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Light reflection is nonlinear optimization

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Abstract

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Keywords

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LIGHT REFLECTION IS NONLINEAR OPTIMIZATION

JIAKUN LIU

Abstract. In this paper, we show that the near field reflector problem is a nonlinear optimization problem. From the corresponding functional and constraint function, we derive the Monge-Ampère type equation for such a problem.

1. Introduction

Optimal transportation, due to its various applications, has been extensively studied in recent years. The modern theory of optimal transportation is mainly built upon Kantorovich's dual functional, which is a linear functional subject to a linear constraint. With his dual functional, Kantorovich introduced linear programming, which is a class of linear optimization problems. An important new application is the reflector design problem. In [14], Xu-Jia Wang showed that the far field case of the reflector design problem is an optimal transportation problem, and so is a linear optimization problem. The purpose of this paper is to show that the general case of the reflector problem is a nonlinear optimization problem. More examples of nonlinear optimization problems and also questions of the existence and regularity of potential functions and optimal mappings will be investigated in [8] and subsequent papers.

Suppose that a point source of light is centered at the origin O and for each $X \in \Omega \subset \mathbb{S}^n$ we issue a ray from O passing through X, which after reflection by a surface Γ will illuminate a point Y on the target surface Ω^* in $\mathbb{R}^{n+1}$. Let $f \in L^1(\Omega), g \in L^1(\Omega^*)$ be the input and gain densities, and $d\mu, d\nu$ denote the surface area elements of Ω, Ω^*, respectively. The near field reflector problem can be formulated as follows: given (Ω, f) and (Ω^*, g) satisfying the energy conservation condition

$$\int_\Omega f d\mu = \int_{\Omega^*} g d\nu, \tag{1.1}$$

find a reflector Γ such that the light emitting from Ω with density f is reflected off Γ to the target Ω^* and the density of reflected light is equal to g.

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In our reflector problem, we assume that both Ω and Ω^* are compact and each has boundary of measure zero. Represent the reflector Γ as a radial graph of function ρ,

$$\Gamma = \{X\rho(X) \,:\, X \in \Omega\}. \tag{1.2}$$

Let $\mathcal{P}(\mu,\nu)$ be the set of measures on $\Omega\times\Omega^*$ with μ,ν as their marginals. Let $\gamma\in\mathcal{P}(\mu,\nu)$. Denote by $C_+(\Omega)$ the set of positive continuous functions on Ω. Define a functional

$$I(u,v) = \int_{\Omega\times\Omega^*} F(X,Y,u,v)d\gamma, \tag{1.3}$$

for $(u,v)\in C_+(\Omega)\times C_+(\Omega^*)$, where

$$F(X,Y,u,v) = f(X)\log u + g(Y)\left(\log v + \log(1-\frac{\langle X,Y\rangle}{v^{-1}+\sqrt{|Y|^2+v^{-2}}})\right) \tag{1.4}$$

and $\langle\ ,\ \rangle$ is the inner product in $\mathbb{R}^{n+1}$. The main result is the following:

Theorem 1.1. *Suppose that f,g are two bounded positive functions on Ω,Ω^*, respectively, such that* (1.1) *is satisfied. Suppose that Ω^* is contained in the cone $\mathcal{C}_V=\{tX \,:\, t>0, X\in V\}$ for a domain $V\subset\mathbb{S}^n$ and*

$$\overline{\Omega}\cap\overline{V}=\emptyset, \tag{1.5}$$

where $\overline{\Omega}$ and $\overline{V}$ denote the closures of Ω and V, respectively. Then there is a dual maximizing pair $(\rho,\eta)\in\mathcal{K}$, which satisfies

$$I(\rho,\eta) = \sup_{(u,v)\in\mathcal{K}} I(u,v),$$

where $I(u,v)$ is given in (1.3)–(1.4)*, and the constraint set $\mathcal{K}$ is given by*

$$\mathcal{K} = \{(u,v)\in C_+(\Omega)\times C_+(\Omega^*) \,:\, \phi(X,Y,u,v)\le 0\},$$

with the constraint function

$$\phi(X,Y,u,v) = \log u + \log v + \log\left(1-\frac{\langle X,Y\rangle}{v^{-1}+\sqrt{|Y|^2+v^{-2}}}\right). \tag{1.6}$$

Moreover, ρ is a solution of the reflector problem with given densities (Ω,f) and (Ω^*,g).

In Theorem 1.1, the functions ρ,η are also called potential functions, and a solution of the reflector problem needs to be understood as a weak solution. The notion of weak solutions was introduced in [6, 7], see §2.2 below. It follows from Remark 2.1 that for each choice of the parameter $c_0>0$, there is a weak solution ρ satisfying $\inf_\Omega \rho\ge c_0$.

Moreover, we show that the function ρ solves a Monge-Ampère type equation. Assume that Ω^* is given implicitly by

$$\Omega^* = \{Z\in\mathbb{R}^{n+1} \,:\, \psi(Z)=0\}. \tag{1.7}$$

Suppose that Ω is a subset of upper unit sphere $\mathbb{S}^n_+ = \mathbb{S}^n \cap \{x_{n+1} > 0\}$. Let $X = (x, x_{n+1})$ be a parameterization of Ω, where $x_{n+1} = \sqrt{1-|x|^2} =: \omega(x)$, and $x = (x_1, \cdots, x_n)$. For simplification, we define some auxiliary functions

$$a = |D\rho|^2 - (\rho + D\rho \cdot x)^2, \tag{1.8}$$

$$b = |D\rho|^2 + \rho^2 - (D\rho \cdot x)^2, \tag{1.9}$$

$$t = \frac{\rho x_{n+1} - y_{n+1}}{\rho x_{n+1}}, \quad \beta = \frac{t}{(Y - X\rho)\cdot \nabla\psi}, \tag{1.10}$$

and denote the matrix

$$\mathcal{N} = \{\mathcal{N}_{ij}\}, \quad \mathcal{N}_{ij} = \delta_{ij} + \frac{x_i x_j}{1-|x|^2}. \tag{1.11}$$

By computing in this local orthonormal frame, we obtain our equation as follows

Theorem 1.2. *The function ρ is a solution of*

$$\left| det\left[D^2\rho - \frac{2}{\rho} D\rho \otimes D\rho - \frac{a(1-t)}{2t\rho}\mathcal{N}\right]\right| = \left|\frac{a^{n+1}}{t^n b\beta}\right| \frac{f}{2^n\rho^{2n+1}\omega^2 g|\nabla\psi|}. \tag{1.12}$$

The equation (1.12) was previously obtained by Karakhanyan and Wang studying the near field reflector problem [6]. One of the main differences in our derivation of (1.12) is that instead of applying the reflection law as in [6], we have differentiated the constraint function (1.6) directly in general cases, see (3.4) below. We remark that our method is more general and can be applied to the study of other reflector and refractor problems, [5, 8].

This paper is arranged as follows. In Section 2, we first introduce a class of nonlinear optimization with potential functions, and then prove Theorem 1.1. In Section 3, we derive the equation for potentials arising in general nonlinear optimization problems, and apply this formula to prove Theorem 1.2. In Remark 3.1, we point out that the far field reflector is a limit case of the near field one and related to a linear optimization problem.

2. Formulation to optimization

2.1. Nonlinear optimization.

In general, we consider a functional

$$I(u,v) = \int_{U\times V} F(x,y,u,v)\,d\gamma, \tag{2.1}$$

for $(u,v) \in C(U)\times C(V)$, where U, V are two compact domains in $\mathbb{R}^n$ or a manifold $\mathcal{M}^n$, F is a function on $U \times V \times \mathbb{R}^2$, and the measure $d\gamma$ has marginals dx, dy, which are the volume elements of U, V, respectively.

