_k}^{-1}$ of A_k into $\mathbb{R}$ combine to give a homomorphism ψ_0 of $K_0(C(X)) = \varinjlim K_0(A_k)$ into $\mathbb{R}$ with range $\mathbb{Z}[\mathbf{n}^{-1}]$. Since the generating automorphism $\tau = \tau_1$ permutes C_k, the kernel of ψ_0 is the range of $\mathrm{id} - \tau_*$ and, hence, ψ_0 induces the required isomorphism ϕ_0 of $K_0(C(X) \rtimes_\phi \mathbb{Z})$ onto $\mathbb{Z}[\mathbf{n}^{-1}]$. To verify the statement about K_1, recall that δ is the coboundary map for the Toeplitz extension of $C(X) \rtimes_\tau \mathbb{Z}$ (see [**21**, §2]) and compute the index of $[i_{\mathbb{Z}}(1)]$ in $K_0(C(X) \otimes \mathcal{K}) \cong K_0(C(X))$. □

Proof of Theorem 3.3. We saw in the proof of Proposition 3.2 and in the paragraph following it that the homeomorphism h of $\prod_{k\geq 0} X_k$ onto the subgroup H_q of $\mathcal{U}(\mathbb{Z}_p)$ satisfies

$$\pi_k(h(\{a_n\})) = \pi_k(q^{a_0 + a_1 o_p(q) + \cdots + a_k o_p(q) p^{k-1}}) \quad \text{for } k \geq 0,$$

and hence carries $J(a_0, \ldots, a_k)$ onto $Z(q^{a_0 + a_1 o_p(q) + \cdots + a_k o_p(q) p^{k-1}})$, where

$$Z_k(n) = \{x \in \mathcal{U}(\mathbb{Z}_p) \mid \pi_k(x) = \pi_k(n)\}.$$

So we deduce from Proposition 3.6 that there is an isomorphism ϕ_0 of $K_0(C(H_q) \rtimes_\sigma \mathbb{Z})$ onto $(1/o_p(q))\mathbb{Z}[p^{-1}]$ such that

$$\phi_0([\chi_{Z_k(m)}]) = \frac{1}{o_p(q)} \frac{1}{p^k}$$

for every integer m which lies in H_q.

Multiplying by the unit $r^{-I(q)l}$ carries $Z_k(m)$ into $Z_k(r^{-I(q)l} m)$ and, hence, $\phi_0 \circ (\gamma_l)_* = \phi_0$. Thus, $(\gamma_l)_*$ is the identity on $K_0(C(H_q) \rtimes_\sigma \mathbb{Z})$. It is also the identity on $K_1(C(H_q) \rtimes_\sigma \mathbb{Z})$ and, hence, the Pimsner–Voiculescu sequence for $((C(H_q) \rtimes_\sigma \mathbb{Z}), \mathbb{Z}, \gamma)$

collapses to the two short exact sequences

$$0 \longrightarrow \frac{1}{o_p(q)}\mathbb{Z}[p^{-1}] \longrightarrow K_0(C(H_q) \rtimes \mathbb{Z}^2) \longrightarrow \mathbb{Z} \longrightarrow 0$$
$$0 \longrightarrow \mathbb{Z} \longrightarrow K_1(C(H_q) \rtimes \mathbb{Z}^2) \longrightarrow \frac{1}{o_p(q)}\mathbb{Z}[p^{-1}] \longrightarrow 0.$$

From this and Corollary 3.5 we can deduce (3.5); since the isomorphism induced by Corollary 3.5 scales the class of [1], we have removed the factor $o_p(q)^{-1}$ by a further scaling to ensure that the final statement does not depend on the order of factors in our decomposition. □

Remark 3.7. The number of simple summands in Theorem 3.3 is $|\mathcal{U}(\mathbb{Z}_p) : H|$ and we can compute this using [**11**, Lemma 3.7]. For example, if p is odd and l is large, we have from (A.1) that

$$\begin{aligned}|\pi_l(H)| &= [o_{p^l}(q), o_{p^l}(r)] = [p^{l-L_p(q)}o_p(q), q^{l-L_p(r)}o_p(r)] \\ &= p^{l-\min(L_p(q),L_p(r))}[o_p(q), o_p(r)];\end{aligned}$$

thus we deduce

$$|\mathcal{U}(\mathbb{Z}_p) : H| = |\mathcal{U}(\mathbb{Z}/p^l\mathbb{Z}) : \pi_l(H)| = \frac{(p-1)p^{\min(L_p(q),L_p(r))-1}}{[o_p(q), o_p(r)]}.$$

We could carry out a similar analysis when $|F| > 1$, although it would not be so easy to work out some of the indices explicitly.

