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## Summability for nonunital spectral triples

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### Abstract

This paper examines the issue of summability for spectral triples for the class of nonunital algebras. For the case of  $(p, -)$  summability, we prove that the Dixmier trace can be used to define a (semifinite) trace on the algebra of the spectral triple. We show this trace is well-behaved, and provide a criteria for measurability of an operator in terms of zeta functions. We also show that all our hypotheses are satisfied by spectral triples arising from eodesically complete Riemannian manifolds. In addition, we indicate how the Local Index Theorem of Connes-Moscovici extends to our nonunital setting.

### Keywords

triples, spectral, nonunital, summability

### Disciplines

Engineering | Science and Technology Studies

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# Summability for Nonunital Spectral Triples

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**Abstract.** This paper examines the issue of summability for spectral triples for the class of nonunital algebras introduced in [23]. For the case of  $(p, \infty)$ -summability, we prove that the Dixmier trace can be used to define a (semifinite) trace on the algebra of the spectral triple. We show this trace is well-behaved, and provide a criteria for measurability of an operator in terms of zeta functions. We also show that all our hypotheses are satisfied by spectral triples arising from geodesically complete Riemannian manifolds. In addition, we indicate how the Local Index Theorem of Connes-Moscovici extends to our nonunital setting.

**AMS Classifications:** 46L80, 46L89.

**Key words:** Dixmier trace, index theorem, noncommutable geometry, spectral triples.

## 1. Introduction

This paper examines the formulation of summability hypotheses for spectral triples over nonunital algebras. Similar ideas have been discussed in [14] and [10]. We focus for the most part on the technical aspects of the  $(p, \infty)$ -summability hypothesis.

In Section 2 we summarise the definitions and results of [23] relevant for this paper. Section 3 reviews the Dixmier trace and its relation to the Wodzicki residue [26], for manifolds in the unital case.

Section 4 extends all these ideas to the nonunital setting, for our local algebras. In particular, we define  $(p, \infty)$ -summability for local spectral triples, and derive the nonunital analogues of the main results in the unital case. These include criteria for measurability in terms of zeta functions, Theorem 12. Section 5 shows that the definition is satisfied for spectral triples arising from (geodesically complete) noncompact manifolds.

In Section 6 we show that our definition also allows an analogue of the theory of distributions for local spectral triples. Various such distributions are defined using the Dixmier trace, the usual trace on Hilbert space, and Connes' pseudodifferential operators [6]. Finally, we look briefly at the Local Index Theorem. The Local Index Theorem [7], remains true for local spectral triples with discrete and finite dimension spectrum, see [7], provided we regard the components of the Chern character as continuous multilinear functionals on elements with 'compact support', giving the result a more distributional flavour.

We do not prove the Local Index Theorem here, referring the reader to [7] and the references therein. Instead we show that interpreted in the above distributional manner, the proof holds with trivial modifications. We present an example in [24] where the index pairing can be computed using the Local Index Theorem for a noncompact space.

Quite recently a new family of examples of nonunital spectral triples was presented in [10]. These were constructed from the Moyal product on smooth rapidly decreasing functions on  $\mathbf{R}^{2N}$ . The main feature of these examples which is relevant to this work is that the algebra  $\mathcal{S}(\mathbf{R}^{2N})$  with the Moyal product does not seem to have a dense ideal with local units. It is however ‘quasilocal’, in that it has a dense subalgebra with local units. Upon reflection, most of the results presented here and in [23] continue to hold for quasilocal algebras. Some results relating to the holomorphic and  $C^\infty$  functional calculus in [23] require modification, as does some of the results relating to distributions in this paper. We will return to this subject in a future work.

## 2. Summary

In this section we will summarise the important definitions and results concerning the smooth algebras we employ and spectral triples over them. The appropriate algebras in the nonunital case are smooth local algebras, as described in the following three definitions.

**DEFINITION 1.** A  $*$ -algebra  $\mathcal{A}$  is smooth if it is Fréchet and  $*$ -isomorphic to a proper dense subalgebra  $i(\mathcal{A})$  of a separable  $C^*$ -algebra  $A$  which is stable under the holomorphic functional calculus.

We will always suppose that we can define the Fréchet topology of  $\mathcal{A}$  using a countable collection of submultiplicative seminorms which includes the  $C^*$ -norm of  $\overline{\mathcal{A}} = A$ , and note that the multiplication is jointly continuous [20, p 24]. By replacing any seminorm  $q$  by  $(1/2)(q(a) + q(a^*))$ , we may suppose that  $q(a) = q(a^*)$  for all  $a \in \mathcal{A}$ . Thus saying that  $\mathcal{A}$  is smooth means that  $\mathcal{A}$  is Fréchet and a pre- $C^*$ -algebra. Asking for  $i(\mathcal{A})$  to be a *proper* dense subalgebra of  $A$  immediately forces the Fréchet topology of  $\mathcal{A}$  to be finer than the  $C^*$ -topology of  $A$  (since Fréchet means locally convex, metrizable and complete). So convergence in the topology of  $\mathcal{A}$  implies convergence in norm.

**DEFINITION 2.** An algebra  $\mathcal{A}$  has local units if for every finite subset of elements  $\{a_i\}_{i=1}^n \subset \mathcal{A}$ , there exists  $\phi \in \mathcal{A}$  such that for each  $i$

$$\phi a_i = a_i \phi = a_i.$$

**DEFINITION 3.** Let  $\mathcal{A}$  be a Fréchet algebra and  $\mathcal{A}_c \subset \mathcal{A}$  be a dense ideal with local units. Then we call  $\mathcal{A}$  a local algebra (when  $\mathcal{A}_c$  is understood).

So a smooth local algebra has a dense ideal whose elements behave like functions with compact support. The basic properties of these local algebras are summarised in the following lemmas.

LEMMA 1. *If  $\mathcal{A}_c \subset \mathcal{A}$  is a local algebra, then  $\exists \{\phi_n\}_{n \geq 1} \subset \mathcal{A}_c$  such that*

- (1)  $\{\phi_n\}_{n \geq 1}$  is an approximate unit for  $\mathcal{A}$ , with  $\phi_n a \rightarrow a$  in the Fréchet topology of  $\mathcal{A}$ .
- (2)  $\forall a \in \mathcal{A}_c \exists i$  such that  $\forall n \geq i \phi_n a = a \phi_n = a$ .
- (3) For all  $i < n$ ,  $\phi_n \phi_i = \phi_i \phi_n = \phi_i$ .
- (4) For all  $n$ ,  $\phi_n = \phi_n^*$ ,  $0 \leq \phi_n \leq 1$ .
- (5)  $\mathcal{A}_c = \bigcup_n \mathcal{A}_n$ , where  $\mathcal{A}_n = \{a \in \mathcal{A} : \phi_n a = a \phi_n = a\}$ .

We call such an approximate unit a local approximate unit.

COROLLARY 2. *Suppose that  $\mathcal{A}_c \subseteq \mathcal{A}$  is a local algebra,  $\{\phi_n\}$  is a local approximate unit and*

$$\mathcal{A}_n = \{a \in \mathcal{A} : \phi_n a = a \phi_n = a\}.$$

*Then each  $\mathcal{A}_n$  is a Fréchet algebra in the topology induced by  $\mathcal{A}$ , and the algebra  $\mathcal{A}_c = \bigcup_n \mathcal{A}_n$  is complete in the inductive limit topology defined by the inclusion maps  $\mathcal{A}_n \hookrightarrow \mathcal{A}$ .*

See [23] for more information on all these results.

DEFINITION 4. A spectral triple  $(\mathcal{A}, \mathcal{H}, \mathcal{D})$  is given by

- (1) A representation  $\pi : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H})$  of a local  $*$ -algebra  $\mathcal{A}$  on the Hilbert space  $\mathcal{H}$ .
- (2) A self-adjoint (unbounded, densely defined) operator  $\mathcal{D} : \text{dom } \mathcal{D} \rightarrow \mathcal{H}$  such that  $[\mathcal{D}, \pi(a)]$  extends to a bounded operator on  $\mathcal{H}$  for all  $a \in \mathcal{A}$  and  $\pi(a)(1 + \mathcal{D}^2)^{-1/2}$  is compact for all  $a \in \mathcal{A}$ .

The triple is said to be even if there is an operator  $\Gamma = \Gamma^*$  such that  $\Gamma^2 = 1$ ,  $[\Gamma, \pi(a)] = 0$  for all  $a \in \mathcal{A}$  and  $\Gamma \mathcal{D} + \mathcal{D} \Gamma = 0$  (i.e.  $\Gamma$  is a  $\mathbf{Z}_2$ -grading such that  $\mathcal{D}$  is odd and  $\pi(\mathcal{A})$  is even). Otherwise the triple is called odd.

*Remark.* We will systematically omit the representation  $\pi$  in future.

DEFINITION 5. If  $(\mathcal{A}, \mathcal{H}, \mathcal{D})$  is a spectral triple, then we define  $\Omega_{\mathcal{D}}^*(\mathcal{A})$  to be the algebra generated by  $\mathcal{A}$  and  $[\mathcal{D}, \mathcal{A}]$ .

DEFINITION 6. A spectral triple  $(\mathcal{A}, \mathcal{H}, \mathcal{D})$  is smooth (or regular) if

$$\mathcal{A} \text{ and } [\mathcal{D}, \mathcal{A}] \subseteq \bigcap_{m \geq 0} \text{dom } \delta^m$$

where for  $x \in \mathcal{B}(\mathcal{H})$ ,  $\delta(x) = [[\mathcal{D}, x]]$ .

The point of contact between smooth algebras and smooth spectral triples is the following Lemma, proved in [23].

LEMMA 3. *If  $(\mathcal{A}, \mathcal{H}, \mathcal{D})$  is a smooth spectral triple, then  $(\mathcal{A}_\delta, \mathcal{H}, \mathcal{D})$  is also a smooth spectral triple, where  $\mathcal{A}_\delta$  is the completion of  $\mathcal{A}$  in the locally convex topology determined by the seminorms*

$$q_{ni}(a) = \|\delta^n d^i(a)\|, \quad n \geq 0, \quad i = 0, 1,$$

where  $d(a) = [\mathcal{D}, a]$ . Moreover,  $\mathcal{A}_\delta$  is a smooth algebra.

We call the topology on  $\mathcal{A}$  determined by the seminorms  $q_{ni}$  of Lemma 3 the  $\delta$ -topology.

DEFINITION 7. A local spectral triple  $(\mathcal{A}, \mathcal{H}, \mathcal{D})$  is a spectral triple such that there exists a local approximate unit  $\{\phi_n\} \subset \mathcal{A}_c$  for  $\mathcal{A}$  satisfying

$$\Omega_{\mathcal{D}}^*(\mathcal{A}_c) = \bigcup_n \Omega_{\mathcal{D}}^*(\mathcal{A})_n, \quad \Omega_{\mathcal{D}}^*(\mathcal{A})_n = \{\omega \in \Omega_{\mathcal{D}}^*(\mathcal{A}) : \phi_n \omega = \omega \phi_n = \omega\}.$$

*Remark.* For a local spectral triple  $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ ,  $\Omega_{\mathcal{D}}^*(\mathcal{A}_c)$  is a dense ideal with local units inside  $\Omega_{\mathcal{D}}^*(\mathcal{A})$ , and so  $\Omega_{\mathcal{D}}^*(\mathcal{A})$  is a local algebra. In [23] the definition of a local spectral triple was taken to be a spectral triple such that  $\Omega_{\mathcal{D}}^*(\mathcal{A}_c)$  is a dense ideal with local units inside  $\Omega_{\mathcal{D}}^*(\mathcal{A})$ , and the existence of a local approximate unit  $\{\phi_n\} \subset \mathcal{A}_c$  for  $\Omega_{\mathcal{D}}^*(\mathcal{A}_c)$  was assumed to be true. This need not be the case, but the results in [23] using this earlier definition are true for the definition presented here.

*Remark.* A local spectral triple has a local approximate unit  $\{\phi_n\}_{n \geq 1} \subset \mathcal{A}_c$  such that  $\phi_{n+1}\phi_n = \phi_n\phi_{n+1} = \phi_n$  and  $\phi_{n+1}[\mathcal{D}, \phi_n] = [\mathcal{D}, \phi_n]\phi_{n+1} = [\mathcal{D}, \phi_n]$ . This is the crucial property we require to prove most of our results.

### 3. Summability for Unital Spectral Triples

This section describes some technical results which are known for the unital case, that will guide us in the nonunital case.

We begin by defining the noncommutative integral given by the Dixmier trace, and relating it to the Wodzicki residue [26]. For more detailed information on these results, see [5, IV.2.β], [13, Chapter 7] and [26]. To define the Dixmier trace and relate it to Lebesgue measure, we require the definitions of several normed ideals of compact operators on Hilbert space. The first of these is

$$\mathcal{L}^{(1, \infty)}(\mathcal{H}) = \left\{ T \in \mathcal{K}(\mathcal{H}) : \sum_{n=0}^N \mu_n(T) = O(\log N) \right\}$$

with norm

$$\|T\|_{1,\infty} = \sup_{N \geq 2} \frac{1}{\log N} \sum_{n=0}^N \mu_n(T).$$

In the above the  $\mu_n(T)$  are the eigenvalues of  $|T| = \sqrt{T^*T}$  arranged in decreasing order and repeated according to multiplicity so that  $\mu_0(T) \geq \mu_1(T) \geq \dots$ . This ideal will be the domain of definition of the Dixmier trace. Related to this ideal are the ideals  $\mathcal{L}^{(p,\infty)}(\mathcal{H})$  for  $1 < p < \infty$  defined as follows

$$\mathcal{L}^{(p,\infty)}(\mathcal{H}) = \left\{ T \in \mathcal{K}(\mathcal{H}) : \sum_{n=0}^N \mu_n(T) = O(N^{1-1/p}) \right\}$$

with norm

$$\|T\|_{p,\infty} = \sup_{N \geq 1} \frac{1}{N^{1-1/p}} \sum_{n=0}^N \mu_n(T).$$

We introduce these ideals because if  $T_i \in \mathcal{L}^{(p_i,\infty)}(\mathcal{H})$  for  $i = 1, \dots, n$  and  $\sum (1/p_i) = 1$ , then [5, p. 304], the product  $T_1, \dots, T_n \in \mathcal{L}^{(1,\infty)}(\mathcal{H})$ . In particular, if the operator  $T \in \mathcal{L}^{(p,\infty)}(\mathcal{H})$  then  $T^p \in \mathcal{L}^{(1,\infty)}(\mathcal{H})$ .

