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Game Theory applied to the Financial Markets

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Abstract

The financial markets refer broadly to any marketplace where the trading of securities occurs, including the stock market, bond market, forex market, and derivatives market, among others, and, as such, they play a vital role in facilitating the smooth operation of capitalist economies by allocating resources and creating liquidity for businesses and entrepreneurs. So, methods to accurately model and predict the financial assets' prices are vital tools for every person and company that are in constant interaction with any kind of market. However, no method so far as been sufficiently accurate in the price prediction, due to the markets' intrinsic volatility.

Time series (forecasting) methods are the better known (and used) mathematical methodology that try to model and predict the financial markets, also it is a hot topic which has many other possible applications, such as weather forecasting, business planning, resources allocation and many others. These methods involve treating the assets' prices as a time series and then proceeding to model it using time series' methods, primarily SARIMA and/or GARCH (or any kind of variation from these), where the forecasting is made with the use of a model which predicts future values based on previously observed ones. These methods are appealing due to its relative simplicity, the adaptability to all kinds of data and the vast research that supports it. However, time series methods involve several assumptions that may not always hold true on the financial markets and, because of the temporal dependencies and high volatility in time series financial data, the obtained models (and predictions) can become unreliable very fast. Also the computer implementation of such methods can be very inefficient.

Nonetheless, there are several other mathematical fields that can be used to model and predict the financial markets, while maintaining (or even improving) the accuracy of time series methods, while avoiding its issues. Game Theory is a very wide and versatile field that can be used for these purposes, because it already has applications in all fields of social science,

as well as in logic, systems science, computer science and much more. So, we will develop a game theoretical model which will be treated as a decision model designed to tell us when it is optimal to buy/sell or not a specific financial asset with relatively More or Less Risk, and all of this will depend on the markets' situation and on the individual's risk to reward levels.

This thesis main goal is to apply the game theoretical decision model together with the common time series approach and then compare its results with a Markov Chain approach, which was specifically designed for this decision model. To this end, we will develop the game theoretical decision model and explain how the Markov chains were obtained to take advantage of this model and its structure. Thus, with the game theoretical decision model, we will compare the accuracy of the time series methodology with the one that we developed using Markov chains. We will accomplish this by applying these procedures to theoretical datasets and to datasets with the (daily and intraday) closing prices of several financial assets. All of this will be done to check if the new decision model is applicable to the financial markets and to check if we can obtain better results, with less assumptions and computer power, than the time series methodology. Additionally, by applying these methodologies to different kinds of data we can withdraw meaningful conclusions about the new methods' accuracy when compared to the commonly used ones. Thus arriving at our goal of exposing and applying new methods that can be st least as accurate and computationally efficient as the existing ones.

We will start this thesis by describing the data (and its sources) that we will be using, then we will present the game theoretical decision model (with the aid of Gibbons (1992), Fudenberg and Tirole (1991) and Shelton (1997)), afterwards we will present the (specifically designed) Markov chain models (with the aid of Bowerman (1974)) and Time Series (with the aid of Shumway and Stoffer (2011) and Brockwell and Davis (2016)), ending with how we will apply each of the models to the data. Finally, we will present the obtained results from applying the described models and then withdraw some conclusions from all of what was discussed. Also, in order to keep the thesis as straightforward and simple as possible, all of the necessary theory and related results are available as appendices.

Keywords: Financial Markets, Stock Exchange, Forecasting, Time Series, SARIMA, GARCH, Game Theory, Markov Chains

Resumo

Os mercados financeiros referem-se amplamente a qualquer mercado em que ocorra a negociação de valores mobiliários, incluindo de ações, títulos, câmbio e derivados, entre outros, e, como tal, desempenham um papel vital na facilitação do bom funcionamento de economias capitalistas, ao alocar recursos e criando liquidez para empresas e empreendedores. Portanto, métodos para modelar e prever com precisão os preços dos ativos financeiros são ferramentas vitais para todas as pessoas e empresas que estão em constante interação com qualquer tipo de mercado. No entanto, nenhum método foi suficientemente preciso na previsão de preços devido à volatilidade intrínseca dos mercados.

A análise (e previsão) de séries temporais é a metodologia matemática mais conhecida (e usada) para tentar modelar e prever os mercados financeiros. Para além disto, é um tópico bastante conhecido, pois tem muitas outras aplicações possíveis, como previsão do tempo, planeamento de negócios, alocação de recursos e muitos outros. Estes métodos envolvem tratar os preços dos ativos como uma série temporal e, em seguida, proceder à modelação usando os métodos das séries temporais, principalmente SARIMA e / ou GARCH (ou qualquer tipo de variação destes). A subsequente previsão é feita com um modelo que prevê os valores futuros com base em valores observados anteriormente. Estes métodos são atraentes devido à sua relativa simplicidade, à adaptabilidade a todos os tipos de dados, à vasta pesquisa que os suporta e à sua rápida implementação computacional. No entanto, os métodos de séries temporais envolvem várias premissas que nem sempre são verdadeiras nos mercados financeiros e, devido às dependências temporais e à alta volatilidade das séries temporais de dados financeiros, os modelos (e previsões) obtidos podem tornar-se pouco confiáveis muito rapidamente, e a implementação computacional de tais métodos pode ser muito ineficiente.

No entanto, existem vários outros campos da matemática que podem ser úteis na modelação e

previsão dos mercados financeiros, que mantêm (ou até melhoram) a precisão dos métodos das séries temporais, evitando os seus problemas. A teoria dos jogos é um campo muito amplo e versátil que pode ser usado para estes fins, porque já possui aplicações em todos os campos das ciências sociais, bem como na lógica, ciência de sistemas, ciência da computação e muito mais. Portanto, desenvolveremos um modelo de teoria de jogos que será tratado como um modelo de decisão, desenhado especificamente para nos dizer quando é ideal comprar/vender ou não um certo ativo financeiro com um risco relativamente maior ou menor, e tudo isto dependerá da situação atual dos mercados e nos níveis de risco de cada individuo.

O objetivo principal desta tese é aplicar o modelo de decisão (baseado em teoria de jogos) juntamente com a abordagem de séries temporais comum e, em seguida, comparar os seus resultados com uma abordagem usando cadeias de Markov, que fram projetadas especificamente para este modelo de decisão. Para isso, desenvolveremos o modelo de decisão com base em teoria de jogos e explicaremos como a abordagem das cadeias de Markov foi obtida de modo a aproveitar este modelo e a sua estrutura. Assim, com este modelo de decisão, compararemos a precisão da metodologia de séries temporais com a que desenvolvemos a usar cadeias de Markov. E conseguiremos isto ao aplicarmos estes procedimentos a conjuntos de dados teóricos e a conjuntos de dados com os preços de fecho (diário e intradiário) de vários ativos financeiros. Tudo isto será feito para verificar se o novo modelo de decisão é aplicável nos mercados financeiros e para verificar se podemos obter melhores resultados, com menos premissas e poder computacional, do que a metodologia de séries temporais. Além disso, ao aplicarmos as diferentes metodologias a diferentes conjuntos de dados, podemos retirar conclusões significativas sobre a precisão dos novos métodos, quando comparados com os métodos mais usados. Chegando assim ao nosso objetivo de expor e aplicar novos métodos que sejam pelo menos tão precisos e eficientes (em termos computacionais) quanto aos existentes.

Começamos a tese por descrever os dados (e as suas fontes) que iremos usar, depois apresentaremos o modelo de decisão baseado em teoria de jogos (com o auxílio de Gibbons (1992), Fudenberg and Tirole (1991) e Shelton (1997)), de seguida apresentamos os modelos de cadeias de Markov projetadas especificamente para este modelo de decisão (com a ajuda de Bowerman (1974)) e os modelos de séries temporais (com a ajuda de Shumway and Stoffer (2011) e

Brockwell and Davis (2016)), terminando com como aplicar cada um dos modelos aos dados. Por fim, apresentaremos os resultados obtidos com a aplicação dos modelos descritos e, em seguida, retiraremos algumas conclusões de tudo o que foi discutido. Além disso, para manter a tese o mais direto e simples possível, toda a teoria necessária e respetivos resultados estão disponíveis nos apêndices.

Keywords: Mercados Financeiros, Bolsa de Valores, Previsão, Séries Temporais, SARIMA, GARCH, Teoria de Jogos, Cadeias de Markov

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Chapter 1

Introduction

In this thesis, we will present and apply methods specifically designed to model the financial market, but, before starting to discuss the models, we need to make a brief introduction to the data that we will be working with.

Our data consists on financial asset prices from several stock exchanges, with special attention to the New York Stock Exchange, the London Stock Exchange and the Lisbon Stock Exchange. Also, we focused on stock and forex (foreign exchange market) prices, because these present higher volatility and volume (i.e., more trades), and the data related to these assets is easier to obtain. Here, volatility is a statistical measure of the dispersion of returns for a given financial asset. It is often measured as either the standard deviation or variance between returns from that same asset.

Moving further we will often use the terms ”stock exchange” and ”financial market” interchangeably, but they slightly differ. The term ”financial market” broadly refers to any marketplace where the trading of securities occurs, including the stock market, bond market, forex market, and derivatives market, among others, whilst the ”stock exchange” is a facility where stockbrokers and traders can buy and sell securities, such as shares of stock, bonds and other financial instruments. However, whenever we refer to the financial market we will be referring to the stock exchange.

The data was retrieved from *Yahoo Finance*, *AlphaVantage* and *WorldTradingData*, but it can be found at https://www.kaggle.com/jfcf0802/daily-and-intraday-stock-data.

We will study daily closing prices, that is, the price of the asset at the end of the day, and also intraday closing prices, i.e., the prices of the asset at the end of each minute. Also, we will look into several assets within multiple stock exchanges. For these reasons, we will obtain all kinds of data, with different characteristics and statistical properties. Moreover, since we cannot access the assets' future prices, we will split the datasets between training and test sets. All the analyzed datasets have 1000 observations and 20% of these will be part of the test sets. This is done so that we can apply our models to the training set and "compare" their predictions with the test set's values.

To exemplify, consider the following datasets from the New York Stock Exchange (plots 1.1 and 1.2) and from the Lisbon Stock Exchange (plots 1.3 and 1.4):

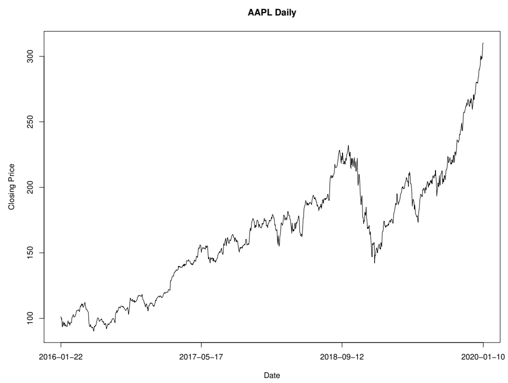

Figure 1.1: AAPL Daily Closing Price from 22/01/2016 to 10/01/2020

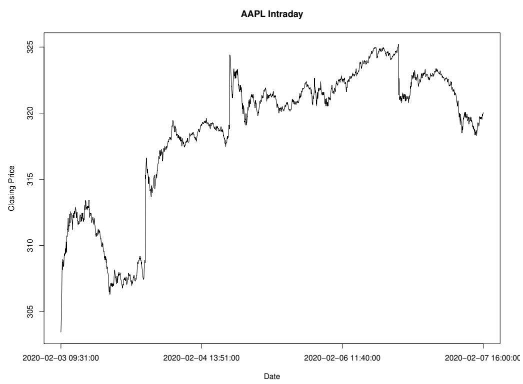

Figure 1.2: AAPL Intraday Closing Price from 03/02/2020 09:31 to 07/02/2020 16:00

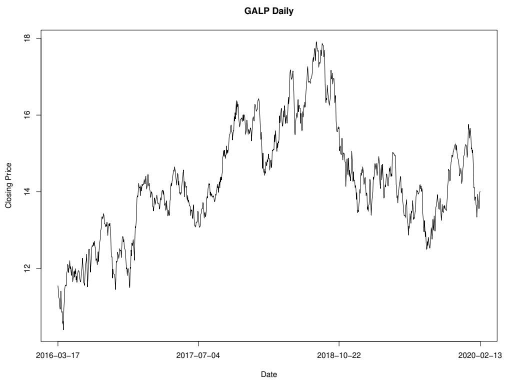

Figure 1.3: GALP Daily Closing Price from 17/03/2016 to 13/02/2020

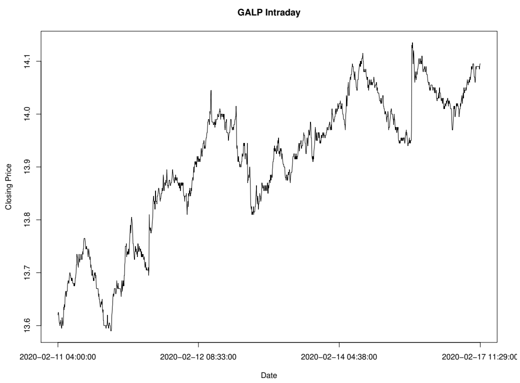

Figure 1.4: GALP Intraday Closing Price from 11/02/2020 04:00 to 17/02/2020 11:29

Measuring past price changes to determine their dispersion should yield a probabilistic result. Additionally, price changes, in stock prices (or in any other financial instruments), usually pattern themselves in a normal distribution (for further details see McDonald (1996), Jackwerth and Rubinstein (1996), Errunza and Losq (1985) and/or Mandelbrot and Taylor (1967)), which is the familiar bell-shaped curve (for further details see, for example, Pishro-Nik (2014)). There are numerous different ways to determine the probability function for a financial instrument. Also, price changes can be measured and quantified empirically, either by the percent change in the instrument's value over specified time intervals or by the change in the logarithm of the price over the time intervals.

Oftentimes when you're thinking in terms of compounding percent changes, the mathematically cleaner concept is to think in terms of log differences. When you're repeatedly multiplying terms together, usually, it's more convenient to work in logs and add terms together. So, let's say our wealth at time T is given by:

$$W_T = \prod_{t=1}^{T}(1+R_t) \iff \log W_T = \sum_{t=1}^{T} r_t,$$

where R_t is the (overall) return at time t and $r_t = log(1+R_t) = logW_t - logW_{t-1}$.

An idea from calculus is that you can approximate a smooth function with a line (for further details see, for example, Stewart (2016)). The linear approximation is simply the first two terms of a Taylor Series. The first order Taylor Expansion of $log(x)$ around $x = 1$ is given by:

$$\log(x) \approx \log(1) + \frac{d}{dx}\log(x)|_{x=1}(x-1).$$

The right hand side simplifies to $0 + \frac{1}{1}(x-1)$ hence:

$$log(x) \approx x - 1.$$

So for x in the neighborhood of 1, we can approximate $log(x)$ with the line $y = x - 1$.

Now consider two variables x_1 and x_2 such that $\frac{x_2}{x_1} \approx 1$. Then the log difference is approximately the percent change $\frac{x_2}{x_1} - 1 = \frac{x_2 - x_1}{x_1}$:

$$\log x_2 - \log x_1 = \log\left(\frac{x_2}{x_1}\right) \approx \frac{x_2}{x_1} - 1.$$

Note that for big percent changes, the log difference is not the same thing as the percent change because approximating the curve $y = log(x)$ with the line $y = x - 1$ gets worse and worse the further away you get from $x = 1$.

Thus we have the following:

- The logarithmic method is well documented. The Black-Scholes formula for option pricing assumes a lognormal dispersion of prices, and there is a theoretical lognormal distribution than can be inferred from the Black-Scholes formula. However, the discussion of the lognormal derivation of price changes is not necessary for this paper (but, for further details, see McMillan (2001) and/or Murphy (1988)).
- Measuring percentage price changes yields a nearly equivalent result to the lognormal method, specially for price changes less than $\approx 15\%$ (for further details see Mandelbrot and Taylor (1967)). Also, this method affords a fair approximation of the real world, while being fairly simple to calculate.

However, if we simply analyzed the price change (between consecutive intervals) of a large sample from some financial instrument, the analysis would be skewed by the change in the price level, hence the need for measuring percentage changes in prices. Thus, any statistical method used to analyze price changes has to be able to account for the increase in the price level of the instrument. This can be taken care by looking at the prices' percentage changes, rather than the actual price changes. Also, there is the added property that percent price changes should (theoretically) follow a normal distribution.

Remark 1. *For more properties on the the percentage change transformation see Sections A.1.4 and A.5.1 from Appendix A.*

However, most real world measurements vary from the standard normal distribution. The theoretical lognormal distribution for stock prices has a slight skew to the positive side, because there is an inherent upward bias in stock prices (for further details see Gottlieb and Kalay (1965)). This is because, since the turn of the century, stocks have appreciated at approximately a $5\% - 10\%$ annual rate, this is partly due to inflation (or even to investor overconfidence), but it is also due to increases in productivity, or the economic surplus society generates (for further details see Scott, Stumpp, and Xu (2003) and/or Royal and Arielle (2020)). Thus, the skewing in the positive side of the theoretic lognormal is understandable. Also note that, factors that have a bearing on assets' prices, such as wars, depression, peace, prosperity, oil shortages, foreign competition, market crashes, pandemics, and so forth, are all contained in its data. So, henceforth, we will consider that all the used data is transformed using the percentage change transformation, i.e., we will apply the Percentage Returns' transformation $U_t = \frac{X_t - X_{t-1}}{X_{t-1}}$ (described in further detail in Section A.5.1 from Appendix A), so each entry on the obtained datasets represents the percentage return from the previous iteration to the present one. Thus, we will apply all of our models to this transformed data. For example, the transformation applied to the previous datasets yields:

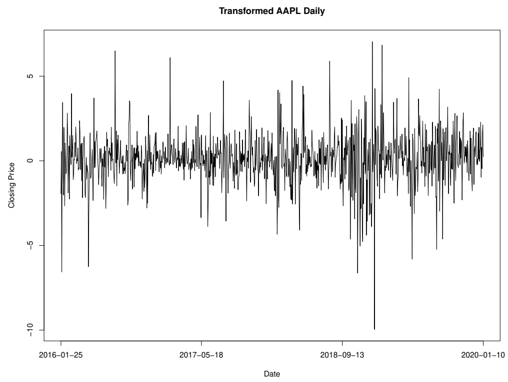

Figure 1.5: Transformed AAPL Daily Closing Price from 25/01/2016 to 10/01/2020

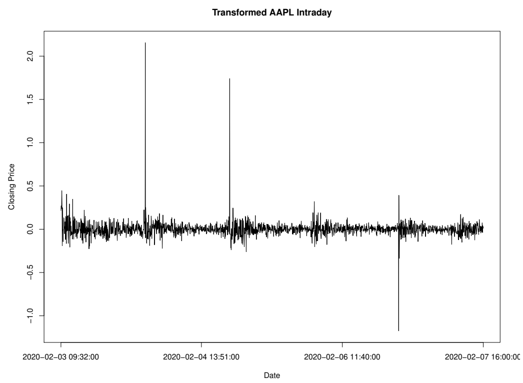

Figure 1.6: Transformed AAPL Intraday Closing Price from 03/02/2020 09:32 to 07/02/2020 16:00

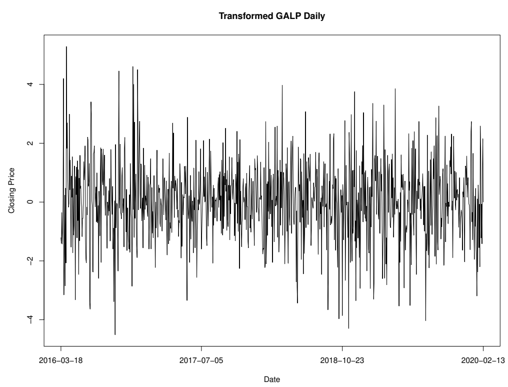

Figure 1.7: Transformed GALP Daily Closing Price from 18/03/2016 to 13/02/2020

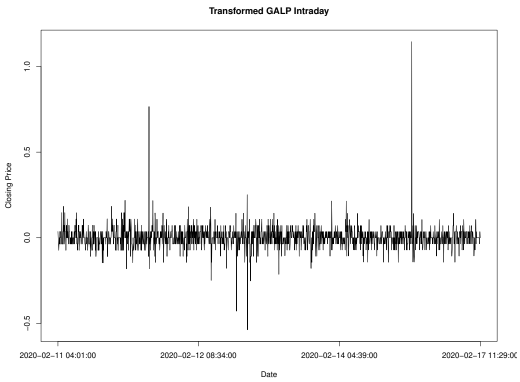

Figure 1.8: Transformed GALP Intraday Closing Price from 11/02/2020 04:01 to 17/02/2020 11:29

Note that we can apply this transformation because all of our values represent asset prices in a stock exchange, thus they are always strictly positive. Also, due to this transformation, we will "lose" one observation, but gain several important properties, which were previously described.

Chapter 2

Models

In this chapter we will make use of the theory (presented in Appendices A-C) in order to design suitable models for financial data (specifically, for the data that was described in the previous chapter), then we will describe how we applied our models using the *R* software.

2.1 The Game Theoretical Model

Since the focus of this thesis is to apply game theory to the financial markets, we will start by presenting the game proposed in Shelton (1997) and the subsequent decision model that we developed from it (with the aid of the theory presented in Appendix C). But, before constructing a game model for the market, we need to understand how the market works, how can we model it and what our goals are. Thus, we will start by identifying what kind of player in the market we will be, because there are two kinds of participants in the financial markets:

- *Investors*: these participants are interested in making a predictable rate of return from their investments, through interest payments dividends and so on.
- *Speculators*: these are interested in trying to profit from changes in the price of an asset.

Thus, since our goal is to predict prices and then act according to our predictions, henceforth we will take the part of a speculator. Also, to be a participant in the market, we must accept some level of risk (high or low risk acceptance level) and we also must have a clear profit objective in mind. Formally, the speculator needs to set a quantity for "Less Risk", "high risk" and "profit objective", always assuming that the asset will be held until the price reaches one of these targets. So, these targets must represent an individual's actual risk and reward appetites, because if they are set randomly, then it is possible that neither are reached or that they are reached sooner than expected. Thus, these must have some basis on reality and the asset should stand a chance of hitting one of them.

Once the decision has been made to take a position in the market (by buying or selling a particular asset), the interaction between the asset's price fluctuation and the speculator's risk acceptance level and profit objective will determine whether or not a profit will be made.

Remark 2. *Note that, this is consistent with game theory, where the outcome is determined by the choices made by both players, not just one.*

Thus, speculators take positions in markets and market prices fluctuate. As such, the speculators' strategies involve determining how much risk to accept, then the market will fluctuate the prices. It is the interaction between the speculator's and the market's actions that determine if a trade is profitable or not. Hence, after setting the profit objective and risk acceptance levels, we have the following scenarios:

- *Zero Adversity*: when there is no price fluctuation against the speculator's position severe enough to cause the trade to hit either risk acceptance levels. In this case, it doesn't matter how much risk is accepted, because the market movement is completely favorable. We will term this pattern of price movement as Zero Adversity.
- *Minor (or Moderate) Adversity*: when the market moves somewhat against the speculator's position, which will cause the speculator to lose money if Less Risk were accepted but would have resulted in a profit if More Risk were accepted. So, any pattern of price movement that will cause a loss if Less Risk is accepted, yet still yield a profit if More Risk is accepted falls into this category, which we will term as Minor Adversity.
- *Major Adversity*: when the market moves completely against both risk acceptance positions, so the Less Risk acceptance position results in a small loss, and the large risk acceptance position results in a large loss. Also, the profit objective was never reached. We will term this pattern of price movement as Major Adversity.

Note that, many different price movement patterns yield the same result and that it is possible to classify all market price movements into these three categories. These classifications are:

- the speculator accepts Less Risk and then the prices move favorably, resulting in a profit to the speculator;
- the speculator accepts More Risk and then the prices move favorably, resulting in a profit to the speculator;
- the speculator accepts Less Risk and the prices move moderately against the position, resulting in a small loss to the speculator;
- the speculator accepts More Risk and the prices move moderately against the position, resulting in a profit to the speculator;
- the speculator accepts Less Risk and the prices move severely against the position, resulting in a small loss to the speculator;
- the speculator accepts More Risk and the prices move severely against the position, resulting in a large loss to the speculator.

Thus, if we quantify our risk acceptance levels and profit objective, the pattern of price fluctuation that subsequently occurs will result in one of the six outcomes previously described. Also, there is no price line that can be drawn that will not yield one of the above six results.
However, even though there are six categories, there are only three possible outcomes that can result from any trade, because the speculator must decide between accepting More Risk or Less Risk on any particular trade, and there are three outcomes associated with either of these actions. In other words, the speculator must decide on how much risk to take, either take More Risk or take Less Risk, then the market decides on how to fluctuate the prices, either fluctuate them so as to cause the speculator zero adversity, minor adversity or major adversity. So, after the speculator's decision, one of three possible states of nature will prevail.
The previous discussion also holds true for short sales. A short sale is where the individual sells a particular asset first, then buys it back at a later date. Typically, the shorted asset is "borrowed" from the brokerage firm, and the broker will require a high margin against the short. Intuitively, a short sale is the inverse of a long position (i.e., a buy-sell position that we have been discussing so far), so short sellers make a profit when the value of an asset declines and

loses money when the prices rise. Thus, the risk acceptance levels are set at prices higher than the price that initiated the trade. However, there is no significant difference in the concepts of risk acceptance levels and profit objectives between being either long or short in the market. But, because of the added costs of being a short seller, profit objectives generally have to be higher in order to recoup margin costs. Thus, henceforth, we will only concentrate on long positions, and its risk acceptance levels and profit objectives.

2.1.1 The Financial Game

To create the game that mimics the financial markets, we need to meet game theory's requirement to have at least two players and that their identities are known, in our case the players are the speculator and the market. However, the market is an abstract entity, thus we enter the subclass of games (presented in Section C.4 from Appendix C), called games against nature, where one of the players is an abstract entity.

In spite of this being a standard game against nature, we must make some important observations and assumptions:

- The market does not come up with prices in a vacuum, rather the prices are the net result of the buying and selling decisions of all the individual participants in the market.
- Generally, an individual has no influence on nature, yet in the financial markets a participant may have an effect on the price movements due to his/hers own actions. Of course that this depends on the individual and on the market. For instance, if the market is small and thinly traded, a large order will tend to move the prices either up or down, or if a person making the order is known (to other participants) to be astute, then his/hers actions may also influence the prices. However, since the majority of the individuals cannot affect "large" markets (such as in the USA, EU, UK markets), we will assume that we are working on a large market and that the effect of any individual is negligible.
- Since the payoffs of each individual are unrelated, then we will assume that the market plays the same game against all participants. This also guarantees that all the individuals are playing against the market separately.

- We will also assume that the goal of the speculator is to make a profit and that the goal of the market is to try and make the speculator lose money.

Remark 3. *Note that with the previous assumptions, we have a game against nature where we assume the Wald's (max-min) Criterion (for more details check Section C.4 from Appendix C and/or Shelton (1997)).*

Here the market "tries" to make the speculator lose money by attempting to fluctuate the prices in such a manner so as to make it impossible to find a good combination of risk acceptance levels and profit objectives. Also, because we are using a theory that will enable an individual to find a way to beat the market, assuming that the market is also trying to beat the individual is the most conservative approach. So, ascribing a motive to the market allows us to analyze the market's strategies as if it is a rational player in the game.

In order to have a game theoretic construction, we need to be able to draw a game matrix outlining the strategies of each player as well as the payoffs. Also, this should be done from the perspective of the individual speculator, because the point of this analysis is to find a set of strategies that will enable an individual to beat the market. Thus, the possible strategies for the speculator are accepting More Risk or relatively Less Risk. And the market's strategies are price movements relative to the speculator's position, i.e., the market can "choose" between three price movements: Zero Adversity, Minor Adversity or Major Adversity.

With this, we have that there are two possible strategies that the speculator can play and three possible strategies the market can play, resulting in six possible outcomes from the interaction between price movements and risk acceptance levels, all of which results in the following game table:

		Speculator	
	\	More Risk (R^+)	Less Risk (R^-)
Market	Zero Adversity (0_A)	Profit	Profit
	Minor Adversity (m_A)	Profit	Small Loss
	Major Adversity (M_A)	Large Loss	Small Loss

Table 2.1: The game table for the financial market game.

Remark 4. *Note that, in the game table, we added between parenthesis some notation so that we can refer to those strategies in a simpler manner.*

Looking at the game table suggests that we should play the strategy of Less Risk, because this column has a minimum of a small loss, which is larger than the minimum in the More Risk column, which is a large loss. Similarly the market will "look" at the payoff table and "decide" to play a strategy that leaves us with the smallest minimum, i.e., the market will choose to play the Major Adversity strategy, because this row's maximum is a small loss, which is the smallest maximum available. Hence the most likely outcome is that the speculator will lose money, which makes this game rather unattractive. However, in the real world a lot of people play the markets and some of them make money (at least some of the time).

Note that the solution Major Adversity, Less Risk is based on the concept of pure strategies. So this solution requires that the speculator always plays the strategy of Less Risk, and the market always plays the strategy of Major Adversity. Thus, this renders the game entirely pointless from the speculator's point of view. But there are some caveats, the market is simultaneously playing against a myriad of players and, as such, it does not know all of the risk acceptance levels, the profit objectives and how many are short sellers or long traders. So, the market has to make its decision on which strategy to play under conditions of both risk and uncertainty.

Given the multitude of players and their strategies, the market will try to fluctuate prices in such a manner so that as many people as possible lose money. Also, from the point of view of any individual speculator, these fluctuations will make it look as if the market is varying its strategy each different time the game is played. All of this (and considering the theory so far) implies that playing each different strategy with some probability is called playing mixed strategies (see Section C.3 from Appendix C).

The speculator may also play mixed strategies, if they vary their risk and reward amounts each time they play the game. Also, they do not know how advantageous it is to play either strategy with any regularity, due to the market's continually changing mixed strategies. But, in the financial markets, the players do not usually change their strategies, i.e., they pick the risk acceptance levels and then wait the assets' prices to hit the corresponding thresholds. So, with this in mind, we will only consider pure strategies for the speculator to play in the financial game.

Now we need to be able to calculate the payoffs to the speculator for any set of strategies he/she plays against any set of mixed strategies that the market may play, in order to determine the merits of playing any one strategy at any particular point in time. Furthermore, this has to be done in the general case, because to have a coherent theory, the solutions must hold true for each and every individual speculator, no matter what strategy they play.

The market will play one of three strategies: fluctuate the prices in a way that causes major adversity to the speculator, fluctuate the prices in a manner that causes minor adversity to the speculator, or fluctuate the prices in a manner favorable to the speculator. Also, the market will choose one of the strategies in an unknown manner to the speculator, so each strategy will have a certain probability of being played. Thus we will use the following notation:

- p_1 := the probability the market plays Minor Adversity;
- p_2 := the probability the market plays Major Adversity;
- p_0 := the probability the market plays Zero Adversity.

This notation is in terms of the probability that either event will occur and, because the market is playing mixed strategies, the sum of the probabilities of playing all of the strategies must equal 1. Therefore, if the market plays Minor Adversity with a probability of p_1 and Major Adversity with probability p_2, then it follows that Zero Adversity occurs with a probability of $p_0 = 1 - p_1 - p_2$.

Regarding the speculator, theoretically, he/she may play two different strategies: More Risk or Less Risk. Analogously to the market, the speculator may play the More Risk strategy with some probability and the Less Risk strategy with some probability. Thus, the speculator is playing mixed strategies, just as the market is. With this, we can define the probabilities of playing the two strategies as follows:

- q = the probability the speculator plays More Risk;
- $1 - q$ = the probability the speculator plays Less Risk.

Once again, the sum of the probabilities of playing both strategies must equal one.

Next we need to make a representation of the payoffs. Recall that there are three different results for this game, a speculator may: make a profit, lose money equal to the Less Risk amount, or lose money equal to the More Risk amount. We will denote this as follows:

- w := profit to the speculator (this corresponds to a "win" to the speculator);

- $-x :=$ loss equal to the "Less Risk" amount (this corresponds to a "small loss" to the speculator);
- $-y :=$ loss equal to the "More Risk" amount (this corresponds to a "large loss" to the speculator).

Here, $w, x, y \in \mathbb{R}^+$ and $w \geq y > x$. So, with this notation we do not need to specify monetary amounts associated with a profit, a small loss, or a large loss, because we have the relative magnitude of these variables. Thus, putting together the above ideas into a game table, we obtain the following:

Speculator

			q	$1-q$
			R^+	R^-
	p_0	0_A	w	w
Market	p_1	m_A	w	$-x$
	p_2	M_A	$-y$	$-x$

Table 2.2: The "updated" game table for the financial market game.

Now, to determine when it is advantageous to play one strategy or the other, we need to start by isolating the pure strategies in terms of their expected profitability, and each of the speculator's strategies must be compared with each of the market's strategies, also all the results must be quantified.

Remark 5. *Even though we presented the probabilities associated with the speculator's strategies, we will not consider them for our model.*

We know that there are three outcomes that can happen after the speculator takes a position in the market: a profit (equal to the profit objective), a small loss (equal to the Less Risk amount) or a large loss (equal to the More Risk amount). And, each of these three outcomes happens with some unknown probability. Also, these events are mutually exclusive, i.e., only one of them can happen at any one point in time (or trade). This is because, if the speculator gets stopped out of the market, he/she made a profit or suffered a loss (large or relatively small), and the highest probability that any event can occur is 100%. Given this, it is possible (although unlikely) that one of the three outcomes happens with 100% probability, but since we want to develop our model in terms of the speculator getting stopped out for either a large loss or a small loss, we will construct a diagram (specifically, a probability triangle) which will reflect these two possibilities.

Remark 6. *The diagram that we will be constructing goes along with the algebraic exposition, in order to make the model much easier to interpret.*

2.1.2 The Probability Triangle

For the diagram, consider the market's probability of playing Major Adversity on the vertical axis, and the market's probability of playing Minor Adversity on the horizontal axis. Also, since the highest value either axis can have is 100% (because neither condition can prevail more than 100% of the time), this implies that all combinations of Major Adversity and Minor Adversity can never sum to more than 100%. This being the case, a diagonal line must be drawn between the 100% mark on both axis, which will contain all possible combinations of the market's strategies of Major Adversity and Minor Adversity. Thus, with all of this, we obtain the following probability triangle:

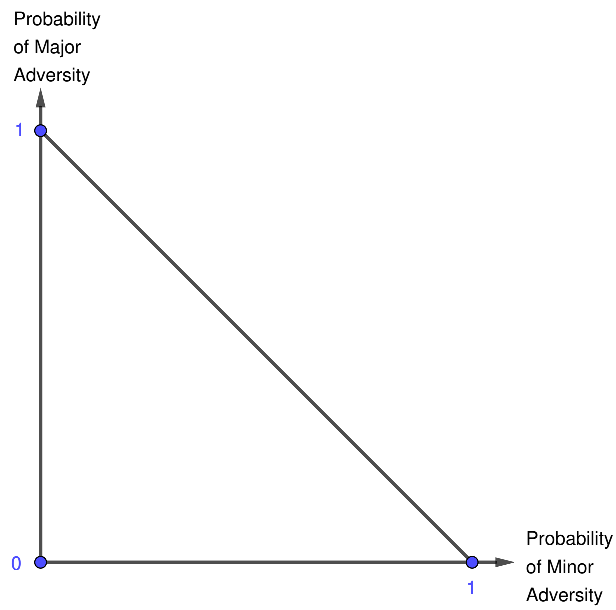

Figure 2.1: The probability triangle showing the likelihood of loss.

We will divide this probability triangle into several regions, which will reflect when it is more advantageous to accept More or Less Risk, or even when it is advantageous not to play the game at all. Furthermore, since game theory gives us methods to determine when a player is guaranteed a certain payoff, we can solve for when it is optimal to accept either More or Less Risk.

So far, we have concentrated on the speculator's strategies which involve taking a position in the market. However, in reality, if we know when to take a position (i.e., when to play the game), we also know when not to take a position in the market (i.e., when not to play the game). So we will develop this model in order to determine when it is advantageous to take a position, along with when it is disadvantageous to do so. Thus, conditions where it is disadvantageous to take a position will correspond to the "Do Not Play" region of the probability triangle.

Now, we can determine, with the aid of the game table 2.2, the expected payoffs from playing each of the speculator's strategies:

- *The Expected Payoff from playing Less Risk (R^-)*:

$$E_S(R^-) = (1 - p_1 - p_2)w + p_1(-x) + p_2(-x) = w - (p_1 + p_2)(w + x) \tag{2.1}$$

- *The Expected Payoff from playing More Risk (R^+)*:

$$E_S(R^+) = (1 - p_1 - p_2)w + p_1 w + p_2(-y) = w - p_2(w + y) \tag{2.2}$$

Equation (2.1) represents the expected payoff from playing the pure strategy Less Risk (R^-) and is written with several variables: the amount that can be won (w), the amount that can be lost due to a small stop (x), and the probability that the market will either give us minor adversity (p_1) or major adversity (p_2). Note that the speculator determines the values of w and x by his/hers risk-to-reward appetite, but the market determines the probabilities p_1 and p_2. If the equation 2.1 is greater than zero, the speculator expects a profit, but if it is less than zero, the speculator expects to lose money. Also, because the speculator is only in control of the variables x and w, we need to express the equation as strict inequality, and solve it in terms of p_1 and p_2. In other words, we need to find out for which market conditions it is always advantageous to accept Less Risk by finding out when the expected payoff from playing Less Risk is greater than zero. Thus we obtain the following:

$$E_S(R^-) > 0 \iff w - (p_1 + p_2)(w + x) > 0 \iff p_1 + p_2 < \frac{w}{w + x} \tag{2.3}$$

Note that we are considering a strict inequality because if $E_S(R^-) = 0$ it is not profitable to play the Less Risk strategy, because its expected payoff is zero.

With all of this, we can incorporate equation 2.3 into the probability triangle yielding the following:

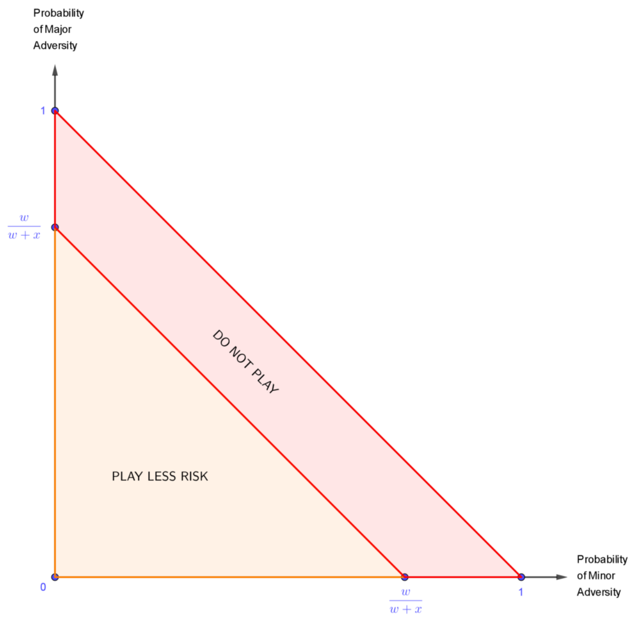

Figure 2.2: The probability triangle divided into two regions: "Play Less Risk" and "Do not Play".

