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Module sectional category of products

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Abstract Adapting a result of Félix–Halperin–Lemaire concerning the Lusternik–Schnirelmann category of products, we prove the additivity of a rational approximation for Schwarz's sectional category with respect to products of certain fibrations.

Keywords Rational homotopy · Sectional category · Topological complexity

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1 Introduction

The sectional category [12] (or Schwarz genus) of a fibration $p : E \to X$, $\mathrm{secat}(p)$, is the smallest integer m such that X admits a cover by $(m + 1)$ open sets on each of which a local section for p exists. This homotopy invariant is a generalization of

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the well-known Lusternik-Schnirelmann (L.-S.) category [10] of a path-connected space X, $\mathrm{cat}(X)$, as the latter is the sectional category of the path fibration $PX \to X$, $\alpha \mapsto \alpha(1)$, where PX is the space of paths starting at the base point.

One of the most important results of [5] says that, if X and Y are simply connected rational spaces of finite type, then $\mathrm{cat}(X \times Y) = \mathrm{cat}(X) + \mathrm{cat}(Y)$. This was achieved by first proving the analogous result for the lower bound module L.-S. category, $\mathrm{mcat}(X)$, of $\mathrm{cat}(X)$ using differential graded (DG) module techniques. It was then lifted to rational category using Hess' theorem [9]. We propose to apply similar DG-module techniques to the lower bound $\mathrm{msecat}(p)$ of $\mathrm{secat}(p)$ called module sectional category and introduced in [7].

Throughout this paper we consider fibrations whose base and total space have the homotopy type of simply connected CW-complexes of finite type. Our main result is

Theorem 1 *Let p and p' be two fibrations. If either p or p' admits a homotopy retraction, then*

$$\mathrm{msecat}(p \times p') = \mathrm{msecat}(p) + \mathrm{msecat}(p').$$

Recall the important particular case of sectional category provided by Farber's (higher) topological complexity [4,11] of a space X, $\mathrm{TC}_n(X) = \mathrm{secat}(\pi_n)$, where the considered fibration $\pi_n \colon X^{[1,n]} \to X^n$ is given by $\pi_n(\alpha) = (\alpha(1), \alpha(2), \ldots, \alpha(n))$. Consequently, the module invariant associated to (higher) topological complexity, i.e.,

$$\mathrm{mTC}_n(X) := \mathrm{msecat}(\pi_n),$$

is additive on products. Namely

Corollary 2 *Let X and Y be two spaces. Then*

$$\mathrm{mTC}_n(X \times Y) = \mathrm{mTC}_n(X) + \mathrm{mTC}_n(Y).$$

These results are improvements over [2] as only one of the two fibrations of Theorem 1 needs a homotopy retraction and the Poincaré duality assumption is no longer required.

2 Preliminaries

This section contains a brief summary of the DG-modules techniques that will be used (see [6] for further details). Let (A, d) be a commutative differential graded algebra over $\mathbb{Q}$ (cdga). An (A, d)-module is a chain complex (M, d) together with a degree 0 action of A satisfying $d(ax) = (da)x + (-1)^{|a|}a(dx)$. A *semifree extension* of an (A, d)-module (M, d) is an (A, d)-module of the form $(M \oplus A \otimes U, d)$ where the action is the one of the direct sum, the differential on M is the differential of (M, d), and U admits a direct sum decomposition $U = \oplus_{i=0}^{\infty} U_i$ such that $d(U_0) \subset M$ and $d(U_n) \subset M \oplus A \otimes (\oplus_{i=0}^{n-1} U_i)$ for $n \geq 1$. A *semifree* (A, d)*-module* is a semifree extension $(A \otimes U, d)$ of the trivial (A, d)-module 0 and the data of a quasi-isomorphism