We want to define the Dixmier trace so that it returns the coefficient of the logarithmically divergent part of the trace of an operator. Unfortunately, since the sequence  $(1/\log N) \sum^N \mu_n(T)$  is in general only bounded, we can not take the limit in a well-defined way. The Dixmier trace is defined in terms of linear functionals  $\omega \in (L^\infty(\mathbf{N}))^*$  on bounded sequences satisfying certain additional properties [5, IV.2.β]. One of these properties is that if the above sequence is convergent, the linear functional returns the limit. For any such functional  $\omega$ , which we call an  $\omega$ -limit, one defines a functional on the positive elements of  $\mathcal{L}^{(1,\infty)}(\mathcal{H})$  by

$$\text{Tr}_\omega(T) = \omega\left(\frac{\sigma_N(T)}{\log N}\right) = \omega - \lim \frac{\sigma_N(T)}{\log N}, \quad T \geq 0,$$

where  $\sigma_N(T) = \sum^N \mu_n(T)$ . For  $T \in \mathcal{L}^{(1,\infty)}(\mathcal{H})$  with  $T \geq 0$ , we say that  $T$  is measurable if  $\text{Tr}_\omega(T)$  is independent of the choice of  $\omega$ -limit. We then write

$$\int T := \text{Tr}_\omega(T)$$

for any  $\omega$ .

It has been shown [5, Proposition 4, p. 306], that for positive  $T \in \mathcal{L}^{(1,\infty)}(\mathcal{H})$ , measurability is equivalent to the following. Denote by  $\zeta_T(s)$  the trace of  $T^s$  for  $s > 1$ . Then

**PROPOSITION 4.** *With  $T$  as above,  $T$  is measurable if and only if the limit*

$$\lim_{s \rightarrow 1^+} (s - 1)\zeta_T(s) = L < \infty, \tag{1}$$

*exists, and in this case,  $L = \int T$ .*

In the next section we obtain a nonunital analogue of Proposition 4 based on results of [2]. It is our strongest means of showing the measurability of operators. One can also show [5, IV.2.β], that for positive operators  $\text{Tr}_\omega$  is additive and positively homogenous, so we extend  $\text{Tr}_\omega$  by linearity to all of  $\mathcal{L}^{(1,\infty)}(\mathcal{H})$ . The space of measurable operators is a closed (in the  $(1, \infty)$  norm) linear space invariant under conjugation by invertible bounded operators and contains  $\mathcal{L}_0^{(1,\infty)}(\mathcal{H})$ , the closure of the finite rank operators in the  $(1, \infty)$  norm.

The following properties are satisfied by all Dixmier traces  $\text{Tr}_\omega$  [5, Proposition 3, p. 306]:

- (1) If  $T \geq 0$  then  $\text{Tr}_\omega(T) \geq 0$ ;
- (2) For all  $S \in \mathcal{B}(\mathcal{H})$  and  $T \in \mathcal{L}^{(1,\infty)}(\mathcal{H})$ , we have  $\text{Tr}_\omega(TS) = \text{Tr}_\omega(ST)$ ;
- (3)  $\text{Tr}_\omega$  vanishes on  $\mathcal{L}_0^{(1,\infty)}(\mathcal{H})$ .

In the unital case the definition of  $(p, \infty)$ -summability for spectral triples is as follows.

**DEFINITION 8.** A spectral triple  $(\mathcal{A}, \mathcal{H}, \mathcal{D})$  with  $\mathcal{A}$  unital is  $(p, \infty)$ -summable, if  $p \geq 1$ , and  $(1 + \mathcal{D}^2)^{-1/2} \in \mathcal{L}^{(p,\infty)}(\mathcal{H})$ .

*Remark.* The alternative definition  $(\mathcal{D} - \lambda)^{-1} \in \mathcal{L}^{(p,\infty)}(\mathcal{H})$ , for some (and hence all)  $\lambda$  in the resolvent set of  $\mathcal{D}$ , is equivalent to the above definition for  $p \geq 1$ . To see this observe that by definition  $(\mathcal{D} - \lambda)^{-1} \in \mathcal{L}^{(p,\infty)}(\mathcal{H})$  if and only if

$$\sqrt{(\mathcal{D} - \bar{\lambda})^{-1}(\mathcal{D} - \lambda)^{-1}} = (\mathcal{D}^2 + |\lambda|^2)^{-1/2} \in \mathcal{L}^{(p,\infty)}(\mathcal{H}).$$

Finally, one can use the resolvent formula to show that replacing  $|\lambda|^2$  by 1 is inessential.

Using the properties of  $\text{Tr}_\omega$  listed above, we obtain standard results like

- $(1 + \mathcal{D}^2)^{-p/2} \in \mathcal{L}^{(1,\infty)}(\mathcal{H})$ , and
- if  $a_i \rightarrow a \in \mathcal{A}$  then  $|\text{Tr}_\omega(a_i - a)(1 + \mathcal{D}^2)^{-p/2}| \leq \|a_i - a\| \|(1 + \mathcal{D}^2)^{-p/2}\|_{(1,\infty)} \rightarrow 0$  for any Dixmier trace  $\text{Tr}_\omega$ .

In general we can not deduce measurability without further information. Examples arising from geometric operators give rise to such additional information. We now discuss the relevant features of these examples.

Let  $P$  be a classical pseudodifferential operator acting on sections of a vector bundle  $E \rightarrow M$  over a compact Riemannian manifold  $(M, g)$  of dimension  $p$ . The operator  $P$  has a symbol  $\sigma(P): T^*M \rightarrow \text{End}(E)$ . If  $P$  is of order  $-p$ , the Wodzicki residue of  $P$  is defined by

$$\text{WRes}(P) = \frac{1}{p(2\pi)^p} \int_{S^*M} \text{Trace}_E \sigma_{-p}(P)(x, \xi) \sqrt{g} \, dx \, d\xi. \quad (2)$$

In the above  $S^*M$  is the cosphere bundle with respect to the metric  $g$ , and  $\sigma_{-p}(P)$  is the part of the symbol of  $P$  homogenous of order  $-p$  in the  $\xi$  variable. Although symbols other than principal symbols are coordinate dependent, the Wodzicki



residue depends only on the conformal class of the metric [6]. We have the following result from Connes [5, 6].

**THEOREM 5 (Connes' Trace Theorem).** *Let  $T$  be a pseudodifferential operator of order  $-p$  acting on sections of a smooth bundle  $E \rightarrow M$  on a  $p$  dimensional compact Riemannian manifold  $M$ . Then as an operator on the Hilbert space  $\mathcal{H} = L^2(M, E)$ ,  $T \in \mathcal{L}^{(1, \infty)}(\mathcal{H})$ ,  $T$  is measurable and  $\int T = \text{WRes}(T)$ .*

It can also be shown that the Wodzicki residue is the unique trace on pseudodifferential operators extending the Dixmier trace [5, 26]. Hence we can define  $\int T$  for any pseudodifferential operator on a manifold by setting  $\int T = \text{WRes}(T)$ , and using Equation (2) to define the Wodzicki residue whatever order the operator is. In particular, if  $T$  is of order strictly less than  $-p = -\dim M$ , then  $\int T = 0$ .

**EXAMPLE.** The principal example where this theory applies is the following. Suppose that  $\mathcal{D}$  is the Dirac operator on (complex) spinors on a compact  $n$ -dimensional Riemannian spin manifold  $X$  with metric  $g$  [21], and that  $f$  is a function on  $X$ . The complex spinor bundle  $S_{\mathbb{C}}$  has rank  $2^{\lfloor n/2 \rfloor}$ , and we denote by  $Id_{S_{\mathbb{C}}}$  the identity operator on  $S_{\mathbb{C}}$  and  $\text{Tr}$  the trace on endomorphisms on  $S_{\mathbb{C}}$ . Then, using Theorem 5, and the Wodzicki residue, we have

$$\begin{aligned} \int f(1 + \mathcal{D}^2)^{-n/2} &= \frac{1}{n(2\pi)^n} \int_X f \sqrt{g} d^n x \int_{S^{n-1}} \text{Tr}(I d_{S_{\mathbb{C}}}) d^{n-1} \xi \\ &= \frac{\text{Vol}(S^{n-1}) 2^{\lfloor n/2 \rfloor}}{n(2\pi)^n} \int_X f(x) \sqrt{g} d^n x. \end{aligned}$$

This is by no means immediate, and we refer to [5, Chapters IV and VI] where this is discussed at length. The statement that  $(1 + \mathcal{D}^2)^{-n/2} \in \mathcal{L}^{(1, \infty)}(L^2(X, S_{\mathbb{C}}))$  and is measurable is essentially Weyl's Theorem, a proof of which can be found in [12, Lemma 1.12.6].

#### 4. Summability for Nonunital Spectral Triples

We now tackle the nonunital case. We begin by dispensing with the Wodzicki residue, so that the appropriate geometric notion of summability is made clear. We will then deal with  $(p, \infty)$ -summability and the trace on a smooth summable spectral triple.

In [9], it was shown that one can analyse the Wodzicki residue of operators on complete noncompact manifolds using the spectral density function. This led to the identification of the Wodzicki residue in the noncompact case as precisely the same thing as in the compact case! That is, if  $T$  is an elliptic operator of order  $-p$  on a  $p$ -dimensional Riemannian manifold  $X$  with metric  $g$ , then for all  $f \in C_c^\infty(X)$  we have

$$\text{WRes}(fT) = \frac{1}{p(2\pi)^p} \int_{S^*X} \sigma_{-p}(fT)(x, \xi) \sqrt{g} dx d\xi.$$

Also, Connes' Trace Theorem, Theorem 5, remains true in this generality, as we will show in the next section, supporting results of [13, p. 297].

DEFINITION 9. A local spectral triple is  $(p, \infty)$ -summable if  $p \geq 1$  and

$$a(\mathcal{D} - \lambda)^{-1} \in \mathcal{L}^{(p, \infty)}(\mathcal{H}) \quad \forall a \in \mathcal{A}_c.$$

We call it  $\theta$ -summable if

$$\text{Trace}(ae^{-t(1+\mathcal{D}^2)}) < \infty$$

for all  $a \in \mathcal{A}_c$  and  $t > 0$ .

*Remark.* If  $\mathcal{A}$  is unital,  $\ker \mathcal{D}$  is finite dimensional. This case is well described in the literature. Note that the summability requirements are only for  $a \in \mathcal{A}_c$ . We do not assume that elements of the algebra  $\mathcal{A}$  are all integrable in the nonunital case. Strictly speaking, this definition describes *local*  $(p, \infty)$ -summability, and this is important. However, we have already overused the word local, and as we will only work with local spectral triples, we will employ the terminology  $(p, \infty)$ -summable to be consistent with the unital case.

Our immediate task is to demonstrate that this definition of summability is well-behaved, in particular that

$$a(1 + \mathcal{D}^2)^{-s/2} \in \mathcal{L}^{(p/s, \infty)}(\mathcal{H}), \quad 1 \leq \text{Re}(s) \leq p, \quad (3)$$

and that for  $\text{Re}(s) > p$  the resulting operator is trace class.

We need a series of technical results. In the following, when we observe that an operator is an element of  $\mathcal{L}^{(p/k, \infty)}(\mathcal{H})$ , and  $p/k < 1$ , then we mean that it is trace class. We also frequently use the resolvent formula

$$[a, (\mathcal{D} - \lambda)^{-1}] = (\mathcal{D} - \lambda)^{-1}[\mathcal{D}, a](\mathcal{D} - \lambda)^{-1}, \quad (4)$$

for the case that  $[\mathcal{D}, a]$  is bounded and  $\lambda$  is in the resolvent set of  $\mathcal{D}$ .

LEMMA 6. *Let  $(\mathcal{A}, \mathcal{H}, \mathcal{D})$  be a local  $(p, \infty)$ -summable spectral triple. For all  $a \in \mathcal{A}_c$ , we have  $a(1 + \mathcal{D}^2)^{-1} \in \mathcal{L}^{(p/2, \infty)}(\mathcal{H})$ , and for all  $\phi \in \mathcal{A}_c$  with  $\phi \geq 0$ ,  $(\phi(1 + \mathcal{D}^2)^{-1}\phi)^{1/2} \in \mathcal{L}^{(p, \infty)}(\mathcal{H})$ .*

*Proof.* Let  $\phi$  be a local unit for  $a \in \mathcal{A}_c$ . Then

$$\begin{aligned} a(\mathcal{D} - \lambda)^{-1} &= a\phi(\mathcal{D} - \lambda)^{-1} \\ &= a(\mathcal{D} - \lambda)^{-1}\phi + a[\phi, (\mathcal{D} - \lambda)^{-1}] \\ &= a(\mathcal{D} - \lambda)^{-1}\phi + a(\mathcal{D} - \lambda)^{-1}[\mathcal{D}, \phi](\mathcal{D} - \lambda)^{-1} \\ &= a(\mathcal{D} - \lambda)^{-1}\phi + a(\mathcal{D} - \lambda)^{-1}[\mathcal{D}, \phi]\psi(\mathcal{D} - \lambda)^{-1}, \end{aligned}$$

where  $[\mathcal{D}, \phi]$  has local unit  $\psi$ . As the second term is in  $\mathcal{L}^{(p/2, \infty)}(\mathcal{H})$ , we find

$$a(\mathcal{D} - \lambda)^{-1} = a\phi(\mathcal{D} - \lambda)^{-1}\phi \text{ mod } \mathcal{L}^{(p/2, \infty)}(\mathcal{H}).$$

Our first statement can now be proved. We have

$$\begin{aligned}
a(1 + \mathcal{D}^2)^{-1} &= a(\mathcal{D} - i)^{-1}(\mathcal{D} + i)^{-1} \\
&= a((\mathcal{D} - i)^{-1}\phi + (\mathcal{D} - i)^{-1}[\mathcal{D}, \phi](\mathcal{D} - i)^{-1})(\mathcal{D} + i)^{-1} \\
&= a(\mathcal{D} - i)^{-1}\phi(\mathcal{D} + i)^{-1} + a(\mathcal{D} - i)^{-1}[\mathcal{D}, \phi](\mathcal{D} - i)^{-1} \times \\
&\quad \times [\mathcal{D}, \psi](\mathcal{D} - i)^{-1}(\mathcal{D} + i)^{-1} + \\
&\quad + a(\mathcal{D} - i)^{-1}[\mathcal{D}, \phi](\mathcal{D} - i)^{-1}\psi(\mathcal{D} + i)^{-1}.
\end{aligned}$$