We want to maximize the functional I among all pairs (u,v) in a constraint set

$$\mathcal{K} = \{(u,v) \in C(U)\times C(V) \,:\, \phi(x,y,u,v) \le 0 \text{ in } U\times V\}, \tag{2.2}$$

where ϕ is the constraint function defined on $U \times V \times \mathbb{R}^2$.

Note that when

$$F(x,y,u,v)=\frac{1}{|V|}f(x)u(x)+\frac{1}{|U|}g(y)v(y),\quad d\gamma=dxdy, \tag{2.3}$$

for some $f>0,\in L^1(U)$, $g>0,\in L^1(V)$ satisfying $\int_U f=\int_V g$, and

$$\phi(x,y,u,v)=u(x)+v(y)-c(x,y), \tag{2.4}$$

we have a linear optimization problem related to an optimal transportation with the cost function c, and mass densities f,g supported on U,V, respectively. See [1, 2, 11, 12].

Definition 2.1. *A pair $(u,v)\in\mathcal{K}$ is called* dual pair *with respect to ϕ, if*

$$\begin{aligned} u(x) &= \sup\{t : \phi(x,y,t,v(y))\le 0,\quad \forall y\in V\},\\ v(y) &= \sup\{s : \phi(x,y,u(x),s)\le 0,\quad \forall x\in U\}. \end{aligned} \tag{2.5}$$

If furthermore $I(u,v)=\sup_{\mathcal{K}} I$, (u,v) is called dual maximizing pair *of I. In such a case, u,v are also called* potential functions *in the nonlinear optimization* (2.1)–(2.2)*.*

Write $F=F(x,y,t,s)$ and $\phi=\phi(x,y,t,s)$, where x,y,t,s are independent variables. Use the subscripts to denote the partial derivatives, i.e. $F_t=\partial F/\partial t$, $\phi_s=\partial\phi/\partial s$, etc. We always assume that F is C^1 smooth in t,s and integrable in x,y; ϕ is C^2 smooth in all variables. Moreover, we assume the following conditions on F and ϕ:

(i) $F(x,y,t,s)$ is monotone increasing in t,s, namely

$$F_t\ge 0,\quad F_s\ge 0,\qquad \forall(x,y,t,s)\in U\times V\times\mathbb{R}\times\mathbb{R}. \tag{2.6}$$

(ii) $\phi(x,y,t,s)$ is strictly increasing in t,s, namely for a constant $\delta_0>0$

$$\phi_t\ge\delta_0,\quad \phi_s\ge\delta_0,\qquad \forall(x,y,t,s)\in U\times V\times\mathbb{R}\times\mathbb{R}. \tag{2.7}$$

(iii) for any pair $(u,v)\in\mathcal{K}$, the balance condition holds:

$$\int_{U\times V}\left\{-F_t(x,y,u(x),v(y))+F_s\frac{\phi_t}{\phi_s}(x,y,u(x),v(y))\right\}d\gamma=0. \tag{2.8}$$

Lemma 2.1. *Under the above assumptions and* (2.6)–(2.8)*, $I(u,v)$ in* (2.1) *has a dual maximizing pair $(\bar u,\bar v)\in\mathcal{K}$, where $\mathcal{K}$ is the constraint set given in* (2.2)*.*

Proof. The proof is inspired by [1, 2]. Given any pair $(u,v)\in\mathcal{K}$, we claim that $I(u,v)$ does not decrease if v is replaced by

$$v^*(y)=\sup\{s : \phi(x,y,u(x),s)\le 0,\quad \forall x\in U\}. \tag{2.9}$$

In fact, by the continuity of ϕ and u, for each $y\in V$ there is some $x\in\overline{U}$ such that

$$\phi(x,y,u(x),v^*(y))=0\ge\phi(x,y,u(x),v(y)),$$

since $(u,v)\in\mathcal{K}$. By (2.7), $v^*\ge v$. Furthermore, $\phi(x,y,u(x),v^*(y))\le 0$ for all $(x,y)\in U\times V$, so $(u,v^*)\in\mathcal{K}$.

Since $v^* \geq v$, by (2.6) we have

$$I(u, v^*) \geq I(u, v).$$

Similarly, if we define

$$u^*(x) = \sup\{t \,:\, \phi(x, y, t, v^*(y)) \leq 0, \quad \forall y \in V\}, \tag{2.10}$$

then $(u^*, v^*) \in \mathcal{K}$ and

$$I(u^*, v^*) \geq I(u, v^*) \geq I(u, v).$$

Thus we do not decrease $I(u, v)$ by replacing (u, v) by (u^*, v^*). The claim is proved.

Define $\mathcal{K}_{C_0} = \mathcal{K} \cap \{u \geq C_0\}$, where C_0 is a constant. The constant C_0 may be chosen negative and sufficiently small in the following context. We show that u^* and v^* are uniformly bounded if $(u, v) \in \mathcal{K}_{C_0}$. Since $v^* \geq v, u \geq C_0$, by (2.7) we have for each $y \in V$, $s := v^*(y)$,

$$\phi(x, y, C_0, s) \leq \phi(x, y, u(x), s) \leq 0, \qquad \text{for all } x \in U.$$

Then by (2.7) again, there exists a constant C_1 such that $s \leq C_1$. This implies that

$$v \leq v^* \leq C_1, \tag{2.11}$$

we may choose C_1 such that $\sup_V v^* = C_1$. By a similar argument, there is another constant $\tilde{C}_0$ depending on ϕ and C_1 such that $\inf_U u^* = \tilde{C}_0$. The constant $\tilde{C}_0 \geq C_0$, since $u^* \geq u$ in U, and so $(u^*, v^*) \in \mathcal{K}_{C_0}$.

We next deduce the lower bound of v^* and the upper bound of u^* by showing that u^* and v^* are locally Lipschitz functions. Consider two points in U, $x_1 \neq x_2$ and $|x_1 - x_2| < \varepsilon$ is sufficiently small. There are two points $y_1, y_2 \in \overline{V}$ such that

$$\begin{aligned} \phi(x_1, y_1, u^*(x_1), v^*(y_1)) &= 0, \\ \phi(x_2, y_2, u^*(x_2), v^*(y_2)) &= 0. \end{aligned}$$

Then we have

$$\begin{aligned} 0 =& \phi(x_2, y_2, u^*(x_2), v^*(y_2)) - \phi(x_1, y_2, u^*(x_1), v^*(y_2)) \\ &+ \phi(x_1, y_2, u^*(x_1), v^*(y_2)) - \phi(x_1, y_1, u^*(x_1), v^*(y_1)) \\ =& \phi_t(\hat{x}, y_2, \hat{u}^*, v^*)(u^*(x_2) - u^*(x_1)) - \phi_x(\hat{x}, y_2, \hat{u}^*, v^*) \cdot (x_2 - x_1) \\ &+ \phi(x_1, y_2, u^*(x_1), v^*(y_2)), \end{aligned}$$

where $\hat{u}^* = \theta u^*(x_1) + (1-\theta) u^*(x_2), \hat{x} = \bar{\theta} x_1 + (1-\bar{\theta}) x_2$, for some $\theta, \bar{\theta} \in (0, 1)$. Noting that $\phi(x_1, y_2, u^*(x_1), v^*(y_2)) \leq 0$, we have

$$u^*(x_2) - u^*(x_1) \geq -\frac{C_2}{C_3} |x_2 - x_1|,$$

where the constants $C_2 = \sup(|\partial_x \phi| + |\partial_y \phi|)$, and $C_3 = \min\{\inf \partial_t \phi, \inf \partial_s \phi\}$. Due to (2.7), the constant $C_3 \geq \delta_0$ is positive. On the other hand, replacing $\phi(x_1, y_2, u^*(x_1), v^*(y_2))$ by

$\phi(x_2, y_1, u^*(x_2), v^*(y_1))$ in the above calculation, we have

$$u^*(x_2) - u^*(x_1) \le \frac{C_2}{C_3}|x_2 - x_1|.$$

Therefore, the Lipschitz constant of u^* on U is controlled by

$$\|u^*\|_{Lip(U)} \le C_4, \tag{2.12}$$

where the constant $C_4 = C_2/C_3$. A similar argument holds for v^* as well, which implies that $\|v^*\|_{Lip(V)} \le C_4$. Hence, we have $u^* \le \tilde{C}_0 + C_4\text{diam}(U)$ and $v^* \ge C_1 - C_4\text{diam}(V)$ because of (2.11).