Remark 3.8. Theorem 2.1 implies, in particular, that $C(H) \rtimes_\sigma \mathbb{Z}^2$ satisfies the hypotheses of the classification theorem of Elliott for AT-algebras [**23**, Theorem 3.2.6]. We can tell from the computation of K-theory in Theorem 3.3 that $C(H) \rtimes_\sigma \mathbb{Z}^2$ is not a Bunce–Deddens algebra, but it is still closely related to an odometer. The homeomorphism of $\prod_{k\geq 0} X_k$ onto H_q identifies the action of the first copy of $\mathbb{Z}$ (multiplication by q on H_q) with an odometer (addition of 1 with carry over). The action of the second copy of $\mathbb{Z}$ (multiplication by r on H_q) also acts as a permutation on each X_k: it moves X_0 around in a different order and this action carries over into X_1 when the marker in X_0 returns to the starting point. So we can think of the action of $\mathbb{Z}^2$ as two independent odometers on the same set. We can normalize the scale so that either copy of $\mathbb{Z}$ acts by addition of 1 with carry over, but not so that both act this way at once.

4. *The Bost–Connes algebra*

The Hecke C^*-algebra $\mathcal{C}_\mathbb{Q}$ of Bost and Connes [**2**] is isomorphic to the semigroup crossed product $C^*(\mathbb{Q}/\mathbb{Z}) \rtimes_\alpha \mathbb{N}^*$. The Fourier transform takes $C^*(\mathbb{Q}/\mathbb{Z})$ onto the algebra of continuous functions on the compact group $\mathcal{Z} := \prod_{p\in\mathcal{P}} \mathbb{Z}_p$ and carries α into the action given by (see [**7**, §3.1])

$$\alpha_n(f)(x) = \begin{cases} f(x/n) & \text{if } n \text{ divides } x \\ 0 & \text{otherwise.}\end{cases}$$

Lemma 2.2 is valid with F replaced by $\mathcal{P}$ and $\mathbb{Z}_F$ by $\mathcal{Z}$. Thus, for $S \subset \mathcal{P}$, an application of [**10**, Theorem 1.7] gives that $J_S := C_0(\mathcal{Z} \setminus \mathcal{Z}_S) \rtimes_\alpha \mathbb{N}^*$ is an ideal of $\mathcal{C}_\mathbb{Q} = C(\mathcal{Z}) \rtimes \mathbb{N}^*$, with quotient isomorphic to $C(\mathcal{Z}_S) \rtimes \mathbb{N}^*$. Choose $a \in \mathcal{Z}$ such that $a_p = 0 \iff p \in S$. Then $\{\mathbb{Q}_+^* a \cap \mathcal{Z}\}$ has closure $\mathcal{Z}_S$, so $C_0(\mathcal{Z} \setminus \mathcal{Z}_S)$ is the kernel of the representation π_a considered in [**9**, p. 440], and it follows from [**11**, Lemma 4.2] that J_S is the kernel of the representation $\pi_a \times V$ described in [**9**, p. 440]. We can now deduce that $S \mapsto J_S$, as S runs through the proper subsets of $\mathcal{P}$, is the parametrization of $(\operatorname{Prim} \mathcal{C}_\mathbb{Q}) \setminus \widehat{\mathbb{Q}_+^*}$ given in [**9**, Theorem 2.8].

