LISBON SCHOOL OF ECONOMICS & MANAGEMENT
UNIVERSIDADE DE LISBOA

MASTER

MATHEMATICAL FINANCE

MASTER´S FINAL WORK

DISSERTATION

AMERICAN OPTIONS AND THE BLACK-SCHOLES MODEL

AFONSO VALENTE RICARDO DE SEABRA COELHO

OCTOBER - 2020

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SUPERVISION:

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Abstract

Option pricing problems have been one of the main focuses in the field of Mathematical Finance since the creation of this concept in the 1970s. More specifically, American options are of great interest in this area of knowledge because they are much more complex mathematically than the standard European options and the Black-Scholes model cannot give an explicit formula to value this style options in most cases.

In this dissertation, we show how pricing American options leads to free boundary problems because of the possibility of early exercise, where our main goal is to find the optimal exercise price. We also present how to reformulate the problem into a linear complementarity problem and a parabolic variational inequality. Moreover, we also address the probabilistic characterization of American options based on the concept of stopping times. These formulations, here viewed from the analytical and probabilistic point of view, can be very useful for applying numerical methods to the problem of pricing American style options since, in most cases, it is almost impossible to find explicit solutions.

Furthermore, we use the Binomial Tree Method, which is a very simple numerical method from the mathematical point of view, to illustrate some aspects of the theory studied throughout this thesis and to compare American options with European and Bermudan Options, by means of a few numerical examples.

KEYWORDS: American Options; Black-Scholes Model; Free Boundary Problems; Linear Complementarity Problems; Parabolic Variational Inequalities; Binomial Tree Method.

Sumário

Os problemas de apreçamento de opções têm sido um dos principais assuntos de em Matemática Financeira, desde a criação desse conceito nos anos 70. Mais especificamente, as opções americanas são de grande interesse nesta área do conhecimento porque são matematicamente muito mais complexas do que as opções europeias padrão e o modelo de Black-Scholes não fornece, na maioria dos casos, uma fórmula explícita para a determinação do preço deste tipo de opções.

Nesta dissertação, mostramos como o estudo de opções americanas conduz à análise de problemas de fronteira livre devido à possibilidade de exercício antecipado, onde nosso principal objetivo é encontrar o preço de exercício ótimo. Também apresentamos a reformulação do problema em termos de um problema de complementaridade linear e de desigualdade variacional parabólica. Além disso, também abordamos a caracterização probabilística das opções americanas com base no conceito de tempos de paragem ótima. Essas formulações, aqui tratadas em termos analíticos ou probabilísticos, podem ser muito úteis na aplicação de métodos numéricos ao problema de precificação de opções do estilo americano, uma vez que, na maioria dos casos, é quase impossível encontrar soluções explícitas.

Além disso, utilizamos o Método da Árvore Binomial, que é um método numérico muito simples do ponto de vista matemático, para ilustrar alguns aspectos da teoria estudada ao longo desta tese e para comparar as opções americanas com as opções europeias e bermudas, por meio de alguns exemplos numéricos.

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1. Introduction

Pricing derivatives has been one of the most important issues of Mathematical Finance since the creation of this concept in the 1970s. A derivative is a financial security whose value depends entirely on (derives from) the price of an underlying asset or a benchmark - group of assets. There is a great variety of derivatives, for example, there are swaps, futures, forwards, and options. The latter are the main topic of this master thesis/dissertation, more specifically, American options: how they can be priced and what is their relation with the standard European options and with a more uncommon type of option, the Bermudan option.

The European style options are the most well-known and the simplest example of these kind of derivatives. Recall that a European option is a contract that gives its holder the right to purchase, if it is a call option, or to sell, if it is a put option, a predetermined amount of the underlying asset S for a given strike price K at a certain future date T, the maturity. The Black-Scholes model, proposed by Fisher Black and Myron Scholes in [2], in 1973, and later complemented by Robert Merton in [13] and [12], assumes that the underlying asset price $S(t)$ follows a geometric Brownian motion

$$dS(t) = rS(t)dt + \sigma S(t)d\widetilde{W}(t),$$

where r is the risk-free interest rate and σ is the volatility of $S(t)$, both strictly positive and constant, and $\widetilde{W}(t)$ is a Brownian motion under the risk-neutral probability measure $\widetilde{\mathbb{P}}$. Hence, the price process of a European call option paying no dividends $V(S,t)$ satisfies the Black-Scholes Partial Differential Equation

$$\frac{\partial V(S,t)}{\partial t} + \frac{\sigma^2}{2}S^2\frac{\partial V(S,t)}{\partial S^2} + rS\frac{\partial V(S,t)}{\partial S} - rV(S,t) = 0.$$

This model provides an explicit formula to calculate the price of European call and put options, the Black-Scholes formula:

$$\begin{cases} V(S,t) = Se^{-q(T-t)}N(d_1) - Ke^{-r(T-t)}N(d_2) & \text{for call options,} \\ V(S,t) = Ke^{-r(T-t)}N(-d_2) - Se^{-q(T-t)}N(-d_1) & \text{for put options,} \end{cases}$$

where $d_1 = \frac{\ln\left(\frac{S}{K}\right)+(r+\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}$ and $d_2 = d_1 - \sigma\sqrt{T-t}$.

The mathematical analysis of American options is much more complicated than that of European options. American options can be exercised anytime until the expiry date, that is what differs them from the European options, and this possibility of early exercise usually leads to a free boundary problem, where our main concern is to find the optimal exercise price. However, it is almost impossible to find an explicit solution to any given free boundary problem. But, related to these problems, linear complementarity problems and variational inequalities can be considered, which are crucial to a successful numerical approach of American options. The thinking behind this, since it is difficult to deal with free boundary problems, is that it is worth the effort to reformulate the problem in order to eliminate any dependence on the free boundary, and thus the latter will not interfere with the solution process, and it can be recovered after the solution is found.

In this dissertation, we will also approach option pricing with a numerical method. The Binomial Tree Method, first proposed by John Cox, Stephen Ross, and Mark Rubinstein in [4], in 1979, has become one of the most popular approaches to pricing options due to its simplicity and flexibility. Obviously, it is a discrete model of the option pricing problem, but it is a very useful tool when pricing American (and not only) options.

After this introduction, we proceed to Chapter 2, which is dedicated solely to definitions, properties, characterizations, and theoretical analysis of American options, which are mainly taken from [5], [13] and [14]. Firstly, in Section 2.1 we present some general concepts about American call and put options, where we show what differs them from the standard European calls and puts. Next, Section 2.2 consists on the definitions and properties of American style options as free boundary problems, which are fundamental for the study of these derivatives. Then, there is section 2.3, where we reformulate the free boundary problems presented in the previous section into linear complementary problems and variational inequalities. Section 2.4 focuses on a specific free boundary problem where the option pays no dividends, the American Put Problem, and its probabilistic characteristics, where the use of stopping times with martingales was introduced by Doob in [6]. We end Chapter 2 with section 2.5, which introduces the Bermudan options and shows their connection to the American type. In Chapter 3, we use the Binomial Tree Method, more specifically, the Cox, Ross and Rubenstein version of it, to present some numerical results and examples of pricing European, American and Bermudan options and to verify some of the theory exposed in Chapter 2. Chapter 4 is the final segment, where we present our conclusions and some thoughts on the matter in hands.

