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The strong relevance logics

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The tautology p - q - p is not a theorem of the various relevance logics (see Anderson and Belnap [1]) because q is not considered to be relevant in the derivation of final p. We can take this lack of relevance to mean simply that p-q-p could have been proved without q and its -, i.e., p-p. By the same criterion we could say that in ((p-p) -q) -q p-p is not relevant. In general we will say that any theorem A of an implicational logic is strongly relevant if there is no subpart B ! which can be removed from A, leaving the rest still a theorem of the same logic. Such a subpart B - is said to be superfluous.

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THE STRONG RELEVANCE LOGICS

Introduction

The tautology

$$p \to q \to p$$

is not a theorem of the various relevance logics (see Anderson and Belnap [1]) because q is not considered to be relevant in the derivation of final p. We can take this lack of relevance to mean simply that $p \to q \to p$ could have been proved without q and its $\to$, i.e., $p \to p$.

By the same criterion we could say that in

$$((p \to p) \to q) \to q$$

$p \to p$ is not relevant.

In general we will say that any theorem A of an implicational logic is **strongly relevant** if there is no subpart $B \to$ which can be removed from A, leaving the rest still a theorem of the same logic. Such a subpart $B \to$ is said to be **superfluous**.

The strongly relevant form of a logic

If L is an implicational logic, the theorems of the **strongly relevant** form $SR(L)$ of L are obtained from the theorems of L by reducing them to strongly relevant theorems by means of the algorithm given below.

The algorithm requires the notion of **depth**. A wfA is said to have **depth** 0 in A.

If $B = B_1 \to \ldots \to B_m \to p$ has **depth** d in A any B_i has depth $d+1$ in A.

The relevance algorithm

To change a theorem A of a logic L to its strongly relevant form, $SR(A)$, in the logic $SR(L)$, proceed in the following way for $d = 1, 2, \ldots$

Remove all superfluous $B \to s$ of depth d from A from the left. Then remove any superfluous $B \to s$ of levels less than $d+1$ from the reduced A, starting from depth 1.

Here are some examples from Classical Logic.

1. In $(p \to q \to r) \to (p \to q) \to p \to r$ there are no superfluous subparts of depth 1 and the only one of depth 2 is $q \to$. The removal leaves

$$(p \to r) \to (p \to q) \to p \to r.$$

Now $(p \to q) \to$ of depth 1 is superfluous. Removing this yields:

$$(p \to r) \to p \to r.$$

Now the first $p \to$ (depth 2) is superfluous and when removed gives

$$r \to p \to r,$$

which then is reduced to

$$r \to r.$$

2. In $((p \to q) \to p) \to p$ the $(p \to q) \to$ of depth 2 is all that can be removed yielding

$$p \to p.$$

Strongly relevant forms of logics

We will name logics by the combinators associated with their axioms:

I	$\vdash p \to p$
B	$\vdash (p \to q) \to (r \to p) \to r \to q$
B′	$\vdash (p \to q) \to (q \to r) \to p \to r$
C	$\vdash (p \to q \to r) \to q \to p \to r$
S	$\vdash (p \to q \to r) \to (p \to q) \to p \to r$

W $\vdash (p \to p \to q) \to p \to q$
K $\vdash p \to q \to p$

In general, relevance logics are those without **K**. First we need two lemmas

LEMMA 1. If **K, B, B′** and **I** hold in L and
(i) Q is in a positive position in $A(Q)$ then

$$\vdash_L A(Q) \to A(P \to Q);$$

(ii) Q is in a negative position in $A(Q)$ then

$$\vdash_L A(P \to Q) \to A(Q).$$

PROOF. We prove (i) and (ii), where only one instance of Q or $P \to Q$ is being replaced, by induction on the depth d of Q in $A(Q)$.

If $d = 0$ $A(Q) = Q$ and $\vdash_L Q \to (P \to Q)$

If $d > 0$ and $A(Q) = A_1 \to \ldots \to A_i(Q) \to B$, there are 2 cases:

If d is odd, Q is in a negative position in $A(Q)$ and in a positive position in $A_i(Q)$, so by the induction hypothesis

$$\vdash A_i(Q) \to A_i(P \to Q).$$

By **B′** $\vdash (A_i(P \to Q) \to B) \to A_i(Q) \to B$
and by **B** applied i − 1 times we get

$$\vdash A(P \to Q) \to A(Q).$$

If d is odd, Q is in a positive position in $A(Q)$ and in a negative position in $A_i(Q)$, so by the induction hypothesis

$$\vdash A_i(P \to Q) \to A_i(Q)$$

and by **B** and **B′** we obtain as above:

$$\vdash A(Q) \to A(P \to Q).$$

Multiple copies of Q and $P \to Q$ can be replaced in $A(Q)$ and $A(P \to Q)$ by repeating this procedure.

LEMMA 2. If **K, B, B′** and **I** hold in L, then the Relevance Algorithm will reduce any theorem A of L that is not $p \to p$.

PROOF. If A has a negative part of the form $P \to Q$, write $A = B(P \to Q)$, then by Lemma 1 (ii) $P \to$ is superfluous in A.

As A has a superfluous part, the Relevance Algorithm will reduce it (though not necessarily that part first, or even at all).

If A has no negative part of the form $P \to Q$, it must be of the form:

$$A = p_i \to p_2 \ldots \to p_n$$

where at least one $p_i = p_n$.

Unless $n = 1$, the Relevance Algorithm reduces this A to $p_n \to p_n$

THEOREM 1. If **K, B, B′** and **I** hold in L then

$$SR(L) = \{p \to p | \ p \text{ is a variable}\}.$$

PROOF. By Lemma 2, the Relevance Algorithm will reduce the length of any theorem that is not $p \to p$. Thus the algorithm will reduce any theorem to $p \to p$.