The term  $a(\mathcal{D} - i)^{-1}\phi(\mathcal{D} + i)^{-1}$  is in  $\mathcal{L}^{(p/2, \infty)}(\mathcal{H})$ , and as  $[\mathcal{D}, \psi]$  has a local unit, and  $(\mathcal{D} + i)^{-1}$  is bounded, the second term is in  $\mathcal{L}^{(p/3, \infty)}(\mathcal{H})$ . Finally,  $a(\mathcal{D} - i)^{-1}[\mathcal{D}, \phi](\mathcal{D} - i)^{-1}\psi(\mathcal{D} + i)^{-1}$  is in  $\mathcal{L}^{(p/3, \infty)}(\mathcal{H})$ , since  $[\mathcal{D}, \phi]$  has local unit  $\psi$ . Thus the first statement is proved. Taking one more commutator in the first term now gives

$$a(1 + \mathcal{D}^2)^{-1} = a\phi(1 + \mathcal{D}^2)^{-1}\phi + a(\mathcal{D} - i)^{-1}[\phi, (\mathcal{D} + i)^{-1}] \bmod \mathcal{L}^{(p/3, \infty)}(\mathcal{H}).$$

Now

$$\begin{aligned}
a(\mathcal{D} - i)^{-1}[\phi, (\mathcal{D} + i)^{-1}] &= a(\mathcal{D}^2 + 1)^{-1}[\mathcal{D}, \phi](\mathcal{D} + i)^{-1} \\
&= a(\mathcal{D}^2 + 1)^{-1}\psi[\mathcal{D}, \phi]\psi(\mathcal{D} + i)^{-1},
\end{aligned}$$

and this is in  $\mathcal{L}^{(p/3, \infty)}(\mathcal{H})$ . So

$$a(1 + \mathcal{D}^2)^{-1} = a\phi(1 + \mathcal{D}^2)^{-1}\phi \bmod \mathcal{L}^{(p/3, \infty)}(\mathcal{H}). \quad (5)$$

The final statement follows as  $(\mathcal{D} - i)^{-1}\phi \in \mathcal{L}^{(p, \infty)}(\mathcal{H})$  if and only if

$$\sqrt{\phi(\mathcal{D} + i)^{-1}(\mathcal{D} - i)^{-1}\phi} = \sqrt{\phi(1 + \mathcal{D}^2)^{-1}\phi} \in \mathcal{L}^{(p, \infty)}(\mathcal{H}). \quad \square$$

**COROLLARY 7.** *Let  $(\mathcal{A}, \mathcal{H}, \mathcal{D})$  be a local  $(p, \infty)$ -summable spectral triple. If  $\phi$  is a local unit for  $a \in \mathcal{A}_c$  then*

$$a(1 + \mathcal{D}^2)^{-1}(1 - \phi) \in \mathcal{L}^{(p/3, \infty)}(\mathcal{H}). \quad (6)$$

*Indeed, there exists a local unit  $\phi$  for  $a \in \mathcal{A}_c$  such that*

$$a(1 + \mathcal{D}^2)^{-1}(1 - \phi) \in \mathcal{L}^1(\mathcal{H}). \quad (7)$$

*Proof.* For the first statement one just notes that

$$a(1 + \mathcal{D}^2)^{-1} = a\phi(1 + \mathcal{D}^2)^{-1}\phi + a\phi(1 + \mathcal{D}^2)^{-1}(1 - \phi),$$

and applies Equation (5) of the previous lemma. To show that we can get equality modulo trace class operators, let  $\{\phi_n\}_{n \geq 1}$  be a local unit with  $\phi_{n+1}\phi_n = \phi_n$  and  $[\mathcal{D}, \phi_n]\phi_{n+1} = [\mathcal{D}, \phi_n]$  for all  $n$ . It follows, by the Leibniz rule, that  $\phi_n[\mathcal{D}, \phi_{n+1}] = 0$ .

A similar computation works for  $[\mathcal{D}, \phi_{n+1}]\phi_n$ . We employ this by noting that if  $\phi_n a = a\phi_n = a$ , then for  $k \geq 0$

$$\begin{aligned} a(\mathcal{D} - \lambda)^{-1} &= a\phi_{n+k}(\mathcal{D} - \lambda)^{-1} \\ &= a(\mathcal{D} - \lambda)^{-1}\phi_{n+k} + a[\phi_{n+k}, (\mathcal{D} - \lambda)^{-1}] \\ &= a(\mathcal{D} - \lambda)^{-1}\phi_{n+k} + a(\mathcal{D} - \lambda)^{-1}[\mathcal{D}, \phi_{n+k}](\mathcal{D} - \lambda)^{-1}. \end{aligned}$$

However, we can go one better, and notice that for  $k \geq 1$ , we can use the properties of the local approximate unit to show that the second term is ‘very’ summable

$$\begin{aligned} a\phi_{n+k-1}(\mathcal{D} - \lambda)^{-1}[\mathcal{D}, \phi_{n+k}](\mathcal{D} - \lambda)^{-1} \\ &= a(\mathcal{D} - \lambda)^{-1}\phi_{n+k-1}[\mathcal{D}, \phi_{n+k}](\mathcal{D} - \lambda)^{-1} + \\ &\quad + a[\phi_{n+k-1}, (\mathcal{D} - \lambda)^{-1}][\mathcal{D}, \phi_{n+k}](\mathcal{D} - \lambda)^{-1} \\ &= a(\mathcal{D} - \lambda)^{-1}[\mathcal{D}, \phi_{n+k-1}](\mathcal{D} - \lambda)^{-1}[\mathcal{D}, \phi_{n+k}](\mathcal{D} - \lambda)^{-1}, \end{aligned}$$

the last line following from  $\phi_{n+k-1}[\mathcal{D}, \phi_{n+k}] = 0$  and Equation (4). Thus by choosing  $k$  sufficiently large (in fact  $k \geq p - 1$ ) and continuing in this fashion, we obtain

$$a(\mathcal{D} - \lambda)^{-1} = a\phi_{n+k}(\mathcal{D} - \lambda)^{-1}\phi_{n+k} + \text{trace class.} \quad \square$$

Before we can show that  $a(1 + \mathcal{D}^2)^{-s/2} \in \mathcal{L}^{(p/s, \infty)}(\mathcal{H})$  for all  $a \in \mathcal{A}_c$ ,  $1 \leq s \leq p$ , we require two more technical results. The first will be used several times.

**LEMMA 8.** *Let  $(\mathcal{A}, \mathcal{H}, \mathcal{D})$  be a smooth, local  $(p, \infty)$ -summable spectral triple with  $p \geq 1$ . Let  $a \in \mathcal{B}(\mathcal{H})$  have local unit  $\psi \in \mathcal{A}_c$  and let  $\phi \in \mathcal{A}_c$  be a local unit for  $\psi$ ,  $[\mathcal{D}, \psi]$ . Then writing  $T_\phi = (1 + \mathcal{D}^2)\phi(1 + \mathcal{D}^2)^{-1}\phi$  we have*

$$\|a(1 + \mathcal{D}^2 + \lambda)^{-1}(1 - T_\phi)\|_{(p/2, \infty)} \leq C(1 + \lambda)^{-1}$$

and

$$\|(1 + \mathcal{D}^2)\psi(1 + \mathcal{D}^2 + \lambda)^{-1}(1 - T_\phi)\| \leq C'(1 + \lambda)^{-1}$$

for some positive constants  $C, C'$ .

*Proof.* First note that as  $(\mathcal{A}, \mathcal{H}, \mathcal{D})$  is smooth,

$$\begin{aligned} [\mathcal{D}^2, \phi](1 + \mathcal{D}^2)^{-1/2} \\ &= |\mathcal{D}|[|\mathcal{D}|, \phi](1 + \mathcal{D}^2)^{-1/2} + [|\mathcal{D}|, \phi]|\mathcal{D}|(1 + \mathcal{D}^2)^{-1/2} \\ &= [|\mathcal{D}|, [|\mathcal{D}|, \phi]](1 + \mathcal{D}^2)^{-1/2} + 2[|\mathcal{D}|, \phi]|\mathcal{D}|(1 + \mathcal{D}^2)^{-1/2} \end{aligned}$$

is bounded, so  $T_\phi$  is bounded. We begin with the following computation

$$\begin{aligned} a(1 + \mathcal{D}^2 + \lambda)^{-1}(1 - T_\phi) \\ &= a\psi(1 + \mathcal{D}^2 + \lambda)^{-1}(1 - T_\phi) \\ &= a(1 + \mathcal{D}^2 + \lambda)^{-1}[\mathcal{D}^2, \psi](1 + \mathcal{D}^2 + \lambda)^{-1}(1 - T_\phi) + \\ &\quad + a(1 + \mathcal{D}^2 + \lambda)^{-1}\psi(1 - T_\phi) \end{aligned}$$

$$\begin{aligned}
&= a(1 + \mathcal{D}^2 + \lambda)^{-2}[\mathcal{D}^2, [\mathcal{D}^2, \psi]](1 + \mathcal{D}^2 + \lambda)^{-1}(1 - T_\phi) + \\
&\quad + a(1 + \mathcal{D}^2 + \lambda)^{-2}[\mathcal{D}^2, \psi](1 - T_\phi) + \\
&\quad + a(1 + \mathcal{D}^2 + \lambda)^{-1}\psi(1 - T_\phi).
\end{aligned} \tag{8}$$

In order to obtain estimates on the  $(p/2, \infty)$  norm, we require several observations. The first is that if  $\phi$  is a local unit for  $\psi$ ,  $[\mathcal{D}, \psi]$ , then

$$\psi(1 - T_\phi) = \psi(1 - \phi^2 - [\mathcal{D}^2, \phi](1 + \mathcal{D}^2)^{-1}\phi) = -\psi[\mathcal{D}^2, \phi](1 + \mathcal{D}^2)^{-1}\phi.$$

Now observe that  $[\mathcal{D}, \psi]\phi = [\mathcal{D}, \psi]$  implies  $\psi[\mathcal{D}, \phi] = 0$ , by the Leibniz rule. So for such a  $\psi$

$$\begin{aligned}
\psi[\mathcal{D}^2, \phi] &= \psi\mathcal{D}[\mathcal{D}, \phi] + \psi[\mathcal{D}, \phi]\mathcal{D} \\
&= -[\mathcal{D}, \psi][\mathcal{D}, \phi] + \mathcal{D}\psi[\mathcal{D}, \phi] \\
&= -[\mathcal{D}, \psi][\mathcal{D}, \phi],
\end{aligned}$$

and this is bounded. Thus

$$\psi(1 - T_\phi) = -\psi[\mathcal{D}^2, \phi](1 + \mathcal{D}^2)^{-1}\phi = [\mathcal{D}, \psi][\mathcal{D}, \phi](1 + \mathcal{D}^2)^{-1}\phi \in \mathcal{L}^{(p/2, \infty)}(\mathcal{H}). \tag{9}$$

An entirely analogous calculation using  $[\mathcal{D}, \psi]\phi = [\mathcal{D}, \psi]$  and  $[\mathcal{D}^2, \psi] = \mathcal{D}[\mathcal{D}, \psi] + [\mathcal{D}, \psi]\mathcal{D}$  shows that

$$[\mathcal{D}^2, \psi](1 - T_\phi) = -[\mathcal{D}, \psi][\mathcal{D}, \phi^2] - [\mathcal{D}^2, \psi][\mathcal{D}^2, \phi](1 + \mathcal{D}^2)^{-1}\phi. \tag{10}$$

Both terms on the right hand side of Equation (10) are bounded. For the second term this follows from the equality (writing  $\delta(\cdot) = [|\mathcal{D}|, \cdot]$  as usual)

$$\begin{aligned}
[\mathcal{D}^2, \psi][\mathcal{D}^2, \phi] &= \delta^2(\psi)\delta^2(\phi) + 2\delta(\psi)\delta^3(\phi) + (\delta(\psi)\delta^2(\phi) + \\
&\quad + 2\delta^2(\psi)\delta(\phi))|\mathcal{D}| + 4\delta(\psi)\delta(\phi)\mathcal{D}^2,
\end{aligned}$$

and the boundedness of the operators  $|\mathcal{D}|(1 + \mathcal{D}^2)^{-1}$ ,  $\mathcal{D}^2(1 + \mathcal{D}^2)^{-1}$  which follows from the functional calculus. Finally, to estimate the first term on the right hand side of Equation (8), we need to know that

$$[\mathcal{D}^2, [\mathcal{D}^2, \psi]] = \delta^4(\psi) + 4\delta^3(\psi)|\mathcal{D}| + 4\delta^2(\psi)\mathcal{D}^2,$$

so that

$$[\mathcal{D}^2, [\mathcal{D}^2, \psi]](1 + \mathcal{D}^2 + \lambda)^{-1} = [\mathcal{D}^2, [\mathcal{D}^2, \psi]](1 + \mathcal{D}^2)^{-1}(1 + \mathcal{D}^2)(1 + \mathcal{D}^2 + \lambda)^{-1} \tag{11}$$

is bounded uniformly in  $\lambda$  (since  $\|(1 + \mathcal{D}^2)(1 + \mathcal{D}^2 + \lambda)^{-1}\| \leq 1$  by the functional calculus). Putting together Equations (9)–(11), along with Equation (8) gives

$$\begin{aligned}
& \|a(1 + \mathcal{D}^2 + \lambda)^{-1}(1 - T_\phi)\|_{(p/2, \infty)} \\
& \leq \|a(1 + \mathcal{D}^2 + \lambda)^{-2}[\mathcal{D}^2, [\mathcal{D}^2, \psi]](1 + \mathcal{D}^2 + \lambda)^{-1}(1 - T_\phi)\|_{(p/2, \infty)} + \\
& \quad + \|a(1 + \mathcal{D}^2 + \lambda)^{-2}[\mathcal{D}^2, \psi](1 - T_\phi)\|_{(p/2, \infty)} + \\
& \quad + \|a(1 + \mathcal{D}^2 + \lambda)^{-1}\psi(1 - T_\phi)\|_{(p/2, \infty)} \\
& \leq \|(1 + \mathcal{D}^2 + \lambda)^{-1}\|((C_1 + C_2)\|a(1 + \mathcal{D}^2 + \lambda)^{-1}\|_{(p/2, \infty)} + \\
& \quad + C_3\|\psi(1 - T_\phi)\|_{(p/2, \infty)}) \\
& \leq (1 + \lambda)^{-1}((C_1 + C_2)\|a(1 + \mathcal{D}^2)^{-1}\|_{(p/2, \infty)}\|(1 + \mathcal{D}^2)\| \times \\
& \quad \times \|(1 + \mathcal{D}^2 + \lambda)^{-1}\| + C_3) \\
& \leq C(1 + \lambda)^{-1}.
\end{aligned}$$