2. American Options

2.1 General concepts

Definition. *An American option is a contract between the writer and the holder that gives the holder the right to purchase, if it is a call option, or to sell, if it is a put option, the underlying asset,* $S(t)$*, at a certain exercise price,* K*, anytime* t *until the maturity,* T*.*

Since an American option can be exercised until maturity, this is where it differs from the European option, which can be exercised at maturity.

In this dissertation we are concerned with the problem of pricing American options, that is, of finding the price of the option according to the evolution of the value of the underlying and the time t, from 0 till maturity T. In particular, the price of such option at $t = 0$ gives the information of how much should the holder pay the writer in order to obtain this kind of derivative security, which is called the premium (as in European options). Moreover, since we can exercise an American option any time before the expiration time, information on the adequate value of the option at any time prior to maturity is also of utmost importance.

Due to the respective exercise conditions, it is clear that American options give the holder more rights than a corresponding European option. So, it should be expected that their price should be higher, i.e.

$$V^{AC}(S,t) \geq V^{EC}(S,t), \qquad\qquad V^{AP}(S,t) \geq V^{EP}(S,t)$$

for any time $t \in [0,T]$ and underlying asset $S \geq 0$. Moreover, under absence of arbitrage, the price of the American call and put options should be greater or equal than their price at maturity given by the pay-off diagram

$$V^{AC}(S,t) \geq V^{EC}(S,T) = (S-K)^{+},$$

$$V^{AP}(S,t) \geq V^{EP}(S,T) = (K-S)^{+},$$

for any time $t \in [0,T]$ and $S \geq 0$, where V^{AC}, V^{AP}, V^{EC} and V^{EP} denote the prices of the American call, the American put, the European call and the European put options, respectively. In fact, if we imagine, for example, that the price $V^{AC}(S,t)$ of an American call option at the time $t < T$ before the maturity T is less by one dollar than its terminal pay-off diagram $(S-K)^{+}$ then, by buying such an option and its immediate exercising (which is allowed for American options) we receive from the writer the underlying asset for the exercise price K. And if we sell it on the market, we receive its spot price S and the holder earns one dollar without bearing any risk, which would obviously lead to an arbitrage opportunity. If, however, by some mispricing, such arbitrage opportunity occurs in the market, then there will be a demand for this kind of options and, since there is such a demand, the market will increase its price to the level that is greater or equal to the pay-off diagram.

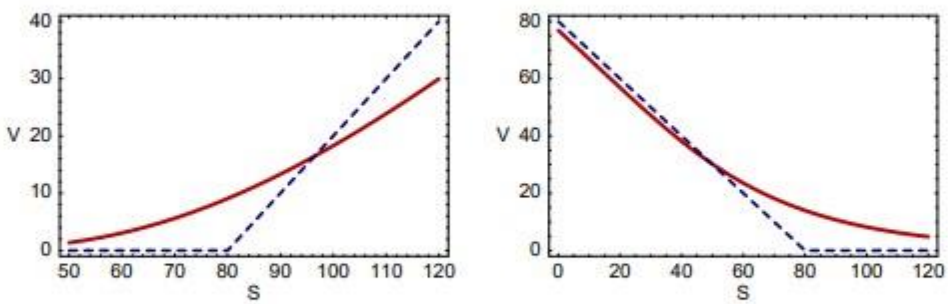

Fig. 1 – Graphs of solutions corresponding to the European call option on asset paying continuous dividends (left) and the European put option on asset paying no dividends (right).

Remarks 1. In Fig. 1, we can see that the price of the European call option on the underlying asset paying continuous dividends with a rate $q > 0$ always intersects the pay-off diagram. This behavior can be easily explained. In fact, from the explicit formula for pricing a European call option

$$V^{EC}(S,t) = Se^{-q(T-t)}N(d_1) - Ke^{-r(T-t)}N(d_2),$$

where $d_1 = \frac{\ln\left(\frac{S}{K}\right)+(r-q+\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}$ and $d_2 = d_1 - \sigma\sqrt{T-t}$, it follows that

$$\lim_{S\to\infty} \frac{V^{EC}(S,t)}{S} = e^{-q(T-t)} < 1.$$

For this reason, $V^{EC}(S,t) < S - K$ for a sufficiently large $S \gg K$ and $0 \le t < T$. Then, since $V^{EC}(0,t) = 0$ the above referred intersection holds.

2. Analogously, for the European put option on the asset with a dividend rate $q \geq 0$, the solution $V^{EP}(S,t)$ always intersects the pay-off diagram of the put option. Similarly, this can be justified by using

$$V^{EP}(S,t) = Ke^{-r(T-t)}N(-d_2) - Se^{-q(T-t)}N(-d_1)$$

and observing that the following inequality holds

$$V^{EP}(0,t) = Ke^{-r(T-t)} \leq K.$$

Also, in Fig. 1, we can see a graph of a solution representing the European put option and its comparison with the pay-off diagram.

Proposition *In the case of an American call option on the underlying asset paying no dividends* $(q = 0)$, *the price is equal to the European one, i.e.,*

- *if* $q = 0$ *then* $V^{AC}(S,t) = V^{EC}(S,t)$, *for each* $S \geq 0, t \in [0,T]$.

In fact, it is not worth to exercise the American call option before the expiry T. If we exercise the option early at the time $t \in [0,T)$ then its value falls to the value given by the pay-off diagram $(S-K)^+$. This means that its value is strictly less than the value of the European call option because $V^{EC}(S,t) > (S-K)^+$ when $q = 0$.

In the case of the American call option on the underlying asset paying dividends $(q > 0)$, the situation is slightly more complicated. In such case, the solution $V^{EC}(S,t)$ intersects the payoff diagram $(S-K)^+$. For that reason, we cannot argue the same way as in the case of $q = 0$. Furthermore, holding an American call option until the expiry $t = T$ would mean that its value is identical with European style of a call option. However, this is not possible because $V^{EC}(S,t) < (S-K)^+$ for large values of the underlying asset price $S \gg K$. Therefore, the price of the American call option is strictly higher than that of the European call option, i.e.

- if $q > 0, r > 0$, then $V^{AC}(S,t) > V^{EC}(S,t)$, for each $S > 0, t \in [0,T)$.

And because the graph of a solution of the European put option always intersects the pay-off diagram of a put option for time $t \geq 0$, we obtain the strict inequality

- if $q \geq 0, r > 0$, then $V^{AP}(S,t) > V^{EP}(S,t)$, for each $S \geq 0, t \in [0,T)$.

2.2 American Options as Free Boundary Problems

American style options give us the possibility of early exercise. Mathematically, pricing American style options involves the study of a free boundary for a parabolic equation, more precisely, for the Black-Scholes equation.

