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Module sectional category of products

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Abstract Adapting a result of Félix–Halperin–Lemaire concerning the Lusternik– Schnirelmann category of products, we prove the additivity of a rational approximation for Schwarz's sectional category with respect to products of certain fibrations.

Keywords Rational homotopy · Sectional category · Topological complexity

Mathematics Subject Classification 55M30 · 55P62

1 Introduction

The sectional category [12] (or Schwarz genus) of a fibration $p : E \to X$, secat(p), is the smallest integer *m* such that *X* admits a cover by (m + 1) open sets on each of which a local section for *p* exists. This homotopy invariant is a generalization of

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the well-known Lusternik-Schnirelmann (L.-S.) category [10] of a path-connected space X, cat(X), as the latter is the sectional category of the path fibration $PX \rightarrow X$, $\alpha \mapsto \alpha(1)$, where PX is the space of paths starting at the base point.

One of the most important results of [5] says that, if X and Y are simply connected rational spaces of finite type, then $cat(X \times Y) = cat(X) + cat(Y)$. This was achieved by first proving the analogous result for the lower bound module L.-S. category, mcat(X), of cat(X) using differential graded (DG) module techniques. It was then lifted to rational category using Hess' theorem [9]. We propose to apply similar DG-module techniques to the lower bound msecat(p) of secat(p) called module sectional category and introduced in [7].

Throughout this paper we consider fibrations whose base and total space have the homotopy type of simply connected CW-complexes of finite type. Our main result is

Theorem 1 Let p and p' be two fibrations. If either p or p' admits a homotopy retraction, then

 $\operatorname{msecat}(p \times p') = \operatorname{msecat}(p) + \operatorname{msecat}(p').$

Recall the important particular case of sectional category provided by Farber's (higher) topological complexity [4,11] of a space X, $TC_n(X) = secat(\pi_n)$, where the considered fibration $\pi_n \colon X^{[1,n]} \to X^n$ is given by $\pi_n(\alpha) = (\alpha(1), \alpha(2), \dots, \alpha(n))$. Consequently, the module invariant associated to (higher) topological complexity, i.e.,

 $\operatorname{mTC}_n(X) := \operatorname{msecat}(\pi_n),$

is additive on products. Namely

Corollary 2 Let X and Y be two spaces. Then

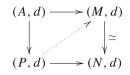
 $mTC_n(X \times Y) = mTC_n(X) + mTC_n(Y).$

These results are improvements over [2] as only one of the two fibrations of Theorem 1 needs a homotopy retraction and the Poincaré duality assumption is no longer required.

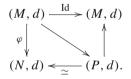
2 Preliminaries

This section contains a brief summary of the DG-modules techniques that will be used (see [6] for further details). Let (A, d) be a commutative differential graded algebra over \mathbb{Q} (cdga). An (A, d)-module is a chain complex (M, d) together with a degree 0 action of A satisfying $d(ax) = (da)x + (-1)^{|a|}a(dx)$. A *semifree extension* of an (A, d)-module (M, d) is an (A, d)-module of the form $(M \oplus A \otimes U, d)$ where the action is the one of the direct sum, the differential on M is the differential of (M, d), and U admits a direct sum decomposition $U = \bigoplus_{i=0}^{\infty} U_i$ such that $d(U_0) \subset M$ and $d(U_n) \subset M \oplus A \otimes (\bigoplus_{i=0}^{n-1} U_i)$ for $n \ge 1$. A *semifree* (A, d)-module is a semifree extension $(A \otimes U, d)$ of the trivial (A, d)-module 0 and the data of a quasi-isomorphism

