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Abstract

For an arbitrary real number with convergents $p_0/q_0, p_1/q_1, p_2/q_2, \dots, p_{n+q_i}/q_{n+q_i}$ is equal to p_i , and so is independent of n , except at a small specified number of values of n . For fixed n , this relation holds for all or for all except a finite number of values of i .

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Self Matching in $\lfloor n\alpha \rfloor$

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Abstract

For an arbitrary real number α with convergents $\frac{p_0}{q_0}, \frac{p_1}{q_1}, \frac{p_2}{q_2}, \dots$, $\lfloor (n+q_i)\alpha \rfloor - \lfloor n\alpha \rfloor$ is equal to p_i , and so is independent of n , except at a small specified number of values of n . For fixed n , this relation holds for all or for all except a finite number of values of i .

1 Introduction

Bunder and Tognetti noted in [3] that any section of the graph of $\lfloor n\tau \rfloor$ (where $\tau = \frac{1}{2}(\sqrt{5} - 1)$) is “matched” for larger values of n . More precisely they proved:

$$\lfloor (n + F_i)\tau \rfloor - \lfloor n\tau \rfloor = F_{i-1}$$

except at $n = kF_{i+1} + \lfloor k\tau \rfloor F_i$ where

$$\lfloor (n + F_i)\tau \rfloor - \lfloor n\tau \rfloor = F_{i-1} - (-1)^i.$$

In this paper we will generalise this result to:

$$\lfloor (n + q_i)\alpha \rfloor - \lfloor n\alpha \rfloor = p_i \quad (\text{possibly } -(-1)^i)$$

where α is an arbitrary positive irrational number and $\frac{p_0}{q_0}, \frac{p_1}{q_1}, \frac{p_2}{q_2}, \dots$ are the convergents of α - just as $\frac{F_0}{F_1}, \frac{F_1}{F_2}, \frac{F_2}{F_3}, \dots$ are the convergents of τ .

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2 Continued Fractions and Convergents

Definition 1 If α is any real number and $\alpha = [a_0, a_1, a_2, \dots]$ in continued fraction form, then the i th convergent of α is given by:

$$\frac{p_i}{q_i} = [a_0, a_1, a_2, \dots, a_i].$$

We quote the following properties of convergents from Khintchine [5]:

Lemma 1 If $\alpha = [a_0, a_1, a_2, \dots]$ then

(i) $p_{-1} = q_{-2} = 1$ and $p_{-2} = q_{-1} = 0$.

(ii) For $i \geq 0$:

(a) $p_i = a_i p_{i-1} + p_{i-2}$

(b) $q_i = a_i q_{i-1} + q_{i-2}$.

3 The Main Result

We will be using work of Fraenkel, Levitt and Shimshoni [4] (their result assumes $1 < \alpha < 2$, but holds for $\alpha \geq 0$). In particular they use a generalisation of the Zeckendorf expansion of an integer (used in [3]). This generalization, as pointed out in Allouche and Shallit [1], is in fact due to Ostrowski 1922 (see [6]).

Theorem 1 Given a positive irrational α with convergents $\frac{p_0}{q_0}, \frac{p_1}{q_1}, \frac{p_2}{q_2}, \dots$, every positive integer n can be represented uniquely by the Ostrowski α -numeration:

$$n = \sum_{i=h}^m k_i q_i$$

where $k_h \neq 0$, $m \geq h \geq 0$ and the k_i satisfy the following conditions:

(i) For each i , $0 \leq k_i \leq a_{i+1}$.

(ii) If $i > 0$ and $k_i = a_{i+1}$, $k_{i-1} = 0$.

(iii) If $h = 0$, $k_h < a_1$.

Note In the remainder of the paper, every such representation of an integer will be assumed to be an Ostrowski α -numeration.

Theorem 2 Given a positive irrational α with convergents $\frac{p_0}{q_0}, \frac{p_1}{q_1}, \frac{p_2}{q_2}, \dots$, if

$$n = \sum_{i=h}^m k_i q_i$$

then

$$\lfloor n\alpha \rfloor = \sum_{i=h}^m k_i p_i + (-1 \text{ if } h \text{ is odd}).$$

Proof Let $\alpha = \alpha' + r$ where $1 < \alpha' < 2$ and r is an integer ≥ -1 . The convergents for α' are $\frac{p'_0}{q_0}, \frac{p'_1}{q_1}, \frac{p'_2}{q_2}, \dots$ where $p'_i + r q_i = p_i$.