Remark 7. *Note that, by definition of w and x:*

$$w < w + x \iff \frac{w}{w+x} < 1 \tag{2.4}$$

The "Play Less Risk" area contains the points where it is profitable to play the strategy of Less Risk, and the "Do Not Play" region contains money-losing strategies. Also, because equation 2.3 was developed as a strict inequality, the line diving the two regions is not included in the "Play Less Risk" area, so the points on the line ($E_S(R^-) = 0$) are included in the area of loss. Again, the line dividing these areas is determined exclusively by the parameters set by the speculator, so this line will vary from individual to individual, always based on each individual's risk-to-reward appetites, also the value yielded by $w/(w + x)$ is not a constant that holds true for all players in the market. But, since we are developing a model in the general case, it must hold true for each and every person, no matter what their individual circumstances are.

Moving forward, we can now focus on determining when it is advantageous to accept More Risk, however it is not as straightforward as it was for Less Risk, because it is only advantageous to accept More Risk when the market is playing Minor Adversity. And, under this condition, a strategy of Less Risk will cause a small loss, but a strategy of More Risk results in a profit. Looking back at the game table 2.2:

- under market conditions of Zero Adversity, both strategies yield a profit, so the speculator is indifferent between the strategies;
- under market conditions of Minor Adversity, a strategy of More Risk generates a profit, and the strategy of Less Risk causes a loss, so it is advantageous to utilize the More Risk strategy;
- if the market conditions correspond to Major Adversity, both the speculator's strategies are unprofitable, but the Less Risk strategy causes a smaller loss than does the More Risk strategy, so it is less advantageous to play More Risk.

We know that, by equations 2.1 and 2.3, if the expected payoff from the Less Risk strategy is positive, then we are "guaranteed" a positive payoff when Less Risk is played. So, to find out when the strategy of More Risk yields a positive payoff when the strategy of Less Risk does not, we need to analyze equation 2.2 while 2.1 is negative.

So, we need to to find out for which market conditions it is always advantageous to accept More Risk by finding out when the Expected Payoff from playing More Risk (R^+) is greater than zero, assuming that $E_S(R^-) < 0$. Thus we obtain the following:

$$E_S(R^+) > 0 \iff w - p_2(w+y) > 0 \iff p_2 < \frac{w}{w+y} \tag{2.5}$$

Note that we are considering a strict inequality because if $E_S(R^+) = 0$ it is not profitable to play the More Risk strategy, since its expected payoff is zero. Also, observe that equation 2.5 is only in terms of Major Adversity (p_2) and it implies that if the probability of Major is greater than $w/(w+y)$ then the trade will lose money, otherwise the trade will make money. In terms of game theory, if the probability of Major Adversity is greater than $w/(w+y)$, then we will not play the game, and if the probability of Major Adversity is less than $w/(w+y)$, then we play the pure strategy of More Risk. Additionally, if the probability of Major Adversity is equal to $w/(w+y)$, then the trade will result in a profit of zero, thus we will also not play the game.

With all of this, we can incorporate equation 2.5 into the probability triangle yielding the following:

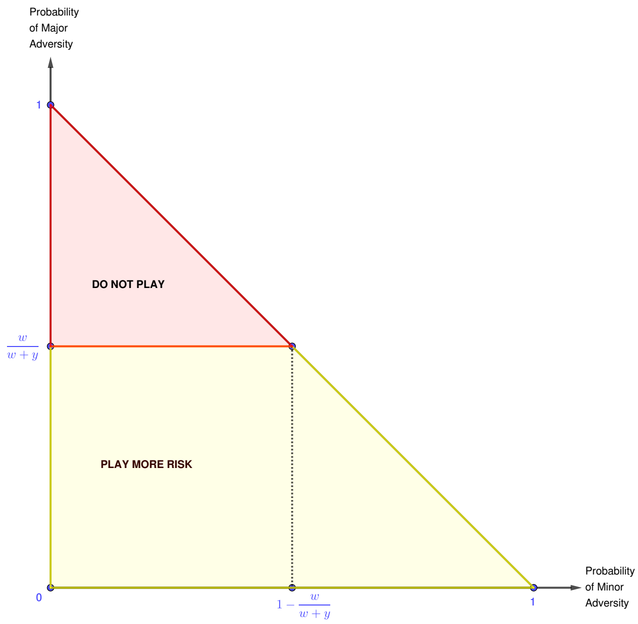

Figure 2.3: The probability triangle divided into two regions: "Play More Risk" and "Do not Play".

Remark 8. *Regarding the previous probability triangle, note that:*

- *by definition of w and y:* $w < w + y \iff \frac{w}{w+y} < 1$;
- $\begin{cases} p_2 = \frac{w}{w+y} \\ p_1 + p_2 = 1 \end{cases} \iff \begin{cases} p_2 = \frac{w}{w+y} \\ p_1 = 1 - \frac{w}{w+y} \end{cases}$.

Here, the lower region contains the conditions where it is advantageous to play the pure strategy of More Risk, and the upper region is where it is disadvantageous to play More Risk. Also, once again, the points in the separating line ($E_S(R^+) = 0$) are included in the Do Not Play area.

The same reasoning used to understand the implications of playing the pure strategy of Less Risk hold true for the strategy of More Risk, i.e., points within the "Play More Risk" area represent profitable trades and points within the "Do Not Play" area represent losses. Also, once again, the solutions must be interpreted in a probabilistic sense.

Now that we know when it is advantageous to play More Risk (assuming that the result of playing Less Risk is negative), we need determine when it is advantageous to play More Risk despite the result of playing Less Risk, because there is a region of the probability triangle where the two strategies overlap. So we still need to determine when it is advantageous to play

More Risk, irrespective of the merit of playing Less Risk. Thus we need to solve the following equations:

- $E_S(R^+) > 0 \iff w - p_2(w+y) > 0 \iff p_2 < \frac{w}{w+y}$; (2.6)
- $E_S(R^-) < E_S(R^+) \iff w - (p_1+p_2)(w+x) < w - p_2(w+y) \iff p_1 > p_2\frac{y-x}{w+x}$. (2.7)

Consider that equation 2.7 was developed as an equality, then if the probability of Minor Adversity is equal to zero (i.e., $p_1 = 0$), then the probability of Major Adversity has to equal to zero as well (i.e., $p_2 = 0$). However, in the inequality, if p_1 were zero, then p_2 would have to be less than zero, but this is in conflict with the variables' definitions, because probabilities can only take values between zero and one, thus they cannot be negative. Also, the probability of Major Adversity occurring in the real world is not less than zero, because, if this were true, all the players in the market would always win. Moreover, since the formula that expresses the slope of the line $((y-x)/(w+x))$ is always a positive number (as the variables x, y and w are all positive numbers), whenever p_1 is zero, then p_2 has to be zero, and vice versa. Also, the line itself represents the boundary where it is equally advantageous to play the pure risk strategies of either Less Risk or More Risk, and the area above the line defines where it is advantageous to play Less Risk.

All of these results are combined in the following probability triangle:

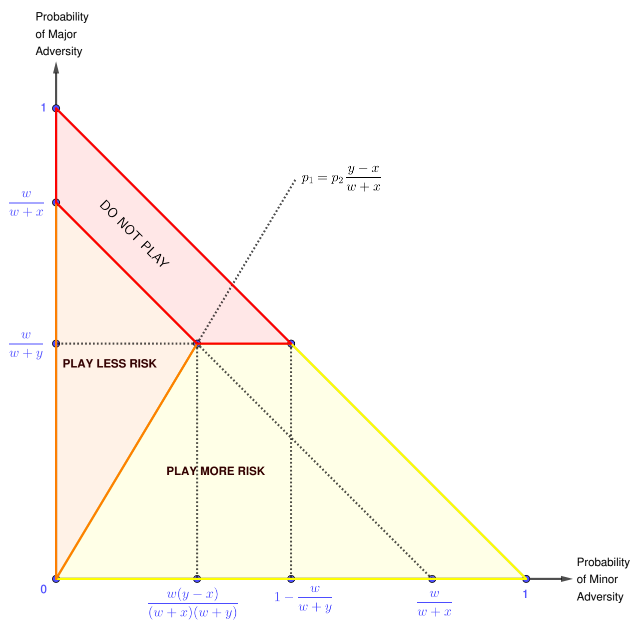

Figure 2.4: The probability triangle with all the analyzes done so far, which is divided into three regions: "Play Less Risk", "Play More Risk" and "Do Not Play".

Remark 9. *Regarding the previous probability triangle, note that:*

- $\frac{w}{w+x} > 1 - \frac{w}{w+y} \iff \frac{w(w+y)+w(w+x)}{(w+x)(w+y)} > 1 \iff \frac{2w^2+wx+wy}{w^2+wy+wx+xy} > 1 \iff 2w^2 + wx + wy > w^2 + wy + wx + xy \iff w^2 > xy$, *which is true because, by definition,* $w \geq y > x$*;*
- $1 - \frac{w}{w+y} > \frac{w(y-x)}{(w+x)(w+y)} \iff 1 > \frac{w}{w+x} \iff x > 0$, *which is true by definition of* x*;*
- $\begin{cases} p_1 = p_2 \frac{y-x}{w+x} \\ p_2 = \frac{w}{w+y} \end{cases} \iff \begin{cases} p_1 = \frac{w(y-x)}{(w+x)(w+y)} \\ p_2 = \frac{w}{w+y} \end{cases}$;
- $\begin{cases} p_1 = p_2 \frac{y-x}{w+x} \\ p_1 + p_2 = \frac{w}{w+x} \end{cases} \iff \begin{cases} p_1 = \frac{w(y-x)}{(w+x)(w+y)} \\ p_2 = \frac{w}{w+y} \end{cases}$;
- $\begin{cases} p_1 + p_2 = \frac{w}{w+x} \\ p_2 = \frac{w}{w+y} \end{cases} \iff \begin{cases} p_1 = \frac{w(y-x)}{(w+x)(w+y)} \\ p_2 = \frac{w}{w+y} \end{cases}$.

In the previous probability triangle, there are three regions: the Do Not Play region, the Less Risk region, and the More Risk region. Also the dotted lines show the location of the original regions, as well some relevant intersection points. Note that all of the interior lines intersect at one point $(p_1 = (w(y-x))/((w+x)(w+y)), p_2 = w/(w+y))$, and that we included the separation line between the Less Risk and More Risk regions (i.e., $p_1 = p_2 \frac{y-x}{w+x}$) in the Less Risk region, but the intersection point between all the interior lines is considered a part of the Do Not Play region.

Finally, observe that, in all of the obtained probability triangles, a "Do Not Play" region has appeared which is not related to any possible strategy (on the presented financial game) that the speculator can choose from. However, the "Do Not Play" strategy is implicit in the game tables 2.1 and 2.2. To see this consider the game table 2.2 and that the speculator has an additional "Do Not Play" strategy. So, if the speculator chooses this strategy, then he/she will not enter the trade, and thus will not lose or win with the trade. Hence, the payoffs from this strategy are always zero independently of the market's strategy. So, the game table 2.2 becomes:

			Speculator		
			q_1	q_2	$1 - q_1 - q_2$
			R^+	R^-	D
	p_0	0_A	w	w	0
Market	p_1	m_A	w	$-x$	0
	p_2	M_A	$-y$	$-x$	0

Table 2.3: The game table for the financial market game including the "Do Not Play" (D) strategy.

However, the payoffs from adding this strategy do not change any of the calculations that we made to determine the several probability triangles, also these would only be relevant if we wanted to determine the best mixed strategy for the speculator to play (specifically, the probabilities q_1, q_2 and $1 - q_1 - q_2$ would be important). But, since we only want to determine the best pure strategy that the speculator should play (i.e., one of the speculator's probabilities will be equal to one) by taking into account the market's probabilities (p_0, p_1 and p_2), the "Do Not Play" strategy being explicit or not in the game table is not relevant, but this strategy is still important to the overall, because it complements the speculator's other two strategies (Play More Risk and Play Less Risk).

So, the complete model, which incorporates all of the previous calculations and graphic representations, has the general form shown by the probability triangle in Figure 2.4. Also, this probability triangle represents the situation a speculator faces in a financial market, because it takes into account the speculator accepting either More Risk or Less Risk, and the market generating conditions of either Zero Adversity, Minor Adversity, or Major Adversity (always with respect to the speculator's position). Additionally, the probability triangle has Minor Adversity and Major Adversity as its axes, yet it also shows the condition of Zero Adversity, which is the complete absence of both Major Adversity and Minor Adversity, which is represented by the origin point on the probability triangle.

Always have in mind that the model has to be interpreted in terms of "if these certain probabilities exist, then we should play a specific strategy". So the model cannot tell us what the probabilities are, it only tells us that if certain probabilities exist, then a particular strategy should be employed. Thus, if we play the financial game repeatedly, under some predetermined circumstances, the wins will outweigh the losses, and the net result of our successive plays will be profitable, this is because we need to interpret the model in the probabilistic sense rather than in an absolute sense. For instance, the model does not suggest that each and every trade that falls within the parameters of $w/(w+x)$ will necessarily be profitable, only that over time the amount won will be greater than the lost.

Now that we have the complete model, we need to estimate the probabilities of the market playing the strategies of Major Adversity and Minor Adversity. Furthermore, we need to make these estimates as accurate as possible, because if they are not, the model will lose its predictive value. And we will accomplish this in the next section, with the aid of Markov Chains.

2.2 The Markov Chains Model

As we have seen in the previous section, playing the markets is an iterated game, so the next important task that we have to address is the (probabilistic) method that we will use to estimate the probabilities of the market playing Zero Adversity, Minor Adversity and Major Adversity (p_0, p_1 and p_2 respectively). However, the financial assets' prices fluctuate from a variety of ranges (but always strictly positive). Thus we need to split the data into classes in order for us to make some kind of probabilistic analysis. For this, consider the standard deviation (α) of a dataset transformed with the percentage change transformation, and define the strategies' thresholds as:

- the Less Risk threshold corresponds to minus two times the standard deviation of the data (-2α);
- the More Risk threshold corresponds to minus three times the standard deviation of the data (-3α);
- the profit threshold corresponds to three times the standard deviation of the data ($3 \cdot \alpha$).

Since different assets from the stock market have different price ranges and levels of volatility, then by defining the thresholds in this manner, we will maintain a certain coherence across all the datasets. Also, note that the less and More Risk thresholds have to be negative, because they correspond to possible losses. Additionally, since the datasets' unit of measure is the percentage change, the standard deviation's unit of measure is also the percentage change.

After defining the thresholds, we can formally say what is the relationship between the market's chosen strategies with an asset's price. Thus, to accomplish this, we will assume that:

- the asset's price drops further than the Less Risk threshold if and only if the market chooses to play the Minor Adversity strategy;
- the asset's price drops further than the More Risk threshold if and only if the market chooses to play the Major Adversity strategy;
- the asset's price increases further than the profit threshold if and only if the market chooses to play the Zero Adversity strategy.

Now, consider that we observed the asset's percentage price change for N successive and mutually independent financial market games, and that we want to determine the mentioned probabilities for the next $(N+1)$ game. Also, the percentage price change of game i is denoted by X_i, $i = 1, \ldots, N+1$. Additionally, assume that if the thresholds of Major Adversity or Zero Adversity are reached in a game, suppose that it was on game $k \in \{1, \ldots, N\}$, then the speculator will not play on the following games, $k+1, \ldots, N$, otherwise the speculator will continue to play. We need to assume this because, if the speculator loses or wins on a game, then we will not continue playing, due to the trade being closed, and if the price does not reach one of the thresholds, the speculator will not win nor lose the game, so he/she needs to keep playing, because the trade is still open.

Remark 10. *Note that if the market chooses to play Minor Adversity, the speculator only has to stop playing if he/she played the Less Risk strategy.*

With all of this, we can start estimating the desired probabilities for the $(N+1)^{th}$ game, knowing that the probability of the market playing a certain strategy at game $N+1$ is related to the probabilities of the market's choices on the N previous games, i.e., we want to determine

$$p_0 = P(X_{N+1} \geq 3 \cdot \alpha | X_1, \ldots, X_N),$$

$$p_2 = P(X_{N+1} \leq -3 \cdot \alpha | X_1, \ldots, X_N),$$

$$p_1 = 1 - p_0 - p_2.$$

Remark 11. *Note that we will not directly estimate p_1, because it is simpler to estimate p_0 and p_2, due to the way we defined these probabilities. Also, we can do this because $p_0 + p_1 + p_2 = 1$. So, moving forward, we will not reference the estimator of p_1 unless we see fit to do so.*

Firstly, suppose that we only consider one game to determine our probabilities, i.e., we will start by considering $N = 1$, so we have the following:

- $p_2 = P(X_2 \leq -3 \cdot \alpha | X_1) = P(X_1 \leq -3 \cdot \alpha);$ (2.8)
- $p_0 = P(X_2 \geq 3 \cdot \alpha | X_1) = P(X_1 \geq 3 \cdot \alpha).$ (2.9)

We can interpret equation 2.8 for the probability of the market playing Major Adversity as follows: if the percentage price change reaches the More Risk threshold in game 1, then the speculator stops playing. So, the probability of the price change reaching the More Risk threshold is obtained by simply calculating the probability of the percentage change reaching the More Risk threshold in the previous game, i.e., $P(X_1 \leq -3 \cdot \alpha)$.

Similarly, the probability of the market playing the Zero Adversity strategy is obtained by calculating the probability of the percentage change reaching the profit objective threshold in the previous game, i.e., $P(X_1 \geq 3 \cdot \alpha)$.

However, since we have access to more historical data of the asset's price, we can determine these probabilities more accurately by taking into account more games. Now, consider that we will use the results of two past games to determine our probabilities, i.e., we will consider $N = 2$, thus we obtain:

$$\begin{aligned} p_2 &= P(X_3 \leq -3 \cdot \alpha | X_2, X_1) = \\ &= P(X_1 \leq -3 \cdot \alpha) + P(-3 \cdot \alpha < X_1 \leq 3 \cdot \alpha \wedge X_2 \leq -3 \cdot \alpha); \end{aligned} \tag{2.10}$$

$$p_0 = P(X_3 \geq 3 \cdot \alpha | X_2, X_1) = P(X_1 \geq 3 \cdot \alpha) + P(X_1 < 3 \cdot \alpha \wedge X_2 \geq 3 \cdot \alpha). \tag{2.11}$$

(2.12)

Thus we can can interpret the new equation 2.10 for the probability of the market playing Major Adversity as follows: the speculator stops playing, if the percentage price change reaches the More Risk threshold in game 1 or if the threshold is only reached in game 2 (implying that, in game 1, no threshold was reached). So, the probability of the price change reaching the More Risk threshold is obtained by adding the probability of the percentage change reaching the More Risk threshold in game 1 to the probability of the percentage change reaching the

More Risk threshold in game 2 without reaching it in game 1. And, a similar interpretation can be given to equation 2.11.

Finally, we can obtain even more accurate probabilities if we consider the results of all the played games (i.e., by considering all the historical price data). Thus, considering the results of N games, the equations for the desired probabilities are:

$$\begin{aligned} p_2 &= P(X_{N+1} \leq -3\cdot\alpha | X_N, \ldots, X_1) = \\ &= P(X_1 \leq -3\cdot\alpha) + P(-3\cdot\alpha < X_1 < 3\cdot\alpha \wedge X_2 \leq -3\cdot\alpha) + \\ &+ P(-3\cdot\alpha < X_1 < 3\cdot\alpha \wedge -3\cdot\alpha < X_2 < 3\cdot\alpha \wedge X_3 \leq -3\cdot\alpha) + \cdots + \\ &+ P(-3\cdot\alpha < X_1 < 3\cdot\alpha \wedge -3\cdot\alpha < X_2 < 3\cdot\alpha \wedge -3\cdot\alpha < X_3 < 3\cdot\alpha \wedge \\ &\wedge \cdots \wedge -3\cdot X_N \leq -3\cdot\alpha). \end{aligned} \tag{2.13}$$

$$\begin{aligned} p_0 &= P(X_{N+1} \geq 3\cdot\alpha | X_N, \ldots, X_1) = P(X_1 \geq 3\cdot\alpha) + P(X_1 < 3\cdot\alpha \wedge X_2 \geq 3\cdot\alpha) + \\ &+ P(X_1 < 3\cdot\alpha \wedge X_2 < 3\cdot\alpha \wedge X_3 \geq 3\cdot\alpha) + \cdots + \\ &+ P(X_1 < 3\cdot\alpha \wedge X_2 < 3\cdot\alpha \wedge X_3 < 3\cdot\alpha \wedge \cdots \wedge X_N \geq 3\cdot\alpha); \end{aligned} \tag{2.14}$$

The intuition behind the obtained equations 2.14 and 2.13 is similar to the one that we used to obtain the equations 2.8 and 2.9. Also, because the N games are mutually independent, from basic probability theory we have that the equations 2.14 and 2.13 are equivalent to:

$$\begin{aligned} p_2 &= P(X_{N+1} \leq -3\cdot\alpha | X_N, \ldots, X_1) = \\ &= P(X_1 \leq -3\cdot\alpha) + P(-3\cdot\alpha < X_1 < 3\cdot\alpha)P(X_2 \leq -3\cdot\alpha) + \\ &+ P(-3\cdot\alpha < X_1 < 3\cdot\alpha)P(-3\cdot\alpha < X_2 < 3\cdot\alpha)P(X_3 \leq -3\cdot\alpha) + \cdots + \\ &+ P(-3\cdot\alpha < X_1 < 3\cdot\alpha)P(-3\cdot\alpha < X_2 < 3\cdot\alpha)P(-3\cdot\alpha < X_3 < 3\cdot\alpha)\cdots \\ &\cdots P(-3\cdot X_N \leq -3\cdot\alpha) \end{aligned} \tag{2.15}$$

$$\begin{aligned} p_0 &= P(X_{N+1} \geq 3\cdot\alpha | X_N, \ldots, X_1) = P(X_1 \geq 3\cdot\alpha) + P(X_1 < 3\cdot\alpha)P(X_2 \geq 3\cdot\alpha) + \\ &+ P(X_1 < 3\cdot\alpha)P(X_2 < 3\cdot\alpha)P(X_3 \geq 3\cdot\alpha) + \cdots + \\ &+ P(X_1 < 3\cdot\alpha)P(X_2 < 3\cdot\alpha)P(X_3 < 3\cdot\alpha)\cdots P(X_N \geq 3\cdot\alpha); \end{aligned} \tag{2.16}$$

From these equations we can see that, for example, to estimate p_0 (and p_2), we would have to estimate $(N(N+1))/2$ probabilities, which would be computationally inefficient and the error from the final estimate would increase due to the large number of individual estimates. So, to overcome these problems we will use Markov chains to estimate the probabilities p_0, p_1 and p_2.

Thus, using the same assumptions and notations as before:

- the asset's percentage price change for N successive financial market games is known;
- the percentage price change of game i is denoted by X_i, $i = 1, \dots, N+1$;
- we want to determine the mentioned probabilities the $N+1^{th}$ game;
- if any of thresholds is reached in a game, suppose that it was on game $k \in \{1, \dots, N\}$, then the speculator will not play on the following games, $k+1, \dots, N$, otherwise the speculator will continue to play.

Now, because we will use Markov Chains, we need to assume that the probabilities associated with each game are related through the Markov property (see Definition 8 from Section B.1 from Appendix B). So, we obtain the following estimators for p_0 and p_2:

- $p_0 = P(X_{N+1} \geq 3 \cdot \alpha | X_N, \dots, X_1) = P(X_{N+1} \geq 3 \cdot \alpha | X_N);$ (2.17)
- $p_2 = P(X_{N+1} \leq -3 \cdot \alpha | X_N, \dots, X_1) = P(X_{N+1} \leq -3 \cdot \alpha | X_N).$ (2.18)

In order for us to be able to use the percentage price change (of a given asset) at game i (i.e., to use X_i) as the underlying stochastic process of the Markov chain, we need to split the data into classes (or states). Also, by defining the Markov chain we will obtain its probability matrix, which will allow us to estimate p_0 and p_2.

Before moving further, we need to note that, for instance, if the price is (at a certain time) on a lower price class (relative to the initial price), then it will have a higher probability of transitioning to a higher price class, due to the nature of the data that we are utilizing, and a similar argument can be made if the price is on a higher class (as we have seen in Chapter 1). However, this is represented by the Markov property, because the probability of the Markov chain being in a certain state at time t only depends on which state the chain was at time $t-1$, so this probability may change according to which states the chain encounters itself in time $t-1$. And this fact will also affect on how we will define the chain's classes.

To define the classes we can utilize the standard deviation (which we previously denoted by α) of the dataset, and since we defined (and used) the strategies' thresholds, we will split the data "around" these thresholds values, also the classes' ranges and distance between them will be α. Additionally, due to the mentioned volatility and wide range of the assets' prices, they may reach one of the thresholds in the first game (or iteration), or they may not reach them at all.

So, for these reasons we will define some intermediate classes between the the classes associated with the thresholds (or market strategies). Thus, with all of this, we obtain that the classes (or states) are:

- the Major Adversity class is $s_M = \{X_t : X_t \leq -2.5 \cdot \alpha\}$; (2.19)
- the Minor Adversity class is $s_m = \{X_t : -2.5 \cdot \alpha < X_t \leq -1.5 \cdot \alpha\}$; (2.20)
- the intermediate classes between Minor Adversity and the Zero Adversity classes are:
  - $s_1 = \{X_t : -1.5 \cdot \alpha < X_t \leq -0.5 \cdot \alpha\}$; (2.21)
  - $s_2 = \{X_t : -0.5 \cdot \alpha < X_t \leq 0.5 \cdot \alpha\}$; (2.22)
  - $s_3 = \{X_t : 0.5 \cdot \alpha < X_t \leq 1.5 \cdot \alpha\}$; (2.23)
  - $s_4 = \{X_t : 1.5 \cdot \alpha < X_t \leq 2.5 \cdot \alpha\}$. (2.24)
- the Zero Adversity (or Profit) class is $s_Z = \{X_t : X_t > 2.5 \cdot \alpha\}$. (2.25)

Remark 12. *Note that, instead of using the previously defined threshold to limit the classes, we chose to define the classes around these thresholds, in order to include them. However, if all the classes maintain a certain coherence according to the thresholds and have the same range (excluding the s_M and s_Z classes), then we will obtain similar results after applying our models.*

As an example, consider the dataset

$$\{45.00, 44.49, 43.44, 40.17, 41.05, 41.53, 41.36, 40.68, 40.46, 38.42\} \quad (2.26)$$

to be the prices of some financial asset for ten consecutive days, then its percentage change transformed dataset (rounded to two decimal cases) is:

$$\{-1.13, -2.36, -7.53, 2.19, 1.17, -0.41, -1.64, -0.54, -5.04\}, \quad (2.27)$$

which was obtained by applying the percentage changes transformation presented in Section A.5.1 (from Appendix A). So, the standard deviation of this transformed dataset is 3.00 (which also is a percentage), i.e., $\alpha = 3.00$. Hence the classes, for this example, are:

- $s_M = \{X_t : X_t \leq -2.5 \cdot \alpha\} = \{X_t : X_t \leq -7.5\}$;
- $s_m = \{X_t : -2.5 \cdot \alpha < X_t \leq -1.5 \cdot \alpha\} = \{X_t : -7.5 < X_t \leq -4.5\}$;
- $s_1 = \{X_t : -1.5 \cdot \alpha < X_t \leq -0.5 \cdot \alpha\} = \{X_t : -4.5 < X_t \leq -1.5\}$;
- $s_2 = \{X_t : -0.5 \cdot \alpha < X_t \leq 0.5 \cdot \alpha\} = \{X_t : -1.5 < X_t \leq 1.5\}$;
- $s_3 = \{X_t : 0.5 \cdot \alpha < X_t \leq 1.5 \cdot \alpha\} = \{X_t : 1.5 < X_t \leq 4.5\}$;

- $s_4 = \{X_t : 1.5 \cdot \alpha < X_t \leq 2.5 \cdot \alpha\} = \{X_t : 4.5 < X_t \leq 7.5\}$;
- $s_Z = \{X_t : X_t > 2.5 \cdot \alpha\} = \{X_t : X_t > 7.5\}$.

2.2.1 Defining the Markov Chains

Before formally defining the necessary Markov Chains, we need to make some observations about the described classes. According to our assumptions, if the market chooses to play Major Adversity or Zero Adversity, the speculator will have to stop playing (which will result in a major a loss or in a profit, respectively) independently of the speculator's chosen strategy, but if the market chooses to play Minor Adversity, the speculator only has to stop playing if he/she chose the Less Risk strategy.

Also, with the aid of game table 2.2 (from the previous section (2.1)), we can see that the results of the market playing the Major Adversity strategy are only noticeable if the speculator chooses to play the More Risk strategy, because if the speculator chooses the Less Risk strategy he/she will stop playing the game immediately after the Less Risk threshold is reached, thus he/she will not know if the price further increased or decreased.

Thus, assuming that the speculator chose the More Risk strategy, we can determine the probability of the market playing the Major Adversity strategy. Also, if we assume that the speculator chose to play the Less Risk strategy, then we can determine the probability of the market playing the Zero Adversity strategy, this is because, in this case, the speculator only has a profit if the market chooses this strategy.

So, for these reasons we will define two Markov Chains, one where we consider that the speculator chose the Less Risk strategy and another where he/she chose to play the More Risk strategy. However, we will always use the same assumptions, strategies' thresholds and data for both the Markov Chains, so that we can utilize the probability matrices from each to estimate the probabilities p_0, p_1 and p_2. Also, we will assume that $s_2 = \{X_t : 0.5 \cdot \alpha < X_t \leq 0.5 \cdot \alpha\}$ is the initial state for both the Markov chains, because when the speculator enters for trade of a certain asset, then the asset's initial percentage price change will be 0% which is an element of the s_2 class.

Regarding the Markov Chain where we assume that the speculator chose to play the More Risk strategy, we have the following observations about its states (or classes):

- The classes $s_M, s_m, s_1, \dots, s_4, s_Z$ will retain the same definitions as before
- To represent the fact that the speculator only stops playing if the price enters the Major Adversity class (s_M) or the Zero Adversity class (s_Z) in the Markov Chain, we simply must define these classes as absorbing states, i.e., if the price enters one of these classes, then it will never exit them (for more details see Section B.3 from Appendix B).
- Since the speculator does not stop playing if the price is in one of the remaining classes, the price may go to any class (including staying in the same class). And, to represent this in terms of Markov Chains, we simply define these classes as transient (for more details see Section B.1 and B.3 from Appendix B). Also, these states ($s_m, s_1, \dots, s_4$) communicate between themselves, thus they form a communicating class in the Markov Chain.

In the terms of Markov Chains (presented in Appendix B), for this Markov Chain, we are considering the stochastic process $\{X_t, t = 1, 2, \dots, N\}$, where X_t is the percentage price change at game t, with a discrete and finite state space $S = \{s_M, s_m, s_1, \dots, s_4, s_Z\}$, where for all states $S_0, S_1, \dots, S_{t-1}, S_i, S_j \in S$ and steps (or games) $t \in \{1, \dots, N\}$:

$$P(X_{t+1} = S_j | X_1 = S_0, \dots, X_{t-1} = S_{t-1}, X_t = S_i) = P(X_{t+1} = S_j | X_t = S_i) =: p_{ij}^{t,t+1}.$$

Here, the steps of the Markov Chain represent each successive game from 1 up until N, and S_i represents a state from the state space $S = \{s_M, s_m, s_Z, s_1, \dots, s_4\}$, which is composed by the classes that we previously defined, thus they have the mentioned properties. Also, the 7×7 transition matrix P_M associated with this chain will be defined as:

$$P_M = \begin{array}{c} \\ s_M \\ s_m \\ s_1 \\ \vdots \\ s_4 \\ s_Z \end{array} \begin{array}{c} \begin{array}{cccccc} s_M & s_m & s_1 & \cdots & s_4 & s_Z \end{array} \\ \begin{pmatrix} p_{11} & p_{12} & p_{13} & \cdots & p_{16} & p_{17} \\ p_{21} & p_{22} & p_{23} & \cdots & p_{26} & p_{27} \\ p_{31} & p_{32} & p_{33} & \cdots & p_{36} & p_{37} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ p_{61} & p_{62} & p_{63} & \cdots & p_{66} & p_{67} \\ p_{71} & p_{72} & p_{73} & \cdots & p_{76} & p_{77} \end{pmatrix} \end{array} \tag{2.28}$$

To visualize this Markov Chain we can use the following diagram:

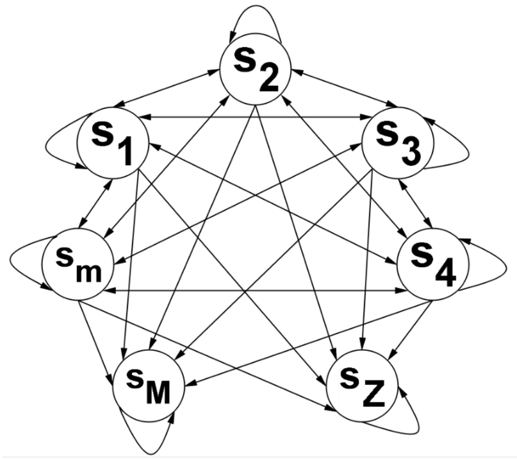

Figure 2.5: The Markov Chain where we assume that the speculator chose to play the More Risk strategy.

Note that we could have simplified the previous diagram by joining the states $s_m, s_1, \ldots, s_4$ in the same communicating class. However, it is useful for us to present the Markov chain in this manner, because it allow us to take more conclusions on how the chain develops as we move forward in time.

So, assuming that the initial state is s_2 (i.e., assuming that $\pi = (0,0,0,1,0,0,0)^T$) and that the transition matrix P_M related to the Markov chain is well defined, the probability of the market playing the Major Adversity strategy (p_2) at time (or game) t is given by the first element of πP_M^t.

Now, regarding the Markov Chain where we assume that the speculator chose to play the Less Risk strategy, we can make similar observations as before, but with some modifications:

- The Major Adversity class is not necessary for this Markov chain, because the speculator will stop playing if the price reaches the Minor Adversity class. So the s_M class will be "included" in the s_m class, thus s_m is altered to $s_m = \{X_t : X_t \leq -1.5 \cdot \alpha\}$ (considering Example 2.26, this class becomes $s_m = \{X_t : X_t \leq -4.5\}$).
- The classes $s_Z, s_1, \ldots, s_4$ are defined as before.
- To represent the fact that the speculator stops playing if the price enters the Minor Adversity class (s_m) or the Zero Adversity class (s_Z) in the Markov Chain, we simply must define these classes as absorbing states, i.e., if the price enters one of these classes,

then it will never exit them (for more details see Section B.3 from Appendix B).

- Since the speculator does not stop playing if the price is in one of the remaining classes, the price may go to any class (including staying in the same class). And, to represent this in terms of Markov Chains, we simply define these classes as transient (for more details see Section B.1 and B.3 from Appendix B).

As before, in the terms of Markov Chains presented in Appendix B, we are considering the same stochastic process $\{X_t, t = 1, 2, \ldots, N\}$, where X_t is the percentage price change at game t, with a discrete and finite state space $S = \{s_m, s_Z, s_1, \ldots, s_4\}$, where for all states $S_0, S_1, \ldots, S_{t-1}, S_i, S_j \in S$ and steps (or games) $t \in \{1, \ldots, N\}$:

$$P(X_{t+1} = S_j | X_1 = S_0, \ldots, X_{t-1} = S_{t-1}, X_t = S_i) = P(X_{t+1} = S_j | X_t = S_i) =: p_{ij}^{t,t+1}.$$

Here, the steps of the Markov Chain represent each successive game from 1 up until N, and S_i represents a state from the state space $S = \{s_m, s_Z, s_1, \ldots, s_4\}$, which is composed by the classes that we previously defined, thus they have the mentioned properties. Also, the 6×6 transition matrix P_L associated with this chain will be defined as:

$$P_L = \begin{array}{c} \\ s_m \\ s_1 \\ \vdots \\ s_4 \\ s_Z \end{array} \begin{array}{c} \begin{array}{cccccc} s_m & s_1 & \cdots & s_4 & s_Z & \end{array} \\ \begin{pmatrix} p_{11} & p_{12} & p_{13} & \cdots & p_{16} & p_{16} \\ p_{21} & p_{22} & p_{23} & \cdots & p_{26} & p_{26} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ p_{51} & p_{52} & p_{53} & \cdots & p_{56} & p_{56} \\ p_{61} & p_{62} & p_{63} & \cdots & p_{66} & p_{66} \end{pmatrix} \end{array} \tag{2.29}$$

To visualize this Markov Chain we can use the following diagram:

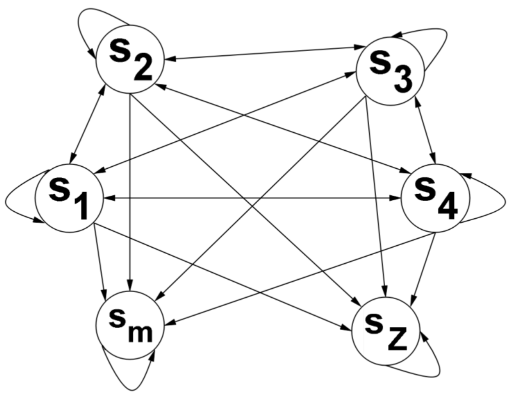

Figure 2.6: The Markov Chain where we assume that the speculator chose to play the Less Risk strategy.

Regarding this diagram, note that is its similar to the previous one (2.5), however, in this one, the state s_M is included in the state s_m. Additionally, we could have simplified the diagram by joining the states $s_1, \ldots, s_4$ in the same communicating class. But again, it is useful for us to present the Markov chain in this manner, for the same reasons as before.

So, assuming that the initial state is s_2 (i.e., assuming that $\pi = (0, 0, 1, 0, 0, 0)^T$) and that the transition matrix P_L related to the Markov chain is well defined, the probability of the market playing the Zero Adversity strategy (p_0) at time/game t is given by the last element of πP_L^t.

With all of this, we have the necessary methods to estimate the probabilities of the market playing Zero Adversity, Minor Adversity and Major Adversity, thus we also have a method on how to choose the best strategy for a certain dataset. However, the estimation method for the market's probabilities is not complete, because we still have to estimate the transition probability matrix for each of the defined Markov chains. So, this is what we will focus on until the end of this section. But, before moving further, note that:

- we will always use the same (percentage change transformed) dataset for all of the estimations;
- since the s_Z state is absorbing in both of the chains, then we do not need to estimate its transition probabilities, i.e., the last row of both the transition matrices (2.28 and 2.29) is of the form $(0, \ldots, 0, 1)$;
- the state s_M in the Markov chain related to the More Risk strategy, like the s_Z state, is

absorbing, thus the first row of the transition matrix 2.28 is of the form $(1, 0, \ldots, 0)$;

- the state s_m in the Markov chain related to the Less Risk strategy, like the s_Z state, is absorbing, thus the first row of the transition matrix 2.29 is of the form $(1, 0, \ldots, 0)$.