We conclude, therefore, that any pair $(u, v) \in \mathcal{K}_{C_0}$ may be replaced by a bounded, Lipschitz pair $(u^*, v^*) \in \mathcal{K}_{C_0}$ without decreasing I. We now choose a sequence $\{(u_k, v_k)\} \subset \mathcal{K}_{C_0}$ such that

$$I(u_k, v_k) \to \sup_{(u,v)\in\mathcal{K}_{C_0}} I(u, v).$$

By the above considerations we may assume that each (u_k, v_k) is a bounded, uniformly Lipschitz pair, uniformly with respect to k, so there is a subsequence converging uniformly to a bounded, Lipschitz, maximizing pair $(\bar{u}, \bar{v}) \in \mathcal{K}_{C_0}$.

Last, we show that when $C_0 < 0$ is sufficiently small,

$$\sup_{(u,v)\in\mathcal{K}_{C_0}} I(u, v) = \sup_{(u,v)\in\mathcal{K}} I(u, v),$$

or equivalently, $\sup_{\mathcal{K}_{C_0}} I$ is independent of C_0. By definition, one has $\sup_{\mathcal{K}_{C_0-1}} I \ge \sup_{\mathcal{K}_{C_0}} I$. So, it suffices to show the reverse inequality. Let $(u, v) \in \mathcal{K}_{C_0-1}$ be a maximizer such that $I(u, v) = \sup_{\mathcal{K}_{C_0-1}} I$, and $\{x_k\}_{k=1,\cdots,N}$ be a set of points in U. For a small constant $\varepsilon > 0$, define

$$\tilde{u} = \begin{cases} u & \text{in } U - \cup_N B_\varepsilon(x_k), \\ u + 2 & \text{in } \cup_N B_\varepsilon(x_k). \end{cases}$$

Note that we may replace $\tilde{u}$ by its mollification $\tilde{u}_h = \rho_h * \tilde{u}$, where ρ_h is the standard mollifier function [3]. For simplicity, we assume $\tilde{u}$ continuous in the sense that for $h > 0$ sufficiently small,

$$I(\tilde{u}_h, v) = I(u, v) + O(N\varepsilon^n).$$

Define

$$\begin{aligned} \tilde{v}^*(y) &= \sup\{s : \phi(x, y, \tilde{u}(x), s) \le 0, \quad \forall x \in U\}, \\ \tilde{u}^*(x) &= \sup\{t : \phi(x, y, t, \tilde{v}^*(y)) \le 0, \quad \forall y \in V\}. \end{aligned}$$

Since the constraint function ϕ is smooth and by (2.7), except a set $E \subset U$ and a set $E' \subset V$ of measure $|E| = |E'| = O(N\varepsilon^n)$,

$$\begin{aligned} \tilde{v}^* &= v - 2\frac{\phi_t}{\phi_s} + O(\delta) \quad \text{in } V \setminus E', \\ \tilde{u}^* &= u + 2 + O(\delta) \quad \text{in } U \setminus E, \end{aligned}$$

where $\delta := \min_{i\neq j}\{\text{dist}(x_i, x_j)\}$. Therefore, by (2.8) and the mean value theorem we have

$$\begin{aligned} I(\tilde{u}^*, \tilde{v}^*) &= I(u,v) + 2\int_{(U\setminus E)\times(V\setminus E')} \left\{F_t - F_s\frac{\phi_t}{\phi_s}\right\} d\gamma + O(\delta) + O(N\varepsilon^n) \\ &\geq I(u,v) - C\delta - CN\varepsilon^n. \end{aligned}$$

As $(u,v)\in\mathcal{K}_{C_0-1}$, we may assume that $\inf_U u = C_0 - 1$. Otherwise, one has $\inf_U u = C_0 - \tau_0$ for some constant $\tau_0 < 1$. This implies that $\sup_{\mathcal{K}_{C_0-1}} I = \sup_{\mathcal{K}_{C_0-\tau_0}} I$, namely $\sup_{\mathcal{K}_{C_0}} I$ is independent of C_0, and the proof is finished. By the definition, δ will become small if the number of points N is sufficiently large so that we have $(\tilde{u}^*, \tilde{v}^*)\in\mathcal{K}_{C_0}$ and

$$\sup_{\mathcal{K}_{C_0}} I \geq I(\tilde{u}^*, \tilde{v}^*) \geq \sup_{\mathcal{K}_{C_0-1}} I - C\delta - CN\varepsilon^n.$$

Then, choosing $\varepsilon > 0$ sufficiently small we have

$$\sup_{\mathcal{K}_{C_0-1}} I \leq \sup_{\mathcal{K}_{C_0}} I,$$

by letting $\delta\to 0, \varepsilon\to 0$, which implies that $\sup_{\mathcal{K}_{C_0}} I$ is independent of C_0, and the proof is finished. □

Remark 2.1. From the proof of Lemma 2.1, we conclude that there exist infinitely many maximizing pairs. In fact, if (u,v) is a maximizer and $C_0 = \inf_U u$, then there is another maximizer in $\mathcal{K}_{C_0+1}$, which is different from (u,v).

Lemma 2.2. *Let $(u,v)\in\mathcal{K}$ be a dual maximizing pair in Lemma 2.1. The equation*

$$\phi(x, T(x), u(x), v(T(x))) = 0 \tag{2.13}$$

can be solved by a mapping $T : U \to V$ implicitly determined by the formula

$$\phi_x(x, T(x), u, v) + \phi_t(x, T(x), u, v)Du(x) = 0, \tag{2.14}$$

at any differentiable point of u. Furthermore, if for any $(x,y,t,s)\in U\times V\times\mathbb{R}^2$,

$$det\,[\phi_{xy} + \phi_{yt}\otimes Du + \phi_{xs}\otimes Dv + \phi_{ts}Du\otimes Dv] \neq 0, \tag{2.15}$$

the mapping T is uniquely determined by (2.14).

The mapping T in Lemma 2.2 is called the *optimal mapping* associated to the dual maximizing pair (u,v). The inequality (2.15) is a generalization of (A2) condition in optimal transportation [9].

Proof. Since u satisfies (2.5) and v,ϕ are continuous, for each $x\in U$, there exists some $y =: T(x)\in\overline{V}$ such that

$$\begin{aligned} \phi(x, y, u(x), v(y)) &= 0, \\ \phi(x', y, u(x'), v(y)) &\leq 0, \end{aligned} \tag{2.16}$$

for any other $x' \in U$. Let $x \in U$ be a differentiable point of u, by differentiation we have

$$\phi_x(x, y, u, v) + \phi_t(x, y, u, v)Du(x) = 0.$$

If there exists another $\tilde{y} \neq y$ in $\overline{V}$ such that

$$\phi_x(x, \tilde{y}, u, v) + \phi_t(x, \tilde{y}, u, v)Du(x) = 0.$$

By the mean value theorem,

$$(\phi_{xy} + \phi_{yt} \otimes Du + \phi_{xs} \otimes Dv + \phi_{ts} Du \otimes Dv) \cdot (\tilde{y} - y) = 0,$$

where the matrix is valued at $(x, \hat{y}, u(x), v(\hat{y}))$ with $\hat{y} = \alpha\tilde{y} + (1 - \alpha)y$ for some $\alpha \in (0, 1)$. This is a contradiction with (2.15), since $\tilde{y} \neq y$. □

Moreover, such an obtained optimal mapping T satisfies the following property, which is a kind of conservation of energy.