2.2.1 American call option paying dividends

First, we will consider the case of a call option on the underlying asset paying continuous dividends $q > 0$. Solving the problem means that we must find a function $V = V^{AC}(S,t)$ and, also, the free boundary position, that is, the function $S_f(t)$ depending on time $t \in [0,T]$, called the optimal exercise price. This function creates two regions. More precisely,

1. For S such that $0 < S < S_f(t)$, with $t \in [0,T]$, we have $V^{AC}(S,t) > (S-K)^+$; in this case we hold the call option because from the model we obtain a value $V^{AC}(S,t)$ strictly higher than the pay-off diagram of the call option. In order to evaluate the option price, we make use of the Black-Scholes model equation. More precisely, for $0 < t < T$ and $S < S_f(t)$, it holds true

$$\begin{cases} V^{AC}(S,t) > S - K \\ \dfrac{\partial V^{AC}}{\partial t} + \dfrac{\sigma^2}{2} S^2 \dfrac{\partial V^{AC}}{\partial S^2} + (r-q)S\dfrac{\partial V^{AC}}{\partial S} - rV^{AC} = 0. \end{cases}$$

2. If, for some $t \in [0,T]$ and some S, we have $S \geq S_f(t)$ then $V^{AC}(S,t) = (S-K)^+$; in this case, we should exercise the call option because its value coincides with the terminal pay-off diagram. Mathematically, $V^{AC}(S,t)$ satisfies

$$\begin{cases} V^{AC}(S,t) = S - K \\ \dfrac{\partial V^{AC}}{\partial t} + \dfrac{\sigma^2}{2} S^2 \dfrac{\partial V^{AC}}{\partial S^2} + (r-q)S\dfrac{\partial V^{AC}}{\partial S} - rV^{AC} < 0. \end{cases}$$

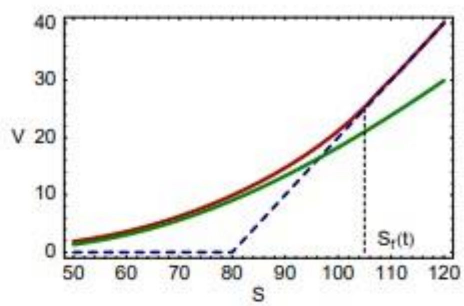

Fig. 2 – Solutions of the European and American call options at some time $0 \leq t < T$.

Thus:

Definition *The free boundary problem for pricing the American call option consists of finding a function* $V = V^{AC}(S,t)$ *and a function* $S_f(t)\colon [0,T] \to \mathbb{R}$ *determining the early exercise boundary with the following properties:*

1. *The function* $V^{AC}(S,t)$ *is a solution to the Black-Scholes partial differential equation:*
$$\frac{\partial V^{AC}(S,t)}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial V^{AC}(S,t)}{\partial S^2} + (r-q)S \frac{\partial V^{AC}(S,t)}{\partial S} - rV^{AC}(S,t) = 0$$
defined on a time dependent domain $0 < S < S_f(t)$, $0 < t < T$.
2. *It satisfies the terminal pay-off diagram:*
$$V^{AC}(S,T) = (S-K)^+$$
3. *and the boundary conditions:*
$$V^{AC}(0,t) = 0, \qquad V^{AC}\big(S_f(t),t\big) = S_f(t) - K, \qquad \frac{\partial V^{AC}}{\partial S}\big(S_f(t),t\big) = 1,$$
at $S = 0$ *and* $S = S_f(t)$, *as mentioned.*

Remark The boundary condition $\frac{\partial V}{\partial S}\big(S_f(t),t\big) = 1$, imposed on a solution at the point $S = S_f(t)$ of the early exercise of a call option, has a financial meaning. This condition and the continuity condition $V^{AC}(S_f(t)\,,t) = S_f(t)\, - K$ guarantee the C^1 continuity of the function $V^{AC}(S,t)$ in the S variable at the point $S = S_f(t)$, for each $t \in [0,T]$. It is obvious that the determination of the Dirichlet boundary conditions $V^{AC}(0,t) = 0$ (at $S = 0$) and $V^{AC}(S_f(t)\,,t) = S_f(t)\, - K$ at $(S = S_f(t))$ is not enough for the free boundary problem to have a unique solution. Indeed, it follows from the basic properties of solutions to parabolic equations (see [16], for example) that for any function $t \mapsto S_f(t)$ we can find a unique solution to the Black-Scholes equation that satisfies the Dirichlet conditions mentioned above at $S = 0$ and $S = S_f(t)$. Hence, we would have no other condition determining the free boundary profile $t \mapsto S_f(t)$. Therefore, we see that an additional condition on the free boundary position $S_f(t)$ is still needed.

Guaranteeing C^1 continuity of the contact of a solution $V^{AC}(S,t)$ and its pay-off diagram $(S-K)^+$, the condition $\frac{\partial V}{\partial S}\big(S_f(t),t\big) = 1$ is indeed the boundary condition fulfilled by an American call option. To show this, we will follow the idea of derivation of the boundary condition due to Merton that can be seen in [9] and which is based on a financial argument that states that the price $V^{AC}(S,t)$ of an American call option should be given as the maximal value among all call option prices whose early exercise boundary is determined by a continuous function of time. More precisely,

$$V^{AC}(S,t) = \max_{\eta} V(S,t;\eta)$$

where the maximum is taken over all positive continuous functions $\eta : [0,T] \to \mathbb{R}^+$. Here $V(S,t;\eta)$ denotes the price of a call option given by a solution to the Black-Scholes equation on a time dependent domain $0 < t < T$, $0 < S < \eta(t)$ and satisfying the Dirichlet boundary conditions $(0,t;\eta) = 0$, $V(\eta(t),t;\eta) = \eta(t) - K$, for $t \in [0,T]$. The early exercise boundary function S_f is then the argument of maximum of the above variational problem.

2.2.2 American put option paying dividends

Definiton The free boundary problem for pricing the American put option consists of finding a function $V = V^{AP}(S,t)$ together with the function $S_f(t)\colon [0,T] \to \mathbb{R}$ determining the early exercise boundary with the following properties:

1. The function $V^{AP}(S,t)$ is a solution to the Black-Scholes partial differential equation:
$$\frac{\partial V^{AP}}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial V^{AP}}{\partial S^2} + (r-q) S \frac{\partial V^{AP}}{\partial S} - rV^{AP} = 0$$
defined on a time dependent domain $S > S_f(t)$, $0 < t < T$.
2. It satisfies the terminal pay-off diagram:
$$V^{AP}(S,T) = (K-S)^+$$
3. and the boundary conditions:

$$V^{AP}(+\infty,t)=0, \qquad V^{AP}\big(S_f(t),t\big)=K-S_f(t), \qquad \frac{\partial V^{AP}}{\partial S}\big(S_f(t),t\big)=-1,$$

for $S=\infty$ and $S=S_f(t)$.