2.2.2 Estimation of the Transition Probabilities

To estimate the transition probabilities for each of the Markov chains, lets start by considering the one where we assume that the speculator chose to play the More Risk strategy, represented by the following transition matrix (similar to the previously presented matrix 2.28):

$$P_M = \begin{array}{c} \\ s_M \\ s_m \\ s_1 \\ \vdots \\ s_4 \\ s_Z \end{array} \begin{array}{c} \begin{array}{cccccc} s_M & s_m & s_1 & \cdots & s_4 & s_Z \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ p_{21} & p_{22} & p_{23} & \cdots & p_{26} & p_{27} \\ p_{31} & p_{32} & p_{33} & \cdots & p_{36} & p_{37} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ p_{61} & p_{62} & p_{63} & \cdots & p_{66} & p_{67} \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \end{array}. \tag{2.30}$$

Now, lets consider that we are departing from state $s_2 = \{X_t : 0.5 \cdot \alpha < X_t \leq 0.5 \cdot \alpha\}$ (the assumed initial state of the chain), so to estimate the transition probabilities $\{p_{41}, p_{42}, p_{43}, \cdots, p_{47}\}$, we will just determine the relative frequency of each of the sates using the dataset, and we will use these frequencies on the corresponding row of the transition matrix.

Utilizing the Example 2.26, the relative frequencies for the transformed dataset for these classes are:

s_M	s_m	s_1	s_2	s_3	s_4	s_Z
$\frac{1}{9}$	$\frac{1}{9}$	$\frac{2}{9}$	$\frac{4}{9}$	$\frac{1}{9}$	$\frac{0}{9}$	$\frac{0}{9}$

Table 2.4: Relative frequencies table considering that the starting state is s_2.

And, replacing in the transition matrix (related to Example 2.26's Markov chain), we obtain:

$$P_M = \begin{array}{c} \\ s_M \\ s_m \\ s_1 \\ s_2 \\ s_3 \\ s_4 \\ s_Z \end{array} \begin{array}{c} \begin{array}{ccccccc} s_M & s_m & s_1 & s_2 & s_3 & s_4 & s_Z \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ p_{21} & p_{22} & p_{23} & p_{24} & p_{25} & p_{26} & p_{27} \\ p_{31} & p_{32} & p_{33} & p_{34} & p_{35} & p_{36} & p_{37} \\ 1/9 & 1/9 & 2/9 & 4/9 & 1/9 & 0 & 0 \\ p_{51} & p_{52} & p_{53} & p_{54} & p_{55} & p_{56} & p_{57} \\ p_{61} & p_{62} & p_{63} & p_{64} & p_{65} & p_{66} & p_{67} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}. \tag{2.31}$$

As it was previously observed, the probabilities of transitioning from state s_2 (to any other state) are not the same as if we considered that we started from a different state. Thus, in order to take this into account and to still use the relative frequency "method" to estimate the transition probabilities, we need to slightly alter the classes on which we will determine the relative frequencies.

For example, consider the classes obtained from Example 2.26, if the price increased 3% (of its initial price) at the first iteration of the chain, i.e., the price went from 100% to 103% of its (initial) value (which translates in a $((103-100)/100)*100 = 3\%$ percentage change in price), then the chain moved from state s_2 to the state s_3. However, if the price is now at the state s_3 and it further increased 3% (comparing to the initial price), the chain will not move from the s_3 state to the s_4 state, because , in this case, the price went from 103% to 106% of its (initial) value, so the percentage change in price is $((106-103)/103)*100 \approx 2.91\%$, which is not a member of the s_4 state, thus the chain will remain in the s_3 state. So, the transition from the s_3 state to all of the other states is not the same (in terms of percentage change) as the transition from s_2 to all of the other states. And, a similar argument can be made if we considered that we started from any state different from s_2.

With this in mind, if we want to use the relative frequencies of the dataset to estimate the transitions from any state to any other, then we need to "re-calculate" the classes in order for the estimation to be coherent with what we assumed and defined. So, to accomplish this, we need to consider the percentage change of price regarding the previous iteration of the chain, and not the percentage change regarding the initial price.

Again, for example, to estimate the transition from the s_3 state to the s_4 state, we need to assume that the initial state is the s_3 state and that we want to transition to the s_4 state, i.e., we need to assume the percentage price change (relative to the s_2) is at 103% and that we want to know what is the percentage price change if the percentage price change transitioned to 106% (relative to the s_2), which would be $((106 - 103)/103) * 100 \approx 2.91\%$. Also, because we are dealing with classes, this obtained percentage change between classes will be used as the "new" α to determine the limits (and ranges) of the classes, this is because s_3 and s_4 are consecutive classes in terms of their range of values (as s_2 and s_3 were in the base classes). Thus, in this case, the s_4 state (or class) becomes $s_4 = \{X_t : 1.46 < X_t \leq 4.37\}$. So, we need to use these "re-calculated" classes to obtain the relative frequencies table, which will be the estimation for the transition probabilities if we consider that we started from state s_3.

To generally define the classes which we will use in the relative frequencies table, we need to consider the direct correspondence $f : \{s_m, s_1, s_2, s_3, s_4\} \to \{2, 1, 0, -1, -2\}$ defined as:

$$f(s_m) = 2$$

$$f(s_1) = 1$$

$$f(s_2) = 0$$

$$f(s_3) = -1$$

$$f(s_4) = -2$$

So, the "altered" classes (or states) obtained considering that we started from state $s \in \{s_m, s_1, s_2, s_3, s_4\}$ are:

- $s_M = \{X_t : X_t \leq (-2.5 + f(s)) \cdot \alpha\};$ (2.32)
- $s_m = \{X_t : (-2.5 + f(s)) \cdot \alpha < X_t \leq (-1.5 + f(s)) \cdot \alpha\};$ (2.33)
- $s_1 = \{X_t : (-1.5 + f(s)) \cdot \alpha < X_t \leq (-0.5 + f(s)) \cdot \alpha\};$ (2.34)
- $s_2 = \{X_t : (-0.5 + f(s)) \cdot \alpha < X_t \leq (0.5 + f(s)) \cdot \alpha\};$ (2.35)
- $s_3 = \{X_t : (0.5 + f(s)) \cdot \alpha < X_t \leq (1.5 + f(s)) \cdot \alpha\};$ (2.36)
- $s_4 = \{X_t : (1.5 + f(s)) \cdot \alpha < X_t \leq (2.5 + f(s)) \cdot \alpha\};$ (2.37)
- $s_Z = \{X_t : X_t > (2.5 + f(s)) \cdot \alpha\}.$ (2.38)

Note that, the value of α used in the equations of the new classes, also needs to be re-calculated, which we will see how to do so after determining the "re-calculated" classes for the Example 2.26 considering that we started from the s_3 state and with $\alpha = 2.91$, which are:

- $s_M = \{X_t : X_t \leq -10.185\}$;
- $s_m = \{X_t : -10.185 < X_t \leq -7.275\}$;
- $s_1 = \{X_t : -7.275 < X_t \leq -4.365\}$;
- $s_2 = \{X_t : -4.365 < X_t \leq -1.455\}$;
- $s_3 = \{X_t : -1.455 < X_t \leq 1.455\}$;
- $s_4 = \{X_t : 1.455 < X_t \leq 4.365\}$;
- $s_Z = \{X_t : X_t > 4.365\}$.

Utilizing the dataset from Example 2.26, we have the following relative frequencies table for these classes:

s_M	s_m	s_1	s_2	s_3	s_4	s_Z
$\frac{0}{9}$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{2}{9}$	$\frac{4}{9}$	$\frac{1}{9}$	$\frac{0}{9}$

Table 2.5: Relative frequencies table considering that the starting state is s_3.

Replacing in the transition matrix 2.31 (related to Example 2.26's Markov chain), we obtain:

$$P_M = \begin{array}{c} \\ s_M \\ s_m \\ s_1 \\ s_2 \\ s_3 \\ s_4 \\ s_Z \end{array} \begin{array}{c} \begin{array}{ccccccc} s_M & s_m & s_1 & s_2 & s_3 & s_4 & s_Z \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ p_{21} & p_{22} & p_{23} & p_{24} & p_{25} & p_{26} & p_{27} \\ p_{31} & p_{32} & p_{33} & p_{34} & p_{35} & p_{36} & p_{37} \\ 1/9 & 1/9 & 2/9 & 4/9 & 1/9 & 0 & 0 \\ 0 & 1/9 & 1/9 & 2/9 & 4/9 & 1/9 & 0 \\ p_{61} & p_{62} & p_{63} & p_{64} & p_{65} & p_{66} & p_{67} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array} . \tag{2.39}$$

Now, for the general case, consider that we want to determine the relative frequencies assuming that we are departing from the $s_i \in \{s_m, s_1, \ldots, s_4\}$ state, then we need to determine the range of each class, i.e., we need to determine the α that we will use in the previously presented formulas 2.32-2.38. For this, we need to consider s_i's consecutive class, which is the class that contains the values immediately before the lower limit of s_i or after the upper limit of s_i, and we will denote it as s_j. Also, it is not relevant which of the two that we choose. For instance, if $s_i = s_m$, then its consecutive classes are s_M and s_1, so s_j can be either s_M or s_m; likewise, the (only) consecutive class of $s_i = s_M$ is $s_j = s_m$. Thus, after obtaining the consecutive class, consider m_i and m_j to be the midpoints of s_i and s_j, respectively. But, if s_j is s_M or s_Z, m_j will be the $m_i + (inf(s_i) - sup(s_i))$ or $m_i + (sup(s_i) - inf(s_i))$, respectively.

So, the α value is obtained by:

$$\alpha = \left| \frac{m_j - m_i}{m_i + 100} 100 \right| . \tag{2.40}$$

Remark 13. *Note that we did not include s_M and s_Z into the set of possible states that s_i can be, this is because the probabilities of departing from these states are fixed, as we saw when we built the transition matrix 2.30. Also, in the calculation of the α, we need to consider the absolute value, in case s_j is related to lower limit of s_i.*

Afterwards, we simply have to determine the classes by replacing the obtained α in all of the equations 2.32-2.38, which we will use to calculate the relative frequencies table of the (same transformed) dataset. Finally, we just replace the obtained relative frequencies in the row of the transition matrix related to the s_i state (or class).

Applying all of this to example 2.26, we obtain the following transition matrix:

$$P_M = \begin{array}{c} \\ s_M \\ s_m \\ s_1 \\ s_2 \\ s_3 \\ s_4 \\ s_Z \end{array} \begin{array}{c} \begin{array}{ccccccc} s_M & s_m & s_1 & s_2 & s_3 & s_4 & s_Z \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4/9 & 4/9 & 1/9 & 0 & 0 & 0 & 0 \\ 2/9 & 2/9 & 4/9 & 1/9 & 0 & 0 & 0 \\ 1/9 & 1/9 & 2/9 & 4/9 & 1/9 & 0 & 0 \\ 0 & 1/9 & 1/9 & 2/9 & 4/9 & 1/9 & 0 \\ 0 & 0 & 1/9 & 1/9 & 2/9 & 4/9 & 1/9 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array} . \tag{2.41}$$

All the presented estimators and examples are related to the Markov Chain where we assume that the speculator chose to play the More Risk strategy. So, to estimate the transition probabilities for the Markov Chain where we assume that the speculator chose to play the Less Risk strategy, which is represented by the following transition matrix (similar to the previously presented matrix 2.29):

$$P_L = \begin{array}{c} s_m \\ s_1 \\ \vdots \\ s_4 \\ s_Z \end{array} \overset{\begin{array}{ccccc} s_m & s_1 & \cdots & s_4 & s_Z \end{array}}{\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ p_{21} & p_{22} & p_{23} & \cdots & p_{25} & p_{26} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ p_{51} & p_{52} & p_{53} & \cdots & p_{55} & p_{56} \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}}. \tag{2.42}$$

And, like in the P_M case, we will determine the relative frequency tables considering that the chain started from each of the sates $s_1, \ldots, s_4$. So, again assume that we are departing from the $s_i \in \{s_1, \ldots, s_4\}$ state, then we need to determine the range of each class, i.e., we need to determine the α that we will use in formulas similar to the previously presented ones (2.32-2.38). So, as before, consider a consecutive class to s_i, denoted as s_j. For instance, if $s_i = s_1$, then its consecutive classes are s_m and s_2, so s_j can be either s_m or s_2; likewise, the (only) consecutive class of $s_i = s_m$ is $s_j = s_1$. After obtaining the consecutive class, consider m_i and m_j to be the midpoints of s_i and s_j, respectively. But, if s_j is s_m or s_Z, m_j will be the $m_i + (inf(s_i) - sup(s_i))$ or $m_i + (sup(s_i) - inf(s_i))$, respectively.
So, the α value is obtained by:

$$\alpha = \left| \frac{m_j - m_i}{m_i + 100} 100 \right|. \tag{2.43}$$

Remark 14. *Note that the equation to obtain α in the P_L case is the same as the previous equation 2.40. Also, we did not include s_m and s_Z into the set of possible states that s_i can be, this is because the probabilities of departing from these states are fixed, as we observed when we built the transition matrix 2.29.*

As in the P_M case, the transition probabilities are not the same as if we considered that the chain started from different states. Thus, in order to take this into account and to still use the relative frequency "method" to estimate the transition probabilities we need to slightly alter the classes on which we will determine the relative frequencies. So, again consider the direct correspondence $f : \{s_1, \ldots, s_4\} \to \{2, 1, 0, -1, -2\}$ defined as:

$$f(s_1) = 1$$

$$f(s_2) = 0$$

$$f(s_3) = -1$$

$$f(s_4) = -2$$

So, the "altered" classes (or states) for the P_L matrix considering that we started from a state $s \in \{s_1, \ldots, s_4\}$ are:

- $s_m = \{X_t : X_t \leq (-1.5 + f(s)) \cdot \alpha\};$ (2.44)
- $s_1 = \{X_t : (-1.5 + f(s)) \cdot \alpha < X_t \leq (-0.5 + f(s)) \cdot \alpha\};$ (2.45)
- $s_2 = \{X_t : (-0.5 + f(s)) \cdot \alpha < X_t \leq (0.5 + f(s)) \cdot \alpha\};$ (2.46)
- $s_3 = \{X_t : (0.5 + f(s)) \cdot \alpha < X_t \leq (1.5 + f(s)) \cdot \alpha\};$ (2.47)
- $s_4 = \{X_t : (1.5 + f(s)) \cdot \alpha < X_t \leq (2.5 + f(s)) \cdot \alpha\}.$ (2.48)
- $s_Z = \{X_t : X_t > (2.5 + f(s)) \cdot \alpha\}.$ (2.49)

Now, we will estimate the P_L matrix for the same dataset

$$\{45.00, 44.49, 43.44, 40.17, 41.05, 41.53, 41.36, 40.68, 40.46, 38.42\}$$

from Example 2.26, which resulted into the transformed dataset

$$\{-1.13, -2.36, -7.53, 2.19, 1.17, -0.41, -1.64, -0.54, -5.04\}.$$

Considering that we started from the s_3 state (with the consecutive state s_4), i.e., considering that:

$$\alpha = \left| \frac{m_j - m_i}{m_i + 100} 100 \right| = \left| \frac{6 - 3}{3 + 100} 100 \right| \approx 2.91.$$

We obtain the classes

- $s_m = \{X_t : X_t \leq -7.275\};$
- $s_1 = \{X_t : -7.275 < X_t \leq -4.365\};$
- $s_2 = \{X_t : -4.365 < X_t \leq -1.455\};$
- $s_3 = \{X_t : -1.455 < X_t \leq 1.455\};$
- $s_4 = \{X_t : 1.455 < X_t \leq 4.365\};$
- $s_Z = \{X_t : X_t > 4.365\}.$

And, by replacing the relative frequencies, P_L is:

$$P_L = \begin{array}{c} \\ s_m \\ s_1 \\ s_2 \\ s_3 \\ s_4 \\ s_Z \end{array} \begin{array}{c} \begin{array}{cccccc} s_m & s_1 & s_2 & s_3 & s_4 & s_Z \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 4/9 & 4/9 & 1/9 & 0 & 0 & 0 \\ 2/9 & 2/9 & 4/9 & 1/9 & 0 & 0 \\ 1/9 & 1/9 & 2/9 & 4/9 & 1/9 & 0 \\ 0 & 1/9 & 1/9 & 2/9 & 4/9 & 1/9 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}. \tag{2.50}$$

2.2.3 Estimating the Market's Probabilities

Now, we have everything that we need to estimate the probabilities of the market playing Zero Adversity (p_0), Minor Adversity (p_1) and Major Adversity (p_2). And, to accomplish this, we will use two Markov chains to estimate p_2 and p_0, as it was previously explained.

To estimate p_2 we will make the use of the Markov Chain where we assumed that the speculator chose to play the More Risk strategy, which is represented by the transition matrix 2.30:

$$P_M = \begin{array}{c} \\ s_M \\ s_m \\ s_1 \\ \vdots \\ s_4 \\ s_Z \end{array} \begin{array}{c} \begin{array}{ccccc} s_M & s_m & s_1 & \cdots & s_Z \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ p_{21} & p_{22} & p_{23} & \cdots & p_{27} \\ p_{31} & p_{32} & p_{33} & \cdots & p_{37} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ p_{61} & p_{62} & p_{63} & \cdots & p_{67} \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \end{array}. \tag{2.51}$$

Also, as before, we will assume that the initial state of the chain is s_2, i.e., the probability distribution of X_0 (the first percentage price change of the chain) is given by the $\pi_0^M = (0, 0, 0, 1, 0, 0, 0)^T$.

Since we want to predict what will happen to an asset's price after we buy it, that is, we want to know if we will have a profit or a loss (according to the financial game that we established) after we enter a trade, then it is sensible to consider what will happen immediately after we buy the asset and/or what is the asset's price tending to. So, to this end, we will consider two separate estimators and analyze the obtained results. Thus, p_2 will be estimated by:

- the probability of the chain reaching the s_M state after one iteration;
- the long-run probability of the chain being at state s_M.

Regarding the first estimator, we will just compute the probability of the chain being at state s_M after one iteration of the chain, so we will compute:

$$\pi_1^M = \pi_0^M P_M = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ p_{21} & p_{22} & p_{23} & p_{24} & p_{25} & p_{26} & p_{27} \\ p_{31} & p_{32} & p_{33} & p_{34} & p_{35} & p_{36} & p_{37} \\ p_{41} & p_{42} & p_{43} & p_{44} & p_{45} & p_{46} & p_{47} \\ p_{51} & p_{52} & p_{53} & p_{54} & p_{55} & p_{56} & p_{57} \\ p_{61} & p_{62} & p_{63} & p_{64} & p_{65} & p_{66} & p_{67} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} \overset{s_M}{p_{41}} & \overset{s_m}{p_{42}} & \overset{s_1}{p_{43}} & \overset{s_2}{p_{44}} & \overset{s_3}{p_{45}} & \overset{s_4}{p_{46}} & \overset{s_Z}{p_{47}} \end{pmatrix}.$$

Where, after the matrix multiplication, we obtained a 1×7 vector π_1^M, which is the probability distribution of the chain after one iteration. Here, note that π_1^M is simply the transition probabilities starting from the s_2 state, which makes sense considering that the initial state is s_2 and we only want to know the probability distribution after one iteration of the chain. Thus, the first entry of π_1^M is the probability of the chain being in state s_M after one iteration and our estimator for p_2 is: $p_2 \approx p_{41}$.

As we can see, this estimator is fairly simple, both in theoretical and in practical terms. So, to try to understand how the percentage price will evolve, we will also consider a estimator related to the long term distribution of the chain. However, we need to note that this probability distribution may not exist, because our chain is not irreducible. So, we cannot use Theorem B.2.2 (from Section B.2 of Appendix B) to guarantee that such distribution exists. Also, if such distribution is to exist, we know (from Section B.3 of Appendix B) that the chain will tend to its absorbing states, thus, in our case, the long run probability distribution would be a 1×7 vector π where one of the absorbing states (s_M or s_Z) has a probability of one. But, we do not know when this will happen or which state will have probability one. Hence, to overcome these issues, we will compute the probability distribution of the chain after $n < \infty$ iterations:

$$\pi_n^M = \pi_0^M P_M^n = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ p_{21} & p_{22} & p_{23} & p_{24} & p_{25} & p_{26} & p_{27} \\ p_{31} & p_{32} & p_{33} & p_{34} & p_{35} & p_{36} & p_{37} \\ p_{41} & p_{42} & p_{43} & p_{44} & p_{45} & p_{46} & p_{47} \\ p_{51} & p_{52} & p_{53} & p_{54} & p_{55} & p_{56} & p_{57} \\ p_{61} & p_{62} & p_{63} & p_{64} & p_{65} & p_{66} & p_{67} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}^n = \tag{2.52}$$

$$= \begin{pmatrix} \overset{s_M}{\pi_{n1}} & \overset{s_m}{\pi_{n2}} & \overset{s_1}{\pi_{n3}} & \overset{s_2}{\pi_{n4}} & \overset{s_3}{\pi_{n5}} & \overset{s_4}{\pi_{n6}} & \overset{s_Z}{\pi_{n7}} \end{pmatrix}. \tag{2.53}$$

Thus, after the matrix multiplication, we obtain a 1×7 vector π_n^M, and its first entry is the probability of the chain being in state s_M after n iterations, so our estimator for p_2 is: $p_2 \approx \pi_{n1}$.

Observe that we cannot apply the Theorem B.2.2 (from Section B.2 of the Appendix B) to determine the long-run probability distribution π of the chain, because it is not irreducible. So, we do need to compute the n matrix multiplications.

Now, similarly to the estimator of p_2, we will estimate p_0 using the Markov Chain where we assumed that the speculator chose to play the Less Risk strategy, which is represented by the transition matrix 2.42:

$$P_L = \begin{array}{c} s_m \\ s_1 \\ \vdots \\ s_4 \\ s_Z \end{array} \overset{\begin{array}{ccccc} s_m & s_1 & \cdots & s_4 & s_Z \end{array}}{\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ p_{21} & p_{22} & p_{23} & \cdots & p_{27} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ p_{51} & p_{52} & p_{53} & \cdots & p_{57} \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}}. \tag{2.54}$$

As before, we will assume that the initial state of the chain is s_2, i.e., we will assume that $\pi_0^L = (0,0,1,0,0,0)^T$. So, for the same reasons as before, p_0 will be estimated by:

- the probability of the chain reaching the s_Z state after one iteration;
- the long-run probability of the chain being at state s_Z.

Regarding the first estimator, we will just compute the probability of the chain being at state s_Z after one iteration of the chain, i.e. we will compute:

$$\pi_1^L = \pi_0^L P_L = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ p_{21} & p_{22} & p_{23} & p_{24} & p_{25} & p_{26} \\ p_{31} & p_{32} & p_{33} & p_{34} & p_{35} & p_{36} \\ p_{41} & p_{42} & p_{43} & p_{44} & p_{45} & p_{46} \\ p_{51} & p_{52} & p_{53} & p_{54} & p_{55} & p_{56} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} \overset{s_m}{p_{41}} & \overset{s_1}{p_{42}} & \overset{s_2}{p_{43}} & \overset{s_3}{p_{44}} & \overset{s_4}{p_{45}} & \overset{s_Z}{p_{46}} \end{pmatrix}.$$

Where, after the matrix multiplication, we obtain a 1×6 vector π_1^L. And, again, note that π_1 is simply the transition probabilities starting from the s_2 state. Also, the last entry of π_1^L is the probability of the chain being in state s_Z after one iteration, so our estimator for p_0 is: $p_0 \approx p_{46}$.

As before, this estimator is fairly simple, and because this chain is also not irreducible, we will compute the probability distribution (π_n^L) of the chain again after $n < \infty$ iterations:

$$\pi_n^L = \pi_0^L P_L^n = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ p_{21} & p_{22} & p_{23} & p_{24} & p_{25} & p_{26} \\ p_{31} & p_{32} & p_{33} & p_{34} & p_{35} & p_{36} \\ p_{41} & p_{42} & p_{43} & p_{44} & p_{45} & p_{46} \\ p_{51} & p_{52} & p_{53} & p_{54} & p_{55} & p_{56} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}^n =$$

$$= \begin{pmatrix} \overset{s_m}{p_{n1}} & \overset{s_1}{p_{n2}} & \overset{s_2}{p_{n3}} & \overset{s_3}{p_{n4}} & \overset{s_4}{p_{n5}} & \overset{s_Z}{p_{n6}} \end{pmatrix},$$

which, after the matrix multiplication, yields a 1×6 vector π_n. And, its last entry is the probability of the chain being in state s_Z after n iterations, so our estimator for p_0 is: $p_0 \approx p_{n6}$. Observe that we had the same issues in both estimators, because the chains were not irreducible, also we used the same number of iterations n (to determine the long-run estimator) in both chains, so that we can compare the obtained results from the different chains. Finally, we need to note that these estimators (for p_0 and p_2) sum up to a value ≤ 1, because, by Section 2.2, the theoretical probabilities that we are estimating have this property, and by the fact that we are under-estimating the market's probabilities, since theoretically we should determine the long-run estimator by using infinite iterations (and not only n).

Remark 15. *Even though the used notations for both estimators are similar, the obtained estimated probabilities result from (n iterations of) different chains, so they represent different probabilities. Additionally, the estimator for p_1 is simply $p_1 = 1 - p_0 - p_2$, for both cases.*

So, with all of this, we can estimate the probabilities of the market playing a certain strategy and thus choose the speculator's optimal strategy according to the previously presented financial game.

To finalize this section we will just pick up the dataset from the Example 2.26 (from the previous section) and compute the estimators for the market's probabilities. For this, recall that the obtained transition matrix related to the chain where we assumed the More Risk strategy is:

$$P_M = \begin{array}{c} \\ s_M \\ s_m \\ s_1 \\ s_2 \\ s_3 \\ s_4 \\ s_Z \end{array} \begin{array}{c} \begin{array}{ccccccc} s_M & s_m & s_1 & s_2 & s_3 & s_4 & s_Z \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4/9 & 4/9 & 1/9 & 0 & 0 & 0 & 0 \\ 2/9 & 2/9 & 4/9 & 1/9 & 0 & 0 & 0 \\ 1/9 & 1/9 & 2/9 & 4/9 & 1/9 & 0 & 0 \\ 0 & 1/9 & 1/9 & 2/9 & 4/9 & 1/9 & 0 \\ 0 & 0 & 1/9 & 1/9 & 2/9 & 4/9 & 1/9 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}. \tag{2.55}$$

The obtained transition matrix related to the chain where we assumed the Less Risk strategy is:

$$P_L = \begin{array}{c|cccccc} & s_m & s_1 & s_2 & s_3 & s_4 & s_Z \\ \hline s_m & 1 & 0 & 0 & 0 & 0 & 0 \\ s_1 & 4/9 & 4/9 & 1/9 & 0 & 0 & 0 \\ s_2 & 2/9 & 2/9 & 4/9 & 1/9 & 0 & 0 \\ s_3 & 1/9 & 1/9 & 2/9 & 4/9 & 1/9 & 0 \\ s_4 & 0 & 1/9 & 1/9 & 2/9 & 4/9 & 1/9 \\ s_Z & 0 & 0 & 0 & 0 & 0 & 1 \end{array}. \tag{2.56}$$

So, assuming $\pi_0^M = (0,0,0,1,0,0,0)^T$, we have that:

$$\pi_1^M = \pi_0^M P_M = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4/9 & 4/9 & 1/9 & 0 & 0 & 0 & 0 \\ 2/9 & 2/9 & 4/9 & 1/9 & 0 & 0 & 0 \\ 1/9 & 1/9 & 2/9 & 4/9 & 1/9 & 0 & 0 \\ 0 & 1/9 & 1/9 & 2/9 & 4/9 & 1/9 & 0 \\ 0 & 0 & 1/9 & 1/9 & 2/9 & 4/9 & 1/9 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} \overset{s_M}{1/9} & \overset{s_m}{1/9} & \overset{s_1}{2/9} & \overset{s_2}{4/9} & \overset{s_3}{1/9} & \overset{s_4}{0} & \overset{s_Z}{0} \end{pmatrix}.$$

And with $\pi_0^M = (0,0,0,1,0,0,0)^T$ we have:

$$\pi_1^L = \pi_0^L P_L = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 4/9 & 4/9 & 1/9 & 0 & 0 & 0 \\ 2/9 & 2/9 & 4/9 & 1/9 & 0 & 0 \\ 1/9 & 1/9 & 2/9 & 4/9 & 1/9 & 0 \\ 0 & 1/9 & 1/9 & 2/9 & 4/9 & 1/9 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} \overset{s_m}{2/9} & \overset{s_1}{2/9} & \overset{s_2}{4/9} & \overset{s_3}{1/9} & \overset{s_4}{0} & \overset{s_Z}{0} \end{pmatrix}.$$

So the one iteration estimators for the market's probabilities, for this example, are:

$$p_0 = 0$$

$$p_1 = 8/9 \approx 0.89$$

$$p_2 = 1/9 \approx 0.11$$

Regarding the long run estimator with $n = 10$ iterations, we have that:

$$\pi_n^M = \pi_0^M P_M^n = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4/9 & 4/9 & 1/9 & 0 & 0 & 0 & 0 \\ 2/9 & 2/9 & 4/9 & 1/9 & 0 & 0 & 0 \\ 1/9 & 1/9 & 2/9 & 4/9 & 1/9 & 0 & 0 \\ 0 & 1/9 & 1/9 & 2/9 & 4/9 & 1/9 & 0 \\ 0 & 0 & 1/9 & 1/9 & 2/9 & 4/9 & 1/9 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}^n =$$

$$= \begin{pmatrix} \overset{s_M}{0.9} & \overset{s_m}{0.03} & \overset{s_1}{0.03} & \overset{s_2}{0.02} & \overset{s_3}{0.01} & \overset{s_4}{0.01} & \overset{s_Z}{} \end{pmatrix}$$

$$\pi_n^L = \pi_0^L P_L^n = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 4/9 & 4/9 & 1/9 & 0 & 0 & 0 \\ 2/9 & 2/9 & 4/9 & 1/9 & 0 & 0 \\ 1/9 & 1/9 & 2/9 & 4/9 & 1/9 & 0 \\ 0 & 1/9 & 1/9 & 2/9 & 4/9 & 1/9 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}^n =$$

$$= \begin{pmatrix} \overset{s_m}{0.95} & \overset{s_1}{0.02} & \overset{s_2}{0.01} & \overset{s_3}{0.01} & \overset{s_4}{0.01} & \overset{s_Z}{} \end{pmatrix}$$

Remark 16. *Note that the presented values are rounded with two decimal cases.*

So the long-run estimators for the market's probabilities, for this example, are:

$$p_0 = 0.01$$

$$p_1 = 0.09$$

$$p_2 = 0.9$$

Note that the estimated probabilities, in this example, change drastically from one estimator to the other, also they suggest that the market (as the iterations of the chain increase) is increasing its probability of choosing the Major Adversity strategy.

To finalize, the previously presented game table 2.2 related to the financial game, for this example, becomes:

			Speculator q	Speculator $1-q$
			R^+	R^-
	p_0	0_A	9	9
Market	p_1	m_A	9	-6
	p_2	M_A	-9	-6

Table 2.6: Example 2.26's game table for the financial market game.

These payoffs (or strategy thresholds) were obtained by applying the theory on Section 2.1 and considering the standard deviation of the transformed dataset (i.e., considering $\alpha = 3$), where we obtained $w = 9$, $x = 6$ and $y = 9$.

Now, with the one iteration estimators and considering the probability triangle (also presented in Section 2.1, but considering these new values), for this case, the speculator should choose to play the More Risk strategy, because:

$$p_2 = 0.11 < \frac{w}{w+y} = 0.5 \text{ and } \frac{p_1}{p_2} \approx 8.09 > \frac{y-x}{w+x} = 0.2. \tag{2.57}$$

And, considering the long run estimator and the same probability triangle, the speculator should choose not to play, because:

$$p_1 + p_2 = 0.99 >= \frac{w}{w+y} = 0.5 \text{ and } p_2 = 0.9 > \frac{w}{w+y} = 0.5. \tag{2.58}$$

Thus, as we can see the two estimators yield different strategies for the speculator to choose. All of this because the market "changes" its behavior as the iterations increase.

2.3 The SARIMA and GARCH Models

Now that we have discussed the specific game theoretical and Markov chains models that we will use, it is time to describe how we will use the SARIMA and GARCH models to predict the market's behavior, and then compare the accuracy of the three approaches. However, we cannot simply apply the time series models to the raw dataset and make a prediction for the future value of the time series, because, in order to make the comparison of the models possible, we need to apply all the models to the same dataset and try to predict the same objects, which in our case means predicting the strategies that the market will choose. Thus, we will apply the time series models to the same percentage change transformed datasets that we have been using on the previous sections. So, if we make predictions based on these models, we obtain percentage change transformed predictions of the asset's price (which is useful, but it is not our ultimate goal).

To obtain a prediction of the market's strategy, firstly we will estimate the optimal time series models for the dataset (for further details see Appendix A). Then, using these estimated models, we will perform K simulations each with N observations, thus obtaining K simulations of percentage change prices for each of the models and each one starting on the last observation of the transformed dataset.

Remark 17. *All of this will be done with the aid of the R software, which we will elaborate further on Section 2.4.*

Now, as in the previous sections, consider the speculator's More Risk strategy thresholds in terms of the dataset's standard deviation α:

- the Profit Objective threshold: $s_P \geq 3 \cdot \alpha$;
- the More Risk threshold: $s_R \leq -3 \cdot \alpha$.

Finally, for each of the K simulations, we need to check which of the thresholds was reached first, because the speculator will exit the trade (or the game) when one of these is reached. And with this we obtain the absolute frequencies of each of the thresholds, and also its relative frequencies if we divide by K.

Remark 18. *Note that, as we are performing simulations involving a model which includes a probability distribution, if we ran the same code several times, we would obtain different results*

after each run. However, these results will not have major differences between them.
Hence, we will estimate the probability of the market playing the Major Adversity strategy (p_2) with the relative frequency related to the More Risk threshold, and similarly the probability of the market playing the Zero Adversity strategy (p_0) with the relative frequency related to the Profit Objective threshold. Also, by default, the estimation for probability of the market playing the Minor Adversity strategy (p_1) is simply $1 - p_2 - p_0$. Additionally, since we have the market's probabilities, then we can choose the speculator's optimal strategy according to the probability triangle presented in Section 2.1.2. But, before moving on, note that we need to determine these probabilities for both the SARIMA and the GARCH models, so we need to make an estimation for each of these models (but always using the same dataset), i.e., we need to perform K simulations for each model estimation and then determine the probabilities for each set of estimations. So, we will obtain two optimal strategies, one for each of the models.
Thus, for all the models presented so far (specifically, Markov Chain, SARIMA and GARCH), the speculator will obtain a optimal strategy for each of them, which is done by estimating the market's probabilities (which may differ for each model) and then we will apply the same probability triangle for each set of probabilities.

2.4 Procedures

Now that we have all the necessary models and estimators, it is time to describe how we will use each model to choose the optimal strategy for a certain dataset. Also, we need to explain how we will check if the predictions were accurate and how accurate.
Consider an abstract dataset composed by strictly positive values, which will represent the price of a certain financial asset for $n+1$ consecutive iterations (it can be $n+1$ consecutive minutes, days,...). Since we worked with percentage change data in the game theoretical model, then we will apply the percentage change transformation to the dataset, obtaining a transformed dataset $C = \{c_1, \ldots, c_n\}$ composed with percentage changes of $n+1$ consecutive iterations. So, we will apply all of our models to this dataset C. Also, in order to check the accuracy of our models, we will split the dataset into training (C_1) and test (C_2) sets, where the training set

will be composed by the first 80% of the observations and the remaining will belong to the test set. Thus, considering the set $C = \{c_1, \ldots, c_n\}$, the training set will be $C_1 = \{c_1, \ldots, c_k\}$ and the test set $C_2 = \{c_{k+1}, \ldots, c_n\}$, for $k < n$.

The general procedure applied to a (transformed and divided) dataset C, consists on estimating the market's probabilities for each of the models (Markov chains, SARIMA and GARCH), thus obtaining three "pairs" of probabilities, then we will use them to determine the speculator's optimal strategy, also obtaining three optimal strategies. Afterwards, we will use the test set to check if the obtained strategies were accurate predictions for the current training set.

To accomplish this, consider the speculator's More Risk strategy thresholds (as we did in the previous section) in terms of the dataset's standard deviation α:

- the Profit Objective threshold: $s_P = 3\alpha$;
- the More Risk threshold: $s_R = -3\alpha$.

Then, using the test set, we will check which of the thresholds was reached first. So, this information together with the chosen optimal strategies, gives us the accuracy of the predictions, specifically:

- considering that optimal chosen strategy was the More Risk strategy, then we will consider that strategy to be accurate if the first threshold to be reached in the test set was the Profit Objective threshold, otherwise the strategy will be considered to be not accurate;
- considering that optimal chosen strategy was the Less Risk strategy, then we will consider that strategy to be accurate if the first threshold to be reached in the test set was the Profit Objective threshold, otherwise the strategy will be considered to be not accurate;
- considering that optimal chosen strategy was the Do Not Play strategy, then we will consider that strategy to be accurate if the first threshold to be reached in the test set was the More Risk threshold, otherwise the strategy will be considered to be not accurate.

Note that the Less Risk threshold was not necessary to determine the accuracy of the strategies. Also, due the nature of the data, none of the thresholds may be reached, so, in this case, we will not consider the strategy to be accurate nor inaccurate. Hence, in this situation, we can decrease the thresholds and recalculate the optimal strategies, or we can just consider that the accuracy cannot be determined due to the nature of the data.

Finally, in order to have more samples to analyze, we will increase the training set by one observation and decrease the test set by one observation, thus obtaining the sets $C_1 = \{c_1, \ldots, c_k, c_{k+1}\}$ and $C_2 = \{c_{k+2}, \ldots, c_n\}$. Then we will redo what we described before, but considering these new sets as training and test sets, respectively. Thus, we will obtain new accuracy data for the new optimal strategies.

To summarize, consider the transformed dataset $C = \{c_1, \ldots, c_n\}$ split between a training set $C_1 = \{c_1, \ldots, c_k\}$ and a test set $C_2 = \{c_{k+1}, \ldots, c_n\}$, then the procedure to be applied is:

(1) Considering the training set C_1:
- estimate the market's probabilities using the Markov chains model and determine the optimal strategy for speculator using the game theoretical model; using the code described in Sections D.2 and D.1 from Appendix D, respectively;
- estimate the optimal SARIMA model, estimate the market's probabilities using the model's simulations and determine the optimal strategy for the speculator using the game theoretical model; using the code described in Sections D.3 and D.1 from Appendix D, respectively;
- estimate the optimal GARCH model, estimate the market's probabilities using the model's simulations and determine the optimal strategy for the speculator using the game theoretical model; using the code described in Sections D.4 and D.1 from Appendix D, respectively.

(2) Considering the test set C_2:
- check the accuracy of the three obtained optimal strategies, using the previously described method; using the code described in Section D.5 from Appendix D;
- store the accuracy results for each of the models.