Lemma 2.3. *Let T be the optimal mapping associated to a dual maximizing pair (u, v). Assume the constraint function $\phi = \phi(x, y, t, s)$ is smooth and satisfies* (2.7). *Then for any $h \in C(V)$, there holds*

$$0 = \int_{U\times V} \left\{ -F_t \frac{\phi_s}{\phi_t} h(T(x)) + F_s h(y) \right\} d\gamma. \tag{2.17}$$

Proof. Let $h \in C(V)$ and $|\epsilon| < 1$ sufficiently small. Define

$$v_\epsilon(y) = v(y) + \epsilon h(y) \tag{2.18}$$

and

$$u_\epsilon(x) = \sup\{t \,:\, \phi(x, y, t, v_\epsilon(y)) \leq 0, \quad \forall y \in V\}. \tag{2.19}$$

Then $(u_\epsilon, v_\epsilon) \in \mathcal{K}$ and $(u_0, v_0) = (u, v)$.

Since (u, v) satisfies (2.5), by Lemma 2.2, for every $x \in U$ the supremum (2.5) is attained at point $y_0 = T(x)$. We claim that at these points we have

$$u_\epsilon(x) - u(x) = -\epsilon \frac{\phi_s}{\phi_t} h(T(x)) + o(\epsilon). \tag{2.20}$$

To prove (2.20), first we show that $LHS \leq RHS$.

$$\begin{aligned}
0 &= \phi(x, y_0, u(x), v(y_0)) \\
&= \phi(x, y_0, u(x), v_\epsilon(y_0) - \epsilon h(y_0)) \\
&= \phi(x, y_0, u(x), v_\epsilon(y_0)) - \epsilon\phi_s h(y_0) + o(\epsilon) \\
&= \phi(x, y_0, u_\epsilon(x) + u(x) - u_\epsilon(x), v_\epsilon(y_0)) - \epsilon\phi_s h(y_0) + o(\epsilon) \\
&= \phi(x, y_0, u_\epsilon(x), v_\epsilon(y_0)) + \phi_t(u(x) - u_\epsilon(x)) - \epsilon\phi_s h(y_0) + o(\epsilon) \\
&\leq \phi_t(u(x) - u_\epsilon(x)) - \epsilon\phi_s h(y_0) + o(\epsilon).
\end{aligned}$$

By (2.7) we have

$$u_\epsilon(x) - u(x) \le -\epsilon \frac{\phi_s}{\phi_t} h(y_0) + o(\epsilon).$$

To show $LHS \ge RHS$ we use the fact that for any such $x \in U$ there are points $y_\epsilon \in \overline{V}$ such that the supremum in (2.19) is attained. Thus

$$\begin{aligned}
0 &\ge \phi(x, y_\epsilon, u(x), v(y_\epsilon)) \\
&= \phi(x, y_\epsilon, u(x), v_\epsilon(y_\epsilon) - \epsilon h(y_\epsilon)) \\
&= \phi(x, y_\epsilon, u(x), v_\epsilon(y_\epsilon)) - \epsilon \phi_s h(y_\epsilon) + o(\epsilon) \\
&= \phi(x, y_\epsilon, u_\epsilon(x) + u(x) - u_\epsilon(x), v_\epsilon(y_\epsilon)) - \epsilon \phi_s h(y_\epsilon) + o(\epsilon) \\
&= \phi(x, y_\epsilon, u_\epsilon(x), v_\epsilon(y_\epsilon)) + \phi_t(u(x) - u_\epsilon(x)) - \epsilon \phi_s h(y_\epsilon) + o(\epsilon) \\
&= \phi_t(u(x) - u_\epsilon(x)) - \epsilon \phi_s h(y_\epsilon) + o(\epsilon).
\end{aligned}$$

Then by (2.7) we have

$$\begin{aligned}
u_\epsilon(x) - u(x) &\ge -\epsilon \frac{\phi_s}{\phi_t} h(y_\epsilon) + o(\epsilon) \\
&= -\epsilon \frac{\phi_s}{\phi_t} h(y_0) + \epsilon \frac{\phi_s}{\phi_t} \left(h(y_0) - h(y_\epsilon)\right) + o(\epsilon).
\end{aligned}$$

Since the supremum in (2.5) is attained at y_0, we have $y_\epsilon \to y_0$ as $\epsilon \to 0$, and therefore, since $h \in C(V)$,

$$\epsilon \frac{\phi_s}{\phi_t} \left(h(y_0) - h(y_\epsilon)\right) = o(\epsilon).$$

This implies that $LHS \ge RHS$, and (2.20) follows.

Next, since $(u, v) = (u_0, v_0)$ maximizes I among all pairs in $\mathcal{K}$, we obtain

$$\begin{aligned}
0 &= \lim_{\epsilon \to 0} \frac{I(u_\epsilon, v_\epsilon) - I(u, v)}{\epsilon} \\
&= \lim_{\epsilon \to 0} \int_{U \times V} \frac{F(x, y, u_\epsilon, v_\epsilon) - F(x, y, u, v)}{\epsilon} d\gamma \\
&= \int_{U \times V} \left\{ -F_t \frac{\phi_s}{\phi_t} h(T(x)) + F_s h(y) \right\} d\gamma.
\end{aligned}$$

□

2.2. **Formulation of reflector problem.** In order to formulate the near field reflector problem to an optimization problem, we need the notion of *ellipsoid of revolution*, which has a special reflection property: the light rays from one focus are always reflected to the other focus.

In the polar coordinate system, an ellipsoid of revolution $E = E(Y, p)$ with one focus at the origin, the other focus at Y, and *focal parameter* $p \in (0, \infty)$ can be represented as

$E = \{X\rho_e(X) \,:\, X \in \mathbb{S}^n\}$ by a radial function

$$\rho_e(X) = \frac{p}{1 - \epsilon(p)\langle X, \frac{Y}{|Y|}\rangle} \tag{2.21}$$

and

$$\epsilon(p) = \sqrt{1 + \frac{p^2}{|Y|^2}} - \frac{p}{|Y|} \tag{2.22}$$

is the *eccentricity*, see [7]. Note that any such ellipsoid is uniquely determined by Y and p. If we regard $p = p(Y)$ as a *focal function* on Ω^*, we then have a family of ellipsoids.

We recall that [6], for an *admissible* reflector Γ_ρ, at each point $X\rho(X) \in \Gamma_\rho$ there exists a *supporting ellipsoid*, namely, for some $Y \in \Omega^*$

$$\begin{cases} \rho(X) = \frac{p(Y)}{1-\epsilon(p(Y))\langle X, \frac{Y}{|Y|}\rangle}, \\ \rho(X') \leq \frac{p(Y)}{1-\epsilon(p(Y))\langle X', \frac{Y}{|Y|}\rangle}, \quad \forall X' \in \Omega. \end{cases} \tag{2.23}$$

In the following context, we also say ρ is admissible if Γ_ρ is admissible.