2.2.3 On the early exercise boundary position

In this section, we present several useful facts concerning the early exercise boundary position for American call and put options. First, we will consider the case of a call option. Notice that the early exercise boundary position should be greater or equal than the exercise price K. In fact, it is not rational to exercise a call option with the expiration price K when the spot price S of the underlying asset is less than K.

Because the function $S \mapsto V(S,t)$ is continuously differentiable with respect to the S variable at $S=S_f(t)$, we obtain, by differentiating the identity $V\big(S_f(t),t\big)=S_f(t)-K$ with respect to time t, the identity: $\frac{\partial V}{\partial S}\big(S_f(t),t\big)\dot{S}_f(t)+\frac{\partial V}{\partial S}\big(S_f(t),t\big)=\dot{S}_f(t)$. Considering the boundary conditions $\frac{\partial V}{\partial S}(S,t)=1$ for $S=S_f(t)$, we conclude that

$$\frac{\partial V}{\partial S}\big(S_f(t),t\big)=0, \qquad \text{for each } t\in(0,T).$$

Using the above expression and the fact that the Black-Scholes equation is valid within the interval $0<S<S_f(t)$, we obtain, by passing to the limit $S\to S_f(t)$:

$$qS_f(t)-rK = -(r-q)S_f(t)\frac{\partial V}{\partial S}\big(S_f(t),t\big)+rV\big(S_f(t),t\big)$$

$$=\frac{\sigma^2}{2}S_f(t)^2\frac{\partial V}{\partial S^2}\big(S_f(t),t\big)\geq 0$$

because the function $S\to V(S,t)$ has nonnegative second derivative at $S=S_f(t)$. In fact, if $\frac{\partial^2 V}{\partial S^2}(S,t)<0$ at $S=S_f(t)$ then, with regard to the boundary condition $\frac{\partial V}{\partial S}\big(S_f(t),t\big)=1$, we would obtain $V(S,t)<(S-K)^+$for all $S<S_f(t)$, where S is close to $S_f(t)$, a contradiction. Now, it follows from the expression above that

$$S_f(t)\geq K\max\left(\frac{r}{q},1\right), \text{ for each } t\in[0,T].$$

It remains to determine the terminal value $S_f(T)$ at the expiration T. Either $S_f(T) = K$ or $S_f(T) > K$. If $S_f(T) > K$ then, concerning the limit $V(S,t) \to S - K$ for $t \to T$, we can deduce that the second derivative $\frac{\partial^2 V}{\partial S^2}$ converges to zero for $S = S_f(t) > K$ as $t \to T$. And, considering the former identity once again, we obtain, in the limit $t \to T$, $K < S_f(T) = rK/q$. But this is possible only if $r > q > 0$. In both cases, we conclude

$$S_f(T) = K \max\left(\frac{r}{q}, 1\right).$$

Similarly, in the case of an American put option, we can show that the early exercise boundary position $S_f(t)$ has the following properties:

$$S_f(T) = K, \qquad S_f(t) \leq K, \text{ for each } t \in [0, T].$$

One of the most important problems in mathematic finance is the analysis of the early exercise boundary $S_f(t)$ and the optimal stopping time (an inverse function to $S_f(t)$) for American call and put options on assets paying a continuous dividend yield with a rate $q > 0$ (or $q \geq 0$). However, an exact analytical expression for the free boundary profile is not known yet.

2.3 American Options as Linear Complementary Problems

In this section, we will focus on the analysis of the Black-Scholes partial differential equation for the entire range of values $0 < S < \infty$ of the underlying asset price. It will be shown that the Black-Scholes inequality holds true for American options (which does not happen with European style options). For the American call option, the following partial differential inequality holds true:

$$\frac{\partial V^{AC}}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial V^{AC}}{\partial S^2} + (r - q) S \frac{\partial V^{AC}}{\partial S} - r V^{AC} \leq 0,$$

for each $0 < S < \infty$, $0 < t < T$.

We know that the Black-Scholes equation is satisfied on the time dependent interval $0 < S < S_f(t)$ in which we hold the option, that is, the expression above but with an equality sign. Meanwhile, for such values of the underlying asset S we have the strict inequality $V(S,t) > (S - K)^+$. However, if $S \geq S_f(t)$ then $(S,t) = (S - K)^+ = (S - K)$ because $S_f(t) \geq K$. Now, if we insert the linear function $S - K$ into the Black-Scholes equation, then we obtain

$$\frac{\partial V^{AC}}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial V^{AC}}{\partial S^2} + (r - q) S \frac{\partial V^{AC}}{\partial S} - r V^{AC}$$

$$= (r - q)S - r(S - K) = rK - qS \leq rK - qS_f(t) \leq 0,$$

because $S_f(t) \geq K \max\left(\frac{r}{q}, 1\right)$.

Analogously, for an American put option on the underlying asset paying no dividends ($q = 0$) we have that in the continuation interval $S > S_f(t)$ where we hold the put option, the Black-Scholes equation is satisfied and therefore the equality holds true. At the same time, we have the strict inequality $V^{AP}(S,t) > (K - S)^+$. If $0 < S \leq S_f(t)$ then $V^{AP}(S,t) = (K - S)^+ = K - S$ because $S_f(t) \leq K$. And if we insert the linear function $K - S$ in the Black-Scholes equation, then we obtain:

$$\frac{\partial V^{AP}}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial V^{AP}}{\partial S^2} + rS \frac{\partial V^{AP}}{\partial S} - r V^{AP}$$

$$= rS - r(S - K) = -rK < 0$$

In short, we have shown the following property which is called *Linear complementarity formulation for American options.*

Proposition *A solution to the problem of pricing the American style of call and put options satisfies:*

$$\frac{\partial V}{\partial t}+\frac{\sigma^2}{2}S^2\frac{\partial V}{\partial S^2}+(r-q)S\frac{\partial V}{\partial S}-rV\leq 0,$$

$$V(S,t)\geq \bar{V}(S),$$

$$\left(\frac{\partial V}{\partial t}+\frac{\sigma^2}{2}S^2\frac{\partial V}{\partial S^2}+(r-q)S\frac{\partial V}{\partial S}-rV\right)\left(V(S,t)-\bar{V}(S)\right)=0,$$

for any $0<S<\infty, 0<t<T$, *where* $\bar{V}$ *denotes the terminal pay-off diagram:*

$$\bar{V}(S)=\begin{cases}(S-K)^+, & \textit{for the call option}\\ (K-S)^+, & \textit{for the put option.}\end{cases}$$

Moreover, it can be stated the following result.