(3) Increase the training set C_1 by one observation and shorten the test set C_2 also by one observation, thus we will now consider the training set to be $C_1 = \{c_1, \ldots, c_k, c_{k+1}\}$ and the test set to be $C_2 = \{c_{k+2}, \ldots, c_n\}$.

(4) Perform all of the previous steps considering the "new" training and test sets, but end the procedure when the test set only has one observation remaining.

After applying this procedure, we need to analyze the obtained results, which we will do next.

Chapter 3

Results

Now, we can put what was presented into practice with some real-time data from the financial markets, compare the models' accuracy results and thus derive some conclusions from them. Firstly, we will make our analysis for some controlled datasets, with the objective to check how the models perform in "well-behaved" scenarios, and then move on to datasets with daily and intraday data. But, before moving further, let us recall that a model is said to be accurate if the speculator's obtained optimal strategies were the correct ones (when comparing to the test set) after the procedure described in Section 2.4 (from Chapter 2) ended. Likewise, the model is said to be inaccurate if the speculator's obtained optimal strategies were the incorrect ones (when comparing to the test set) after the same procedure ended. However, if a model's accuracy could not be determined (at a certain time) then it is said to have null accuracy. Additionally, we will also present (and analyze) the following characteristics obtained from applying the described procedures:

- The percentage of times that the several models obtained the same strategies. This was done in order to check how often different approaches would lead to the same optimal strategies. Also, we will present these results regarding pairs of models, for instance we will present the percentage of times that the Markov chains and the SARIMA models obtained the same strategies.
- The average time (in the same units as the corresponding dataset) that it took for the trade to close (in the test set), after a strategy was given. This average time was obtained by determining the number necessary iterations for the several test sets to reach one of the speculator's thresholds.

- The percentage of the speculator's obtained strategies that were "Play Less Risk", "Play More Risk" and/or "Do Not Play".
- We used the obtained strategies and entered a fictional market with an initial monetary value of 10000, where we only bought one item of each financial asset. This was done in order to see the profit that we would obtain if we entered a financial market and used the speculator's obtained strategies (for each of the models) to enter (and then exit) a trade.

To facilitate the presentation, we will round all of the results up to two decimal cases, but, if needed to, we will display some results in scientific notation. Thus, the values that will be presented are approximations of the actual results. Also, the tables resulting from the application of the previously discussed models and procedures are presented in Appendix E.

Remark 19. *In order to make the text lighter, we will refer to the Markov chains model considering the one iteration estimator as the MC1 model, and to the Markov chains model considering the long-run estimator as the MCn model.*

Finally, the following sections will be structured in the same manner, that is, a brief explanation of the dataset(s), followed by the presentation of each model's obtained accuracy results (and related conclusions), ending with the analysis of some characteristics resultant from the models' appliance.

3.1 Controlled Datasets

For this section, we will start by explaining how we constructed each dataset and make an overall analysis of the obtained results.

The first dataset ("Dataset 1" from Section E.1 of Appendix E) was constructed with the purpose to check how the models perform in a "mild" Major Adversity scenario, i.e., the price of the asset will not always be decreasing but its trend will. And to obtain such a dataset we followed the presented steps until we obtained 1000 observations:

(1) defined the first value of the dataset as 1000;

(2) the second value of the dataset is just an increase of 3% of the previous one;

(3) the third value of the dataset is a decrease of 9% of the previous one;

(4) the even observations are obtained with an increase of 3% of the previous value;

(5) the odd observations are obtained with a decrease of 9% of the previous value.

We constructed the dataset in this manner in order to mimic an event of Major Adversity, so for the models to "perform well" in this dataset, the speculator's obtained optimal strategy must always be "Do Not Play", because the market, ultimately, is choosing to decrease the asset's price in the long-run.

Remark 20. *To check the results for this dataset, see tables' values (from Section E.1 of Appendix E) related to "Dataset 1".*

The second dataset ("Dataset 2" from Section E.1 of Appendix E) was constructed with the purpose to check how the models perform in an "extreme" Major Adversity scenario, i.e., the price of the asset will always be decreasing. To obtain such a dataset, we just defined it as 1000 observations starting from 1000, always decreasing by 3% of the previous value and adding a random value from a standard normal distribution. We constructed the dataset in this manner in order to mimic an extreme event of Major Adversity. So, for the models to "perform well" in this dataset, the speculator's obtained optimal strategy must always be "Do Not Play", because the market will always choose the Major Adversity strategy.

Remark 21. *To check the results for this dataset, see tables' values (from Section E.1 of Appendix E) related to "Dataset 2".*

The third dataset ("Dataset 3" from Section E.1 of Appendix E) was constructed with the purpose to check how the models perform in a "mild" Zero Adversity scenario, i.e., the price of the asset will not always be increasing but its trend will. And to obtain such a dataset we followed the presented steps until we obtained 1000 observations:

(1) defined the first value of the dataset as 1000;

(2) the second value of the dataset is just an decrease of 3% of the previous one;

(3) the third value of the dataset is a increase of 9% of the previous one;

(4) the even observations are obtained with a decrease of 3% of the previous value;

(5) the odd observations are obtained with an increase of 9% of the previous value.

We constructed the dataset in this manner in order to mimic an event of Zero Adversity, so for the models to "perform well" in this dataset, the speculator's obtained optimal strategy must either be "Play Less Risk" or "Play More Risk", because the market, ultimately, is choosing to increase the asset's price in the long-run.

Remark 22. *To check the results for this dataset, check tables' values (from Section E.1 of Appendix E) related to "Dataset 3".*

The last dataset ("Dataset 4" from Section E.1 of Appendix E) was constructed with the purpose to check how the models perform in an "extreme" Zero Adversity scenario, i.e., the price of the asset will always be increasing. To obtain such a dataset, we just defined it as 1000 observations starting from 1000, always increasing by 3% of the previous value and adding a random value from a standard normal distribution. We constructed the dataset in this manner in order to mimic an extreme event of Zero Adversity. So, for the models to "perform well" in this dataset, the speculator's obtained optimal strategy must always be "Play Less Risk" (or even "Play More Risk"), because the market will always choose the Zero Adversity strategy.

Remark 23. *To check the results for this dataset, see tables' values (from Section E.1 of Appendix E) related to "Dataset 4".*

Now that we have explained how each dataset was constructed, we will make a global analysis of the obtained results for the controlled datasets (considering that 20% of the data belongs to the test set), obtaining:

- The highest standard deviation of the transformed datasets was $\alpha \approx 3$ (obtained in Datasets 1 and 3) and the lowest was $\alpha \approx 0$ (in Dataset 2).
- The MC1 model was 75% more accurate then the other models, i.e., on 3 datasets this model had higher (or equal) accuracy results than all the other models. Also, the highest accuracy result was 100% (obtained in Datasets 2 and 4), while the lowest was 0% (on Dataset 1).
- The MCn, SARIMA and GARCH models were 100% more accurate then the other models. Additionally, the highest accuracy result was 100% (obtained in Datasets 2 and 4), while the lowest was 98.5% (on Dataset 3).

From these results we can see that the MCn, SARIMA and GARCH models are the models with the best accuracy results, which means that if the speculator used these models (for these datasets), he/she would obtain more strategies that would result in a profit (or at least a smaller loss). Consequently, the MC1 model obtained the worst accuracy results. Also we need to note that the lowest accuracy results were always obtained in the same datasets, while the highest was almost always in the same one.

Finally, 50% of the datasets presented no null accuracy results for all the models, meaning that the thresholds were always reached in all of the test sets. In Dataset 1, 0.5% of the models resulted in null accuracy results, which was the same for all the models. While this percentage was 1.5% for Dataset 3.

Moving to the obtained characteristics (which resulted from the explained procedures) for these controlled datasets were:

- The percentage of times that the several models obtained the same strategies:
  - The MCn model fully coincided in all the datasets with the time series models, i.e., in all the dataset these models always resulted in the same strategies
  - The MC1 model fully coincided in 50% of the datasets with all the other models, but on the other hand, it never coincided with any model in the other datasets.
- The average time (in iterations) for all of the datasets was the same across all of the models. Also, the highest average time was 3.475 iterations (in Dataset 3) and the lowest was 1 iteration (on Datasets 2 and 4).
- The percentage of the speculator's obtained strategies for each of the models was:
  - For the MC1 model:
    * in 50% of the datasets always chose the More Risk strategy;
    * in 25% of the datasets always chose the Less Risk strategy;
    * in 25% of the datasets always chose the Not Play strategy.
  - For the MCn, SARIMA and GARCH models:
    * in 50% of the datasets always chose the Less Risk strategy;
    * in 50% of the datasets always chose the Not Play strategy.

- The obtained possible profits using each model were:
  - For the MC1 model:
    * negative in 25% of the datasets, null in 25% and positive in the remaining ones;
    * the lowest profit (or highest loss) was in Dataset 1;
    * the highest profit was in Dataset 3.
  - For the MCn, SARIMA and GARCH models:
    * null in 50% of the datasets and positive in the remaining datasets;
    * the lowest profit (or highest loss) was in Datasets 1 and 2;
    * the highest profit was in Dataset 3.

As it was said before, the MCn, SARIMA and GARCH models fully coincided between them, in terms of accuracy results and chosen strategies. But, on the other hand, the MC1 model never coincided with all the other models in 50% of the datasets.

The MC1 model in 50% of the datasets always chose the More Risk strategy, and then switched between all the strategies in the remaining datasets. Meanwhile, the other models always chose the Less Risk strategy in 50% of the datasets, and then the Not play strategy on the other datasets.

In terms of possible profits, the MCn, SARIMA and GARCH models obtained positive profits in 50% of the datasets and null profits on the other ones, while the MC1 model obtained negative profits in 25% of the datasets, null profits in 25% and the remaining were positive.

From all of these results, we can see that the MCn, SARIMA and GARCH models obtained better accuracy and profits results, because they chose the expected optimal strategies for each of the datasets. While the MC1 model was the worst in all of the same aspects.

Finally, the average time it took the models to reach a threshold always coincided between models and its range was approximately from 1 to 3 iterations. Also, we need to note that both the highest and lowest profits were always obtained in the same datasets.

3.2 Daily Datasets

For this section, we will analyze datasets which are only composed with daily closing prices of several financial assets, this means that we are going to analyze the assets' prices at the end of each day (specifically, at the closing of the financial market).

We applied our models to 100 different datasets, but we will not analyze each of them, rather we will make a global analysis of the results. Also, note that whenever we refer to a specific dataset, we are actually referring to the price data that we obtained for a certain financial asset. Additionally, all the datasets have exactly 1000 observations of the closing price at the end of the day.

Thus, after applying our models to these datasets, considering that 20% of the data belongs to the test set, we obtained that:

- The highest standard deviation of the transformed datasets was $\alpha \approx 3.44$ (obtained in dataset TNXP) and the lowest was $\alpha \approx 0$ (in dataset PSON).
- The MC1 model was 41% more accurate then the other models, i.e., on 41 datasets this model had higher (or equal) accuracy results than all the other models. Also, the highest accuracy result was 66.5% (obtained in dataset AAPL), while the lowest was 33.5% (on dataset TNXP).
- The MCn model was 50% more accurate then the other models. And, the highest accuracy result was 66.5% (obtained in datasets AAPL and TNXP), while the lowest was 34.5% (on dataset NOS).
- The SARIMA model was 42% more accurate then the other models. And, the highest accuracy result was 66.5% (obtained in dataset AAPL), while the lowest was 35% (on dataset GFS).
- The GARCH model was 40% more accurate then the other models. And, the highest accuracy result was 66.5% (obtained in dataset AAPL), while the lowest was 35.5% (on dataset NOS).

Remark 24. *To check the results for these datasets, see tables' values from Section E.2 of Appendix E.*

From these results we can see that the MCn model is the one with the best accuracy results, which means that if the speculator used this model (for these datasets), he/she would obtain more strategies that would result in a profit (or at least a smaller loss). Additionally, the MC1, SARIMA and GARCH models obtained very similar accuracy results.

Regarding the time series models, the SARIMA model obtained slightly higher accuracy results than the GARCH model, which is the opposite of what was expected, since the GARCH models were specifically developed for this kind of data, as such it would be expected for them to perform better in terms of accuracy.

Finally, the obtained null accuracy results were the same across all the models and were almost always zero, also the highest percentage of null models was 6% (obtained in dataset TWTR). Additionally, we need to note that the highest accuracy results were always obtained in the same dataset.

The obtained characteristics (which resulted from the explained procedures) for these datasets were:

- The percentage of times that the several models obtained the same strategies:
  - The Markov chains models fully coincided between them in 81% of the datasets, i.e., in 81 datasets these two models always resulted in the same strategies. Also they never coincided in 5% of the datasets and, on the other datasets, the percentage of coinciding models ranged from 2% to 99%.
  - The MC1 model never coincided (in all the datasets) with the time series models.
  - The MCn model never coincided in 81% of the datasets with the SARIMA model and, on the other datasets, the percentage of coinciding models ranged from 1% to 98%.
  - The MCn model never coincided in 82% of the datasets with the GARCH model and, on the other datasets, the percentage of coinciding models ranged from 0.5% to 98.5%.
  - The SARIMA model fully coincided with the GARCH model in 6% of the datasets and, on the other datasets, the percentage of coinciding models ranged from 51.5% to 99.5%.

- The average time (in days) for all the datasets was the same across all the models. Also, the highest average time was 9.35 days (in dataset GFS) and the lowest was 1.015 days (on dataset PSON).
- The percentage of the speculator's obtained strategies for each of the models was:
  - The MC1 model always chose to play the More Risk strategy on all of the datasets.
  - For the MCn model:
    - in 81% of the datasets always chose the More Risk strategy;
    - in 1% of the datasets always chose the Less Risk strategy;
    - in 4% of the datasets always chose the Not Play strategy;
    - in 1% of the datasets chose between the play More Risk and Less Risk strategies;
    - in 13% of the datasets only chose between the More Risk and Not Play strategies.
  - For the SARIMA model:
    - in 5% of the datasets always chose the Less Risk strategy;
    - in none of the datasets always chose the More Risk and Not Play strategies;
    - in the remaining datasets only chose between the Less Risk and Not Play strategies. Also, the percentage of times the Less Risk strategy was chosen (instead of the Not Play strategy) is less than 50% in only 3% of all the datasets.
  - For the GARCH model:
    - in 26% of the datasets always chose the Less Risk strategy;
    - in none of the datasets always chose the More Risk and Not Play strategies;
    - in the remaining datasets only chose between the Less Risk and Not Play strategies. Also, the percentage of times the Less Risk strategy was chosen (instead of the Not Play strategy) is less than 50% in only 1% of all the datasets.
- The obtained possible profits using each model were:
  - For the MC1 model:
    - negative in 18% of the datasets and positive in the remaining datasets;
    - the lowest profit (or highest loss) was in dataset CCL;
    - the highest profit was in dataset AZN.
  - For the MCn model:

    * negative in 16% of the datasets, null in 4% of the datasets and positive in the remaining datasets;
    * the lowest profit (or highest loss) was in dataset CCL;
    * the highest profit was in dataset AZN.
  - For the SARIMA model:
    * negative in 25% of the datasets and positive in the remaining datasets;
    * the lowest profit (or highest loss) was in dataset CCL;
    * the highest profit was in dataset AVV.
  - For the GARCH model:
    * negative in 19% of the datasets and positive in the remaining datasets;
    * the lowest profit (or highest loss) was in dataset CCL;
    * the highest profit was in dataset AZN.

From the first item we can see that no model fully coincided in terms of chosen strategies with another one, but, on the other hand, the MC1 model never coincided with the time series models (SARIMA and GARCH), similarly the MCn model almost never coincided with the time series models. Also, the Markov chains models almost always coincided between them. Regarding the time series models, they almost always coincided between them, even though they only fully coincided in 6% of the datasets.

The MC1 model always chose the More Risk strategy across all of the datasets, while this only happened in 81% of the datasets for the MCn model. But, unlike the Markov chains models, the time series models never chose the More Risk strategy.

From all of these results, we can see that the MCn model performed better both in terms of accuracy results and of possible profits. Meanwhile, the MC1 model performed similarly to the time series models, both in terms of accuracy results and of possible profits. Also, regarding the time series models, the SARIMA model had slightly higher accuracy results than the GARCH model, however it had the highest percentage of unprofitable datasets.

Finally, the average time it took the models to reach a threshold always coincided between models and its range was from approximately a day to two weeks (each week in the financial markets is composed by five days). Also, we need to note that the lowest profit was always obtained in the same dataset, while the highest one was almost always obtained in the same one.

3.3 Intraday Datasets

For this section, we will analyze datasets which are only composed with 1000 observations of intraday closing prices of several financial assets, this means that we are going to analyze the assets' prices at the end of each minute for several days. Also, we applied our models to 100 different datasets, but we will not analyze each of them, rather we will make a global analysis of the results. Also, note that whenever we refer to a specific dataset, we are actually referring to the price data that we obtained for a certain financial asset.
Thus, after applying our models to these datasets, considering that 20% of the data belongs to the test set, we obtained that:

- The highest standard deviation of the transformed datasets was $\alpha \approx 2.13$ (obtained in dataset TNXP), also this value was the only value for the standard deviation greater than 1. Furthermore, all the other values for the standard deviation were smaller than 0.4, where the lowest was $\alpha \approx 0$ (in dataset Z).
- The MC1 was 27% more accurate then the other models, i.e., on 27 datasets this model had higher (or equal) accuracy results than all the other models. Also, the highest accuracy result was 68.5% (obtained in dataset NOS), while the lowest was 0.5% (on dataset TWTR).
- The MCn model was 43% more accurate then the other models. And, the highest accuracy result was 78.5% (obtained in dataset FCX), while the lowest was 0.5% (on dataset TWTR).
- The SARIMA model was 43% more accurate then the other models. And, the highest accuracy result was 74% (obtained in dataset BCP), while the lowest was 2.5% (on dataset TWTR).
- The GARCH model was 33% more accurate then the other models. And, the highest accuracy result was 67.5% (obtained in dataset O), while the lowest was 14% (on dataset TWTR).

Remark 25. *To check the results for these datasets, see tables' values from Section E.3 of Appendix E.*

From these results we can see that the MCn and SARIMA models were the ones with the best accuracy results, which means that if the speculator used one of these models (for these datasets), he/she would obtain more strategies that would result in a profit (or at least a smaller loss). Meanwhile, the MC1 model was the one with the lowest accuracy results.

Regarding the time series models, the SARIMA model obtained higher accuracy results than the GARCH model, which is the opposite of what was expected, since the GARCH models were specifically developed for this kind of data.

Finally, the obtained null accuracy results were the same across all the models and were almost always zero, where the highest percentage of null models was 12.5% (obtained in dataset TWTR). Also, we need to note that the smallest accuracy results were always obtained in the same dataset.

The obtained characteristics (which resulted from the explained procedures) for these controlled datasets were:

- The percentage of times that the several models obtained the same strategies:
  - The Markov chains models fully coincided between them in 61% of the datasets, i.e., in 61 datasets these two models always resulted in the same strategies. Also, they never coincided in 17% of the datasets and, on the other datasets, the percentage of coinciding models ranged from 1% to 98.5%.
  - The MC1 model never coincided in all the datasets with the time series models.
  - The MCn model fully coincided with the SARIMA model in 1% of the datasets, never coincided in 67% of the datasets and, on the other datasets, the percentage of coinciding models ranged from 1% to 87.5%.
  - The MCn model never coincided with the GARCH model in 76% of the datasets and, on the other datasets, the percentage of coinciding models ranged from 0.5% to 95.5%.
  - The SARIMA model fully coincided with the GARCH model in 4% of the datasets and, on the other datasets, the percentage of coinciding models ranged from 12.5%

to 99.5%.

- The average time (in minutes) for all the datasets was the same across all the models. And, the highest average time was 52.56 minutes (in dataset TWTR), while the lowest was 1.035 minutes (on dataset Z).
- The percentage of the speculator's obtained strategies for each of the models was:
  - The MC1 model always chose to play the More Risk strategy on all of the datasets.
  - For the MCn model:
    * in 61% of the datasets always chose the More Risk strategy;
    * in 1% of the datasets always chose the Less Risk strategy;
    * in 16% of the datasets always chose the Not Play strategy;
    * in 22% of the datasets only chose between the More Risk and Not Play strategies.
  - For the SARIMA model:
    * in 5% of the datasets always chose the Less Risk strategy;
    * in none of the datasets always chose the More Risk and Not Play strategies;
    * in the remaining datasets only chose between the Less Risk and Not Play strategies. Also, the percentage of times the Less Risk strategy was chosen (instead of the Not Play strategy) is less than 50% in only 7% of all the datasets.
  - For the GARCH model:
    * in 27% of the datasets always chose the Less Risk strategy;
    * in none of the datasets always chose the More Risk;
    * in the remaining datasets, only chose between the Less Risk and Not Play strategies. Also, the percentage of times the Less Risk strategy was chosen (instead of the Not Play strategy) is less than 50% in only 3% of all the datasets.
- The obtained profits using each model were:
  - For the MC1 model:
    * negative in 41% of the datasets and positive in the remaining datasets;
    * the lowest profit (or highest loss) was in dataset RB;
    * the highest profit was in dataset AZN.
  - For the MCn model:

    * negative in 37% of the datasets, null in 16% and positive in the remaining datasets;
    * the lowest profit (or highest loss) was in dataset RB;
    * the highest profit was in dataset AHT.
  - For the SARIMA model:
    * negative in 40% of the datasets and positive in the remaining datasets;
    * the lowest profit (or highest loss) was in dataset RB;
    * the highest profit was in dataset AZN.
  - For the GARCH model:
    * negative in 43% of the datasets and positive in the remaining datasets;
    * the lowest profit (or highest loss) was in dataset RB;
    * the highest profit was in dataset AZN.

From the first item we can see that no model fully coincided in terms of chosen strategies with another one, but the MC1 model never coincided with the time series models (SARIMA and GARCH), and the MCn model almost never coincided with the time series models. Also, the Markov chains models almost always coincided between them. Similarly, the time series models almost always coincided between them, even though they only fully coincided in 4% of the datasets.

The MC1 model always chose the More Risk strategy across all the datasets, while this only happened in 61% of the datasets for the MCn model. But, unlike the Markov chains models, the time series models never chose the More Risk strategy.

The average time it took the models to reach a threshold always coincided between models and its range was from approximately 1 to 53 minutes. Also, the lowest profit was always obtained in the same dataset, while the highest was almost always obtained in the same one.

Finally, from all of these results, we can see that the MCn and SARIMA models were the ones with the best accuracy results, however, the MCn model had the lowest percentage of unprofitable datasets and was a model which resulted in all kinds of strategies. Additionally, regarding the time series models, the SARIMA model had better accuracy and profit results than the GARCH model.

Chapter 4

Conclusions

Now that we have all the results for all the datasets, we can make a summary of what we obtained and then withdraw some conclusions from it.

4.1 Conclusions

Firstly, from all the obtained results, we can note that the lowest standard deviation of the transformed datasets was $\alpha \approx 0$, while the highest was $\alpha \approx 3$. Also, we need to note that, in the Intraday datasets, 99% of the transformed datasets had a standard deviation lower than 0.4.

Regarding the obtained accuracy results for the models, we can see that the MCn model obtained the best accuracy results for each type of datasets (Controlled, Daily and Intraday), but on the Controlled datasets it tied with the time series models, while this happened with the SARIMA model in the Intraday datasets. About the maximum and minimum accuracy results, we obtained that:

- for the Controlled datasets, the lowest accuracy result was almost always obtained in the same dataset, while the highest was always obtained in the same one;
- for the Daily datasets, only the highest accuracy result was always obtained in the same dataset;
- for the Intraday datasets, only the lowest accuracy result was always obtained in the same dataset.

Additionally, in all of the datasets both the null accuracy results and the average closing times were the same across the models. However, we need to note that, in terms of maximum results, the intraday datasets resulted in higher null results and higher average closing times.

For the percentage of equal strategies across the models we obtained that:

- For the controlled datasets, the MCn model always coincided with the time series models. Meanwhile, the MC1 model fully coincided with the other models in 50% of the datasets, whilst it never coincided in the other datasets.
- For both the daily and intraday datasets, the Markov chains models almost always coincided between them. But the MC1 model never coincided with the time series models, and, consequently, the MCn model almost never coincided with the time series models. Regarding the time series models, they almost always had a high percentage of coinciding strategies between them, but they almost never fully coincided between them.

Now, regarding the percentage of chosen strategies for each of the models we obtained that:

- For the controlled datasets, the MC1 model almost always chose the More Risk strategy (sometimes switching to the other strategies), while the other models chose between the Less Risk and Not Play strategies.
- For both the daily and intraday datasets: the MC1 model always chose the More Risk strategy; the MCn model almost always chose the More Risk strategy, but it also chose between the other strategies; the time series models chose between the Less Risk and Not Play strategies.

Finally, regarding the obtained possible profits resulting from applying the different models we can see that the MCn model obtained the least percentage of unprofitable datasets for each type of datasets, but on the Controlled datasets it tied with the time series models, on the other hand, the MC1 model obtained the highest percentage of strictly positive profits in the Daily datasets, while the same happened for the Intraday datasets with the SARIMA model.

About the maximum and minimum obtained possible profits, we have that:

- for the Controlled datasets, the lowest and highest possible profits were always obtained in the same datasets;
- for both the Daily and Intraday datasets, the lowest possible profit was always obtained in the same dataset (CCL and RB, respectively) and the highest was almost always obtained in the same one (AZN for both types).

Thus, gathering all these results, we can see that the Markov chains model (considering the long-run estimator) behaved better both in terms of accuracy and possible profits, than the other models, additionally, this model resulted in all kinds of strategies, unlike all the other models. So, with the Markov chains model (considering the long-run estimator), we obtained better results than all the other presented models.

Finally, the game theoretical model that we used as the decision model to make predictions (and to buy and sell financial assets) is a useful and accurate tool (both for the Markov chains models and the time series models), because it gives us an optimal strategy chosen in terms of these market probabilities, also these strategies are the same as the ones commonly used by the markets' investors (and speculators). So, instead of directly predicting a financial asset's price (and then acting upon this predictions), we can obtain a probabilistic model that lets us see how the financial markets behaved and where they may be going in terms of the assets' prices.

4.2 Future Work

From the presented theory and subsequently results, a number of possible extensions can be made, such as:

- create a new decision model that incorporates both the Markov chains and time series models;
- add more classes to the Markov chains model, add more strategies to the game theoretical model and/or alter the existing classes/strategies;
- study on how these models can be adapted to all kinds of data.

Also, we can study the possible relationships between the volatility, type and/or length of the datasets to:

- the obtained optimal model (both in terms of accuracy and possible profit results);
- the obtained values for the accuracy and possible profit results;
- the standard deviations, the average closing time and the percentage of null models.

All of this should be studied in order to better model and predict the financial markets, and then extend to all kinds of data.

Appendices

Appendix A

Time Series Theory

Time series modeling is a dynamic research area which has attracted attention from the researchers' community over the last few decades. The main aim of time series modeling is to carefully collect and rigorously study the past observations of a time series to develop an appropriate model which describes its inherent structure. This model is then used to generate future values for the series, i.e., to make forecasts. Time series forecasting thus can be termed as the act of predicting the future by understanding the past. Due to the indispensable importance of time series forecasting in numerous practical fields such as business, economics, finance, science, engineering, etc., proper care should be taken to fit an adequate model to the underlying time series. It is obvious that a successful time series forecasting depends on an appropriate model fitting. So, a lot of efforts have been done by researchers over many years for the development of efficient models to improve the forecasting accuracy. As a result, various important time series forecasting models have been developed in literature.

One of the most popular and frequently used time series models is the Autoregressive Integrated Moving Average (ARIMA) model, where the basic assumption made to implement this model is that the considered time series is linear and follows a particular known statistical distribution, such as the normal distribution. Also, the ARIMA model has its own subclasses of models: the Autoregressive (AR), Moving Average (MA) and Autoregressive Moving Average (ARMA). And, for seasonal time series forecasting, Box and Jenkins (1970) proposed a quite successful variation of ARIMA model, the Seasonal ARIMA (SARIMA). The popularity of the SARIMA models is mainly due to its flexibility to represent several varieties of time series with simplicity, as well as the associated Box-Jenkins methodology for optimal model building process. But the

severe limitation of these models is the pre-assumed linear form of the associated time series which becomes inadequate in many practical situations. So, to overcome this drawback, various non-linear stochastic models have been proposed in literature, however, from the implementation point of view, these are not so straight-forward and simple as the SARIMA models.
Hence, in this chapter, we will explore all of these concepts relying on Shumway and Stoffer (2011) and Brockwell and Davis (2016).

A.1 Basic Concepts

A.1.1 Definition of a Time Series

A time series is a set of observations x_t, each one being recorded at a specific time $t \in T$, where T is the set which contains the information of when observations are made. A discrete-time time series (the type to which this paper is devoted to) is one in which the set T is a discrete set, as is the case when observations are made at fixed time intervals (e.g., when $T = \{0, \pm 1, \pm 2, \dots\}$). Continuous-time time series are obtained when observations are recorded continuously over some time interval (e.g., when $T = [0, 1]$).

Remark 26. *Henceforth, whenever we refer to a time series, we mean a discrete-time time series with $T = \{0, \pm 1, \pm 2, \dots\}$, unless otherwise stated.*

To model a time series we need to select a suitable probability model (or class of models) for the data, and to allow for the possibly unpredictable nature of future observations, it is natural to suppose that each observation x_t is a realized value of a certain random variable X_t. So, formally, we define a time series as:

Definition 1. *A time series model for the observed data $\{x_t\}_{t \in T}$ is a specification of the joint distributions (or possibly only the means and covariances) of a sequence of random variables $\{X_t\}_{t \in T}$ of which $\{x_t\}_{t \in T}$ is postulated to be a realization.*

Remark 27. *Note that, in literature, the notation for the process (or the model) and for the time series data can be different, but we will always use $\{x_t\}_t \equiv \{x_t\}_{t \in T} \equiv \{x_t, t \in T\}$ to denote both the process and the time series. We shall frequently use the term time series to mean* *both the data and the process of which it is a realization. But, if the concept that we are referring to is not clear form the context, we will use the notation* $\{X_t\}_t \equiv \{X_t\}_{t\in T} \equiv \{X_t, t \in T\}$ *for the process.*

A complete probabilistic time series model for the sequence of random variables $\{X_1, X_2, \ldots\}$ specifies all of the joint distributions of the random vectors $(X_1, \ldots, X_n)\,, n = 1, 2, \ldots,$ or equivalently all of the probabilities

$$P\left(X_1 \leq x_1, \ldots, X_n \leq x_n\right), \quad -\infty < x_1, \ldots, x_n < \infty, \quad n = 1, 2, \ldots.$$

Such a specification is rarely used in time series analysis (unless the data are generated by some well-understood simple mechanism), because in general it will contain far too many parameters to be estimated from the available data. Instead we specify only the first and second-order moments of the joint distributions, i.e., the expected values $E(X_t)$ and the expected products $E(X_{t+h}X_t)$, $t = 1, 2, \ldots,$ $h = 0, 1, 2, \ldots.$ Thus mainly focusing on properties of the sequence $\{X_t\}_t$ that depend on these, and such properties are referred to as second-order properties.
In the particular case where all the joint distributions are multivariate normal, the second-order properties of $\{X_t\}_t$ completely determine the joint distributions, giving a complete probabilistic characterization of the sequence. But, in general, we shall lose a certain amount of information by looking at time series "through second-order spectacles". However, the theory of minimum mean squared error linear prediction depends only on the second-order properties, thus providing further justification for the use of the second-order characterization of time series models (see Brockwell and Davis (2016) for more details).
Lastly, a time series containing records of a single variable is termed as univariate, and if records of more than one variable are considered, it is termed as multivariate. But, for the purposes of this paper, we will only consider univariate time series.

A.1.2 Components of a Time Series

Generally, a time series is supposed to be affected by four main components, which can be obtained from the observed data. These components are:

- *Trend*: the general tendency of a time series to increase, decrease or stagnate over a long period of time. So we can say that the trend is a long term movement in a time series.

- *Seasonal*: the seasonal variations in a time series are fluctuations that occur at specific regular intervals, usually less than a year, such as weekly, monthly, or quarterly. The important factors causing seasonal variations are: the weather, vacation, and holidays and consists of periodic, repetitive, and generally regular and predictable patterns in the levels of a time series.
- *Cyclical*: the cyclical variation in a time series describes the medium-term changes in the series, caused by circumstances, which repeat in cycles. The duration of a cycle extends over longer period of time, usually two or more years.
- *Random*: the random variations in a time series are caused by unpredictable influences, which are not regular and also do not repeat in a particular pattern. Also, there is no defined statistical technique for measuring random fluctuations in a time series.

Note that instead of including a cyclical component, we can consider a seasonal component with a larger interval, which is why, usually, both of them are not present in the models.

Now, considering the effects of these four components, two different types of models are commonly used for a time series:

- *Multiplicative Model*: $X_t = T_t \times S_t \times C_t \times R_t$.
- *Additive Model*: $X_t = T_t + S_t + C_t + R_t$.

Here X_t is the time series and T_t, S_t, C_t and R_t are, respectively the trend, seasonal, cyclical and random components, all considered at time t.

The Multiplicative model is based on the assumption that the four components of a time series are not necessarily independent and they can affect one another; whereas in the additive model it is assumed that the four components are independent of each other.

Remark 28. *These models representations are referred to as the classical decomposition models.*

A.1.3 Time Series and Stochastic Processes

As stated in Section A.1.1 a time series is non-deterministic in nature, i.e. we cannot predict with certainty what will occur in future. Generally a time series $\{x_t, t = 0, 1, 2, \dots\}$ is assumed to follow certain probability model which describes the joint distribution of the random variable

X_t, and the mathematical expression describing the probability structure of a time series is termed as a stochastic process. Thus the sequence of observations of the series is actually a sample realization of the stochastic process that produced it.

Remark 29. *Again, because it will be clear from the context of our discussions, we use the term time series whether we are referring generically to the process or to a particular realization and make no notational distinction between the two concepts. Additionally, the used notation is* $\{x_t\}_t \equiv \{x_t\}_{t=0,1,2,\dots} \equiv \{x_t, t = 0, 1, 2, \dots\}$, *unless otherwise stated.*

A usual assumption is that the time series variables x_t are independent and identically distributed (i.i.d) following the normal distribution. However, an interesting point is that time series are in fact not exactly i.i.d; they follow more or less some regular pattern in long term. This is the reason why time series forecasting using a proper technique, yields result close to the actual value.

A.1.4 The Concept of Stationarity

Loosely speaking, a time series $\{x_t, t = 0, \pm 1, \dots\}$ is said to be stationary if it has statistical properties similar to those of the "time-shifted" series $\{x_{t+h}, t = 0, \pm 1, \dots\}$, for each integer h. Restricting attention to those properties that depend only on the first and second-order moments of $\{x_t\}_t$, we can make this idea precise with the following definitions.

Definition 2. *Let* $\{x_t\}_t$ *be a time series with* $E(x_t^2) < \infty$. *Then the mean function of* $\{x_t\}_t$ *is*

$$\mu_x(t) = E(x_t).$$

The covariance function of $\{x_t\}$ *is*

$$\gamma_x(r, s) = Cov(x_r, x_s) = E[(x_r - \mu_x(r))(x_s - \mu_x(s))]$$

for all integers r *and* s.

Definition 3. $\{x_t\}_t$ *is (weakly) stationary if*

(1) $\mu_x(t)$ *is independent of* t;

(2) $\gamma_x(t + h, t)$ *is independent of* t *for each* h.

Remark 30. *Strict stationarity of a time series* $\{x_t, t = 0, \pm 1, \dots\}$ *is defined by the* *condition that* $(x_1, \ldots, x_n)$ *and* $(x_{1+h}, \ldots, x_{n+h})$ *have the same joint distributions for all integers* h *and* $n > 0$. *And we can check that if* $\{x_t\}_t$ *is strictly stationary and* $E(x_t^2) < \infty$ *for all* t, *then* $\{x_t\}_t$ *is also weakly stationary. So, henceforth, whenever we use the term stationary we shall mean weakly stationary as in Definition 3, unless we specifically indicate otherwise.*

Remark 31. *In view of condition (2), whenever we use the term covariance function with reference to a stationary time series* $\{x_t\}_t$ *we shall mean the function* γ_x *of one variable, defined by* $\gamma_x(h) := \gamma_x(h, 0) = \gamma_x(t + h, t)$. *Also, the function* $\gamma_x(\cdot)$ *will be referred to as the autocovariance function and* $\gamma_x(h)$ *as its value at lag* h, *i.e., its value regarding the observation* x_{t+h}.

Definition 4. *Let* $\{x_t\}_t$ *be a stationary time series. The autocovariance function (ACVF) of* $\{x_t\}_t$ *at lag* h *is*

$$\gamma_x(h) := Cov(x_{t+h}, x_t).$$

The autocorrelation function (ACF) of $\{x_t\}_t$ *at lag* h *is*

$$\rho_x(h) \equiv \frac{\gamma_x(h)}{\gamma_x(0)} = Cor(x_{t+h}, x_t).$$

From these definitions we obtain the following properties:

- $\gamma_x(\cdot)$ has the linearity property of covariances, that if $E(x^2) < \infty$, $E(Y^2) < \infty$, $E(Z^2) < \infty$ and a, b and c are any real constants, then

$$Cov(ax + bY + c, Z) = aCov(x, Z) + bCov(Y, Z);$$

- $\gamma_x(0) = Var(x_t)$ is the variance of $\{x_t\}_t$;
- $\rho_x(\cdot)$ is dimensionless and so is independent of the scale of measurement;
- $-1 \leq \rho_x(\cdot) \leq 1$.

Concluding, the concept of stationarity is a mathematical idea constructed to simplify the theoretical and practical development of stochastic processes. To design and adequate model for forecasting, the underlying time series is expected to be stationary. Unfortunately, it is not always the case, because the greater the time span of historical observations, the greater is the chance that the time series will exhibit non-stationary characteristics (as stated in **Hipel and McLeod (1994)**). However, for relatively short time span, one can reasonably model the series using a stationary stochastic process. Usually, time series showing trend or seasonal patterns are non-stationary in nature. Also, in such cases, differencing and power transformations are often used to remove the trend and to make the series stationary, both which will be discussed later on.

A.1.5 Model Parsimony

While building a proper time series model we have to consider the principle of parsimony, which states that the model with smallest possible number of parameters is always the one to be selected so as to provide an adequate representation of the underlying time series data. In other words, out of a number of suitable models, one should consider the simplest one, still maintaining an accurate description of inherent properties of the time series.

The idea of model parsimony is similar to the famous Occam's razor principle. Additionally, as discussed by Hipel and McLeod (1994), one aspect of this principle is that when faced with a number of competing and adequate explanations, pick the simplest one. Moreover, the more complicated the model, the more possibilities will arise for departure from the actual model assumptions. Also, with the increase of model parameters, the risk of overfitting subsequently increases.

An over fitted time series model may describe the training data very well, but it may not be suitable for forecasting. Additionally, because potential overfitting affects the ability of a model to forecast well, parsimony is often used as a guiding principle to overcome this issue. Thus in summary it can be said that, while making time series forecasts, genuine attention should be given to select the most parsimonious model among all of the possibilities.