THEOREM 4.1. *Suppose that S is a proper subset of $\mathcal{P}$.*

(a) *If $\mathcal{P} \setminus S$ is infinite, then $J_S = \bigcap_{p \notin S} J_{S \cup \{p\}}$.*

(b) *If $0 < |\mathcal{P} \setminus S| < \infty$, then*

$$\left(\bigcap_{p \notin S} J_{S \cup \{p\}} \right) \Big/ J_S \cong (C(\mathcal{U}(\mathbb{Z}_{\mathcal{P} \setminus S})) \rtimes_\sigma \mathbb{Z}^S) \otimes \mathcal{K}(l^2(\mathbb{N}^{\mathcal{P} \setminus S})).$$

(c) *$\mathcal{C}_\mathbb{Q} / J_\mathcal{P}$ is isomorphic to $C^*(\mathbb{Q}_+^*) = C(\widehat{\mathbb{Q}_+^*})$.*

Moreover, $C(\mathcal{U}(\mathbb{Z}_{\mathcal{P} \setminus S})) \rtimes_\sigma \mathbb{Z}^S$ is a simple AT-algebra with real rank zero and a unique tracial state.

It follows from [**9**, §2] that every basic open neighbourhood of J_S has the form

$$U_G = \{J_T \mid T \subset \mathcal{P}, T \cap G = \emptyset\}$$

for some finite subset G of $\mathcal{P} \setminus S$. When $\mathcal{P} \setminus S$ is infinite, there are always lots of $J_{S \cup \{p\}}$ in U_G, and thus $J_S \in \overline{\{J_{S \cup \{p\}} \mid p \notin S\}}$; this states precisely that $\bigcap_{p \notin S} J_{S \cup \{p\}} \subset J_S$. The other inclusion is trivial and (a) follows. Part (c) is true because $\mathcal{Z}_\mathcal{P} = \{0\}$. To prove (b) we just need to replace F by $\mathcal{P}$ and $\mathbb{Z}_F$ by $\mathcal{Z}$ in the proof of Lemma 2.6.

It remains to prove the statements about $C(\mathcal{U}(\mathbb{Z}_{\mathcal{P} \setminus S})) \rtimes_\sigma \mathbb{Z}^S$. The Chinese Remainder Theorem implies that $\mathbb{Z}$ is dense in $\mathbb{Z}_{\mathcal{P} \setminus S}$ and, hence, $\mathbb{Z}^S = \mathbb{Z} \cap \mathcal{U}(\mathbb{Z}_{\mathcal{P} \setminus S})$ is dense in $\mathcal{U}(\mathbb{Z}_{\mathcal{P} \setminus S})$. Thus, $\mathbb{Z}^S$ acts minimally and freely on $\mathcal{U}(\mathbb{Z}_{\mathcal{P} \setminus S})$ and $C(\mathcal{U}(\mathbb{Z}_{\mathcal{P} \setminus S})) \rtimes_\sigma \mathbb{Z}^S$ is simple. However, since $|S| = \infty$, we cannot apply the results of [**19**] to conclude that $C(\mathcal{U}(\mathbb{Z}_{\mathcal{P} \setminus S})) \rtimes_\sigma \mathbb{Z}^S$ has real rank zero and stable rank one, as we did in §2 for $C(H) \rtimes_\sigma \mathbb{Z}^S$. Instead we aim to use Theorems 1 and 2 of [**1**], and to do this we need to show that $C(\mathcal{U}(\mathbb{Z}_{\mathcal{P} \setminus S})) \rtimes_\sigma \mathbb{Z}^S$ is an AH-algebra with the extra property of slow dimension growth.

Since $\mathcal{P} \setminus S$ is finite, we have as in (2.7) that $\mathcal{U}(\mathbb{Z}_{\mathcal{P} \setminus S})$ is the inverse limit of the finite groups $\mathcal{U}\big(\mathbb{Z}/\big(\prod_{p \in \mathcal{P} \setminus S} p^{l_p}\big)\mathbb{Z}\big)$ over $l = (l_p) \in \mathbb{N}^{\mathcal{P} \setminus S}$. Hence, $\mathcal{U}(\mathbb{Z}_{\mathcal{P} \setminus S}) = \varprojlim F_n$, where

$$F_n := \mathcal{U}\Big(\mathbb{Z} \Big/ \Big(\prod_{p \in \mathcal{P} \setminus S} p^n \Big) \mathbb{Z} \Big). \tag{4.1}$$

