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Perelman's l-distrance

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Abstract
This talk is a preparation of the necessary tools for proving the non-collapsing results. The L-length defined by Perelman is the analog of an energy path, but defined in a Riemannian manifold context. The length is used to define the l reduced distance and later on, the reduced volume. So far the properties of the l-length have two applications in the proof of the Poincare conjecture. Associated with the notion of reduced volume, they are used to prove non-collapsing results and also to study the K- solutions.

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Perelman's *l*-distance

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This talk is a preparation of the necessary tools for proving the non-collapsing results. The $\mathcal{L}$-length defined by Perelman is the analog of an energy path, but defined in a Riemannian manifold context. The length is used to define the l reduced distance and later on, the reduced volume. So far the properties of the l-length have two applications in the proof of the Poincaré conjecture. Associated with the notion of reduced volume, they are used to prove non-collapsing results and also to study the κ- solutions.

The first step is introducing the reduced distance by means of the $\mathcal{L}$-length defined by Perelman, [2].

Consider a backward solution of the Ricci flow $(M, g(\tau))$:

$$\frac{\partial}{\partial \tau} g_{ij}(\tau) = 2Ric_{ij}(g(\tau)),$$

where $\tau = T - t$ (T is the final time). Let $\gamma : [\tau_1, \tau_2] \to M$ be a curve on the manifold parametrized by backward time.

Definition 1 ([2])**.** *The $\mathcal{L}$-length of a curve γ is*

$$\mathcal{L}(\gamma) := \int_{\tau_1}^{\tau_2} \sqrt{\tau}(R(\gamma(\tau)) + |\dot{\gamma}(\tau)|^2) d\tau,$$

where $R(\gamma(\tau))$ is the scalar curvature at the point $\gamma(\tau)$.

Considering a variation of the curve γ, $\tilde{\gamma}(s, \tau)$, $s \in (-\epsilon, +\epsilon)$, $t \in [\tau_1, \tau_2]$ we can define the tangential and variational vector fields by $X = \frac{\partial \tilde{\gamma}}{\partial \tau}$ and $Y = \frac{\partial \tilde{\gamma}}{\partial s}$.

The first properties obtained are the (Euler-Lagrange) equations of $\mathcal{L}$-geodesics:

Proposition 1 ([2])**.** *There holds*

$$\nabla_X X - \frac{1}{2}\nabla R + 2Ric(X, \cdot) + \frac{1}{2\tau} X = 0.$$

The proof comes easily from the first variation formula for the $\mathcal{L}$-length.

Let $\tau_1 = 0$ and $\tau_2 = \overline{\tau}$ we consider furthermore variations of curves on M, connecting points $p, q \in M$ with fixed starting point p and moving end point $q = q(\overline{\tau})$.

Definition 2 ([2])**.** *Denote by $L(q, \overline{\tau})$ the $\mathcal{L}$-length of the $\mathcal{L}$- shortest curve $\gamma(\tau)$, $0 \leq \tau \leq \overline{\tau}$ connecting p and q.*
The reduced length is defined as $l(q, \tau) = \frac{L(q, \tau)}{2\sqrt{\tau}}$.

From the definition one can see that properties of the reduced distance can be easily obtained if we have the corresponding ones for the $\mathcal{L}$-length. We are concentrating the last mentioned ones in two main propositions:

Proposition 2 ([2]). *There holds*

$$L_{\overline{\tau}}(q,\overline{\tau}) = 2\sqrt{\overline{\tau}}R(q) - \frac{1}{\overline{\tau}}K - \frac{1}{2\overline{\tau}}L(q,\overline{\tau}),$$
$$|\nabla L|^2(q,\overline{\tau}) = -4\overline{\tau}R(q) + \frac{2}{\sqrt{\overline{\tau}}}L(q,\overline{\tau}) - \frac{4}{\sqrt{\overline{\tau}}}K.$$

where $K = K(\gamma,\overline{\tau}) = \int_0^{\overline{\tau}} \tau^{\frac{3}{2}} H(X(\tau))d\tau$ *and* $H(X)$ *is the trace of the expression appearing in Hamilton's Harnack inequality,* [4].

Proposition 3 ([2]). *There holds*

$$\Delta L \leq \frac{n}{\sqrt{\overline{\tau}}} - 2\sqrt{\overline{\tau}}R - \frac{1}{\overline{\tau}}\int_0^{\overline{\tau}} H(X)d\tau.$$

For the proof of the last one we have followed the detailed steps of [1, 3]. One starts by computing the second variation and then the Hessian of the $\mathcal{L}$-length. We define the $\mathcal{L}$-Jacobi fields along $\mathcal{L}$-geodesics and prove that they are minimizers of Hessian of the $\mathcal{L}$-length. Then making a special choice of orthonormal basis for $T_{\gamma(\tau)}M$ we obtain the result.

Using the above properties of the $\mathcal{L}$-length we can also obtain the reduced length $l(q,\tau)$ properties, which will be used in the following to prove monotonicity of reduced volume and non-collapsing results:

Proposition 4 ([2]). *One has*

$$l_{\overline{\tau}} - \Delta l + |\nabla l|^2 - R + \frac{n}{2\overline{\tau}} \geq 0$$
$$2\Delta l - |\nabla l|^2 + R + \frac{l-n}{\overline{\tau}} \leq 0$$
$$\min_{\overline{\tau}} l(\cdot,\overline{\tau}) \leq \frac{n}{2}$$
$$\frac{d}{d\tau}|_{\tau=\overline{\tau}}|\tilde{Y}|^2 \leq \frac{1}{\overline{\tau}} - \frac{1}{\sqrt{\overline{\tau}}}\int_0^{\overline{\tau}} \sqrt{\tau}H(X,\tilde{Y})d\tau,$$

where $\tilde{Y}$ *is any* $\mathcal{L}$*- Jacobi field along* $\gamma(\tau)$.

References

[1] Bruce Kleiner, John Lott. Notes on Pereman's papers. http://arxiv.org/abs/math.DG/0605667

[2] Grisha Perelman. The entropy formula for the Ricci flow and its geometric application. arxiv.org/abs/math.DG/0211159

[3] John Morgan, Gang Tian. Ricci flow and the Poincaré conjecture. arxiv.org/abs/math.DG/0607607

[4] Richard Hamilton. The Harnack estimate for the Ricci flow. J. Differential Geom. 41 (1995),215-226.