Proposition *Pricing an American call or put option by means of a solution to the linear complementarity problem can be mathematically done by finding a continuously differentiable function* $V(S,t)$ *such that it is a solution to the linear complementarity formulation and it satisfies the terminal pay-off diagram and corresponding boundaries.*

We can write the linear complementarity problem for pricing American call or put options in terms of a solution to a parabolic variational inequality. We can transform the Black-Scholes equation by using the following change of independent variables:

$$S=Ke^x, \qquad\qquad t=T-\tau,$$

where $x\in(0,\infty),\ \tau\in(0,T)$ and transformed function

$$V(S,t)=Ke^{-\alpha x-\beta\tau}u(x,\tau),$$

where

$$\alpha=\frac{r-q}{\sigma^2}-\frac{1}{2}, \qquad \beta=\frac{r+q}{2}+\frac{\sigma^2}{8}+\frac{(r-q)^2}{2\sigma^2}$$

After some calculations, we can deduce that the Black-Scholes equation can be rewritten in the form:

$$\frac{\partial u}{\partial \tau} = \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2}$$

for each $x \in \mathbb{R}, \tau \in (0,T)$. Since the American call or put option should satisfy the condition $V(S,t) \geq V(S,T) \equiv \bar{V}(S)$ we get the following condition for the transformed function:

$$u(x,\tau) \geq g(x,\tau),$$

for each $x \in \mathbb{R},\ \tau \in (0,T)$, where the function g corresponds to the transformed pay-off diagram of the call or put option, that is,

$$g(x,\tau) = e^{\alpha x+\beta\tau}\max(e^x - 1,0)\text{, for a call option,}$$

$$g(x,\tau) = e^{\alpha x+\beta\tau}\max(1 - e^x,0)\text{, for a put option,}$$

with the initial condition

$$u(x,0) = g(x,0),$$

for each $x \in \mathbb{R}$. For a call option we obtain the following boundary conditions:

$$u(-\infty,\tau) = g(-\infty,\tau) = 0, \quad \lim_{x\to\infty} u(x,\tau)/g(x,\tau) = 1,$$

for each $\tau \in (0,T)$. For a put option, we have

$$\lim_{x\to-\infty} u(x,\tau)/g(x,\tau) = 1, \quad u(+\infty,\tau) = g(+\infty,\tau) = 0,$$

for each $\tau \in (0,T)$.

Concisely, we can state:

Proposition *The linear complementarity problem for pricing the American call or put option can be written in the form of a parabolic variational inequality:*

$$\frac{\partial u}{\partial \tau} - \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2} \geq 0$$

$$u(x,\tau) - g(x,\tau) \geq 0$$

$$\left(\frac{\partial u}{\partial \tau} - \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2}\right)\big(u(x,\tau) - g(x,\tau)\big) = 0$$

for each $x \in \mathbb{R}$, $0 < \tau < T$.

Moreover, to solve the linear complementarity problem for pricing the American call or put option is to find a function $u: \mathbb{R} \times (0, T) \to \mathbb{R}$ *such that* u *is a continuously differentiable function satisfying the transformed linear complementarity inequation above and corresponding initial and boundary conditions.*

2.4 American Put Problem (paying no dividends)

2.4.1 Analytical formulation

For American options, the American put problem is probably the most studied as a free boundary problem. In fact, when the underlying asset pays no dividends, the American call option price is the same as the European call option price, as we saw in section 2.1, and therefore it becomes less interesting from the mathematical point of view.

We recall what was said in section 2.2 but considering now that no dividends are paid. It was shown that the Black-Scholes formula for a European put cannot give the correct price for an American put, since it predicts values below the payoff. Consider the Black-Scholes partial differential equation for the valuation of a European put:

$$\frac{\partial V^{EP}(S,t)}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial V^{EP}(S,t)}{\partial S^2} + rS \frac{\partial V^{EP}(S,t)}{\partial S} - rV^{EP}(S,t) = 0$$

with payoff

$$V^{EP}(S,T) = (K-S)^+.$$

We know that, moreover, a solution P satisfies the following boundary conditions:

$$V^{EP}(0,t) = Ke^{-r(T-t)}, \qquad V^{EP}(S,t) \to 0 \text{ as } S \to \infty.$$

The value of the European put falls below its intrinsic value for some values of S. We can easily see this by taking into consideration the value of the put option at $S = 0$. Here, the intrinsic value of the option is K but, from the boundary condition, $V^{EP}(0,t) = Ke^{-r(T-t)} \leq K$. Therefore, the value of the option is less than its intrinsic value for $t < T$. If we valued the American put option according to the European put option formula, there would be arbitrage possibilities. So, we must impose the condition

$$V^{EP}(S,T) \geq (K-S)^+$$

for the American put option.

It was shown that a free boundary condition must exist since the European put option formula does not satisfy the above condition. And now assume that $V^{EP} = K - S$ for some $S < K$. If this is the case, then $V^{EP}(S,t)$ clearly does not satisfy the Black-Scholes equation (unless $r = 0$) since

$$\frac{\partial}{\partial t}(K-S) + \frac{\sigma^2}{2}S^2\frac{\partial}{\partial S^2}(K-S) + rS\frac{\partial}{\partial S}(K-S) - r(K-S) = -rK < 0,$$

However, V^{EP} does satisfy the inequality. When $V^{EP} = K - S$ the return from the portfolio is less than the return from an equivalent bank deposit, so the exercise of the option is optimal.

In line with what we have already referred in the previous sections, at any given time t, we must divide the S axis into two distinct regions, one where early exercise is optimal:

$$V^{EP} = K - S, \qquad \frac{\partial V^{EP}}{\partial t} + \frac{\sigma^2}{2}S^2\frac{\partial V^{EP}}{\partial S^2} + rS\frac{\partial V^{EP}}{\partial S} - rV^{EP} < 0,$$

and the other, where early exercise is not optimal

$$V^{EP} > K - S, \qquad \frac{\partial V^{EP}}{\partial t} + \frac{\sigma^2}{2}S^2\frac{\partial V^{EP}}{\partial S^2} + rS\frac{\partial V^{EP}}{\partial S} - rV^{EP} = 0.$$

Let $S_f(t)$ be defined to be the largest value of S, at time t, for which we have $V^{EP}(S,t) = (K-S)^+$. Then

$$V^{EP}(S_f(t),t) = (K - S_f(t))^+$$

but

$$V^{EP}(S,t) > (K-S)^+, \qquad \text{if } S > S_f(t).$$

This defines the free boundary $S_f(t)$.

2.4.2 Probabilistic Characterization

The following theorems and proofs are mostly taken from [6] and [14].

First, in order to approach the probabilistic characterization of an American put option, we remind the concept of stopping times. A stopping time τ is a random variable taking values in $[0,\infty]$ and satisfying

$$\{\tau \leq t\} \in \mathcal{F}(t) \text{ for all } t \geq 0.$$

By this definition, a stopping time τ has the property that the decision to stop at time t must be based on information available at time t. The stopping times we shall face in this subject are the times at which an American option is exercised. The decision of an agent to exercise this option may depend on all the information available at that time but may not depend on future information.

Theorem (optional sampling). *A martingale stopped at a stopping time is a martingale. A supermartingale (or a submartingale) stopped at a stopping time is a supermartingale.(or submartingale, respectively).*

While the proof of this theorem will not be given here, the intuition is logical. If $M(t)$ is a martingale, then the stopped process $M(t \wedge \tau)$ agrees with $M(t)$ before time τ and thus is also a martingale. After time τ, the stopped process is frozen, that is, it no longer changes with time, and this is a trivial martingale. The only way the martingale property could be violated is if the stopping decision looked ahead.