We denote the canonical surjection of $\mathcal{U}(\mathbb{Z}_{\mathcal{P} \setminus S})$ onto F_n by π_n. The analogue of Lemma 2.7 for F_n and the canonical action of $\mathbb{Z}^S$ by multiplication on F_n implies that $C(\mathcal{U}(\mathbb{Z}_{\mathcal{P} \setminus S})) \rtimes_\sigma \mathbb{Z}^S$ is the closed union of C^*-subalgebras isomorphic to $C(F_n) \rtimes \mathbb{Z}^S$.

Towards applying Proposition 2.8, we note that the infinite direct sum $\mathbb{Z}^S$ is the union of the subgroups $\mathbb{Z}^E$ associated to finite subsets E of S. Thus, by using an argument similar to that in Lemma 2.7 we have that $C(F_n) \rtimes \mathbb{Z}^S$ is the closed union of subalgebras isomorphic to $C(F_n) \rtimes \mathbb{Z}^E$. However, by choosing a particular sequence E_n of finite subsets of S, we can show that $C(\mathcal{U}(\mathbb{Z}_{\mathcal{P}\setminus S})) \rtimes_\sigma \mathbb{Z}^S$ has slow dimension growth.

Indeed, since $\mathbb{Z}/\big(\prod_p p^n\big)\mathbb{Z} \cong \prod_p \mathbb{Z}/p^n\mathbb{Z}$, we have $F_n \cong \prod_{p\in\mathcal{P}\setminus S} \mathcal{U}(\mathbb{Z}/p^n\mathbb{Z})$. Thus, F_n is a product of at most $|\mathcal{P}\setminus S|+1$ cyclic groups (the $+1$ allows for the possibility that $2 \in \mathcal{P}\setminus S$) and, hence, has a generating set $\{x_{n,i}\}$ with at most $|\mathcal{P}\setminus S|+1$ elements. By Dirichlet's Theorem, there are primes $q_{n,i}$ such that

$$q_{n,i} \equiv x_{n,i} \quad \Big(\mathrm{mod} \prod_{p\in\mathcal{P}\setminus S} p^n\Big),$$

and each $q_{n,i}$ belongs to S because it is a unit modulo $\prod_{p\in\mathcal{P}\setminus S} p^n$. Now let $E'_n := \{q_{n,i}\}$, list the primes in S as $\{r_n \mid n \in \mathbb{N}\}$ and take

$$E_n := \Big(\bigcup_{m\le n} E'_m\Big) \cup \{r_1, \ldots, r_n\}.$$

We then have $\pi_n(\mathbb{Z}^{E_n}) = F_n$, $E_m \subset E_n$ for $m \le n$, and $\bigcup E_n = S$.

We have now realized $C(\mathcal{U}(\mathbb{Z}_{\mathcal{P}\setminus S})) \rtimes_\sigma \mathbb{Z}^S$ as the closure of an increasing union $\bigcup_{n\in\mathbb{N}} B_n$ in which B_n is isomorphic to the crossed product $C(F_n) \rtimes \mathbb{Z}^{E_n}$ by a transitive action of $\mathbb{Z}^{E_n}$. By an argument identical to that at the end of §2 we conclude that $C(\mathcal{U}(\mathbb{Z}_{\mathcal{P}\setminus S})) \rtimes \mathbb{Z}^S$ has a unique tracial state.

We prove next that $C(\mathcal{U}(\mathbb{Z}_{\mathcal{P}\setminus S})) \rtimes \mathbb{Z}^S$ is an AH-algebra with real rank zero. Proposition 2.8 implies that $B_n \cong C(F_n) \rtimes \mathbb{Z}^{E_n} \cong C(\mathbb{T}^{|E_n|}, M_{|F_n|}(\mathbb{C}))$. However,

$$\frac{|E_n|}{|F_n|} \le \frac{n(|\mathcal{P}\setminus S|+2)}{\prod_p (p-1)p^{n-1}} \to 0 \quad \text{as } n \to \infty,$$