A.2 Time Series Models

The selection of a proper model is extremely important as it reflects the underlying structure of the series, also this fitted model is in turn used for forecasting. A time series model is said to be linear or non-linear depending on whether the current value of the series is a linear or non-linear function of past observations.

In general, models for time series data can have many forms and represent different stochastic processes. Also, as it was said before, there are two widely used linear time series models in literature, namely Autoregressive (AR) and Moving Average (MA) models. Additionally, combining these two, the Autoregressive Moving Average (ARMA) and Autoregressive Integrated Moving Average (ARIMA) models have been proposed in literature. For seasonal time series forecasting, a variation of ARIMA, the Seasonal Autoregressive Integrated Moving Average (SARIMA) model is used.

Linear models have drawn much attention due to their relative simplicity in understanding and implementation. However, many practical time series show non-linear patterns. As such, various nonlinear models have been suggested in literature, like the Generalized Autoregressive Conditional Heteroscedasticity (GARCH).

Now we shall discuss about the important linear and non-linear stochastic time series models with their different properties.

A.2.1 The Autoregressive Moving Average (ARMA) Models

An $ARMA(p,q)$ model is a combination of $AR(p)$ and $MA(q)$ models and is suitable for univariate time series modeling. In an $AR(p)$ model the future value of a variable is assumed to be a linear combination of p past observations and a random error together with a constant term. Mathematically the $AR(p)$ model can be expressed as:

$$x_t = \varphi_1 x_{t-1} + \varphi_2 x_{t-2} + \cdots + \varphi_p x_{t-p} + \epsilon_t = \sum_{i=1}^{p} \varphi_i x_{t-i}, \tag{A.1}$$

where x_t and ϵ_t are, respectively, a stationary time series (with zero mean) and the random error at time period t, and the model's parameters are $\varphi_i, i = 0, 1, 2, \ldots, p$, with $\varphi_p \neq 0$. The integer constant p is known as the order of the model. And, for estimating the parameters of an AR model, using a given time series, the Yule-Walker equations are used (for further details see Brockwell and Davis (2016) and Hipel and McLeod (1994)).

Note that, if the mean μ of x_t is not zero we can replace x_t by $x_t - \mu$ in equation A.1, obtaining

$$x_t - \mu = \varphi_1(x_{t-1} - \mu) + (\varphi_2 x_{t-2} - \mu) + (\cdots + \varphi_p x_{t-p} - \mu) + \epsilon_t.$$

Thus obtaining a time series with zero mean and an $AR(p)$ model without a constant term. So, for simplicity, we will always consider that the time series has zero mean, because otherwise we can subtract it.

Now, just as an $AR(p)$ model regresses using past values of the series, a $MA(q)$ model uses past errors as the explanatory variables, and it is formulated as:

$$x_t = \epsilon_t + \theta_1\epsilon_{t-1} + \theta_2\epsilon_{t-2} + \cdots + \theta_q\epsilon_{t-q} = \epsilon_t + \sum_{j=1}^{q} \theta_j\epsilon_{t-j},$$

where x_t and ϵ_t are as before, the model's parameters are $\theta_j, j = 1, \ldots, q$, with $\theta_q \neq 0$, and q is the order of the model. Thus, conceptually, a moving average model is a linear regression of the current observation of the time series against the random shocks of one or more prior observations. And, as before, if the mean of the time series is not zero, then we can subtract it to the time series (to each observed value), thus obtaining a time series with zero mean, which gives us an $MA(q)$ model without a constant term. So, for simplicity, we will always consider that the time series has zero mean, because otherwise we can subtract it.

Also, in the MA model, the random error is assumed to be a white noise process (i.e., a sequence of independent and identically distributed (i.i.d) normal random variables with mean 0 and a constant variance 1), but this assumption is not necessary for the models' definition. Thus, henceforth, whenever we refer to the random error we will assume it to be a white noise process with zero mean and constant variance.

Remark 32. *Note that, fitting an MA model to a time series is more complicated than fitting an AR model, because in the former one the random error terms are not foreseeable.*

Autoregressive (AR) and moving average (MA) models can be effectively combined together to form a general and useful class of time series models, known as the ARMA models. Mathematically, an $ARMA(p, q)$ model is defined as:

$$x_t = \sum_{i=1}^{p} \varphi_i x_{t-i} + \epsilon_t + \sum_{j=1}^{q} \theta_j\epsilon_{t-j}, \tag{A.2}$$

where x_t and ϵ_t are as before, $\varphi_p \neq 0$, $\theta_q \neq 0$, and the model's orders (p, q), refer to p autoregressive and q moving average terms.

Remark 33. *Again, if x_t has a nonzero mean μ, we set $\alpha = \mu(1 - \varphi_1 - \cdots - \varphi_p)$ and write* *the model as*

$$x_t = \alpha + \sum_{i=1}^{p} \varphi_i x_{t-i} + \epsilon_t + \sum_{j=1}^{q} \theta_j \epsilon_{t-j}.$$

Usually ARMA models are presented using the backshift operator notation, which is defined as $B(x_t) = x_{t-1}$, with $B^i(x_t) = x_{t-i}$. Thus, the lag (or characteristic) polynomials are used to represent ARMA models as follows:

- $AR(p)$ *model*: $\epsilon_t = \varphi(B)x_t$.
- $MA(q)$ *model*: $x_t = \theta(B)\epsilon_t$.
- $ARMA(p, q)$ *model*: $\varphi(B)x_t = \theta(B)\epsilon_t$.

Where $\varphi(B) = 1 - \sum_{i=1}^{p} \varphi_i B^i$ and $\theta(B) = 1 + \sum_{j=1}^{q} \theta_j B^j$ are referred the lag polynomials. Additionally, $\varphi(z) = 1 - \sum_{i=1}^{p} \varphi_i z^i$ and $\theta(z) = 1 + \sum_{j=1}^{q} \theta_j z^j$ are the AR and MA polynomials, respectively; where z is a complex number.

A.2.2 Models' Important Properties

Before moving further, we encounter have the following problems:

- parameter redundant models;
- stationary AR models that depend on the future;
- MA models that are not unique;
- may not obtain stationary models.

To address the first problem, we will henceforth refer to an $ARMA(p, q)$ model to mean that it is in its simplest form. That is, in addition to the original definition given in equation A.2, we will also require that $\varphi(z)$ and $\theta(z)$ have no common factors.

Now, to address the problem of future-dependent models, we formally introduce the concept of causality.

Definition 5. *An $ARMA(p, q)$ model is said to be causal, if the time series $\{x_t\}_t$ can be written as a one-sided linear process:*

$$x_t = \sum_{j=0}^{\infty} \psi_j \epsilon_{t-j} = \psi(B)\epsilon_t,$$

where $\psi(B) = \sum_{j=0}^{\infty} \psi_j B^j$, $\sum_{j=0}^{\infty} |\psi_j| < \infty$, and $\psi_0 = 1$.

So, an $ARMA(p,q)$ model is causal if and only if $\varphi(z) \neq 0$, for $|z| \leq 1$ (for more details see Brockwell and Davis (2016)). In other words, an ARMA process is causal only when the roots of $\varphi(z)$ lie outside the unit circle; that is, $\varphi(z) = 0$ only when $|z| > 1$.
To address the problem of uniqueness, we choose the model that allows an infinite autoregressive representation.

Definition 6. *An $ARMA(p,q)$ model is said to be invertible, if the time series $\{x_t\}_t$ can be written as*

$$\pi(B)x_t = \sum_{j=0}^{\infty} \pi_j x_{t-j} = \epsilon_t,$$

where $\pi(B) = \sum_{j=0}^{\infty} \pi_j B^j$, $\sum_{j=0}^{\infty} |\pi_j| < \infty$, and $\pi_0 = 1$.

Thus, an $ARMA(p,q)$ is invertible if and only if $\theta(z) \neq 0$ for $|z| \leq 1$ (for more details see Brockwell and Davis (2016)). In other words, an ARMA process is invertible only when the roots of $\theta(z)$ lie outside the unit circle; that is, $\theta(z) = 0$ only when $|z| > 1$.

Remark 34. *In Hipel and McLeod (1994) it is shown that, invertibility is an important property of an $AR(p)$ process, this means that an $AR(p)$ process can always be written in terms of a $MA(\infty)$ process. Whereas, for an $MA(q)$ process to be invertible, all the unit roots of the $\theta(B)$ polynomial must lie outside the unit circle.*

Finally, to address the stationarity issue, when an $AR(p)$ process is represented as $\epsilon_t = \varphi(B)x_t$, then, by Box and Jenkins (1970), a necessary and sufficient condition for the $AR(p)$ process to be stationary is that all the roots of the characteristic polynomial must fall outside the unit circle. And, by Hipel and McLeod (1994), an $MA(q)$ process is always stationary, irrespective of the values the MA parameters.
Additionally, the conditions regarding stationarity, causality and invertibility of AR and MA processes also hold for an ARMA process. An $ARMA(p,q)$ process is stationary and causal if all the roots of the AR polynomial lie outside the unit circle. Similarly, if all the roots of the MA polynomial lie outside the unit circle, then the $ARMA(p,q)$ process is invertible and can be expressed as a pure AR process. **Remark 35.** *For the proofs of the previous statements see Shumway and Stoffer (2011) and/or Brockwell and Davis (2016).*

A.2.3 Autocorrelation and Partial Autocorrelation Functions (ACF and PACF)

To determine a proper model for a given time series data, it is necessary to carry out the ACF and PACF analysis. These statistical measures reflect how the observations in a time series are related to each other. For modeling and forecasting purposes it is often useful to plot the ACF and PACF against consecutive time lags. These plots help in determining the order of AR and MA terms.

Before moving further, let us recall the definitions of the Autocovariance (ACVF) and Autocorrelation (ACF) functions given in Section A.1.4. Considering a time series $\{x_t, t = 0, 1, 2, \dots\}$ we have the following definitions:

- *Autocovariance function at lag h*: $\gamma_x(h) = Cov(x_t, x_{t+h}) = E[(x_t - \mu)(x_{t+h} - \mu)]$;
- *Autocorrelation function at lag h*: $\rho_x(h) = \frac{\gamma_x(h)}{\gamma_x(0)}$.

Here μ is the mean of the time series, i.e., $\mu = E(x_t)$, and $\gamma_x(0)$ is the autocovariance at lag 0, i.e., $\gamma_x(0) = Var(x_t)$.

Another important measure is the Partial Autocorrelation Function (PACF), which is used to measure the the correlation between x_{t+h} and x_t with the linear dependence of $\{x_{t+1}, \dots, x_{t+h-1}\}$ on each, removed. Thus we obtain the formal definition:

Definition 7. *The partial autocorrelation function (PACF) of a stationary process, x_t, denoted $\phi_x(h)$, for $h = 1, 2, \dots,$ is defined as*

$$\phi_x(1) = \rho_x(1) \text{ and } \phi_x(h) = Corr(x_{t+h} - \hat{x}_{t+h}, x_t - \hat{x}_t), h \geq 2.$$

Here $\hat{x}_t$ denotes the regression of x_t on $\{x_{t+1}, x_{t+2}, \dots, x_{t+h-1}\}$, i.e., $\hat{x}_t = \beta_1 x_{t+1} + \beta_2 x_{t+2} + \dots + \beta_{h-1} x_{t+h-1}$, and $\hat{x}_{t+h}$ denotes the regression of x_{t+h} on $\{x_{t+h-1}, x_{t+h-2}, \dots, x_{t+1}\}$, i.e., $\hat{x}_{t+h} = \beta_1 x_{t+h-1} + \beta_2 x_{t+h-2} + \dots + \beta_{h-1} x_{t+1}$.

Remark 36. *Note that if the process is stationary, then the coefficients of $\hat{x}_{t+h}$ and $\hat{x}_t$ are the same (to check this result see Shumway and Stoffer (2011)). Also, both $(x_{t+h} - \hat{x}_{t+h})$ and $(x_t - \hat{x}_t)$ are uncorrelated with $\{x_{t+1}, x_{t+2}, \dots, x_{t+h-1}\}$.*

Normally, the stochastic process governing a time series is unknown and so it is not possible to determine the actual or theoretical ACF and PACF values. So these values have to be estimated from the training data (or from the known time series at hand). Hence, for simplicity, whenever we refer to the ACF and PACF, we mean the estimated (or sample) ACF and PACF, unless stated otherwise.

From Hipel and McLeod (1994), the most appropriate sample estimate for the ACVF at lag h is

$$c_x(h) = \frac{1}{n}\sum_{t=1}^{n-h}(x_t - \mu)(x_{t+h} - \mu).$$

And the estimate for the sample ACF at lag h is given by

$$r_x(h) = \frac{c_x(h)}{c_x(0)}.$$

Here $\{x_t, t = 0, 1, 2, \dots\}$ is the sampled time series of size n and with mean μ.

As explained by Box and Jenkins (1970), the sample ACF plot is useful in determining the type of model to fit to a time series of length n. Since ACF is symmetrical about lag zero, it is only required to plot the sample ACF for positive lags, from lag one onwards to a maximum lag of about $n/4$. The sample PACF plot helps in identifying the maximum order of an AR process. The methods for calculating ACF and PACF for ARMA models are described in Hipel and McLeod (1994).

A.2.4 Autoregressive Integrated Moving Average (ARIMA) Models

The previously described ARMA models can only be used for stationary time series data. However, in practice, many time series show non-stationary behavior. Additionally, time series which contain trend and seasonal patterns, are also non-stationary in nature. Thus, from the application view point, ARMA models are inadequate to properly describe non-stationary time series. For this reason the ARIMA model is proposed, which is a generalization of an ARMA model to include the case of non-stationarity as well.

In ARIMA models a non-stationary time series is made stationary by applying finite differencing of the data points, i.e., by applying a finitely many times the (lag-1) difference operator $\nabla x_t = x_t - x_{t-1} = (1 - B)x_t$ to the data, where B is the backward shift operator, $B(x_t) = x_{t-1}$. And powers of the operator B and ∇ are defined as $B^j(x_t) = x_{t-j}$ and $\nabla^j(x_t) = \nabla(\nabla^{j-1}(x_t)) = (1 - B)^j x_t, j \geq 1$, with $\nabla^0(x_t) = x_t$, respectively. And thus we obtain the mathematical formulation of the $ARIMA(p,d,q)$ model using lag polynomials:

$$\varphi(B)\nabla^d x_t = \theta(B)\epsilon_t \iff \left(1-\sum_{i=1}^{p}\varphi_i B^i\right)(1-B)^d x_t = \left(1+\sum_{j=1}^{q}\theta_j B^j\right)\epsilon_t, \tag{A.3}$$

where x_t is a time series, ϵ_t is the random error (as it was previously defined), and p, d and q are integers greater than or equal to zero that refer to the order of the autoregressive, integrated, and moving average parts of the model, respectively

Furthermore, note that:

- The integer d controls the level of differencing. Generally, we only need at most to use $d = 2$, and when $d = 0$, then it reduces to an $ARMA(p,q)$ model.
- An $ARIMA(p,0,0)$ is nothing but the $AR(p)$ model and $ARIMA(0,0,q)$ is the $MA(q)$ model.
- The $ARIMA(0,1,0)$ model, $x_t = x_{t-1} + \epsilon_t$, is a special case known as the Random Walk model, and it is widely used for non-stationary data.

A.2.5 Seasonal Autoregressive Integrated Moving Average (SARIMA) Models

The ARIMA model A.3 is for non-seasonal non-stationary data. Box and Jenkins (1970) have generalized this model to deal with seasonality (of period s), which is known as the Seasonal ARIMA (SARIMA) model. In this model, seasonal differencing of appropriate order is used to remove non-stationarity from the series, i.e., the lag-s differencing operator $\nabla_s x_t = x_t - x_{t-s} = (1-B^s)x_t$ is applied to the data (note that this operator should not be confused with the operator $\nabla^s = (1-B)^s$ defined earlier). But after removing the seasonality, we may still obtain a non-stationary model, thus we need to apply the difference operator, or we can apply the seasonal difference operator $\nabla_s^D x_t = (1-B^s)^D x_t$.

This model is generally termed as the $SARIMA(p,d,q)\times(P,D,Q)_s$ model and its mathematical formulation in terms of lag polynomials is:

$$\Phi(B^s)\varphi(B)\nabla^d\nabla_s^D\nabla^d x_t = \Theta(B^s)\theta(B)\epsilon_t, \tag{A.4}$$

where:

- x_t is a time series and ϵ_t is the random error (as it was previously defined);
- s is the seasonal length;

- the ordinary autoregressive and moving average components are represented by polynomials $\varphi(B) = (1 - \sum_{i=1}^{p} \varphi_i B^i)$ and $\theta(B) = \left(1 + \sum_{j=1}^{q} \theta_j B^j\right)$ of orders p and q, respectively (as in the model A.3);
- the seasonal autoregressive and moving average components are represented by $\Phi(B^s) = \left(1 - \sum_{k=1}^{P} \Phi_k (B^s)^k\right)$ and $\Theta(B^s) = \left(1 + \sum_{l=1}^{Q} \Theta_l (B^s)^l\right)$ of orders P and Q, respectively;
- the ordinary and seasonal difference components are represented by $\nabla^d = (1 - B)^d$ and $\nabla_s^D = (1 - B^s)^D$, respectively.

A.2.6 Nonlinear Time Series Models

So far we have discussed about linear time series models. However, as it was mentioned earlier, nonlinear models should also be considered for better time series analysis and forecasting, because all the previously discussed models assume a constant variance. Thus, models such as the autoregressive conditionally heteroscedastic (ARCH) model, first introduced by Engle (1982), were developed to model changes in volatility, or variability, of a time series (specifically financial time series). These models were later extended to generalized ARCH (GARCH) models by Bollerslev (1986).

Consider a strictly positive time series x_t, then its return time series is defined as

$$y_t = \frac{x_t - x_{t-1}}{x_{t-1}}. \tag{A.5}$$

The previous definition implies that $x_t = (1 + y_t)x_{t-1}$. Thus, based in Shumway and Stoffer (2011), if the return represents a small (in magnitude) percentage change then

$$\nabla[log(x_t)] \approx y_t.$$

So, either value, $\nabla[log(x_t)]$ or $(x_t - x_{t-1})/x_{t-1}$, will be called the return, and will be denoted by y_t. Furthermore, the study of y_t is the focus of ARCH, GARCH and other volatility models. Typically, for financial series, the return y_t, does not have a constant conditional variance, and highly volatile periods tend to be clustered together. In other words, there is a strong dependence of sudden bursts of variability in a return on the series own past. Thus we obtain the simplest ARCH model ($ARCH(1)$), which models the returns as

$$y_t = \sigma_t \epsilon_t \tag{A.6}$$

$$\sigma_t^2 = \alpha_0 + \alpha_1 y_{t-1}^2, \tag{A.7}$$

where ϵ_t is the random noise and $\alpha_1 \geq 0$ (because otherwise σ_t^2 may be negative). And, as with the previous models, the random error is assumed to be a white noise process, but it is not necessary.

Thus, the $ARCH(1)$ models return as a white noise process with non-constant conditional variance, and that conditional variance depends on the previous return (for further details see Shumway and Stoffer (2011)). In addition, it is possible to write the $ARCH(1)$ model as a (non-Gaussian) $AR(1)$ model in the square of the returns y_t^2 as:

$$y_t^2 = \alpha_0 + \alpha_1 y_{t-1}^2 + v_t, \tag{A.8}$$

where $v_t = \sigma_t^2(\epsilon_t^2 - 1)$. Also, because ϵ_t^2 is the square of a $N(0,1)$ random variable, $\epsilon_t^2 - 1$ is a shifted (to have zero mean), χ_1^2 random variable.

Furthermore, from the previous definitions we can characterize an $ARCH(1)$ process by the following properties (which are explored in great detail by Shumway and Stoffer (2011)):

- if $\alpha_1 < 1$, the process y_t itself is white noise and its unconditional distribution is symmetrically distributed around zero;
- if, in addition, $3\alpha_1^2 < 1$, the square of the process, y_t^2, follows a causal $AR(1)$ model with ACF given by $\rho_{y^2}(h) = \alpha_1^h \geq 0$, for all $h > 0$. If $3\alpha_1 \geq 1$, but $\alpha_1 < 1$, then y_t^2 is strictly stationary with infinite variance.

The $ARCH(1)$ model can be extended to the general $ARCH(m)$, that is, Equation A.6, $y_t = \sigma_t \epsilon_t$, is retained, but Equation A.7 is extended to

$$\sigma_t^2 = \alpha_0 + \alpha_1 y_{t-1}^2 + \cdots + \alpha_m y_{t-m}^2. \tag{A.9}$$

Another extension of ARCH is the generalized ARCH (GARCH) model developed by Bollerslev (1986). The $GARCH(m, r)$ model retains Equation A.6 and extends A.7 to

$$\sigma_t^2 = \alpha_0 + \sum_{j=1}^{m} \alpha_j y_{t-j}^2 + \sum_{j=1}^{r} \beta_j \sigma_{t-j}^2, \tag{A.10}$$

where y_t, σ_t and ϵ_t remain as before.

Remark 37. *Estimation of the parameters of the $ARCH(m)$ and $GARCH(m, r)$ models is typically accomplished by conditional Maximum Likelihood Estimation (MLE), with some minor differences between them (as we can see in Shumway and Stoffer (2011)).*

A.3 Estimation

After choosing the best theoretical model for the data (using the theory presented so far), we need to estimate its parameters. Thus, we will present some methods that will aid us in estimating our models' parameters.

The method chosen to estimate the SARIMA models is maximum likelihood. It is first assumed that the SARIMA model with parameters $\boldsymbol{\theta}$ have been identified. Further, it is assumed that the number of observations must be at least 50 and preferably 100 for efficient estimation (Box and Jenkins (1976)).

The first part of the maximum likelihood estimation is to specify the probability density function implied by the chosen model. Also, it is assumed that the error term of the model is distributed as Gaussian white noise. The estimation procedure is then performed in two steps. First the likelihood function is derived and then the value of the parameter vector $\boldsymbol{\theta}$ is specified to maximize the value of that function. The maximum likelihood estimate $\hat{\boldsymbol{\theta}}$ is then interpreted as having the value which maximizes the probability for observing this specific sample of observations (Hamilton (1994)). Afterwards, the fit of the model is evaluated by diagnostic checks of the residuals. The residuals should behave like Gaussian white noise, i.e. appear random, homoscedastic and normally distributed (for further details see Box and Jenkins (1976)).

The first part is accomplished with a graphical check of the standardized residuals (i.e., the residuals divided by their standard deviation), and these should look random and homoscedastic. Also, the number of outliers is important and a good indicator would be that about 95 percent of the residuals lie inside their 95% confidence interval ± 1.96 (see Brockwell and Davis (2016) for further details).

The next step is to evaluate the assumption of randomness by using the sample autocorrelation function of the residuals. The autocorrelations of interest are those that are significantly different from zero, that is those who lie outside the sample size dependent approximately 95% confidence interval $\pm 2/\sqrt{T}$ (Hamilton (1994)). Those significant lags suggest some kind of inconsistency in the residuals, but there is no reason to worry if only about five percent of the autocorrelations are significant (Brockwell and Davis (2016)).

The most important part of the diagnostic checking is the use of tests (and/or observe the ACF and PACF plots) to possibly acquire statistically significant results which would imply a rejection of the fitted model. The chosen tests are the following:

- If the ACF and PACF plots do not present significant lags, then the data may be a white noise process. Because, for large n, the sample autocorrelations of an i.i.d. sequence $x_1, \ldots, x_n$ with finite variance are approximately i.i.d. with distribution $N(0, 1/n)$ (see Brockwell and Davis (2016)). Hence, if $x_1, \ldots, x_n$ is a realization of such an sequence, about 95% of the sample autocorrelations should fall between the bounds $\pm 1.96/\sqrt{n}$. If we compute the sample autocorrelations up to lag 40 and find that one or more values fall outside the bounds, we therefore reject the i.i.d. hypothesis.
- Instead of checking to see whether each sample autocorrelation $\hat{\rho}(h)$ falls inside the bounds defined above, it is also possible to apply a Ljung-Box (LB) Test (where H_0 is that the data are independently distributed, i.e. the correlations in the population from which the sample is taken are 0, so that any observed correlations in the data result from randomness of the sampling process).
- To verify if the data is stationary , we will apply an Augmented Dickey–Fuller (ADF) test[1] (where H_0 is that a unit root is present in the time series sample).
- To verify if the data is homoscedastic, we will apply a Breusch-Pagan (BP) test (where H_0 is that the errors of a regression model are homoscedastic).
- To verify if the data is normally distributed, we will apply a Jarque-Bera (JB) test (where H_0 is that the data is normally distributed).

Note that, all the previous tests can also be applied to the "raw" or to the transformed data (instead of the residuals of the model) to check if the obtained data appears to be a white noise process.

[1] We didn't apply a Kwiatkowski–Phillips–Schmidt–Shin (KPSS) test because has a high rate of Type I errors (it tends to reject the null hypothesis too often)

The methods used to estimate GARCH models are similar to the ones describe for the SARIMA models (for further details see Shumway and Stoffer (2011) and Brockwell and Davis (2016)), but we also have to into account the squared residuals of the model. Thus, after obtaining the optimal SARIMA model, we need to check the residuals and the squared residuals of the model (by analyzing the ACF and PACF plots, and performing the tests previously described).

A.4 Forecasting

Now that we have the model and its (theoretical or estimated) parameters, we can focus on forecasting, where the goal is to predict future values of a time series, x_{n+h}, $h = 1, 2, \ldots$, based on the data collected to the present, $\boldsymbol{x} = \{x_n, x_{n-1}, \ldots, x_1\}$, and on the chosen models (with theoretical or estimated parameters). However, to better understand the forecasting results, we need to develop some theory.

A.4.1 SARIMA Forecasting

Firstly, we will discuss the $SARIMA(p, d, q) \times (P, D, Q)_s$ model applied to the time series x_t, which has the following expression (previously presented A.4):

$$\Phi(B^s)\varphi(B)\nabla^d\nabla_s^D\nabla^d x_t = \Theta(B^s)\theta(B)\epsilon_t, \tag{A.11}$$

where all the definitions remain the same, but assume, for simplicity, $\epsilon_t \sim N(0, \sigma_t^2)$.
Notice that the difference operators are applied to transform the observed non-stationary time series x_t to the stationary process x_t^* with the following equation (Brockwell and Davis (2016))

$$x_t^* = \nabla^d\nabla_s^D x_t = (1 - B)^d(1 - B^s)^D x_t. \tag{A.12}$$

Further, recall that $\varphi(B)$ and $\theta(B)$ are the lag polynomials, and the seasonal lag polynomials are $\Phi(B^s)$ and $\Theta(B^s)$, all this as it was defined in Equation A.4:

$$\varphi(B) = \left(1 - \sum_{i=1}^{p} \varphi_i B^i\right), \tag{A.13}$$

$$\theta(B) = \left(1 + \sum_{j=1}^{q} \theta_j B^j\right), \tag{A.14}$$

$$\Phi(B^s) = \left(1 - \sum_{k=1}^{P} \Phi_k (B^s)^k\right), \tag{A.15}$$

$$\Theta(B^s) = \left(1 + \sum_{l=1}^{Q} \Theta_l (B^s)^l\right). \tag{A.16}$$

For forecasting of integrated processes the fact that the observed variable x_t can be replaced by the differentiated variable x_t^* as in Equation A.12 is used. Box, G.Jenkins, and Reinsel (2008) proposed that the $SARIMA(p,d,q) \times (P,D,Q)_s$ for x_t can be seen as a special form of the equivalent representation of y_t^* as an $ARMA(p+sP, q+sQ)$ written as

$$\varphi(B)^* x_t^* = \theta(B)^* \epsilon_t, \tag{A.17}$$

where the AR part, $\varphi(B)^*$, is derived by multiplying the autoregressive lag polynomials $\varphi(B)$ and $\Phi(B^s)$, i.e., $\varphi(B)^* = \varphi(B)\Phi(B^s)$, and the MA part, $\theta(B)^*$, is derived by multiplying the moving average lag polynomials $\theta(B)$ and $\Theta(B^s)$, i.e., $\theta(B)^* = \theta(B)\Theta(B^s)$.

Thus, assuming that x_t^* is stationary with mean zero (otherwise we replace x_t^* by $x_t^* - \mu_x$), the 1-step-ahead forecast function for x_t^* is written as

$$\hat{x}^*_{n+1|t} = \varphi(B)^* \hat{x}^*_n + \theta(B)^* \hat{\epsilon_n}, \tag{A.18}$$

where $\hat{\epsilon}_t = x_t - \hat{x}^*_{t|t-1}$.

Furthermore, the h-step-ahead forecast, meaning the time that follows after the last observed information, is then derived by

$$\hat{x}^*_{n+h|\boldsymbol{x^*}} = \varphi(B)^* \hat{(x)}^*_{n+h-1|\boldsymbol{x^*}} + \theta(B)^* \epsilon_{n+\hat{h}-1}. \tag{A.19}$$

So, as we can see, the h-step-ahead forecast is obtained using: the previously observed values of x_t^*, the previous forecasts of $\hat{x}_t^*$ (i.e., $\boldsymbol{x^*} = \{x_n^*, x_{n-1}^*, \ldots, x_1^*\}$) and the residuals $\hat{\epsilon}_t$ which have been determined for all time points up to the last observed information but are equal to zero for the ones where the real values have yet not been observed (for further details see

Hamilton (1994)).

Also, note that (by Shumway and Stoffer (2011), Brockwell and Davis (2016) and Hamilton (1994)):

- the forecasts are unbiased, i.e. $E(\hat{x}^*_{t+s|t}) = x^*_{t+s}$;
- the variance of the prediction error and the $\alpha\%$ confidence interval for $\hat{\epsilon}_t$ depend on: the AR coefficients, σ^2_t and the horizon h;
- the variance of the prediction error increases with h;
- the variance of the prediction error tends to the variance of the process.

Here, depending on the distribution of ϵ_t, some properties may change.

Additionally, these properties hold, because we are considering the minimum mean square error predictor of x_{n+m}, i.e., $\hat{x}_{n+h} = E(x_{n+h}|\boldsymbol{x})$, because the conditional expectation minimizes the mean square error $E(x_{n+h} - g(\boldsymbol{x}))^2$, where $g(\boldsymbol{x})$ is a function of the observations $\boldsymbol{x}$ (which, in this section, we considered a SARIMA model).

A.4.2 GARCH Forecasting

Since we already have a strong theoretical background from Sections A.2.6 and A.5.1, so the conditional mean equation A.6 can be considered a stationary ARMA model, thus we can forecast using the methods presented in the previous section. The h-step-ahead forecast for the conditional variance equation A.10 is derived by

$$\hat{\sigma}^2_{t+h|t} = \alpha_0 + \sum_{j=1}^{m} \alpha_j \hat{y}^2_{t+h-j} + \sum_{j=1}^{r} \beta_j \hat{\sigma}^2_{t+h-j}. \tag{A.20}$$

Note that, here, y_t is return time series of x_t, as it was defined in Section A.2.6.

So, as in the SARIMA case, we assume that the time series is stationary (otherwise we can differentiate it) and that $\epsilon_t \sim N(0, \sigma^2_\epsilon)$.

A.5 A General Approach to Time Series Modeling

A.5.1 The Box-Jenkins Methodology

After describing various time series models, the next issue to our concern is how to select an appropriate model that can produce accurate forecast based on a description of historical pattern in the data and how to determine the optimal model orders. Box and Jenkins (1970) developed a practical approach to build ARIMA model, which best fit to a given time series and also satisfy the parsimony principle. Their concept has fundamental importance on the area of time series analysis and forecasting. Additionally, this method can be used to find the best model in all the time series literature, however we will only use it to find the optimal model between the previously discussed ones.

The Box-Jenkins methodology does not assume any particular pattern in the historical data of the series to be forecasted. Rather, it uses a three step iterative approach of model identification, parameter estimation and diagnostic checking to determine the best parsimonious model from a general class of ARIMA models. This three-step process is repeated several times until a satisfactory model is finally selected. Then this model can be used for forecasting future values of the time series.

The Box-Jenkins methodology is schematically shown in Figure A.1:

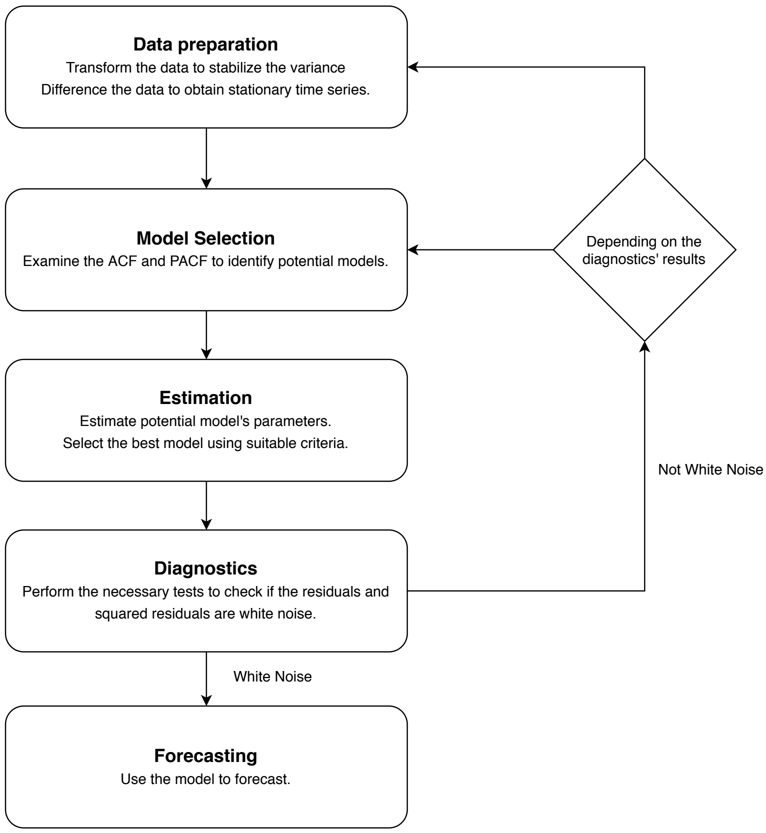

Figure A.1: The Box-Jenkins methodology for optimal model selection

If the time series is not stationary nor homoscedastic (i.e., constant variance), then we need to apply a data transformation and differentiation to the time series, so that we can get these properties and then estimate our models.

To this end, we can apply some data transformations (in order to achieve homoscedasticity) such as:

- Box-Cox transformation: choose λ that minimizes the variance of the data,

$$U_t = \begin{cases} \frac{X_t^{\lambda}-1}{\lambda} & \text{if } \lambda \neq 0 \\ log(X_t) & \text{if } \lambda = 0 \end{cases};$$

- Logarithmic transformation: $U_t = log(X_t)$;
- Percentage Returns (or Changes) transformation: $U_t = \frac{X_t - X_{t-1}}{X_{t-1}}$.

And then differentiate (at most with two degrees) the obtained data to achieve stationarity. Thus concluding the first step of the Box-Jenkins methodology.

Now that we have our transformed data, we will need to select a class of models (such as SARIMA or GARCH) in order to start fitting a model to the data. We can accomplish this by looking at the plot of the "raw" data, and its ACF and PACF plots:

	ACF	PACF
$AR(p)$	Exponentially decreasing or damped sine wave	Spikes to lag p then zero
$MA(q)$	Spikes to lag q then zero	Exponentially decreasing or damped sine wave
$ARMA(p, q)$	Exponentially decreasing or damped sine wave after $q - p$ lags	Exponentially decreasing or damped sine wave after $p - q$ lags
$SAR(P)_s$	Exponentially decreasing or damped sine wave for all lags times s	Spikes for lag Ps then zero
$SMA(Q)_s$	Spikes for lag Qs then zero	Exponentially decreasing or damped sine wave for all lags times s
$SARMA(P, Q)_s$	Exponentially decreasing or damped sine wave for all lags times s after lags $(Q - P)s$	Exponentially decreasing or damped sine wave for all lags times s after lags $(P - Q)s$

Table A.1: ACF and PACF to identify the orders of $SARM(p, q) \times (P, Q)_s$, only positive lags are of interest

After obtaining the optimal SARIMA model, we need to check the residuals and the squared residuals of the model (by analyzing the ACF and PACF plots, and performing the necessary tests):

- If both the residuals and the squared residuals appear to be white noise, then we have a viable model for our data.
- If the residuals do not appear to be a white noise process, then we need to find a more suitable SARIMA model.
- If the residuals appear to be white noise, but the squared residuals do not, then we will need to find a GARCH model for the data (on top of the SARIMA model).

In the case that we need to find a GARCH model, we also need to check the residuals and the squared residuals of the select GARCH model. And, if they both seem to be a white noise process, then we have a viable model for our data, otherwise we will need to find a more suitable GARCH model.

Afterwards, we can estimate the model's parameters (which were described in Section A.3) and then perform the diagnostics of the residuals and the squared residuals, that is we can observe the ACF and PACF plots, and/or apply some tests to check if they are a white noise process, i.e., if it is a sequence of i.i.d. normal random variables with mean 0 and a constant variance 1. Thus we need to check if the data to be tested is i.i.d. normal, stationary and homoscedastic (all of which was described in Section A.3).

Finally, after all the necessary tests for the residuals (and squared residuals) have been made, we will need to choose the optimal model of all (or in the same class) for the data. For this purpose, we can use the criterion that the sample ACF and PACF, calculated from the training data should match with the corresponding theoretical or actual values. Or, other widely used measures for model identification are the the information criteria, such as the Akaike Information Criterion (AIC), the AIC with a correction (AICc) and the Bayesian Information Criterion (BIC) which are defined as:

$$AIC = -2\log[L(\tilde{\Psi})] + 2n \tag{A.21}$$

$$AICc = AIC + \frac{2n(n+1)}{T-n+1} \tag{A.22}$$

$$BIC = -2\log[L(\tilde{\Psi})] + \text{nlog}(T) \tag{A.23}$$

Here $L(\tilde{\Psi})$ is the likelihood function, $\tilde{\Psi}$ is the maximum likelihood estimates of the parameters for the model, n is the number of parameters in the model and T is the number of effective observations (i.e. the sample size). The optimal model order is chosen by the number of model parameters, which minimizes either AIC, AICc or BIC (or any other information criterion).

Remark 38. *Depending on the model being considered, the previous information criteria may be defined differently (for further details see Brockwell and Davis (2016) and Shumway and Stoffer (2011)).*

Furthermore, we divided the data between training and test sets [2], so that we can choose the best model with the aid of the following accuracy measures:

- the mean absolute error: $MAE = mean(|e_i|)$;
- the root mean squared error: $RMSE = \sqrt{mean(e_i^2)}$;
- the mean absolute percentage error: $MAPE = mean(|100e_t/x_t|)$;
- the mean absolute scaled error: $MASE = mean\left(\frac{|e_i|}{\frac{1}{T-1}\sum_{i=2}^{T}|x_i - x_{i-1}|}\right)$.[3]

Where $x_1, \ldots, x_T$ represents the time series and $e_t = x_t - \hat{x}_{t|N}, t = N+1, \ldots, T$[4] the forecasting errors.