 $(A \otimes U, d) \xrightarrow{\simeq} (M, d)$ is called a *semifree resolution* of (M, d). The category of (A, d)-modules is a proper closed model category in which semifree extensions are cofibrations (see, for instance, [7, Theorem 4.1]). Two (A, d)-module morphisms $\phi, \psi: (M, d) \to (N, d)$ are *homotopic* if there is an A-linear map $\theta: M \to N$ of degree -1 such that $\phi - \psi = d\theta + \theta d$. We will frequently use the fact that any (A, d)-module morphism $\varphi: (M, d) \to (N, d)$ can be decomposed as (the inclusion of) a semifree extension followed by a quasi-isomorphism as well as the following lifting lemma. Given a solid arrow commutative diagram of (A, d)-modules of the form



in which the morphism $(A, d) \rightarrow (P, d)$ is a semifree extension, there is an (A, d)module morphism $(P, d) \rightarrow (M, d)$ making commutative the upper triangle and
homotopy commutative (rel. A) the lower triangle. A morphism of (A, d)-modules $\varphi: (M, d) \rightarrow (N, d)$ is said to have a homotopy retraction if there exists a commutative
diagram of (A, d)-modules,



If *M* is an (A, d)-module, the module $M^{\#} = \hom(M, \mathbb{Q})$ admits an (A, d)-module structure with action $(a\varphi)(x) = (-1)^{|\alpha| \cdot |\varphi|} \varphi(ax)$ and differential $d\varphi = (-1)^{|\varphi|} \varphi \circ d$. If *N* is an (A, d)-module, then the module $M \otimes_A N$ admits an (A, d)-module structure with action $a(m \otimes n) = (am) \otimes n$ and differential $d(m \otimes n) = dm \otimes n + (-1)^{|m|} m \otimes dn$. If *P* is (A, d)-semifree and if η is a quasi-isomorphism of (A, d)-modules then $\eta \otimes_A \operatorname{Id}_P$ and $\operatorname{Id}_P \otimes_A \eta$ are also quasi-isomorphisms.

The following lemma is an adaptation of a central idea of [5].

Lemma 3 Let φ : $(A, d) \to (B, d)$ be a surjective cdga morphism with kernel K and A of finite type. The morphism φ admits a homotopy retraction of (A, d)-modules if and only if for any (A, d)-semifree resolution $\eta: P \xrightarrow{\simeq} A^{\#}$, the projection

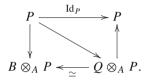
$$\varrho\colon P\longrightarrow \frac{P}{K\cdot P}$$

is injective in homology.

Proof Suppose that φ admits a homotopy retraction of (A, d)-modules. This means that there exists a homotopy commutative diagram of (A, d)-modules of the form



where Q is an (A, d)-semifree resolution of B. Now let $\eta : P \xrightarrow{\simeq} A^{\#}$ be an (A, d)-semifree resolution. By applying $- \bigotimes_A P$ to the diagram above, we get

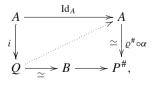


Since *B* and $\frac{A}{K}$ are isomorphic cdgas, we have $B \otimes_A P = \frac{P}{K \cdot P}$. Hence the left hand morphism is simply the projection $\varrho: P \to \frac{P}{K \cdot P}$. The diagram shows that ϱ admits a homotopy retraction of (A, d)-modules. Hence it is injective in homology.

Conversely, suppose that ρ is injective in homology. Since A is of finite type, $\eta^{\#} \colon A \to P^{\#}$ is also an (A, d)-semifree resolution. Moreover,

$$\varrho^{\#} \colon \left(\frac{P}{K \cdot P}\right)^{\#} \to P^{\#}$$

is surjective in homology. Hence there exists a cycle $\gamma \in \left(\frac{P}{K \cdot P}\right)^{\#}$ such that $[\gamma \circ \varrho] = [\eta^{\#}(1)]$. Now define an (A, d)-module morphism $\alpha \colon A \to \left(\frac{P}{K \cdot P}\right)^{\#}$ by setting $\alpha(1) = \gamma$. Then $\varrho^{\#} \circ \alpha$ is a quasi-isomorphism. To finish the proof, we observe that $K \cdot \left(\frac{P}{K \cdot P}\right)^{\#} = 0$. Hence the map $\varrho^{\#} \circ \alpha$ factors through φ as B = A/K. Let $A \stackrel{i}{\to} Q \stackrel{\simeq}{\to} B$ be a decomposition of φ as a semifree extension followed by a quasi-isomorphism. Applying the lifting lemma to the solid arrow commutative diagram



we obtain the desired homotopy (A, d)-module retraction for φ .