and, thus, the sequence $B_n \cong C(F_n) \rtimes \mathbb{Z}^{E_n}$ of subalgebras of $C(\mathcal{U}(\mathbb{Z}_{\mathcal{P}\setminus S})) \rtimes \mathbb{Z}^S$ has slow dimension growth. It now follows from [**1**, Theorem 1] that $C(\mathcal{U}(\mathbb{Z}_{\mathcal{P}\setminus S})) \rtimes \mathbb{Z}^S$ has topological stable rank one. Since the projections in $C(\mathcal{U}(\mathbb{Z}_{\mathcal{P}\setminus S})) \rtimes \mathbb{Z}^S$ trivially separate the unique tracial state, [**1**, Theorem 2] implies that $C(\mathcal{U}(\mathbb{Z}_{\mathcal{P}\setminus S})) \rtimes \mathbb{Z}^S$ has real rank zero. The K-groups of $C(\mathcal{U}(\mathbb{Z}_{\mathcal{P}\setminus S})) \rtimes \mathbb{Z}^S$ are inductive limits of torsion-free groups and, hence, are themselves torsion-free, so it follows as in §2 that $C(\mathcal{U}(\mathbb{Z}_{\mathcal{P}\setminus S})) \rtimes \mathbb{Z}^S$ is an AT-algebra.

This completes the proof of Theorem 4.1.

A. *Appendix. The orders of a prime in groups of units*

For p prime and $m \in \mathbb{N}$ such that $(m, p) = 1$, we denote by $o_{p^l}(m)$ the order of m in $\mathcal{U}(\mathbb{Z}/p^l\mathbb{Z})$. It was shown in [**11**, Theorem 3.1] that if p is odd, there is a positive integer $L_p(m)$ such that

$$o_{p^l}(m) = \begin{cases} o_p(m) & \text{if } 1 \le l \le L_p(m) \\ p^{l-L_p(m)} o_p(m) & \text{if } l > L_p(m); \end{cases} \tag{A.1}$$

the proof uses that the groups $\mathcal{U}(\mathbb{Z}/p^l\mathbb{Z})$ are cyclic. We now show how to modify the arguments of [**11**, §3] to obtain an analogue of (A.1) for $p = 2$, in which case $\mathcal{U}(\mathbb{Z}/2^l\mathbb{Z})$ are no longer cyclic.

PROPOSITION A.1. *If m is an odd integer and $m \equiv 1 \pmod 4$, then there exists a positive integer $K = L_2(m)$ such that*

$$o_{2^l}(m) = \begin{cases} 1 & \text{if } 1 \leq l \leq K \\ 2^{l-K} & \text{if } l > K; \end{cases} \tag{A.2}$$

if $m \equiv 3 \pmod 4$, then there exists a positive integer $L = L_2(m)$ such that

$$o_{2^l}(m) = \begin{cases} 1 & \text{if } l = 1 \\ 2 & \text{if } 1 < l \leq L \\ 2^{l-(L-1)} & \text{if } l > L. \end{cases} \tag{A.3}$$

To prove Proposition A.1 we use general properties of cyclic groups as in [**11**, §3]. We begin with a lemma.

LEMMA A.2. *Suppose $l \geq 3$. Then the group $\{n \in \mathcal{U}(\mathbb{Z}/2^l\mathbb{Z}) \mid n \equiv 1 \pmod 4\}$ is the cyclic subgroup $\langle 5 \rangle_l$ of $\mathcal{U}(\mathbb{Z}/2^l\mathbb{Z})$ generated by 5.*

Proof. Theorem $2'$ of [**5**, Ch. 4.1] states that $|\langle 5 \rangle_l| = 2^{l-2}$. For $k \geq 0$ we have

$$5^k = (4+1)^k = \sum_{n=0}^{k} \binom{k}{n} 4^n = 4 \sum_{n=1}^{k} \binom{k}{n} 4^{n-1} + 1,$$

so $5^k \equiv 1 \pmod 4$. Hence, if $n \equiv 5^k \pmod{2^l}$ for some $0 \leq k < 2^{l-2}$, then $n \equiv 1 \pmod 4$. Since the order of $\{n \in \mathcal{U}(\mathbb{Z}/2^l\mathbb{Z}) \mid n \equiv 1 \pmod 4\}$ is also 2^{l-2}, the result follows. □