Theorem 1 gives us the unique numeration for n , which is the same for α and for α' , and by Fraenkel, Levitt and Shimshoni [4]:

$$\lfloor n\alpha' \rfloor = \sum_{i=h}^m k_i p'_i + (-1 \text{ if } h \text{ is odd}).$$

So,

$$\begin{aligned} \lfloor n\alpha \rfloor &= nr + \lfloor n\alpha' \rfloor \\ &= \sum_{i=h}^m r k_i q_i + \sum_{i=h}^m k_i (p_i - r q_i) + (-1 \text{ if } h \text{ is odd}) \\ &= \sum_{i=h}^m k_i p_i + (-1 \text{ if } h \text{ is odd}). \end{aligned}$$

We can now prove our main result.

Theorem 3 If α is a positive irrational with convergents $\frac{p_0}{q_0}, \frac{p_1}{q_1}, \frac{p_2}{q_2}, \dots$ and $n = \sum_{i=h}^m k_i q_i$, then

$$\lfloor (n + q_j)\alpha \rfloor - \lfloor n\alpha \rfloor = \begin{cases} p_j & \text{if } h \leq j \text{ or } j + h \text{ is even,} \\ p_j + (-1)^j & \text{if } j < h \text{ and } j + h \text{ is odd.} \end{cases}$$

Proof Let $n = \sum_{i=h}^m k_i q_i$. In each of the cases below we find the corresponding Ostrowski α -enumeration of $n + q_j$ and then use Theorem 2.

Case 1 If $j < h - 1$ or $j = h - 1$ and $k_h < a_{h+1}$ then

$$n + q_j = q_j + \sum_{i=h}^m k_i q_i.$$

Then $\lfloor (n + q_j)\alpha \rfloor - \lfloor n\alpha \rfloor = p_j + (-1 \text{ if } j \text{ is odd}) + (1 \text{ if } h \text{ is odd})$, which gives the result.

Case 2 If $j = h - 1$ and $k_h = a_{h+1}$,

$$n = \sum_{k=0}^r a_{h+2k+1} q_{h+2k} + \sum_{i=h+2r+1}^m k_i q_i$$

where $r \geq 0$, $k_{h+2r+1} < a_{h+2r+2}$, and $k_{h+2r+2} < a_{h+2r+3}$ or $m = h + 2r$. Then

$$n + q_j = (k_{h+2r+1} + 1) q_{h+2r+1} + \sum_{i=h+2r+2}^m k_i q_i.$$

Then $j + h$ is odd and:

$$\begin{aligned} \lfloor (n + q_j)\alpha \rfloor - \lfloor n\alpha \rfloor &= p_{h+2r+1} + (-1 \text{ if } h \text{ is even}) - \sum_{k=0}^r a_{h+2k+1} p_{h+2k} + (1 \text{ if } h \text{ is odd}) \\ &= p_j + (-1)^j. \end{aligned}$$

Case 3 If $m \geq j \geq h$ and either $0 < k_j < a_{j+1} - 1$, $0 < k_j = a_{j+1} - 1$ and $k_{j-1} = 0$, or $k_j = 0$ and $k_{j+1} < a_{j+2}$, then,

$$n + q_j = \sum_{i=h}^{j-1} k_i q_i + (k_j + 1)q_j + \sum_{i=j+1}^m k_i q_i$$

and

$$\lfloor (n + q_j)\alpha \rfloor - \lfloor n\alpha \rfloor = p_j.$$

Case 4 If $m \geq j \geq h$, $k_j = a_{j+1} - 1$, $k_{j+1} < a_{j+2}$ and $k_{j-1} > 0$ then,

$$n + q_j = \sum_{i=h}^{j-2} k_i q_i + (k_{j-1} - 1)q_{j-1} + (k_{j+1} + 1)q_{j+1} + \sum_{i=j+2}^m k_i q_i$$

and

$$\lfloor (n + q_j)\alpha \rfloor - \lfloor n\alpha \rfloor = -p_{j-1} + p_{j+1} - k_j p_j = p_j.$$