$(A \otimes U, d) \xrightarrow{\simeq} (M, d)$ is called a *semifree resolution* of (M, d). The category of (A, d)-modules is a proper closed model category in which semifree extensions are cofibrations (see, for instance, [7, Theorem 4.1]). Two (A, d)-module morphisms $\phi, \psi\colon (M, d) \to (N, d)$ are *homotopic* if there is an A-linear map $\theta : M \to N$ of degree -1 such that $\phi - \psi = d\theta + \theta d$. We will frequently use the fact that any (A, d)-module morphism $\varphi\colon (M, d) \to (N, d)$ can be decomposed as (the inclusion of) a semifree extension followed by a quasi-isomorphism as well as the following lifting lemma. Given a solid arrow commutative diagram of (A, d)-modules of the form

$$\begin{array}{ccc} (A, d) & \longrightarrow & (M, d) \\ \downarrow & \nearrow & \downarrow{\scriptstyle \simeq} \\ (P, d) & \longrightarrow & (N, d) \end{array}$$

in which the morphism $(A, d) \to (P, d)$ is a semifree extension, there is an (A, d)-module morphism $(P, d) \to (M, d)$ making commutative the upper triangle and homotopy commutative (rel. A) the lower triangle. A morphism of (A, d)-modules $\varphi\colon (M, d) \to (N, d)$ is said to have a homotopy retraction if there exists a commutative diagram of (A, d)-modules,

$$\begin{array}{ccc} (M, d) & \xrightarrow{\mathrm{Id}} & (M, d) \\ {\scriptstyle \varphi}\downarrow & \searrow & \uparrow \\ (N, d) & \xleftarrow[\simeq]{} & (P, d). \end{array}$$

If M is an (A, d)-module, the module $M^\# = \hom(M, \mathbb{Q})$ admits an (A, d)-module structure with action $(a\varphi)(x) = (-1)^{|a|\cdot|\varphi|}\varphi(ax)$ and differential $d\varphi = (-1)^{|\varphi|}\varphi \circ d$. If N is an (A, d)-module, then the module $M \otimes_A N$ admits an (A, d)-module structure with action $a(m\otimes n) = (am)\otimes n$ and differential $d(m\otimes n) = dm\otimes n + (-1)^{|m|} m\otimes dn$. If P is (A, d)-semifree and if η is a quasi-isomorphism of (A, d)-modules then $\eta \otimes_A \mathrm{Id}_P$ and $\mathrm{Id}_P \otimes_A \eta$ are also quasi-isomorphisms.

The following lemma is an adaptation of a central idea of [5].

Lemma 3 *Let $\varphi\colon (A, d) \to (B, d)$ be a surjective cdga morphism with kernel K and A of finite type. The morphism φ admits a homotopy retraction of (A, d)-modules if and only if for any (A, d)-semifree resolution $\eta\colon P \xrightarrow{\simeq} A^\#$, the projection*

$$\varrho\colon P \longrightarrow \frac{P}{K \cdot P}$$

is injective in homology.

Proof Suppose that φ admits a homotopy retraction of (A, d)-modules. This means that there exists a homotopy commutative diagram of (A, d)-modules of the form

$$\begin{array}{ccc} A & \xrightarrow{\mathrm{Id}_A} & A \\ {\scriptstyle\varphi}\downarrow & \overset{i}{\searrow} & \uparrow{\scriptstyle r} \\ B & \underset{\simeq}{\longleftarrow} & Q, \end{array}$$

where Q is an (A, d)-semifree resolution of B. Now let $\eta : P \xrightarrow{\simeq} A^{\#}$ be an (A, d)-semifree resolution. By applying $- \otimes_A P$ to the diagram above, we get

$$\begin{array}{ccc} P & \xrightarrow{\mathrm{Id}_P} & P \\ \downarrow & \searrow & \uparrow \\ B \otimes_A P & \underset{\simeq}{\longleftarrow} & Q \otimes_A P. \end{array}$$

Since B and $\frac{A}{K}$ are isomorphic cdgas, we have $B \otimes_A P = \frac{P}{K \cdot P}$. Hence the left hand morphism is simply the projection $\varrho\colon P \to \frac{P}{K \cdot P}$. The diagram shows that ϱ admits a homotopy retraction of (A, d)-modules. Hence it is injective in homology.