COROLLARY A.3. *An element of $\mathcal{U}(\mathbb{Z}/2^l\mathbb{Z})$ is congruent to $3 \pmod 4$ if and only if it is congruent to $-5^k \pmod{2^l}$ for some k satisfying $0 \leq k < 2^{l-2}$.*

COROLLARY A.4. *Suppose $m \in \mathbb{Z}$ satisfies $m \equiv 1 \pmod 4$. Then for every $l > 0$ we have*

$$o_{2^l}(m) = \begin{cases} o_{2^{l+1}}(m) & \text{if 2 does not divide } o_{2^{l+1}}(m) \\ o_{2^{l+1}}(m)/2 & \text{if 2 divides } o_{2^{l+1}}(m). \end{cases} \tag{A.4}$$

Proof. Since a number is coprime to 2^l if and only if it is coprime to 2^{l+1}, the reduction map $\pi : \mathcal{U}(\mathbb{Z}/2^{l+1}\mathbb{Z}) \to \mathcal{U}(\mathbb{Z}/2^l\mathbb{Z})$ is a surjective homomorphism. Lemma A.2 implies that $m \equiv 5^r \pmod{2^{l+1}}$, where $r = o_{2^{l+1}}(5)/o_{2^{l+1}}(m) = 2^{l-1}/o_{2^{l+1}}(m)$. Hence, by applying [**11**, Lemma 3.2] to the restriction of π to a homomorphism of $\langle 5 \rangle_{l+1}$ onto $\langle 5 \rangle_l$, we have

$$\begin{aligned} o_{2^l}(m) &= o(\pi(5^r)) \\ &= \begin{cases} 2^{l-1}/(2^{l-1}/o_{2^{l+1}}(m), 2^{l-1}) & \text{if } 2^{l-1} \text{ divides } 2^{l-1}/o_{2^{l+1}}(m) \\ 2^{l-1}/(2(2^{l-1}/o_{2^{l+1}}(m), 2^{l-1})) & \text{if } 2^{l-1} \text{ does not divide } 2^{l-1}/o_{2^{l+1}}(m), \end{cases} \end{aligned}$$

which simplifies to (A.4). □

Proof of Proposition A.1. Suppose first that $m \equiv 1 \pmod 4$. For fixed N, there exists an $l \in \mathbb{N}$ satisfying $m^N < 2^l$. Then $o_{2^l}(m) > N$ and, hence, the sequence $\{o_{2^l}(m) \mid l \in \mathbb{N}\}$ must be unbounded. In particular, $\{o_{2^l}(m)\}$ is not a constant sequence. Let K be the first integer such that $o_{2^K}(m) < o_{2^{K+1}}(m)$. Then $o_{2^l}(m) = o_2(m) = 1$ for $1 \le l \le K$ and by Corollary A.4 we have $o_{2^{K+1}}(m) = 2o_2(m) = 2$. Since $o_{2^{K+1}}(m)$ divides $o_{2^l}(m)$ for all $l > K$, it follows that 2 divides $o_{2^l}(m)$ for all $l > K$. We now apply Corollary A.4 $l - K$ times to deduce that $o_{2^l}(m) = 2^{l-K} o_{2^K}(m) = 2^{l-K}$.

Now suppose that $m \equiv 3 \pmod 4$. Certainly $o_2(m) = 1$. For $l > 1$, Corollary A.3 tells us that $m \equiv -5^k$ for some $0 \le k < 2^{l-2}$. Thus, $m^2 \equiv 5^{2k} \pmod{2^l}$ and, therefore, $m^2 \in \langle 5 \rangle_l$. Let L be the first integer such that $o_{2^L}(m^2) < o_{2^{L+1}}(m^2)$. Applying Corollary A.4 to m^2 and repeating the argument of the preceding paragraph gives (A.3) because $o_{2^l}(m) = 2o_{2^l}(m^2)$. □

We now need to extend these results to cover actions on $\mathcal{U}(\mathbb{Z}_F)$ for an arbitrary finite set F of primes. We write $F = \{p_1, \ldots, p_n\}$ and fix a prime q which is not in F. We denote by $o_{(l_1,\ldots,l_n)}(q)$ the order of $(q, \ldots, q)$ in $\prod_{i=1}^n \mathcal{U}(\mathbb{Z}/{p_i}^{l_i}\mathbb{Z})$.