3 The invariant msecat(*p*)

Let us denote by $p_m: J_X^m(E) \to X$ the join of m + 1 copies of a fibration $p: E \to X$. As is well-known [12], secat $(p) \le m$ if and only if p_m admits a homotopy section. By definition, msecat(p) is the smallest m such that $A_{PL}(p_m)$ admits a homotopy retraction of $A_{PL}(X)$ -modules, where A_{PL} denotes Sullivan's functor of piecewise linear forms [13].

Let $\varphi \colon (A, d) \to (B, d)$ be any cdga model of p and

$$(A,d) \hookrightarrow (A \otimes (\mathbb{Q} \oplus U), d) \xrightarrow{\varsigma} (B,d).$$
(1)

a factorization in the category of (A, d)-modules of φ as the inclusion of a semifree extension followed by a quasi-isomorphism ξ . We refer to the inclusion as a semifree model of p. For $x \in U$, we write $dx = d_0x + d_+x$, where $d_0x \in A$ and $d_+x \in A \otimes U$. We notice that, if φ is surjective, then the quasi-isomorphism ξ can be constructed to satisfy $\xi(U) = 0$, which implies that $d_0x \in \ker \varphi$ for $x \in U$. Recall that the n^{th} -suspension $s^{-n}V$ of a graded vector space V is defined by $(s^{-n}V)^i = V^{i-n}$.

According to [7] (Thm 5.4, p.135), msecat(p) is the least m such that the following (A, d)-semifree model of p_m

$$j_m \colon (A, d) \to \underbrace{(A \otimes (\mathbb{Q} \oplus s^{-m}U^{\otimes m+1}), D)}_{J_m}.$$

admits a retraction of (A, d)-modules, where the differential D is given by

$$D(s^{-m}x_0 \otimes \cdots \otimes x_m) = (-1)^{\sum_{k=1}^{m} (k|x_{m-k}|+k-1)} d_0 x_0 \cdots d_0 x_m + \sum_{i=0}^{m} \sum_{j_i} (-1)^{(|a_{ij_i}|+1)(|x_0|+\cdots+|x_{i-1}|+m)} a_{ij_i} \otimes s^{-m} x_0 \otimes \cdots \otimes x_{ij_i} \otimes \cdots \otimes x_m,$$

for $x_0, ..., x_m \in U$ and $d_+x_i = \sum_{j_i} a_{ij_i} \otimes x_{ij_i}$ with $a_{ij_i} \in A$ and $x_{ij_i} \in U$.

Using the following notation (suggested by the standard rules of signs)

$$s^{-m}x_0\otimes\cdots\otimes d_+x_i\otimes\cdots\otimes x_m:=\sum_{j_i}\sigma_{ij_i}a_{ij_i}\otimes s^{-m}x_0\otimes\cdots\otimes x_{ij_i}\otimes\cdots\otimes x_m,$$

we can write $D_+(s^{-m}x_0\otimes\cdots\otimes x_m)$ as

$$D_+(s^{-m}x_0\otimes\cdots\otimes x_m)=(-1)^m\sum_{i=0}^m\sum_{j_i}\tau_is^{-m}x_0\otimes\cdots\otimes d_+x_i\otimes\cdots\otimes x_m,$$

where $\sigma_{ij_i} := (-1)^{|a_{ij_i}|(|x_0|+\cdots+|x_{i-1}|+m)}$ and $\tau_i := (-1)^{(|x_0|+\cdots+|x_{i-1}|)}$.

When the fibration $p : E \to X$ is endowed with a homotopy retraction, there exists a surjective cdga model of p which is a retraction of a cdga cofibration (see,

for instance, [3, Section 5.1] for an explicit construction). Such a model is called an *s-model*. We will use the following result from [1].

Theorem 4 ([1, Theorem 3.3]) Let p be a fibration endowed with a homotopy retraction. For any s-model $\varphi: A \to \frac{A}{K}$ of p, msecat(p) is the smallest m for which the projection $\rho_m: A \to \frac{A}{K^{m+1}}$ admits a homotopy retraction of (A, d)-modules.