Here we have again used the estimate  $\|(1 + \mathcal{D}^2)(1 + \mathcal{D}^2 + \lambda)^{-1}\| \leq 1$  as well as the estimate  $\|(1 + \mathcal{D}^2 + \lambda)^{-1}\| \leq (1 + \lambda)^{-1}$ . This proves the first statement. To obtain the operator norm estimate, we begin with Equation (8) and find

$$\begin{aligned}
& \|(1 + \mathcal{D}^2)\psi(1 + \mathcal{D}^2 + \lambda)^{-1}(1 - T_\phi)\| \\
& \leq \|(1 + \mathcal{D}^2)(1 + \mathcal{D}^2 + \lambda)^{-2}[\mathcal{D}^2, [\mathcal{D}^2, \psi]](1 + \mathcal{D}^2 + \lambda)^{-1}(1 - T_\phi)\| + \\
& \quad + \|(1 + \mathcal{D}^2)(1 + \mathcal{D}^2 + \lambda)^{-2}[\mathcal{D}^2, \psi](1 - T_\phi)\| + \\
& \quad + \|(1 + \mathcal{D}^2)(1 + \mathcal{D}^2 + \lambda)^{-1}\psi(1 - T_\phi)\| \\
& \leq \|(1 + \mathcal{D}^2 + \lambda)^{-1}\|((C_1 + C_2)\|(1 + \mathcal{D}^2)(1 + \mathcal{D}^2 + \lambda)^{-1}\| + \\
& \quad + C_3\|(1 + \mathcal{D}^2)\psi(1 - T_\phi)\|) \\
& \leq C(1 + \lambda)^{-1}.
\end{aligned}$$

Here we used the boundedness of  $(1 + \mathcal{D}^2)\psi(1 - T_\phi) = (1 + \mathcal{D}^2)[\mathcal{D}, \psi][\mathcal{D}, \phi](1 + \mathcal{D}^2)^{-1}\phi$  which follows easily from the smoothness assumption.  $\square$

**LEMMA 9.** *Let  $(\mathcal{A}, \mathcal{H}, \mathcal{D})$  be a smooth, local  $(p, \infty)$ -summable spectral triple with  $p \geq 1$ . Let  $a \in \mathcal{B}(\mathcal{H})$  be such that  $\psi a = a\psi = a$  for some  $\psi \in \mathcal{A}_c$ . If  $\phi \in \mathcal{A}_c$  is a local unit for  $\psi$  and  $[\mathcal{D}, \psi]$ , then for  $0 < s < 1$ , we have*

$$a(1 + \mathcal{D}^2)^{-s} - a(\phi(1 + \mathcal{D}^2)^{-1}\phi)^s \in \mathcal{L}^{(p/2, \infty)}(\mathcal{H}).$$

Moreover, for some positive constant  $C$

$$\|a(1 + \mathcal{D}^2)^{-s} - a(\phi(1 + \mathcal{D}^2)^{-1}\phi)^s\|_{(p/2, \infty)} \leq C$$

independent of  $s \in (0, 1)$ .

*Proof.* To simplify the notation, set

$$\eta_\phi = \phi(1 + \mathcal{D}^2)^{-1}\phi \quad \text{and} \quad T_\phi = (1 + \mathcal{D}^2)\phi(1 + \mathcal{D}^2)^{-1}\phi.$$

For  $0 < \operatorname{Re}(s) < 1$  and  $B \geq 0$  a bounded positive operator on  $\mathcal{H}$ , the functional calculus gives

$$B^s = \frac{\sin(s\pi)}{\pi} \int_0^\infty \lambda^{-s} B(1 + \lambda B)^{-1} d\lambda,$$

where the integral converges in the operator norm. For  $B = (1 + \mathcal{D}^2)^{-1}$  we get

$$B(1 + \lambda B)^{-1} = (1 + \mathcal{D}^2 + \lambda)^{-1},$$

while for  $B = \phi(1 + \mathcal{D}^2)^{-1}\phi$  we find

$$\begin{aligned} B(1 + \lambda B)^{-1} &= \phi(1 + \mathcal{D}^2)^{-1}\phi(1 + \lambda\phi(1 + \mathcal{D}^2)^{-1}\phi)^{-1} \\ &= \phi(1 + \mathcal{D}^2)^{-1}\phi(1 + \lambda\phi(1 + \mathcal{D}^2)^{-1}\phi(1 + \mathcal{D}^2)(1 + \mathcal{D}^2)^{-1})^{-1} \\ &= \phi(1 + \mathcal{D}^2)^{-1}\phi((1 + \mathcal{D}^2 + \lambda\phi(1 + \mathcal{D}^2)^{-1}\phi(1 + \mathcal{D}^2))(1 + \mathcal{D}^2)^{-1})^{-1} \\ &= \phi(1 + \mathcal{D}^2)^{-1}\phi(1 + \mathcal{D}^2)((1 + \mathcal{D}^2) + \lambda\phi(1 + \mathcal{D}^2)^{-1}\phi(1 + \mathcal{D}^2))^{-1} \\ &= T_\phi^*(1 + \mathcal{D}^2 + \lambda T_\phi^*)^{-1} \\ &= (1 + \mathcal{D}^2 + \lambda T_\phi)^{-1}T_\phi, \end{aligned}$$

the last line following since  $B(1 + \lambda B)^{-1}$  is self-adjoint. Now by adding and subtracting  $(1 + \mathcal{D}^2 + \lambda)^{-1}T_\phi$ , we find

$$\begin{aligned} &(1 + \mathcal{D}^2 + \lambda)^{-1} - (1 + \mathcal{D}^2 + \lambda T_\phi)^{-1}T_\phi \\ &= (1 + \mathcal{D}^2 + \lambda)^{-1} - (1 + \mathcal{D}^2 + \lambda)^{-1}T_\phi + (1 + \mathcal{D}^2 + \lambda)^{-1}T_\phi - \\ &\quad - (1 + \mathcal{D}^2 + \lambda T_\phi)^{-1}T_\phi \\ &= (1 + \mathcal{D}^2 + \lambda)^{-1}(1 - T_\phi) + \lambda(1 + \mathcal{D}^2 + \lambda)^{-1}(T_\phi - 1) \times \\ &\quad \times (1 + \mathcal{D}^2 + \lambda T_\phi)^{-1}T_\phi \\ &= (1 + \mathcal{D}^2 + \lambda)^{-1}(1 - T_\phi)(1 - \lambda(1 + \mathcal{D}^2 + \lambda T_\phi)^{-1}T_\phi) \\ &= (1 + \mathcal{D}^2 + \lambda)^{-1}(1 - T_\phi)(1 - (1 + \lambda\eta_\phi)^{-1}\lambda\eta_\phi) \\ &= (1 + \mathcal{D}^2 + \lambda)^{-1}(1 - T_\phi)(1 + \lambda\eta_\phi)^{-1}. \end{aligned}$$

Thus the difference  $(1 + \mathcal{D}^2)^{-s} - (\phi(1 + \mathcal{D}^2)^{-1}\phi)^s$  is given by

$$\frac{\sin(s\pi)}{\pi} \int_0^\infty \lambda^{-s} (1 + \mathcal{D}^2 + \lambda)^{-1} (1 - T_\phi) (1 + \lambda\eta_\phi)^{-1} d\lambda. \quad (12)$$

Now the functional calculus shows us that  $\|(1 + \lambda\eta_\phi)^{-1}\| \leq 1$ , so Lemma 8 gives us, for some positive constant  $C$ ,

$$\begin{aligned} &\|a(1 + \mathcal{D}^2)^{-s} - a(\phi(1 + \mathcal{D}^2)^{-1}\phi)^s\|_{(p/2, \infty)} \\ &\leq \frac{\sin(s\pi)}{\pi} \int_0^\infty \lambda^{-s} \|a(1 + \mathcal{D}^2 + \lambda)^{-1} (1 - T_\phi) (1 + \lambda\eta_\phi)^{-1}\|_{(p/2, \infty)} d\lambda \\ &\leq C \frac{\sin(s\pi)}{\pi} \int_0^\infty \lambda^{-s} (1 + \lambda)^{-1} d\lambda, \end{aligned}$$

and this is finite since  $s > 0$ . As [8],

$$\int_0^\infty \lambda^{-s} \frac{1}{1+\lambda} d\lambda = \frac{\pi}{\sin(s\pi)}, \quad (13)$$

we have

$$\|a(1 + \mathcal{D}^2)^{-s} - a(\phi(1 + \mathcal{D}^2)^{-1}\phi)^s\|_{(p/2, \infty)} \leq C. \quad \square$$

**PROPOSITION 10.** *Let  $(\mathcal{A}, \mathcal{H}, \mathcal{D})$  be a smooth, local  $(p, \infty)$ -summable spectral triple. Let  $a \in \mathcal{B}(\mathcal{H})$  satisfy  $a\phi = \phi a = a$  for some  $\phi \in \mathcal{A}_c$ . For all  $s$  with  $1 \leq \operatorname{Re}(s) \leq p$ ,*

$$a(1 + \mathcal{D}^2)^{-s/2} \in \mathcal{L}^{(p/s, \infty)}(\mathcal{H}).$$

*For  $\operatorname{Re}(s) > p$ ,  $a(1 + \mathcal{D}^2)^{-s/2}$  is trace class.*

*Proof.* For purely imaginary  $s$ ,  $(1 + \mathcal{D}^2)^{-s}$  is bounded, so we suppose that  $s$  is real. If  $s/2$  is integral, choose  $\phi_1$  so that  $a(1 + \mathcal{D}^2)^{-1}(1 - \phi_1)$  is trace class,  $\phi_2$  so that  $\phi_1(1 + \mathcal{D}^2)^{-1}(1 - \phi_2)$  is trace class, and so on up to  $\phi_{s/2}$ . Then by Corollary 7,

$$\begin{aligned} a(1 + \mathcal{D}^2)^{-s/2} &= a(\phi_1(1 + \mathcal{D}^2)^{-1}\phi_1)(\phi_2(1 + \mathcal{D}^2)^{-1}\phi_2) \\ &\quad \cdots (\phi_{s/2}(1 + \mathcal{D}^2)^{-1}\phi_{s/2}) \bmod \mathcal{L}^1(\mathcal{H}) \end{aligned}$$

and so is in  $\mathcal{L}^{(p/s, \infty)}(\mathcal{H})$ . Now suppose that  $s$  is not an even integer and let  $b$  be the greatest even integer less than or equal to  $s$ . Then

$$\begin{aligned} a(1 + \mathcal{D}^2)^{-s/2} &= a(1 + \mathcal{D}^2)^{-b/2}(1 + \mathcal{D}^2)^{-(s-b)/2} \\ &= K\phi_{b/2}(1 + \mathcal{D}^2)^{-(s-b)/2}, \end{aligned}$$

where  $K$  is in  $\mathcal{L}^{(p/b, \infty)}(\mathcal{H})$ . Now for  $s$  not an even integer,  $0 < (s - b)/2 < 1$ , so we may apply the results of the last Lemma. So, for suitable  $\psi \in \mathcal{A}_c$ ,

$$\phi_{b/2}(1 + \mathcal{D}^2)^{-(s-b)/2} = \phi_{b/2}(\psi(1 + \mathcal{D}^2)^{-1}\psi)^{(s-b)/2} \bmod \mathcal{L}^{(p/2, \infty)}(\mathcal{H}),$$

and  $(\psi(1 + \mathcal{D}^2)^{-1}\psi)^{(s-b)/2} \in \mathcal{L}^{(p/(s-b), \infty)}(\mathcal{H})$ . As  $s - b < 2$ ,  $\mathcal{L}^{(p/2, \infty)}(\mathcal{H}) \subseteq \mathcal{L}^{(p/(s-b), \infty)}(\mathcal{H})$ , and we see that  $\phi_{b/2}(1 + \mathcal{D}^2)^{-(s-b)/2} \in \mathcal{L}^{(p/(s-b), \infty)}(\mathcal{H})$ , since the error terms are also in  $\mathcal{L}^{(p/(s-b), \infty)}(\mathcal{H})$ . So modulo trace class errors

$$a(1 + \mathcal{D}^2)^{-s/2} = K\phi_{b/2}(1 + \mathcal{D}^2)^{-(s-b)/2} \in \mathcal{L}^{(p/s, \infty)}(\mathcal{H}).$$

A similar argument shows that for  $s > p$  the operator  $a(1 + \mathcal{D}^2)^{-s/2}$  is trace class.  $\square$

**COROLLARY 11.** *Let  $(\mathcal{A}, \mathcal{H}, \mathcal{D})$  be a smooth, local  $(p, \infty)$ -summable spectral triple. Then for any Dixmier trace  $\operatorname{Tr}_\omega$ , the function*

$$a \longrightarrow \operatorname{Tr}_\omega(a(1 + \mathcal{D}^2)^{-p/2})$$

*defines a trace on  $\mathcal{A}_c \subset \mathcal{A}$ .*



*Proof.* Let  $a, b \in \mathcal{A}_c$ , and let  $\phi$  be a local unit for  $b$ . Then the trace property of the Dixmier trace gives us

$$\begin{aligned} \mathrm{Tr}_\omega(ab(1 + \mathcal{D}^2)^{-p/2}) &= \mathrm{Tr}_\omega(b(1 + \mathcal{D}^2)^{-p/2}a) \\ &= \mathrm{Tr}_\omega(b[(1 + \mathcal{D}^2)^{-p/2}, a]) + \mathrm{Tr}_\omega(ba(1 + \mathcal{D}^2)^{-p/2}) \\ &= \mathrm{Tr}_\omega(b[(1 + \mathcal{D}^2)^{-p/2}, a]\phi) + \mathrm{Tr}_\omega(ba(1 + \mathcal{D}^2)^{-p/2}). \end{aligned}$$

Now, by the standard resolvent formulae and the Leibniz rule, we have

$$\begin{aligned} [(1 + \mathcal{D}^2)^{-p/2}, a] &= -(1 + \mathcal{D}^2)^{-p/2}[(1 + \mathcal{D}^2)^{p/2}, a](1 + \mathcal{D}^2)^{-p/2} \\ &= -(1 + \mathcal{D}^2)^{-p/2} \sum_{j=0}^{p-1} (1 + \mathcal{D}^2)^{j/2} [(1 + \mathcal{D}^2)^{1/2}, a] \times \\ &\quad \times (1 + \mathcal{D}^2)^{(p-j-1)/2} (1 + \mathcal{D}^2)^{-p/2} \\ &= - \sum_{j=0}^{p-1} (1 + \mathcal{D}^2)^{-(p-j)/2} [(1 + \mathcal{D}^2)^{1/2}, a] (1 + \mathcal{D}^2)^{-(j+1)/2}. \end{aligned}$$

By Proposition 10 and the boundedness of  $[(1 + \mathcal{D}^2)^{1/2}, a]$ , we have for  $0 \leq j \leq p-1$

$$\begin{aligned} b(1 + \mathcal{D}^2)^{-(p-j)/2} [(1 + \mathcal{D}^2)^{1/2}, a] (1 + \mathcal{D}^2)^{-(j+1)/2} \phi &\in \mathcal{L}^{(p/(p-j), \infty)}(\mathcal{H}) \\ &\cdot \mathcal{L}^{(p/(j+1), \infty)}(\mathcal{H}) \subset \mathcal{L}^1(\mathcal{H}). \end{aligned}$$

Thus

$$\begin{aligned} \mathrm{Tr}_\omega(ab(1 + \mathcal{D}^2)^{-p/2}) &= \mathrm{Tr}_\omega(b[(1 + \mathcal{D}^2)^{-p/2}, a]\phi) + \mathrm{Tr}_\omega(ba(1 + \mathcal{D}^2)^{-p/2}) \\ &= - \sum_{j=0}^{p-1} \mathrm{Tr}_\omega(b(1 + \mathcal{D}^2)^{-(p-j)/2} [(1 + \mathcal{D}^2)^{1/2}, a] (1 + \mathcal{D}^2)^{-(j+1)/2} \phi) + \\ &\quad + \mathrm{Tr}_\omega(ba(1 + \mathcal{D}^2)^{-p/2}) \\ &= \mathrm{Tr}_\omega(ba(1 + \mathcal{D}^2)^{-p/2}). \quad \square \end{aligned}$$

We finish this section with a measurability criteria, using [2, Theorem 5.6].