Analogous intuition applies to supermartingales: a stopped supermartingale is a supermartingale before being frozen, and after being frozen it is a martingale, which is still a supermartingale. Again, the stopping must be done at a stopping time. Looking ahead to make the stopping decision can ruin the supermartingale property.

Throughout this section, we will also consider an American put on a stock whose price is the geometric Brownian motion

$$dS(t) = rS(t)dt + \sigma S(t)d\widetilde{W}(t),$$

where the interest rate r and the volatility σ are strictly positive constants and $\widetilde{W}(t)$ is a Brownian motion under the risk-neutral probability measure $\widetilde{\mathbb{P}}$.

Let $0 \leq t \leq T$ and $x \geq 0$ be given. Assume $S(t) = x$. Let $\mathcal{F}_u^{(t)}$, $t \leq u \leq T$, denote the σ-algebra generated by the process $S(v)$ as v ranges over $[t, u]$, and let $\mathcal{T}_{t,T}$ denote the set of stopping times for the filtration $\mathcal{F}_u^{(t)}$, $t \leq u \leq T$, taking values in $[t, T]$ or taking the value ∞. In other words, $\{\tau \leq u\} \in \mathcal{F}_u^{(t)}$ for every $u \in [t, T]$; a stopping time in $\mathcal{T}_{t,T}$ makes the decision to stop at a time $u \in [t, T]$ based only on the path of the stock price between times t and u. The price at time t of the American put option expiring at time T is defined to be

$$V(t,x) = \max_{\tau \in \mathcal{T}_{t,T}} \widetilde{\mathbb{E}}\big[e^{-r(\tau-t)}\big(K - S(\tau)\big)\big|S(t) = x\big]. \tag{1}$$

If $\tau = \infty$, we interpret $e^{-r(\tau-t)}(K - S(\tau))$ to be zero. This is the case when the put expires unexercised.

Theorem. *Let $S(u)$, $t \le u \le T$, be the stock price, a geometric Brownian motion, starting at $S(t) = x$ and with the stopping set $\mathcal{S}$ defined by*

$$\mathcal{S} = \{(t,x); V(t,x) = (K - x)^+\}.$$

Let

$$\tau_* = \min\{u \in [t,T]; \big(u, S(u)\big) \in \mathcal{S}\},$$

where we interpret τ_* to be ∞ if $\big(u, S(u)\big)$ does not enter $\mathcal{S}$ for any $u \in [t,T]$. Then

$e^{-ru}V(u, S(u), t \le u \le T$

is a supermartingale under $\widetilde{\mathbb{P}}$, and the stopped process $e^{-r(u\wedge\tau_*)}V(u, S(u \wedge \tau_*))$, $t \le u \le T$, is a martingale.

Proof. The Itô-Doeblin formula applies to $e^{-ru}V\big(u, S(u)\big)$, even though $V_u(u,x)$ and $V_{xx}(u,x)$ are not continuous along the curve $x = L(t-u)$ because the process $S(u)$ spends zero time on this curve. All that is needed for the Itô-Doeblin formula to apply is that $V_x(u,x)$ be continuous, and this follows from the boundary condition $V_x(t,x) = -1$ for $x = L(T-t)$, $0 \le t \le T$, where $L(T-t)$ corresponds to the level at or below K that the stock price must fall to before the option is worth being exercised. Thus, we can compute

$$d\big[e^{-ru}V\big(u,S(u)\big)\big] =$$

$$= e^{-ru}\left[-rV\big(u,S(u)\big)du + V_u\big(u,S(u)\big)du + V_x\big(u,S(u)\big)dS(u) + \frac{1}{2}V_{xx}\big(u,S(u)\big)dS(u)dS(u)\right]$$

$$= e^{-ru}\left[-rV\big(u,S(u)\big) + V_u\big(u,S(u)\big) + rS(u)V_x\big(u,S(u)\big) + \frac{1}{2}\sigma^2 S(u)^2 V_{xx}\big(u,S(u)\big)\right]du$$
$$+ e^{-ru}\sigma S(u)V_x\big(u,S(u)\big)d\widetilde{W}(u).$$

The du term is $-e^{-ru}rK\mathbb{I}_{\{S(u)<L(T-u)\}}$. This is nonpositive, and so $e^{-ru}V(u,S(u)$ is a supermartingale under $\widetilde{\mathbb{P}}$. In fact, starting from $u = t$ and up until time τ_*, we have

$S(u) > L(T-u)$. So, the du term is zero. Therefore, the stopped process

$e^{-r(u\wedge\tau_*)}V\big(u\wedge\tau_*, S(u\wedge\tau_*)\big)$, $t < u < T$, is a martingale. ■

Corollary. *Consider an agent with initial capital $X(0) = V(0, S(0))$, the initial put price. Suppose the agent uses the portfolio process $\Delta(u) = V_x(u, S(u))$ and consumes cash at rate $C(u) = rK\mathbb{I}_{\{S(u)<L(T-u)\}}$ per unit time. Then $X(u) = V(u, S(u))$ for all times u between $u = 0$ and the time the option is exercised or expires. In particular, $S(u) \geq (K - S(u))^+$ for all times u until the option is exercised or expires, so the agent can pay off a short position regardless of when the option is exercised.*

Proof. The differential of the agent's portfolio value process is

$$dX(t) = \Delta(t)dS(t) + r\big(X(t) - \Delta(t)S(t)\big)dt - C(t)dt.$$

So, the differential of the discounted portfolio value process is

$$\begin{aligned} d\big(e^{-rt}X(t)\big) &= e^{-rt}\big(-rX(t)dt + dX(t)\big) \\ &= e^{-rt}(\Delta(t)dS(t) - r\Delta(t)S(t)dt - C(t)dt) \\ &= e^{-rt}\big(\Delta(t)\sigma S(t)d\widetilde{W}(t) - C(t)dt\big). \end{aligned}$$

Substituting for $\Delta(u)$ and $C(u)$ in this equation and comparing it to $d\big[e^{-ru}V\big(u, S(u)\big)\big]$, we see that $d[e^{-ru}X(u)] = d\big[e^{-ru}V\big(u, S(u)\big)\big]$. Integrating this equation and using $X(0) = V\big(0, S(0)\big)$, we obtain $X(t) = V\big(t, S(t)\big)$ for all times t prior to exercise or expiration. ■

Observation. The previous proofs are so that the Itô-Doeblin formula can be applied. Here we show that the only function $V(t, x)$ satisfying these conditions is the function $V(t, x)$ defined by (1). To do this, we first fix t with $0 \leq t \leq T$. The supermartingale property for $e^{-rt}V\big(t, S(t)\big)$, presented by the previous theorems, implies that

$$e^{-r(t\wedge\tau)}V\big(t\wedge\tau, S(t\wedge\tau)\big) \geq \widetilde{\mathbb{E}}\big[e^{-r(T\wedge\tau)}V\big(T\wedge\tau, S(T\wedge\tau)\big)\big|\mathcal{F}(t)\big].$$