PROPOSITION A.5. *There exist positive integers $K_1, \ldots, K_n$ and d such that*

$$o_{(K_1+l_1,\ldots,K_n+l_n)}(q) = d\, {p_1}^{l_1} \ldots {p_n}^{l_n} \tag{A.5}$$

for every $(l_1, \ldots, l_n) \in \mathbb{N}^F$.

Proof. Suppose first that $p_1, \ldots, p_n$ are distinct odd primes, and let $L_{p_i}(q)$ be as in (A.1). Let

$$z_i := \max\{z \mid {p_i}^z \text{ divides } o_{p_j}(q) \text{ for some } j \in \{1, \ldots, n\}\},$$

and define $K_i := L_{p_i}(q) + z_i$ and $d := [o_{p_1}(q), \ldots, o_{p_n}(q)]$, where $[r_1, \ldots, r_n]$ is the least common multiple of the integers r_i. In general, if g_i are elements of order r_i in finite groups G_i, then the order of $(g_1, \ldots, g_n)$ in $G_1 \times \cdots \times G_n$ is $[r_1, \ldots, r_n]$. Thus, from the properties of $L_{p_i}(q)$ we obtain

$$\begin{aligned} o_{(K_1+l_1,\ldots,K_n+l_n)}(q) &= [p_1^{(K_1+l_1)-L_{p_1}(q)} o_{p_1}(q), \ldots, p_n^{(K_n+l_n)-L_{p_n}(q)} o_{p_n}(q)] \\ &= [p_1^{z_1+l_1} o_{p_1}(q), \ldots, p_n^{z_n+l_n} o_{p_n}(q)] \\ &= {p_1}^{l_1} \ldots {p_n}^{l_n} [o_{p_1}(q), \ldots, o_{p_n}(q)], \end{aligned} \tag{A.6}$$

which is (A.5).

Now suppose that $2 \in F$, say $p_1 = 2$. If $q \equiv 1 \pmod 4$, we let

$$z_i := \max\{z \mid {p_i}^z \text{ divides } o_{p_j}(q) \text{ for some } j \in \{2, \ldots, n\}\},$$

and define $K_1 := L_2(q) + z_1$, $K_i := L_{p_i}(q) + z_i$ for $i > 1$ and $d := [o_{p_2}(q), \ldots, o_{p_n}(q)]$. Reasoning as in (A.6) gives (A.5).

If $q \equiv 3 \pmod 4$, we let

$$\begin{aligned} z_1 &= \max(1, \max\{z \mid 2^z \text{ divides } o_{p_j}(q) \text{ for some } j \in \{2, \ldots, n\}\}), \\ z_i &= \max\{z \mid {p_i}^z \text{ divides } o_{p_j}(q) \text{ for some } j \in \{2, \ldots, n\}\} \end{aligned}$$

for $i > 1$, and define $K_1 := L_2(q) + z_1 - 1$, $K_i := L_{p_i}(q) + z_i$ for $i > 1$ and $d := [2, o_{p_2}(q), \ldots, o_{p_n}(q)]$. Again, reasoning as in (A.6) gives (A.5). □

COROLLARY A.6. *The closure H of $q^{\mathbb{Z}}$ in $\mathcal{U}(\mathbb{Z}_F)$ is a subgroup of finite index*

$$|\mathcal{U}(\mathbb{Z}_F) : H| = \left(\prod_{i=1}^{n}(p_i - 1)p_i^{K_i-1}\right)\Big/ d.$$

Proof. Apply Proposition A.5 to $\mathbf{l} = (l, l, \ldots, l)$ to see that $|\pi_l(H)| = d\left(\prod_{i=1}^{n} p_i^l\right)$ for large l and the result follows from [**11**, Lemma 3.7]. □

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