Note that the errors e_t are on the same scale as the data, so the accuracy measures (MAE, RMSE) that are based directly on them are scale-dependent, therefore they cannot be used to

[2] Note that most software already makes this division on order to choose the optimal models for the data.

[3] This is the non seasonal MASE definition, because we are not considering any kind of seasonality.

[4] The h-step-ahead forecast can be written as $\hat{x}_{N+h|N}$ (the "hat" notation indicates that it is an estimate rather than an observed value, and the subscript indicates that we are estimating x_{N+h} using all the data observed up to and including time N).

make comparisons between series that are on different scales.
As a scale independent measure, the MAPE is frequently used to compare forecast performance between different data sets. However, sometimes we may see a very large value of MAPE even though the model appears to fit the data well and this happens if any data values are close to 0. Because MAPE divides the absolute error by the actual data, values close to 0 can greatly inflate the MAPE. However, the MASE isn't inflated by values very close to 0, while maintaining the same properties.
Finally, after we obtain the optimal model for the data, we can use this model for forecasting and/or simulating. Where both of them are made with the aid of the theory presented so far and a suitable software (in our case the R software).
Finally, we can make forecasts using the obtained model (see Section A.4.1) and with the aid of a suitable software (in our case R).

A.5.2 Model Selection with the HK-algorithm

The Box-Jenkins methodology for the model selection can be very inefficient in terms of the required computer calculations needed, thus the Hyndman-Khandakar (HK) algorithm was developed by Hyndman and Khandakar (2008) and can be applied in R with the function *auto.arima* from the **forecast** package. They suggest an iterative time-saving procedure where the model with the smallest value of some information criteria AIC, AICc or BIC (see Equations A.21-A.23) will be found much faster, so it is now found without comparing every possible model.
The HK-algorithm performs an iterative procedure to select the model that minimizes the value of each criterion. It begins with estimation of the following four models:

- $SARIMA(2, d, 2) \times (1, D, 1)_s$
- $SARIMA(0, d, 0) \times (0, D, 0)_s$
- $SARIMA(1, d, 0) \times (1, D, 0)_s$
- $SARIMA(0, d, 1) \times (0, D, 1)_s$

where d and D are assumed to have been found previously and a constant is included in the models if $d + D \leq 1$. The model which attains the smallest value for the chosen information criterion is then selected and the procedure continues with varying the parameters in the following ways:

- Let each of p, q, P and Q vary with ± 1.
- Let both p and q vary with ± 1 at the same time.
- Let both P and Q vary with ± 1 at the same time.
- Include the intercept if previously not included otherwise do the opposite.

This step of the procedure will be repeated until none of these variations decreases the value of the criterion.

There are some constraints that follow with the use of this method. These are used to check that the model is reasonable and well-fitted and are the following:

- The maximum orders of p and q are five.
- The maximum orders of P and Q are two.
- All non-invertible or non-causal models are rejected. These are found by computing the roots of the lag polynomials $\varphi(B)\Phi(B)$ and $\theta(B)\Theta(B)$, if any root is smaller than 1.001 then the model is rejected.
- If errors arise when fitting the model with the non-linear optimization routine (used by the software) then the model is rejected.

At this stage the final model is found and the Box-Jenkins procedure can continue to its next step, meaning estimation.

Regarding the GARCH model selection, there is not a similar algorithm. However, we can adapt the previous (also with the aim to find the best model considering the smallest information criterion). We can start by estimation the following models:

- $GARCH(1,1)$
- $GARCH(0,0)$
- $GARCH(1,0)$
- $GARCH(0,1)$

Then model which attains the smallest value for the chosen information criterion is selected and the procedure continues with varying the parameters in the following way: let both m and

r vary with ± 1 at the same time.

As the previous algorithm, there are some constraints that follows with the use of this method. These are used to check that the model is reasonable and well-fitted and are the following:

- The maximum orders of m and r are five.
- The maximum orders of P and Q are two.
- All non-invertible or non-causal models are rejected.
- If errors arise when fitting the model with the non-linear optimization routine (used by the software) then the model is rejected.

As before, the final model is found and the Box-Jenkins procedure can continue to its next step, meaning estimation.

Appendix B

Markov Chains Theory

Consider a process which has a (finite) number of possible states. A decision maker periodically observes the current state of the process and chooses one of a number of available actions. The result is that the next state of the system is chosen according to transition probabilities depending on the current state, the action chosen in the current state, and the next state. Furthermore, a cost (or reward) is incurred by the decision maker, which depends on the current and previous states, and on the decision maker's chosen action. So, the decision maker's task is to find an "optimal policy", i.e., a way to choose actions which minimizes (or maximizes) some appropriate measure of over-all cost (or cost criterion).

The previously discussed process falls in the general area of Markov decision processes. If the transition probabilities or costs change with time, we call the Markov decision process non-stationarity. If both the transition probabilities and costs are independent of time, we call the Markov decision process stationary. If the decision maker knows that the system of transition probabilities and costs governing the current realization of the process is one of a known family of such systems, but he does not know which, the Markov decision process is called Bayesian [1]. If the action chosen by the decision maker who finds the process in a particular state at any time depends only upon the state (i.e., if the policy is state stationary), the sequence of states observed by the decision maker forms a Markov chain. Also, a Markov chain is non-stationary if the transition probabilities change with time, and it is called stationary if the transition probabilities are independent of time. If the action chosen depends only upon the current state

[1] For the purposes of this paper we do not need to explore the Bayesian Markov decision processes, but for further details see Bowerman (1974)

and the current time (i.e., if the policy is time dependent state stationary), the sequence of states forms a non-stationary Markov chain. Also, it turns out that if the sequence of states forms a Markov chain, then the long-run proportion of time that the process occupies each state (the long-run distribution) can often be found. Knowledge of the long-run distribution is very important in determining an optimal policy for a Markov decision process under the expected average cost criterion.

For our purposes, we will study topics in stationary and non-stationary Markov chains related to Markov decision processes. So, we will define a Markov chain and formally describe the difference between stationary and non-stationary Markov chains. Finally, we present some of the basic, classical definitions and results for stationary Markov chains (all of this, relying mainly on Bowerman (1974)).

B.1 Basic Concepts

A Markov process is a stochastic process (as it was defined Section A.1.3) that satisfies the Markov property (sometimes characterized as memorylessness). In simpler terms, a Markov process is a process for which one can make predictions for its future based solely on its present state just as well as one could knowing the process's full history.

A Markov chain is a type of Markov process that has either a discrete state space or a discrete index set (often representing time), but the precise definition of a Markov chain varies. For example, it is common to define a Markov chain as a Markov process in either discrete or continuous time with a countable state space (thus regardless of the nature of time), but it is also common to define a Markov chain as having discrete time, regardless of the state space (so it can a countable or continuous state space).

The system's state space and time parameter index need to be specified. The following Table B.1 (from Markov Chain (2020)) gives an overview of the several kinds of Markov processes for different levels of state space generality and for discrete or continuous time:

	Countable state space	Continuous or general state space
Discrete-time	(discrete-time) Markov chain on a countable or finite state space	Markov chain on a measurable state space
Continuous-time	Continuous-time Markov process or Markov jump process	Any continuous stochastic process with the Markov property

Table B.1: Different types of Markov chains

For our purposes, we will reserve the term "Markov Chain" for discrete-time Markov chain, as in the following definition:

Definition 8 (Markov Property). *A stochastic process* $\{x_t, t = 0, 1, 2, \dots\}$ *with a discrete and finite (or countable) state space* S *is said to be a Markov chain if for all states* $i_0, i_1, \dots, i_{t-1}, i, j$ *and (steps)* $t \geq 0$*:*

$$P(x_{t+1} = j | x_0 = i_0, \dots, x_{t-1} = i_{t-1}, x_t = i) = P(x_{t+1} = j | x_t = i) = p_{ij}^{t,t+1}.$$

Furthermore, the matrix of one-step-transition probabilities (or transition matrix) from time t to $t+1$, which we denote by $P^{t,t+1}$, is defined for $t \geq 0$ to be:

$$P^{t,t+1} = \begin{pmatrix} p_{11}^{t,t+1} & p_{12}^{t,t+1} & \cdots \\ p_{21}^{t,t+1} & p_{22}^{t,t+1} & \cdots \\ \vdots & \vdots & \\ p_{i1}^{t,t+1} & p_{i2}^{t,t+1} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix},$$

where for $t \geq 0$:

- $p_{ij}^{t,t+1} \geq 0, \quad \forall i \in S, \quad \forall j \in S;$
- $\sum_{j \in S} p_{ij}^{t,t+1} = 1, \quad \forall i \in S.$

Regarding this transition matrix, intuitively, we have that:

- the rows represent the present (states and/or times);
- the columns represent the future (states and/or times);
- the entry (i, j) is the conditional probability that in the future we are in state j, given that in the present we are in state i, thus it represents the probability of going from state i to state j.

Also, we can note that:

- the transition matrix must list all possible states in the state space S;

- the transition matrix is a square matrix which depends on the size of the state space, because X_t and X_{t+1} both take values in the same state space;
- each of the transition matrix's rows should sum up to one, which means that X_{t+1} must take one of the stated values;
- the transition matrix's columns do not in general sum up to one.

Now, we can define a (time-)homogeneous Markov chain as:

Definition 9. *A Markov chain is said to be (time-)homogeneous if for all states i and j:*

$$P(x_{t+1} = j|x_t = i) = P(x_{t+n+1} = j|x_{t+n} = i),$$

for $t = 0, 1, 2, \dots$ and $n \geq 0$. Otherwise the Markov chain is said to be non-homogeneous.

Markov chains are frequently illustrated graphically through diagrams, where circles or ovals are used to represent states, single-step transition probabilities are represented by directed arrows, which are frequently, but not always, labeled with the values of the transition probabilities. The absence of a directed arrow indicates that no single-step transition is possible. As an example (from Stewart (2009)), consider a homogeneous, discrete-time Markov chain that describes the daily weather pattern in Belfast (Northern Ireland), which we we will consider that only has three types of patterns: rainy, cloudy, and sunny. These three weather conditions describe the three states of our Markov chain: state 1 (R) represents a rainy day; state 2 (C), a cloudy day; and state 3 (S), a sunny day. The weather is observed daily, and on any given rainy day: the probability that it will rain the next day is estimated at 0.8; the probability that the next day will be cloudy is 0.15, while the probability that the next day will be sunny is only 0.05. Similarly, probabilities may be assigned when a particular day is cloudy or sunny as shown in following transition probability matrix P:

$$P = \begin{pmatrix} 0.8 & 0.15 & 0.05 \\ 0.7 & 0.2 & 0.1 \\ 0.5 & 0.3 & 0.2 \end{pmatrix}.$$

The diagram for this Markov chain is the following:

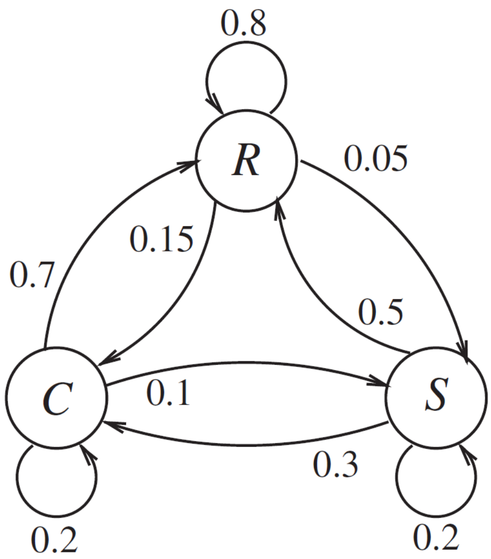

Figure B.1: Transition diagram for the weather at Belfast (from Stewart (2009)).

Moving forward, we can define stationary and non-stationary Markov chains:

Definition 10. *If $p_{ij}^{t,t+1}$ is independent of t, then the Markov chain is said to possess stationary transition probabilities and is called a stationary Markov chain. In this case, we write $p_{ij}^{t,t+1}$ as p_{ij}, for all $t \geq 0$. Otherwise, if $p_{ij}^{t,t+1}$ is dependent upon t , then the Markov chain is said to possess non-stationary transition probabilities and is called a non-stationary Markov chain.*

Intuitively, a Markov chain process is said to be stationary when it is invariant under an arbitrary shift of the time origin, but this does not mean that the transitions are not allowed to depend on the current situation, i.e., the evolution of the process may change over time but this evolution will be the same irrespective of when the process was initiated.
With this, we can extend the previous definitions to:

Definition 11.
- *the transition probability from i to j after n time steps: $p_{ij}^{m,m+n}$;*
- *probability that the first visit to j, starting from state i at time m, occurs at time t: $f_{ij}^{m,t}$.*

Thus obtaining the following theorem:

Theorem B.1.1.
- *$p_{ij}^{m,t} = \sum_{k \in S} p_{ik}^{m,r} p_{kj}^{r,t}$, where $0 \leq m < r < t$;*
- *$p_{ij}^{m,m+n}$ is the $(i,j)^{th}$ element of the n-step-transition matrix $P^{m,m+n}$, and*

$$P^{m,m+n} = P^{m,m+1} \cdot P^{m+1,m+2} \cdots P^{m+n-1,m+n}, \text{ for } n \geq 1.$$

Proof 1. *For the proof of the theorem see Bowerman (1974).*

It should be noted that if the Markov chain is homogeneous and it possesses stationary transition probabilities, then, for $n \geq 1$, $P^{m,m+n} = P \cdot P \cdots P = P^n$, and hence, since $p_{ij}^{m,m+n}$ and $f_{ij}^{m,m+n}$ are independent of m, we write them respectively p_{ij}^n and f_{ij}^n. Thus, henceforth, we will consider that the Markov chain is both stationary and homogeneous, unless we state otherwise.

Remark 39. *Note that we are only utilizing the operators commonly used in matrix addition and multiplication, thus we maintain its properties (for further details see Bowerman (1974) and Fette (2009)).*

B.2 Definitions and Results for Stationary Markov Chains

We now give some basic definitions and results for stationary Markov chains. Since a stationary Markov chain is described by a single transition matrix P, we will sometimes talk about the Markov chain by talking about P.

Definition 12. *State j is said to be accessible from state i if, for some $n \geq 1$, $p_{ij}^n > 0$. This is denoted by $i \to j$.*
Two states i and j are said to communicate, if they are accessible from each other. This is denoted by $i \leftrightarrow j$.

The relation defined by communication satisfies the following conditions:

- all states communicate with themselves, i.e., $p_{ii}^0 = 1$;
- if $i \leftrightarrow j$, then $j \leftrightarrow i$;
- if $i \leftrightarrow k$ and $k \leftrightarrow j$, then $k \leftrightarrow j$.

Thus, the above conditions imply that communication is an example of an equivalence relation (for further details see, for example, Kobayashi, Mark, and Turin (2012) and/or Stewart (2009)). With these definitions we can present the following (and useful) proposition:

Proposition B.2.1. *For each Markov chain, there exists a unique decomposition of the state space S into a sequence of disjoint subsets $C_1, C_2, \ldots,$ $S = \cup_{i=1}^{\infty} C_i$, in which each subset has* *the property that all states within it communicate. Each such subset is called a communication class of the Markov chain.*

Proof 2. *For the proof of the theorem see, for example, Kobayashi, Mark, and Turin (2012) and/or Stewart (2009)).*

From this proposition we can define an irreducible Markov chain:

Definition 13. *A Markov chain for which there is only one communication class is called an irreducible Markov chain.*

Moving forward, we will make the use of the following definitions:

Definition 14. *State i is said to have period d if the only n such that $p_{ii}^n \neq 0$ are multiples of d, and d is the greatest integer with this property.*

Definition 15. *Let $f_{ij}^* = \sum_{n=1}^{\infty} f_{ij}^n$ be the probability of ever visiting state j from state i.*
If $f_{ij}^* = 1$, we call the state i recurrent (or persistent). In this case, we define $\mu_{ii} = \sum_{n=1}^{\infty} n f_{ij}^n$, then state i is called positive recurrent if $\mu_{ii} < \infty$, and null recurrent if $\mu_{ii} = \infty$.
If $f_{ij}^* < 1$, we call the state i transient.

With this, we have the following theorem:

Theorem B.2.1. *If a Markov chain with transition matrix P is irreducible, then:*

- *All of its states are of the same type in that they have the same period and are all either positive recurrent, null recurrent, or transient.*
- *If $S = \{1, \ldots, n\}$, that is if S is finite (or countable), all of the sates are positive recurrent.*
- *If $S = \{1, \ldots, n\}$ and I is the $n \times n$ identity matrix, then the rank of $(P - I)$ is $(n - 1)$.*
- *If $S = \{1, \ldots, n\}$ and I is the $n \times n$ identity matrix, then the rank of the $(n-1) \times (n-1)$ submatrix of $(P - I)$, formed by deleting the n^{th} row and n^{th} column of $(P - I)$, is $n - 1$.*
- *if P is positive recurrent , then there exists an unique solution $\pi = (\pi_1, \pi_2, \ldots)$ to the system of equations:*

$$\begin{cases} \pi P = \pi \\ \sum_{i \in S} \pi_i = 1 \end{cases} \tag{B.1}$$

  where $\pi_i > 0, \forall i \in S$.

  Furthermore, if $S = \{1, \ldots, n\}$ the system of equations has an unique solution.

Proof 3. *For the proof of the theorem see Bowerman (1974).*

Before moving forward to an important theorem, let us give the following definitions:

Definition 16. *A Markov chain is called aperiodic if each of its states has period* 1.
Also, Markov chain which is both aperiodic and irreducible is called regular.

Definition 17. *The statement* $R_n \xrightarrow{n\to\infty} Q$ *means that as* $n \to \infty$ *the* $(i,j)^{th}$ *element of the matrix* R_n *converges in a pointwise (and not in necessarily a stronger) fashion to the* $(i,j)^{th}$ *element of the matrix* Q.

Thus arriving to the theorem:

Theorem B.2.2. *If a Markov chain is irreducible, positive recurrent and aperiodic, then:*

$$\lim_{n\to\infty} p_{ij}^n = \lim_{n\to\infty} = P(x_{m+n} = j | x_m = i) = \pi_j > 0$$

where $\pi = (\pi_1, \pi_2, \dots)$ *is the unique solution to the system of equations B.1.*
Hence:

$$P^n = \begin{pmatrix} p_{11}^n & p_{12}^n & \cdots & p_{1r}^n & \cdots \\ p_{21}^n & p_{22}^n & \cdots & p_{2r}^n & \cdots \\ \vdots & \vdots & & \vdots & \\ p_{m1}^n & p_{m2}^n & \cdots & p_{mr}^n & \cdots \\ \vdots & \vdots & & \vdots & \ddots \end{pmatrix} \xrightarrow{n\to\infty} \begin{pmatrix} \pi_1 & \pi_2 & \cdots & \pi_r & \cdots \\ \pi_1 & \pi_2 & \cdots & \pi_r & \cdots \\ \vdots & \vdots & & \vdots & \\ \pi_1 & \pi_2 & \cdots & \pi_r & \cdots \\ \vdots & \vdots & & \vdots & \ddots \end{pmatrix} = Q$$

where Q *is a matrix wherein each row is* π.

Proof 4. *For the proof of the theorem see Bowerman (1974).*

Intuitively, in the previous theorem, π_i is the long-run proportion of transitions that are made into state i, so we call $\pi = (\pi_1, \pi_2, \dots)$ the long-run (or limiting) probability distribution of the Markov chain [2] (and, to find it, we only need to solve the system of equations B.1). Also, consider a Markov chain $\{x_t, t = 0, 1, 2, \dots\}$ with a $N \times N$ transition matrix P, and let the probability distribution of x_0 be given by the $1 \times N$ vector π. Then the probability distribution of x_n is given by the $1 \times N$ vector πP^n. Thus the i^{th} element of πP^n represents the probability of x_n being on state i after t iterations.

Remark 40. *The distribution of* x_n *is* πP^n, *and, similarly, the distribution of* x_{n+m} *is* πP^{n+m}.

[2] In some literature, π is called the stationary distribution of the Markov chain

Note that there are several important theorems regarding periodic Markov chains, but, since they are not relevant for our purposes, we will not present them (for further details see Bowerman (1974)).

B.3 Absorbing Markov chains

For our purposes, it is very important to know what is and what properties do Absorbing Markov chains have. For this we will need the following definitions:

Definition 18. *A state i is said to be absorbing if:*

- $p_{ij} = 0, \forall j \neq i$;
- $p_{ii} = 1$.

Definition 19. *A Markov chain is said to be an absorbing Markov chain if:*

- *It has at least one absorbing state.*
- *From every state in the Markov chain there exists a sequence of state transitions with nonzero probability that lead to an absorbing state. These non-absorbing states are called transient states.*

Remark 41. *Note that the prior definition 15 of transient state is equivalent to the previously given definition 19. Also, an absorbing state must be recurrent, while a recurrent state is not necessarily absorbing (for further details see Bowerman (1974) and Fette (2009)).*

Theorem B.3.1. *In a Markov chain with transition matrix P,a state is absorbing if and only if $p_{ii} = 1$.*

Proof 5. *For the proof of the theorem see Bowerman (1974) and Fette (2009).*

Intuitively, an absorbing state is a fixed point or steady state that, once reached, the system never leaves. Similarly, valuable convergence insights can also be gained when the system can be modeled as an absorbing Markov chain.

Definition 20. *Let an absorbing Markov chain with transition matrix P have k transient states* *and r absorbing states. Then the canonical form of P is given by*

$$P = \begin{pmatrix} Q & R \\ \mathbf{0} & I_r \end{pmatrix}, \tag{B.2}$$

where

- *Q is a $k \times k$ matrix, representing the state transitions between the non-absorbing states of the chain;*
- *R is a nonzero $k \times r$ matrix, representing the matrix of state transition probabilities from non-absorbing states to absorbing states;*
- *I_r is the $r \times r$ identity matrix, corresponding to the state transitions between the absorbing states of the chain;*
- *$\mathbf{0}$ is an $r \times k$ zero matrix, representing the probability of transition from absorbing states to non-absorbing states.*

Definition 21. *Given an absorbing chain with a modified transition matrix, as in Equation B.2:*

- *the fundamental matrix is given by*

$$N = \sum_{n=0}^{\infty} Q^n = (I_r - Q)^{-1}; \tag{B.3}$$

- *the matrix of the expected number of steps is given by*

$$t = N\mathbf{1}; \tag{B.4}$$

- *the absorbing matrix is given by*

$$B = NR. \tag{B.5}$$

Where I_r, Q and R are as before, and $\mathbf{1}$ is a column vector of all ones,.

Remark 42. *Consider, for simplicity, that: n_{ij} corresponds to $(i,j)^{th}$ entry of the N matrix, t_i corresponds to $(i,1)^{th}$ entry of the t matrix, and b_{ij} corresponds to $(i,j)^{th}$ entry of the B matrix.*

So, given P, Markov theory provides us with information on convergence and the expected frequency that the system visits a transitory state.

Theorem B.3.2.

- $\lim_{n\to\infty} Q^n = 0$, *which implies that the probability of the system not terminating in one of the absorbing states of the chain goes to zero;*
- n_{km} *gives the expected number of times that the system will pass through state* m *given that the system starts in state* k*;*
- t_k *gives the expected number of iterations before the state is absorbed when the system starts in state* k*;*
- b_{km} *specifies the probability the system ends up in absorbing state* m *if the system starts in state* k.

Proof 6. *For the proof of the theorem see Kemeny and Shell (1960) and Fette (2009).*

Thus, once we obtain an absorbing Markov chain with transition matrix P, the following insights are readily gained:

- Steady states for the system can be identified by finding those states m for which $p_{mm} = 1$.
- Convergence to one of these steady states is assured, and the expected distribution of states can be found by solving for B.
- Given an initial state, m, convergence rate information is given by solving for t.

Note that, other properties can be derived from these matrices, but, since they are not necessary for our purposes, we will not present them (for more details see Fette (2009) and Kemeny and Shell (1960)).

To finalize this section we present a diagram of a Markov chain which includes all of the previously defined states:

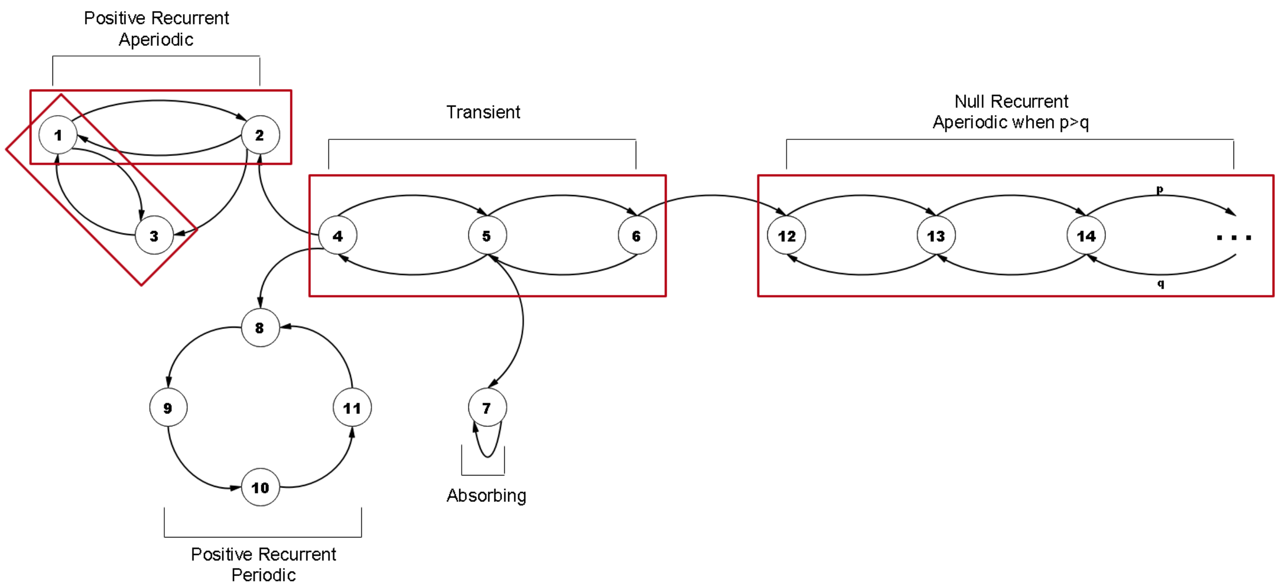

Figure B.2: An example of a Markov chain with various states (from Kobayashi, Mark, and Turin (2012)).

Note that, in the previous diagram, the Markov chain is not irreducible, however the boxes around the states represent the possible communicating classes.

B.4 Simulation and Estimation of Markov Chains

There are several ways to estimate and simulate Markov chain transition probabilities, and these may vary depending on the software that we are using. But the most basic methods involve Monte Carlo Simulation (Fismen (1997), Welton and Ades (2005)), Gibbs Sampling (Fismen (1997), Welton and Ades (2005)), Metropolis-Hastings Algorithm (Fismen (1997), Welton and Ades (2005)), always with the aid of Classical and Bayesian Statistics (Welton and Ades (2005)).

Since we are going to utilize the *R* software, we will briefly present its methods (and packages) regarding Markov chains:

- the **msm** package (Jackson (2011))handles Multi-State Models for panel data.
- the **mcmcR** package (Geyer and Johnson (2013)) implements Monte Carlo Markov Chain approach.
- the **hmm** package (Himmelmann and www.linhi.com (2010)) fits hidden Markov models

with covariates.

- the **mstate** package (Wreede, Fiocco, and Putter (2011)) fits Multi-State Models based on Markov chains for survival analysis.

Also, the most complete package for discrete Markov chains is the **markovchain** package (Spedicato et al. (2017)) which gives more flexibility in handling discrete Markov chains than other existing solutions, providing methods for both homogeneous and non-homogeneous Markov chains, as well as methods suited to perform statistical and probabilistic analysis. Furthermore, this package uses the presented theory as its foundation and the applied methods (and algorithms) are "converted" from the *MATLAB* software.

Appendix C

Game Theory

This chapter is only a brief review of the necessary theory (relying mainly on Gibbons (1992)). However the field of Game Theory is much richer than what we will present (for further insights on this vast field see, for example, Gibbons (1992)).
To this end, we will explore the relevant theory and then apply it to some problems, by formulating them as game theoretical models. And we will restrict our attention to simultaneous-move games, where the players simultaneously choose their actions at the start, and then they receive payoffs that depend on the combination of the chosen actions. Also, within this class of games, we restrict attention to games of complete information, that is, each player's payoff function (the function that determines the player's payoff from the combination of actions chosen by the players) is common knowledge among all players.

C.1 Basic Concepts

In the normal-form representation of a game, each player simultaneously chooses a strategy, and the combination of the chosen strategies determines the payoff for each player. A game can be represented using the following normal-form representation:

- the players in the game;
- the strategies available to each player;
- the payoff received by each player for each combination of strategies that could be chosen by the players.

As an example of the normal-form representation, consider the classic game known as the Prisoners' Dilemma:

- two suspects are arrested and charged with a crime;
- the police lack sufficient evidence to convict the suspects, unless at least one confesses;
- the police hold the suspects in separate cells and explain the consequences that will follow from the actions they could take;
- if neither confesses, then both will be convicted of a minor offense and sentenced to one month in jail;
- if both confess, then both will sentenced to jail for six months;
- if one confesses but the other does not, then the confessor will be released immediately, but the other will be sentenced to nine months in jail.

This problem can be represented by a payoff bi-matrix (i.e., a matrix that can have an arbitrary number of rows or columns, where each of its cells has two numbers representing the payoffs of the two players:

		Prisoner 2	
		Confess	Not Confess
Prisoner 1	Confess	-6,-6	0,-9
	Not Confess	-9,0	-1,-1

Table C.1: Prisoners' Dilemma

Remark 43. *Often in the literature, the game's payoff bi-matrix can be referred to as the game table, so we will use these terms interchangeably.*

In this game, each player has two strategies available: Confess and Not Confess. The payoffs of the two players when a particular pair of strategies is chosen are given in the appropriate cell of the previous bi-matrix. And, by convention, the payoff to the so-called row player (in this case, Prisoner 1) is the first payoff given, followed by the payoff to the column player (here, Prisoner 2). Thus, for example, if Prisoner 1 chooses Confess and Prisoner 2 chooses Not Confess, then Prisoner 1 receives the payoff 0 (representing immediate release) and Prisoner 2 receives the payoff -9 (representing nine months in jail).

Generally, in a n-player game, in which the players are numbered form 1 to n, an arbitrary player is called player i. Also, we will use the following notations:

- S_i denotes the set of strategies available to player i (often called i's strategy space);
- s_i denotes an arbitrary member of S_i, and we write $s_i \in S_i$ to indicate that the strategy s_i is a member of the set of strategies S_i;
- $(s_1, \ldots, s_n)$ denotes a combination of strategies, one for each player;
- u_i denotes player i's payoff function, i.e., $u_i(s_1, \ldots, s_n)$ is the payoff to player i if the players choose the strategies $(s_1, \ldots, s_n)$.

Remark 44. *The strategy space of all the other players except player i is denoted by $S_{-i} = S_1, \ldots, S_{i-1}, S_{i+1}, \ldots, S_n$, and $s_{-i} \in S_{-i}$ is strategy from this space.*

Collecting all of this information together, we have:

Definition 22. *The normal-form representation of an n-player game specifies the players' strategy spaces $S_1, \ldots, S_n$ and their payoff functions $u_1, \ldots, u_n$. We denote this game by $G = \{S_1, \ldots, S_n; u_1, \ldots, u_m\}$.*

Note that, in a normal-form game, the players choose their strategies simultaneously, this does not imply that the parties necessarily act simultaneously, because it suffices that each player chooses his/hers action without the knowledge of the others' choices, as would be the case if the prisoners reached decisions at arbitrary times while in their separate cells.

Furthermore, although the normal-form representation is usually used for static games (in which the players all move without knowing the other players' choices), it is also possible to use it to represent sequential-move games (for further details see Gibbons (1992)).

Having described one way to represent a game, we need to describe how to solve a game theoretical problem. For this, consider the Prisoners' Dilemma, with the idea that all the players are rational[1]. Thus, in the Prisoners' Dilemma:

- if one suspect is going to play Confess, then the other would prefer to play Confess and so be in jail for six months, instead of playing Not Confess and so be in jail for nine months;
- if one suspect is going to play Not Confess, then the other would prefer to play Confess and so be released immediately, rather than play Not Confess and so be in jail for one

[1] We will briefly address the concept of rationality later on, but for further details see Aumann (1976) and/or Gibbons (1992)

month.

So, for the player i, playing Not Confess results in a better payoff then playing Confess, because, for each strategy that prisoner j could choose, the payoff to prisoner i from playing Not Confess is less than the payoff to i from playing Confess. Generally, we have the following definition:

Definition 23. *In the normal-form game $G = \{S_1, \ldots, S_n; u_1, \ldots, u_n\}$, let s_i' and s_i'' be feasible strategies for player i (i.e., s_i' and s_i'' are members if S_i). So, strategy s_i' is strictly dominated by strategy s_i'' if, for each feasible combination of the other players' strategies, i's payoff from playing s_i' is strictly less then is's payoff from playing s_i'':*

$$u_i(s_1, \ldots, s_{i-1}, s_i', s_{i+1}, \ldots, s_n) < u_i(s_1, \ldots, s_{i-1}, s_i'', s_{i+1}, \ldots, s_n),$$

for each $(s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n)$ that can be constructed from the other players' strategy spaces $S_1, \ldots, S_{i-1}, S_{i+1}, \ldots, S_n$.

Rational players do not play strictly dominated strategies, because there is no belief that a player could hold (about the strategies the other players will choose) such that it would be optimal to play such a strategy. And, in the Prisoners' Dilemma case, a rational player will choose Confess, so (Confess,Confess) will be the outcome reached by two rational players, even though (Confess,Confess) results in worse payoffs for both players than would (Not Confess,Not Confess).

For now, consider the idea that rational players not playing strictly dominated strategies leads to the solution of other games. And, lets analyze the following abstract game:

Player 2

	\	Left	Middle	Right
Player 1	Up	1,0	1,2	0,1
	Down	0,3	0,1	2,0

Table C.2: Abstract Game

Here:

- Player 1 has two strategies and Player 2 has three strategies: $S_1 = \{\text{Up, Down}\}$ and $S_2 = \{\text{Left, Middle, Right}\}$.

- For Player 1, neither Up nor Down is strictly dominated.
- For Player 2, Right is strictly dominated by Middle, so a rational Player 2 will not play right.
- Thus, if Player 2 knows that Player 2 is rational, then Player 1 can eliminate Right from Player 2's strategy space. So we can eliminate the column associated with playing Right from the matrix C.2, and thus obtaining a "smaller" game.
- Now, Down is strictly dominated by Up for Player 1, so if Player 1 is rational (and Player 1 knows that Player 2 is rational) then Player 1 will not play Down.
- So, if Player 2 knows that Player 1 is rational, and Player 2 knows that Player 1 knows that Player 2 is rational, then Player 2 can eliminate Down from Player 1's strategy space. And thus reducing the game once more, by eliminating the row associated with playing Down.
- Now, Left is strictly dominated by Middle for Player 2, leaving (Up,Middle) as the outcome of the game.

This process is called Iterated Elimination of Strictly Dominated Strategies (IESDS). Although, it is based on the idea that rational players do not play strictly dominated strategies, the process has two drawbacks.

Firstly, in each step requires a further assumption about what the players know about each others' rationality. So if we want to be able to apply the process for an arbitrary number of steps, we need to assume that it is common knowledge that the players are rational, that is, we need to assume not only that all players are rational, but also that all the players know that all the players are rational, and that all the players know that all the players know that all the players are rational, and so on, *ad infinitum* (see Aumann (1976) for further details).

The second drawback of IESDS is that the process often produces a very imprecise prediction about the outcome of the game. For instance, if in a game there are no strictly dominated strategies to be eliminated, then all the strategies in the game survive IESDS, so the process produces no prediction whatsoever about the outcome of the game. Thus, we need a solution concept that produces much tighter predictions in a very broad class of games, i.e., the Nash Equilibrium.

C.2 Motivation and Definition of Nash Equilibrium

Suppose that game theory makes an unique prediction about the strategy each player will choose. So, in order for this prediction to be correct, it is necessary that each player is willing to choose the strategy predicted by the theory. Thus, each players' predicted strategy must be that player's best response to the predicted strategies of the other players. Such prediction could be called strategically stable, because no single player wants to deviate from his/hers predicted strategy, and such prediction is called a Nash Equilibrium:

Definition 24. *In the n-player normal-form game $G = \{S_1, \ldots, S_n; u_1 \ldots, u_n\}$, the strategies $(s_1^*, \ldots, s_n^*)$ are a Nash Equilibrium if, for each player i, s_i^* is (at least tied for) player i's best response to the strategies specified for the $n-1$ other players, $(s_1^*, \ldots, s_{i-1}^*, s_i^*, s_{i+1}^*, \ldots, s_n^*)$:*

$$u_i(s_1^*, \ldots, s_{i-1}^*, s_i^*, s_{i+1}^*, \ldots, s_n^*) \geq u_i(s_1^*, \ldots, s_{i-1}^*, s_i, s_{i+1}^*, \ldots, s_n^*) \qquad \text{(NE)}$$

for every feasible strategy $s_i \in S_i$, in other words, s_i^* solves

$$\max_{s_i \in S_i} u_i(s_1^*, \ldots, s_{i-1}^*, s_i, s_{i+1}^*, \ldots, s_n^*).$$

Another way to motivate the definition of Nash equilibrium is to argue that if game theory is to provide an unique solution to a game theoretical problem then the solution must be a Nash equilibrium (for further details see Gibbons (1992)).

Again, consider the Prisoners' Dilemma. A brute force approach to finding a game's Nash equilibria is simply to check whether each possible combination of strategies satisfies condition NE in Definition 24. So, in a two-player game, this approach begins as follows:

- For each player, and for each feasible strategy for that player, determine the other player's best response to that strategy. We can do this by highlighting the payoff to player j's best response to each of player i's feasible strategies in the cells of the bi-matrix.
- A pair of strategies satisfies condition NE if each player's strategy is a best response to the other's. In other words, if both payoffs are highlighted in the corresponding cell of the bi-matrix.

So, in the Prisoners' Dilemma (Confess, Confess) is the only strategy pair that satisfies NE, and (Up, Middle) is also the the only strategy pair that satisfies NE in the abstract game C.2. Now, we need to explore the relationship between Nash equilibrium and the IESDS process.

Proposition C.2.1. *If IESDS eliminates all but the strategies* $(s_1^*, \ldots, s_n^*)$*, then these strategies are the unique Nash Equilibrium of the game.*

Proof 7. *For the proof of the proposition see Gibbons (1992).*

However, the IESDS process frequently does not eliminate all but a single combination of strategies, on the other hand we have the following:

Proposition C.2.2. *If the strategies* $(s_1^*, \ldots, s_n^*)$ *are a Nash equilibrium then they survive IESDS, but there can be strategies that survive IESDS which are not part of any Nash equilibrium.*

Proof 8. *For the proof of the proposition see Gibbons (1992).*

So, Nash equilibrium is a stronger solution concept than IESDS, but we must verify if the Nash equilibrium is too strong a solution concept. That is, we need to be sure that a Nash equilibrium exists. And Nash (1950) proved that in any finite game (i.e., a game in which the number of players n and the strategy sets $S_1, \ldots, S_n$ are all finite) there exists at least one Nash equilibrium.