For $\tau \in \mathcal{T}_{t,T}$, we have $t \wedge \tau = t$, whereas $T \wedge \tau = \tau$ if $\tau < \infty$ and $T \wedge \tau = T$ if $\tau = \infty$. Therefore, for $\tau \in \mathcal{T}_{t,T}$,

$$e^{-rt}V\big(t, S(t)\big) \geq \widetilde{\mathbb{E}}\big[e^{-r\tau}V\big(\tau, S(\tau)\big)\mathbb{I}_{\{\tau<\infty\}} + e^{-rT}V\big(T, S(T)\big)\mathbb{I}_{\{\tau=\infty\}}\big|\mathcal{F}(t)\big]$$

$$\geq \tilde{\mathbb{E}}\left[e^{-r\tau}V\big(\tau, S(\tau)\big)\middle|\mathcal{F}(t)\right],$$

where, as usual, we interpret $e^{-r\tau}V\big(\tau, S(\tau)\big) = 0$ if $\tau = \infty$. Inequality $V\big(t, S(t)\big) \geq \big(K - S(t)\big)^{+}$ and the fact that $(K - S(t))^{+} \geq K - S(t)$ imply that

$$\tilde{\mathbb{E}}\left[e^{-r\tau}V\big(\tau, S(\tau)\big)\middle|\mathcal{F}(t)\right] \geq \tilde{\mathbb{E}}[e^{-r\tau}(K - S(\tau))|\mathcal{F}(t)].$$

Putting this last two expressions together, we conclude that

$$e^{-rt}V\big(t, S(t)\big) \geq \tilde{\mathbb{E}}\left[e^{-r\tau}\big(K - S(\tau)\big)\middle|\mathcal{F}(t)\right].$$

Because $S(t)$ is Markov process, the right-hand side is a function of t and $S(t)$. Specifically, if we denote the value of $S(t)$ by x, we may rewrite this as

$$e^{-rt}V(t, x) = \tilde{\mathbb{E}}\left[e^{-r\tau}\big(K - S(\tau)\big)\middle|S(t) = x\right].$$

Since this holds for any $\tau \in \mathcal{T}_{t,T}$, we conclude that

$$V(t, x) \geq \max_{\tau \in \mathcal{T}_{t,T}} \tilde{\mathbb{E}}\left[e^{-r\tau}\big(K - S(\tau)\big)\middle|S(t) = x\right].$$

For the reverse inequality, we recall (from the Theorem of the supermartingales) that the stopped process $e^{-r(t\wedge\tau_*)}V\big(t \wedge \tau_*, S(t \wedge \tau_*)\big)$ is a martingale, where τ_*, defined earlier, is such that $V\big(\tau_*, S(\tau_*)\big) = K - S(\tau_*)$ if $\tau_* < \infty$. Replacing τ by τ_*, we turn the first inequality into an equality. If $\tau_* = \infty$, we have $\big(T, S(T)\big) \in \mathcal{C}$ (that is, $S(T) > K$), so $V\big(T, S(T)\big)\mathbb{I}_{\{\tau_* = \infty\}} = 0$. This makes the second inequality into an equality. Finally, because $V\big(\tau, S(\tau)\big) = K - S(\tau)$ on $\mathbb{I}_{\{\tau < \infty\}}$, the third inequality is an equality, and therefore we get

$$V(t, x) = \tilde{\mathbb{E}}\left[e^{-r(\tau_* - t)}\big(K - S(\tau_*)\big)\middle|S(t) = x\right].$$

This equation shows that the equality must hold in the last inequality, and this corresponds to the price at time t of the American put expiring at time T that we defined in the beginning of the section.

2.5 Bermudan Options

A Bermudan option is an intermediate option between an American and a European that may be exercised only on one of a finite set of dates and its value is always equal or greater than the value of an European option and equal or less than the value of an American option. We shall denote by $\mathcal{B}_\Delta$ the Bermuda put option with allowable exercise times $\left\{k\Delta: k = 1,2, \dots, \left[\frac{T}{\Delta}\right]\right\} \cup \{T\}$, and by $V(t, S_t; T, \Delta)$ its value at time t.

If the set of allowable exercise times is just $\{T\}$ (the maturity) then the option reduces to the European type. On the other hand, if this set is $\left\{k\Delta: k = 1,2, \dots, [\frac{T}{\Delta}]\right\} \cup \{T\}$ where $\Delta > 0$ is small, then the option approximates an American option.

Proposition. *As* $\Delta \downarrow 0$, *the time-zero value* $V(0, S_0; T, \Delta)$ *of the Bermudan put option* $\mathcal{B}_\Delta$ *converges to the value* $V^{AP}(0, S_0)$ *of the American put option.*

Bermudan options are of interest in part because they are not just theoretical and really are traded but also because we can use an interactive scheme called dynamic programming (or backward induction) to numerically compute their arbitrage prices. The main idea is that if one decides not to exercise the option at the first possible exercise time, then it is converted to another Bermudan option, but with one fewer possible exercise date. Thus, we can relate the price of the original Bermudan option to that of a Bermudan option with one less allowable exercise time. The price of this option may, similarly, be related to that of yet another Bermudan option with still on less allowable exercise time, and so on, until finally all prices are related to that of a put option with just one allowable exercise time, which will be basically a European put option and its price is given by the Black-Scholes formula.

3. Binomial Tree Method

The European, Bermudan and American options provide their owners successively larger sets of possibilities, and so their values must be ordered as follows

$$V^E(t) \leq V^B(t) \leq V^A(t)$$

and we can easily show this by using the binomial tree method, instead of the more complex and analytical Black-Scholes model.

The binomial tree method is a numerical method and one of the most popular approaches for evaluating the price of options because of its simplicity and flexibility. Since the model is binomial, there are only two possible outcomes: a move up, or a move down, that is, the underlying asset can only be worth one of two possible values in one time period, which is not realistic, as assets can take any number of values within any time range.

The greatest advantage of the binomial option pricing model is that it is relatively simple from the mathematical point of view. This model reduces possibilities of price changes and removes any possibility for arbitrage, and it can be a useful tool to value American options (and embedded options) by means of iteration using multiple periods.

Unlike the Black-Scholes model, which provides a numerical result based on inputs, the binomial model allows for the calculation of the asset and the option for multiple periods along with the range of possible results for each period. This multi-period view allows the user to visualize the change in asset price from period to period and evaluate the option based on decisions made at different points in time. For an American option, which can be exercised at any time before the expiration date, the binomial model can clarify when exercising the option may be the best choice and when it should be held. By looking at the binomial tree of values, a trader can determine in advance when a decision on an exercise may occur.

3.1 Cox, Ross and Rubinstein Model

Let us consider a stock with initial price S_0 undergoing a random walk. Over a time step Δt, the stock has a probability p of rising by a factor u and a probability of $1 - p$ of falling by a factor d.