Case 5 If $m \geq j = h = 0$ and $k_0 = a_1 - 1$,

$$n = (a_1 - 1)q_0 + \sum_{k=1}^r a_{2k+1} q_{2k} + \sum_{i=2r+1}^m k_i q_i$$

where $k_{2r+1} < a_{2r+2}$ and $r \geq 0$ (as $h = 0, k_0 > 1$). Then,

$$n + q_j = (k_{2r+1} + 1)q_{2r+1} + \sum_{i=2r+2}^m k_i q_i$$

and

$$\lfloor (n + q_j)\alpha \rfloor - \lfloor n\alpha \rfloor = p_{2r+1} - 1 - (a_1 - 1)p_0 - \sum_{k=1}^r a_{2k+1} p_{2k} = p_1 - 1 - a_1 p_0 + p_0 = p_0$$

Case 6 If $m \geq j \geq h$, $k_j = a_{j+1} - 1$ and $k_{j+1} = a_{j+2}$ then $a_{j+1} = 1$ and

$$n = \sum_{i=h}^{j-1} k_i q_i + \sum_{k=0}^r a_{j+2k+2} q_{j+2k+1} + \sum_{i=j+2r+2}^m k_i q_i$$

where $r \geq 0$ and $k_{j+2r+2} < a_{j+2r+3}$ or $j + 2r + 1 = m$, then,

$$n + q_j = \sum_{i=h}^{j-1} k_i q_i + (k_{j+2r+2} + 1)q_{j+2r+2} + \sum_{i=j+2r+3}^m k_i q_i,$$

and

$$[(n + q_j)\alpha] - [n\alpha] = p_{j+2r+2} - \sum_{k=0}^r a_{j+2k+2} p_{j+2k+1} = p_j.$$

Case 7 If $m \geq j \geq h$ and $k_j = a_{j+1}$ then $j > 0$ and $k_{j-1} = 0$ or $j = h$ and

$$n = \sum_{i=h}^{j-2r-2} k_i q_i + \sum_{k=0}^{r+s} a_{j+2k-2r+1} q_{j+2k-2r} + \sum_{i=j+2s+1}^m k_i q_i$$

where $r, s \geq 0$, $h \leq j - 2r - 2$ or $h = j - 2r$, and $j + 2s \leq m$, there are several cases. If $m = j + 2s$, the last summation is zero. If $m > j + 2s$, $k_{j+2s+1} < a_{j+2s+2}$. If $j - 2r - 2 < h$ the first summation is zero. If $j - 2r - 2 \geq h$, $k_{j-2r-2} < a_{j+2r-1}$. If $j - 2r = 0$, as $q_{-1} = 0$, the second summation sums to q_{j+2s+1} , i.e. $h = j + 2s + 1$, which is impossible. So $j - 2r > 0$. If $j - 2r > 1$, $k_{j-2r-2} + 1 < a_{j-2r-1}$ or $k_{j-2r-2} + 1 = a_{j-2r-1}$, $j - 2r - 2 > 0$ and either $k_{j-2r-3} = 0$ or $h = j - 2r - 2$,

$$n + q_j = \sum_{i=h}^{j-2r-3} k_i q_i + (k_{j-2r-2} + 1)q_{j-2r-2} + (a_{j-2r} - 1)q_{j-2r-1} + \sum_{k=1}^r a_{j-2r+2k} q_{j-2r+2k-1} + (k_{j+2s+1} + 1)q_{j+2s+1} + \sum_{i=j+2s+2}^m k_i q_i.$$

If $h = j - 2r > 1$, we have the same but with zero for the first summation and k_{j-2r-2} .