Remark 45. *Note that, the equilibrium in Nash's theorem may involve mixed strategies, which we will address in Section C.3, where we will also formally state Nash's theorem (Theorem C.3.1).*

Lets now consider another classic example, The Battle of the Sexes. This example shows that a game can have multiple Nash equilibria, thus it will be useful in the discussions of the mixed strategies in Section C.3.

Consider that, while at separate workplaces, Chris and Pat must choose to attend either the Opera or a (prize) Fight. And both players would rather spend the time together than apart, but Pat would prefer they be together at the Fight, while Chris would prefer they be together at the Opera. Thus we obtain the following bi-matrix of the game:

Pat

Chris	\	Opera	Fight
	Opera	2,1	0,0
	Fight	0,0	1,2

Table C.3: The Battle of the Sexes

From which we can see that both (Opera, Opera) and (Fight, Fight) are Nash equilibria. However, we previously argued that if game theory is to provide an unique solution to a game then the solution must be a Nash equilibrium. However, this argument ignores the possibility of games in which game theory does not provide an unique solution. And, if a convention is to develop about how to play a given game, then the strategies prescribed by the convention must be a Nash equilibrium, but this argument also ignores the possibility of games for which a convention will not develop. So, we need to expand our theory in an effort to identify such a compelling game equilibrium in different classes.

Thus, the existence of multiple Nash equilibria is not a problem in and on itself. In the Battle of the Sexes, however, (Opera, Opera) and (Fight, Fight) seem equally compelling, which suggests that there may be games for which game theory does not provide an unique solution and no convention will develop [2]. So, in such games, the concept of Nash equilibrium loses much of its appeal as a prediction of the game's outcome.

To conclude this section, we will mention some applications of the presented theory, however we will not discuss them, since it is not the purpose of this paper. Among the several practical applications of this theory, the following are the better known:

- *Cournot Model of Duopoly*: which is one of the cornerstones of the theory of industrial organization (for further details see Cournot (1897) and Gibbons (1992));
- *Bertrand Model of Duopoly*: which is a different model (from the previous one) of how two duopolists might interact (for further details see Bertrand (1883) and Gibbons (1992));
- *Final-Offer Arbitration*: which is a very used way of settling many disputes (for further details see Farber (1980) and Gibbons (1992));

[2] In Section C.3 we describe a third Nash equilibrium of the Battle of the Sexes involving mixed strategies.

- *The Problem of the Commons*: which addresses the problem that if citizens respond only to private incentives, public goods will be under-provided and public resources over-utilized (for further details see Hume (1739), Hardin (1968) and Gibbons (1992)).

C.3 Mixed Strategies

In Section C.1, we defined S_i to be the set of strategies available to player i, and the combination of strategies $(s_1^*, \ldots, s_n^*)$ to be a Nash equilibrium if, for each play i, s_i^* is player i's best response to the strategies of the $n-1$ other player, i.e.,

$$u_i(s_1^*, \ldots, s_{i-1}^*, s_i^*, s_{i+1}^*, \ldots, s_n^*) \geq u_i(s_1^*, \ldots, s_{i-1}^*, s_i, s_{i+1}^*, \ldots, s_n^*) \tag{NE}$$

for every feasible strategy $s_i \in S_i$, which remains the same as in Definition 24.

So, consider the game, known as Matching Pennies, where we have two player, each with the strategy space {Heads, Tails} and the payoffs are represented in the following bi-matrix:

		Player 2	
	\	Heads	Tails
Player 1	Heads	-1,1	1,-1
	Tails	1,-1	-1,1

Table C.4: Matching Pennies

By definition NE, there is no Nash equilibrium in this game, because no pair of strategies can satisfy NE:

- if the players' strategies match (i.e., if we have (Heads, Heads) or (Tails, Tails)), then Player 1 prefers to switch strategies;
- if the players' strategies do not match (i.e., if we have (Heads, Tails) or (Tails, Heads)), then Player 2 prefers to switch strategies.

Thus, the distinguishing feature of Matching Pennies is that each player would like to outguess the other. **Remark 46.** *Note that, versions of this game also arise in poker, baseball, battle, and other settings (for further details see Gibbons (1992)).*

But, in any kind of game in which each player would like to outguess the others, there is no Nash equilibrium (as it was defined in Section C.1), because the solution to such a game necessarily involves uncertainty about what the players will do.

To address this, we introduce the notion of mixed strategy, which we will interpret in terms of one player's uncertainty about what the other will do (for further details about this interpretation see Harsanyi (1973) and Gibbons (1992)). So, we will extend the definition of Nash equilibrium to include mixed strategies, thereby capturing the uncertainty inherent in the solution to such games as Matching Pennies.

Definition 25. *A mixed strategy for player i is a probability distribution over (some or all of) the strategies in S_i.*

In all of the games analyzed (in this section and on the previous ones), a player's pure strategies are all the different actions actions the player can take. Thus, hereafter, the strategies in S_i will be referred to as player i's pure strategies.

For example, in the Matching Pennies game, S_i consists of the two pure strategies Heads and Tails, and a mixed strategy for player i is the probability distribution $(q, 1-q)$, where q is the probability of playing Heads, $1-q$ is the probability of the player playing Tails, where $0 \leq q \leq 1$. So, the mixed strategy $(0, 1)$ is simply the pure strategy Tails, while the mixed strategy $(1, 0)$ corresponds to the pure strategy Heads.

To generalize, suppose that player i has K pure strategies, $S_i = \{s_{i1}, \ldots, s_{iK}\}$, then a mixed strategy for player i is a probability distribution $(p_{i1}, \ldots, p_{iK})$, where p_{ik} is the probability that player i will play strategy s_{ik}, for $k = 1, \ldots, K$. Since p_{ik} is a probability, we require $0 \leq p_{ik} \leq 1$, for $k = 1, \ldots, K$, and $p_{i1} + \cdots + p_{iK} = 1$. Also, we will use p_i to denote an arbitrary mixed strategy from the set of probability distributions over S_i, just as we use s_i to denote an arbitrary pure strategy from S_i.

Definition 26. *In the normal-form game $G = \{S_1, \ldots, S_n; u_1, \ldots, u_n\}$, suppose $S_i = \{s_{i1}, \ldots, s_{iK}\}$. Then a mixed strategy for player i is a probability distribution $p_i = (p_{i1}, \ldots, p_{iK})$, where $0 \leq p_{ik} \leq 1$, for $k = 1, \ldots, K$ and $p_{i1} + \cdots + p_{ik} = 1$.*

Now that we have the formal definition of a mixed strategy, we can use these probabilities to calculate expected payoffs for uncertain outcomes. Consider a two-player game with the following payoff bi-matrix:

	Player 2	
$\backslash$	s_{21}	s_{22}
Player 1 s_{11}	u_{11}, u_{21}	u_{13}, u_{23}
s_{12}	u_{12}, u_{22}	u_{14}, u_{24}

Table C.5: Abstract Game.

Here Player 1's mixed strategy is $p = (p_1, p_2)$ (which means that p_1 is the probability that 1 chooses s_{11} and p_2 is the probability that 1 chooses s_{12}) and, similarly, Player 2's mixed strategy is $q = (q_1, q_2)$. So, given all of this, for example, Player 1's expected payoffs from playing the strategies s_{11} and s_{12} are given by, respectively:

$$E_1(s_{11}) = q_1 u_{11} + q_2 u_{13}, \tag{C.1}$$

$$E_1(s_{12}) = q_1 u_{12} + q_2 u_{14}. \tag{C.2}$$

Thus, the expected value of Player 1's payoff is $E_1 = p_1 E_1(s_{11}) + p_2 E_1(s_{12})$. Note that we can determine the expected payoff via matrix multiplication, i.e., if we define Player 1's payoff matrix as A_1 and Player 2's payoff matrix as A_2, then the expected value of Player 1's payoff is given by

$$E_1 = \begin{pmatrix} p_1 & p_2 \end{pmatrix} \begin{pmatrix} E_1(s_{11}) \\ E_1(s_{12}) \end{pmatrix} = \begin{pmatrix} p_1 & p_2 \end{pmatrix} \begin{pmatrix} u_{11} & u_{13} \\ u_{12} & u_{14} \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = pA_1q.$$

Thus, generally, we can define the expected payoff for a two-player game as:

Definition 27. *In the normal-form two-player game $G = \{S_1, S_2; u_1, u_2\}$, suppose $S_i = \{s_{i1}, \ldots, s_{iK}\}$ and that player i's mixed strategy is $p_i = (p_{i1}, \ldots, p_{iK})$ (where $0 \leq p_{ik} \leq 1$, for $k = 1, \ldots, K$ and $p_{i1} + \cdots + p_{ik} = 1$). Also, let A_i denote player i's payoff matrix. Then player i's expected payoff is given by:*

$$E_i = \begin{pmatrix} p_{i1} & \cdots p_{iK} \end{pmatrix} \begin{pmatrix} E_i(s_{i1}) \\ \vdots \\ E_i(s_{iK}) \end{pmatrix} = \begin{pmatrix} p_1 & \cdots p_{iK} \end{pmatrix} A_i \begin{pmatrix} p_{j1} \\ \vdots \\ p_{jK} \end{pmatrix} = p_i A_i p_j, \text{ where } j \neq i.$$

Remark 47. *We only gave the formal definition for a two-player game, because the definition for a n-player game is more complicated and it is not necessary for our purposes. But for further details see Gibbons (1992) and/or Fudenberg and Tirole (1991).*

Moving further, recall, from Section C.1, that if a strategy s_i is strictly dominated then there is no belief that player i could hold (about the strategies the other players will choose) such that it would be optimal to play s_i. The converse is also true, provided we allow for mixed strategies:

Proposition C.3.1. *If there is no belief that player i could hold (about the strategies the other players will choose) such that it would be optimal to play strategy s_i, then there exists another strategy that strictly dominates s_i.*

Proof 9. *For the proof of the proposition see Pearce (1984) and Gibbons (1992).*

Consider the game represented by the following bi-matrix:

Player 2

	\	L	R
Player 1	T	3,__	0,__
	M	0,__	3,__
	B	1,__	1,__

Table C.6: Abstract Game

This game shows that the converse of Proposition C.3.1 would be false if we restricted attention to pure strategies. Furthermore, this game shows that a given pure strategy may be strictly dominated by a mixed strategy, even if the pure strategy is not strictly dominated by any other pure strategy. In this game, for any belief $(q, 1-q)$ that Player 1 could hold about 2's play, 1's best response is either T (if $q \geq 1/2$) or M (if $q \leq 1/2$), but never B. Yet B is not strictly dominated by either T or M. The key is that B is strictly dominated by a mixed strategy: if Player 1 plays T with probability 1/2 and M with probability 1/2, then 1's expected payoff is 3/2 no matter what (pure or mixed) strategy 2 plays, and 3/2 exceeds the payoff of 1 that playing B surely produces. Thus, this example illustrates the role of mixed strategies in finding another strategy that strictly dominates s_i.

Now we need to discuss the existence of Nash equilibrium, by exploring the following:

- extend the given definition of Nash equilibrium to allow for mixed strategies;

- apply this definition to Matching Pennies;
- use of a graphical argument to show that any two-player game in which each players has two pure strategies has a Nash equilibrium (possibly involving mixed strategies);
- formally state and discuss Nash's Theorem C.3.1, which guarantees that any finite game has a Nash equilibrium (again, possibly involving mixed strategies).

Recall that the definition of Nash equilibrium given in Section C.2 guarantees that each player's pure strategy is a best response to the other players' pure strategies. To extend this definition to include mixed strategies, we simply require that each player's mixed strategy be a best response to the other players' mixed strategies. And, since any pure strategy can be represented as the mixed strategy that puts zero on all of the player's other pure strategies, this extended definition subsumes the earlier one.

Now, we will use Matching Pennies as an example (with the previously presented bi-matrix C.4). Suppose that Player 1 believes that Player 2 will play Heads with probability q and Tails with probability $1-q$. In other words, Player 1 believes that 2 will play the mixed strategy $(q, 1-q)$. Given this belief, Player 1's expected payoffs are $q \cdot (-1) + (1-q) \cdot 1 = 1 - 2q$ from playing Heads and $q \cdot 1 + (1-q) \cdot (-1) = 2q - 1$ from playing Tails. Since $1 - 2q > 2q - 1$ if and only if $q < 1/2$, Player 1's best pure-strategy response is Heads if $q < 1/2$ and Tails if $q > 1/2$, and Player 1 is indifferent between Heads and Tails of $q = 1/2$. So it remains to consider possible mixed-strategy responses by Player 1.

Let $(r, 1-r)$ denote the mixed strategy in which Player 1 plays Heads with probability r. For each value of q between zero and one, we now compute the values of r, denote $r^*(q)$, such that $(r, 1-r)$ is a best response for Player 1 to $(q, 1-q)$ by Player 2. Thus, Player 1's expected payoff from playing $(r, 1-r)$ when 2 plays $(q, 1-q)$ is

$$r \cdot q \cdot (-1) + r \cdot (1-q) \cdot 1 + (1-r) \cdot q \cdot 1 + (1-r) \cdot (1-q) \cdot (-1) = (2q-1) + r \cdot (2-4q), \quad \text{(C.3)}$$

where $r \cdot q$ is the probability of (Heads, Heads), $r \cdot (1-q)$ the probability of (Heads, Tails), and so on.

Remark 48. *The events A and B are independent if $P(A|B) = P(A) \cdot P(B)$. Thus, in writing $r \cdot q$ for the probability that 1 plays Heads and 2 plays Heads, we are assuming that 1 and 2 make their choices independently, as befits the description we gave of simultaneous-move* *games (for further details see Gibbons (1992) and Aumann (1974)).*

Since Player 1's expected payoff is increasing in r if $2 - 4q > 0$ and decreasing in r if $2 - 4q < 0$, Player 1's best response is $r = 1$ (i.e., Heads) if $q < 1/2$, and $r = 0$ (i.e., Tails) if $q > 1/2$. Which makes this statement stronger than the closely related statement given previously, where we considered only pure strategies, and found that if $q < 1/2$, then Heads is the best pure strategy, and if $q > 1/2$ then Tails is the best pure strategy. Here we considered all pure and mixed strategies, but again find that if $q < 1/2$, then Heads is the best of all (pure or mixed) strategies, and if $q > 1/2$, then Tails is the best of all strategies.

The nature of Player 1's best response to $(q, 1-q)$ changes when $q = 1/2$. Also, as noted earlier, when $q = 1/2$ Player 1 is indifferent between the pure strategies Heads and Tails. Furthermore, because Player 1's expected payoff in Equation C.3 is independent of r when $q = 1/2$, Player 1 is also indifferent among all mixed strategies $(r, 1 - r)$. That is, when $q = 1/2$ the mixed strategy $(r, 1-r)$ is a best response to $(q, 1-q)$ for nay value of r between zero and one. Thus, $r^*(1/2)$ is the entire interval $[0, 1]$.

Now, to generally derive player i's best response to player j's mixed strategy, and to give a formal statement of the extended definition of Nash equilibrium, we now restrict attention to the two player case, which captures the main ideas as simply as possible.

Consider a two-player game $G = \{S_1, S_2; u_1, u_2\}$, let J denote the number of pure strategies in S_1 and K the number of strategies in S_2. For simplicity, we will write $S_1 = \{s_{11}, \ldots, s_{1J}\}$ and $S_1 = \{s_{21}, \ldots, s_{2K}\}$, and we will use s_{1j} and s_{2k} to denote arbitrary pure strategies from S_1 and S_2, respectively.

If Player 1 believes that Player 2 will play the strategies $(s_{21}, \ldots, s_{2K})$ with probabilities $(p_{21}, \ldots, p_{2K})$, then Player 1's expected payoff from playing the pure strategy s_{1j} is

$$\sum_{k=1}^{K} p_{2k} u_1(s_{1j}, s_{2k}), \tag{C.4}$$

and Player 1's expected payoff from playing the mixed strategy $p_1 = (p_{11}, \ldots, p_{1J})$ is

$$v_1(p_1, p_2) = \sum_{j=1}^{J} p_{1j} \left[\sum_{k=1}^{K} p_{2k} u_1(s_{1j}, s_{2k}) \right] \tag{C.5}$$

$$= \sum_{j=1}^{J} \sum_{k=1}^{K} p_{1j} p_{2k} u_1(s_{1j}, s_{2k}) \tag{C.6}$$

where $p_{1j}p_{2k}$ is the probability that 1 plays s_{1j} and 2 plays s_{2k}.

So, Player 1's expected payoff from the mixed strategy p_1, given in Equation C.5, is the weighted sum of the expected payoff for each of the pure strategies $\{s_{11}, \dots, s_{1J}\}$, given in Equation C.4, where the weights are the probabilities $(p_{11}, \dots, p_{1J})$. Thus, for the mixed strategy $(p_{11}, \dots, p_{1J})$ to be a best response for Player 1 to Player 2's mixed strategy p_2, it must be that $p_{1j} > 0$ only if

$$\sum_{k=1}^{K} p_{2k} u_1(s_{1j'}, s_{2k})$$

for every $s_{1j'}$ in S_1. That is, for a mixed strategy to be a best response to p_2, it must put positive probability on a given pure strategy only if the pure strategy is itself a best response to p_2. Conversely, if Player 1 has several pure strategies that are best response to p_2, then any mixed strategy that puts all its probability on some or all of these pure-strategy best responses (and zero probability on all other pure strategies) is also a best response for Player 1 to p_2.

Now, before giving a formal statement of the extended definition of Nash equilibrium, we need to compute Player 2's expected payoff when players 1 and 2 play the mixed strategies p_1 and p_2, respectively.

Thus, if Player 2 believes that Player 1 will play the strategies $(s_{11}, \dots, s_{1J})$ with probabilities $(p_{11}, \dots, p_{1J})$, then Player 2's expected payoff from playing the strategies $(s_{21}, \dots, s_{2K})$ with probabilities $(p_{21}, \dots, p_{2K})$ is

$$v_2(p_1, p_2) = \sum_{k=1}^{K} p_{2k} \left[\sum_{j=1}^{J} p_{1j} u_2(s_{1j}, s_{2k}) \right] \tag{C.7}$$

$$= \sum_{k=1}^{K} \sum_{j=1}^{J} p_{1j} p_{2k} u_2(s_{1j}, s_{2k}). \tag{C.8}$$

So, given $v_1(p_1, p_2)$ and $v_2(p_1, p_2)$, we can restate the requirement of Nash equilibrium that each player's mixed strategy be a best response to the other player's mixed strategy: for the pair of mixed strategies (p_1^*, p_2^*) to be a Nash equilibrium, p_1^* must satisfy

$$v_1(p_1^*, p_2^*) \geq v_1(p_1, p_2^*) \tag{C.9}$$

for every probability distribution p_1 over S_1, and p_2^* must satisfy

$$v_2(p_1^*, p_2^*) \geq v_2(p_1^*, p_2) \tag{C.10}$$

for every probability distribution p_2 over S_2.

All of this results in the following definition:

Definition 28. *In the two-player game $G = \{S_1, S_2; u_1, u_2\}$, the mixed strategies (p_1^*, p_2^*) are a Nash equilibrium if each player's mixed strategy is a best response to the other player's mixed strategy: Equations C.9 and C.10 must hold.*

We next apply this definition to Matching Pennies. To do so, we compute the values of q, denoted $q^*(r)$, such that $(q, 1-q)$ is a best response for Player 2 to $(r, 1-r)$ by Player 1, obtaining that:

- if $r < 1/2$ then 2's best response is Tails, so $q^*(r) = 0$;
- if $r > 1/2$ then 2's best response is Heads, so $q^*(r) = 1$;
- if $r = 1/2$ then 2 is indifferent not only between Heads and Tails but also among all mixed strategies $(q, 1-q)$, so $q^*(1/2)$ is the entire interval $[0, 1]$.

All of this can be done graphically, by plotting the intervals for the mixed strategies and the best response functions $q^*(r)$ and $r^*(q)$ of each player. And the intersections of the best response functions yields the same (mixed-strategy) Nash equilibrium that we obtained for the Matching Pennies: if player i plays $(1/2, 1/2)$ then $(1/2, 1/2)$ is a best response for player j, as required for the Nash equilibrium (for further details see Gibbons (1992)).

Also, note that, such mixed-strategy Nash equilibrium does not rely on any player choosing a strategy at random. Rather, we interpret player j's mixed strategy as a statement of player i's uncertainty about player j's choice of a (pure) strategy. In other words, the idea is to endow player j with a small amount of private information such that, depending on the realization of the private information, player j slightly prefers one of the relevant pure strategies. However, since player i does not observe j's private information, i remains uncertain about j's choice, and we represent i's uncertainty by j's mixed strategy (for further details see Gibbons (1992)).

Remark 49. *To check the extended definition of Nash equilibrium applied to the Battle of the Sexes see Gibbons (1992).*

In any game, a Nash equilibrium (involving pure or mixed strategies) appears as an intersection of the players' best-response correspondences, even when there are more than two players, and even when some or all of the players have more than two pure strategies. Unfortunately, the only games in which the players' best-response correspondences have simple graphical rep-

resentations are the two-player games in which each player has only two strategies.
We conclude this section with a discussion of the existence of a Nash equilibrium in more general games:

Theorem C.3.1 (Nash (1950)). *In the* n*-player normal form game* $G = \{S_1, \ldots, S_n; u_1, \ldots, u_n\}$*, if both* n *and* S_i *are finite, for every* i*, then there exists at least one Nash equilibrium, possibly involving mixed strategies.*

Proof 10. *For the proof of the theorem see Nash (1950) and Gibbons (1992).*

Remark 50. *In the proof of the previous theorem there are some interesting results about the behavior of the best-response functions, but we will not present them because it is not the goal of this paper (for further details see Gibbons (1992)).*

Nash's theorem guarantees that an equilibrium exists in a broad class of games, but there are simple games (such as the practical applications presented in the list C.2 on Section C.2) that do not belong in this class of games, because, for example, they have an infinite strategy space. So, this shows that the hypotheses of Nash's theorem are sufficient but not necessary conditions for an equilibrium to exist. However, there are many games that do not satisfy the hypotheses of the theorem but nonetheless have one or more Nash equilibria.

C.4 Games Against Nature

For our purposes, we need to discuss a subclass of games known as Games Against Nature, which are two player games where one of the players is an "entity".
Usually, game theory is concerned with analyzing conflict situations where the participants of the game are conscious, rational entities seeking to reach some objective. However, very often one of the players cannot be regarded as a conscious individual having his own preferences and objectives. Thus the other players cannot count on its rational behavior. So, the participant representing the forces uncontrollable by the players usually is referred to as Nature. Therefore games of this type are called games against nature.
We are going to deal with only "two-person games" where Player 1 is called the Decision Maker

(DM) and Player 2 is the "entity" Nature. Both players are assumed to have finitely many pure strategies (m and n, respectively) and the $m \times n$ payoff matrix A is known. Of course, the payoff matrix is meaningful only for Player 1 and a_{ij} is assumed to represent the gain obtained by Player 1 if he/she applies his/hers i^{th} strategy while Nature is in state j, i.e., a_{ij} corresponds to $(i,j)^{th}$ entry of the A matrix.

Remark 51. *Nature's strategies are called states, because it cannot choose what strategy to play.*

It is our basic assumption that nothing is known about the probability distribution governing nature's "selection" of states. Statistical decision theory deals with decision making problems where these probabilities come into the picture (but for further details on statistical decision theory see, for example, Luce and Raiffa (1957) and/or Wald (1950)).

The main question here is how should a rational DM choose his/hers strategy in a game against nature? This question cannot be answered definitely, because it depends on the criteria the DM applies, on his attitude towards risk, on his ideas about gain and loss, etc. We mention four well-known and simple concepts for strategy selection.

- *Laplace's Criterion* says that all states of nature should be regarded as equally probable because nothing is known about the real probabilities. Thus the DM chooses strategy i^* if
$$\sum_{j=1}^{n} \frac{1}{n} a_{i^*j} \geq \frac{1}{n} \sum_{j=1}^{n} a_{ij}, \text{ for } i = 1, \ldots, m. \tag{C.11}$$
Among the severe conceptual deficiencies of the idea we only mention its sensitivity to the definition of possible states.
- *Wald's (max-min) Criterion* says that Nature is supposed to be acting against the DM who adopts a max-min strategy, i.e., an optimal strategy of Player 1 in a matrix game A against nature as Player 2.
There are situations where using this criterion is justified (e.g., choosing medication for someone whose disease is not exactly diagnosed) but in most cases the DM is willing to take some risk in the hope of increasing his gain.
Note that, with this criterion, we can find the optimal strategy for Player 1 by applying

the same methods, previously described, to find a Nash Equilibrium.

- *Max-max Criterion* says, opposing to Wald's criterion, that the DM should pick the matrix A's row where the maximal element is maximal, therefore the name max-max.
- *Hurwicz's Criterion*, which is a kind of combination of the max-min and max-max criteria, defines a number α between 0 and 1 called "the index of optimism", which is supposed to measure the attitude of the DM toward risk. If $\alpha = 1$, then the DM is most optimistic (i.e., we are considering the max-max criterion), and if $\alpha = 0$ then he is most pessimistic (i.e., we are considering Wald's criterion). Let s_i and S_i be the least and greatest elements, respectively, in row i of A and define

$$h_i = \alpha S_i + (1 - \alpha)s_i, \text{ for } i = 1, \dots, m.$$

  Then the DM chooses the strategy giving the greatest h_i.

  Hurwicz's criterion can be criticized in that the choice of α is very subjective. In addition it does not meet other "rationality" requirements in which a decision making criterion is supposed to satisfy (e.g., if $\alpha > 0$, then the convex combination of two optimal decisions is not necessarily optimal).
- *Savage's Criterion* proposes to set up a "regret matrix", R, where r_{ij} is the loss incurred when the DM chooses his/hers i^{th} strategy and nature is in state j. Then, in the matrix game R the DM should apply the max-min (Wald's) criterion.

With these concepts we can proceed to our models. But, for more details about this type of games see Szép and Forgó (1985) and/or Biswas (1997).

Appendix D

R Code and Functions

This appendix is only a display of the most important functions of the *R* software that were used to apply the theory. So, since it is only a "translation" of the given theory to the *R* language and also it is not the purpose of this paper to produce and/or explore the necessary coding for the theory, the functions will not be explored in great detail. However, the inputs and outputs of each function will be explained, and some relevant observations will be made along the codes.

D.1 Game Theory

To determine the speculator's optimal strategy given the market's (estimated or not) probabilities p_1 and p_2, the following *Estrategia* function was used:

```
Estrategia=function(last.price,class.range,p1,p2,lucro.range=3,M.range=3,m.range=2){


  w=last.price*(lucro.range*class.range/100)
  x=last.price*(m.range*class.range/100)
  y=last.price*(M.range*class.range/100)


  p1=round(p1,4)
  p2=round(p2,4)
  if(p1+p2>1 || p1<0 || p2<0){
    print("Invalid Parameters")
    print(paste(p1,p2))
    print(ficheiro)
    stop()
```

```
    return(-3)
  }
  if(w+x==0 || w+y==0){return(0)}
  if(p1+p2<w/(w+x) && (p1/p2<=(y-x)/(w+x) || p1==0)){
    return(-1)
  } else if(p2<w/(w+y) && (p1/p2>(y-x)/(w+x) || p2==0)) {
    return(+1)
  } else {
    return(0)
  }
}
```

Listing D.1: *Estrategia* Function

Here, the function's inputs are:

- *last.price* is the last recorded percentage change price of a financial asset, which, for our purposes, this input is the last observation of the transformed training set.
- *class.range* is the range of the classes which we are using to split the data, we defined this range as the standard deviation of the dataset α.
- *p1* and *p2* are the market's probabilities p_1 and p_2, respectively.
- *lucro.range*, *M.range* and *m.range* are the multipliers which define the Zero Adversity, Major Adversity and Minor Adversity thresholds in the division of the classes, respectively. However, since we defined these thresholds in a specific way (see Section 2.1 from Chapter 2), so these inputs are not necessary, but can be altered.

The function's only output is the optimal strategy that the speculator should play based on the given inputs (which is determined as it was described in Section 2.1 from Chapter 2). So, if the function's output is:

- 0, then "Do Not Play" is the speculator's obtained optimal strategy;
- -1, then "Less Risk" is the speculator's obtained optimal strategy;
- $+1$, then "More Risk" is the speculator's obtained optimal strategy;
- -3, then the input parameters were invalid and a optimal strategy could not be determined.

D.2 Markov Chains

To determine the market's probabilities utilizing the described Markov chains model, only standard *R* functions are used (such as *hist*, *matrix*, ...), so we will not present in order not to extend this paper. However, the code used is a simple implementation of the described theory. Thus we will only present the pseudo-code for this method:

```
Markov.Chains=function(data,range,iterations=20,nclasses=0,lucro.range=2,M.range=3,m.range
  =2,last.price){
  #determine the stratagies' thresholds
  last.price=dados[length(dados)]
  lucro=(1+lucro.range*range/100)*last.price
  rmenos=(1-m.range*range/100)*last.price
  rmais=(1-M.range*range/100)*last.price

  #determine the transition probability matrix and its classes considering the Less Risk
  strategy
  m.matrix #the transition matrix
  m.classes #the classes

  #determine the transition probability matrix and its classes considering the More Risk
  strategy
  M.matrix #the transition matrix
  M.classes #the classes

  #determine the 1 iteration probability distribution for each matrix
  m.dp1 #the distribution for the Less Risk chain
  M.dp1 #the distribution for the More Risk chain

  #use the previous probability distributions to determine the 1 iteration estimators
  p0.1=m.dp1[length(m.dp1)] #estimator for the probability of Zero Adversity
  p2.1=M.dp1[1] #estimator for the probability of Major Adversity
  p1.1=1-p0.1-p2.1 #estimator for the probability of Minor Adversity

  #determine the n iteration probability distribution for each matrix
  m.dpn #the distribution for the Less Risk chain
  M.dpn #the distribution for the More Risk chain

  #use the previous probability distributions to determine the n iteration estimators
```

```
  p0.n=m.dpn[length(m.dpn)] #estimator for the probability of Zero Adversity
  p2.n=M.dpn[1] #estimator for the probability of Major Adversity
  p1.n=1-p0.n-p2.n #estimator for the probability of Minor Adversity

  #determine thecpeculator's optimal strategy for each of the estimators
  prev.1=Estrategia(last.price,range,p1.1,p2.1,lucro.range,M.range,m.range) #optimal
  strategy for the 1 iteration estimator
  prev.n=Estrategia(last.price,range,p1.n,p2.n,lucro.range,M.range,m.range) #optimal
  strategy for the n iteration estimator

  return(list(prev.1,prev.n,p1.1,p2.1,p1.n,p2.n,m.matrix,M.matrix))
}
```

Listing D.2: Pseudo-code determine the market's probabilities with the Markov chains model

The inputs for this routine are:

- *data* is the transformed dataset.
- *range* is the range of the classes which we are using to split the data, we defined this range as the standard deviation of the dataset α.
- *iterations* is the number n of iterations that we want to consider for the long run estimator.
- *nclasses* is the number of extra intermediate classes that we want to add to the chains in addition to the ones that we previously defined.
- *lucro.range*, *M.range* and *m.range* are the parameters that define the Zero Adversity, Major Adversity and Minor Adversity thresholds, according to the presented theory.
- *last.price* is the last "raw" price of a financial asset, which, in our case, is the last observation of the "raw" training set.

And this function's outputs is a list which contains:

- *prev.1* and *prev.n* is the speculator's optimal strategy for this dataset, which was obtained using the *Estrategia* function, considering the one iteration estimators and the long-run estimators, respectively.
- *p1.1* and *p2.1* are the probabilities of Minor Adversity (p_1) and Major Adversity (p_2), respectively, which were obtained considering the one iteration estimators.
- *p1.n* and *p2.n* are the probabilities of Minor Adversity (p_1) and Major Adversity (p_2), respectively, which were obtained considering the long run iteration estimators.

- *m.matrix* and *M.matrix* are the transition matrices for the Markov chain considering the Less Risk strategy and the chain considering the More Risk strategy, respectively.

D.3 SARIMA

To determine the market's probabilities utilizing an estimated SARIMA model and its simulations, the following functions are used:

- the *auto.arima* function (from the **forecast**) package is used to find the optimal SARIMA model for a certain dataset utilizing the Hyndman-Khandakar (HK) algorithm described in Section A.5.2 from Appendix A, but for further details see Hyndman and Khandakar (2008);
- the *simulate* function (from the **stats**) package is used to make simulations given a previously defined model, and for further details see *simulate v3.6.2*;
- the *Estrategia* function previously described in D.1;
- the *Series* function described in D.3.

The method described to determine the market's probabilities using an estimated SARIMA model and its simulations is represented in the following code:

```
Series=function(data,range2,n.obs=200,n.sim=1000,lucro.range=2,M.range=3,m.range=2,last.
  price){
  lucro=(1+lucro.range*range/100)*last.price
  rmenos=(1-m.range*range/100)*last.price
  rmais=(1-m.range*range/100)*last.price

  #estimated model
  x2=last.price
  modelo.arima = auto.arima(data,stepwise = TRUE)

  #simulations
  n=0
  n.rmenos=0
  n.rmais=0
  n.lucro.m=0
  n.lucro.M=0
  n.lucro=0
```

```
s1=0
s2=0
while (n<n.sim) {
  print(n/n.sim)
  s1=c(s1,n)
  s2=s2+1
  if(sum(s1)==0 && s2==100){
    print("Low volatility, then range of the classes will be decreased.")
    range2=range2/2
    last.price=data[length(data)]
    lucro=(1+lucro.range*range2/100)*last.price
    rmenos=(1-m.range*range2/100)*last.price
    rmais=(1-M.range*range2/100)*last.price
    s1=0
    s2=0
  }

  sim=simulate(modelo.arima,n.obs)
  sim=Undo.dif.per(sim,x2)
  sim=sim[2:length(sim)]
  temp=sim
  l.index=which(sim>=lucro)[1]
  rM.index=which(sim<=rmais)

  temp = temp[!(temp %in% sim[rM.index])]
  rm.index=which(temp<=rmenos)[1]
  rm.index=which(sim==temp[rm.index])[1]
  rM.index=which(sim<=rmais)[1]

  if(is.na(l.index) && is.na(rM.index) && is.na(rm.index)){next}
  if(is.na(rM.index) && is.na(rm.index)){
    n.lucro=n.lucro+1
    n=n+1
    next
  }

  l.index=l.index[!is.na(l.index)]
  rm.index=rm.index[!is.na(rm.index)]
  rM.index=rM.index[!is.na(rM.index)]
```

```
    if(length(l.index)==0 || is.na(l.index)){
      l.index=length(sim)+1
    }
    if(length(rm.index)==0 || is.na(rm.index)){
      rm.index=length(sim)+1
    }
    if(length(rM.index)==0 || is.na(rM.index)){
      rM.index=length(sim)+1
    }

    indices=c(l.index,rm.index,rM.index)

    prim=which(indices==min(indices))
    temp=sort(indices)
    seg=which(indices==temp[2])

    if(prim==1){
      n.lucro=n.lucro+1
      n=n+1
      next
    } else if(prim==3 || (prim==2 && seg==3)){
      n.rmais=n.rmais+1
      n=n+1
      next
    } else if(prim==2 && seg==1){
      n.rmenos=n.rmenos+1
      n=n+1
      next
    } else {
      stop()
    }
  }

  #probabilities
  p2.st=n.rmais/n #major adversity probability
  p1.st=n.rmenos/n #minor adversity probability

  #choosing the optimal strategy
```

```
    if(length(p1.st)<=0 || length(p2.st)<=0){
      print(p1.st)
      print(p2.st)
      print("Model Error")
    }
    prev=Estrategia(last.price,range2,p1.st,p2.st,lucro.range,M.range,m.range)
    return(list(prev,p1.st,p2.st,range2))
  }
```

Listing D.3: Used function to determine the market's probabilities with the SARIMA model's estimation and simulations

The inputs for this routine are:

- *data* is the transformed dataset.
- *range2* is the range of the classes which we are using to split the data, we defined this range as the standard deviation of the dataset α.
- *n.obs* is the number of observations of each of the simulations.
- *n.sim* is the number of simulations.
- *lucro.range*, *M.range* and *m.range* are the parameters that define the Zero Adversity, Major Adversity and Minor Adversity thresholds, according to the presented theory.
- *last.price* is the last "raw" price of a financial asset, which, in our case, is the last observation of the "raw" training set.

And this function's outputs is a list which contains:

- *prev* is the speculator's optimal strategy for this dataset, which was obtained using the *Estrategia* function.
- *p1.st* and *p2.st* are the probabilities of Minor Adversity (p_1) and Major Adversity (p_2), respectively, which were obtained using the described method.
- *range2* is again the range of the classes used to split the data, and it is given as an output as well because this range may change (when the function is running) if the data has a low volatility.

D.4 GARCH

To determine the market's probabilities utilizing an estimated GARCH model and its simulations, the following functions are used:

- the *auto.arima* function (from the **forecast**) package is used to find the optimal SARIMA model for a certain dataset utilizing the Hyndman-Khandakar (HK) algorithm described in Section A.5.2 from Appendix A, but for further details see Hyndman and Khandakar (2008);
- the *simulate* function (from the **stats**) package is used to make simulations given a previously defined model, and for further details see *simulate v3.6.2*;
- the *ugarchspec* and *ugarchfit* functions (from the **rugarch** package) are used to find the optimal GARCH model for a certain dataset (and given an estimated SARIMA model) utilizing the algorithm described in Section A.5.2 from Appendix A;
- the *Estrategia* function previously described in D.1;
- the *Melhor.Garch* function described in D.4;
- the *Garch* function described in D.5.