**THEOREM 12.** *Let  $(\mathcal{A}, \mathcal{H}, \mathcal{D})$  be a smooth, local  $(p, \infty)$ -summable spectral triple with  $p \geq 1$ . Suppose that  $T \in \mathcal{B}(\mathcal{H})$  is such that  $\psi T = T\psi = T$  for some  $\psi \geq 0$ ,  $\psi \in \mathcal{A}_c$ . If the limit*

$$\lim_{s \rightarrow p/2^+} (s - p/2) \mathrm{Trace}(T(1 + \mathcal{D}^2)^{-s}) \quad (14)$$

*exists, then it is equal to*

$$\frac{p}{2} \int T(1 + \mathcal{D}^2)^{-p/2}.$$

*Proof.* Let  $\psi T = T\psi = T$ , and choose a local unit  $\phi \in \mathcal{A}_c$  for  $\psi$  and  $[\mathcal{D}, \psi]$ . Then as  $\phi \in \mathcal{A}_c$ , Lemma 6 shows that  $\phi(1 + \mathcal{D}^2)^{-1}\phi \in \mathcal{L}^{(p/2, \infty)}(\mathcal{H})$ , and

$$T(\phi(1 + \mathcal{D}^2)^{-1}\phi)^s \in \begin{cases} \mathcal{L}^{(p/2s, \infty)}(\mathcal{H}) & 1 \leq s \leq p/2 \\ \mathcal{L}^1(\mathcal{H}) & s > p/2 \end{cases}$$

Let  $n = [p/2]$  denote the integer part of  $p/2$  and set  $r = p/2 - [p/2]$ , so that  $0 \leq r < 1$ , and  $p/2 = n + r$ . Observe that

$$\phi(1 + \mathcal{D}^2)^{-1}\phi = \phi^2(1 + \mathcal{D}^2)^{-1} + K,$$

where  $K = \phi[(1 + \mathcal{D}^2)^{-1}, \phi] \in \mathcal{L}^{(p/3, \infty)}$ . To see that this is so, choose a local unit  $\chi \in \mathcal{A}_c$  for  $\phi$ ,  $[\mathcal{D}, \phi]$ , and a local unit  $\rho \in \mathcal{A}_c$  for  $\chi$ ,  $[\mathcal{D}, \chi]$ . Then we have the following computation

$$\begin{aligned} K &= \phi[(1 + \mathcal{D}^2)^{-1}, \phi] \\ &= -\phi(1 + \mathcal{D}^2)^{-1}[\mathcal{D}^2, \phi](1 + \mathcal{D}^2)^{-1} \\ &= -\phi(1 + \mathcal{D}^2)^{-1}\mathcal{D}[\mathcal{D}, \phi](1 + \mathcal{D}^2)^{-1} - \phi(1 + \mathcal{D}^2)^{-1}[\mathcal{D}, \phi]\mathcal{D}(1 + \mathcal{D}^2)^{-1} \\ &= -\phi(1 + \mathcal{D}^2)^{-1}\mathcal{D}[\mathcal{D}, \phi]\chi(1 + \mathcal{D}^2)^{-1} - \phi(1 + \mathcal{D}^2)^{-1}[\mathcal{D}, \phi]\chi\mathcal{D}(1 + \mathcal{D}^2)^{-1} \\ &= -\phi(1 + \mathcal{D}^2)^{-1}[\mathcal{D}^2, \phi]\chi(1 + \mathcal{D}^2)^{-1} + \phi(1 + \mathcal{D}^2)^{-1}[\mathcal{D}, \phi][\mathcal{D}, \chi](1 + \mathcal{D}^2)^{-1} \\ &= -\phi(1 + \mathcal{D}^2)^{-1/2}((1 + \mathcal{D}^2)^{-1/2}[\mathcal{D}^2, \phi])\chi(1 + \mathcal{D}^2)^{-1} + \\ &\quad + \phi(1 + \mathcal{D}^2)^{-1}[\mathcal{D}, \phi][\mathcal{D}, \chi]\rho(1 + \mathcal{D}^2)^{-1}. \end{aligned}$$

As  $(1 + \mathcal{D}^2)^{-1/2}[\mathcal{D}^2, \phi]$  is bounded, along with  $[\mathcal{D}, \phi]$  and  $[\mathcal{D}, \chi]$ , we may now use Proposition 10 to see that  $K \in \mathcal{L}^{(p/3, \infty)}(\mathcal{H})$  as claimed. This is a mild refinement of Corollary 7.

Hence for positive integers  $m, m'$ ,  $(1 + \mathcal{D}^2)^{-m}K \in \mathcal{L}^{(p/(2m+3), \infty)}(\mathcal{H})$  and  $K(1 + \mathcal{D}^2)^{-m'} \in \mathcal{L}^{(p/(2m'+3), \infty)}(\mathcal{H})$ , using Proposition 10. This follows because multiplication on the left gives factors  $(1 + \mathcal{D}^2)^{-m}\phi \cdots = (1 + \mathcal{D}^2)^{-m}\chi\phi \cdots$ , while multiplication on the right gives factors of  $\cdots\chi(1 + \mathcal{D}^2)^{-m'-1}$  or  $\cdots\rho(1 + \mathcal{D}^2)^{-m'-1}$ . Thus  $(1 + \mathcal{D}^2)^{-m}K(1 + \mathcal{D}^2)^{-m'} \in \mathcal{L}^{(p/(2m+2m'+3), \infty)}(\mathcal{H})$ , and we may iterate this argument to products

$$(1 + \mathcal{D}^2)^{-m_1}K(1 + \mathcal{D}^2)^{-m_2}K \cdots K(1 + \mathcal{D}^2)^{-m_j} \in \mathcal{L}^{(p/(3(j-1)+2\sum m_j), \infty)}(\mathcal{H}),$$

$$0 \leq m_j.$$

We also note that subsequent commutators  $[K, \phi^{2i}]$ ,  $i = 1, 2, \dots$ , lie in successively smaller ideals, by repeated application of the above arguments using local approximate units. In fact we only really require that these commutators remain in  $\mathcal{L}^{(p/3, \infty)}(\mathcal{H})$ , which is immediate since it is an ideal. The form of  $K$  then allows us to apply Proposition 10 as above to  $[K, \phi^{2i}]$  to see that  $(1 + \mathcal{D}^2)^{-m}[K, \phi^{2i}](1 + \mathcal{D}^2)^{-m'} \in \mathcal{L}^{(p/(2m+2m'+3), \infty)}(\mathcal{H})$ .

Thus we can write

$$\begin{aligned} (\phi(1 + \mathcal{D}^2)^{-1}\phi)^n &= (\phi^2(1 + \mathcal{D}^2)^{-1} + K)^n \\ &= \phi^{2n}(1 + \mathcal{D}^2)^{-n} + K_1 + \cdots + K_n, \end{aligned}$$

where each  $K_j$  is a sum of products of  $j$  factors of  $K$  and  $n - j$ -factors of  $(1 + \mathcal{D}^2)^{-1}$ . Two points should be made. By the first observation of the last paragraph, the position of the  $K$ 's in the product is irrelevant. Secondly, we have ignored higher order commutators  $[K, \phi^2]$  which arise when we pull all the  $\phi$ 's to the left, and have simply counted terms  $K' = [(1 + \mathcal{D}^2)^{-1}, \phi^2]$  as another  $K$ . This will not affect the following discussion, as  $K'$  lies in the same ideal as  $K$ .

Next observe that  $K_j \in \mathcal{L}^{(p/(2n+j), \infty)}(\mathcal{H})$ , and in all cases

$$2n + j > p - 2 + j.$$

So for  $j = 2, \dots, n$ ,  $K_j \in \mathcal{L}^1(\mathcal{H})$ .

The first case we consider is  $n = p/2 = [p/2]$  is an integer. Then  $2n + j > p$  for all  $j \geq 1$ , and in this case  $K_1 \in \mathcal{L}^1(\mathcal{H})$  also. So for  $0 < s < 1$

$$\begin{aligned} \text{Trace}(T(\phi(1 + \mathcal{D}^2)^{-1}\phi)^{p/2+s}) \\ &= \text{Trace}(T\phi(\phi(1 + \mathcal{D}^2)^{-1}\phi)^{p/2+s}) \\ &= \text{Trace}(T\phi(1 + \mathcal{D}^2)^{-p/2}(\phi(1 + \mathcal{D}^2)^{-1}\phi)^s) + C(s). \end{aligned}$$

Here  $C(s)$  is the sum of the traces of  $K_j(\phi(1 + \mathcal{D}^2)^{-1}\phi)^s$ ,  $j = 1, \dots, n$ . As the  $K_j$  are all trace class, and  $(\phi(1 + \mathcal{D}^2)^{-1}\phi)^s$  is bounded in norm as  $s \rightarrow 0$ , the function  $C(s)$  is bounded as  $s \rightarrow 0$ . So now write

$$\begin{aligned} \text{Trace}(T(\phi(1 + \mathcal{D}^2)^{-1}\phi)^{p/2+s}) &= \text{Trace}(T\phi(1 + \mathcal{D}^2)^{-p/2}((\phi(1 + \mathcal{D}^2)^{-1}\phi)^s - \\ &\quad - (1 + \mathcal{D}^2)^{-s} + (1 + \mathcal{D}^2)^{-s})) + C(s), \end{aligned}$$

and observe that each of the three products is trace class, by applications of Lemma 6 and Proposition 10. Since  $T$  is bounded, we can use the cyclicity of the trace to obtain

$$\begin{aligned} \text{Trace}(T(\phi(1 + \mathcal{D}^2)^{-1}\phi)^{p/2+s}) &= \text{Trace}(T\phi(1 + \mathcal{D}^2)^{-p/2-s} + \\ &\quad + B(s)\phi(1 + \mathcal{D}^2)^{-p/2}) + C(s). \end{aligned}$$

The operator  $B(s)$  is the difference

$$(\phi(1 + \mathcal{D}^2)^{-1}\phi)^s T - (1 + \mathcal{D}^2)^{-s} T,$$

the adjoint of which we studied in Lemma 9. Using the self-adjointness of the difference  $(1 + \mathcal{D}^2)^{-s} - (\phi(1 + \mathcal{D}^2)^{-1}\phi)^s$ , the proof of Lemma 9 shows that (writing, as before,  $T_\phi = (1 + \mathcal{D}^2)\phi(1 + \mathcal{D}^2)^{-1}\phi = (1 + \mathcal{D}^2)\eta_\phi$ )

$$\begin{aligned} B(s)\phi(1 + \mathcal{D}^2)^{-p/2} \\ &= -\frac{\sin(s\pi)}{\pi} \int_0^\infty \lambda^{-s} (1 + \lambda\eta_\phi)^{-1} (1 - T_\phi^*) (1 + \mathcal{D}^2 + \lambda)^{-1} \times \\ &\quad \times T\phi(1 + \mathcal{D}^2)^{-p/2} d\lambda \\ &= -\frac{\sin(s\pi)}{\pi} \int_0^\infty \lambda^{-s} (1 + \lambda\eta_\phi)^{-1} (1 - T_\phi^*) (1 + \mathcal{D}^2 + \lambda)^{-1} \psi \times \\ &\quad \times T\phi(1 + \mathcal{D}^2)^{-p/2} d\lambda \end{aligned}$$

$$= -\frac{\sin(s\pi)}{\pi} \int_0^\infty \lambda^{-s} (1 + \lambda\eta_\phi)^{-1} (1 - T_\phi^*) (1 + \mathcal{D}^2 + \lambda)^{-1} \times \\ \times \psi(1 + \mathcal{D}^2) (1 + \mathcal{D}^2)^{-1} T\phi (1 + \mathcal{D}^2)^{-p/2} d\lambda.$$

Since  $(1 + \mathcal{D}^2)^{-1} T\phi (1 + \mathcal{D}^2)^{-p/2}$  is trace class, the trace norm of  $B(s)\phi(1 + \mathcal{D}^2)^{-p/2}$  is bounded by

$$\|B(s)\phi(1 + \mathcal{D}^2)^{-p/2}\|_1 \\ \leq \frac{\sin(s\pi)}{\pi} \int_0^\infty \lambda^{-s} \|(1 + \mathcal{D}^2)^{-1} T\phi (1 + \mathcal{D}^2)^{-p/2}\|_1 \times \\ \times \|(1 + \lambda\eta_\phi)^{-1} (1 - T_\phi^*) (1 + \mathcal{D}^2 + \lambda)^{-1} \psi(1 + \mathcal{D}^2)\| d\lambda \\ \leq C \frac{\sin(s\pi)}{\pi} \int_0^\infty \lambda^{-s} \|(1 - T_\phi^*) (1 + \mathcal{D}^2 + \lambda)^{-1} \psi(1 + \mathcal{D}^2)\| d\lambda \\ \leq C \frac{\sin(s\pi)}{\pi} \int_0^\infty \lambda^{-s} (1 + \lambda)^{-1} d\lambda \\ = C,$$

the second last line following from the norm estimate of Lemma 8, and the last equality following from Equation (13). Thus

$$\|B(s)\phi(1 + \mathcal{D}^2)^{-p/2}\|_1 \leq C \quad \text{for all } 0 < s < 1.$$

Hence for  $p/2$  integral and  $0 < s < 1$

$$\text{Trace}(T(\phi(1 + \mathcal{D}^2)^{-1}\phi)^{s+p/2}) = \text{Trace}(T(1 + \mathcal{D}^2)^{-s-p/2}) + b(s),$$

where  $b(s)$  is bounded as  $s \rightarrow 0$ .