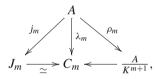
By using this result together with Lemma 3, we obtain the following new characterization of msecat(p) when p admits a homotopy retraction.

Proposition 5 Let $p : E \to X$ be a fibration endowed with a homotopy retraction, $\varphi : A \to \frac{A}{K}$ an s-model for p and $(A, d) \to (A \otimes (\mathbb{Q} \oplus U), d)$ a semifree extension for φ , as in (1). Let also $\eta : P \xrightarrow{\simeq} A^{\#}$ be an (A, d) semifree resolution. Then the following are equivalent

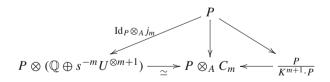
(*i*) msecat(p) $\leq m$,

(ii) the morphism $\operatorname{Id}_P \otimes_A j_m \colon P \to P \otimes (\mathbb{Q} \oplus s^{-m} U^{\otimes m+1})$ is injective in homology, (iii) the projection $P \to \frac{P}{K^{m+1} \cdot P}$ is injective in homology.

Proof It is clear that (*i*) implies (*ii*). From the proof of [1, Theorem 3.3], there is a diagram



where the map $\lambda_m : A \to C_m$ is a model of $p_m : J_X^m(E) \to X$, the left hand triangle is commutative up to a homotopy of (A, d)-modules, and the right hand triangle is strictly commutative. Applying $\mathrm{Id}_P \otimes_A -$ to the previous diagram, we get the following diagram of (A, d)-modules:



where the left hand triangle is commutative up to a homotopy of (A, d)-modules and the right hand triangle is strictly commutative, which yields $(ii) \Rightarrow (iii)$. Finally the implication $(iii) \Rightarrow (ii)$ follows from Lemma 3 applied to ρ_m .

4 The main result

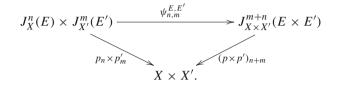
Finally, we present a proof of the additivity of module sectional category when only one of the fibrations admits homotopy retraction.

We first notice that one of the inequalities of Theorem 1 follows in general:

Proposition 6 Let $p: E \to X$ and $p': E' \to X'$ be two fibrations. We have

 $\operatorname{msecat}(p \times p') \leq \operatorname{msecat}(p) + \operatorname{msecat}(p').$

Proof In [8, Section 7.2], maps $\psi_{n,m}^{E,E'}$ producing a commutative diagram of the following form are constructed:



By applying A_{PL} to this diagram, we can establish that, if $msecat(p) \le m$ and $msecat(p') \le n$ then $msecat(p \times p') \le m + n$.

In order to prove our main result (Theorem 1), it remains to establish the inequality $\operatorname{msecat}(p \times p') \ge \operatorname{msecat}(p) + \operatorname{msecat}(p')$ under the additional assumption that one of the fibration, say p, admits a homotopy retraction. We notice that, if both fibrations would admit a homotopy retraction, a direct adaptation of the strategy of [5] together with Proposition 5 would give a proof of this inequality. The following less immediate adaptation of [5] provides a proof when only p admits a homotopy retraction.

Proof (*Proof of Theorem* 1) Take an s-model φ for p and an (A, d)-semifree extension $(A \otimes (\mathbb{Q} \oplus U), d)$ of φ such that $d_0(x) \in K = \ker \varphi$ for $x \in U$. Let also $(B, d) \rightarrow (B \otimes (\mathbb{Q} \oplus V), d)$ be a (B, d)-semifree model of p'. Then $p \times p'$ is modeled by the tensor product of the two semifree extensions which gives a semifree extension of $(A \otimes B, d)$ -modules that we write as follows

 $A \otimes B \to A \otimes B \otimes (\mathbb{Q} \oplus Z)$, where $Z = U \oplus V \oplus U \otimes V$.

In order to prove the statement, we suppose msecat(p) = m and $msecat(p \times p') = m + n$ and show that $msecat(p') \le n$.