To estimate the optimal GARCH model that fits a certain dataset (using the methods described in the theory), the following function is used:

```
Melhor.Garch=function(dados,gp.max=2,gq.max=2){
  #determine the optimal SARIMA model
  ARIMA=FALSE
  modelo.arima=auto.arima(dados)
  order=arimaorder(modelo.arima)
  p=order['p']
  d=order['d']
  q=order['q']
  inter=modelo.arima$coef['intercept']
  gp=1
  gq=1
  if(is.na(inter)){
    inter=FALSE
  } else {
    inter=TRUE
  }
```

```
#determine the optimal GARCH model
specs.garch=ugarchspec(variance.model = list(model='sGARCH',garchOrder=c(1,1)),
  mean.model = list(armaOrder=c(p,q),include.mean=inter), distribution.model = 'norm')
ok=c(p,q,1,1,10**10,inter,FALSE)

for(gp in 0:gp.max){
  for (gq in 0:gq.max) {
    specs.garch=ugarchspec(variance.model = list(model='sGARCH',garchOrder=c(gp,gq)),
      mean.model = list(armaOrder=c(p,q),include.mean=inter), distribution.model = 'norm')
    tryCatch({
      modelo.garch=withTimeout(ugarchfit(spec=specs.garch,data=dados),timeout = 240)
    }, error=function(e){})
    if(has_error(infocriteria(modelo.garch),silent = T)){
      next
    } else {
      aic=infocriteria(modelo.garch)
      aic=aic['Akaike',]
      if(aic<=ok[5]){
        ok=c(p,q,gp,gq,aic,inter,FALSE)
      } else {
        next
      }
    }
  }
}
specs.garch=ugarchspec(variance.model = list(model='sGARCH',garchOrder=c(ok[3],ok[4])),
  mean.model = list(armaOrder=c(ok[1],ok[2]),include.mean=ok[6]), distribution.model = '
norm')

tryCatch({
  modelo.garch=withTimeout(ugarchfit(spec=specs.garch,data=dados),timeout = 240)
}, error=function(e){})
if(has_error(infocriteria(modelo.garch),silent = T)){
  modelo.garch=modelo.arima
  aic=modelo.garch$aic
  ARIMA=TRUE
} else {
  aic=infocriteria(modelo.garch)
```

```
      aic=aic['Akaike',]
      ARIMA=FALSE
    }
    lista=list(ARIMA,modelo.garch,specs.garch)
    return(lista)
}
```

Listing D.4: Code to determine the optimal GARCH model for a certain dataset

The inputs for this routine are:

- *dados* is the dataset, which can be transformed or not.
- *gp.max* and *gq.max* are the maximum order $GARCH(p,q)$ that we will allow the function to reach its search.

And this function's output is *list* is a list that contains:

- *ARIMA* object which tells us if the found model is a pure SARIMA model or not.
- *modelo.garch* is the estimation of the (found) optimal GARCH model.
- *specs.garch* is the model's specifications, which allows to obtain the same estimation for the found model.

The method described to determine the market's probabilities using an estimated GARCH model and its simulations is represented in the following code:

```
Garch=function(data,range2,n.obs=200,n.sim=1000,gp.max=2,gq.max=2,lucro.range=2,M.range=3,m.
    range=2,last.price){
    lucro=(1+lucro.range*range/100)*last.price
    rmenos=(1-m.range*range/100)*last.price
    rmais=(1-M.range*range/100)*last.price

    x2=last.price
    modelo.arima=auto.arima(dados)
    order=arimaorder(modelo.arima)
    p=order['p']
    d=order['d']
    q=order['q']
    inter=modelo.arima$coef['intercept']

    if(d>=1){
    ARIMA=TRUE
```

```
modelo.garch=modelo.arima
} else {
  ARIMA=FALSE
  if(is.na(inter)){
    specs.garch=ugarchspec(variance.model = list(model='sGARCH',garchOrder=c(1,1)),
      mean.model = list(armaOrder=c(p,q),include.mean=FALSE), distribution.model = 'norm')
  } else {
    specs.garch=ugarchspec(variance.model = list(model='sGARCH',garchOrder=c(1,1)),
      mean.model = list(armaOrder=c(p,q)), distribution.model = 'norm')
  }
  modelo.garch=ugarchfit(spec=specs.garch,data=dados)
  if(has_error(infocriteria(modelo.garch),silent = T)){
    print("Model Error")
    stop()
  }
}

n=0
n.rmenos=0
n.rmais=0
n.lucro.m=0
n.lucro.M=0
n.lucro=0
s1=0
s2=0
while (n<n.sim) {
  s1=c(s1,n)
  s2=s2+1
  if(sum(s1)==0 && s2==100){
    print("Low volatility, the range of the classes will be decreased")
    range2=range2/2
    last.price=data[length(data)]
    lucro=(1+lucro.range*range2/100)*last.price
    rmenos=(1-m.range*range2/100)*last.price
    rmais=(1-M.range*range2/100)*last.price
    s1=0
    s2=0
  }
  if(isTRUE(ARIMA)){
```

```
    sim=simulate(modelo.garch,n.obs)
    sim=Undo.dif.per(sim,x2)
    sim=sim[2:length(sim)]
  } else {
    sim=ugarchsim(modelo.garch,n.sim = n.obs)
    sim=sim@simulation[["seriesSim"]]
    sim=Undo.dif.per(sim,x2)
    sim=sim[2:length(sim)]
  }

  temp=sim
  l.index=which(sim>=lucro)[1]
  rM.index=which(sim<=rmais)
  temp = temp[!(temp %in% sim[rM.index])]
  rm.index=which(temp<=rmenos)[1]
  rm.index=which(sim==temp[rm.index])[1]
  rM.index=which(sim<=rmais)[1]

  if(is.na(l.index) && is.na(rM.index) && is.na(rm.index)){next}
  if(is.na(rM.index) && is.na(rm.index)){
    n.lucro=n.lucro+1
    n=n+1
    next
  }

  l.index=l.index[!is.na(l.index)]
  rm.index=rm.index[!is.na(rm.index)]
  rM.index=rM.index[!is.na(rM.index)]

  if(length(l.index)==0 || is.na(l.index)){
    l.index=length(sim)+1
  }
  if(length(rm.index)==0 || is.na(rm.index)){
    rm.index=length(sim)+1
  }
  if(length(rM.index)==0 || is.na(rM.index)){
    rM.index=length(sim)+1
  }
```

```
      indices=c(l.index,rm.index,rM.index)

      prim=which(indices==min(indices))
      temp=sort(indices)
      seg=which(indices==temp[2])

      if(prim==1){
        n.lucro=n.lucro+1
        n=n+1
        next
      } else if(prim==3 || (prim==2 && seg==3)){
        n.rmais=n.rmais+1
        n=n+1
        next
      } else if(prim==2 && seg==1){
        n.rmenos=n.rmenos+1
        n=n+1
        next
      } else {
      stop()
      }
    }

    p2.st=n.rmais/n #prob de major adversity
    p1.st=n.rmenos/n #prob de minor adversity

    prev=Estrategia(last.price2,range2,p1.st,p2.st,lucro.range,M.range,m.range)
    return(list(prev,p1.st,p2.st,range2))
  }
```

Listing D.5: Code to determine the market's probabilities using the GARCH model's estimation and simulations

The inputs for this routine are:

- *data* is the transformed dataset.
- *range2* is the range of the classes which we are using to split the data, we defined this range as the standard deviation of the dataset α.
- *n.obs* is the number of observations of each of the simulations.

- *n.sim* is the number of simulations.
- *lucro.range*, *M.range* and *m.range* are the parameters that define the Zero Adversity, Major Adversity and Minor Adversity thresholds, according to the presented theory.
- *last.price* is the last "raw" price of a financial asset, which, in our case, is the last observation of the "raw" training set.

And this function's output is a list which contains:

- *prev* is the speculator's optimal strategy for this dataset, which was obtained using the *Estrategia* function.
- *p1.st* and *p2.st* are the probabilities of Minor Adversity (p_1) and Major Adversity (p_2), respectively, which were obtained using the described method.
- *range2* is again the range of the classes used to split the data, and it is given as an output as well because this range may change (when the function is running) if the data has a low volatility.

D.5 Strategy Accuracy

To check the accuracy of the speculator's obtained strategy the following function is used:

```
Resultado=function(data.trei,data.teste,range,estra,lucro.range=2,M.range=3,m.range=2){
  #define the thresholds
  last.price=data.trei[length(data.trei)]
  lucro=(1+lucro.range*range/100)*last.price
  rmenos=(1-m.range*range/100)*last.price
  rmais=(1-M.range*range/100)*last.price

  #check the accuracy
  i=-2
  j=1
  #if the optimal strategy is More Risk
  if(estra==1){
    lim.sup=lucro
    lim.inf=rmais
    while(i==-2){
      if(data.teste[j]>=lim.sup){
        i=1
```

```
        } else if(data.teste[j]<=lim.inf){
          i=-1
        } else {
          if(j<length(data.teste)){
            j=j+1
          } else {
            i=0
          }
        }
      }
#if the optimal strategy is Less Risk
} else if(estra==-1){
    lim.sup=lucro
    lim.inf=rmenos
    while(i==-2){
      if(data.teste[j]>=lim.sup){
        i=1
      } else if(data.teste[j]<=lim.inf){
        i=-1
      } else {
        if(j<length(data.teste)){
          j=j+1
        } else {
          i=0
        }
      }
    }
#if the optimal strategy is Do Not Play
} else if(estra==0){
    lim.sup=lucro
    lim.inf=rmais
    while(i==-2){
      if(data.teste[j]>=lim.sup){
        i=-1
      } else if(data.teste[j]<=lim.inf){
        i=1
      } else {
        if(j<length(data.teste)){
          j=j+1
```

```
          } else {
            i=0
          }
        }
      }
    }

    #result
    return(list(i,j))
}
```

Listing D.6: Code to check the accuracy of the speculator's obtained strategy

The inputs for this routine are:

- *data.trei* and *data.teste* which are the "raw" training and test sets, respectively.
- *range* is the range of the classes which we are using to split the data, we defined this range as the standard deviation of the dataset α.
- *estra* is the speculator's obtained optimal strategy, which was obtained using one of the models.
- *lucro.range*, *M.range* and *m.range* are the parameters that define the Zero Adversity, Major Adversity and Minor Adversity thresholds, according to the presented theory.

And this function's output is a list which contains:

- i which is the outcome of the accuracy check. And if this outcome is:
  - 1, if the optimal strategy was correctly chosen for the given sets, i.e., it yielded a profit for the speculator;
  - -1, if the optimal strategy was wrongly chosen for the given sets, i.e., it yielded a loss for the speculator;
  - 0, if neither of the thresholds was reached, i.e., the speculator did not get a profit nor a loss. This can happen if the data has low volatility, if the classes' range is to high or if we did not have enough data for the accuracy check.
- j which is the time (or game) where the asset's price reached a threshold.

Appendix E

Datasets and Tables

This appendix only contains the tables resulting from the application of the presented models and procedures, which were analyzed in Chapter 3. The used notation is:

- column "Dataset" refers to the data set we are dealing with;
- column "Standard Deviation" refers to the standard deviation of the transformed training set;
- column "Training Set" states the size of the initial training set;
- column "Test Set" states the size of the initial test set;
- columns named with "MC1" present the results related to the Markov chains model considering the one iteration estimator;
- columns named with "MCn" present the results related to the Markov chains model considering the long-run estimator;
- columns named with "SAR" present the results related to the SARIMA model;
- columns named with "GAR" present the results related to the GARCH model;
- columns named with "Accurate" present the percentage of the model's strategies which were accurate;
- columns named with "Inaccurate" present the percentage of the model's strategies which were inaccurate;
- columns named with "Null" present the percentage of the model's strategies which were null;
- columns named with "Time" present the model's average time to reach a threshold;
- columns named with "LR" present the percentage of the model's strategies which were

"Play Less Risk";

- columns named with "LMR" present the percentage of the model's strategies which were "Play More Risk";
- columns named with "Profit" present profit obtained following the model's strategies;
- columns named with two models present the percentage of strategies which were equal.

Although we referenced percentages, to obtain the actual percentages we need to multiply the tables' values by 100%.

E.1 Controlled Datasets

The resulting tables for the controlled datasets are:

Dataset	Standard Deviation	Training Set	Test Set
Dataset 1	3.001877	800	200
Dataset 2	8.81E-14	800	200
Dataset 3	3.001877	800	200
Dataset 4	9.79E-14	800	200

Table E.1: Models' "inputs" for each of the datasets.

Dataset	MC1.Accurate	MC1.Null	MC1.Inaccurate	MCn.Accurate	MCn.Null	MCn.Inaccurate
Dataset 1	0	0.005	0.995	0.995	0.005	0
Dataset 2	1	0	0	1	0	0
Dataset 3	0.985	0.015	0	0.985	0.015	0
Dataset 4	1	0	0	1	0	0

Table E.2: The Markov chains models' accuracy results for each of the datasets.

Dataset	SAR.Accurate	SAR.Null	SAR.Inaccurate	GAR.Accurate	GAR.Null	GAR.Inaccurate
Dataset 1	0.995	0.005	0	0.995	0.005	0
Dataset 2	1	0	0	1	0	0
Dataset 3	0.985	0.015	0	0.985	0.015	0
Dataset 4	1	0	0	1	0	0

Table E.3: The Time Series models' accuracy results for each of the datasets.

Dataset	MC1.MCn	MC1.SAR	MC1.GAR	MCn.SAR	MCn.GAR	SAR.GAR
Dataset 1	0	0	0	1	1	1
Dataset 2	1	1	1	1	1	1
Dataset 3	0	0	0	1	1	1
Dataset 4	1	1	1	1	1	1

Table E.4: The percentage of coinciding strategies.

Dataset	Time.MC1	Time.MCn	Time.SAR	Time.GAR
Dataset 1	1.495	1.495	1.495	1.495
Dataset 2	1	1	1	1
Dataset 3	3.475	3.475	3.475	3.475
Dataset 4	1	1	1	1

Table E.5: Each of the models' average time results for each of the datasets.

Dataset	MC1.LR	MC1.MR	MCn.LR	MCn.MR	SAR.LR	SAR.MR	GAR.LR	GAR.MR
Dataset 1	0	1	0	0	0	0	0	0
Dataset 2	0	0	0	0	0	0	0	0
Dataset 3	0	1	1	0	1	0	1	0
Dataset 4	1	0	1	0	1	0	1	0

Table E.6: Each of the models' obtained strategies for each of the datasets.

Dataset	**MC1.Profit**	**MCn.Profit**	**SAR.Profit**	**GAR.Profit**
Dataset 1	-1.13E-08	0	0	0
Dataset 2	0	0	0	0
Dataset 3	3.47E+15	3.47E+15	3.47E+15	3.47E+15
Dataset 4	641.4531	641.4531	641.4531	641.4531

Table E.7: Each of the models' resulting profits for each of the datasets.

E.2 Daily Datasets

The resulting tables for the daily closing data are:

Dataset	**Standard Deviation**	**Training Set**	**Test Set**
A	1.134719	800	200
AAP	1.041148	800	200
AAPL	0.766705	800	200
ABT	0.611033	800	200
AC	0.953992	800	200
ADM	0.630121	800	200
ADS	0.792818	800	200
ADSK	1.068369	800	200
AGRO	0.957508	800	200
AGTC	2.220972	800	200
AHT	0.892525	800	200
AMD	2.17187	800	200
APA	1.165366	800	200
AUTO	0.824818	800	200
AVV	1.14684	800	200
AZN	0.73209	800	200
BAC	0.769647	800	200
BBBY	1.354673	800	200
BCP	1.487428	800	200
BLL	1.10205	800	200
CCL	0.73268	800	200

Dataset	MC1.Accurate	MC1.Null	MC1.Inaccurate	MCn.Accurate	MCn.Null	MCn.Inaccurate
A	0.425	0	0.575	0.425	0	0.575
AAP	0.48	0.005	0.515	0.48	0.005	0.515
AAPL	0.665	0.005	0.33	0.665	0.005	0.33
ABT	0.425	0.025	0.55	0.425	0.025	0.55
AC	0.535	0.025	0.44	0.535	0.025	0.44
ADM	0.445	0	0.555	0.445	0	0.555
ADS	0.565	0	0.435	0.565	0	0.435
ADSK	0.525	0	0.475	0.525	0	0.475
AGRO	0.45	0.01	0.54	0.45	0.01	0.54
AGTC	0.415	0.025	0.56	0.4	0.025	0.575
AHT	0.475	0	0.525	0.475	0	0.525
AMD	0.53	0.045	0.425	0.53	0.045	0.425
APA	0.44	0	0.56	0.57	0	0.43
AUTO	0.44	0.005	0.555	0.44	0.005	0.555
AVV	0.56	0.01	0.43	0.56	0.01	0.43
AZN	0.545	0.01	0.445	0.545	0.01	0.445
BAC	0.475	0	0.525	0.475	0	0.525
BBBY	0.44	0.005	0.555	0.495	0.005	0.5
BCP	0.46	0.005	0.535	0.46	0.005	0.535
BLL	0.44	0.02	0.54	0.44	0.02	0.54
CCL	0.39	0	0.61	0.39	0	0.61
CHK	0.38	0.005	0.615	0.575	0.005	0.42
CNA	0.425	0	0.575	0.575	0	0.425
COST	0.47	0.025	0.505	0.47	0.025	0.505
CRDA	0.54	0	0.46	0.54	0	0.46
CTXS	0.645	0	0.355	0.645	0	0.355
DGE	0.485	0.005	0.51	0.485	0.005	0.51
EBAY	0.49	0.005	0.505	0.49	0.005	0.505
ECL	0.475	0.01	0.515	0.475	0.01	0.515
EDP	0.655	0.005	0.34	0.655	0.005	0.34
ETN	0.55	0.005	0.445	0.55	0.005	0.445
EVR	0.41	0.015	0.575	0.41	0.015	0.575
EXPN	0.535	0.005	0.46	0.535	0.005	0.46
EZJ	0.445	0.005	0.55	0.445	0.005	0.55
FB	0.475	0.055	0.47	0.475	0.055	0.47
FCX	0.46	0.015	0.525	0.46	0.015	0.525
FERG	0.535	0.01	0.455	0.535	0.01	0.455
FEYE	0.51	0.005	0.485	0.51	0.005	0.485
GALP	0.4	0.005	0.595	0.4	0.005	0.595

Dataset	SAR.Accurate	SAR.Null	SAR.Inaccurate	GAR.Accurate	GAR.Null	GAR.Inaccurate
A	0.415	0	0.585	0.425	0	0.575
AAP	0.475	0.005	0.52	0.505	0.005	0.49
AAPL	0.665	0.005	0.33	0.665	0.005	0.33
ABT	0.41	0.025	0.565	0.425	0.025	0.55
AC	0.515	0.025	0.46	0.535	0.025	0.44
ADM	0.495	0	0.505	0.46	0	0.54
ADS	0.565	0	0.435	0.565	0	0.435
ADSK	0.525	0	0.475	0.525	0	0.475
AGRO	0.405	0.01	0.585	0.46	0.01	0.53
AGTC	0.41	0.025	0.565	0.45	0.025	0.525
AHT	0.41	0	0.59	0.475	0	0.525
AMD	0.53	0.045	0.425	0.53	0.045	0.425
APA	0.435	0	0.565	0.42	0	0.58
AUTO	0.4	0.005	0.595	0.43	0.005	0.565
AVV	0.555	0.01	0.435	0.56	0.01	0.43
AZN	0.515	0.01	0.475	0.54	0.01	0.45
BAC	0.495	0	0.505	0.47	0	0.53
BBBY	0.385	0.005	0.61	0.425	0.005	0.57
BCP	0.455	0.005	0.54	0.45	0.005	0.545
BLL	0.44	0.02	0.54	0.425	0.02	0.555
CCL	0.415	0	0.585	0.4	0	0.6
CHK	0.44	0.005	0.555	0.4	0.005	0.595
CNA	0.43	0	0.57	0.435	0	0.565
COST	0.485	0.025	0.49	0.47	0.025	0.505
CRDA	0.535	0	0.465	0.54	0	0.46
CTXS	0.605	0	0.395	0.64	0	0.36
DGE	0.48	0.005	0.515	0.495	0.005	0.5
EBAY	0.505	0.005	0.49	0.49	0.005	0.505
ECL	0.495	0.01	0.495	0.47	0.01	0.52
EDP	0.62	0.005	0.375	0.655	0.005	0.34
ETN	0.555	0.005	0.44	0.54	0.005	0.455
EVR	0.38	0.015	0.605	0.375	0.015	0.61
EXPN	0.485	0.005	0.51	0.535	0.005	0.46
EZJ	0.435	0.005	0.56	0.455	0.005	0.54
FB	0.45	0.055	0.495	0.465	0.055	0.48
FCX	0.455	0.015	0.53	0.465	0.015	0.52
FERG	0.54	0.01	0.45	0.53	0.01	0.46
FEYE	0.465	0.005	0.53	0.52	0.005	0.475
GALP	0.385	0.005	0.61	0.395	0.005	0.6

Dataset	MC1.MCn	MC1.SAR	MC1.GAR	MCn.SAR	MCn.GAR	SAR.GAR
A	1	0	0	0	0	0.8
AAP	1	0	0	0	0	0.79
AAPL	1	0	0	0	0	0.97
ABT	1	0	0	0	0	0.925
AC	1	0	0	0	0	0.89
ADM	1	0	0	0	0	0.725
ADS	1	0	0	0	0	1
ADSK	1	0	0	0	0	1
AGRO	0.66	0	0	0.11	0.005	0.78
AGTC	0.33	0	0	0.14	0.06	0.75
AHT	1	0	0	0	0	0.855
AMD	1	0	0	0	0	1
APA	0.02	0	0	0.125	0.03	0.855
AUTO	1	0	0	0	0	0.805
AVV	1	0	0	0	0	0.995
AZN	1	0	0	0	0	0.915
BAC	1	0	0	0	0	0.875
BBBY	0.06	0	0	0.185	0.015	0.8
BCP	1	0	0	0	0	0.78
BLL	1	0	0	0	0	0.845
CCL	1	0	0	0	0	0.965
CHK	0.05	0	0	0.13	0.055	0.9
CNA	0	0	0	0.665	0.42	0.655
COST	1	0	0	0	0	0.985
CRDA	1	0	0	0	0	0.995
CTXS	1	0	0	0	0	0.775
DGE	1	0	0	0	0	0.855
EBAY	1	0	0	0	0	0.945
ECL	1	0	0	0	0	0.83
EDP	1	0	0	0	0	0.875
ETN	1	0	0	0	0	0.755
EVR	1	0	0	0	0	0.985

Dataset	Time.MC1	Time.MCn	Time.SAR	Time.GAR
A	2.745	2.745	2.745	2.745
AAP	4	4	4	4
AAPL	3.735	3.735	3.735	3.735
ABT	3.565	3.565	3.565	3.565
AC	3.33	3.33	3.33	3.33
ADM	4.25	4.25	4.25	4.25
ADS	4.71	4.71	4.71	4.71
ADSK	5.25	5.25	5.25	5.25
AGRO	2.775	2.775	2.775	2.775
AGTC	5.485	5.485	5.485	5.485
AHT	3.84	3.84	3.84	3.84
AMD	7.37	7.37	7.37	7.37
APA	2.515	2.515	2.515	2.515
AUTO	3.36	3.36	3.36	3.36
AVV	4.895	4.895	4.895	4.895
AZN	3.58	3.58	3.58	3.58
BAC	4.22	4.22	4.22	4.22
BBBY	2.16	2.16	2.16	2.16
BCP	7.345	7.345	7.345	7.345
BLL	5.17	5.17	5.17	5.17
CCL	2.435	2.435	2.435	2.435
CHK	2.225	2.225	2.225	2.225
CNA	2.765	2.765	2.765	2.765

Dataset	MC1.LR	MC1.MR	MCn.LR	MCn.MR	SAR.LR	SAR.MR	GAR.LR	GAR.MR
A	0	1	0	1	0.76	0	0.89	0
AAP	0	1	0	1	0.845	0	0.925	0
AAPL	0	1	0	1	0.97	0	1	0
ABT	0	1	0	1	0.925	0	1	0
AC	0	1	0	1	0.89	0	1	0
ADM	0	1	0	1	0.73	0	0.965	0
ADS	0	1	0	1	1	0	1	0
ADSK	0	1	0	1	1	0	1	0
AGRO	0	1	0	0.66	0.795	0	0.985	0
AGTC	0	1	0	0.33	0.815	0	0.915	0
AHT	0	1	0	1	0.855	0	0.98	0
AMD	0	1	0	1	1	0	1	0
APA	0	1	0	0.02	0.875	0	0.97	0
AUTO	0	1	0	1	0.805	0	0.96	0
AVV	0	1	0	1	0.995	0	1	0
AZN	0	1	0	1	0.93	0	0.985	0
BAC	0	1	0	1	0.82	0	0.945	0
BBBY	0	1	0	0.06	0.815	0	0.985	0
BCP	0	1	0	1	0.81	0	0.95	0
BLL	0	1	0	1	0.99	0	0.855	0
CCL	0	1	0	1	0.765	0	0.78	0
CHK	0	1	0	0.05	0.87	0	0.94	0
CNA	0	1	0	0	0.335	0	0.58	0
COST	0	1	0	1	0.985	0	1	0
CRDA	0	1	0	1	0.995	0	1	0
CTXS	0	1	0	1	0.79	0	0.985	0
DGE	0	1	0	1	0.845	0	0.97	0
EBAY	0	1	0	1	0.975	0	0.97	0
ECL	0	1	0	1	0.825	0	0.995	0
EDP	0	1	0	1	0.905	0	0.96	0
ETN	0	1	0	1	0.765	0	0.97	0
EVR	0	1	0	1	0.69	0	0.705	0
EXPN	0	1	0.01	0.99	0.94	0	1	0
EZJ	0	1	0	1	0.97	0	0.96	0
FB	0	1	0	1	0.97	0	0.99	0
FCX	0	1	0	1	0.945	0	0.965	0
FERG	0	1	0	1	0.905	0	0.985	0

Dataset	MC1.Profit	MCn.Profit	SAR.Profit	GAR.Profit
A	2.819323	2.819323	0.259453	4.332698
AAP	92.58677	92.58677	71.56485	102.6871
AAPL	526.1841	526.1841	521.4618	526.1841
ABT	15.21986	15.21986	8.171348	15.21986
AC	201.2238	201.2238	171.9453	201.2238
ADM	552.8049	552.8049	527.3544	612.5782
ADS	342.3404	342.3404	342.3404	342.3404
ADSK	225.9676	225.9676	225.9676	225.9676
AGRO	2.826181	1.028172	0.089719	3.101177
AGTC	1.258847	-4.3973	-0.93481	2.231855
AHT	1378.759	1378.759	576.5388	1377.929
AMD	111.4884	111.4884	111.4884	111.4884
APA	-5.30463	0.538399	-6.31652	-8.52792
AUTO	117.9251	117.9251	-21.8288	78.89634
AVV	7560.404	7560.404	7459.321	7560.404
AZN	8083.97	8083.97	6936.009	7863.37
BAC	13.63498	13.63498	12.87642	12.4478
BBBY	2.811963	-4.85908	-2.61811	1.59865
BCP	0.179942	0.179942	0.111975	0.150418
BLL	26.24047	26.24047	25.65617	10.92661
CCL	-693.513	-693.513	-773.382	-865.054
CHK	-4.85193	-0.2716	-4.61273	-4.853
CNA	5.078159	0	-31.2617	-15.3869
COST	133.2965	133.2965	144.1551	133.2965

E.3 Intraday Datasets

The resulting tables for the intraday closing price data are:

Dataset	Standard Deviation	Training Set	Test Set
A	0.140081	800	200
AAP	0.107842	800	200
AAPL	0.026676	800	200
ABT	0.036912	800	200
AC	0.063778	800	200
ADM	0.061712	800	200
ADS	0.210131	800	200
ADSK	0.133863	800	200
AGRO	0.257083	800	200
AGTC	0.18214	800	200
AHT	0.061699	800	200
AMD	0.135243	800	200
APA	0.239824	800	200
AUTO	0.052282	800	200
AVV	0.095003	800	200
AZN	0.042593	800	200
BAC	0.087655	800	200
BBBY	0.133421	800	200
BCP	0.36024	800	200
BLL	0.049984	800	200
CCL	0.134498	800	200

Dataset	MC1.Accurate	MC1.Null	MC1.Inaccurate	MCn.Accurate	MCn.Null	MCn.Inaccurate
A	0.41	0.025	0.565	0.46	0.025	0.515
AAP	0.305	0.015	0.68	0.305	0.015	0.68
AAPL	0.5	0	0.5	0.5	0	0.5
ABT	0.3	0.03	0.67	0.3	0.03	0.67
AC	0.36	0.045	0.595	0.36	0.045	0.595
ADM	0.3	0.07	0.63	0.3	0.07	0.63
ADS	0.335	0.04	0.625	0.405	0.04	0.555
ADSK	0.485	0	0.515	0.485	0	0.515
AGRO	0.525	0.07	0.405	0.405	0.07	0.525
AGTC	0.265	0.045	0.69	0.265	0.045	0.69
AHT	0.62	0.03	0.35	0.62	0.03	0.35
AMD	0.325	0.025	0.65	0.65	0.025	0.325
APA	0.51	0.035	0.455	0.51	0.035	0.455
AUTO	0.39	0.02	0.59	0.39	0.02	0.59
AVV	0.465	0.03	0.505	0.465	0.03	0.505
AZN	0.525	0.015	0.46	0.46	0.015	0.525
BAC	0.22	0.01	0.77	0.77	0.01	0.22
BBBY	0.385	0.035	0.58	0.385	0.035	0.58
BCP	0.285	0	0.715	0.285	0	0.715
BLL	0.52	0.025	0.455	0.52	0.025	0.455
CCL	0.375	0.025	0.6	0.54	0.025	0.435
CHK	0.415	0.01	0.575	0.575	0.01	0.415
CNA	0.39	0.02	0.59	0.29	0.02	0.69
COST	0.57	0.015	0.415	0.475	0.015	0.51
CRDA	0.415	0.065	0.52	0.335	0.065	0.6
CTXS	0.44	0.005	0.555	0.44	0.005	0.555
DGE	0.345	0.005	0.65	0.345	0.005	0.65
EBAY	0.59	0	0.41	0.59	0	0.41
ECL	0.29	0.02	0.69	0.29	0.02	0.69
EDP	0.445	0.005	0.55	0.445	0.005	0.55
ETN	0.465	0.025	0.51	0.465	0.025	0.51
EVR	0.53	0.055	0.415	0.53	0.055	0.415
EXPN	0.42	0.05	0.53	0.42	0.05	0.53
EZJ	0.32	0.02	0.66	0.65	0.02	0.33
FB	0.455	0	0.545	0.455	0	0.545
FCX	0.21	0.005	0.785	0.785	0.005	0.21
FERG	0.36	0.04	0.6	0.35	0.04	0.61
FEYE	0.24	0.025	0.735	0.24	0.025	0.735
GALP	0.35	0.08	0.57	0.35	0.08	0.57

Dataset	SAR.Accurate	SAR.Null	SAR.Inaccurate	GAR.Accurate	GAR.Null	GAR.Inaccurate
A	0.36	0.025	0.615	0.41	0.025	0.565
AAP	0.315	0.015	0.67	0.275	0.015	0.71
AAPL	0.42	0	0.58	0.5	0	0.5
ABT	0.395	0.03	0.575	0.305	0.03	0.665
AC	0.385	0.045	0.57	0.375	0.045	0.58
ADM	0.405	0.07	0.525	0.485	0.07	0.445
ADS	0.35	0.04	0.61	0.365	0.04	0.595
ADSK	0.51	0	0.49	0.465	0	0.535
AGRO	0.375	0.07	0.555	0.53	0.07	0.4
AGTC	0.5	0.045	0.455	0.365	0.045	0.59
AHT	0.6	0.03	0.37	0.575	0.03	0.395
AMD	0.32	0.025	0.655	0.415	0.025	0.56
APA	0.51	0.035	0.455	0.51	0.035	0.455
AUTO	0.465	0.02	0.515	0.405	0.02	0.575
AVV	0.47	0.03	0.5	0.45	0.03	0.52
AZN	0.515	0.015	0.47	0.525	0.015	0.46
BAC	0.32	0.01	0.67	0.355	0.01	0.635
BBBY	0.46	0.035	0.505	0.395	0.035	0.57
BCP	0.74	0	0.26	0.285	0	0.715
BLL	0.505	0.025	0.47	0.52	0.025	0.455
CCL	0.395	0.025	0.58	0.375	0.025	0.6
CHK	0.4	0.01	0.59	0.525	0.01	0.465
CNA	0.395	0.02	0.585	0.395	0.02	0.585
COST	0.51	0.015	0.475	0.555	0.015	0.43
CRDA	0.445	0.065	0.49	0.4	0.065	0.535
CTXS	0.445	0.005	0.55	0.44	0.005	0.555
DGE	0.37	0.005	0.625	0.345	0.005	0.65
EBAY	0.595	0	0.405	0.59	0	0.41
ECL	0.36	0.02	0.62	0.29	0.02	0.69
EDP	0.4	0.005	0.595	0.405	0.005	0.59
ETN	0.485	0.025	0.49	0.47	0.025	0.505
EVR	0.505	0.055	0.44	0.435	0.055	0.51
EXPN	0.415	0.05	0.535	0.41	0.05	0.54
EZJ	0.38	0.02	0.6	0.35	0.02	0.63
FB	0.455	0	0.545	0.455	0	0.545
FCX	0.27	0.005	0.725	0.315	0.005	0.68
FERG	0.38	0.04	0.58	0.36	0.04	0.6
FEYE	0.255	0.025	0.72	0.24	0.025	0.735

Dataset	MC1.MCn	MC1.SAR	MC1.GAR	MCn.SAR	MCn.GAR	SAR.GAR
A	0.195	0	0	0.53	0	0.47
AAP	1	0	0	0	0	0.91
AAPL	1	0	0	0	0	0.75
ABT	1	0	0	0	0	0.695
AC	1	0	0	0	0	0.965
ADM	1	0	0	0	0	0.535
ADS	0.63	0	0	0.03	0.015	0.915
ADSK	1	0	0	0	0	0.855
AGRO	0	0	0	0.635	0.135	0.49
AGTC	1	0	0	0	0	0.56
AHT	1	0	0	0	0	0.845
AMD	0	0	0	0.035	0.14	0.835
APA	1	0	0	0	0	1
AUTO	1	0	0	0	0	0.72
AVV	1	0	0	0	0	0.66
AZN	0	0	0	0.165	0	0.835
BAC	0	0	0	0.15	0.205	0.835
BBBY	1	0	0	0	0	0.72
BCP	1	0	0	0	0	0.385
BLL	1	0	0	0	0	0.955
CCL	0.515	0	0	0.06	0	0.81
CHK	0.05	0	0	0.365	0.405	0.435
CNA	0.72	0	0	0.04	0	0.86
COST	0.655	0	0	0.115	0	0.855
CRDA	0.8	0	0	0.025	0.035	0.62
CTXS	1	0	0	0	0	0.995
DGE	1	0	0	0	0	0.965
EBAY	1	0	0	0	0	0.995
ECL	1	0	0	0	0	0.905
EDP	1	0	0	0	0	0.965
ETN	1	0	0	0	0	0.855
EVR	1	0	0	0	0	0.405

Dataset	Time.MC1	Time.MCn	Time.SAR	Time.GAR
A	5.85	5.85	5.85	5.85
AAP	6.77	6.77	6.77	6.77
AAPL	5.23	5.23	5.23	5.23
ABT	8.095	8.095	8.095	8.095
AC	5.335	5.335	5.335	5.335
ADM	7.19	7.19	7.19	7.19
ADS	4.195	4.195	4.195	4.195
ADSK	5.44	5.44	5.44	5.44
AGRO	5.78	5.78	5.78	5.78
AGTC	11.16	11.16	11.16	11.16
AHT	7.275	7.275	7.275	7.275
AMD	5.385	5.385	5.385	5.385
APA	10.845	10.845	10.845	10.845
AUTO	6.705	6.705	6.705	6.705
AVV	10.285	10.285	10.285	10.285
AZN	6.065	6.065	6.065	6.065
BAC	5.12	5.12	5.12	5.12
BBBY	6.28	6.28	6.28	6.28
BCP	8.23	8.23	8.23	8.23
BLL	9.52	9.52	9.52	9.52
CCL	5.475	5.475	5.475	5.475
CHK	8.175	8.175	8.175	8.175
CNA	9.14	9.14	9.14	9.14

Dataset	MC1.LR	MC1.MR	MCn.LR	MCn.MR	SAR.LR	SAR.MR	GAR.LR	GAR.MR
A	0	1	0	0.195	0.47	0	1	0
AAP	0	1	0	1	0.785	0	0.825	0
AAPL	0	1	0	1	0.75	0	1	0
ABT	0	1	0	1	0.71	0	0.985	0
AC	0	1	0	1	0.93	0	0.945	0
ADM	0	1	0	1	0.655	0	0.52	0
ADS	0	1	0	0.63	0.935	0	0.96	0
ADSK	0	1	0	1	0.875	0	0.97	0
AGRO	0	1	0	0	0.365	0	0.865	0
AGTC	0	1	0	1	0.6	0	0.83	0
AHT	0	1	0	1	0.88	0	0.955	0
AMD	0	1	0	0	0.965	0	0.86	0
APA	0	1	0	1	1	0	1	0
AUTO	0	1	0	1	0.755	0	0.895	0
AVV	0	1	0	1	0.755	0	0.865	0
AZN	0	1	0	0	0.835	0	1	0
BAC	0	1	0	0	0.85	0	0.795	0
BBBY	0	1	0	1	0.73	0	0.92	0
BCP	0	1	0	1	0.385	0	1	0
BLL	0	1	0	1	0.955	0	1	0
CCL	0	1	0	0.515	0.81	0	1	0
CHK	0	1	0	0.05	0.595	0	0.59	0
CNA	0	1	0	0.72	0.865	0	0.995	0
COST	0	1	0	0.655	0.785	0	0.9	0
CRDA	0	1	0	0.8	0.685	0	0.885	0
CTXS	0	1	0	1	0.995	0	1	0
DGE	0	1	0	1	0.975	0	0.99	0
EBAY	0	1	0	1	0.995	0	1	0
ECL	0	1	0	1	0.905	0	1	0
EDP	0	1	0	1	0.875	0	0.91	0
ETN	0	1	0	1	0.9	0	0.955	0
EVR	0	1	0	1	0.965	0	0.4	0
EXPN	0	1	0	1	0.985	0	0.99	0
EZJ	0	1	0	0.01	0.93	0	0.93	0
FB	0	1	0	1	0.97	0	0.98	0
FCX	0	1	0	0	0.865	0	0.84	0
FERG	0	1	0	0.97	0.825	0	1	0

Dataset	MC1.Profit	MCn.Profit	SAR.Profit	GAR.Profit
A	0.268886	-0.49585	-0.85414	0.268886
AAP	-12.5237	-12.5237	-14.5324	-16.763
AAPL	8.796666	8.796666	3.074455	8.796666
ABT	-2.98795	-2.98795	-2.27364	-2.95415
AC	-2.52993	-2.52993	-1.86534	-2.30646
ADM	-102.714	-102.714	-71.3646	-35.8532
ADS	-24.3726	-23.1046	-23.7397	-19.16
ADSK	20.85166	20.85166	20.86672	17.64757
AGRO	2.002062	0	0.21739	1.882981
AGTC	-1.11367	-1.11367	-0.37508	-0.80182
AHT	362.9774	362.9774	328.4228	320.5697
AMD	-3.99696	0	-4.3586	-2.10044
APA	4.644873	4.644873	4.644873	4.644873
AUTO	-0.62975	-0.62975	3.163704	-1.49857
AVV	293.062	293.062	213.0482	212.6501
AZN	482.3404	0	404.9148	482.3404
BAC	-4.60278	0	-3.67541	-3.36241
BBBY	-0.01713	-0.01713	0.090436	-0.04394
BCP	-0.06717	-0.06717	0.030168	-0.06717
BLL	4.596844	4.596844	4.172863	4.596844
CCL	-22.643	27.81635	-36.5024	-22.643
CHK	0.453549	0.157664	-1.38426	1.165271
CNA	-0.17616	-4.52818	-0.78003	-0.06313
COST	51.50693	28.17051	37.18558	46.49731

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