Conversely, suppose that ϱ is injective in homology. Since A is of finite type, $\eta^{\#}\colon A \to P^{\#}$ is also an (A, d)-semifree resolution. Moreover,

$$\varrho^{\#}\colon \left(\frac{P}{K \cdot P}\right)^{\#} \to P^{\#}$$

is surjective in homology. Hence there exists a cycle $\gamma \in \left(\frac{P}{K \cdot P}\right)^{\#}$ such that $[\gamma \circ \varrho] = [\eta^{\#}(1)]$. Now define an (A, d)-module morphism $\alpha\colon A \to (\frac{P}{K \cdot P})^{\#}$ by setting $\alpha(1) = \gamma$. Then $\varrho^{\#} \circ \alpha$ is a quasi-isomorphism. To finish the proof, we observe that $K \cdot \left(\frac{P}{K \cdot P}\right)^{\#} = 0$. Hence the map $\varrho^{\#} \circ \alpha$ factors through φ as $B = A/K$. Let $A \xrightarrow{i} Q \xrightarrow{\simeq} B$ be a decomposition of φ as a semifree extension followed by a quasi-isomorphism. Applying the lifting lemma to the solid arrow commutative diagram

$$\begin{array}{ccccc} A & & \xrightarrow{\mathrm{Id}_A} & & A \\ {\scriptstyle i}\downarrow & & \nearrow & & \downarrow{\scriptstyle \simeq\ \varrho^{\#}\circ\alpha} \\ Q & \underset{\simeq}{\longrightarrow} & B & \longrightarrow & P^{\#}, \end{array}$$

we obtain the desired homotopy (A, d)-module retraction for φ. □

3 The invariant msecat(p)

Let us denote by $p_m : J_X^m(E) \to X$ the join of $m+1$ copies of a fibration $p: E \to X$. As is well-known [12], secat$(p) \le m$ if and only if p_m admits a homotopy section. By definition, msecat(p) is the smallest m such that $A_{PL}(p_m)$ admits a homotopy retraction of $A_{PL}(X)$-modules, where A_{PL} denotes Sullivan's functor of piecewise linear forms [13].

Let $\varphi: (A, d) \to (B, d)$ be any cdga model of p and

$$(A, d) \hookrightarrow (A \otimes (\mathbb{Q} \oplus U), d) \xrightarrow{\xi} (B, d). \tag{1}$$

a factorization in the category of (A, d)-modules of φ as the inclusion of a semifree extension followed by a quasi-isomorphism ξ. We refer to the inclusion as a semifree model of p. For $x \in U$, we write $dx = d_0x + d_+x$, where $d_0x \in A$ and $d_+x \in A \otimes U$. We notice that, if φ is surjective, then the quasi-isomorphism ξ can be constructed to satisfy $\xi(U) = 0$, which implies that $d_0x \in \ker\varphi$ for $x \in U$. Recall that the n^{th}-*suspension* $s^{-n}V$ of a graded vector space V is defined by $(s^{-n}V)^i = V^{i-n}$.

According to [7] (Thm 5.4, p.135), msecat(p) is the least m such that the following (A, d)-semifree model of p_m

$$j_m : (A, d) \to \underbrace{(A \otimes (\mathbb{Q} \oplus s^{-m}U^{\otimes m+1}), D)}_{J_m}.$$

admits a retraction of (A, d)-modules, where the differential D is given by

$$D(s^{-m}x_0 \otimes \cdots \otimes x_m) = (-1)^{\sum_{k=1}^{m}(k|x_{m-k}|+k-1)} d_0x_0 \cdot \cdots \cdot d_0x_m$$
$$+ \sum_{i=0}^{m} \sum_{j_i} (-1)^{(|a_{ij_i}|+1)(|x_0|+\cdots+|x_{i-1}|+m)} a_{ij_i} \otimes s^{-m}x_0 \otimes \cdots \otimes x_{ij_i} \otimes \cdots \otimes x_m,$$

for $x_0, \ldots, x_m \in U$ and $d_+x_i = \sum_{j_i} a_{ij_i} \otimes x_{ij_i}$ with $a_{ij_i} \in A$ and $x_{ij_i} \in U$.