Next we define a set-valued mapping $T_\rho : \Omega \to \Omega^*$. For any $X \in \Omega$,

$$T_\rho(X) = \{Y \in \Omega^* \,:\, Y \text{ is the focus of a supporting ellipsoid of } \Gamma_\rho \text{ at } X\rho(X)\}. \tag{2.24}$$

Note that at any differentiable point X of ρ, $T_\rho(X)$ is single valued and is exactly the reflection mapping. For any subset $G \subset \Omega$, we denote $T_\rho(G) = \bigcup_{X \in G} T_\rho(X)$. Therefore, we can define a measure $\mu_\# = \mu_{\rho,g}$ in Ω such that for any Borel set $G \subset \Omega$,

$$\mu_\#(G) = \int_{T_\rho(G)} g d\nu. \tag{2.25}$$

Definition 2.2. *An admissible function ρ is called a weak solution of the reflector problem if $\mu_{\rho,g} = f d\mu$ as measures, namely for any Borel set $G \subset \Omega$,*

$$\int_G f d\mu = \int_{T_\rho(G)} g d\nu. \tag{2.26}$$

The above definition was introduced in [6]. Obviously an admissible smooth solution is a weak solution, in that case, the reflector Γ_ρ is naturally an envelope of a family of confocal ellipsoids of revolution. Therefore, the radial function ρ satisfies

$$\rho(X) = \inf_{Y \in \Omega^*} \frac{p(Y)}{1 - \epsilon(p(Y))\langle X, \frac{Y}{|Y|}\rangle}, \quad X \in \Omega, \tag{2.27}$$

and for each $Y \in \Omega^*$ the ellipsoid $E_{Y,p(Y)}$ is supporting to Γ_ρ, we also have the focal function p satisfies

$$p(Y) = \sup_{X \in \Omega} \rho(X) \left[1 - \epsilon(p(Y))\langle X, \frac{Y}{|Y|}\rangle\right], \quad Y \in \Omega^*. \tag{2.28}$$

Note that in (2.27) for each $X \in \Omega$ the infimum is achieved at some $Y \in \Omega^*$ and in (2.28) for each $Y \in \Omega^*$ the supremum is achieved at some $X \in \Omega$.

The relations (2.27)–(2.28) between the radial and focal functions of a reflector Γ_ρ are analogous to the classical relations between the radial and support functions for convex bodies, for example, see [10]. Inspired by that and [14], we set $\eta = 1/p$. Then the pair (ρ, η) satisfies the following dual relation

$$
\begin{aligned}
\rho(X) &= \inf_{Y\in\Omega^*} \frac{1}{\eta(Y)\left(1-\epsilon(\eta(Y))\langle X, \frac{Y}{|Y|}\rangle\right)}, \\
\eta(Y) &= \inf_{X\in\Omega} \frac{1}{\rho(X)\left(1-\epsilon(\eta(Y))\langle X, \frac{Y}{|Y|}\rangle\right)},
\end{aligned} \tag{2.29}
$$

where η is a Legendre type transform of ρ, [4].

Similarly to [14], we can now formulate the reflector problem to a nonlinear optimization (2.1)–(2.2) as follows. Set the functional

$$
\begin{aligned}
I(\rho,\eta) &= \int_{\Omega\times\Omega^*} F(X,Y,\rho,\eta) \\
&= \int_{\Omega\times\Omega^*} f(X)\log\rho + g(Y)\left(\log\eta + \log(1-\frac{\langle X,Y\rangle}{\eta^{-1}+\sqrt{|Y|^2+\eta^{-2}}})\right),
\end{aligned} \tag{2.30}
$$

and the constraint set

$$
\mathcal{K} = \{(\rho,\eta)\in C_+(\Omega)\times C_+(\Omega^*) \,:\, \phi(X,Y,\rho,\eta)\le 0\},
$$

with the constraint function

$$
\phi(X,Y,\rho,\eta) = \log\rho + \log\eta + \log\left(1-\epsilon(\eta(Y))\langle X, \frac{Y}{|Y|}\rangle\right). \tag{2.31}
$$

In fact, by (2.22) and $\eta = 1/p$ it is easy to see that

$$
\epsilon(\eta(Y))\langle X, \frac{Y}{|Y|}\rangle = \frac{\langle X,Y\rangle}{\eta^{-1}+\sqrt{|Y|^2+\eta^{-2}}}. \tag{2.32}
$$

Lemma 2.4. *Let (ρ,η) be a dual maximizing pair of (2.30)–(2.31), and T be the associated optimal mapping. Then $T = T_\rho$ at any differentiable point of ρ, where T_ρ is the reflection mapping in (2.24).*

Proof. We first introduce some geometric notation. By restricting to a subset we may assume that Ω is in the north hemisphere. Let $X = (x, x_{n+1})$ be a smooth parameterization of $\Omega \subset \mathbb{S}^n$, where $x_{n+1} = \sqrt{1-|x|^2}$ and $x = (x_1, \cdots, x_n)$.

Denote $\partial_i = \partial/\partial x_i$, $e_i = \partial_i X$, and the metric $g_{ij} = \langle e_i, e_j\rangle$, where $\langle\, ,\rangle$ is the inner product of $\mathbb{R}^{n+1}$. By direct computations, for $i, j, k, l = 1, \cdots, n$,

$$e_i = \left(0, \cdots, 1, \cdots, 0, \frac{-x_i}{\sqrt{1-|x|^2}}\right), \qquad 1 \text{ is in the } i\text{th coordinate}, \tag{2.33}$$

$$g_{ij} = \delta_{ij} + \frac{x_i x_j}{1-|x|^2}, \qquad g^{ij} = \delta_{ij} - x_i x_j, \quad \text{where } (g^{ij}) = (g_{ij})^{-1}, \tag{2.34}$$

and the Christoffel symbols are

$$\Gamma_{ij}^k = \frac{1}{2} g^{kl}\left(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}\right) = x_k\left(\delta_{ij} + \frac{x_i x_j}{1-|x|^2}\right). \tag{2.35}$$

Denote $e_{n+1} = -X$, the unit inner normal of $\mathbb{S}^n$ at X. The Gauss formula is

$$\partial_j e_i = \Gamma_{ij}^k e_k + h_{ij} e_{n+1},$$

where the second fundamental form

$$\begin{aligned} h_{ij} &= \delta_{ij} + \frac{x_i x_j}{1-|x|^2} \\ &= g_{ij}. \end{aligned} \tag{2.36}$$

Namely, one has that

$$\partial_j e_i = \Gamma_{ij}^k e_k - g_{ij} X. \tag{2.37}$$

The above equalities (2.33)–(2.37) can all be obtained by basic computations.

Let ρ be a function defined on Ω. The tangential gradient of ρ is defined by

$$\nabla\rho = \sum_{i,j=1}^n g^{ij} e_i \partial_j \rho. \tag{2.38}$$

Note that $\nabla\rho(X) \in T_X\mathbb{S}^n$, the tangent space, so $\langle\nabla\rho, X\rangle = 0$. By direct calculation

$$\begin{aligned} \langle\nabla\rho, e_i\rangle &= \langle g^{jk} e_j \partial_k\rho, e_i\rangle \\ &= g^{jk} g_{ij}\partial_k\rho = \delta_{ik}\partial_k\rho = \partial_i\rho, \end{aligned} \tag{2.39}$$

for all $1 \le i \le n$. Let $D\rho = (\partial_1\rho, \cdots, \partial_n\rho)$ be the standard gradient of ρ. From (2.33), (2.34) and (2.38), we have

$$\nabla\rho \;=\; (D\rho, 0) - (D\rho\cdot x)\, X, \tag{2.40}$$

$$|\nabla\rho|^2 \;=\; \langle\nabla\rho, \nabla\rho\rangle = |D\rho|^2 - (D\rho\cdot x)^2. \tag{2.41}$$

Let $\Gamma_\rho = \{X\rho(X) : X \in \Omega\}$ be the graph of ρ over Ω. We claim that the unit normal of Γ_ρ at $X\rho(X)$ is

$$\gamma = \frac{\nabla\rho - \rho X}{\sqrt{\rho^2 + |\nabla\rho|^2}}. \tag{2.42}$$

Indeed, for $i = 1, \cdots, n$, the tangential of Γ_ρ at $X\rho(X)$ is

$$\tau_i = \partial_i(X\rho(X)) = \rho e_i + (\partial_i\rho) X.$$

From (2.39), for any $i = 1, \cdots, n$, the following holds:

$$\begin{aligned}\langle \tau_i, \gamma \rangle &= \frac{1}{\sqrt{\rho^2 + |\nabla \rho|^2}} \langle \rho e_i + (\partial_i \rho) X, \nabla \rho - \rho X \rangle \\ &= \frac{1}{\sqrt{\rho^2 + |\nabla \rho|^2}} (\rho \partial_i \rho - \rho \partial_i \rho) = 0.\end{aligned}$$

It is obvious that $|\gamma| = 1$, thus γ is the unit normal.