For  $p/2$  nonintegral,  $K_1 \notin \mathcal{L}^1(\mathcal{H})$ . Set  $r = p/2 - [p/2] > 0$  and consider  $s$  with  $0 < s < 1 - r$ . Then

$$\text{Trace}(T(\phi(1 + \mathcal{D}^2)^{-1}\phi)^{n+s+r}) \\ = \text{Trace}(T((1 + \mathcal{D}^2)^{-n} + K_1)(\phi(1 + \mathcal{D}^2)^{-1}\phi)^{s+r}) + C(s)$$

where  $C(s)$  arises from the trace of the terms  $K_j(\phi(1 + \mathcal{D}^2)^{-1}\phi)^{s+r}$ , and we observe that  $C(s)$  is bounded as  $s \rightarrow 0$ . Recalling that  $K_1 \in \mathcal{L}^{(p/(2n+1), \infty)}(\mathcal{H})$ , each product arising from expanding

$$\phi((1 + \mathcal{D}^2)^{-n} + K_1)((\phi(1 + \mathcal{D}^2)^{-1}\phi)^{s+r} - (1 + \mathcal{D}^2)^{-s-r} + (1 + \mathcal{D}^2)^{-s-r}),$$

is trace class. For  $K_1(1 + \mathcal{D}^2)^{-s-r}$  this follows from the same argument that showed  $K_1 \in \mathcal{L}^{(p/(2n+1), \infty)}(\mathcal{H})$ . Thus we have, using the cyclicity of the trace,

$$\text{Trace}(T(\phi(1 + \mathcal{D}^2)^{-1}\phi)^{n+s+r}) - C(s) \\ = \text{Trace}(T\phi((1 + \mathcal{D}^2)^{-n} + K_1)((\phi(1 + \mathcal{D}^2)^{-1}\phi)^{s+r} - (1 + \mathcal{D}^2)^{-s-r} + \\ + (1 + \mathcal{D}^2)^{-s-r})) \\ = \text{Trace}(\phi(1 + \mathcal{D}^2)^{-n} B(s+r) + \phi K_1 B(s+r) + \phi(1 + \mathcal{D}^2)^{-n-s-r} T + \\ + \phi K_1 (1 + \mathcal{D}^2)^{-s-r} T) \\ = \text{Trace}(B(s+r)\phi(1 + \mathcal{D}^2)^{-n} + B(s+r)\phi K_1 + \phi(1 + \mathcal{D}^2)^{-p/2-s} T + \\ + \phi K_1 (1 + \mathcal{D}^2)^{-s-r} T).$$

First observe that

$$\|\phi K_1(1 + \mathcal{D}^2)^{-s-r} T\|_1 \leq \|\phi K_1(1 + \mathcal{D}^2)^{-r}\|_1 \|(1 + \mathcal{D}^2)^{-s} T\|,$$

and this is obviously bounded as  $s \rightarrow 0$ . Next, familiar calculations show that both

$$B(r)\phi K_1 \quad \text{and} \quad B(r)\phi(1 + \mathcal{D}^2)^{-n}$$

are trace class. We show that

$$B(s+r)\phi K_1 \rightarrow B(r)\phi K_1 \quad \text{and} \quad B(s+r)\phi(1 + \mathcal{D}^2)^{-n} \rightarrow B(r)\phi(1 + \mathcal{D}^2)^{-n}$$

in the trace norm topology. Since  $(1 + \mathcal{D}^2)^{-1} T \phi K_1$  is trace class, we may use the computations of Lemma 9 and the norm estimate of Lemma 8 to obtain

$$\begin{aligned} & \| (B(s+r) - B(r))\phi K_1 \|_1 \\ &= \left\| \int_0^\infty \left( \frac{\lambda^{-s-r} \sin((s+r)\pi)}{\pi} - \frac{\lambda^{-r} \sin(r\pi)}{\pi} \right) \times \right. \\ & \quad \left. \times (1 + \lambda \eta_\phi)^{-1} (1 - T_\phi^*) (1 + \mathcal{D}^2 + \lambda)^{-1} T \phi K_1 \, d\lambda \right\|_1 \\ &\leq C \int_0^\infty \left| \frac{\lambda^{-s-r} \sin((s+r)\pi)}{\pi} - \frac{\lambda^{-r} \sin(r\pi)}{\pi} \right| \| (1 + \lambda \eta_\phi)^{-1} \times \\ & \quad \times (1 - T_\phi^*) (1 + \mathcal{D}^2 + \lambda)^{-1} \psi (1 + \mathcal{D}^2) (1 + \mathcal{D}^2)^{-1} T \phi K_1 \|_1 \, d\lambda \\ &\leq C \| (1 + \mathcal{D}^2)^{-1} T \phi K_1 \|_1 \int_0^\infty \frac{\sin(r\pi)}{\pi} \left| \cos(s\pi) \lambda^{-s-r} + \right. \\ & \quad \left. + \frac{\cos(r\pi) \sin(s\pi)}{\sin(r\pi)} \lambda^{-s-r} - \lambda^{-r} \right| \frac{1}{1 + \lambda} \, d\lambda \\ &\leq C \| (1 + \mathcal{D}^2)^{-1} T \phi K_1 \|_1 \int_0^\infty \frac{\sin(r\pi)}{\pi} O(s) \lambda^{-s-r} \frac{1}{1 + \lambda} \, d\lambda + \\ & \quad + C \| (1 + \mathcal{D}^2)^{-1} T \phi K_1 \|_1 \int_0^\infty \frac{\sin(r\pi)}{\pi} \left| \lambda^{-s-r} - \lambda^{-r} \right| \frac{1}{1 + \lambda} \, d\lambda \\ &= O(s) \| (1 + \mathcal{D}^2)^{-1} T \phi K_1 \|_1 \frac{\sin(r\pi)}{\sin((s+r)\pi)} + \\ & \quad + C \| (1 + \mathcal{D}^2)^{-1} T \phi K_1 \|_1 \left( \int_0^\infty (\lambda^{-r} - \lambda^{-s-r}) \frac{1}{1 + \lambda} \, d\lambda + \right. \\ & \quad \left. + 2 \int_0^1 (\lambda^{-s-r} - \lambda^{-r}) \frac{1}{1 + \lambda} \, d\lambda \right). \end{aligned}$$

In the last equality we have used Equation (13) for the first term, and written the integral of the absolute value as

$$\int_0^\infty |\lambda^{-s-r} - \lambda^{-r}| \frac{1}{1 + \lambda} \, d\lambda$$

$$\begin{aligned}
&= \int_1^\infty (\lambda^{-r} - \lambda^{-s-r}) \frac{1}{1+\lambda} d\lambda + \int_0^1 (\lambda^{-s-r} - \lambda^{-r}) \frac{1}{1+\lambda} d\lambda \\
&= \int_0^\infty (\lambda^{-r} - \lambda^{-s-r}) \frac{1}{1+\lambda} d\lambda + 2 \int_0^1 (\lambda^{-s-r} - \lambda^{-r}) \frac{1}{1+\lambda} d\lambda.
\end{aligned}$$

The first of these integrals is given by

$$\frac{\pi}{\sin(\pi r)} - \frac{\pi}{\sin((s+r)\pi)} \longrightarrow 0 \quad \text{as } s \rightarrow 0.$$

The second is given by [8, p. 211],

$$\begin{aligned}
2 \int_0^1 (\lambda^{-s-r} - \lambda^{-r}) \frac{1}{1+\lambda} d\lambda &= \frac{2s}{(1-r)(1-s-r)} - \frac{2s}{(2-r)(2-s-r)} + \\
&\quad + \frac{2s}{(3-r)(3-s-r)} - \dots
\end{aligned}$$

This too goes to zero as  $s \rightarrow 0$ . Putting these estimates together yields

$$B(s+r)\phi K_1 \longrightarrow B(r)\phi K_1 \quad \text{in } \mathcal{L}^1(\mathcal{H}).$$

An entirely analogous calculation shows that

$$\begin{aligned}
&\|(B(s+r) - B(r))\phi(1 + \mathcal{D}^2)^{-n}\|_1 \\
&\leq C \|(1 + \mathcal{D}^2)^{-1} T\phi(1 + \mathcal{D}^2)^{-n}\|_1 \\
&\quad \int_0^\infty \left| \frac{\lambda^{-s-r} \sin((s+r)\pi)}{\pi} - \frac{\lambda^{-r} \sin(r\pi)}{\pi} \right| \frac{1}{1+\lambda} d\lambda \longrightarrow 0.
\end{aligned}$$

Hence in all cases and for  $0 < s - p/2 < 1 - r$

$$\text{Trace}(T(\phi(1 + \mathcal{D}^2)^{-1}\phi)^s) = \text{Trace}(T(1 + \mathcal{D}^2)^{-s}) + b(s),$$

where  $b(s)$  is some function of  $s$  bounded as  $s \rightarrow p/2$ . By [2, Theorem 5.6], if the limit exists we have

$$\begin{aligned}
&\lim_{s \rightarrow p/2^+} \left( s - \frac{p}{2} \right) \text{Trace}(T(1 + \mathcal{D}^2)^{-s}) \\
&= \lim_{s \rightarrow p/2^+} \left( s - \frac{p}{2} \right) \text{Trace}(T(\phi(1 + \mathcal{D}^2)^{-1}\phi)^s) + \left( s - \frac{p}{2} \right) b(s) \\
&= \frac{p}{2} \int T(\phi(1 + \mathcal{D}^2)^{-1}\phi)^{p/2} \\
&= \frac{p}{2} \int T(1 + \mathcal{D}^2)^{-p/2}.
\end{aligned}$$

The final equality follows from an argument similar to those used throughout this proof and the proof of Proposition 10, to show that  $a(1 + \mathcal{D}^2)^{-p/2} = a(\phi(1 + \mathcal{D}^2)^{-1}\phi)^{p/2}$  modulo operators in  $\mathcal{L}^1(\mathcal{H})$ .  $\square$

### 5. $(p, \infty)$ -Summability for Complete Manifolds

We now show that  $(p, \infty)$ -summability holds for Euclidean spaces, and then lift this result to manifolds. This will show that our generalisations are reasonable, and clear up a persistent ‘folk area’ of noncommutative geometry. A similar result, for the Laplacian on Euclidean spaces, has been shown in [4], but to be able to employ our results we need to know that the Dirac operator (or Hodge-de Rham operator) satisfies  $(p, \infty)$ -summability in the sense of Definition 9.

**PROPOSITION 13.** *If  $f \in C_c^\infty(\mathbf{R}^p)$  then for  $\lambda \notin \mathbf{R}$   $f(\mathcal{D} - \lambda)^{-1} \in \mathcal{L}^{(p, \infty)}(L^2(\mathbf{R}^p, S))$ , where  $\mathcal{D}$  is the Dirac operator acting on sections of the spinor bundle  $S$ , and  $f$  acts as a multiplication operator on  $L^2(\mathbf{R}^p, S)$ .*

*Proof.* Our proof is an adaptation of the ideas in [4, pp. 16–17]. We first recall that if  $B \subseteq \mathbf{R}^p$  is a box, then the Dirac operator with periodic boundary conditions gives rise to a densely defined self-adjoint operator on  $L^2(T_B^p, S)$  which we denote by  $\mathcal{D}_B$ . Here  $T_B^p$  is the torus obtained by identifying the opposite faces of  $B$ , and  $S$  is the restriction of the spinor bundle on  $\mathbf{R}^p$  to  $T_B^p$ . This makes sense as  $S$  is a trivial bundle, and so the fibres over opposite faces can be canonically identified.

Now for  $p \neq 1$ , and writing  $\mu_n$  for the  $n$ th singular value we have

$$\begin{aligned} (\mathcal{D}_B - \lambda)^{-1} \in \mathcal{L}^{(p, \infty)}(L^2(T_B^p, S)) &\Leftrightarrow \mu_n(\mathcal{D}_B - \lambda)^{-1} = \mathcal{O}(n^{-1/p}) \\ &\Leftrightarrow \mu_n(1 + \mathcal{D}_B^2)^{-1/2} = \mathcal{O}(n^{-1/p}). \end{aligned}$$

For  $p = 1$  we have

$$\begin{aligned} (\mathcal{D}_B - \lambda)^{-1} \in \mathcal{L}^{(1, \infty)}(L^2(T_B^1, S)) &\Leftrightarrow \frac{1}{\log(N)} \sum_{n=1}^N \mu_n((\mathcal{D}_B - \lambda)^{-1}) \text{ bounded} \\ &\Leftrightarrow \mu_n(1 + \mathcal{D}_B^2)^{-1/2} = \mathcal{O}(n^{-1}). \end{aligned}$$

In all cases  $p \geq 1$  integral, it is well-known that  $\mu_n(1 + \mathcal{D}_B^2)^{-1/2} = \mathcal{O}(n^{-1/p})$  (this is a special case of Weyl’s Theorem [12, Lemma 1.12.6]). Thus  $(\mathcal{D}_B - \lambda)^{-1} \in \mathcal{L}^{(p, \infty)}(L^2(T_B^p, S))$ .