Using the following notation (suggested by the standard rules of signs)

$$s^{-m}x_0 \otimes \cdots \otimes d_+x_i \otimes \cdots \otimes x_m := \sum_{j_i} \sigma_{ij_i} a_{ij_i} \otimes s^{-m}x_0 \otimes \cdots \otimes x_{ij_i} \otimes \cdots \otimes x_m,$$

we can write $D_+(s^{-m}x_0 \otimes \cdots \otimes x_m)$ as

$$D_+(s^{-m}x_0 \otimes \cdots \otimes x_m) = (-1)^m \sum_{i=0}^{m} \sum_{j_i} \tau_i s^{-m}x_0 \otimes \cdots \otimes d_+x_i \otimes \cdots \otimes x_m,$$

where $\sigma_{ij_i} := (-1)^{|a_{ij_i}|(|x_0|+\cdots+|x_{i-1}|+m)}$ and $\tau_i := (-1)^{(|x_0|+\cdots+|x_{i-1}|)}$.

When the fibration $p : E \to X$ is endowed with a homotopy retraction, there exists a surjective cdga model of p which is a retraction of a cdga cofibration (see, for instance, [3, Section 5.1] for an explicit construction). Such a model is called an *s-model*. We will use the following result from [1].

Theorem 4 ([1, Theorem 3.3]) *Let p be a fibration endowed with a homotopy retraction. For any s-model $\varphi\colon A \to \frac{A}{K}$ of p,* $\mathrm{msecat}(p)$ *is the smallest m for which the projection $\rho_m\colon A \to \frac{A}{K^{m+1}}$ admits a homotopy retraction of (A, d)-modules.*

By using this result together with Lemma 3, we obtain the following new characterization of $\mathrm{msecat}(p)$ when p admits a homotopy retraction.

Proposition 5 *Let $p : E \to X$ be a fibration endowed with a homotopy retraction, $\varphi\colon A \to \frac{A}{K}$ an s-model for p and $(A, d) \to (A \otimes (\mathbb{Q} \oplus U), d)$ a semifree extension for φ, as in (1). Let also $\eta\colon P \xrightarrow{\simeq} A^{\#}$ be an (A, d) semifree resolution. Then the following are equivalent*

(i) $\mathrm{msecat}(p) \leq m$,
(ii) the morphism $\mathrm{Id}_P \otimes_A j_m\colon P \to P \otimes (\mathbb{Q} \oplus s^{-m} U^{\otimes m+1})$ *is injective in homology,*
(iii) the projection $P \to \frac{P}{K^{m+1}\cdot P}$ is injective in homology.

Proof It is clear that (i) implies (ii). From the proof of [1, Theorem 3.3], there is a diagram

$$\begin{array}{ccccc} & & A & & \\ & \swarrow^{j_m} & \downarrow{\scriptstyle \lambda_m} & \searrow^{\rho_m} & \\ J_m & \xrightarrow[\simeq]{} & C_m & \longleftarrow & \frac{A}{K^{m+1}}, \end{array}$$

where the map $\lambda_m : A \to C_m$ is a model of $p_m\colon J^m_X(E) \to X$, the left hand triangle is commutative up to a homotopy of (A, d)-modules, and the right hand triangle is strictly commutative. Applying $\mathrm{Id}_P \otimes_A -$ to the previous diagram, we get the following diagram of (A, d)-modules:

$$\begin{array}{ccccc} & & P & & \\ & \swarrow^{\mathrm{Id}_P \otimes_A j_m} & \downarrow & \searrow & \\ P \otimes (\mathbb{Q} \oplus s^{-m} U^{\otimes m+1}) & \xrightarrow[\simeq]{} & P \otimes_A C_m & \longleftarrow & \frac{P}{K^{m+1}\cdot P} \end{array}$$

where the left hand triangle is commutative up to a homotopy of (A, d)-modules and the right hand triangle is strictly commutative, which yields $(ii) \Rightarrow (iii)$. Finally the implication $(iii) \Rightarrow (ii)$ follows from Lemma 3 applied to ρ_m. □

4 The main result

Finally, we present a proof of the additivity of module sectional category when only one of the fibrations admits homotopy retraction.

We first notice that one of the inequalities of Theorem 1 follows in general:

Proposition 6 *Let $p\colon E \to X$ and $p'\colon E' \to X'$ be two fibrations. We have*

$$\mathrm{msecat}(p \times p') \leq \mathrm{msecat}(p) + \mathrm{msecat}(p').$$

Proof In [8, Section 7.2], maps $\psi_{n,m}^{E,E'}$ producing a commutative diagram of the following form are constructed:

$$\begin{array}{ccc} J_X^n(E) \times J_{X'}^m(E') & \xrightarrow{\ \psi_{n,m}^{E,E'}\ } & J_{X\times X'}^{m+n}(E \times E') \\ & {\scriptstyle p_n \times p'_m} \searrow \quad \swarrow {\scriptstyle (p\times p')_{n+m}} & \\ & X \times X'. & \end{array}$$

By applying A_{PL} to this diagram, we can establish that, if $\mathrm{msecat}(p) \leq m$ and $\mathrm{msecat}(p') \leq n$ then $\mathrm{msecat}(p \times p') \leq m + n$. □

In order to prove our main result (Theorem 1), it remains to establish the inequality $\mathrm{msecat}(p \times p') \geq \mathrm{msecat}(p) + \mathrm{msecat}(p')$ under the additional assumption that one of the fibration, say p, admits a homotopy retraction. We notice that, if both fibrations would admit a homotopy retraction, a direct adaptation of the strategy of [5] together with Proposition 5 would give a proof of this inequality. The following less immediate adaptation of [5] provides a proof when only p admits a homotopy retraction.

Proof (*Proof of Theorem* 1) Take an s-model φ for p and an (A, d)-semifree extension $(A \otimes (\mathbb{Q} \oplus U), d)$ of φ such that $d_0(x) \in K = \ker \varphi$ for $x \in U$. Let also $(B, d) \to (B \otimes (\mathbb{Q} \oplus V), d)$ be a (B, d)-semifree model of p'. Then $p \times p'$ is modeled by the tensor product of the two semifree extensions which gives a semifree extension of $(A \otimes B, d)$-modules that we write as follows

$$A \otimes B \to A \otimes B \otimes (\mathbb{Q} \oplus Z), \quad \text{where} \quad Z = U \oplus V \oplus U \otimes V.$$

In order to prove the statement, we suppose $\mathrm{msecat}(p) = m$ and $\mathrm{msecat}(p \times p') = m + n$ and show that $\mathrm{msecat}(p') \leq n$.