At the differentiable point X of ρ, by (2.24), $Y = T_\rho(X)$ is the focus of the supporting ellipsoid of Γ_ρ at $X\rho(X)$. Denote the reflected direction by $Y_r = \frac{Y - X\rho}{|Y - X\rho|}$. By (2.42) and the reflection law,

$$\begin{aligned}Y_r &= X - 2\langle X, \gamma \rangle \gamma \\ &= \frac{2\rho \nabla \rho + (|\nabla \rho|^2 - \rho^2) X}{|\nabla \rho|^2 + \rho^2}.\end{aligned} \tag{2.43}$$

Denote the length of reflected ray

$$d := |Y - X\rho|. \tag{2.44}$$

Hence, we have

$$\begin{aligned}Y &= T_\rho(X) = X\rho + Y_r d \\ &= \frac{2\rho \nabla \rho + (|\nabla \rho|^2 - \rho^2) X}{|\nabla \rho|^2 + \rho^2} d + X\rho,\end{aligned} \tag{2.45}$$

and

$$\langle X, Y \rangle = d \frac{|\nabla \rho|^2 - \rho^2}{|\nabla \rho|^2 + \rho^2} + \rho. \tag{2.46}$$

On the other hand, by differentiating the constraint function in (2.31) and the formula (2.14), we obtain

$$\frac{\partial_i \rho}{\rho} = \frac{\epsilon \langle e_i, Y_e \rangle}{1 - \epsilon \langle X, Y_e \rangle}, \tag{2.47}$$

where $Y_e = T(X)/|T(X)|$, $T(X) = Y$ is the optimal mapping. By noting that $e_i \perp X$, $\nabla \rho \perp X$ and $\langle \nabla \rho, e_i \rangle = \partial_i \rho$, from (2.47) we have the decomposition

$$Y_e = \frac{1 - \epsilon \langle X, Y_e \rangle}{\epsilon \rho} \nabla \rho + \langle X, Y_e \rangle X. \tag{2.48}$$

From (2.13), (2.29) and (2.31), observe that at differentiable point X of ρ, there exists a unique supporting ellipsoid E of Γ_ρ at $X\rho(X)$, with foci O, Y and eccentricity ϵ. Note that the sum of length $\rho = |X\rho(X) - O|$ and length $d = |Y - X\rho(X)|$ equals to the diameter of E, i.e.

$$\rho + d = \mathrm{diam}(E). \tag{2.49}$$

By the definition of eccentricity ϵ,

$$\epsilon = \frac{|Y|}{\mathrm{diam}(E)} = \frac{|Y|}{\rho + d}. \tag{2.50}$$

Combining (2.48) and (2.50), one obtains the following equation for $Y = T(X)$,

$$Y = \frac{\rho + d - \langle X, Y\rangle}{\rho}\nabla\rho + \langle X, Y\rangle X. \tag{2.51}$$

It then suffices to show that $Y = T_\rho(X)$ in (2.45) is a solution of (2.51). In fact, by (2.46) we have

$$\begin{aligned} T(X) &= \left(d - d\frac{|\nabla\rho|^2 - \rho^2}{|\nabla\rho|^2 + \rho^2}\right)\nabla\rho/\rho + X\rho + Xd\frac{|\nabla\rho|^2 - \rho^2}{|\nabla\rho|^2 + \rho^2} \\ &= X\rho + \frac{2\rho\nabla\rho + (|\nabla\rho|^2 - \rho^2)X}{|\nabla\rho|^2 + \rho^2}d \\ &= X\rho + Y_r d = T_\rho(X). \end{aligned} \tag{2.52}$$

□

Proof of Theorem 1.1. The proof essentially follows from [14]. Let $u = \log\rho$, $v = \log\eta$. In order to apply Lemma 2.1, we need first to verify that F and ϕ in (2.30)–(2.31) satisfy the conditions (2.6)–(2.8). For the constraint function ϕ in (2.31), it is easy to see that $\phi_t = 1 > 0$. By (2.22) and $\eta = 1/p$,

$$\begin{aligned} \phi_s &= \eta\left(\frac{1}{\eta} - \frac{\frac{\partial\epsilon}{\partial\eta}\langle X, Y_e\rangle}{1 - \epsilon\langle X, Y_e\rangle}\right) \\ &= 1 - \frac{1}{\sqrt{1 + \eta^2|Y|^2}}\frac{\epsilon\theta}{1 - \epsilon\theta}, \end{aligned}$$

where $\theta = \langle X, Y_e\rangle \in [-1, 1)$ due to (1.5). Since ϕ_s is decreasing in θ, we have

$$\phi_s > 1 - \frac{1}{\sqrt{1 + \eta^2|Y|^2}}\frac{\epsilon\theta_0}{1 - \epsilon\theta_0},$$

where the constant $\theta_0 < 1$ depends on domains Ω, Ω^*. Set $\tau = 1/(\eta|Y|)$, $\epsilon = \sqrt{1 + \tau^2} - \tau$. One has the second term in the above inequality

$$\frac{1}{\sqrt{1 + \eta^2|Y|^2}}\frac{\epsilon\theta_0}{1 - \epsilon\theta_0} < \frac{\tau(\sqrt{1 + \tau^2} - \tau)}{\sqrt{1 + \tau^2}(1 - \sqrt{1 + \tau^2} + \tau)} =: h(\tau),$$

where the function h is decreasing in τ. Thus, by the Taylor expansion of $\sqrt{1 + \tau^2}$ near $\tau = 0$,

$$h(\tau) < \lim_{\tau\to 0} h(\tau) = 1, \quad \text{for } \tau > 0.$$

Hence, we otain $\phi_s > \delta_0$ and (2.7) holds, for a positive constant δ_0. From (2.30)–(2.32), one can see that $F_t = \phi_t f(X)$ and $F_s = \phi_s g(Y)$. Since f and g are both positive, we have the condition (2.6) satisfied. The condition (2.8) is an equivalent to the assumption (1.1).

Therefore, from Lemma 2.1, we obtain a dual maximizing pair $(\rho, \eta) \in \mathcal{K}$ in Theorem 1.1. Then by the dual relation (2.5) and (2.31), one can see that ρ is admissible (2.23), and η is the Legendre type transform of ρ as in (2.29). From Lemma 2.1, one knows that ρ is Lipschitz continuous. Actually, since an admissible function has supporting ellipsoid at any point of its graph, it is semi-convex and twice differentiable almost everywhere [6]. Hence, by Lemma 2.4 $T_\rho = T$ a.e., where T_ρ is the mapping defined in (2.24).

Next, we show that T satisfies the mearsure preserving condition (2.26). Since $F_t = \phi_t f(X)$ and $F_s = \phi_s g(Y)$, by (2.7) and applying Lemma 2.3 to T, we obtain that

$$\int_\Omega f(X)h(T(X))d\mu = \int_{\Omega^*} g(Y)h(Y)d\nu,$$

for arbitrary test functions $h \in C(\Omega^*)$. Therefore, since $T_\rho = T$ a.e., we see that T_ρ satisfies (2.26), namely ρ is a weak solution of the reflector problem. □

3. Derivation of equation

We first derive the partial differential equation for the nonlinear optimization problem (2.1)–(2.2) in general. Let (u, v) be a dual maximizing pair of I. Assume that all the functions are smoothly differentiable at this stage. By a second differentiation of (2.14) we obtain

$$\begin{aligned} 0 =&\phi_{xx} + \phi_{xy}DT + 2\phi_{xt} \otimes Du + (\phi_{xs} \otimes Dv)DT \\ &+ (\phi_{yt} \otimes Du)DT + \phi_{tt}Du \otimes Du + (\phi_{ts}Dv \otimes Du)DT + \phi_t D^2u, \end{aligned} \tag{3.1}$$

where each side is regarded as an $n \times n$ matrix valued at (x, y), $y = T(x)$.