Cox, Ross and Rubinstein were who first proposed a method for computing p, u and d, and their model is actually the most popular amongst binomial models. Over a small period of time, the binomial model acts similarly to an asset that exists in a risk neutral world. This results in the following equation, which implies that the effective return of the binomial model is equal to the risk-free rate

$$pu + (1 - p)d = e^{r\Delta t}$$

Furthermore, the variance of a risk-neutral asset and an asset in a risk neutral world match. This gives the following equation

$$pu^2 + (1 - p)d^2 - (e^{r\Delta t})^2 = \sigma^2 \Delta t$$

The Cox, Ross and Rubinstein model suggests the following relationship between the upside and downside factors

$$u = \frac{1}{d}.$$

Rearranging these equations, we get the following equations for p, u and d

$$p = \frac{e^{r\Delta t} - d}{u - d},$$

$$u = e^{\sigma\sqrt{\Delta t}},$$

$$d = e^{-\sigma\sqrt{\Delta t}}.$$

The values of p, u and d given by the Cox, Ross and Rubenstein model ensure that the underlying initial asset price is symmetric for a multi-step binomial model.

We can now present the backward induction expression of the options prices, which are the main focus of the Binomial Tree Method. Considering that $T > 0$ is the maturity, set N to be the number of discrete time points (nodes), $\Delta t = T/N$ to be a time interval, $t_n = n\Delta t$, $n = 0, 1, \dots, N$. Denote $V^i = V^i(S, t)$, $i = e, a, b$, the option price with underlying asset value S for each type of option, European, American and Bermudan, respectively. Set $V_{n,j}^i = V^i(S_j, t_n)$. It is assumed that S_j will jump either up to $S_j u$ with probability p or down to $S_j d$ with probability $(1 - p)$.

As we know, European options are exercised at maturity:

$$\phi_j = V_{N,j}^e = \begin{cases} (K - S_j)^+ \text{for put options,} \\ (S_j - K)^+ \text{ for call options,} \end{cases}$$

and their backward induction option pricing is given by:

$$V_{n,j}^e = e^{-(r-q)\Delta t}\left(pV_{j+1}^{n+1} + (1-p)V_{j-1}^{n+1}\right).$$

In the case of American options, their exercise can occur anytime until maturity. Thus, their pricing takes this feature into account. Therefore, it is given by:

$$\begin{cases} V_{n,j}^a = \max\left\{e^{-(r-q)\Delta t}\left(pV_{j+1}^{n+1} + (1-p)V_{j-1}^{n+1}\right), \phi_j\right\}, \qquad 0 \le n \le N-1, \\ \phi_j = V_{N,j}^a = \begin{cases} (K - S_j)^+ \text{for put options,} \\ (S_j - K)^+ \text{ for call options.} \end{cases} \end{cases}$$

Finally, in the case of Bermudan options, their exercise can occur in predefined points t_k (including maturity T). Then, their option pricing is given by

$$\begin{cases} V_{n,j}^b = \max\left\{e^{-(r-q)\Delta t}\left(pV_{j+1}^{n+1} + (1-p)V_{j-1}^{n+1}\right), \phi_j\right\}, 0 \le n \le N-1, \text{for } t = t_k \\ V_{n,j}^b = e^{-(r-q)\Delta t}\left(pV_{j+1}^{n+1} + (1-p)V_{j-1}^{n+1}\right), \qquad 0 \le n \le N-1, \text{for others } t \\ \phi_j = V_{N,j}^b = \begin{cases} (K - S_j)^+ \text{for put options,} \\ (S_j - K)^+ \text{ for call options.} \end{cases} \end{cases}$$

3.2 Numerical Results: Pricing European, American and Bermudan Options

In this section, we will put into practice the Cox, Ross and Rubinstein version of the Binomial Tree Method and illustrate some aspects of the theory presented in Chapter 2, by showing some numerical examples. These examples will consist on comparing the prices of the different types of option when applying the same parameters to all.

In the end of section 2.1, for example, we saw that the American call option paying no dividends ($q = 0$) has the same value as the European option (and as the Bermudan option, since $V^{EC} \leq V^{BC} \leq V^{AC}$). So, for the first example, we chose to calculate the value of a European, an American and a Bermudan call options paying no dividends with the following parameters:

Parameters	Values
Stock Price (S)	$100
Strike Price (K)	$100
Maturity (T)	5 years
Volatility (σ)	30%
Risk Free Interest Rate (r)	5%
Dividend Yield (q)	0%
t_k	1y, 2y, 3y, 4y

Table 1 – Parameters.

From them, after some computations, we get the following values for u, d and p:

Table 2 – Values of u, d and p.

Up Movement (u)	1.16183
Down Movement ($d = 1/u$)	0.86071
Up Probability (p)	0.5043

Finally, by means of the backward induction expressions, we get the following option values:

Table 3 – Call option prices ($q = 0$)

European	Bermudan	American
$35.65	$35.65	$35.65

In this case, the option prices are the same for all the style options, as expected. However, the same does not happen in the case of a put option or when a dividend yield $q > 0$ is introduced.

The next example consists of a call option with the same parameters but now the dividend yield is 10%. We then get:

Table 4 – Call option prices ($q = 0.1$)

European	Bermudan	American
\$10.77	\$14.70	\$15.47

Once again, we were able to corroborate the theoretical assumption $V^{EC} \leq V^{BC} \leq V^{AC}$. As we can see, the value of the Bermudan call option lies between the European and American call options.

4. Conclusion

Pricing American style options is one of the most discussed subjects, not only because of its financial interest but also due to its rich mathematical structure, either analytical or probabilistic. In fact, the Black-Scholes model used for pricing derivatives leads in this case to the mathematical study of a free boundary problem hard to solve analytically, with explicit solution known only for very particular cases. American options also admit a probabilistic characterization based on the concept of stopping times.

In this dissertation, after reviewing some basic concepts, we presented the formulation of the Black-Scholes model for an American call or put option problem as a free boundary problem, and then we showed that it can be reduced to a linear complementarity problem and a parabolic variational inequality. These reformulations of the American option pricing problem give us the advantage of not having an explicit mention of the free boundary. Therefore, if we can solve either one, then we find the optimal exercise boundary. Besides this analytical structure, we also mentioned the probabilistic characterization of American options. All these formulations can be very useful in more sophisticated situations where there is no explicit solution and numerical methods have to be applied for pricing. In the last part of this dissertation, we introduced Bermudan options and then, after reviewing the binomial tree method, we made a comparison between America, Bermudan and European options. We ended with a numerical example that illustrated the theory.

Our study has always been developed in the framework of the classical Black-Scholes model. As future work, we intend to study American options assuming existence of transaction costs, which implies studying a generalization of the Black-Scholes model. Mathematically, this will lead us to consider nonlinear free-boundary problems. It is currently a very important research topic in option pricing. Since the work of Leland [10], and also of Avellaneda and Paras [1], a large number of papers appeared concerning generalizations of the Black-Scholes model (see, for example, [3], [7], [8] and [15], and the references there contained). They contributed to a better understanding of how to overcome the disadvantages that appear in the financial markets due to the restrictive and unreal conditions of the classic Black-Scholes model.

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