It can probably easily be shown that if **K, B, B′** (but not **I**) hold in L, then

$$SR(L) = \{p \to q \to p | \ p, q \text{ are variables}\}.$$

The same holds if L has **K** and **B** or **K** and **B′**, but not **I** nor even **K**.

THEOREM 2. $SR(\mathbf{KI}) = \{p \to p | \ p \text{ is a variable}\}$.

PROOF. It is easy to show that every theorem T of **KI**-logic is of the form

$$T_1 = B_1 \to \ldots \to B_n \to A \to A$$

or

$$T_2 = B_1 \to \ldots \to B_n \to A \to B \to A.$$

We can also assume that A in T_1, and A and $B \to A$ in T_2 are not theorems of **KI** logic, since in that case we would have:

$B \to A$ or $A = C_1 \to \ldots \to C_k \to A_1 \to A_1$
or A or $B \to A = C_1 \to \ldots \to C_k \to A_1 \to B_1 \to A$,
so that $T_1 = B_1 \to \ldots \to B_n \to A \to C_1 \to \ldots \to C_k \to A_1 \to A_1$,
$T_2 = B_1 \to \ldots \to B_n \to A \to B \to C_1 \to \ldots \to C_k \to A_1 \to A_1$,
$T_1 = B_1 \to \ldots \to B_n \to A \to C_1 \to \ldots \to C_k \to A_1 \to B_1 \to A_1$
$T_2 = B_1 \to \ldots \to B_n \to A \to B \to C_1 \to \ldots \to C_k \to A_1 \to B_1 \to A_1$
$T_2 = B_1 \to \ldots \to B_n \to A \to C_1 \to \ldots \to C_k \to A_1 \to A_1$
or $T_2 = B_1 \to \ldots \to B_n \to A \to C_1 \to \ldots \to C_k \to A_1 \to B_1 \to A_1$
which are in the above forms but with A_1 smaller than A.

Now

$$\begin{aligned} SR(T_2) &= SR(A \to B \to A) \\ &= SR(A \to A) \\ SR(T_1) &= SR(A \to A). \end{aligned}$$

Let

$$A = A_1 \to A_2,$$

$$
\begin{aligned}
\text{then} \qquad SR(A \to A) &= SR((A_1 \to A_2) \to A_1 \to A_2) \\
&= SR((A_1 \to A_2) \to A_2) \\
&= SR(A_2 \to A_2) \\
\text{or} \qquad SR(A \to A) &= A_2 \to (A_1 \to A_2) \\
&= SR(A_2 \to A_2).
\end{aligned}
$$

We can continue this reduction till we get $p \to p$ for some variable p.

The same result probably holds for **KBI** and **KB′I**.

For logics without **K** the situation is much more complex as is shown below:

LEMMA 4.

(i) $SR(\mathbf{BB'IW}) \not\subseteq SR(\mathbf{BCI}) \cup SR(\mathbf{BCIW}) \cup SR(\mathbf{BB'I})$;

(ii) $SR(\mathbf{BB'I}) \cap SR(\mathbf{BCI}) \not\subseteq SR(\mathbf{BCIW}) \cup SR(\mathbf{BB'IW})$;

(iii) $SR(\mathbf{BCIW}) \cap SR(\mathbf{BCI}) \not\subseteq SR(\mathbf{BB'IW}) \cup SR(\mathbf{BB'I})$;

(iv) $SR(\mathbf{BCIW}) \not\subseteq SR(\mathbf{BCI})$;

(v) $SR(\mathbf{BB'I}) \not\subseteq SR(\mathbf{BCI})$.

PROOF.

$$((p \to q) \to p) \to (p \to q) \to (p \to q) \to q$$

is a theorem of $SR(\mathbf{BB'I})$ and $SR(\mathbf{BCI})$ but not of $SR(\mathbf{BCIW})$ nor $SR(\mathbf{BB'IW})$ wherein it is reduced to

$$((p \to q) \to p) \to (p \to q) \to q.$$

Hence (ii) holds.

The last formula above is a theorem of $SR(\mathbf{BB'IW})$ but not of $SR(\mathbf{BCIW})$ where it is reduced to

$$p \to (p \to q) \to q.$$

Neither is it a theorem of $SR(\mathbf{BCI})$ or $SR(\mathbf{BB'I})$. Hence (i) holds.

This last formula above is a theorem of $SR(\mathbf{BCIW})$ and $SR(\mathbf{BCI})$, but not of $SR(\mathbf{BB'IW})$ or $SR(\mathbf{BB'I})$, so (iii) holds.

$$(p \to (p \to (p \to q))) \to p \to q$$

is a theorem of $SR(\mathbf{BCIW})$ but not of $SR(\mathbf{BCI})$, so (iv) holds.

$$(p \to r \to q) \to ((p \to q) \to r) \to p \to (p \to q) \to q.$$

is a theorem of $SR(\mathbf{BB'I})$ but not of $SR(\mathbf{BCIW})$.

THEOREM 3. The systems $SR(\mathbf{BCIW})$, $SR(\mathbf{BB'IW})$, $SR(\mathbf{BCI})$, and $SR(\mathbf{BB'I})$ are mutually independent.

PROOF. By Lemma 4.

We should note that the relevance requirements here, although similar, are stronger than those in [2] where effectively only superfluous subparts of depth 1 have been removed.

The work can be extended to logics with the connectives $\wedge$ and $\vee$ where parts $\wedge B, B\wedge, B\vee$ and $\vee B$ can be superfluous.

Again all theorems of positive classical, intuitionistic and **BCK** logics reduce to the form $p \to p$. For relevance logics, as before, the situation is more complex.

References

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