Note that for every $x \in U$, the equality (2.13) holds at point $y = T(x)$, and for any other $y' \in V$ we have

$$\phi(x, y', u(x), v(y')) \le 0,$$

since $(u, v) \in \mathcal{K}$. Thus, at $(x, T(x))$ we have

$$\frac{d\phi}{dy} = \phi_y + \phi_s Dv = 0.$$

By the assumption (2.7), $\phi_s > 0$, we get

$$Dv = -\frac{\phi_y}{\phi_s}. \tag{3.2}$$

Combining (3.1) and (3.2), we obtain the equation

$$\begin{aligned} &\left|\phi_t D^2u + \phi_{tt}Du \otimes Du + 2\phi_{xt} \otimes Du + \phi_{xx}\right| \\ =&\left|\phi_{xy} + \phi_{xs} \otimes Dv + \phi_{yt} \otimes Du + \phi_{ts}Dv \otimes Du\right| |DT| \\ =&\left|\phi_{xy} - \frac{1}{\phi_s}\phi_{xs} \otimes \phi_y + \phi_{yt} \otimes Du - \frac{\phi_{ts}}{\phi_s}\phi_y \otimes Du\right| |DT|, \end{aligned} \tag{3.3}$$

hence by (2.7),

$$\begin{aligned} &\left|\det\left[D^2u + \frac{\phi_{tt}}{\phi_t}Du \otimes Du + \frac{2}{\phi_t}\phi_{xt} \otimes Du + \frac{1}{\phi_t}\phi_{xx}\right]\right| \\ =&\frac{1}{\phi_t^n}\left|\det\left[\phi_{xy} - \frac{1}{\phi_s}\phi_{xs} \otimes \phi_y + \phi_{yt} \otimes Du - \frac{\phi_{ts}}{\phi_s}\phi_y \otimes Du\right]\right| |\det DT|. \end{aligned} \tag{3.4}$$

Equation (3.4) is a second order fully nonlinear PDE of general Monge-Ampère type [3]. In the special case of optimal transportation (2.3)–(2.4), equation (3.4) becomes

$$\left|\det\left[D^2u - D^2_{xx}c\right]\right| = \left|\det D^2_{xy}c\right| \frac{\rho}{\rho^*\circ T}. \tag{3.5}$$

For the derivation of the optimal transportation equation (3.5), see [9] for more.

Using the notation from the proof of Lemma 2.4, we can now derive the PDE in the near field reflector problem by using the formula (3.4) and constraint function (2.31).

Denote $Y_e = Y/|Y|$, $D\rho = (\partial_1\rho, \cdots, \partial_n\rho)$ the gradient of ρ, and $D^2\rho = (\partial_i\partial_j\rho)$ the Hessian of ρ. By differentiating (2.31),

$$\begin{aligned}
&\phi_t = \frac{1}{\rho}, \qquad \phi_{tt} = -\frac{1}{\rho^2}, \qquad \phi_{xt} = 0,\\
&\phi_{x_i} = -\frac{\epsilon\langle e_i, Y_e\rangle}{1-\epsilon\langle X, Y_e\rangle}, \qquad \phi_{x_ix_j} = -\frac{\epsilon\langle \partial_j e_i, Y_e\rangle}{1-\epsilon\langle X, Y_e\rangle} - \frac{\epsilon^2\langle e_i, Y_e\rangle\langle e_j, Y_e\rangle}{(1-\epsilon\langle X, Y_e\rangle)^2}.
\end{aligned}$$

As in (2.47), at $Y = T(X)$, where T is the optimal mapping in (2.13), we have

$$\frac{\partial_i\rho}{\rho} = \frac{\epsilon\langle e_i, Y_e\rangle}{1-\epsilon\langle X, Y_e\rangle}. \tag{3.6}$$

Therefore,

$$\phi_{x_ix_j} = -\frac{\epsilon\langle \partial_j e_i, Y_e\rangle}{1-\epsilon\langle X, Y_e\rangle} - \frac{1}{\rho^2}\partial_i\rho\partial_j\rho, \tag{3.7}$$

and the LHS of equation (3.4) becomes

$$\begin{aligned}
M(\rho) &:= \left|\det\left[D^2\rho + \frac{\phi_{tt}}{\phi_t}D\rho\otimes D\rho + \frac{2}{\phi_t}\phi_{xt}\otimes D\rho + \frac{1}{\phi_t}\phi_{xx}\right]\right|\\
&= \left|\det\left[\partial_i\partial_j\rho - \frac{2}{\rho}\partial_i\rho\partial_j\rho - \rho\frac{\epsilon\langle \partial_j e_i, Y_e\rangle}{1-\epsilon\langle X, Y_e\rangle}\right]\right|.
\end{aligned} \tag{3.8}$$

From (2.33),

$$\partial_j e_i = \left(0, \cdots, 0, -\frac{\delta_{ij}}{\sqrt{1-|x|^2}} - \frac{x_ix_j}{(1-|x|^2)\sqrt{1-|x|^2}}\right). \tag{3.9}$$

In the special case $\Omega^* \subset \{y_{n+1} = 0\}$,

$$Y_e = \frac{Y}{|Y|} = (Y_{e,1}, \cdots, Y_{e,n}, 0).$$

So, $\langle \partial_j e_i, Y_e\rangle = 0$ and (3.8) becomes

$$M(\rho) = \left|\det\left[D^2\rho - \frac{2}{\rho}D\rho\otimes D\rho\right]\right|.$$

Let $u = 1/\rho$. We have the standard Monge-Ampère operator as

$$M(u) = |\det D^2u|.$$

In the general case when Ω^* is given by (1.7), let us now calculate the term $\rho\frac{\epsilon\langle\partial_j e_i, Y_e\rangle}{1-\epsilon\langle X, Y_e\rangle}$ in (3.8). Let E be the supporting ellipsoid of Γ_ρ at $X\rho(X)$, with foci O, Y and eccentricity ϵ. Recall that we have the relation (2.50).

Therefore,

$$
\begin{aligned}
\rho\frac{\epsilon\langle\partial_j e_i, Y_e\rangle}{1-\epsilon\langle X, Y_e\rangle} &= \frac{\rho\epsilon\langle\partial_j e_i, Y\rangle}{|Y|-\epsilon\langle X, Y\rangle} \\
&= \frac{\rho\langle\partial_j e_i, Y\rangle}{\rho+d-\langle X, Y\rangle} \\
\text{from (3.9)} \qquad &= \frac{-\rho}{\rho+d-\langle X, Y\rangle}\left(\frac{y_{n+1}}{x_{n+1}}\right)\left(\delta_{ij}+\frac{x_i x_j}{1-|x|^2}\right).
\end{aligned} \tag{3.10}
$$

Combining (2.46) into (3.10), we obtain

$$
\frac{\rho\epsilon\langle\partial_j e_i, Y_e\rangle}{1-\epsilon\langle X, Y_e\rangle} = -\frac{|\nabla\rho|^2+\rho^2}{2\rho d}\left(\frac{y_{n+1}}{x_{n+1}}\right)\mathcal{N}_{ij}, \tag{3.11}
$$

where $\{\mathcal{N}_{ij}\}$ is in (1.11). Actually, as one can see from (2.36), $\mathcal{N}_{ij} = g_{ij} = h_{ij}$ is equal to the metric and the second fundamental form under the *projection coordinates* (2.33).