Next we observe that while multiplication by  $f \in C_c^\infty(\mathbf{R}^p)$  gives a bounded operator on  $L^2(\mathbf{R}^p, S)$ , we may also regard it as a bounded operator

$$f: L^2(\mathbf{R}^p, S) \longrightarrow L^2(T_B^p, S),$$

where  $\text{supp}(f)$  is contained in the interior of the box  $B$ . Moreover,  $f$  maps the domain of  $\mathcal{D}$ , the Dirac operator on  $\mathbf{R}^p$ , to the domain of  $\mathcal{D}_B$  described above. The function  $f$  also naturally defines a multiplication operator on  $L^2(T_B^p, S)$ . This allows us to compute in  $\mathcal{B}(L^2(\mathbf{R}^p, S), L^2(T_B^p, S))$ , for  $\lambda \notin \mathbf{R}$

$$\begin{aligned} f(\mathcal{D} - \lambda)^{-1} - (\mathcal{D}_B - \lambda)^{-1}f &= (\mathcal{D}_B - \lambda)^{-1}(\mathcal{D}_B f - f\mathcal{D})(\mathcal{D} - \lambda)^{-1} \\ &= (\mathcal{D}_B - \lambda)^{-1}(df \cdot)(\mathcal{D} - \lambda)^{-1}. \end{aligned}$$

Here, by a slight abuse, we have written  $df \cdot$  for Clifford multiplication [21], by  $df$  as a map from  $L^2(\mathbf{R}^p, S)$  to  $L^2(T_B^p, S)$ . Thus

$$f(\mathcal{D} - \lambda)^{-1} = (\mathcal{D}_B - \lambda)^{-1}(df \cdot)(\mathcal{D} - \lambda)^{-1} + (\mathcal{D}_B - \lambda)^{-1}f.$$

Composing this with the isometric inclusion  $\iota: L^2(T_B^p, S) \rightarrow L^2(\mathbf{R}^p, S)$  (i.e. by regarding  $f$  as a multiplication operator on  $L^2(\mathbf{R}^p, S)$ ) we get

$$\begin{aligned} & \|\iota f(\mathcal{D} - \lambda)^{-1}\|_{(p, \infty)}^{\mathbf{R}^p} \\ &= \left\| \iota(\mathcal{D}_B - \lambda)^{-1} \left( f + df \cdot (\mathcal{D} - \lambda)^{-1} \right) \right\|_{(p, \infty)}^{\mathbf{R}^p} \\ &\leq \|\iota(\mathcal{D}_B - \lambda)^{-1}f\|_{(p, \infty)}^{\mathbf{R}^p} + \|\iota(\mathcal{D}_B - \lambda)^{-1}df \cdot\|_{(p, \infty)}^{\mathbf{R}^p} \|(\mathcal{D} - \lambda)^{-1}\|_{\mathbf{R}^p} \end{aligned}$$

since  $(\mathcal{D} - \lambda)^{-1}$  is a bounded operator. Now as the inclusion map  $\iota$  is isometric, and the norms of  $f, df \cdot$  (as operators from  $L^2(\mathbf{R}^p, S)$  to  $L^2(B, S)$ ) are attained on (classes of) spinors with support in  $B$ , we have

$$\|\iota(\mathcal{D}_B - \lambda)^{-1}f\|_{(p, \infty)}^{\mathbf{R}^p} \leq \|(\mathcal{D}_B - \lambda)^{-1}\|_{(p, \infty)}^{T^p} \|f\|_{\mathbf{R}^p \rightarrow T^p},$$

and similarly for  $df \cdot$ . So

$$\begin{aligned} \|f(\mathcal{D} - \lambda)^{-1}\|_{(p, \infty)}^{\mathbf{R}^p} &\leq \|(\mathcal{D}_B - \lambda)^{-1}\|_{(p, \infty)}^{T^p} \|f\|_{\mathbf{R}^p \rightarrow T^p} + \\ &\quad + \|(\mathcal{D}_B - \lambda)^{-1}\|_{(p, \infty)}^{T^p} \|df \cdot\|_{\mathbf{R}^p \rightarrow T^p} \|(\mathcal{D} - \lambda)^{-1}\|_{\mathbf{R}^p} \\ &< \infty. \end{aligned}$$

Thus  $f(\mathcal{D} - \lambda)^{-1} \in \mathcal{L}^{(p, \infty)}(L^2(\mathbf{R}^p, S))$ .  $\square$

**COROLLARY 14.** *The tuple  $(C_c^\infty(\mathbf{R}^p), L^2(\mathbf{R}^p, S), \mathcal{D})$  is a smooth, local  $(p, \infty)$ -summable spectral triple. For  $f \in C_c^\infty(\mathbf{R}^p)$  acting by multiplication on spinors and with  $\mathcal{D}$  the Dirac operator acting on the (complex) spinor bundle  $S$ , we have  $f(1 + \mathcal{D}^2)^{-p/2} \in \mathcal{L}^{(1, \infty)}(L^2(\mathbf{R}^p, S))$ . Furthermore, it is measurable and*

$$\int f(1 + \mathcal{D}^2)^{-p/2} = \frac{2^{\lfloor p/2 \rfloor} \text{Vol}(S^{p-1})}{p(2\pi)^p} \int_{\mathbf{R}^p} f(x) d^p x.$$

*Proof.* That  $(C_c^\infty(\mathbf{R}^p), L^2(\mathbf{R}^p, S), \mathcal{D})$  is a smooth spectral triple follows from [23, Proposition 20] and the example of [23, pp. 18–19]. Locality follows from the fact that  $\mathcal{D}$  preserves supports (being a differential operator) so that if  $\phi$  is a local unit for a compactly supported function  $f$ ,  $\phi$  is also a local unit for  $[\mathcal{D}, f] = df \cdot$ . Finally the  $(p, \infty)$ -summability follows from Proposition 13.

We may now apply Proposition 10, which gives us for  $s > p$

$$\text{Trace}(f(1 + \mathcal{D}^2)^{-s/2}) < \infty.$$

Indeed

$$\text{Trace}(f(1 + \mathcal{D}^2)^{-s/2}) = \int_{\mathbf{R}^p} K_s(x, x) d^p x$$



where  $K_s(x, y) = 2^{\lfloor p/2 \rfloor} (2\pi)^{-p/2} f(x) g_s^F(x - y)$  is the kernel of  $\text{Trace}_S f (1 + \mathcal{D}^2)^{-s/2}$  [22, Theorem IX.29] and [25, Theorem 3.9]. Here  $\text{Trace}_S$  is the (matrix) trace of endomorphisms on the spinor bundle  $S$ ,  $g_s^F$  is the inverse Fourier transform of  $g_s = (1 + \|x\|^2)^{-s/2}$ , and the factor of  $2^{\lfloor p/2 \rfloor}$  arises from the rank of the spinor bundle. So

$$\begin{aligned} & \text{Trace}(f(x)(1 + \mathcal{D}^2)^{-s/2}) \\ &= 2^{\lfloor p/2 \rfloor} (2\pi)^{-p/2} \int f(x) g_s^F(0) d^p x \\ &= 2^{\lfloor p/2 \rfloor} (2\pi)^{-p} \int f(x) \left( \int g_s(\xi) d^p \xi \right) d^p x \\ &= \frac{2^{\lfloor p/2 \rfloor} \text{Vol}(S^{p-1})}{(2\pi)^p} \int f(x) d^p x \int_0^\infty (1 + r^2)^{-s/2} r^{p-1} dr \\ &= \frac{2^{\lfloor p/2 \rfloor} \text{Vol}(S^{p-1}) \Gamma(p/2) \Gamma((s-p)/2)}{2(2\pi)^p \Gamma(s/2)} \int f(x) d^p x, \end{aligned}$$

where the last line comes from evaluating the integral over  $r$  explicitly using the Laplace transform. It is now apparent that there is a simple pole at  $s = p$ , and we have

$$\begin{aligned} \lim_{s \rightarrow p} \left( \frac{s}{2} - \frac{p}{2} \right) \text{Trace}(f(1 + \mathcal{D}^2)^{-s/2}) &= \frac{2^{\lfloor p/2 \rfloor} \text{Vol}(S^{p-1})}{2(2\pi)^p} \int f(x) d^p x \\ &= \frac{p}{2} \int f(1 + \mathcal{D}^2)^{-p/2}, \end{aligned}$$

where the last line follows from Corollary 12.  $\square$

**PROPOSITION 15.** *Let  $X$  be a geodesically complete  $p$ -dimensional Riemannian spin manifold. Let  $f: X \rightarrow \mathbf{C}$  be a smooth compactly supported function and  $\mathcal{D}$  the Dirac operator on spinors. Then*

$$f(\mathcal{D} - \lambda)^{-1} \in \mathcal{L}^{(p, \infty)} \quad f(1 + \mathcal{D}^2)^{-p/2} \in \mathcal{L}^{(1, \infty)} \text{ is measurable}$$

$$\begin{aligned} \int f(1 + \mathcal{D}^2)^{-p/2} &= \text{WRes}(f(1 + \mathcal{D}^2)^{-p/2}) \\ &= \frac{2^{\lfloor p/2 \rfloor} \text{Vol}(S^{p-1})}{p(2\pi)^p} \int_X f(x) d^p x =: c(p) \int_X f(x) d^p x. \end{aligned}$$

*Proof.* The proof is very similar to the  $\mathbf{R}^p$  case, but as we are no longer dealing with constant coefficient differential operators, we do not have recourse to periodic boundary conditions and can not reduce the problem to the case of a torus.

Choose coordinate charts  $U^i$  which are contractible (so the spinor bundle is trivial) with compact closure, and coordinates which map

$$x_i: U^i \xrightarrow{\cong} V^i \subset \mathbf{R}^p,$$

where  $V^i$  has compact closure. Using a partition of unity we can reduce the general case of  $f: X \rightarrow \mathbf{C}$  compactly supported to the case of  $f$  with support contained in the interior of a single chart  $U^i$ , and so reduce to the case of  $\mathbf{R}^p$ . The only real difference is we are now dealing with an operator on  $V^i \subset \mathbf{R}^p$  with nonconstant coefficients.

Let  $W \subset V^i$  be a closed set with the support of  $f$  contained in the interior of  $W$  and such that  $W$  has smooth boundary  $\partial W$ . To prove the first statement, it suffices to show that  $f(\mathcal{D} - \lambda)^{-1} \in \mathcal{L}^{(p,\infty)}(L^2(W, S))$ , where  $\mathcal{D}$  is of the form

$$\sum_{i=1}^p \gamma^i \partial_i + \omega,$$

where for  $1 \leq i \leq p$ ,  $\gamma^i$  are nonconstant Clifford variables on  $W$ , the  $\partial_i = \partial/\partial x^i$  are partial derivative operators, and  $\omega$  is the connection form on  $W$ .

Let  $\mathcal{D}_I$  be the invertible double of  $\mathcal{D}$  defined on the closed manifold  $\tilde{W} = W \cup_{\partial W} (-W)$  [1, Chapter 9], with spinor bundle  $\tilde{S}$ . Then by Weyl's Theorem [12, Lemma 1.12.6], for  $\lambda \notin \mathbf{R}$ ,  $(\mathcal{D}_I - \lambda)^{-1} \in \mathcal{L}^{(p,\infty)}(L^2(\tilde{W}, \tilde{S}))$ . We extend  $f$  to a function on all of  $\tilde{W}$  by extending by zero.

We now proceed as in the case of  $\mathbf{R}^p$ , and compute in the space of bounded operators from  $L^2(\tilde{W}, \tilde{S})$  to  $L^2(W, S)$ , regarding multiplication by  $f$  as such an operator,

$$f(\mathcal{D}_I - \lambda)^{-1} - (\mathcal{D} - \lambda)^{-1}f = (\mathcal{D} - \lambda)^{-1}(f\mathcal{D}_I - \mathcal{D}f)(\mathcal{D}_I - \lambda)^{-1}.$$

So letting  $\iota: L^2(W, S) \rightarrow L^2(\tilde{W}, \tilde{S})$  be the isometric inclusion and writing  $df \cdot$  for Clifford multiplication by  $df$  from the spinor bundle on  $\tilde{W}$  to the spinor bundle on  $W$ , we have

$$\begin{aligned} \|(\mathcal{D} - \lambda)^{-1}f\|_{(p,\infty)}^W &\leq \|df \cdot ((\mathcal{D}_I - \lambda)^{-1}\iota)\|_{(p,\infty)}^W \|(\mathcal{D} - \lambda)^{-1}\|_W + \\ &\quad + \|f(\mathcal{D}_I - \lambda)^{-1}\iota\|_{(p,\infty)}^W, \end{aligned}$$

similarly to the  $\mathbf{R}^p$  case. The proof now follows as in the  $\mathbf{R}^p$  case by noting that

$$\|f(\mathcal{D}_I - \lambda)^{-1}\iota\|_{(p,\infty)}^W \leq \|(\mathcal{D}_I - \lambda)^{-1}\|_{(p,\infty)}^{\tilde{W}} \|f\|_{\tilde{W} \rightarrow W}.$$

The measurability of  $f(1 + \mathcal{D}^2)^{-p/2}$  follows because Weyl's Theorem gives an estimate on the singular values  $\mu_n$  of  $(\mathcal{D}_I - \lambda)^{-1}$  of the form [12, Lemma 1.12.6],

$$\mu_n = Cn^{-1/p} + o(n^{-1/p}),$$

for some positive constant  $C$ . The equality with the Wodzicki residue is from [9], and this provides the value of the Dixmier trace of  $f(1 + \mathcal{D}^2)^{-p/2}$ . Completeness of the manifold is only used here to ensure that the Dirac operator is essentially self-adjoint, to justify the use of the functional calculus. Weaker conditions exist to ensure the self-adjointness of  $\mathcal{D}$  [16, p. 274].  $\square$

**COROLLARY 16.** *If  $X$  is a geodesically complete Riemannian spin manifold, then the triple  $(C_c^\infty(X), L^2(X, S), \mathcal{D})$  is a smooth, local  $(p, \infty)$  summable spectral triple. Here  $S$  is the (complex) spinor bundle on  $X$  and  $\mathcal{D}$  is the Dirac operator.*

*Proof.* As for  $\mathbf{R}^p$ , the fact that  $(C_c^\infty(X), L^2(X, S), \mathcal{D})$  is a smooth spectral triple follows from [23, Proposition 20], and locality follows because  $\mathcal{D}$  is a differential operator. The  $(p, \infty)$ -summability comes from Proposition 15.  $\square$

## 6. Distributions and the Local Index Theorem

Having shown that the Dixmier trace remains an effective tool in the nonunital case, we turn to other functionals on  $\mathcal{A}_c$  defined using the Dixmier trace as well as the usual trace.

We follow the lead of ordinary distribution theory [22]. If  $X$  is a paracompact manifold, the distributions on  $X$  are defined to be the continuous linear forms on  $C_c^\infty(X)$ , where the compactly supported smooth functions carry the inductive limit topology (defined by any increasing sequence of compact sets whose union is  $X$ ).

**DEFINITION 10.** Let  $(\mathcal{A}, \mathcal{H}, \mathcal{D})$  be a smooth spectral triple with  $\mathcal{A}$  complete in the  $\delta$ -topology (c.f. Lemma 3). The distributions on  $(\mathcal{A}, \mathcal{H}, \mathcal{D})$  are the continuous linear functionals on the topological algebra  $\mathcal{A}_c$ , where  $\mathcal{A}_c$  carries its natural inductive limit topology obtained from the smooth topology on  $\mathcal{A}$  and any local approximate unit. Denote the linear space of distributions by  $\mathcal{A}'_c$ .