Then in these cases:

$$[(n + q_j)\alpha] - [n\alpha] = p_{j-2r-2} + (a_{j-2r} - 1)p_{j-2r-1} + \sum_{k=1}^r a_{j-2r+2k} p_{j-2r+2k-1} + p_{j+2s+1} - \sum_{k=0}^{r+s} a_{j+2k-2r+1} p_{j+2k-2r} = p_j.$$

If $j - 2r > 1$, $k_{j-2r-2} + 1 = a_{j-2r-1}$, $h \leq j - 2r - 3$ and $k_{j-2r-3} > 0$,

$$n + q_j = \sum_{i=h}^{j-2r-4} k_i q_i + (k_{j-2r-3} + 1)q_{j-2r-3} + \sum_{k=0}^r a_{j-2r+2k} q_{j-2r+2k-1} + (k_{j+2s+1} + 1)q_{j+2s+1} + \sum_{i=j+2s+2}^m k_i q_i.$$

If $h = j - 2r - 2 = 0$ and $k_0 + 1 = a_1$, the expansion of $n + q_j$ is the same but with zero for the first sum and for the q_{j-2r-3} term.

In these cases:

$$[(n + q_j)\alpha] - [n\alpha] = -p_{j-2r-3} + \sum_{k=0}^r a_{j-2r+2k} p_{j-2r+2k-1} + p_{j+2s+1} - \sum_{k=0}^{r+s} a_{j+2k-2r+1} p_{j+2k-2r} - k_{j-2r-2} p_{j-2r-2} = p_j.$$

If $j - 2r = 1$, h is 1,

$$n + q_j = (a_1 - 1)q_0 + \sum_{k=1}^r a_{2k+1} q_{2k} + (k_{2r+2s+2} + 1)q_{2r+2s+2} + \sum_{i=2r+2s+3}^m k_i q_i$$

and

$$\lfloor (n + q_j)\alpha \rfloor - \lfloor n\alpha \rfloor = (a_1 - 1)p_0 + \sum_{k=1}^r a_{2k+1}p_{2k} + p_{2r+2s+2} - \sum_{k=0}^{r+s} a_{2k+2}p_{2k+1} + 1 = p_j.$$

Case 8 If $j > m + 1$ or $j = m + 1$ then $n + q_j = \sum_{i=h}^m k_i q_i + q_j$ and $\lfloor (n + q_j)\alpha \rfloor - \lfloor n\alpha \rfloor = p_j$.

Case 9 If $j = m + 1$ and $a_{m+2} = 1$ then $n + q_j = \sum_{i=h}^{m-1} k_i q_i + (k_m - 1)q_m + q_{m+2}$ and $\lfloor (n + q_j)\alpha \rfloor - \lfloor n\alpha \rfloor = -p_m + p_{m+2} = p_j$.

4 The Mismatch Points

It follows that the values of n (called j -mismatch points in [3]) where $\lfloor (n + q_j)\alpha \rfloor - \lfloor n\alpha \rfloor \neq p_j$ are those with $n = \sum_{i=j+2r+1}^m k_i q_i$, where $r \geq 0$.

If $n = \sum_{i=h}^m k_i q_i$ is fixed, $\lfloor (n + q_j)\alpha \rfloor - \lfloor n\alpha \rfloor \neq p_j$ only when $j = h - 1, h - 3, \dots, h - 2\lfloor \frac{h-1}{2} \rfloor - 1$.

If $h = 0$, $\lfloor (n + q_j)\alpha \rfloor - \lfloor n\alpha \rfloor = p_j$ for all j .

5 A Special Case

In the special case where a_i is a constant i.e., $\alpha = [a, a, a, \dots] = \frac{1}{2}(a + (a^2 + 4)^{1/2})$, it is easy to show from Lemma 1 that $p_i = q_{i+1}$.

We now show that the numbers n where

$$\lfloor (n + q_j)\alpha \rfloor - \lfloor n\alpha \rfloor = p_j + (-1)^j$$

(the j -mismatch points) are exactly the numbers of the form

$$n = kq_{j+1} + \lfloor k\alpha \rfloor q_j.$$

First we need a lemma:

Lemma 2 If $\alpha = [a, a, a, \dots]$ and $t, i \geq 0$ then $q_i q_t + q_{i+1} q_{t+1} = q_{i+t+2}$.