Next, let us now calculate the length $d = |Y - X\rho|$ appearing in (3.11). Recall that $\nabla\rho = g^{ij}e_i\partial_j\rho$ satisfies (2.40)–(2.41). Thus, from (2.45), we have

$$
\begin{aligned}
y_{n+1} &= \frac{d}{|\nabla\rho|^2+\rho^2}\left(-2\rho(D\rho\cdot x)x_{n+1} + (|\nabla\rho|^2-\rho^2)x_{n+1}\right) + \rho x_{n+1} \\
&= \frac{d}{|\nabla\rho|^2+\rho^2}\left(|D\rho|^2-(\rho+D\rho\cdot x)^2\right)x_{n+1} + \rho x_{n+1}.
\end{aligned} \tag{3.12}
$$

Therefore,

$$
d = \left(\frac{y_{n+1}}{x_{n+1}}-\rho\right)\frac{|\nabla\rho|^2+\rho^2}{|D\rho|^2-(\rho+D\rho\cdot x)^2}. \tag{3.13}
$$

Finally, combining (3.13) into (3.11) we obtain

$$
\frac{\rho\epsilon\langle\partial_j e_i, Y_e\rangle}{1-\epsilon\langle X, Y_e\rangle} = \frac{|D\rho|^2-(\rho+D\rho\cdot x)^2}{2\rho}\left(\frac{y_{n+1}}{\rho x_{n+1}-y_{n+1}}\right)\mathcal{N}_{ij} \tag{3.14}
$$

Using the notation (1.8)–(1.11), $a = |D\rho|^2 - (\rho + D\rho\cdot x)^2$ and $\mathcal{N} = (\mathcal{N}_{ij})$. We get the LHS of equation (3.4)

$$
\begin{aligned}
M(\rho) &= \left|\det\left[D^2\rho - \frac{2}{\rho}D\rho\otimes D\rho - \frac{a y_{n+1}}{2\rho(\rho x_{n+1}-y_{n+1})}\mathcal{N}\right]\right| \\
&= \left|\det\left[D^2\rho - \frac{2}{\rho}D\rho\otimes D\rho - \frac{a(1-t)}{2t\rho}\mathcal{N}\right]\right|.
\end{aligned} \tag{3.15}
$$

To compute the RHS of (3.4), one can directly differentiate the constraint function ϕ in (2.31), but the computations are rather complicated. Instead, we recall a result in [6] in the following: the Jacobian determinant of the reflection mapping T_ρ is equal to

$$
\begin{aligned}
|\det DT_\rho| &= 2^n\rho^{2n+1}x_{n+1}|\nabla\psi|\left|\frac{t^n b\beta}{a^{n+1}}\right|\left|\det\left[D^2\rho-\frac{2}{\rho}D\rho\otimes D\rho-\frac{a(1-t)}{2t\rho}\mathcal{N}\right]\right| \\
&= 2^n\rho^{2n+1}x_{n+1}|\nabla\psi|\left|\frac{t^n b\beta}{a^{n+1}}\right|M(\rho),
\end{aligned}
\tag{3.16}
$$

where b,β are defined in (1.9)–(1.10) and ψ is the defining function of Ω^* in (1.7). Alternatively, one can obtain (3.16) by differetiating the mapping T_ρ in (2.45).

On the other hand, from (3.4)

$$
|\det DT| = M(\rho)\phi_t^n\left|\det\left[\phi_{xy}-\frac{1}{\phi_s}\phi_{xs}\otimes\phi_y+\phi_{yt}\otimes Du-\frac{\phi_{ts}}{\phi_s}\phi_y\otimes Du\right]\right|^{-1}. \tag{3.17}
$$

By Lemma 2.4 and Theorem 1.1, $|\det DT| = |\det DT_\rho|$. Thus

$$
\frac{1}{\phi_t^n}\left|\det\left[\phi_{xy}-\frac{\phi_{xs}}{\phi_s}\otimes\phi_y+\phi_{yt}\otimes Du-\frac{\phi_{ts}}{\phi_s}\phi_y\otimes Du\right]\right| = \left|\frac{a^{n+1}}{t^n b\beta}\right|\frac{1}{2^n\rho^{2n+1}x_{n+1}|\nabla\psi|}. \tag{3.18}
$$

Note that we projected $\Omega\subset\mathbb{S}^n$ on the n dimensional space $(x_1,\cdots,x_n)$ in (2.33), $dx = \omega d\mu$, where $d\mu$ is the surface area element of Ω, $\omega=\sqrt{1-|x|^2}$. By Lemma 2.4 and (2.26),

$$
|\det DT| = |\det DT_\rho| = \frac{f}{\omega g}. \tag{3.19}
$$

Therefore, combining (3.18)–(3.19) into (3.4), we obtain the equation

$$
M(\rho) = \left|\frac{a^{n+1}}{t^n b\beta}\right|\frac{f}{2^n\rho^{2n+1}\omega^2 g|\nabla\psi|}. \tag{3.20}
$$

This completes the proof of Theorem 1.2. However, note that since we calculate the absolute value for the determinant, the matrix in $M(\rho)$, (3.15) has a different sign to that in [6].

Remark 3.1. Another special case of the reflector problem is the far field case [13]. Suppose a ray X is reflected off by Γ_ρ to a direction Y. Set the functional and constraint function in (2.30)–(2.31) to be

$$
I(\rho,\eta) = \int_\Omega \log\rho(X)f(X)+\int_{\Omega^*}\log\eta(Y)g(Y), \tag{3.21}
$$

$$
\phi(X,Y,\rho,\eta) = \log\rho+\log\eta+\log(1-\langle X,Y\rangle). \tag{3.22}
$$

Similarly to Theorem 1.1, one can show that if (ρ,η) is a dual maximizing pair of I, then ρ is a solution of the far field reflector problem. This formulation was previously obtained by Wang in [14].

The equation in the far field case can be directly obtained by using the formula (3.4) and differentiating the constraint function (3.22). Here we remark that the far field equation is a limit case of (3.15) for the near field one, [6].

To see this, using our notations (1.8)–(1.10), from (3.13) we have the length of reflected ray $d = |Y - X\rho|$ is equal to

$$d = \left(\frac{y_{n+1} - \rho x_{n+1}}{x_{n+1}}\right) \frac{|\nabla\rho|^2 + \rho^2}{|D\rho|^2 - (\rho + D\rho\cdot x)^2} = -t\rho\frac{b}{a}. \tag{3.23}$$

Let's regard the target $\Omega_r^* = \{rZ \,:\, Z \in \Omega_1^*\}$, where r is sufficiently large, and Ω_1^* is a domain in the south hemisphere of $\mathbb{S}^n$. In this case, the defining function in (1.7) will be $\psi(Z) = r^2 - |Z|^2$. Let g_r be the light distribution on Ω_r^* under the same reflector Γ. Then when r is sufficiently large, $r^n g_r \to g$, and

$$\beta|\nabla\psi| = \frac{t|\nabla\psi|}{(Y - X\rho)\cdot\nabla\psi} \to \frac{-t}{d} = \frac{a}{\rho b}, \tag{3.24}$$

$$\frac{r}{t} = \frac{|Y|}{t} \to \frac{d}{t} = -\frac{\rho b}{a}. \tag{3.25}$$

Sending $r \to \infty$, from (1.12) we obtain the equation for the far field case

$$\left|\det\left[D^2\rho - \frac{2}{\rho}D\rho\otimes D\rho + \frac{a}{2\rho}\mathcal{N}\right]\right| = \frac{|b|^n f}{2^n\rho^n\omega^2 g}. \tag{3.26}$$

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