**LEMMA 17.** *If  $(\mathcal{A}, \mathcal{H}, \mathcal{D})$  is a smooth, local  $(p, \infty)$ -summable spectral triple with  $\mathcal{A}$  complete in the  $\delta$ -topology, then the map*

$$a \rightarrow \mathrm{Tr}_\omega(a(1 + \mathcal{D}^2)^{-p/2})$$

*is a distribution, for any Dixmier trace  $\mathrm{Tr}_\omega$ .*

*Proof.* Since  $\mathcal{A}_c$  is a strict inductive limit, the Dixmier trace is continuous on  $\mathcal{A}_c$  if and only if it is continuous when restricted to each of the subalgebras  $\mathcal{A}_n = \{a \in \mathcal{A}_c: \phi_n a = a \phi_n = a\}$ , for any local approximate unit  $\{\phi_n\}$  [20, 22]. So if the sequence  $a_i \rightarrow a \in \mathcal{A}_n$ , then we have

$$\begin{aligned} & |\mathrm{Tr}_\omega((a - a_i)(1 + \mathcal{D}^2)^{-p/2})| \\ &= |\mathrm{Tr}_\omega((a - a_i)\phi(1 + \mathcal{D}^2)^{-p/2})| \\ &\leq \|a - a_i\| |\mathrm{Tr}_\omega(\phi(1 + \mathcal{D}^2)^{-p/2})| \rightarrow 0. \end{aligned} \tag{15}$$

Since convergence in the  $\delta$ -topology implies convergence in norm, we are done.  $\square$

Now suppose that  $\mathcal{A} \hookrightarrow \mathcal{A}_b$  is an embedding of  $\mathcal{A}$  as a closed essential ideal in the smooth unital algebra  $\mathcal{A}_b$ ; i.e. a unitization. Then [23],  $\mathcal{A}_c$  is also an essential ideal in  $\mathcal{A}_b$ , and

COROLLARY 18. *If  $(\mathcal{A}, \mathcal{H}, \mathcal{D})$  is a smooth, local  $(p, \infty)$ -summable spectral triple with  $\mathcal{A}$  complete in the  $\delta$ -topology, the map  $F: \mathcal{A}_b \hookrightarrow \mathcal{A}'_c$  is a continuous linear injection, where*

$$F(a)(b) = \text{Tr}_\omega(ab(1 + \mathcal{D}^2)^{-p/2}), \quad a \in \mathcal{A}_b, \quad b \in \mathcal{A}_c.$$

*Proof.* That  $F$  is an injection follows from  $\mathcal{A}_c$  being an essential ideal in  $\mathcal{A}_b$ . Continuity follows from the continuity of the multiplication in  $\mathcal{A}_b$  and Lemma 17.  $\square$

COROLLARY 19. *If  $(\mathcal{A}, \mathcal{H}, \mathcal{D})$  is a smooth, local  $(p, \infty)$ -summable spectral triple with  $\mathcal{A}$  complete in the  $\delta$ -topology, the map  $F_T: \mathcal{A}_b \hookrightarrow \mathcal{A}'_c$  is a continuous linear injection, where*

$$F_T(a)(b) = \text{Trace}(abe^{-t\mathcal{D}^2}), \quad a \in \mathcal{A}_b, \quad b \in \mathcal{A}_c.$$

*Similarly, if  $\text{Re}(z) > p$  then*

$$\zeta_a(b) = \text{Trace}(ab(1 + \mathcal{D}^2)^{-z/2}), \quad a \in \mathcal{A}_b, \quad b \in \mathcal{A}_c,$$

*is a distribution. For fixed  $a, b$ , the former is  $C^\infty$  for all  $t > 0$  and the latter is holomorphic for all  $z$  with  $\text{Re}(z) > p$ .*

*Proof.* We must first show that finitely summable spectral triples are  $\theta$ -summable. This is almost the same as the unital version [13], namely for all  $a \in \mathcal{A}_c$ ,  $t > 0$  and  $s > p$

$$\begin{aligned} |\text{Trace}(ae^{-t(1+\mathcal{D}^2)})| &= |\text{Trace}(a(1 + \mathcal{D}^2)^{-s/2}(1 + \mathcal{D}^2)^{s/2}e^{-t(1+\mathcal{D}^2)})| \\ &\leq \|(1 + \mathcal{D}^2)^{s/2}e^{-t(1+\mathcal{D}^2)}\| |\text{Trace}(a(1 + \mathcal{D}^2)^{-s/2})|. \end{aligned}$$

The result now follows by using the functional calculus to show that for fixed  $t, s$  as above,  $(1 + \mathcal{D}^2)^{s/2}e^{-t(1+\mathcal{D}^2)}$  is bounded. Both functionals  $F_T(a)$  and  $\zeta_a$  are continuous on  $\mathcal{A}_c$  by an argument similar to that in Lemma 17.

So we are left with the smoothness and continuity. So for fixed  $a \in \mathcal{A}_b$  and  $b \in \mathcal{A}_c$ , we want to show that  $\text{Trace}(abe^{-t\mathcal{D}^2})$  is a smooth function of  $t > 0$ . Observe that for fixed  $t > 0$ ,  $b \in \mathcal{A}_c$ , and any bounded operator  $A$ ,  $Abe^{-t\mathcal{D}^2/2} \in \mathcal{L}^1(\mathcal{H})$ . Moreover, the function

$$B \longrightarrow \text{Trace}(Abe^{-t\mathcal{D}^2}B), \quad B \in \mathcal{B}(\mathcal{H}),$$

is continuous in the strong operator topology. The final ingredient is to note that for all  $\xi \in \text{dom}\mathcal{D}^2$ ,

$$\lim_{\epsilon \rightarrow 0} e^{-t\mathcal{D}^2/2} \left( \frac{e^{-\epsilon\mathcal{D}^2} - Id}{\epsilon} \right) \xi = -e^{-t\mathcal{D}^2/2} \mathcal{D}^2 \xi,$$

by the definition of the generator of a contraction semigroup. Moreover  $\|\mathcal{D}^2 e^{-t\mathcal{D}^2/2}\| < \infty$ , by the functional calculus. So for all  $t > 0$  the function

$$\epsilon \longrightarrow e^{-t\mathcal{D}^2/2} \left( \frac{e^{-\epsilon\mathcal{D}^2} - Id}{\epsilon} \right)$$

is a strongly continuous function to the bounded operators. Hence

$$\begin{aligned} & \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \text{Trace}(Abe^{-(t+\epsilon)\mathcal{D}^2} - Abe^{-t\mathcal{D}^2}) \\ &= \text{Trace} \left( Abe^{-t\mathcal{D}^2/2} \lim_{\epsilon \rightarrow 0} \left( e^{-t\mathcal{D}^2/2} \frac{e^{-\epsilon\mathcal{D}^2} - Id}{\epsilon} \right) \right) \\ &= \text{Trace}(Abe^{-t\mathcal{D}^2/2} (-e^{-t\mathcal{D}^2/2} \mathcal{D}^2)) \\ &= -\text{Trace}(Ab\mathcal{D}^2 e^{-t\mathcal{D}^2}). \end{aligned}$$

The higher derivatives may be shown to exist, and computed, in the same way, using the fact that  $Ab\mathcal{D}^{2n} e^{-t\mathcal{D}^2/2} \in \mathcal{L}^1(\mathcal{H})$  for all  $n \geq 0$ .

To show that  $\text{Trace}(ab(1 + \mathcal{D}^2)^{-z/2})$  is holomorphic for  $\text{Re}(z) > p$ , suppose both  $z$  and  $z_0$  have real part greater than  $p$ , and suppose that  $\text{Re}(z_0) > \text{Re}(z)$ . Then, since  $(1 + \mathcal{D}^2)^{-1/2}$  is bounded, positive, and has trivial kernel

$$\begin{aligned} & \lim_{z_0 \rightarrow z} \frac{1}{z_0 - z} (\text{Trace}(ab(1 + \mathcal{D}^2)^{-z_0/2}) - \text{Trace}(ab(1 + \mathcal{D}^2)^{-z/2})) \\ &= \lim_{z_0 \rightarrow z} \frac{1}{z_0 - z} \text{Trace}(ab(1 + \mathcal{D}^2)^{-z/2} (e^{(z_0 - z) \log((1 + \mathcal{D}^2)^{-1/2})} - 1)) \\ &= \text{Trace}(ab(1 + \mathcal{D}^2)^{-z/2} \log((1 + \mathcal{D}^2)^{-1/2})). \end{aligned}$$

As  $\log((1 + \mathcal{D}^2)^{-1/2})$  is a bounded operator, the result follows.  $\square$

The distributions we are most interested in, which include those above, are those defined using pseudodifferential operators in the sense of Connes-Moscovici [17].

**DEFINITION 11.** Let  $(\mathcal{A}, \mathcal{H}, \mathcal{D})$  be a smooth spectral triple, and suppose that  $\mathcal{A} \hookrightarrow \mathcal{A}_b$  is a smooth unitization in the sense above. Let  $\mathcal{B}(\mathcal{A}_b)$  be the algebra of polynomials in the operators  $\delta^n([\mathcal{D}, a])$  and  $\delta^n(a)$ , for all  $a \in \mathcal{A}_b$  and  $n \geq 0$ . Also, define the algebra of pseudodifferential operators  $\Psi^*(\mathcal{A})$  to be the algebra of operators possessing an expansion

$$P \sim b_q(1 + \mathcal{D}^2)^{q/2} + b_{q-1}(1 + \mathcal{D}^2)^{q/2-1/2} + \dots, \quad b_q \in \mathcal{B}(\mathcal{A}_b),$$

where the tilde indicates that

$$P - \sum_{-N < n \leq q} b_n(1 + \mathcal{D}^2)^{n/2} \in \text{OP}^{-N}$$

and

$$P \in \text{OP}^\alpha \Leftrightarrow (1 + \mathcal{D}^2)^{-\alpha/2} P \in \bigcap_{m \geq 1} \text{dom } \delta^m.$$

In this definition we should use  $\delta(x) = [(1 + \mathcal{D}^2)^{1/2}, x]$ , and this derivation has the same domain as  $[|\mathcal{D}|, \cdot]$  since the functional calculus shows that  $|\mathcal{D}| - (1 + \mathcal{D}^2)^{1/2}$  is bounded. To see that  $\Psi^*(\mathcal{A})$  is indeed an algebra, we recall that there is an expansion [7],

$$(1 + \mathcal{D}^2)^{\alpha/2} b \sim \sum_{k=0}^{\infty} c_{\alpha,k} \delta^k(b) (1 + \mathcal{D}^2)^{\alpha/2-k/2},$$

where  $c_{\alpha,k}$  is the coefficient of  $\epsilon^k$  in

$$(1 + \epsilon)^\alpha = \sum_0^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!} \epsilon^k.$$

That this is true for nonintegral  $\alpha$  follows from results in [6, 7]. We will accept  $\Psi^*(\mathcal{A})$  as the algebra of pseudodifferential operators on  $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ , based largely on the ability to employ asymptotic expansions within  $\Psi^*(\mathcal{A})$ . Further justification and information can be found in [6, 7, 13]. The next result follows easily from what we have shown thus far.

**PROPOSITION 20.** *If  $P \in \Psi^*(\mathcal{A})$  is a pseudodifferential operator on the smooth  $(p, \infty)$ -summable spectral triple  $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ , then*

$$a \rightarrow \text{Trace} \left( a P (1 + \mathcal{D}^2)^{-\frac{z+\text{deg}(P)}{2}} \right), \quad \text{Re}(z) > p, \quad a \in \mathcal{A}_c$$

*is a distribution.*

Distributions of this form appear in the statement and proof of the Local Index Theorem. Connes and Moscovici's Local Index Theorem allows one to compute (the distributional form of) the Chern character in cyclic cohomology. The tools developed in the previous two sections allow one to extend all the necessary ingredients to the nonunital case, by defining and computing the cyclic cocycles on the 'compactly supported' elements of our local algebra.

In order to render the computations tractable, one requires additional information. In particular, the asymptotic expansions for pseudodifferential operators are only effective in this regard when we can remove all but finitely many terms. The assumption employed by Connes-Moscovici (in the unital case), see [7, Definition II.1] to do this is that the functions

$$\zeta_b(z) = \text{Trace}(b(1 + \mathcal{D}^2)^{-z/2}), \quad \text{Re}(z) > p, \quad b \in \mathcal{B}(\mathcal{A})$$

have meromorphic extensions as functions of  $z$ . The poles of all of these extensions must have uniformly bounded order, and all lie in the same discrete set  $Sd$ . This 'discrete and finite dimension spectrum' hypothesis [7], extends to local algebras

simply by replacing the hypothesis  $b \in \mathcal{B}(\mathcal{A})$  by  $b \in \mathcal{B}(\mathcal{A}_c)$ , where  $\mathcal{B}(\mathcal{A}_c)$  is defined analogously to  $\mathcal{B}(\mathcal{A}_b)$  in Definition 11. Alternative proofs of the (unital) Local Index Theorem appear in [3, 15].

For local spectral triples, all the estimates necessary to prove the Local Index Theorem, and show that the JLO cocycle is entire, are essentially the same as in [11, 7], with minor adjustments employing local units. The only serious adjustment necessary is that one must consider the continuous cyclic cohomology of  $\mathcal{A}_c$ , where  $\mathcal{A}_c$  carries the inductive limit topology. There is no loss of generality in this as far as index theory is concerned, because  $\mathcal{A}_c$  is stable under the holomorphic functional calculus [23], so  $K_*(\mathcal{A}_c) \cong K_*(\mathcal{A})$ ; see [23]. We also note that the algebra  $\mathcal{A}_c$  is H-unital because it has local units [23, 19].

With these minor modifications, the Local Index Theorem can be used to compute index pairings. A class of examples (with simple dimension spectrum) and a detailed computation using this extended Local Index Theorem appears in [24].

## 7. Conclusion

The main result of this paper is that the various summability hypotheses in non-commutative geometry can be extended to the nonunital case in the context of local algebras. This allows us to show that most of the ‘summability type’ results in the unital case (trace on the algebra of a spectral triple, relation between measurability and zeta functions, Local Index Theorem) all have reasonable analogues in the nonunital case. The restriction that the ‘Clifford algebra’,  $\Omega_{\mathcal{D}}^*(\mathcal{A}_c)$ , has local units in  $\mathcal{A}_c$  is an extra locality requirement that we have found necessary to impose throughout.

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