Let $P \xrightarrow{\simeq} A^\#$ be an (A, d)-semifree resolution. Since $\mathrm{msecat}(p) = m$ we know from Proposition 5 that there exists $\Omega \in H(K^m \cdot P)$ which is not trivial in $H(P)$. Then there exist a cocyle $\omega \in K^m \cdot P$ representing Ω in $H(P)$ and $\theta \in P \otimes s^{-(m-1)}U^{\otimes m}$ such that $d\theta = \omega$. As a chain complex, we can write $P = \omega \cdot \mathbb{Q} \oplus S$ where $d(S) \subset S$, and we define the following linear map of degree $-|\omega|$:

$$I_\omega\colon P \to \mathbb{Q}, \quad I_\omega(\omega) = 1, \quad I_\omega(S) = 0.$$

This map commutes with differentials. Now write the element $\theta \in P \otimes s^{-(m-1)}U^{\otimes m}$ as

$$\theta = \sum_i q_i \otimes s^{-(m-1)}x_i$$

with $q_i \in P$ and $x_i \in U^{\otimes m}$. Since $d\theta = \omega$ we have $d_+\theta = 0$ and $d_0\theta = \omega$.

Let $\psi : B \otimes (\mathbb{Q} \oplus s^{-n} V^{\otimes n+1}) \to P \otimes B \otimes (\mathbb{Q} \oplus s^{-m-n} Z^{\otimes m+n+1})$ be the B-linear map of degree $|\omega|$ given by $\psi(1) = \omega \otimes 1$ and, for $y \in V^{\otimes n+1}$,

$$\psi(s^{-n} y) = -(-1)^{n|\omega|} \sum_i (-1)^{(n+1)|q_i|} q_i \otimes 1 \otimes s^{-m-n} x_i \otimes y$$

and extended to $B \otimes (\mathbb{Q} \oplus s^{-n} V^{\otimes n+1})$ by the rule $\psi(b \cdot x) = (-1)^{|b||\omega|} b \cdot \psi(x)$. Notice that the structure of (B, d)-module on $P \otimes B \otimes (\mathbb{Q} \oplus s^{-m-n} Z^{\otimes m+n+1})$ is given by $b \cdot (q \otimes b' \otimes z) = (-1)^{|q||b|} q \otimes bb' \otimes z$. In particular $\psi(b) = \omega \otimes b$. Let us now see that ψ commutes with differentials, that is $\psi \circ d = (-1)^{|\omega|} d \circ \psi$. Since ψ is B-linear and since ω is a cocycle we only have to see that

$$d\psi(s^{-n} y) = (-1)^{|\omega|} \psi(ds^{-n} y),$$

for each $y \in V^{\otimes n+1}$. Writing the differential of $P \otimes B \otimes (\mathbb{Q} \oplus s^{-m-n} Z^{\otimes m+n+1})$ as

$$d = d_0 + d_+ \in P \otimes B \oplus P \otimes B \otimes s^{-m-n} Z^{\otimes m+n+1}$$

we can check that

- $d_0 \psi(s^{-n} y) = (-1)^{|\omega|} \psi(d_0 s^{-n} y)$ using the fact that $d_0\theta = \omega$, and
- $d_+ \psi(s^{-n} y) = (-1)^{|\omega|} \psi(d_+ s^{-n} y)$ using the fact that $d_+\theta = 0$.

From the assumption $\mathrm{msecat}(p \times p') = m + n$ we know that the morphism

$$j_{m+n}^{A \otimes B} : A \otimes B \to A \otimes B \otimes (\mathbb{Q} \oplus s^{-m-n} Z^{\otimes m+n+1}).$$

admits a retraction r of $(A \otimes B, d)$-modules. Finally the composite

$$\begin{array}{ccc} B \otimes (\mathbb{Q} \oplus s^{-n} V^{\otimes n+1}) & \xrightarrow{\psi} & P \otimes B \otimes (\mathbb{Q} \oplus s^{-m-n} Z^{\otimes m+n+1}) \\ & & \Big\downarrow {\scriptstyle P \otimes_A r} \\ & & P \otimes B \xrightarrow{\quad I_\omega \otimes \mathrm{Id} \quad} B. \end{array}$$

gives a morphism (of degree 0) of (B, d)-module which is a retraction for the inclusion $B \to B \otimes (\mathbb{Q} \oplus s^{-n} V^{\otimes n+1})$. This proves that $\mathrm{msecat}(p') \leq n$. □

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