Proof $q_i q_t + q_{i+1} q_{t+1} = q_i q_t + (aq_i + q_{i-1})q_{t+1}$
 $= q_{i-1} q_{t+1} + q_i (aq_{t+1} + q_t)$
 $= q_{i-1} q_{t+1} + q_i q_{t+2}$

$$\begin{aligned}
&= q_{i-2}q_{t+2} + q_{i-1}q_{t+3} \\
&= \dots \\
&= q_{-1}q_{t+i+1} + q_0q_{t+i+2} \\
&= q_{i+t+2} \text{ as } q_{-1} = 0 \text{ and } q_0 = aq_{-1} + q_{-2} = 1.
\end{aligned}$$

Theorem 4 Given $\alpha = [a, a, a, a \dots] = \frac{1}{2} (a + (a^2 + 4)^{1/2})$,

(a) If n is not of the form $kq_{j-1} + [k\alpha]q_j$, then $[(n + q_j)\alpha] - [n\alpha] = p_j$.

(b) If n is of the form $kq_{j-1} + [k\alpha]q_j$, then $[(n + q_j)\alpha] - [n\alpha] = p_j + (-1)^j$.

Proof (a) If $n = \sum_{i=h}^m k_i q_i$ and $[(n + q_j)\alpha] - [n\alpha] \neq p_j$, then $j < h$ and $j + h$ (and so $h - j$) is odd by Theorem 3.

Let $k = \sum_{i=h}^m k_i q_{i-j-1}$, then by Theorem 3, using $p_{i-j-1} = q_{i-j}$ and Lemma 2,

$$\begin{aligned}
kq_{j-1} + [k\alpha]q_j &= \sum_{i=h}^m k_i (q_{j-1}q_{i-j-1} + q_jq_{i-j}) \\
&= \sum_{i=h}^m k_i q_i = n.
\end{aligned}$$

Hence if n is not of the form $kq_{j-1} + [k\alpha]q_j$ then $[(n + q_j)\alpha] - [n\alpha] = p_j$.

(b) Let $k = \sum_{i=h_1}^m k_i q_i$, then, as above, if $n = kq_{j-1} + [k\alpha]q_j$

$$n = \sum_{i=h_1}^m k_i q_{i+j+1} + (-q_j \text{ if } h_1 \text{ is odd}).$$

If h_1 is even we have $j < h_1 + j + 1 = h$ (for n) and $j + h$ is odd.

If h_1 is odd $q_{h_1+j+1} - q_j = a \sum_{r=j+1}^{h_1+j} q_r$ so h (for n) = $j + 1 > j$ and $j + h$ is odd.

So in either case, by Theorem 3:

$$[(n + q_j)\alpha] - [n\alpha] = p_j + (-1)^j.$$

The j -mismatch points for $\alpha = [b, a, a, a, \dots]$ can be shown to be $kq_{j-1} + [k(\alpha + a - b)]q_j$, but the result does not generalize, in an obvious way, to as representable as other repeated continued fractions.

6 An Alternative to Theorem 2

The $0 < \alpha < 1$, and so $p_0 = 0$, case of the following alternative to Theorem 2 appears in Brown [2] and in Allouche and Shallit [1]. It follows easily from our Theorems 2 and 3.

Theorem 5 If α is a positive irrational number with convergents $\frac{p_0}{q_0}, \frac{p_1}{q_1}, \frac{p_2}{q_2}, \dots$ and n has Ostrowski α -numeration $\sum_{i=h}^m k_i q_i$, then $\lfloor (n+1)\alpha \rfloor = \sum_{i=h}^m k_i p_i + p_0$.

Proof By Theorems 3 and 2, as $q_0 = 1$:

$$\begin{aligned} \lfloor (n+1)\alpha \rfloor &= \lfloor n\alpha \rfloor + \begin{cases} p_0 & \text{if } h \text{ is even,} \\ p_0 + 1 & \text{if } h \text{ is odd.} \end{cases} \\ &= \sum_{i=h}^m k_i p_i + p_0 \end{aligned}$$

The results in Theorems 2 and 5 look quite different, however we could have used (the $0 < \alpha < 1$ case of) the latter, instead of Theorem 2, to prove Theorem 3 in a similar way to the above.

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