Let $P \xrightarrow{\simeq} A^{\#}$ be an (A, d)-semifree resolution. Since msecat(p) = m we know from Proposition 5 that there exists $\Omega \in H(K^m \cdot P)$ which is not trivial in H(P). Then there exist a cocyle $\omega \in K^m \cdot P$ representing Ω in H(P) and $\theta \in P \otimes s^{-(m-1)}U^{\otimes m}$ such that $d\theta = \omega$. As a chain complex, we can write $P = \omega \cdot \mathbb{Q} \oplus S$ where $d(S) \subset S$, and we define the following linear map of degree $-|\omega|$:

$$I_{\omega}: P \to \mathbb{Q}, \quad I_{\omega}(\omega) = 1, \quad I_{\omega}(S) = 0.$$

This map commutes with differentials. Now write the element $\theta \in P \otimes s^{-(m-1)}U^{\otimes m}$ as

$$\theta = \sum_{i} q_i \otimes s^{-(m-1)} x_i$$

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with $q_i \in P$ and $x_i \in U^{\otimes m}$. Since $d\theta = \omega$ we have $d_+\theta = 0$ and $d_0\theta = \omega$.

Let $\psi: B \otimes (\mathbb{Q} \oplus s^{-n}V^{\otimes n+1}) \to P \otimes B \otimes (\mathbb{Q} \oplus s^{-m-n}Z^{\otimes m+n+1})$ be the *B*-linear map of degree $|\omega|$ given by $\psi(1) = \omega \otimes 1$ and, for $y \in V^{\otimes n+1}$,

$$\psi(s^{-n}y) = -(-1)^{n|\omega|} \sum_{i} (-1)^{(n+1)|q_i|} q_i \otimes 1 \otimes s^{-m-n} x_i \otimes y$$

and extended to $B \otimes (\mathbb{Q} \oplus s^{-n}V^{\otimes n+1})$ by the rule $\psi(b \cdot x) = (-1)^{|b||\omega|} b \cdot \psi(x)$. Notice that the structure of (B, d)-module on $P \otimes B \otimes (\mathbb{Q} \oplus s^{-m-n}Z^{\otimes m+n+1})$ is given by $b \cdot (q \otimes b' \otimes z) = (-1)^{|q||b|} q \otimes bb' \otimes z$. In particular $\psi(b) = \omega \otimes b$. Let us now see that ψ commutes with differentials, that is $\psi \circ d = (-1)^{|\omega|} d \circ \psi$. Since ψ is *B*-linear and since ω is a cocycle we only have to see that

$$d\psi(s^{-n}y) = (-1)^{|\omega|}\psi(ds^{-n}y),$$

for each $y \in V^{\otimes n+1}$. Writing the differential of $P \otimes B \otimes (\mathbb{Q} \oplus s^{-m-n}Z^{\otimes m+n+1})$ as

$$d = d_0 + d_+ \in P \otimes B \oplus P \otimes B \otimes s^{-m-n} Z^{\otimes m+n+1}$$

we can check that

$$- d_0\psi(s^{-n}y) = (-1)^{|\omega|}\psi(d_0s^{-n}y) \text{ using the fact that } d_0\theta = \omega, \text{ and} \\ - d_+\psi(s^{-n}y) = (-1)^{|\omega|}\psi(d_+s^{-n}y) \text{ using the fact that } d_+\theta = 0.$$

From the assumption msecat $(p \times p') = m + n$ we know that the morphism

$$j_{m+n}^{A\otimes B} \colon A\otimes B \to A\otimes B\otimes (\mathbb{Q}\oplus s^{-m-n}Z^{\otimes m+n+1}).$$

admits a retraction r of $(A \otimes B, d)$ -modules. Finally the composite

$$B \otimes (\mathbb{Q} \oplus s^{-n} V^{\otimes n+1}) \xrightarrow{\psi} P \otimes B \otimes (\mathbb{Q} \oplus s^{-m-n} Z^{\otimes m+n+1})$$
$$\downarrow^{P \otimes_A r}_{P \otimes B} \xrightarrow{I_{\omega} \otimes \mathrm{Id}} B.$$

gives a morphism (of degree 0) of (B, d)-module which is a retraction for the inclusion $B \to B \otimes (\mathbb{Q} \oplus s^{-n} V^{\otimes n+1})$. This proves that $\operatorname{msecat}(p') \leq n$.

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